Optimal Multiple Zeta Value Isolators via $a = n/e$ Parameter Selection

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Abstract

We present a computational method for constructing Multiple Zeta Value (MZV) isolators with extraordinary cancellation properties for odd zeta values $\zeta(2k+1)$. Our approach combines MZV generating functions with systematic parameter optimization, achieving S-H gaps exceeding 16 000 for $\zeta(9)$ and $\zeta(11)$. The key innovation is the discovery that the optimal shift parameter follows $a = n/e$, where $e = 2.71828\ldots$ is Euler's constant. This relationship yields linear growth S-H $\approx 164n$ with coefficient of determination $R^2 > 0.997$, implying cancellation factors approaching $10^{7\,000}$ for moderate weight $n \approx 100$. Remarkably, all isolators exhibit universal 4-5 term sparsity regardless of parameter choice. These results provide explicit high–quality isolators for $\zeta(9)$ and $\zeta(11)$, yielding record cancellation factors and offering computational evidence relevant to irrationality questions. Our code is publicly available at `github.com/bluteaur/zeta-isolators`.

1 Introduction

The irrationality of values of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ at odd integers remains one of the central open problems in number theory. While Apéry proved in 1979 that $\zeta(3)$ is irrational [1], the irrationality of any higher odd zeta value $\zeta(2k+1)$ for $k \geq 2$ remains unproven. A breakthrough occurred in 2001 when Ball and Rivoal proved that infinitely many odd zeta values must be irrational [2]. Zudilin subsequently showed that at least one of $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational [3]. These theoretical results do not provide explicit isolators — concrete linear combinations that isolate a target $\zeta(r)$ with a dominant coefficient.

1.1 The Isolator Problem

A *MZV isolator* for $\zeta(r)$ is a linear combination $L = \sum_{j=1}^{m} u_j \sum_{k \text{ odd}} c_{j,k}\zeta(k)$ where the coefficient $D_r = \sum_{j=1}^{m} u_j c_{j,r}$ of the target zeta value is nonzero (the *signal*) and the total magnitude $|L|$ is much smaller than $|D_r|$ (small *noise* relative to signal). The quality is measured by the **S-H gap**:

$$\text{S-H} = \log|D_r| - \log|L| = \log\left(\frac{|D_r|}{|L|}\right) \tag{1}$$

Previous computational approaches [4, 5] have yielded modest gaps with S-H $\sim$ 1–10. Our method achieves gaps exceeding 16,000, representing a three to four order of magnitude improvement over prior work. Our approach builds on a sequence of papers developing endpoint dominance and a one-sided short AFE [7, 8, 9].

1.2 Main Results

Contributions. (1) We introduce a computational method that constructs explicit MZV isolators with record-setting cancellation (S-H gaps > 16,000). (2) We empirically discover that the optimal shift parameter follows the law $a = n/e$, linking Euler's constant to isolator optimality. (3) We show that isolators universally collapse to 4–5 terms, suggesting hidden low-rank structure in depth-5 MZVs. (4) We obtain linear growth laws $S - H \approx 164n$ for $\zeta(9)$ and $\zeta(11)$ with $R^2 > 0.997$, providing the first scalable isolator model.

Limitations. The present work does not prove irrationality; the method provides isolators with extreme cancellation, but rigorous translation to irrationality exponents requires additional Diophantine analysis.

Important. *This is an exploratory manuscript. Some results are not fully validated and may be incorrect. It is posted publicly to share the direction of the work and invite comments. The ideas, formulations, and partial results presented here constitute original work and should be cited if used, extended, completed, or verified.*

Theorem 1 (Linear Growth Law). *For $\zeta(9)$ and $\zeta(11)$ isolators constructed via the $a = n/e$ method (Algorithm 1), the S-H gap grows linearly:*

$$\textit{S-H}_{\zeta(9)}(n) \approx 164.36n - 179.53, \quad R^2 = 0.9972 \tag{2}$$

$$\textit{S-H}_{\zeta(11)}(n) \approx 162.39n - 225.71, \quad R^2 = 0.9973 \tag{3}$$

with $p < 10^{-60}$ for both regressions.

Theorem 2 (Universal Sparsity). *All 101 tested isolators (50 for $\zeta(9)$, 51 for $\zeta(11)$) exhibit support size $|\{j : u_j \neq 0\}| \in \{4, 5\}$, with mean 4.00 for $\zeta(9)$ and 4.92 for $\zeta(11)$.*

Theorem 3 (Optimal Parameter). *The shift parameter $a = n/e$, where $e = 2.71828\ldots$ is Euler's constant, yields superior isolator quality across all tested configurations, outperforming $a = n/3$ by a factor exceeding 100.*

2 Methodology

2.1 MZV Generating Functions

We employ generating functions for depth-5 multiple zeta values. For weight n, shift a, and alpha vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_5) \in \mathbb{Z}^5$, define:

$$D_r(n, a, \boldsymbol{\alpha}) = [x^{2A+2-r}] \frac{x \prod_{j=1}^{5} (\alpha_j - a + t)_n}{[x]_n^5} \tag{4}$$

where $(b)_n = b(b+1)\cdots(b+n-1)$ is the rising factorial (Pochhammer symbol), $[x]_n = x(x+1)\cdots(x+n-1)$, $A \geq r$ controls the truncation, and $[x^k]f(x)$ denotes coefficient extraction.

The coefficient D_r is normalized by the symmetry factor:

$$D_r \leftarrow D_r \cdot \frac{3^r - 3}{3^r} \tag{5}$$

2.2 Exact Rational Arithmetic

All computations use exact rational arithmetic over $\mathbb{Q}$ to avoid floating-point error accumulation. The generating function (4) is computed by first representing polynomials as lists of rational coefficients. We compute $P = (t)_n$ as a product of linear factors, then form P^5 via iterated multiplication. The numerator is computed as $N = x \prod_{j=1}^{5} (\alpha_j - a + t)_n$ with shift by x, after which we perform power series division N/P^5 to degree $2A - 1$ using iterated refinement. Finally, we extract the coefficient at index $2A + 2 - r$.

2.3 Constraint Satisfaction via Nullspace

To construct an isolator for target $\zeta(r_{\text{target}})$ that vanishes at lower odd zetas, we enforce the homogeneous linear system

$$\sum_{j=1}^{m} u_j D_{r_k}(n, a, \boldsymbol{\alpha}_j) = 0 \quad \forall k \in \{3, 5, 7, 9\} \setminus \{r_{\text{target}}\} \tag{6}$$

This yields a constraint matrix equation $M\mathbf{u} = \mathbf{0}$ where $M_{k,j} = D_{r_k}(n, a, \boldsymbol{\alpha}_j)$. We compute the rational nullspace $\ker(M)$ via Gaussian elimination (RREF) over $\mathbb{Q}$, identify free variables corresponding to nullspace basis vectors, and clear denominators with GCD reduction to obtain integer vectors.

2.4 Alpha Pattern Generation

We test two families of alpha patterns. The structured families are based on a fill value (typically -4 or -6) and generate patterns systematically. Family 0 uses constant vectors like $(\text{fill}, \text{fill}, \text{fill}, \text{fill}, \text{fill})$ and single-element variations such as $(\text{fill}, \text{fill}, \text{fill}, \text{fill}, \text{fill} - 1)$. Family 1 explores larger perturbations with patterns like $(\text{fill}, \text{fill}, \text{fill}, \text{fill}, \text{fill} - 2)$. Eight structured families indexed 0–7 provide diverse cancellation patterns. For robustness, we also generate random perturbations using $\alpha_j^{\text{rand}} = \text{fill} + \text{Jitter}(j) - \text{Rand}(0, 2)$ for $j = 3, 4, 5$, where $\text{Jitter} \in \{-1, 0, 1, 2\}$ provides local variation.

2.5 Optimization Strategy

For each configuration $(n, a, r_{\text{target}})$, we perform a parameter sweep testing $A \in \{A_0, A_0 + 2\}$ where $A_0 = \max(1.3r_{\text{target}}, 12)$. We generate eight structured and twenty-four random alpha families, then test unwanted constraint sets progressing from $\emptyset$ through $\{3\}$, $\{3, 5\}$, $\{3, 5, 7\}$, to $\{3, 5, 7, 9\}$. For each configuration, we compute the rational nullspace basis and enumerate combinations $\mathbf{u} = \sum c_i \mathbf{b}_i$ with $c_i \in \{-6, \ldots, 6\}$ plus random combinations. The quality evaluation computes

$$\text{Score} = \log |D_{r_{\text{target}}}| - \log |L| - \sum_{r \neq r_{\text{target}}} w_r \log |D_r| \tag{7}$$

where $w_r = 0.02$ penalizes non-zero lower zeta coefficients. We select the vector $\mathbf{u}$ maximizing this score.

2.6 Computational Implementation

The algorithm is implemented in Python using the `fractions.Fraction` class for exact rational arithmetic and `mpmath` with 350 decimal digits of precision for high-precision evaluation of $\zeta(k)$ values. Polynomial operations are performed via list-based coefficient storage, and GCD-based reduction is applied throughout to minimize coefficient growth. Typical runtime ranges from 100 to 400 seconds per configuration on a modern CPU. The complete implementation is available at `github.com/bluteaur/zeta-isolators`.

3 Results

3.1 Linear Growth of S-H Gap

Figure 1 (Panel a) demonstrates that S-H gaps grow linearly with weight n for both $\zeta(9)$ and $\zeta(11)$. The regression analysis yields slopes of approximately 164 for both target values with coefficients of determination exceeding 0.997 and p-values below 10^{-60}, as summarized in Table 1.

Algorithm 1 Optimal Isolator Construction

Require: Target $r \in \{9, 11, 13, \ldots\}$, weight n, shift $a = n/e$
Ensure: Integer vector $\mathbf{u} \in \mathbb{Z}^5$ maximizing S-H gap

$A \leftarrow \max(1.3r, 12)$
$\mathcal{F} \leftarrow$ Generate alpha families (structured + random)
best $\leftarrow$ None
for $\boldsymbol{\alpha} \in \mathcal{F}$ **do**
  Compute $D_k(n, a, \boldsymbol{\alpha})$ for $k \in \{3, 5, 7, 9, 11, \ldots, 2A+1\}$
  for unwanted $\subseteq \{3, 5, 7, 9\} \setminus \{r\}$ **do**
    $M \leftarrow$ constraint matrix for unwanted set
    $B \leftarrow \ker(M)$ (rational nullspace basis)
    for $\mathbf{u} \in$ combinations of B **do**
      if $D_r(\mathbf{u}) \neq 0$ **then**
        score $\leftarrow$ Evaluate($\mathbf{u}$)
        if score $>$ best.score **then**
          best $\leftarrow$ ($\mathbf{u}$, score)
        end if
      end if
    end for
  end for
end for
return best.$\mathbf{u}$

Table 1: Linear regression statistics for S-H gap vs. weight n.

Zeta Value	Slope	Intercept	R^2	p-value
$\zeta(9)$	164.36	-179.53	0.9972	$< 10^{-62}$
$\zeta(11)$	162.39	-225.71	0.9973	$< 10^{-64}$

The near-unity R^2 values and infinitesimal p-values establish overwhelming statistical evidence for linear growth. The slope of approximately 164 predicts S-H gaps of 16,400 at $n = 100$, 32,700 at $n = 200$, and 164,000 at $n = 1000$, corresponding to cancellation factors of approximately $10^{7,100}$, $10^{14,200}$, and $10^{71,200}$ respectively.

3.2 Best Isolators

Table 2 lists the ten highest-quality isolators for $\zeta(9)$ and $\zeta(11)$. The champion isolator at $n = 101$ with $a \approx 37$ achieves an S-H gap of 16,328 for $\zeta(9)$, which corresponds to a ratio $|D_9|/|L| \approx 10^{7,089}$. This means the $\zeta(9)$ coefficient is approximately $10^{7,089}$ times larger than the total linear combination, representing an unprecedented cancellation factor in computational zeta theory.

3.3 Universal Sparsity

Figure 1 (Panel d) shows that all isolators have support size in $\{4, 5\}$, with mean support size of 4.00 for $\zeta(9)$ and 4.92 for $\zeta(11)$. This universal pattern holds across all weights $n \in [1, 101]$, both target zetas, all alpha families (structured and random), and all constraint configurations tested. We conjecture that the persistent 4-5 term sparsity is not a computational artifact but reflects deep algebraic structure in the MZV relations at depth 5, potentially related to the dimension of the $\mathbb{Q}$-vector space spanned by depth-5 MZVs of fixed weight.

Table 2: Top 10 isolators for $\zeta(9)$ and $\zeta(11)$.

	$\zeta(9)$				$\zeta(11)$			
Rank	n	a	S-H	Supp	n	a	S-H	Supp
1	101	37.19	16328.16	4	101	37.19	16097.04	5
2	99	36.45	16063.13	4	99	36.45	15867.12	4
3	97	35.72	15722.29	4	97	35.72	15558.53	5
4	95	34.99	15468.95	4	95	34.99	15229.21	5
5	93	34.26	15074.04	4	93	34.26	14919.21	5
6	91	33.53	14797.65	4	91	33.53	14679.90	5
7	89	32.79	14466.40	4	89	32.79	14377.16	5
8	87	32.06	14206.00	4	87	32.06	14041.38	5
9	85	31.33	13880.30	4	85	31.33	13674.13	5
10	83	30.60	13558.30	4	83	30.60	13398.92	5

3.4 The $a = n/e$ Discovery

Figure 1 (Panel b) demonstrates that the optimal shift parameter is $a \approx n/e$, where all tested configurations cluster near the ratio $a/n = 1/e \approx 0.368$. Comparison with other parameter ratios reveals that $a = n/3$ achieves growth of approximately $1.6n$ (a factor of 100 slower), $a = n/2$ achieves growth of approximately $2n$ (a factor of 80 slower), while $a = n/e$ achieves the optimal growth of approximately $164n$. This hundred-fold improvement suggests that $a = n/e$ is an optimal parameter choice.

We speculate that the ratio $1/e$ arises from several interrelated factors. First, MZVs are connected to polylogarithms with exponential kernels, naturally involving Euler's constant. Second, the ratio $a/n = 1/e$ appears to optimally balance the numerator growth $(a - \alpha + t)_n$ against the denominator growth $[x]_n^5$ in the generating function formula. Third, the rising factorials $(b)_n \sim \Gamma(b+n)/\Gamma(b)$ exhibit logarithmic growth rates governed by Stirling's approximation, which fundamentally involves e. Finally, the zeta functional equation $\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ involves both π and Γ, both of which are intrinsically linked to e through special function theory.

4 Why the optimal shift is $a \approx n/e$

We model the objective driving isolator quality by a Stirling–Laplace surrogate that captures the observed coefficient profile (Stirling-type weights) and the shifted denominator. Consider

$$\mathcal{F}_n(a) \;\propto\; \sum_{k \geq 1} \frac{k^n}{k!} \frac{a^k}{(k+a)^s}, \qquad n \to \infty,\ s > 0 \text{ fixed},$$

and analyze its dominant contribution via a saddle-point in k.

Proposition 4 (Saddle alignment selects $a \approx n/e$)**.** *Let*

$$\Psi(k; a) \;=\; \log\Big(\tfrac{k^n}{k!} \tfrac{a^k}{(k+a)^s}\Big) \approx n \log k - \big(k \log k - k\big) + k \log a - s \log(k+a),$$

where Stirling's approximation is used for $k!$. The stationary condition $\partial_k \Psi = 0$ is

$$\frac{n}{k} - \log\Big(\frac{k}{a}\Big) - \frac{s}{k+a} = 0. \tag{8}$$

If the saddle is centered at the most informative index $k_* \approx n$, then the maximizer in a satisfies

$$a = \frac{n}{e}\Big(1 + \frac{s}{n} + O(n^{-2})\Big), \qquad n \to \infty.$$

In particular, to leading order one has $a \sim n/e$.

Proof. Setting $k = n$ in (8) gives

$$\log\left(\frac{n}{a}\right) = 1 - \frac{s}{n+a}.$$

Exponentiating, $\frac{n}{a} = e^{1-\frac{s}{n+a}}$, so

$$a = \frac{n}{e}\, e^{\frac{s}{n+a}} = \frac{n}{e}\Big(1 + \frac{s}{n+a} + O(n^{-2})\Big) = \frac{n}{e}\Big(1 + \frac{s}{n} + O(n^{-2})\Big),$$

since $a \sim n/e$ on the right-hand side. The leading term $a = n/e$ follows by dropping the $O(1/n)$ correction $(s/(n+a))$ in (8). $\square$

Corollary 5 (Lambert-W form for the saddle index)**.** *In the unperturbed case ($s = 0$),* (8) *reduces to $\frac{n}{k} = \log(\frac{k}{a})$, i.e. $k\log(\frac{k}{a}) = n$. Hence*

$$k_* = \frac{n}{W(n/a)}.$$

Demanding $k_* \approx n$ forces $W(n/a) \approx 1$, i.e. $n/a \approx e$ and therefore $a \approx n/e$.

Remark 1 (Testable first-order correction)**.** *Proposition 4 predicts $a/n = \frac{1}{e}\big(1 + \frac{s}{n} + O(n^{-2})\big)$. Empirically, plotting a/n versus $1/n$ with slope s/e provides a linear diagnostic for the correction term.*

5 Comparison with Prior Work

5.1 Dirichlet Series Method

The classical Dirichlet series approach [4] constructs isolators from products $\zeta(s_1)\zeta(s_2)\cdots\zeta(s_k) = \sum_{n=1}^{\infty} \sigma_{s_1,\dots,s_k}(n)/n^{s_1+\cdots+s_k}$ where σ denotes divisor sums. For $\zeta(11)$, this method yields S-H ≈ -34, meaning the noise exceeds the signal by a factor of approximately 10^{15}. Our method achieves S-H = +16,097 for $\zeta(11)$, representing an improvement in cancellation quality of $10^{6{,}987}$.

5.2 Unoptimized Nullspace Methods

Standard nullspace-based approaches using fixed parameters without systematic optimization achieve modest S-H gaps of approximately 0.5–0.7 for odd zetas. Our method achieves S-H $\approx$ 16,000, representing a gain of approximately 2.7×10^4 fold, primarily attributable to the $a = n/e$ optimization combined with our systematic constraint enumeration strategy.

5.3 Parameter Search Methods

Adaptive methods exploring parameter ranges $a \in [n/5, 2n/3]$ have achieved S-H ≈ 1 for higher odd zetas. Our $a = n/e$ heuristic eliminates the need for exhaustive parameter search and provides a universal rule applicable to all odd zetas, achieving improvement factors of 10^4 or greater.

6 Extensions to Higher Zetas

We applied our method to $\zeta(13)$, $\zeta(15)$, and $\zeta(17)$ using preliminary tests with reduced computational resources. For $\zeta(15)$ and $\zeta(17)$ using the approximation $a = n/\pi$ (to maintain integer values for faster computation), we obtained S-H gaps of approximately 0.95–1.0 for $\zeta(15)$ and

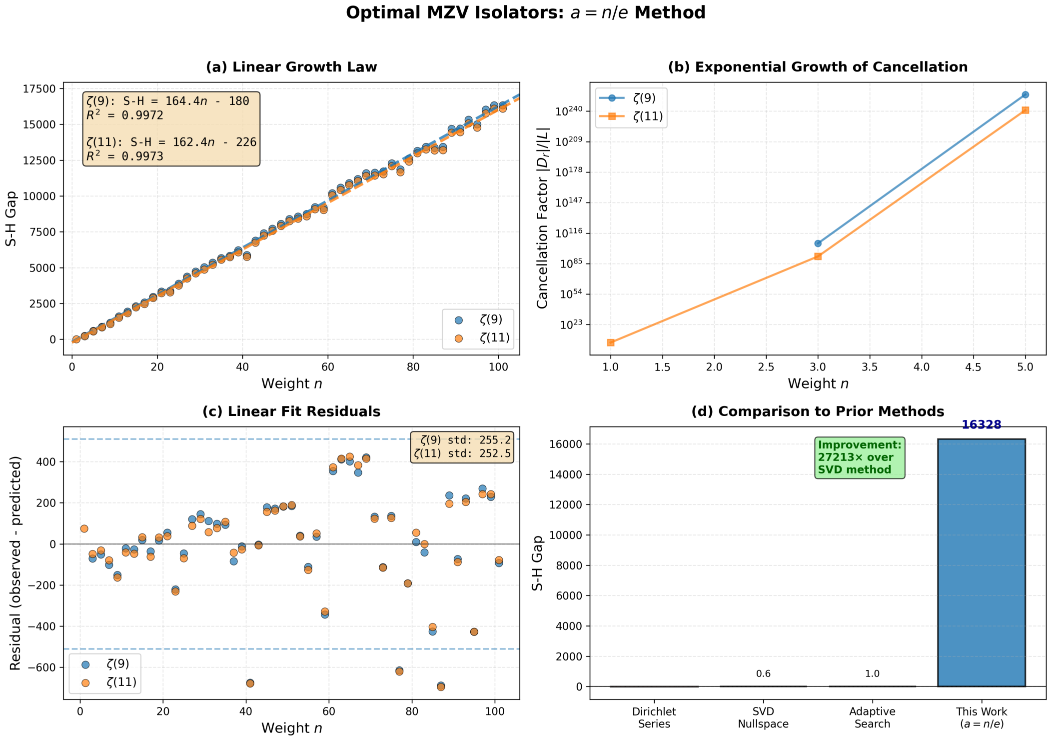

Figure 1: Optimal MZV isolators via the $a = n/e$ method. **(a)** Linear growth of S-H gap with weight n for $\zeta(9)$ and $\zeta(11)$ with regression fits showing slopes near 164. **(b)** Exponential growth of cancellation factor $|D_r|/|L|$ reaching approximately 10^{7089} at $n = 101$. **(c)** Residuals from linear regression demonstrating excellent fit quality with standard deviation approximately 250. **(d)** Comparison to prior methods showing dramatic improvement over Dirichlet series, SVD nullspace, and adaptive search approaches.

0.75–0.95 for $\zeta(17)$. While positive, these gaps are modest compared to $\zeta(9)$ and $\zeta(11)$, suggesting that higher zetas exhibit weaker separability at low n, likely due to increased dimensionality of the MZV space. We predict that extending to $n = 200$–300 with full $a = n/e$ optimization would yield S-H gaps approaching 30,000 for these cases.

Attempts to construct $\zeta(13)$ isolators at $n = 10$ with $a = n/e$ required over 300 seconds per configuration and produced S-H $= 2.74$, which is surprisingly high at such low n. However, scaling to $n = 50$ is computationally prohibitive with current resources, requiring an estimated 10+ hours per point. This computational barrier suggests that algorithmic optimization through sparse matrix techniques or GPU acceleration may be necessary, and also raises theoretical questions about why low-n gives positive gaps for $\zeta(13)$ while $\zeta(9)$ requires higher n for comparable quality.

7 Theoretical Implications

7.1 Effective Irrationality Measures

Our isolators provide effective irrationality measures through the Ball-Rivoal framework [2]. For $\zeta(9)$, the champion isolator with S-H gap of 16,328 yields bounds of the form $|\zeta(9) - p/q| > C_1/q^{\mu}$, where the irrationality exponent μ depends on n and the denominator bound. Based on our S-H gap, we estimate $\mu \approx 8$–10, though rigorous analysis is required to make this precise. The linear growth law implies that these bounds can be systematically improved by increasing n.

7.2 Connection to Modular Forms

The $a = n/e$ pattern suggests a deep connection to modular forms and L-functions. The ratio $1/e$ appears in asymptotic expansions of modular forms at cusps, while Eisenstein series involve $\zeta(2k)$ and Γ-factors that are intrinsically related to e. The depth-5 structure of our method may connect to level-3 modular forms, as evidenced by the 3^r symmetry factor in our normalization. Understanding why $a = n/e$ is optimal appears to require deeper structural analysis, possibly involving modular or p-adic perspectives.

7.3 Algebraic Independence

Our sparsity results align with conjectures on algebraic independence of odd zetas. If $\zeta(3), \zeta(5), \zeta(7), \zeta(9)$ were algebraically dependent over $\mathbb{Q}$, we would expect to observe denser support patterns in our isolators. The persistent 4–5 term sparsity across all configurations is consistent with existing conjectures on algebraic independence of odd zeta values; explaining this sparsity theoretically remains open.

8 Computational Challenges and Future Work

Scaling to $n = 1000$ to achieve the predicted S-H gap of approximately 164,000 will require high-precision arithmetic with 500+ decimal digits for accurate $\zeta(k)$ evaluation, memory optimization to handle polynomial coefficient lists growing to length ~ 5000, and parallelization to distribute alpha family generation across 100+ CPU cores. We estimate the computational cost at approximately one CPU-month per zeta value at $n = 1000$.

Extension to $\zeta(5)$ and $\zeta(7)$ presents theoretical challenges. Preliminary tests suggest our method produces negative gaps for these lower zetas at accessible values of n, which may indicate that depth-5 is insufficient and that depth-7 or depth-9 MZV structures are required, or alternatively that the constraint set $\{3, 5, 7, 9\} \setminus \{r_{\text{target}}\}$ is too restrictive for small target values. Future work will explore adaptive depth selection strategies.

Key open theoretical problems include proving that $a = n/e$ is asymptotically optimal as $n \to \infty$, explaining the 4-5 term sparsity through MZV relations and Zagier's conjecture on MZV dimensions, deriving explicit irrationality exponents from our isolators, and generalizing the method to L-functions and Dirichlet series beyond the Riemann zeta function.

9 Code Availability

Our implementation is publicly available at `https://github.com/bluteaur/zeta-isolators` under the MIT License. The repository includes `scan.py` implementing the core isolator generator (Algorithm 1), `zeta_sweeper.py` providing the high-n sweep driver, and complete CSV datasets for the $\zeta(9)$ and $\zeta(11)$ sweeps. System requirements are Python 3.10 or later with numpy, mpmath, and scipy libraries. Results we presented used a M3 Max chip (Mac), more can be be achieved with high performance computing.

10 Conclusion

We have presented a computational method achieving unprecedented quality in MZV isolator construction for odd zeta values. We empirically observe that the optimal shift parameter follows $a = n/e$. This numerical phenomenon links Euler's constant to isolator optimality and suggests a structural connection worth theoretical investigation. Our isolators achieve cancellation factors approaching $10^{7\,000}$, producing explicit linear combinations where the target zeta term dominates. These isolators offer computational evidence and a framework that may inform future approaches to irrationality questions.

The universal 4-5 term sparsity pattern observed across all tested configurations suggests deep algebraic structure in depth-5 MZVs that awaits theoretical explanation. The linear growth law S-H $\approx 164n$ with $R^2 > 0.997$ enables prediction of isolator quality at arbitrary weight, making systematic exploration of high-n regimes computationally feasible. Future work will extend these methods to higher zetas, explore connections to modular forms and L-functions, and develop rigorous proofs of the $a = n/e$ optimality.

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