대기수과제 2

2-(a) $A = \begin{pmatrix}1&1\\1&0\end{pmatrix}$

① $\sigma(A) = \sqrt{\lambda(A^TA)}$

$A^TA = \begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix} = \begin{pmatrix}2&1\\1&1\end{pmatrix}$.

$\det(A^TA-\lambda I) = \begin{vmatrix}2-\lambda&1\\1&1-\lambda\end{vmatrix} = \lambda^2-3\lambda+1=0$

$\lambda(A^TA) = \frac{3\pm\sqrt5}{2}$, $\sigma(A) = \sqrt{\frac{3\pm\sqrt5}{2}} = \frac{\sqrt5\pm1}{2}$

i) $\lambda(A^TA) = \frac{3+\sqrt5}{2}$

$$\begin{pmatrix}\frac{1-\sqrt5}{2}&1\\1&\frac{-1-\sqrt5}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow V_1 = \sqrt{\frac{2}{5+\sqrt5}}\begin{pmatrix}\frac{1+\sqrt5}{2}\\1\end{pmatrix}$$

ii) $\lambda(A^TA) = \frac{3-\sqrt5}{2}$

$$\begin{pmatrix}\frac{1+\sqrt5}{2}&1\\1&\frac{-1+\sqrt5}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow V_2 = \sqrt{\frac{2}{5-\sqrt5}}\begin{pmatrix}\frac{1-\sqrt5}{2}\\1\end{pmatrix}$$

② $AV_1 = \sigma_1 u_1$, $AV_2 = \sigma_2 u_2$

$$AV_1 = \begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{5+\sqrt5}{10}}\\\sqrt{\frac{5-\sqrt5}{10}}\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{5+\sqrt5}{10}}+\sqrt{\frac{5-\sqrt5}{10}}\\\sqrt{\frac{5+\sqrt5}{10}}\end{pmatrix}$$

$$= \frac{\sqrt5+1}{2}\begin{pmatrix}\sqrt{\frac{5-\sqrt5}{10}}+\sqrt{\frac{5-2\sqrt5}{5}}\\\sqrt{\frac{5-\sqrt5}{10}}\end{pmatrix} = \sigma_1 u_1$$

$$AV_2 = \begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{5-\sqrt5}{10}}\\\sqrt{\frac{5+\sqrt5}{10}}\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{5-\sqrt5}{10}}+\sqrt{\frac{5+\sqrt5}{10}}\\\sqrt{\frac{5-\sqrt5}{10}}\end{pmatrix}$$

$$= \frac{\sqrt5-1}{2}\begin{pmatrix}\sqrt{\frac{5+\sqrt5}{10}}+\sqrt{\frac{5+2\sqrt5}{5}}\\\sqrt{\frac{5+\sqrt5}{10}}\end{pmatrix} = \sigma_2 u_2$$

$A = U\Sigma V^T$

$$= \begin{pmatrix}\sqrt{\frac{5-\sqrt5}{10}}+\sqrt{\frac{5-2\sqrt5}{5}} & \sqrt{\frac{5+\sqrt5}{10}}+\sqrt{\frac{5+2\sqrt5}{5}}\\\sqrt{\frac{5-\sqrt5}{10}} & \sqrt{\frac{5+\sqrt5}{10}}\end{pmatrix}$$

$$\times\begin{pmatrix}\frac{\sqrt5+1}{2}&0\\0&\frac{\sqrt5-1}{2}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{5+\sqrt5}{10}}&\sqrt{\frac{5-\sqrt5}{10}}\\\sqrt{\frac{5-\sqrt5}{10}}&\sqrt{\frac{5+\sqrt5}{10}}\end{pmatrix}$$

2-(b) $B = \begin{pmatrix}0&1&1\\\sqrt2&2&0\\0&1&1\end{pmatrix}$

① $\sigma(B) = \sqrt{\lambda(B^TB)}$

$$B^TB = \begin{pmatrix}0&\sqrt2&0\\1&2&1\\1&0&1\end{pmatrix}\begin{pmatrix}0&1&1\\\sqrt2&2&0\\0&1&1\end{pmatrix} = \begin{pmatrix}2&2\sqrt2&0\\2\sqrt2&6&2\\0&2&2\end{pmatrix}$$

$$\det(B^TB-\lambda I) = \begin{vmatrix}2-\lambda&2\sqrt2&0\\2\sqrt2&6-\lambda&2\\0&2&2-\lambda\end{vmatrix}$$

$$= -\lambda(\lambda-2)(\lambda-8) = 0.$$

$\lambda(B^TB) = 8, 2, 0$ $\quad\sigma(A) = 2\sqrt2, \sqrt2, 0$

i) $\lambda(B^TB) = 8$

$$\begin{pmatrix}-6&2\sqrt2&0\\2\sqrt2&-2&2\\0&2&-6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow V_1 = \frac{1}{2\sqrt3}\begin{pmatrix}\sqrt2\\3\\1\end{pmatrix}$$

ii) $\lambda(B^TB) = 2$

$$\begin{pmatrix}0&2\sqrt2&0\\2\sqrt2&4&2\\0&2&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow V_2 = \frac{1}{\sqrt3}\begin{pmatrix}1\\0\\-\sqrt2\end{pmatrix}$$

iii) $\lambda(B^TB) = 0$

$$\begin{pmatrix}2&2\sqrt2&0\\2\sqrt2&6&2\\0&2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow V_3 = \frac{1}{2}\begin{pmatrix}\sqrt2\\-1\\1\end{pmatrix}$$

② $BV_1 = \sigma_1 u_1$, $BV_2 = \sigma_2 u_2$, $BV_3 = \sigma_3 u_3$

$$BV_1 = \begin{pmatrix}0&1&1\\\sqrt2&2&0\\0&1&1\end{pmatrix}\begin{pmatrix}\frac{\sqrt6}{6}\\\frac{\sqrt3}{2}\\\frac{\sqrt3}{6}\end{pmatrix} = \begin{pmatrix}\frac23\sqrt3\\\frac43\sqrt3\\\frac23\sqrt3\end{pmatrix} = 2\sqrt2\begin{pmatrix}\frac{\sqrt6}{6}\\\frac{\sqrt6}{3}\\\frac{\sqrt6}{6}\end{pmatrix} = \sigma_1 u_1$$

$$BV_2 = \begin{pmatrix}0&1&1\\\sqrt2&2&0\\0&1&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt3}\\0\\-\frac{\sqrt6}{3}\end{pmatrix} = \begin{pmatrix}-\frac{\sqrt6}{3}\\\frac{\sqrt6}{3}\\-\frac{\sqrt6}{3}\end{pmatrix} = \sqrt2\begin{pmatrix}-\frac{\sqrt3}{3}\\\frac{\sqrt3}{3}\\-\frac{\sqrt3}{3}\end{pmatrix} = \sigma_2 u_2$$

$$BV_3 = \begin{pmatrix}0&1&1\\\sqrt2&2&0\\0&1&1\end{pmatrix}\begin{pmatrix}\frac{\sqrt2}{2}\\-\frac12\\\frac12\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix} = \sigma_3 u_3$$

↳ u_3 : u_1, u_2에 수직관계

$u_3 = \begin{pmatrix}a\\b\\c\end{pmatrix}$ $\quad u_1u_3 = \frac{\sqrt6}{6}a + \frac{\sqrt6}{3}b + \frac{\sqrt6}{6}c = 0$

$u_2u_3 = -\frac{\sqrt3}{3}a + \frac{\sqrt3}{3}b - \frac{\sqrt3}{3}c = 0$

$B = U\Sigma V^T$

$$= \begin{pmatrix}\frac{\sqrt6}{6}&-\frac{\sqrt3}{3}&\frac{1}{\sqrt2}\\\frac{\sqrt6}{3}&\frac{\sqrt3}{3}&0\\\frac{\sqrt6}{6}&-\frac{\sqrt3}{3}&-\frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}2\sqrt2&0&0\\0&\sqrt2&0\\0&0&0\end{pmatrix}\begin{pmatrix}\frac{\sqrt6}{6}&\frac{\sqrt3}{2}&\frac{\sqrt3}{6}\\\frac{1}{\sqrt3}&0&-\frac{\sqrt6}{3}\\\frac{\sqrt2}{2}&-\frac12&\frac12\end{pmatrix}$$

3.

(a) $f(x,y) = e^{xy^2}$

$\frac{\partial f}{\partial x} = y^2 e^{xy^2}$, $\frac{\partial f}{\partial y} = 2xye^{xy^2}$

(b) $z = \arctan\left(\frac{x+y}{1-xy}\right)$

$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \times \frac{1+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2+(x+y)^2}$

$= \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2}$

$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \times \frac{1+x^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2+(x+y)^2}$

$= \frac{1+x^2}{(1+x^2)(1+y^2)} = \frac{1}{1+y^2}$

(c) $yz + x\log y = z^3$

$y \cdot \frac{\partial z}{\partial x} + \log y = 3z^2 \cdot \frac{\partial z}{\partial x}$ $\quad \frac{\partial z}{\partial x} = \frac{\log y}{3z^2-y}$

$z + y \cdot \frac{\partial z}{\partial y} + \frac{x}{y} = 3z^2 \cdot \frac{\partial z}{\partial y}$ $\quad \frac{\partial z}{\partial y} = \frac{z+\frac{x}{y}}{3z^2-y}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\log y}{3z^2-y}\right) = 0$

(d) $u = e^{r\theta}\cos\theta$

$\frac{\partial^3 u}{\partial r \partial \theta^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\right)$

$= \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\left(r\theta^{r\theta}\cos\theta - e^{r\theta}\sin\theta\right)\right)$

$= \frac{\partial}{\partial r}\left(r^2e^{r\theta}\cos\theta - re^{r\theta}\sin\theta - re^{r\theta}\sin\theta - e^{r\theta}\cos\theta\right)$

$= \frac{\partial}{\partial r}\left((r^2-1)e^{r\theta}\cos\theta - 2re^{r\theta}\sin\theta\right)$

$= 2re^{r\theta}\cos\theta + (r^2-1)\theta e^{r\theta}\cos\theta$
$\quad -2e^{r\theta}\sin\theta - 2r\theta e^{r\theta}\sin\theta$

$= ((r^2-1)\theta + 2r)e^{r\theta}\cos\theta - (2+2r\theta)e^{r\theta}\sin\theta$

4. $u(x,t) = (x-at)^6 + (x+at)^6$

$u_{tt} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)$

$= \frac{\partial}{\partial t}\left(-6a(x-at)^5 + 6a(x+at)^5\right)$

$= 30a^2(x-at)^4 + 30a^2(x+at)^4$

$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(6(x-at)^5 + 6(x+at)^5\right)$

$= 30(x-at)^4 + 30(x+at)^4$

$\therefore u_{tt} = a^2 u_{xx}$.