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SIMILARITY RELATIONS:

SOLVING THE PROBLEMS WITH EXPECTED UTILITY THEORY

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EC 329: Decision: Theories and Experiments

December 21, 13

The theory of expected utility is similar to its predecessor, expected value, which is calculated by multiply a lottery’s outcomes by their respective probabilities, and summing the results. The figure that is found is the given lottery’s expected value. In expected utility theory, however, calculating a lottery’s expected utility involves an additional layer: the decision-maker’s utility function. An example of which could be:

Instead of simply multiplying outcomes and probabilities, one multiplies the *utility* of an outcome by the probability of the occurrence of that outcome. This can be seen in the following example:

Unfortunately, expected utility does not explain all of the instances of preference found in experimentation. For example, the Allais paradox, first noted by Maurice Allais in 1953, illustrates a decision-maker facing risk. The paradox is found in the fact that the “vast majority” of participants prefer lotteries in accordance with expected utility theory in one experiment, but ignore the theory in another experiment. How could this be? This will be the focus of much of this paper.

Fortunately, much advancement in the field of decision theory has occurred since the problems with expected utility were uncovered. One such mark of progress can be seen in the work of Ariel Rubinstein. In his paper, “Similarity and Decision-making under Risk,” he outlines an alternative to expected utility. It is focused on the idea of the similarity between the outcomes and the probabilities of two lotteries and how that information can be used in determining which of the two lotteries is more favorable.

*Section 1*

Daniel Bernoulli spearheaded the concept of expected utility in 1738. A child of expected value, the theory approximates an individual’s preferences under uncertainty. Refer to the expected value calculation, below:

Imagine two lotteries:

* An outcome of $1000 with a probability of 0.9, and $0, otherwise
* An outcome of $900 with a probability of 1.0

The expected values of these lotteries are (respectively):

* $900
* $900

That’s correct. By weighting the outcomes by the probabilities of the occurrences of those outcomes, we have determined that each of the two lotteries yield the same expected value. In different language, the average outcome of both lotteries is the same. Does this mean that these lotteries are the same? Based on only the expected value calculation, we can see that one would, in theory, be indifferent between the two lotteries.

While expected value can be a useful tool, Bernoulli’s advancement, namely expected utility, proves to deliver better estimations of preference. The rationale was simple: the value of a lottery shouldn’t be determined solely based on its monetary value. Rather, it should be derived from the utility that the given monetary value provides to the individual. A decision-maker is assigned a utility function, such as, , with being any given outcome. The function yields the utility gained from outcome, . For example, the utility from an outcome of $1000 could be explained as such:

Similarly, the utility derived from an outcome of perhaps $500 can be seen as

In the (above) example, it is clear that one derives more value from the first dollars he or she receives, relative to the last dollars. Given that the difference between the utility of $1000 and $500 is

we can observe that the first $500 received is worth 22.36 “utility units” (herein denoted as “utils”) but the second $500 (when the subject receives $1000 in total) is only worth 9.26 utils. This idea can be seen as an example of the theory of diminishing marginal utility. Additionally, considering that individuals don’t necessarily value outcomes and their respective utilities proportionately, expected value may not be an adequate means for determining preference. Such an example can be seen, below:

While the expected values of the two lotteries are the same, expected utility yields a much different result.

Although more useful to economists than its predecessor, expected utility has some issues. One such flaw is known as the Allais paradox. Let’s take a look.

Given the following lotteries,

most subjects stated their preference as such:

This poses an interesting problem, given that the expected utilities can be computed as:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Lottery** | **X** | **P** | **EV** | **EU** |
| 1 | 4000 | 0.8 | *3200* | 50.60 |
| 2 | 3000 | 1.0 | 3000 | *54.77* |
| 3 | 4000 | 0.2 | *800* | 12.65 |
| 4 | 3000 | 0.25 | 750 | *13.69* |

This table gives us considerable information. First, we can see that with respect to expected value, lotteries 1 and 3 should be preferred over their counterparts. When considering our other heuristic, expected utility, the preferences are switched. Lotteries 2 and 4 are now preferable. Unfortunately, these calculations do not agree with the results of experimentation. When deciding between lotteries 1 and 2, the decision makers favored the second, validating expected utility’s result. The choice between lotteries 3 and 4, however, results in the vast majority of participants choosing lottery 3, voiding the very heuristic just observed.

The dilemma depicted in the (above) example is a topic of interest amongst economists because without a proper explanation, the loose threads seen in this experimentation hinder further development. Although the two sets of lotteries, [L1, L2] and [L3, L4] appear to be proportionate with respect to expected utility, as can be seen, below,

the vast majority of subjects decide their preference within the two sets in different manners. Clearly, this is an issue that is asking to be explained. Now, we can look to the work of Ariel Rubinstein, who provides an interesting method with which to grapple with this irregularity.

*Section 2*

In his paper, “Similarity and Decision-making under Risk”, Ariel Rubinstein explains a solution to some of the problems with expected utility theory. He understands that the Allais paradox reveals properties of the decision-making scheme used in certain situations: if outcomes, x1 and x2 are “similar”, the decision making process is then reverted to the relationship between p1 and p2. If the given probabilities are also similar, a different heuristic is used. If not, the lottery with the more favorable probability is preferred. This process makes intuitive sense. If one is deciding between two lotteries with the same (or, in this case, similar) probabilities, or likelihoods of occurrence, the sensible route of action would involve deciding based on which payout is most favorable. If one were asked to decide between a 99% chance of receiving $100 and a 99% chance of receiving $200, the individual would, clearly, prefer the latter option.[[1]](#footnote-1)

Now that we have highlighted this method of thinking, let’s take a look at the lotteries from the previous section:

As we can see, the probabilities p1 and p2 are not “similar”. While 80% certainty represents fairly good odds, there is still a 20% chance of winning nothing. It is known that individuals are very much risk averse and the opportunity of a certain bet makes a probability of only 80% appear paltry in comparison. Additionally, the outcomes, x1 and x2, are seen to be distinct such that probabilities wouldn’t become the deciding factor in regards to preference. Because of these parameters, expected utility proves sufficient in determining favorability among lotteries.

Venturing on to the comparison between lottery L3 and L4, we can see that only the probabilities have changed. In fact, they have been divided by four, resulting in p3 and p4 equaling 0.2 and 0.25, respectively. Now, the difference between probabilities is smaller and considering neither a 20% nor a 25% chance is anywhere near “definite”, p3 and p4 can be seen as similar. Because the probabilities can be treated as alike, preference can be focused upon the lotteries’ respective outcomes. Just as before, 4000 is considerably larger than 3000. The significant difference in outcomes causes one to prefer lottery L3.

Based on the thought experiment outlined in the previous paragraphs, Rubinstein presented a procedure[[2]](#footnote-2), denoted as “\*”, which can be seen, below.

1. Let L1=(x1, p1) and L2=(x2, p2) represent two lotteries
2. If x1>x2 and p1>p2, L1>L2
   1. Similarly, if x2>x1 and p2>p1, L2>L1
3. If p1~p2 and x1!~x2 and x1>x2, L1>L2
4. If x1~x2 and p1!~p2 and p1>p2, L1>L2
5. If neither the outcomes nor the probabilities are similar, choose a different heuristic[[3]](#footnote-3)

Put simply, if either the two outcomes or the two probabilities of the two lotteries are similar enough that the decision-maker can treat them as approximately the same, he or she can base his or her preference solely on the other, dissimilar, factors. According to Rubinstein, “The main finding is that *similarities* on both the prize and the probability dimensions result in a “unique” preference which is consistent with similarities and the above procedure.”[[4]](#footnote-4)

If one is to think about how he or she actually makes decisions, expected utility probably does not enter one’s mind. (At least not at first.) Additionally, it is probably unlikely that only a single heuristic is used and is applied uniformly in all circumstances. In the work, *The Adaptive Decision Maker[[5]](#footnote-5)*, by Payne, Bettman, and Johnson, it is said that when an “agent” is tasked with making a decision, he or she has at his or her disposal a wide array of choice heuristics that could prove useful in formulating that decision. Let us illustrate this concept with an example:

Imagine yourself walking into a coffee shop. Upon entrance, you walk to the counter and begin to decide as to what type of drink you wish to purchase. What type of drink do you choose? You have many options, and similarly, there are many routes you can take in choosing the drink you will eventually enjoy (or dislike, for that matter). Here are a few of the seemingly limitless possible decision-making techniques:

* + Choose the same drink that you bought, previously. You enjoyed a 12oz Americano, yesterday, and you think that drink will grant you the most enjoyment, today.
  + Choose a completely new drink that you have never tried, before. While you enjoyed your coffee, yesterday, you are excited to try something new.
  + Select the drink that the barista prefers. He or she spends a lot of time creating coffee drink and would probably have some unique insight into what would be enjoyable e.g. what type of beans are “good”, today.
  + Ask your friend, “What are you getting?” Choose the drink that he or she decides to order. Perhaps you are indecisive.
  + Select the coffee drink based on any of the many metrics available to you. These can include volume, price, perceived caffeine content, temperature, and many other specifications.

Although, in this paper, we are primarily discussing choice in terms of lotteries, this example is helpful in convincing one of the myriad decision-making tools available when forced into a situation familiar to many of us.

Now, let us apply this coffee-choosing example to the thoughts of Rubinstein. As discussed, previously, his methodology is determinant on similarities between the outcomes or probabilities of two, or presumably, more than two, lotteries. This concept proves helpful when outcomes are alike, but probabilities are much different, or vice versa. Such can be seen as logical, considering if one is tasked with the decision between $99 with certainty or $100 dollars with only a 50% likelihood, the best outcome would probably be clear to most people. The idea of losing out on a single dollar isn’t a particularly painful one, especially when contrasted by a 50/50 chance of going home empty-handed. Unfortunately, Rubinstein’s procedure proves fruitless when the lotteries, themselves, are very similar to one another, or when they are particularly dissimilar to one another.[[6]](#footnote-6) This poses a problem.

First, let’s discuss the problem that appears when the lotteries are very similar to each other with respect to both probabilities and outcomes. For example, the lotteries,

are very similar to each other, as can be seen in the following table, with utility described by .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Lottery** | **X** | **P** | **EV** | **EU** |
| A | 1000 | 0.5 | 500 | 22.36 |
| B | 950 | 0.55 | 522.5 | 22.85 |

How is a typical subject taking part in an experiment supposed to decide between these two lotteries? While there are many possibilities (e.g. reverting back to expected utility, choosing the lottery with the simplest numbers, etc.) the likelihood of any individual using the same heuristic is probably quite low. As can be understood through the coffee shop example, some heuristics might seem pleasing on one day, with others seeming appropriate on other days, or in other situations. If lotteries appear to be very similar, one’s corresponding preferences are probably arbitrary.

After having discussed preference in relation to two lotteries that appear to be wholly similar, the opposite can be approached. What if two lotteries are dissimilar with respect to both their outcomes *and* their probabilities? If that were the case, such as in the following example:

one’s preference would be decidedly difficult to predict. Some possibilities include:

* Reverting back, once again, to expected utility theory,
* Deciding whether outcomes are more similar than probabilities, or vice versa, and using Rubinstein’s procedure as if they were actually similar.

It is difficult to know which of these, or if either, would be effective in determining preference. Experimentation would be required to find validity in these methods.

*Section 3*

Why should economists care about the problems set forth by these experiments? This question can be answered by identifying the significance of the shortcomings of expected utility.

As noted previously, expected utility fails to account properly for a decision-maker’s risk aversion, which can be identified in the Allais paradox. Given the following sets of lotteries,

expected utility theory suggests that the preference should be displayed as

In experimentation, however, these preferences do not prove to be accurate. Instead, L4 is preferred to L3 by the “vast majority of subjects.” This problem proves to be a serious one, especially considering the significance of the topic of lotteries to areas such as insurance and gambling.

Although speaking of expected utility’s negative qualities is of interest in its own right, it is also necessary to highlight why solving these problems is relevant. As outlined, previously, in this paper, the conclusions of experimentations such as that of the Allais paradox can be remedied given certain alternatives to pure expected utility. Namely, the invention of similarity relations can be used to more aptly model one’s preferences under risk.

*Section 4*

In Ariel Rubinstein’s paper, his procedure was only used to determine preference between two lotteries. While useful in and of itself, this procedure is limiting in that decisions frequently involve more than two possible choices. Could the procedure be adapted to explain preference amongst three lotteries? In this section, we will attempt to illustrate a possible procedure and the shortcomings of said approach.

First, let’s describe the situation in which we are involving ourselves. Instead of a typical set of two lotteries, a third has been added, as can be seen in the following example:

This set of lotteries yields the following expected value and expected utility calculations:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Lottery** | **X** | **P** | **EV** | **EU** |
| 1 | 4000 | 0.8 | 3200 | 50.60 |
| 2 | 3000 | 1.0 | 3000 | 54.77 |
| 3 | 2000 | 1.0 | 2000 | 44.72 |

Once again, utility is modeled by . Now, let’s attempt to modify Rubinstein’s procedure in deciding preference between these *three* lotteries.

1. None of the outcomes are similar to each other
2. The probabilities, p2 and p3, are the same
3. Given that the probabilities of lotteries L1 and L2 are similar, or rather, the same, the decision-making process can be focused upon the outcomes of those two lotteries
4. Since 3000 is larger than 2000, lottery L2 should be preferred to lottery L3, and additionally, L1

In this case, similarity occurs only with regard to probabilities. In the next example, this will not be the case:

The expected values and expected utilities for this set of lotteries can be calculated as:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Lottery** | **X** | **P** | **EV** | **EU** |
| 4 | 4000 | 0.8 | 3200 | 50.60 |
| 5 | 3000 | 1.0 | 3000 | 54.77 |
| 6 | 3750 | 0.91 | 3412.5 | 55.73 |

Although this second example is not as simple as the first, it can be used to appropriately illustrate our new procedure:

1. The outcomes, x4 and x6, are similar
2. Since p6 is greater than p4, lottery L6 is preferred to lottery L4
3. The probabilities, p5 and p6, are also similar
4. Since x6 is greater than x5, lottery L6 is preferred to lottery L5
5. Lottery L6 is preferred to both lotteries L4 and L5

This new procedure is fairly simple, just as was the case with Rubinstein’s original.

Formally, this procedure can be outlined as the following:

**Step I**

Compare the outcome and probability of each lottery to those of each other

**Step II**

For the lotteries with similar outcomes or probabilities, determine preference between only those two lotteries, using Rubinstein’s original procedure

**Step III**

If it were found that one lottery is preferable to both of its counterparts, that lottery would be preferred in relation to the set as a whole

**Step IV**

If step three results in failure, a different heuristic should be used[[7]](#footnote-7)

In essence, this procedure is the same as Rubinstein’s, except that more comparisons are made. Instead of simply comparing x1 to x2 and p1 to p2, one must compare x1 to both x2 and x3 and compare x2 to x3, and so on and so forth for both outcomes and probabilities. Once these similarity relations are gathered, Rubinstein’s procedure can be applied to those relations, determining which lotteries are preferable on a “micro” scale (i.e. L1 versus L2, L2 versus L3, etc.). After this step, the procedure can be applied on a “macro” scale through which the most preferable two lotteries from the “micro” comparisons are pitted against each other. This final step results in a general “winner”, with one lottery being preferable to the other two.

Given the (above) procedure, we have a fairly comprehensive method for determining preference amongst three lotteries. Conceivably, the same procedure could be applied to more than three lotteries, with the only difference being the addition of more “tiers” of comparison, weeding out lotteries that are clearly unfavorable on each tier, eventually resulting in a final lottery. This is not to say that the procedure is without flaws, however. Just as with Rubinstein’s procedure for two lotteries, this method finds issue with edge cases such as complete dissimilarity between lotteries or a complete lack of similarity between those lotteries. Additionally, the procedure has not been tested through experimentation. With that being said, however, it is possible that it would demonstrate success similar to that of Rubenstein’s original procedure.

*Notes*

I have created a simple web application demonstrating Ariel Rubinstein’s decision-making procedure.[[8]](#footnote-8) It makes several assumptions on the part of the decision-maker, namely the ratios that determine similarity between outcomes and probabilities. Optimally, a “quiz” would allow the application to gain some insight into how the decision-maker defines “similar”. For example, a sliding scale could be used to determine how near a value must be to another value in order for the two to be considered “similar”. Additionally, the application only allows for two lotteries.

1. Of course, this example is incredibly simple. It is used only to describe the concept of choosing the lottery with the higher outcome, if the probabilities of the given outcomes are the same, or similar. [↑](#footnote-ref-1)
2. [↑](#footnote-ref-2)
3. Rubinstein does not specify the nature of this last “step.” [↑](#footnote-ref-3)
4. Ariel Rubinstein. “Similarity and Decision-making under Risk,” *Journal of Economic Theory* (1968) [↑](#footnote-ref-4)
5. John W. Payne, James R. Bettman, and Eric J. Johnson. *The Adaptive Decision Maker* (Cambridge: Cambridge University Press, 1993) [↑](#footnote-ref-5)
6. In other words, if x1 ~ x2 and p1 ~ p2, Rubinstein’s procedure will not work. Similarly, if x1!~x2 and p1!~p2, the procedure will not work. [↑](#footnote-ref-6)
7. Just as in Rubinstein’s original procedure, the procedure used with three lotteries is purposefully open-ended, allowing for the eventual use of other heuristics, if necessary. [↑](#footnote-ref-7)
8. The web application can be found at https://s3.amazonaws.com/blwsk.com/lottery-solver.html [↑](#footnote-ref-8)