Implementation

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Firstly, elimination of dependent dofs gives

$$K^\star \delta u = \delta f^\star \tag{1}$$

Condensing the free nodes and leaving only the prescribed nodes, the system equation is written as,

$$(K^\star_M)\delta u_b = (\delta f^\star_b), \quad \text{with} \quad K^\star_M = K^\star_{bb} - K^\star_{ba}(K^\star_{aa})^{-1}K^\star_{ab} \tag{2}$$

where $u_b = \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix}$. The voigt notation says,

$$\delta\sigma = \begin{bmatrix} \delta\sigma_{11} \\ \delta\sigma_{22} \\ \delta\sigma_{12} \end{bmatrix} = \frac{1}{V_0} \begin{bmatrix} H_1 & H_2 & H_4 \end{bmatrix} \begin{bmatrix} \delta f_1 \\ \delta f_2 \\ \delta f_4 \end{bmatrix} = \frac{1}{V_0} H \, \delta f \tag{3}$$

where,

$$H_q = \begin{bmatrix} x & 0 \\ 0 & y \\ \frac{y}{2} & \frac{x}{2} \end{bmatrix}_q, \quad q = 1, 2, 4 \tag{4}$$

The matrix to link displacement and strain is,

$$u_q = \begin{bmatrix} x & 0 & \frac{y}{2} \\ 0 & y & \frac{x}{2} \end{bmatrix}_q \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = H_q^T \varepsilon_M \tag{5}$$

Therefore, the displacement vector for the controled nodes {1,2,4} can be represented by,

$$u_p = \begin{bmatrix} H_1 & H_2 & H_4 \end{bmatrix}^T \varepsilon_M = H^T \varepsilon_M \tag{6}$$

Considering that,

$$\delta\sigma = \frac{1}{V_0} H \, \delta f = \frac{1}{V_0} H K^\star_M H^T \delta\varepsilon_M \tag{7}$$

Therefore, the consistent tangent is identified as,

$$D = \frac{1}{V_0} H K^\star_M H^T \tag{8}$$

The probing method to compute the tangent,

$$D = \frac{1}{V_0} H K_M^\star H^T = \frac{1}{V_0} H(K_{bb}^\star - K_{ba}^\star (K_{aa}^\star)^{-1} K_{ab}^\star) H^T \tag{9}$$

Obviously, the matrix D is 3×3. Multiplying the D matrix with vector $e_1 = (1, 0, 0)^T$ yields,

$$D^1 = \frac{1}{V_0} H(K_{bb}^\star - K_{ba}^\star (K_{aa}^\star)^{-1} K_{ab}^\star) H^T e_1 = \frac{1}{V_0} H(K_{bb}^\star f - K_{ba}^\star (K_{aa}^\star)^{-1} K_{ab}^\star f) \tag{10}$$

where $f = H^T e_1$. In order to compute the $K_{bb}^\star f$ and $K_{ab}^\star f$, it suffices to do the following matrix-vector multiplication,

$$\begin{bmatrix} K_{aa}^\star & K_{ab}^\star & 0 \\ K_{ba}^\star & K_{bb}^\star & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix} = \begin{bmatrix} K_{ab}^\star f \\ K_{bb}^\star f \\ 0 \end{bmatrix} = \begin{bmatrix} g \\ h \\ 0 \end{bmatrix} \tag{11}$$

Therefore, the g and h can be compute by equ. 11, it is then true that,

$$D^1 = \frac{1}{V_0} H(h - K_{ba}^\star (K_{aa}^\star)^{-1} g) \tag{12}$$

In order to compute $(K_{aa}^\star)^{-1} g$, it suffices to solve the following linear system,

$$\begin{bmatrix} K_{aa}^\star & K_{ab}^\star & 0 \\ K_{ba}^\star & K_{bb}^\star & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

After geting the value of u, it is then true that,

$$D^1 = \frac{1}{V_0} H(h - K_{ba}^\star u) \tag{14}$$

In order to compute $K_{ba}^\star u$, the following matrix-vector muliplication is used,

$$\begin{bmatrix} K_{aa}^\star & K_{ab}^\star & 0 \\ K_{ba}^\star & K_{bb}^\star & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_{aa}^\star u \\ K_{ba}^\star u \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ l \\ 0 \end{bmatrix} \tag{15}$$

To summarize, the first column of the consistent tangent is,

$$D^1 = \frac{1}{V_0} H(h - l) \tag{16}$$