固体力学实验作业

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Introduction

We consider the telegraph equation in one-space dimension, given by

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t} + cu = 0, 0 < x < L, t > 0 \qquad (1)$$

where $a, b, c > 0$, u can represent both voltage and current in conductor.

The equation (1) is derived from the study of propagation of electrical signals in a cable of transmission line[7].

Solution

We try to solve the equation in a simplified situation. We consider equation (1) with homogeneous boundary condition

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t} + cu = 0 \\ u|_{t=0} = \phi(x), \frac{\partial u}{\partial t}|_{t=0} = \psi(x) \\ u|_{x=0} = 0, u|_{x=L} = 0 \end{cases} \tag{2}$$

It is obvious that equation (1) is linear. Comparing with the general form of second-order linear equation with two variables, we obtain

$$A = 1, B = 0, C = -a^2, D = 2b, E = 0, F = c, G = 0$$

then $\Delta = B^2 - AC = a^2 > 0$.

Therefore, equation (1) is a hyperbolic equation.

Solution

Now we can apply the method of separation of variables to obtain the general solution(when b is small)[7]

$$u(x,t) = e^{-bt}\sum_{n=1}^{\infty}(C_n cos(q_n t) + D_n sinq_n t)sin\frac{n\pi}{L}x \tag{3}$$

where

$$\begin{aligned} q_n &= \sqrt{|(\frac{n\pi a}{L}^2) + c - b^2|} \\ C_n &= \frac{2}{L}\int_0^L \phi(x)sin\frac{n\pi}{L}xdx \\ D_n &= \frac{b}{q_n}C_n + \frac{2}{Lq_n}\int_0^L \psi(x)sin\frac{n\pi}{L}xdx \end{aligned} \tag{4}$$

It takes the form of wave. The solution to (1) with the nonhomogeneous boundary condition can be found in [1].

Application

We now come to the application of the the equation in traditional Electrical Engineering. From the e^{-bt} we now the signal will vanish very fast. Comparing with the original equation[7]

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= LC\frac{\partial^2 v}{\partial t^2} + (RC + GL)\frac{\partial v}{\partial t} + GRv \\ \frac{\partial^2 i}{\partial x^2} &= LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + GRi \end{aligned} \tag{5}$$

we obtain $b = \frac{1}{2}\frac{RC+GL}{LC}$. Therefore, to minimalize the decay, we need to reduce the ratio of $\frac{R}{L}$ and $\frac{G}{C}$.

Application

However, by discussing the inverse problem, we can exploit the decay to determine several quantities. For instance, when measuring the moisture content of soil, b can be a reflection of the content of water[8].

Application

Moreover, the telegraph equation can describe many other non-linear phenomena in physical, chemical and biological process[10, 9]. It can be used to model the thermal waves[2], and the effect is better than using the heat equation[1]. The telegraph equation is actually a descriptor for what are termed persistent random walks[4].

Application

It can also picture the migration dynamics of fish schools[11]. As the fish will move in the direction of the gradient of temperature field, and the fish in a fish school are discrete in nature, this problem could be similarly modelled like the previous one. Biologists encounter these equations in the study of pulsate blood flow in arteries and in one dimensional random motion of bugs along a hedge[3].

Application

The equation can also be applied in the wave propagation in nonequilibrium media like reaction-diffusion system, which is associated with the relativistic Hamilton/ Lagrange mechancs[5]. When calculating the propagation speed for the traveling wave in the reaction-diffusion system involving the diffusion with a finite velocity, we can derive the telegraph equation when the reaction rate corresponding to the KPP kinetics is 0.

Application

The compressible fluid flows are also governed by the equation[6]. Originally, the equation of the system is

$$c^2\Delta u - \frac{\partial^2 u}{\partial t^2} + \frac{4\mu}{3\rho_0}\frac{\partial \Delta u}{\partial t} = 0 \tag{6}$$

considering the damping term is small, which leads to $\Delta u = -\frac{\omega^2}{c^2}u$, we obtain

$$c^2\Delta u - \frac{\partial^2 u}{\partial t^2} + \frac{4\mu\omega^2}{3\rho_0 c^2}\frac{\partial u}{\partial t} = 0 \tag{7}$$

shares the same form of equation (1).

The equation of motion of aviscoelastic fluid under the Maxwell body theory is telegraph equation as well[6].

Conclusion

Decay

Inverse problem

Discrete

Finite velocity

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Thanks