Def^n partition P of $[a,b]$ is $\{x_0, x_1, \dots x_n\}$ with $x_i \le x_{i+1}$ $i=0,\dots n-1$

Def^n $\Delta x_i = x_i - x_{i-1}$

$M_i = \sup\limits_{x_{i-1} \le x \le x_i} x$ $\qquad U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$

$m_i = \inf\limits_{x_{i-1} \le x \le x_i} x$ $\qquad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$ $\qquad L(P,f) \le U(P,f)$

$L(P,f) \le \int_a^b f\,dx \le U(P,f)$

$\overline{\int_a^b} f\,dx = \inf\limits_P U(P,f)$ $\qquad \underline{\int_a^b} f\,dx = \sup\limits_P L(P,f)$

both equal $\Rightarrow$ Riemann integrable on $[a,b] \Rightarrow f \in \mathcal{R}$

Def^n Let α monotonically increasing

$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$ $\qquad L(P,f,\alpha) \le U(P,f,\alpha)$

$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$ $\qquad L(P,f,\alpha) \le \int_a^b f\,d\alpha \le U(P,f,\alpha)$

$\overline{\int_a^b} f\,d\alpha = \inf\limits_P U(P,f,\alpha)$

$\underline{\int_a^b} f\,d\alpha = \sup\limits_P (L(P,f,\alpha))$

equal $\Rightarrow$ integrable on $[a,b]$ wrt α $\Rightarrow f \in \mathcal{R}(\alpha)$

Def^n P^* refinement of P if $P^* \supset P$.

Thm $L(P,f,\alpha) \le L(P^*,f,\alpha)$

$U(P^*,f,\alpha) \le U(P,f,\alpha)$

Thm $f \in \mathcal{R}(\alpha)$ on $[a,b] \Leftrightarrow \forall \epsilon > 0\ \exists$ partition P. $|U(P,f,\alpha) - L(P,f,\alpha)| < \epsilon$.

Thm f continuous on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,b]$

Thm f monotonic on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$ on $[a,b]$
α increasing + continuous + monotonic

Thm f bounded + finitely many points of discontinuity + α continuous where f isn't $\Rightarrow f \in \mathcal{R}(\alpha)$

Thm $f \in R(\alpha)$ on $[a,b]$, $f : [a,b] \to [m,M]$ (bounded)
ϕ continuous on $[m,M]$ $\Rightarrow h = \phi \circ f \in R(\alpha)$ on $[a,b]$

Thm $f \in R(\alpha)$ $g \in R(\alpha) \Rightarrow fg \in R(\alpha)$
" $\Rightarrow |f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

Thm α increasing monotonically.
$\alpha' \in R$ on $[a,b]$. f bounded real function on $[a,b]$. Then
$f \in R(\alpha) \Leftrightarrow f\alpha' \in R$ and $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$

INTEGRATION AND DIFFERENTIATION

Thm $f \in R$ on $[a,b]$. For $a \leq x \leq b$, let $F(x) = \int_a^x f(t)\,dt$
Then F is continuous on $[a,b]$ and
f continuous at $x_0 \in [a,b]$ then F differentiable at x_0

Integration by parts:

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

RECTIFIABLE

Def^n $\Lambda(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$ for $\gamma : [a,b] \to \mathbb{R}^k$

$$\underbrace{\Lambda(\gamma)}_{\text{length of the curve}} = \sup_P \Lambda(P,\gamma)$$

Thm γ' continuous on $[a,b] \Rightarrow \Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$

NOTES: ~~$\int_a^b f\,dx$~~ ~~If $f \in R$ and~~