DERIVATIVES OF REAL FUNCTIONS

Def^n ① $f : [a,b] \to \mathbb{R}$.

$\forall x \in [a,b]$ $\quad \phi(t) = \dfrac{f(t) - f(x)}{t - x}$ (Rise over run) $\quad (a < t < b, t \neq x)$

$$f'(x) = \lim_{t \to x} \phi(t)$$

f' is defined at $x \Rightarrow f$ differentiable at x.

Theorem f defined on $[a,b]$.
Then f differentiable at $x \in [a,b] \Rightarrow f$ continuous at x.

Thm ① $(f+g)'(x) = f'(x) + g'(x)$

② $(fg)'(x) = \cancel{fg}\, g'(x) + f'(x)\, g(x)$

③ $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x) f'(x) - g'(x) f(x)}{g^2(x)}$

Thm. $(f \circ g)' = (f' \circ g)\, g'$ (chain rule)

MEAN VALUE THEOREMS

Def^n $f : X \to \mathbb{R}$.
$p \in X$ is $\boxed{\text{a local maxima}}$ if $\exists\, \delta > 0$ st $\forall\, q \in X.\ d(p,q) < \delta \Rightarrow f(q) \leq f(p)$

Thm Let p be local maxima on X and f differentiable. Then $f'(p) = 0$

Thm $f\ g$ continuous, $f, g : [a,b] \to \mathbb{R}$ differentiable in (a,b).
$\Rightarrow \exists\, x \in (a,b)$ st $[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$] generalized MVT.

THE MEAN VALUE THM : $f : [a,b] \to \mathbb{R}$, f continuous on $[a,b]$ and differentiable on (a,b). $\boxed{\exists\, x \in (a,b) \text{ st } f'(x) = \dfrac{f(b) - f(a)}{b - a}}$

Thm f differentiable on (a,b)
$f' \geq 0\ \forall x \in (a,b) \Rightarrow$ monotonically increasing f
$f' = 0\ \forall x \in (a,b) \Rightarrow$ constant f
$f' \leq 0\ \forall x \Rightarrow$ monotonically decreasing

CONTINUITY OF DERIVATIVES

→ f continuous on [a,b] but IVT on f'

Thm f real differentiable on $[a,b]$. Let λ be st $f'(a) < \lambda < f'(b)$. Then $\exists\, x \in (a,b)$ st $f'(x) = \lambda$

Corollary f differentiable on $[a,b] \Rightarrow f$ has no discontinuities of the 1st kind.

L'HOSPITAL'S RULE

Thm f, g real differentiable in (a,b) and $g'(x) \neq 0 \ \forall\, x \in (a,b)$

If $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$ and

either ① $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or

② $g(x) \to +\infty$ as $x \to a$

Then $\dfrac{f(x)}{g(x)} \to A$ as $x \to a$

TAYLOR'S THEOREM

Thm $f : [a,b] \to \mathbb{R}$. $n \in \mathbb{Z}^+$. $f^{(n-1)}$ continuous on $[a,b]$, $f^{n}(t)$ exists $\forall\, t \in (a,b)$

- Let $\alpha, \beta \in (a,b)$, $\alpha \neq \beta$.

  Let $P(t) = \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k$

  Then $\exists\, x$ between α & β st $f(\beta) = P(\beta) + \dfrac{f^{n}(x)}{n!}(\beta - \alpha)^n$

$\boxed{\text{FACT}}$ (by MVT)

$f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$ for $a < x_1 < x < x_2 < b$ and f differentiable on (a,b)

↑ specific $x \in (x_1, x_2)$

IMPT TO SHOW:

① Monotonically ↑: $x_1 < x_2$ then $f(x_1) \leq f(x_2)$ OR show $f' \geq 0$

② $[x_1, x_2]$. Average slope over this is $\dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$ #

③ f is strictly increasing over a domain $\Rightarrow$ f is injective.

Derivative is bounded + f differentiable $\Rightarrow$ f uniformly continuous