2.6 Consider points distributed in a p-dimensional unit ball. The median distance to the point closest to the center of the ball is given by the inverse CDF of the first order statistic of $R \sim Unif[0,1]$

$$median = F^{-1}_{R_{(1)}}$$

Given that the CDF of $R_{(1)}$ is related to $P(R_i \geq r)$ by

$$1 - F_{R_{(1)}}(r) = \prod_{i=1}^{N} P(R_i \geq r) = \prod_{i=1}^{N} 1 - P(R_i \leq r)$$

$P(R_i \leq r)$ for points uniformly distributed in the unit sphere is given by the ratio of the volume of a sphere with radius $r \in [0,1]$ to the volume of the unit sphere, and we previously derived a formula for the volume of a p-dimensional sphere $V_p(r)$, so

$$P(R_i \leq r) = \frac{V_p(r)}{V_p(1)} = \frac{\frac{\pi^{p/2} r^p}{\Gamma(1+\frac{p}{2})}}{\frac{\pi^{p/2} 1^p}{\Gamma(1+\frac{p}{2})}} = r^p$$

Combining the above we have

$$\begin{aligned} F_{R_{(1)}}(r) &= 1 - (1 - r^p)^N \\ F^{-1}_{R_{(1)}}(x) &= (1 - x^{1/N})^{1/p} \\ F^{-1}_{R_{(1)}}\left(\frac{1}{2}\right) &= \left(1 - \frac{1}{2}^{1/N}\right)^{1/p} \quad // \end{aligned}$$

3.3 (a) Let $c^T y$ be an unbiased estimator of $a^T \beta$. Then

$$\begin{aligned} E(c^T y) &= a^T \beta \\ c^T E(y) &= a^T \beta \\ c X^T \beta &= a^T \beta \end{aligned}$$

Which requires that $c^T X = a^T$. Now we can write $Var(c^T y) = Var(c^T y - a^T \hat{\beta} + a^T \hat{\beta})$ and simplify

$$Var(c^T y - a^T \hat{\beta} + a^T \hat{\beta}) = Var(c^T y - a^T \hat{\beta}) + 2Cov(c^T y - a^T \hat{\beta}, a^T \hat{\beta}) + Var(a^T \hat{\beta})$$

Which is greater than or equal to $Var(a^T \hat{\beta})$ if $Cov(c^T y - a^T \hat{\beta}, a^T \hat{\beta}) = 0$.

$$\begin{aligned} Cov(c^T y - a^T \hat{\beta}, a^T \hat{\beta}) &= Cov(c^T y - a^T (X^T X)^{-1} X^T y, a^T \hat{\beta}) \\ &= (c^T - a^T (X^T X)^{-1} X^T) Cov(y) (a^T (X^T X)^{-1} X^T)^T \\ &= \sigma^2 \mathbf{I} (c^T - a^T (X^T X)^{-1} X^T)(X (X^T X)^{-1} a) \\ &= \sigma^2 \mathbf{I} (c^T X (X^T X)^{-1} a - a^T (X^T X)^{-1} X^T X (X^T X)^{-1} a) \\ &= \sigma^2 \mathbf{I} (c^T X (X^T X)^{-1} a - a^T (X^T X)^{-1} a) \\ &= \sigma^2 \mathbf{I} (a^T (X^T X)^{-1} a - a^T (X^T X)^{-1} a) \text{ substituting from above} \\ &= 0 \end{aligned}$$

So $Var(c^T y) \geq Var(a^T \hat{\beta})$ for any unbiased estimator $c^T y$ of $a^T \beta$, where $a^T \hat{\beta}$ is the OLS estimator of $a^T \beta$.

3.7

$$
\begin{aligned}
\pi(\beta|\mathbf{y}) &= \frac{\prod_{i=1}^{N}\prod_{j=1}^{p} f_y(y_i|\beta_j)\prod_{j=1}^{p}\pi(\beta_j)}{m(y)} \\
&\propto exp\left[-\frac{1}{2}\frac{\sum_{i=1}^{N}(y_i-\beta_0-\sum_{j=1}^{p}x_{ij}^T\beta_j)^2}{\sigma^2}\right] + exp\left[-\frac{1}{2}\frac{\sum_{j=1}^{p}\beta_j^2}{\tau^2}\right] \\
&\propto exp\left[-\frac{1}{2}\frac{\sum_{i=1}^{N}(y_i-\beta_0-\sum_{j=1}^{p}x_{ij}^T\beta_i)^2}{\sigma^2} + -\frac{1}{2}\frac{\sum_{j=1}^{p}\beta_i^2}{\tau^2}\right] \\
log(\pi(\beta|\mathbf{y})) &\propto -\frac{1}{2\sigma^2\tau^2}\tau^2\sum_{i=1}^{N}(y_i-\beta_0-\sum_{j=1}^{p}x_{ij}^T\beta_i)^2 + \sigma^2\sum_{j=1}^{p}\beta_i^2 \\
-log(\pi(\beta|\mathbf{y})) &\propto \sum_{i=1}^{N}(y_i-\beta_0-\sum_{j=1}^{p}x_{ij}^T\beta_i)^2 + \frac{\sigma^2}{\tau^2}\sum_{j=1}^{p}\beta_i^2 //
\end{aligned}
$$