Derivation of System of First Order Differential Equations for each Model

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1 Model 1: $V(x) = 0$

The starting point is the given fourth-order non-linear differential equation:

$$-\frac{dV(x)}{dx} - m\ddot{x} - \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\dddot{x}}{\dot{x}^5} + \frac{\ddddot{x}}{2\dot{x}^4}\right) = 0 \tag{1}$$

Since $V(x) = 0$, the equation reduces to:

$$-m\ddot{x} - \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\dddot{x}}{\dot{x}^5} + \frac{\ddddot{x}}{2\dot{x}^4}\right) = 0 \tag{2}$$

Let's start by isolating the term involving $\ddddot{x}$:

$$-\frac{\hbar^2}{2m}\cdot\frac{\ddddot{x}}{2\dot{x}^4} = m\ddot{x} + \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\dddot{x}}{\dot{x}^5}\right) \tag{3}$$

Multiply both sides by $-\frac{2m}{\hbar^2}$ to isolate $\ddddot{x}$:

$$\frac{\ddddot{x}}{2\dot{x}^4} = -\frac{2m^2}{\hbar^2}\ddot{x} - 5\frac{\ddot{x}^3}{\dot{x}^6} + 4\frac{\ddot{x}\dddot{x}}{\dot{x}^5} \tag{4}$$

Now multiply through by $2\dot{x}^4$:

$$\ddddot{x} = 2\dot{x}^4\left(-\frac{2m^2}{\hbar^2}\ddot{x} - 5\frac{\ddot{x}^3}{\dot{x}^6} + 4\frac{\ddot{x}\dddot{x}}{\dot{x}^5}\right) \tag{5}$$

$$\ddddot{x} = -\frac{4m^2}{\hbar^2}\dot{x}^4\ddot{x} - 10\frac{\ddot{x}^3}{\dot{x}^2} + 8\frac{\ddot{x}\dddot{x}}{\dot{x}} \tag{6}$$

Next we define the following auxiliary variables:

$$
\begin{aligned}
x_1 &= x \quad \text{(position)} \\
x_2 &= \dot{x} \quad \text{(velocity)} \\
x_3 &= \ddot{x} \quad \text{(acceleration)} \\
x_4 &= \dddot{x} \quad \text{(jerk)}
\end{aligned}
$$

Using the defined variables, we can rewrite the original fourth-order differential equation as a system of first-order differential equations:

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= x_3, \\
\frac{dx_3}{dt} &= x_4, \\
\frac{dx_4}{dt} &= -\frac{4m^2}{\hbar^2} x_2^4 x_3 - 10\frac{x_3^3}{x_2^2} + 8\frac{x_3 x_4}{x_2}
\end{aligned}
\tag{7}
$$

2 Model 2: $V(x) = \frac{kx^2}{2}$

Given the fourth-order non-linear differential equation:

$$
-\frac{dV(x)}{dx} - m\ddot{x} - \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5} + \frac{\ddddot{x}}{2\dot{x}^4}\right) = 0 \tag{8}
$$

Since $V(x) = \frac{kx^2}{2}$, we have:

$$
-\frac{dV(x)}{dx} = -kx \tag{9}
$$

Thus, the differential equation becomes:

$$
-kx - m\ddot{x} - \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5} + \frac{\ddddot{x}}{2\dot{x}^4}\right) = 0 \tag{10}
$$

We isolate the term involving $\ddddot{x}$:

$$
-\frac{\hbar^2}{2m}\frac{\ddddot{x}}{2\dot{x}^4} = kx + m\ddot{x} + \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5}\right) \tag{11}
$$

Multiply both sides by $-\frac{2m}{\hbar^2}$ to isolate $\ddddot{x}$:

$$\frac{\dddot{\dot{x}}}{2\dot{x}^4} = -\frac{2m}{\hbar^2}(kx + m\ddot{x}) - 5\frac{\ddot{x}^3}{\dot{x}^6} + 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5} \tag{12}$$

Now multiply through by $2\dot{x}^4$:

$$\ddddot{x} = 2\dot{x}^4\left(-\frac{\hbar^2}{2m}(kx + m\ddot{x}) - 5\frac{\ddot{x}^3}{\dot{x}^6} + 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5}\right) \tag{13}$$

Expanding this:

$$\ddddot{x} = -\frac{\hbar^2}{m}(kx + m\ddot{x})\dot{x}^4 - 10\frac{\ddot{x}^3}{\dot{x}^2} + 8\frac{\ddot{x}\,\dddot{x}}{\dot{x}} \tag{14}$$

Next, we define the following auxiliary variables:

$$x_1 = x \quad \text{(position)}, \quad x_2 = \dot{x} \quad \text{(velocity)}, \quad x_3 = \ddot{x} \quad \text{(acceleration)}, \quad x_4 = \dddot{x} \quad \text{(jerk)}$$

Using the defined variables, we can rewrite the original fourth-order differential equation as a system of first-order differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= x_3, \\
\frac{dx_3}{dt} &= x_4, \\
\frac{dx_4}{dt} &= -\frac{\hbar^2}{m}(kx_1 + mx_3)x_2^4 - 10\frac{x_3^3}{x_2^2} + 8\frac{x_3x_4}{x_2}
\end{aligned} \tag{15}$$

3 Model 2: $V(x) = \frac{k}{x}$

Given the fourth-order nonlinear differential equation:

$$-\frac{dV(x)}{dx} - m\ddot{x} - \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5} + \frac{\ddddot{x}}{2\dot{x}^4}\right) = 0$$

Since $V(x) = \frac{k}{x}$:

$$-\frac{dV(x)}{dx} = \frac{k}{x^2}$$

Thus, the differential equation becomes:

$$\frac{k}{x^2} - m\ddot{x} - \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5} + \frac{\ddddot{x}}{2\dot{x}^4}\right) = 0$$

We isolate the term involving $\ddddot{x}$:

$$-\frac{\hbar^2}{2m} \cdot \frac{\ddddot{x}}{2\dot{x}^4} = -\frac{k}{x^2} + m\ddot{x} + \frac{\hbar^2}{2m}\left(5\frac{\ddot{x}^3}{\dot{x}^6} - 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5}\right)$$

Multiply both sides by $-\frac{2m}{\hbar^2}$:

$$\frac{\ddddot{x}}{2\dot{x}^4} = \frac{2m}{\hbar^2}\left(\frac{k}{x^2} - m\ddot{x}\right) - 5\frac{\ddot{x}^3}{\dot{x}^6} + 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5}$$

Multiply through by $2\dot{x}^4$:

$$\ddddot{x} = 2\dot{x}^4 \cdot \left(\frac{2m}{\hbar^2}\left(\frac{k}{x^2} - m\ddot{x}\right) - 5\frac{\ddot{x}^3}{\dot{x}^6} + 4\frac{\ddot{x}\,\dddot{x}}{\dot{x}^5}\right)$$

Expanding this:

$$\ddddot{x} = \frac{4m}{\hbar^2}\dot{x}^4\left(\frac{k}{x^2} - m\ddot{x}\right) - 10\frac{\ddot{x}^3}{\dot{x}^2} + 8\frac{\ddot{x}\,\dddot{x}}{\dot{x}}$$

We define the following auxiliary variables:

$$x_1 = x \quad \text{(position)}, \quad x_2 = \dot{x} \quad \text{(velocity)}, \quad x_3 = \ddot{x} \quad \text{(acceleration)}, \quad x_4 = \dddot{x} \quad \text{(jerk)}$$

Using the defined variables, rewrite the original fourth-order differential equation as a system of first-order differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= x_3, \\
\frac{dx_3}{dt} &= x_4, \\
\frac{dx_4}{dt} &= \frac{4m}{\hbar^2}\dot{x}^4\left(\frac{k}{x_1^2} - mx_3\right) - 10\frac{x_3^3}{x_2^2} + 8\frac{x_3 x_4}{x_2}
\end{aligned} \tag{16}$$