Foundations of Programming Languages 2023

Fundamental theorems of the λ-calculus

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Fundamental theorems of the λ-calculus

- **The Church-Rosser theorem**
- The finiteness of developments
- The conservation theorem for λ_I
- **Standardisation**

The diamond property

A reduction relation satisfies the *diamond property* iff:

$$\begin{array}{ccc} M & \longrightarrow & N_2 \\ \downarrow & & \downarrow \\ N_1 & \longrightarrow & N \end{array}$$

The relation $\to \beta$ does not satisfy the diamond property:

$$\begin{array}{ccccc} M = \omega(II) & & \longrightarrow & & \omega I = N_2 \\ \downarrow & & & & \downarrow \\ N_1 = (II)(II) & \longrightarrow & (II)(I) & \longrightarrow & II = N \end{array}$$

where $\omega = \lambda x.xx$.

The Church-Rosser theorem

A reduction R is confluent (or Church-Rosser), if its reflexive and transitive closure $\twoheadrightarrow_R$ satisfies the *diamond property*.

$$\begin{tikzcd} M \arrow[r, two heads] \arrow[d, two heads] & N_2 \arrow[d, two heads] \\ N_1 \arrow[r, two heads] & N \end{tikzcd}$$

$\twoheadrightarrow_\beta$ satisfies the diamond property therefore β **is Church-Rosser**.

Proving confluence:

- Newman's Lemma
- Tait and Martin-Löf's multi-step reduction
- Using finiteness of developments

Newman's Lemma

Formally, a reduction R is Weak Church-Rosser (WCR), if it satisfies the *weak diamond property*.

A reduction satisfies the *weak diamond property* iff:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N_2 \arrow[d, two heads] \\ N_1 \arrow[r, two heads] & N \end{tikzcd}$$

The relation β satisfies the *weak-diamond property*.

Newman's Lemma

Strong Normalization $\wedge$ Weak-Church Rosser $\Rightarrow$ Church Rosser

Although β is WCR, it is not strongly normalisable.

WCR $\not\Rightarrow$ CR

To prove that β is WCR it is sufficient to consider the redexes Δ_1, Δ_2, respectively contracted in:

$$M \xrightarrow{\Delta_1}_\beta N_1 \qquad M \xrightarrow{\Delta_2}_\beta N_2$$

and construct N such that $N_1 \twoheadrightarrow_\beta N, N_2 \twoheadrightarrow_\beta N$, by contracting the **residuals**[1] of Δ_1, Δ_2 in N_2, N_1, respectively.

However, WCR is not sufficient to show confluence due to the existence of infinite reduction paths.

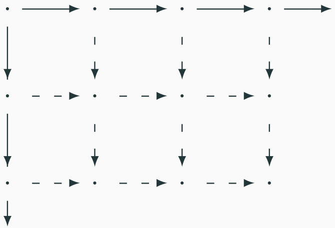
$$\begin{array}{ccccccc} M & \rightarrow & N_2 & \rightarrow & N_3 & \rightarrow & \cdots \\ \downarrow & & \downarrow\!\!\downarrow & & & & \\ N_1 & \twoheadrightarrow & N & & & & \end{array}$$

[1] more on this later...

Tait and Martin-Löf's proof

If a relation R satisfies the diamond property, then its transitive closure R^* also satisfies the diamond property.

As suggested by the following diagram:



To prove that $\twoheadrightarrow_\beta$ satisfies the diamond property:

- define a reduction relation that satisfies the diamond property;
- show that $\twoheadrightarrow_\beta$ is the transitive closure of that relation.

Some insight

Looking again at:

$$\begin{tikzcd} \omega(II) \arrow[rr] \arrow[d] & & \omega I \arrow[d] \\ (II)(II) \arrow[r] & (II)(I) \arrow[r] & II \end{tikzcd}$$

Note that:

$$\begin{tikzcd} \omega(II) \arrow[r, two heads] \arrow[d, two heads] & \omega I \arrow[d, two heads] \\ (II)(II) \arrow[r, two heads] & II \end{tikzcd}$$

Defining the multi-step relation (first attempt)

In the reduction of $\omega(II)$ one cannot merge $(II)(II)$ and ωI to II in one step because of the two redexes in $(II)(II)$.

What if we reduce them in parallel?

(i) $x \Rightarrow x$

(ii) $(\lambda x.M)N \Rightarrow M[N/x]$

(iii) If $M \Rightarrow M'$ then $\lambda x.M \Rightarrow \lambda x.M'$

(iv) If $M \Rightarrow M'$ and $N \Rightarrow N'$ then $MN \Rightarrow M'N'$

Now we have $(II)(II) \Rightarrow II$.

But is this relation good enough?

$\Rightarrow$ and the diamond property

$$
\begin{array}{ccc}
(\lambda x.M)N & \Rightarrow & M[N/x] \\
\Downarrow & & \Downarrow \\
(\lambda x.M')N' & \Rightarrow & \dots
\end{array}
$$

For example:

$$
\begin{array}{ccccc}
(\lambda x.(\lambda y.yy)x)(II) & & \Rightarrow & & (\lambda x.xx)I \\
\Downarrow & & & & \Downarrow \\
(\lambda y.yy)(II) & \Rightarrow & (\lambda y.yy)I & \Rightarrow & II
\end{array}
$$

The parallel reduction $\Rightarrow$ does not satisfy the diamond property

The multi-step relation $\rightrightarrows$[2]

The multi-step relation is defined as:

(i) $x \rightrightarrows x$

(ii) If $M \rightrightarrows M'$ then $\lambda x.M \rightrightarrows \lambda x.M'$

(iii) If $M \rightrightarrows M'$ and $N \rightrightarrows N'$ then $MN \rightrightarrows M'N'$

(iv) If $M \rightrightarrows M'$ and $N \rightrightarrows N'$ then $(\lambda x.M)N \rightrightarrows M'[N'/x]$

Now we have:

$$\begin{array}{ccc}
(\lambda x.(\lambda y.yy)x)(II) & \rightrightarrows & (\lambda x.xx)I \\
\downdownarrows & & \downdownarrows \\
(\lambda y.yy)(II) & \rightrightarrows & II
\end{array}$$

[2] This relation is called parallel by Takahashi.

Properties of $\rightrightarrows$

$\rightrightarrows$ includes the β-identity of λ-terms:

- If $M =_\beta M'$ then $M \rightrightarrows M'$.

$\rightrightarrows$ includes $\to \beta$:

- If $M \to_\beta M'$ then $M \rightrightarrows M'$.

$\rightrightarrows$ is included in $\twoheadrightarrow_\beta$:

- If $M \rightrightarrows M$ then $M \twoheadrightarrow_\beta M'$.

$\rightrightarrows$ is compatible with substitution:

- If $M \rightrightarrows M'$ and $N \rightrightarrows N'$, then $M[N/x] \rightrightarrows M'[N'/x]$.

β is confluent

As consequences of the previous properties...

- $\rightrightarrows$ satisfies the diamond property (Prove!)
- $\twoheadrightarrow_\beta$ is the transitive closure of $\rightrightarrows$:

$$\rightarrow_\beta \subseteq \rightrightarrows \subseteq \twoheadrightarrow_\beta$$

Since $\twoheadrightarrow_\beta$ is this transitive closure of $\rightarrow_\beta$, so it is of $\rightrightarrows$.

Theorem (Church-Rosser)

If $M \twoheadrightarrow_\beta N_1$ and $M \twoheadrightarrow_\beta N_2$, there exists N such that $N_1 \twoheadrightarrow_\beta N$ and $N_2 \twoheadrightarrow_\beta N$. Thus β is Church-Rosser.

A simpler proof due to Takahashi

From M obtain a term M^* by contracting all the existing redexes.

(i) $x^* = x$

(ii) $(\lambda x.M)^* = \lambda x.M^*$

(iii) $(MN)^* = M^*N^*$ if M is not an abstraction

(iv) $((\lambda x.M)N)^* = M^*[N^*/x]$

Takahashi proved the following stronger result for $\rightrightarrows$ (triangle method):

$$\text{If } M \rightrightarrows N \text{ then } N \rightrightarrows N^*$$

Other methods to proof confluence

Recall that $\rightarrow \beta$ satisfies the weak-diamond property, which does not imply confluence.

The following stronger result does imply confluence:

$$\begin{array}{ccc} M & \rightarrow & M' \\ \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow \\ N' & \twoheadrightarrow & N \end{array}$$

This is called the strip lemma, and β is CR follows now from:

$$\begin{array}{ccccccccc} M & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & \cdots & \rightarrow & M' \\ \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & & & \downarrow\!\!\!\!\downarrow \\ \cdot & \twoheadrightarrow & \cdot & \twoheadrightarrow & \cdot & \twoheadrightarrow & & \twoheadrightarrow & N' \end{array}$$

The strip lemma

To prove the strip lemma, suppose that $M \xrightarrow{\Delta}_\beta M'$, then:

- keep track of what happens to Δ in $M \twoheadrightarrow_\beta N'$;
- reduce all the **residuals** of Δ in N' to obtain N.

Extend the set of λ-terms to deal with the bookkeeping.

- Define a new set of terms, where the first abstraction in a redex is marked with an index;
- Extend the notion of β-reduction to deal with both indexed and non-indexed abstractions.

$$
\begin{array}{ll}
\beta_0 : & (\lambda_i x.M)N \to M[N/x] \\
\beta_1 & (\lambda x.M)N \to M[N/x]
\end{array}
$$

Indexed terms

For any given term M and set of redex occurrences $\mathcal{F}$, we define

$$(M, \mathcal{F})$$

as the indexed term obtained from M by indexing the occurrences given by $\mathcal{F}$.

Example

For $M = (\lambda x.(II)x)(II)(Iz)$ and $\mathcal{F} = \{\text{second } II, Iz\}$ then

$$(M, \mathcal{F}) = (\lambda w.(II)w)((\lambda_0 x.x)I)((\lambda_0 x.x)z)$$

Note that $(M, \varnothing)$ is just a normal λ-term.

Residuals

In a reduction

$$\sigma : M \twoheadrightarrow_\beta N$$

The set of residuals of a set of redexes $\mathcal{F} \subseteq M$ in N, relative to σ (denoted $\mathcal{F}/\sigma$) is the set of remaining indexed redexes after σ.

$$\begin{array}{ccc} M' = (M, \mathcal{F}) & \xrightarrow{\sigma'}\!\!\!\!\rightarrow & N' = (N, \mathcal{F}/\sigma) \\ |\,|\downarrow & & \downarrow|\,| \\ M & \xrightarrow{\sigma}\!\!\!\!\rightarrow & N \end{array}$$

where $|M|$ just removes the indexes from M.

Intuitively: one marks a set of redexes in the initial term, and follows them thought the course of σ.

Examples[3]

Consider $\Delta = (\lambda a.a(Ix))(xb)$

σ	$\mathcal{F}/\sigma$
$(\lambda x.xx)a\Delta \to aa\Delta$	the unchanged Δ
$(\lambda x.xx)\Delta \to \Delta\Delta$	two residuals Δ
$(\lambda x.y)\Delta \to y$	no residuals
$(\lambda x.x)\Delta \to (\lambda x.x)(xb(Ix))$	no residuals
$(\lambda x.\Delta)P \to (\lambda a.a(Px))(Pb)$	the residual $\Delta[P/x]$
$\Delta \to (\lambda a.ax)(xb)$	the residual $(\lambda x.ax)(xb)$

[3] taken from [Barendregt94]

The Finiteness of Developments

A *development* of M (relative to a set of redexes $\mathcal{F}$) is a reduction

$$M \xrightarrow{\Delta_1}_\beta M_1 \xrightarrow{\Delta_2}_\beta M_2 \xrightarrow{\Delta_3}_\beta \cdots$$

where Δ_i are redexes in $\mathcal{F}$ or their **residuals** (we only perform β_0).

A *development* of M (relative to a set of redexes $\mathcal{F}$) is complete if the set of residuals of $\mathcal{F}$ becomes empty.

Theorem

All developments of a λ-term M are finite.

For example,

$$(\lambda_0 x.xx)(\lambda x.xx) \to_{\beta_0} (\lambda x.xx)(\lambda x.xx) \in \mathsf{NF}_{\beta_0}$$

How redexes are created

Lévy (1978) defined three ways to create new redexes:

(i) $(\lambda xy.M)NP \to (\lambda y.M[N/x])$

(ii) $(\lambda x.x)(\lambda y.M)P \to (\lambda y.M)P$

(iii) $(\lambda x.C[xN])(\lambda y.M) \to C'[(\lambda y.M)N'$

Note that the only indexed redexes will be residuals of already indexed redexes.

Another CR proof

Definition: If N is a complete development of M then $M \rightrightarrows N$.

We use the same notation for $\rightrightarrows$ because both definitions define the same relation.

For example $\omega(II)$ (as before) we have:

$$\omega(II) \rightrightarrows (II)(II), \qquad \omega(II) \rightrightarrows II, \qquad \omega(II) \rightrightarrows \omega I$$

but not

$$\omega(II) \rightrightarrows I(II), \qquad \omega(II) \rightrightarrows (II)I.$$

To prove CR one just needs to prove that $\twoheadrightarrow_\beta$ is the transitive closure of $\rightrightarrows$ (has defined above) and that $\rightrightarrows$ satisfies the diamond property.

Conservation Theorem for λ_I

Church's initial approach to the λ-calculus considered a restricted set of terms (λ_I), in which abstractions were non-erasing.

An I-redex is a term $(\lambda x.M)N$ where $x \in \mathrm{fv}(M)$.

Theorem (Conservation for λ_I)

If $M \to N$, then $\infty(M) \Rightarrow \infty(N)$.

Or in other words:

$$M \text{ is normalisable } \Rightarrow M \text{ is strongly normalisable.}$$

The general λ-calculus (sometimes denoted λ_K) does not have this property.

Note that $(\lambda xy.x)I\Omega$ has a normal form, but it is not strongly normalisable.

The conservation theorem and the theory of residuals

The conservation theorem is a consequence of the finiteness of developments and the fact that, in the λ_I calculus, when reducing:

$$M \xrightarrow{\Delta_1} N$$

The only redex that has no residual is Δ_1.

This is not true in the λ_K: $(\lambda xy.y)\Omega I \to (\lambda y.y)I$ and Ω has no residual after this reduction step.

Nonetheless, there is a conservation theorem in the λ_K-calculus:

$$\text{If } M \xrightarrow{\Delta} N \text{ and } \Delta \text{ is an } I\text{-redex then } \infty(M) \to \infty(N).$$

Standardisation

The standardisation theorem states that, if $M \twoheadrightarrow N$, then there is a standard reduction from M to N.

A standard reduction (denoted $\underset{s}{\twoheadrightarrow}$) is a sequence

$$\sigma : M_1 \xrightarrow{\Delta_1} M_2 \xrightarrow{\Delta_2} M_3 \xrightarrow{\Delta_3} \cdots$$

in which, for every $i < j$, Δ_j is not a residual of a redex to the left of Δ_i, i.e., all contractions proceed from left to right.

For example:

$$(\lambda x.xx)(II) \to (\lambda x.xx)I \to II \to I$$

is not a standard reduction, since $(\lambda x.xx)I$ is to the left of II, whereas the following is:

$$(\lambda x.xx)(II) \to (II)(II) \to I(II) \to II \to I$$

Head reduction

Every λ-term has one of two forms:

(i) $\lambda x_1 \dots x_n . x M_1 \dots M_m, n, m \geq 0$

(ii) $\lambda x_1 \dots x_n . (\lambda x . M_0) M_1 \dots M_m, n \geq 0, m \geq 1.$

In (i) x is called head variable, and in (ii) $(\lambda x.M_0)M_1$ is called head-redex.

Also, a head normal form is a term of the form (i).

The head reduction of M is the sequence:

$$M = M_1 \xrightarrow{\Delta_1} M_2 \xrightarrow{\Delta_2} \cdots$$

where Δ_i are head-redexes.

Internal reduction

An internal redex of a term M is a redex Δ that is not the head-redex of M.

For example, for $M = \lambda x.(\lambda y.x)(\omega(Ix))$:

- $(\lambda y.x)(\omega(Ix))$ is the head-redex;
- $\omega(Ix)$ and Ix are internal redexes.

For any reduction $M \twoheadrightarrow N$, there exists M' such that

$$M \underset{h}{\twoheadrightarrow} M' \quad \text{and} \quad M' \underset{i}{\twoheadrightarrow} N$$

where $\underset{h}{\twoheadrightarrow}$ (resp. $\underset{i}{\twoheadrightarrow}$) denotes head (resp. internal) reduction.

Again, this is a consequence of the finiteness of developments.

Standardisation theorem

Theorem: If $M \twoheadrightarrow N$, then $M \underset{s}{\twoheadrightarrow} N$.

Proof: By induction on N using the fact that $M \underset{h}{\twoheadrightarrow} M' \underset{i}{\twoheadrightarrow} N$.

- If $N = x$ then $M' = x$ and the reduction is a standard one (h is standard).
- If $N = \lambda x_1 \dots x_n . N_0 N_1 \dots N_m$ $(n + m > 0)$ then

  $$M' = \lambda x_1 \dots x_n . M_0 M_1 \dots M_m$$

  with $M_i \twoheadrightarrow N_i$, for $0 \leq i \leq m$. By induction hypothesis:

  $$\sigma_i : M_i \underset{s}{\twoheadrightarrow} N_i$$

  therefore, for $\sigma : M \underset{h}{\twoheadrightarrow} M'$ then

  $$\sigma + \sigma_1 + \cdots + \sigma_m : M \underset{s}{\twoheadrightarrow} N.$$

Head normal forms and solvability

Solvability captures the notion of meaningful/defined terms.

A λ-term M is solvable if there exists a closure $\lambda x_1 \ldots x_m.M$ and terms $N_1, \ldots, N_n$ such that

$$MN_1 \ldots N_n =_\beta I$$

For example $x\Omega$ is solvable since $(\lambda yx.x)\Omega \to I$ whereas Ω itself is not solvable.

A consequence of the standardisation theorem is:

M has a head normal form $\Leftrightarrow$ the head reduction of M terminates.

Wadsworth [1971] proved that:

M is solvable $\Leftrightarrow$ M has a head normal form.

Exercises

1. Recall the inductive definition of $\rightrightarrows$. Prove the following properties:
    1.1 If $M =_\beta M'$ then $M \rightrightarrows M'$.
    1.2 If $M \to_\beta M'$ then $M \rightrightarrows M'$.
    1.3 If $M \rightrightarrows M$ then $M \twoheadrightarrow_\beta M'$.
    1.4 If $M \rightrightarrows M'$ and $N \rightrightarrows N'$, then $M[N/x] \rightrightarrows M'[N'/x]$.
2. Prove that β is WCR.
3. Prove that both definitions of $\rightrightarrows$ define the same relation.