BASIC ENGINEERING SCIENCE

Donald E. Richards
Rose-Hulman Institute of Technology

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Donald E. Richards

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Acknowledgements

No textbook springs fully formed as an original contribution from the mind of an author, and this book is no exception. Although the author had long considered the use of an integrated approach in teaching thermodynamics, heat transfer, and fluid mechanics, the Texas A&M group broadened his conception of how an integrated approach could unify engineering science education. Charles Glover and Louis Everett were especially helpful in sharing ideas and encouragement.

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Last but definitely not least, the author would like to thank his wife and daughters—Martha, Abby, and Emma—for their love and support.

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Fall 2007 Edition: Although this textbook has been in continuous use from its first printing, this represents the first printed version with corrections in the typeset copy. The comments of students and colleagues have been incorporated in this version. Special thanks go to Kevin T. Nelson, Stephen DuPaul, and T. J. Kelly—three students who carefully and critically read the text.

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*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

Preface

> *"Engineers, unlike physicists, are after useful artifacts and must predict the performance of the objects they design."*
>
> *"Organization according to control-volume ideas is thus not only simpler but brings clearer understanding of the physical principles common to otherwise disparate situations."*
>
> *"In the end the requirements that have tipped the scales in favor of control-volume analysis lie in the goal or mission of the engineer—to design and produce useful artifacts."*
>
> *"Practicing engineers are always on the lookout for more effective tools with which to think and do."*
>
> *"By organizing knowledge according to physical laws rather than known problems, it aids in recognizing a control-volume problem when met in an unfamiliar disguise."*
>
> *"Control-volume analysis, by setting up an explicit method of bookkeeping for the various flow quantities, provides such a procedure for the many engineers who must deal with fluid-mechanical devices."*
>
> *"Control-volume analysis is useful precisely because it provides a framework and method for thinking clearly about a large class of the often confusing problems that arise in engineering design."*
>
> *— Walter G. Vincenti in What Engineers Know and How They Know It\(^1\)*

The words above were originally written about the development of the control volume[2] as a tool for analysis in thermodynamics and fluid mechanics. However, if you replace the phrases "control-volume ideas" and "control-volume approach" with the phrase "system, accounting, and modeling approach," the words apply equally well to the thrust of this textbook.

The current textbook is based on a different paradigm for organizing an engineering science core—a system, accounting, and modeling approach—that emphasizes the common, underlying concepts of engineering science. Although this approach is not necessarily new, as most graduate students have been struck by this idea sometime during their graduate education, its use as the organizing principle for an undergraduate curriculum is new. By focusing on the underlying concepts and stressing the similarities between subjects that are often perceived by students (and faculty) as unconnected topics, this approach provides students a framework for recognizing and building connections as they learn new material.

Background

In 1988, a group of faculty members at Texas A&M University began work on a new integrated curriculum to replace the core engineering science courses in a typical curriculum. The result was an interdisciplinary sequence of four courses called the Texas A&M/NSF Engineering Core Curriculum[3] and organized around what they called the conservation and accounting principle. Glover, Lundsford, and Fleming produced an introductory textbook[4] that used this approach. More recently Holtzapple and Reece have introduced this approach in a freshman text.[5] Recently, the author has also learned of a similar approach being promoted and developed by Prof. W. C. Reynolds at Stanford University for a course called ME10: Introduction to Engineering Analysis.[6] Calls to consider a systems approach have also come from physicists.[7, 8]

In 1993, seven schools came together as the Foundation Coalition (FC) under the auspices of the NSF Engineering Education Coalitions Program. One of the major thrusts of the FC was curriculum integration. Building on the earlier work at Texas A&M, Rose-Hulman developed a new sophomore engineering curriculum—the Rose-Hulman/Foundation-Coalition Sophomore

Engineering Curriculum (SEC). The curriculum is currently required for all Rose-Hulman students majoring in mechanical engineering, electrical engineering, and computer engineering.

The Sophomore Engineering Curriculum

The SEC is a required, eight-course[9] sequence of engineering science and mathematics courses completed during the sophomore year. The SEC covers material traditionally taught in dynamics, fluid mechanics, thermodynamics, electrical circuits, system dynamics, differential equations, matrix algebra, and statistics. Two faculty-student teams developed the curriculum and its content during the summers of 1994 and 1995. The curriculum was first taught in the fall of 1995. Currently the SEC consists of the eight courses shown in the table below:

Sophomore Engineering Curriculum	Quarter Credit Hours
Fall Quarter MA 221 Differential Equations and Matrix Algebra I (4) ES 201 Conservation & Accounting Principles (4)	8
Winter Quarter MA 222 Differential Equations and Matrix Algebra II (4) ES 202 Fluid & Thermal Systems (3) ES 203 Electrical Systems (3) ES 204 Mechanical Systems (3)	13
Spring Quarter MA 223 Statistics for Engineers (4) ES 205 Analysis & Design of Engineering Systems (5)	9
Total	**30**

One of the unique features of the SEC is the 1–3–1 sequence for the engineering science courses. The sequence starts with the general course ES 201 in the fall. In the winter, the courses are more discipline/phenomena specific with ES 202, 203, and 204. Finally in the spring, the focus again becomes more general with ES 205.

This Textbook

This textbook is based on over five years of experience in teaching the first engineering science course in the SEC, called ES 201: Conservation and Accounting Principles. ES 201 is taken during the first quarter of the sophomore year and introduces the systems, accounting, and modeling approach as the basis for engineering analysis. The content of ES 201 as mapped to traditional engineering science courses is shown in the table below:

Content Map for ES 201 — Conservation & Accounting Principles

Content Map for ES 201 — Conservation & Accounting Principles	

Fluid Mechanics
• Pressure
  ◦ Absolute vs. gauge pressure
  ◦ Forces due to uniform pressure
• Integral equations for control volumes
  ◦ Reynolds transport equation
  ◦ Conservation of mass
    ▪ mass and volume flow rate
    ▪ continuity equation
  ◦ Conservation of linear momentum
  ◦ Conservation of angular momentum
  ◦ Conservation of energy
Thermodynamics
• Basic concepts: system, property, state
• P-v-T relation for ideal gas: $pV = mRT$
• Simple substance models with constant specific heats
• Ideal gas model
• Incompressible substance model
• Conservation of energy and the First Law of Thermodynamics
  ◦ Mechanical concepts of work and energy
  ◦ Thermodynamic work
  ◦ Energy of a system: internal, kinetic and gravitational potential
  ◦ Energy transfers by work: pdV work, shaft work, electrical work
  ◦ Energy transfer by heat transfer
  ◦ Energy balance for open and closed systems
• Entropy and the second law of thermodynamics
  ◦ Reversible and irreversible processes
  ◦ Second law of thermodynamics
  ◦ Entropy transfer by heat transfer
  ◦ Entropy production in irreversible processes
  ◦ Entropy balance for open and closed systems
• Analysis of simple thermodynamics cycles
  ◦ Power, heat pump and refrigeration cycles
  ◦ Measures of Performance
  ◦ Performance of internally reversible cycle.
Material and Energy Balances
• Molar and mass flow rate
• Mixture composition
• Balanced chemical equations
• Production/consumption of chemical species in chemical reactions
• Chemical species accounting for systems with chemical reactions
	Electrical Circuits
	• Conservation of net charge
	• Kirchhoff's Current Law
	• Node voltages
	• Simple DC circuits
	Engineering Statics
	• Equilibrium of rigid bodies
	Engineering Dynamics
	• Kinematics of particles
	  ◦ Rectilinear and curvilinear motion
	  ◦ Rectangular components
	  ◦ Relative motion
	• Kinetics of particles
	  ◦ Newton's second law of motion
	  ◦ Equations of motion
	  ◦ Dry (Coulomb) friction
	  ◦ Principle of impulse and momentum
	  ◦ Impulsive motion
	  ◦ Mechanical work
	  ◦ Kinetic and potential energy
	  ◦ Principle of work and energy
	• Systems of particles
	  ◦ Mass center
	  ◦ Application of Newton's laws
	  ◦ Linear and angular momentum of a system of particles
	  ◦ Principle of impulse and momentum
	  ◦ Kinetic energy
	  ◦ Principle of work and energy
	  ◦ Variable systems of particles (Open systems or Control volumes)
	  ◦ Steady stream of particles
	  ◦ Systems gaining or losing mass
	• Kinematics of rigid bodies
	  ◦ Translation
	• Kinetics of rigid bodies with translation
	  ◦ Newton's second law of motion
	  ◦ Principle of impulse and momentum
	  ◦ Impulsive motion
	  ◦ Principle of work and energy
	  ◦ Conservation of mechanical energy

After a general discussion of the approach in the first two chapters, six fundamental physical laws are formulated using the systems and accounting framework. The fundamental laws are related to six extensive properties—mass, charge, linear momentum, angular momentum, energy, and entropy. In each case, the physical law is introduced by answering four questions about the pertinent extensive property:

- What is the property in question?
- How can it be stored in a system?
- How can it be transported across the system boundary?
- How can it be generated or consumed inside the system?

The answers to these questions provide the information to formulate each law within a systems and accounting framework. Once these questions are answered, the behavior of the property for a system can be described using an accounting (or balance) equation.

All but one of the physical laws are conservation principles. Although not a conservation principle, the sixth law (entropy accounting) is important because entropy can only be produced or in the limit of an internally reversible process conserved. A summary of the basic physical laws formulated in the systems and accounting framework can be found in the appendix.

Once the governing equations are developed the emphasis shifts to the analysis of system behavior. With the basic laws formulated in a consistent fashion, the problem becomes one of identifying the appropriate system, selecting and applying the pertinent accounting equations, and constructing a problem specific model. Throughout the text a consistent problem solving approach is emphasized regardless of the underlying physical laws. Again this is based on a series of generic questions as shown in the tables below:

Written Format	Typical Questions

• Known
• Find
• Given
• Analysis Strategy
  ∘ Constructing the Model
  ∘ Symbolic Solution
  ∘ Numerical Solution
• Comments
	• What's the system?
	• What properties should we count?
	• What's the time interval?
	• What are the important interactions?
	• How do the basic equations simplify?
	• What are the unknowns?
	• How many equations do I need?
	• What are the important constitutive relations?

This is another benefit of using the systems and accounting framework to organize the material. As an example, all problems involving linear momentum begin from the conservation of linear momentum equation. From this single starting point, problem specific forms can be obtained by applying appropriate modeling assumptions, e.g. closed vs. open system and transient vs. steady state vs. finite time. Using appropriate, problem-specific assumptions, we can quickly recover any of the "standard" forms, e.g. $F = ma$, $\sum F = 0$, the impulse-momentum equation, and the steady-state linear momentum balance for fluid mechanics. In each case, the emphasis is not on the final form of the equation but on the modeling assumptions and how they change the basic equations.

A Request of Students and Faculty

As the first effort to generate a complete textbook from a mushrooming set of notes, there are surely errors and omissions in the text. For these the author takes full credit and asks your help in identifying mistakes in this text. To eliminate these in future editions, you are encouraged to contact the author directly with any errors or omissions you identify.

You are also encouraged to contact the author and share your views about the systems, accounting, and modeling approach that forms the basis for this text. It is the author's firm belief that this approach has much to contribute to engineering education and that we have only begun to explore and exploit its potential impact. A major strength of the approach is in how it forms a foundation for advanced work. Faculty members are encouraged to explore how they can use what students learn from this text as a springboard to learning in related and advanced courses.

Sources

[1] Excerpts from Chapter 4, "A Theoretical Tool for Design: Control-Volume Analysis, 1912-1953," in *What Engineers Know and How They Know It*, The Johns Hopkins University Press, Baltimore, 1990.

[2] A control volume is a region in space as opposed to a fixed quantity of matter that is used for analysis. In mechanics, the use of a control volume is called the Eulerian approach, while using a control mass, or a fixed quantity of matter, is called the Lagrangian approach.

[3] C. J. Glover, C. A. Erdman, "Overview of the TAMU/NSF Engineering Core Curriculum Development," presented at the 1992 ASEE/IEEE Frontiers in Education conference, 11-14 November 1992, Nashville, Tennessee. Also see http://ww-chen.tamu.edu/uesc/ about the Unified Engineering Science Core.

[4] C. J. Glover, K. M. Lunsford and J. A. Fleming, Conservation Principles and the Structure of Engineering, 5th Ed., McGraw-Hill, New York, 1996.

[5] M. T. Holtzapple and W. D. Reece, Foundations of Engineering, McGraw-Hill, Boston, 2000.

[6] W. C. Reynolds, *Introduction to Engineering Analysis: An integrated approach to the fundamental principles that underlie all engineering analysis*. Notes under development by Prof. W. C. Reynolds at Stanford University.

[7] H. Burkhardt, "Systems physics: a uniform approach to the branches of classical physics," *American Journal of Physics*, Vol. 55, pp. 344-350.

[8] H. U. Fuchs, *The Dynamics of Heat*, Springer-Verlag, New York, 1996.

[9] Eight courses on a quarter system for a total of thirty quarter-credit hours spread over three quarters.

CHAPTER OVERVIEW

1: Introduction

1.1: Introduction

Frequently students enter engineering with little knowledge of the engineering profession. If you were to ask students why they selected engineering you might hear several answers:

- "I was good in mathematics and physics (or chemistry)."
- "I like to work on cars."
- "I like to take things apart."
- "My mom (or dad or uncle or sister) is an engineer and it sounded interesting."
- "I want a job with a good salary."

The common thread usually being an interest in making things and a natural ability in mathematics and the sciences.

If you were to ask graduate engineers several years after graduation what they do, you would get varying responses but you might hear the following:

- "Engineers are problem solvers."
- "Engineers design things."
- "Engineers build things."

The common thread here being problem solving and designing and building new things. Few will mention science and mathematics.

Edward Krick provides a fairly concise description of this in his book, *An Introduction to Engineering and Engineering Design* [1] :

> *"An engineer is a problem solver. Ordinarily starting with a broadly expressed function-to-be-filled, the engineer must translate this general statement of what is wanted into the specifications for a device (or structure or process) which will economically fulfill that objective. To arrive at this solution the engineer must apply his or her knowledge and inventiveness to uncover a reasonable proportion of the many alternative solutions to the typical problem. The engineer must evaluate these alternative solutions in the face of numerous intangible and conflicting criteria. The limited time available for proposing a solution precludes an exhaustive exploration of all possible solutions. In lieu of complete information the engineer makes extensive use of judgment. The engineering problem that is not complicated by economic considerations is rare indeed. A private enterprise ordinarily accepts an engineer's solution to a problem only if it shows commercial promise, and a public enterprise insists on an attractive benefit-to-cost ratio."*[2]

Consider: What exactly is a profession?

Reread this passage. What are the various parts of this job description for an engineer?

In the same text, Krick also provides us with a succinct definition of engineering:

"Engineering is a profession concerned primarily with the application of a certain body of knowledge, set of skills, and point of view, in the creation of devices, structures, and processes used to transform resources to forms which satisfy the needs of society." [3]

Notice that this definition is not limited to a specific type of engineering.

Sources

[1] Edward V. Krick. *An Introduction to Engineering and Engineering Design*. John Wiley & Sons, New York, 1965.

[2] Ibid., pg. 30.

[3] Ibid., pg. 40.

1.2: Engineering Analysis and Engineering Design

Engineering analysis and engineering design are two activities common to all engineering disciplines (Figure 1.2.1). **Engineering analysis** is the process by which an engineer develops a model of an engineering problem and uses the model to obtain useful engineering information. **Engineering design** is the *iterative* process through which an engineer creates *new* devices, structures, and processes. Engineering design always includes engineering analysis; however, engineering analysis may be undertaken for many other reasons. In this course, we will focus on engineering analysis; however, do not be mislead into thinking that engineering is only engineering analysis.

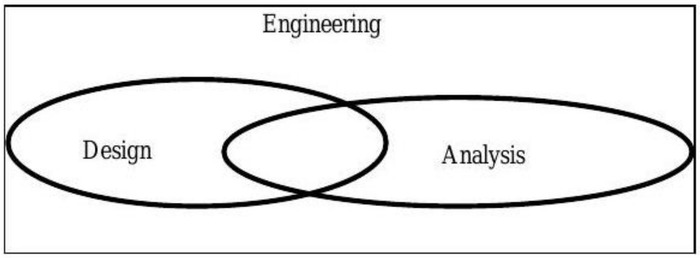


Figure 1.2.1: Engineering Analysis and Engineering Design

1.3: Science, Engineering Science, and Mathematics

The foundation of engineering education begins with mathematics, physics, and chemistry. In recent years, the biological sciences have played an increasingly important role in basic science education.

Building on this foundation is a core of engineering knowledge that all engineers should understand. This core material has typically been taught over several quarters (sometimes years) in several independent courses. These courses - statics, dynamics, mechanics of materials, thermodynamics, fluid mechanics, heat transfer, and electrical circuits - are collectively referred to as the **engineering sciences**.

The engineering sciences have much in common with the sciences as they are based on the same natural laws and use mathematics as a common language. However, there are significant differences in both organization and emphasis.

Differences between the sciences and engineering sciences are partially a result of the different goals of science and engineering. The sciences have grown out of the scientific community where the primary goal is understanding and describing nature. The engineering sciences by contrast have grown out of the engineering community where the primary goal is design - the creation of new things. This means that engineering and the engineering sciences are much more than just applied science. Engineers are often faced with developing new, albeit incomplete, knowledge to bridge gaps in the available science and allow them to keep designing.

The emphasis on design means that engineers face new and unique problems every day. They not only are expected to find the best answer but to clearly demonstrate that their solution is correct. (How many times a day do you take for granted that an engineer did their job correctly?) Because of this, *engineering education emphasizes the methodology and process of problem solving*. (See Appendix A.) Students often fail to recognize this important aspect of engineering education.

When faced with a new problem and a *required* problem solving approach, students frequently object because they know the formula from a similar problem in physics or chemistry. Why bother to *construct* a solution to this new problem? Unfortunately, pattern matching often leads to incorrect solutions and severely limits a student's ability as an engineer. It is precisely because we cannot test your ability to solve every conceivable problem you will meet that we emphasize the methodology and process of the problem solving approach. If you can successfully apply this approach to the various problems we'll tackle in class, we have faith that you can also successfully tackle new problems in the future.

The consistent application of a problem-solving methodology using explicit modeling assumptions to develop models from fundamental principles is the hallmark of engineering courses. This approach will be stressed repeatedly in this course. Throughout this course you will be asked to not only solve a problem but to document your solution and demonstrate that it logically follows from the basic physical laws.

1.4: Modeling and Engineering Analysis

Engineering analysis and modeling are intimately related. In fact, modeling is something we do daily as we solve the problems of every day life. In most engineering science courses, you will be developing your ability to construct mathematical models that serve as a basis for design decisions.

What is a model?

Rather than answer this question directly, let's explore what a model is by doing some modeling. Imagine that you have been asked to describe a typical engineering course to a college friend who is not an engineering student. Surely, you have some model in your mind. Based upon your experience you might describe a typical engineering course as follows:

1. Forty 50-minute classes (including three 50-minute exams).
2. One 4-hour comprehensive final exam.
3. Each class has 50 minutes of lecture interspersed with questions.
4. Two (really hard) homework problems are assigned and collected daily.
5. 10 to 20 pages of reading are assigned daily.
6. Homework must be worked and presented following a prescribed format.
7. Engineering classes are very hard and time consuming.

Now suppose your friend is really impressed and wants to know how hard the course is, specifically, how much time would you expect to spend weekly on one of these courses? Take a minute and predict how much time you would expect to spend.

- How much time did you predict? (16 hours, 24 hours, 32 hours, 40 hours?)
- How did you come up with your number?
- How did you model the courses to answer this question?

There are several ways to answer this question. One approach might be to estimate the time using the relationship

$$t = N \times (T + 1)$$

where t is time spent on the course, N is the total number of credit hours, and T is the hours per week spent outside of class per credit hour. Assuming one 4-credit hour course and say 3 hours per week outside of class for every hour in class, you get a total of 16 hours per week. (If a typical Rose student takes 16 credit hours per quarter, how many hours a week should he or she devote to the course? C'mon try calculating how many hours this would be. Hmmm! This almost sounds like a full-time job.)

Before proceeding, we should recognize that both our models are useful representations of an engineering course. The first model, a *descriptive* model, is a list of attributes of an engineering course. The second model, a *predictive* model, is a mathematical formula to predict the time spent during the week on a course.

How can you have two different models? Why are they different? Don’t they both describe the same engineering course? Herein lies the basis for our definition of a model. True, both models describe the same engineering course, but they were developed for different purposes. They were constructed to answer different questions. In both cases, the modeler took an engineering course in the "real world" and constructed a model of an engineering course in the "model world."

As illustrated in Figure 1.4.1, the real world and the model world appear to be separated. In passing from the real world to the model world, the modeler must carefully apply Occam's razor to carve away all but the essential elements. Occam's razor is named after William of Occam, a fourteenth century English philosopher, who challenged philosophers to keep only the essential elements in any problem.[1]

So what is a model? A **model** is a *purposeful* representation.2 The phrase "purposeful" is an essential part of the definition. As we have shown above, the very nature of a model depends upon its purpose — the reason it was constructed.

Reexamining our two models, we see that they each capture different features of an engineering course. In this way they are both *incomplete*.

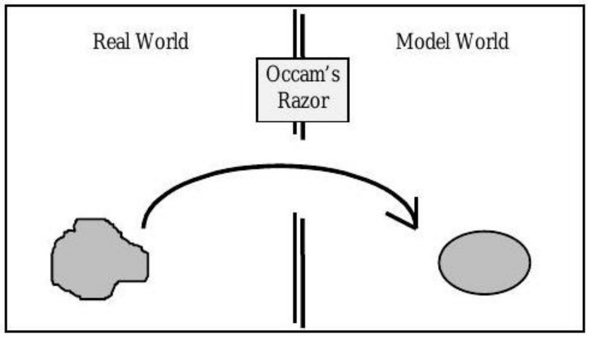


Figure 1.4.1: Occam's Razor

All models are incomplete to some extent. The best models capture only features of the "real world" that are essential to accurately answer the questions being posed. The best engineers judiciously apply Occam's razor to develop models that provide answers within the constraints of the available resources.

Types of Models

There are many different ways to classify models. Three useful classifications are shown in Figure 1.4.2. The first two classifications are fairly self explanatory; however, the third one may require some explanation.

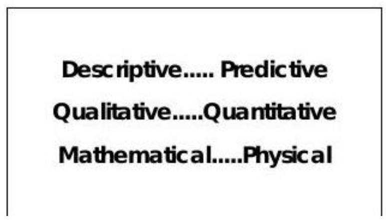


Figure 1.4.2: Ways to Classify Models

Engineers often make use of mathematical and of physical models. With the advent of modern day computers, mathematical models have become increasingly powerful. **Mathematical models** are descriptions of real systems using mathematical expressions that can be used to predict their system behavior. **Physical models** are scale or full-size representations of real systems whose performance is usually measured in the laboratory. Physical models are often used to verify the predictions of mathematical models. In many important technological applications, such as the flow of liquids in a pipe or flow over an aircraft wing, physical models are the best way to predict the behavior of the real system.

Mathematical models can be further classified as either deterministic or stochastic. **Deterministic models** will give you the same answer each time if the inputs are unchanged. A stochastic model, on the other hand, has the element of chance built into it and is only repeatable on average. **Stochastic models** are commonly used in fault-analysis or reliability analysis where a sequence of events must occur for something to happen and each event has a certain probability of occurrence.

While the focus of this course is on developing deterministic, mathematical models of engineering systems, the important role of physical models should not be forgotten. Later courses will illustrate the use of physical models as an engineering analysis tool.

Modeling Heuristics and Algorithms

What makes modeling and engineering problem solving such a challenge is that it is impossible to give you one set procedure that will always give a solution. Any set of steps that will always give you an answer is called an **algorithm**. Unfortunately, there are no algorithms for engineering problem solving.

Successful engineering problem solvers develop the necessary models by applying heuristics. A heuristic is "a plausible or reasonable approach that has often (but not necessarily always) proved to be useful; it is not guaranteed to be useful or to lead to a solution."[3] Some people call this a rule of thumb.

Consider: When you ask your parents for money, do you rely on an algorithm or a heuristic?

During the course of your engineering education you should begin to collect heuristics. In this course we will introduce many heuristics without explicitly calling them by name; however, we will often reflect on why we did something in a particular problem. This is when you should be on the lookout for a useful heuristic.

One of the most useful heuristics we will be using in this course is the problem solving method described in Appendix A. Every problem you solve in this course should be solved using this approach. Occasionally you will find a problem that is so simple that the full approach is unnecessary. But in most cases, it should be followed.

Sources

[1] A. M. Starfield, K. A. Smith, and A. L. Bleloch. *How to Model It: Problems Solving For the Computer Age*. Burgess Publishing 1994, p. 19.

[2] Ibid., p. 8.

[3] Ibid., p. 21.

1.5: Modeling an Engineering System

As discussed above, application of Occam's razor is a useful heuristic for any engineering problem solver, but how does this relate to specific engineering systems. Figure 1.5.1 illustrates the many inputs to the modeling process for engineering systems. All of the items in the central portion will be of interest as we learn to model engineering systems.

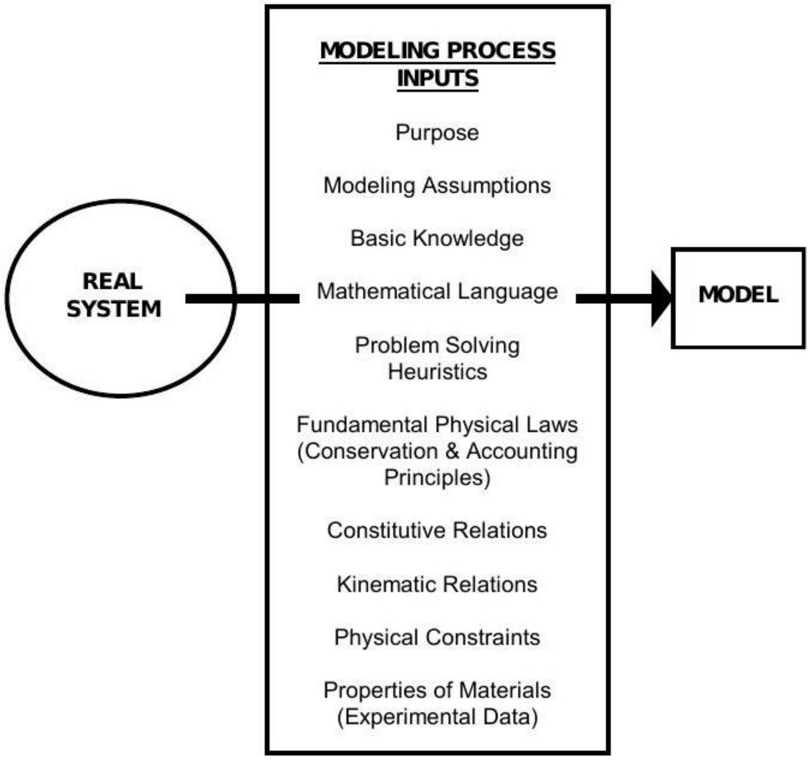


Figure 1.5.1: Modeling an engineering system

1.6: Conservation and Accounting - A Useful Framework

As discussed earlier, the engineering sciences have traditionally been taught in separate, independent courses. In taking this approach, similarities between the various subjects were often obscured. Because of this students failed to appreciate the significant common threads that run through all of these courses. The goal of this course and the Sophomore Engineering Curriculum is to help you see similarities between the various subjects. It is our belief that in stressing these, we can help you become better problem solvers across a broader range of subjects. In doing this, we believe you will be better engineers.

So what is the most significant common thread? The answer is the Accounting Principle. A simplified but correct version of the Accounting Principle is presented in Figure $1.6.1$. We will refer to this principle throughout the year in the Sophomore Engineering Curriculum. Please note that this principle is not the same as financial accounting; but it is a fundamental concept in physical science. You will find that you already have experience with it and naturally, although subconsciously, use the Accounting Principle daily. For example, when was the last time you balanced your check book?

Consider:

Imagine using the accounting principle to reconcile your checkbook with your monthly statement.

- What should you count?
- What's the system?
- What's the time period?
- **Can you produce money? (Legally!)**

$$\underbrace{\begin{bmatrix}\text{Stuff}\\ \textit{accumulated}\\ \text{inside the}\\ \text{system}\\ \text{during a}\\ \text{time period}\end{bmatrix}}_{\text{ACCUMULATION}} = \underbrace{\begin{bmatrix}\text{Stuff}\\ \textit{entering}\\ \text{the system}\\ \text{during a}\\ \text{time period}\end{bmatrix}}_{\text{INPUT}} - \underbrace{\begin{bmatrix}\text{Stuff}\\ \textit{leaving}\\ \text{the system}\\ \text{during a}\\ \text{time period}\end{bmatrix}}_{\text{OUTPUT}} + \underbrace{\begin{bmatrix}\text{Stuff}\\ \textit{produced}\\ \text{inside}\\ \text{the system}\\ \text{during a}\\ \text{time period}\end{bmatrix}}_{\text{PRODUCED}} - \underbrace{\begin{bmatrix}\text{Stuff}\\ \textit{consumed}\\ \text{inside}\\ \text{the system}\\ \text{during a}\\ \text{time period}\end{bmatrix}}_{\text{CONSUMED}}$$

Figure $1.6.1$: The Accounting Principle

Application of the Accounting Principle always begins with three questions:

1. What's the important stuff to count?
2. What's the system of interest?
3. What's the time period for counting?

Why is this single principle so important? Its significance lies in the fact that many of the fundamental laws of physics are based on keeping track of or accounting for extensive properties. When the property counted is mass, charge, linear momentum, angular momentum, energy, or entropy, the accounting principle gives us a common framework to think about and to apply the six fundamental natural laws:

- Conservation of Mass
- Conservation of Charge
- Conservation of Linear Momentum
- Conservation of Angular Momentum
- Conservation of Energy
- Entropy Accounting (The Second Law of Thermodynamics).

As we will show in the next chapter, conservation is a special case of the accounting principle.

The coming chapters will demonstrate how the Accounting Principle provides a common framework for engineering problem solving. Chapter 2 will present the basic concepts that underlie the Accounting Principle and its application to engineering systems.

1.7: Problems

Problem 1.1

(Adapted from Glover, Lunsford, and Fleming)

There are 20 chickens and 2 foxes present on an acre of land on the morning of June 1, 1990. During the day, half of the chickens leave the acre of land to search for food, but both of the foxes stay on the acre of land. By nightfall, all of the chickens that left have returned. At the end of the day, the total number of chickens is 15 and the total number of foxes is 2. Using the accounting principle, explain what happened. Remember to explicitly define the counted property, the system and its boundary, and the time period. Also state any assumptions you made in order to solve the problem.

Problem 1.2

(Adapted from Glover, Lunsford, and Fleming)

A baseball stadium can hold a maximum of 55,000 spectators. Before the gates open, there are only 200 stadium personnel in the stadium. When the gates open, there are 500 times more spectators than media personnel that enter the stadium. There are 50 baseball players that enter the stadium. During the game, one-fourth of the spectators leave. There are 30,000 spectators at the end of the game. No players, stadium personnel, or media personnel leave during the game. Using the accounting principle, calculate the total number of people in the stadium at the end of the game. [Hint: Try making a table to solve this problem]

Problem 1.3

Billy has a job delivering newspapers every morning before school. Billy is industrious and actually works for two different newspapers, the *Herald* and the *Post*. Billy's route covers five different streets. Every house gets either the *Herald* or the *Post* newspaper, but not both. Each morning Billy receives 140 newspapers from the *Herald* and 190 newspapers from the *Post*. The available data for the number of houses and newspapers delivered is given in the following table:

Streets	Total Number of Houses	*Herald* Houses	*Post* Houses
Elm	64	34	?
Park	58	27	?
Oak	37	?	20
Main	84	35	?
1st	75	?	50

Select a suitable system (or systems) and use the accounting principle to determine the total number of newspapers, including how many *Heralds* and *Posts*, Billy accumulates each day. (You will find that setting up a table will help you solve this problem.) Remember to explicitly define the counted property, the system and its boundary, and the time period. Also state any assumptions you made in order to solve the problem.

Problem \1.4

Financial data on Mr. Jones are presented in the table below:

Mr. Jones' Six-Month Financial Data

Transaction	January	February	March	April	May	June
Deposit	$2,600	$2,300	$2,000	$2,100	$2,400	$2,600
Expenses						
Mortgage	1,000	1,000	1,000	1,000	1,000	1,000
Auto	500	500	5000	500	500	500

Transaction	January	February	March	April	May	June
Bills	1,000	700	1,100	900	400	800
Food	190	160	210	180	200	140
Insurance	600	0	0	0	0	0

The initial balance at the beginning of January was $5,000. You have been asked to evaluate his financial health by tracking the monthly balance in his checking account. Use a conservation and accounting framework to solve the problem. [Hint: Set up a spreadsheet showing transports of money in and out of the account and the change in the account. Treat the account as a system.] Consider the following two cases and answer the questions:

Case A --- A non-interest bearing checking account.

Develop a spreadsheet that shows the balance in Mr. Jones' account at the end of each month. Plot the balance in the account as a function of time.

1. Is he ever bankrupt? If so, in what month does this occur? If not, what is the minimum positive balance and when does it occur?
2. What is the net change in the amount of money in the checking account for this six-month period?
3. If there is to be no net change in the amount of money in the account over the six months, what amount, on average, must Mr. Jones deposit monthly to his account? Does this answer depend on the initial balance in the account?

Case B --- An interest bearing checking account.

If at the end of each month, the money added to the amount in the form of interest is given by the following equation

$$\text{Dollars Added} = P \cdot i$$

where: P = the *average* amount of money in the account during the month
$i = 0.005(\$\text{ interest})/(\$\text{ of principal})$ [Equivalent of 6% annually]

Calculate the balance at the end of each month and plot the balance as a function of time. [Should you treat the interest dollars as money produced or money transported? Does it really make any difference?]

1. Is he ever bankrupt? If so, in what month does this occur? If not, what is the minimum positive balance and when does it occur?
2. What is the net change in the amount of money in the checking account for this six-month period?
3. If there is to be no net change in the amount of money in the account over the six months, what amount, on average, must Mr. Jones deposit monthly to his account? Does this answer depend on the initial balance in the account? Compare your answer to that is Case A. Does it make sense?

Problem 1.5

The Food Lion grocery store in Corolla, NC is located on the Outer Banks and serves a local population of residents and tourists of approximately 5,000 people. Only 20% of the people are permanent residents of the Outer Banks. The remaining 80% of the people – the tourists – rent cottages on a weekly basis and stay only one week. One-half of the tourists arrive on Saturday and the rest arrive on Sunday. As manager for the Food Lion store, you must order supplies on a weekly basis. Deliveries are made twice a week on Tuesday and Friday.

A typical family uses the following groceries in a week:

	Used per Family (4 persons)
Produce	14 lbs
Paper Products	1.5 ft^3
Milk	2 gallons
Beverages	56 cans

The residents do all of their shopping for the week on Wednesday to avoid the weekend crowds. The tourists buy 80% of their groceries on the day after they arrive and buy the remaining groceries four days after they arrive. Using the accounting principle, answer the following questions.

1. Develop a table and/or graph showing the daily demand for the groceries starting with Sunday.
   - What day has the peak demand?
   - If you are required to have a minimum of 10% reserve of your average daily sales, how much storage space would you need for paper products?
2. Determine the minimum amount of groceries that must be delivered on Tuesday and Friday to meet customer demand during the summer tourist season.
3. Your suppliers would like to increase the number of deliveries per week; however, they would also like to deliver *approximately* the same amount on each trip. How many deliveries per week would you suggest, what days of the week should they be made on, and how much should be delivered on each trip?

Problem 1.6

Fuel oil is used to fire a small power plant to generate electricity. The fuel oil fires a boiler that generates steam which in turn drives a steam turbine connected to an electric generator. The power plant requires $160\ \mathrm{lbm/h}$ of fuel oil to steadily generate electrical energy at the rate of $290\ \mathrm{kW}$. In addition, the combustion of the fuel requires $15.8\ \mathrm{lbm}$ of air for each pound-mass of fuel supplied to the power plant.

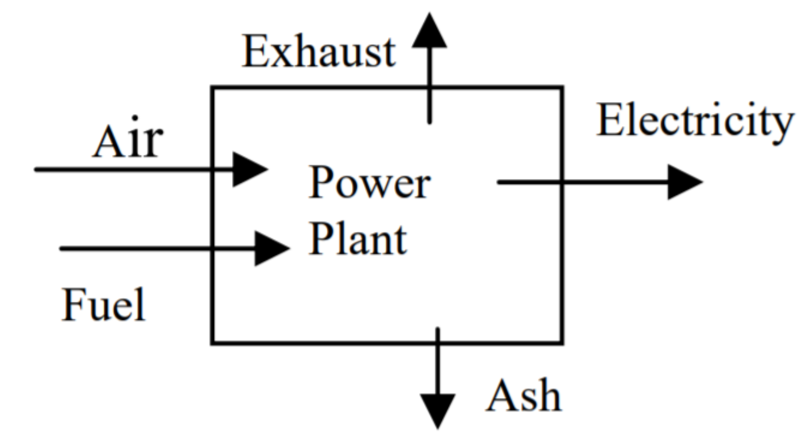


Figure $1.7.1$: Inputs and outputs of the power plant.

Use the accounting concept, clearly stating all assumptions, and determine:

1. the rate at which air must be supplied to the power plant and the rate at which exhaust and ash are produced, in $\mathrm{lbm/h}$.
2. the amount of fuel required and the amount of exhaust and ash produced, in lbm, if the plant operates continuously for 24 hours.

Problem 1.7

GitErDone, Inc. (GED), the premier manufacturer of Indiana widgets, manufactures RH Widgets. The GED factory consists of three areas: a repair shop (RS), a production shop (PS), and an inspection and delivery area (IDA). Over a five-day period, the staff recorded the following operating information:

Day	Damaged widgets returned to RS from customers	Widgets produced in PS and sent to IDA	Widgets rejected in IDA and returned to RS for repair	Good-as-new widgets sent from RS to IDA	Widgets delivered by IDA to customers

Day	Damaged widgets returned to RS from customers	Widgets produced in PS and sent to IDA	Widgets rejected in IDA and returned to RS for repair	Good-as-new widgets sent from RS to IDA	Widgets delivered by IDA to customers
1	33	13,600	1362	1350	14,000
2	52	12,600	1258	1150	14,000
3	47	14,600	1465	1050	14,000
4	28	14,000	1395	1250	14,000
5	40	13,200	1320	1350	14,000

At the beginning of Day 1, 500 damaged widgets were in the repair shop and 21000 widgets were in the inspection and delivery area. No widgets are stored in the production shop. The storage area in the IDA is small and can hold only a five-day delivery inventory.

Apply the generic accounting principle to an appropriate system (the factory, RS, PS, and/or IDA) to answer the following questions. Although hand calculations are acceptable, a spreadsheet helps is ideal for setting up this problem. If you use a spreadsheet, include a copy with your solution.

1. Determine the inventory of widgets and damaged widgets and their location at the end of each day. Show both tabular and graphical results.
2. If the factory inventory of widgets is increasing, determine the number of days to fill the available widget storage area. If the factory inventory of widgets is decreasing, determine the number of days before the inventory of widgets drops to zero. [Hint: Use the average daily values.]
3. Under optimum operating conditions, the average inventory of widgets in the factory is constant. Using the given operating data, estimate how many widgets on-average must be delivered to customers daily to achieve optimum conditions. Under these conditions, what is happening to the inventory of damaged widgets in the repair shop? Is this a problem?

CHAPTER OVERVIEW

2: Basic Concepts

2.1: System, Property, State, and Process

To provide a concrete example for the discussions to follow, let's consider the following problem:

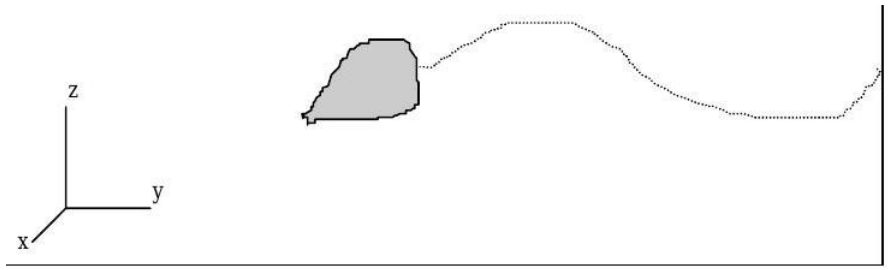


Figure 2.1.1: Problem -- A little girl is playing with a red balloon. Suddenly the string around the neck of the balloon becomes untied and the air begins to rush from the balloon. What is the path of the balloon?

To apply the accounting concept to model the balloon and predict its motion, we must first answer three questions:

- What property (or properties) do we want to count?
- What's the system we want to examine?
- What's the time period of interest?

These three questions should be answered each time we apply the accounting concept. But before we can answer the questions, we need to introduce some terms and concepts.

As a starting point for our discussion, we will begin to develop a vocabulary for describing the behavior of physical systems. These words sound familiar, but be sure that you understand the definitions given here for these terms.

2.1.1 System

The accounting concept can only be applied to a system. So what's a system? A system is any region in space or quantity of matter set aside for analysis. In the balloon problem, there are at least four possible systems:

- System A - the mass of air originally inside the balloon when the string comes untied,
- System B - the volume contained inside the balloon,
- System C - the balloon (just the rubber membrane), and
- System D - the balloon (System C) plus the volume contained inside the balloon (System B).

Notice that System D is really a composite system formed by combining System B and System C. That is, System B and System C are subsystems of System D.

Once you have defined your system, you have also defined the surroundings. The surroundings are everything outside of your system. The system and its surroundings are separated by the system boundary. In some texts, the system boundary is also called a control surface. Figure 2.1.2 shows a sketch of a generic system with the system boundary indicated as a dashed line. This is common practice in sketching systems.

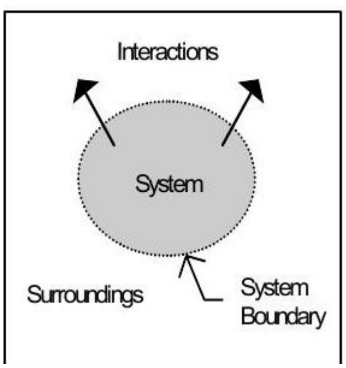


Figure 2.1.2: A simple system sketch.

Control surfaces may coincide with a real, physical boundary in the problem, or they may be imaginary surfaces in space. Imagine your room as a system. If you treat the door opening as a control surface, the surface may be real if it coincides with a shut solid door; however, if the door is open then you might describe the surface as being imaginary.

Every system can be classified as either open or closed. A **closed system** is a system with a fixed quantity of mass, and is often called a *control mass*. An **open system** is a system with a specified volume and is frequently referred to as a *control volume*. The mass of a closed system cannot change; however, other extensive properties including its volume may change. Both the mass and the volume of an open system may change.

Let's return to the balloon problem and classify the four systems we identified earlier:

- **System A** - the air originally inside the balloon when the string comes untied.

  This is a closed system because we have identified the mass as the system. To better visualize how this system will behave as time proceeds imagine that all of the air initially in the balloon is stained green. Then imagine what this air looks like as the balloon zooms around and leaves a trail of "green" air. The system boundary (or control surface) would be the sharp interface between the stained and unstained air.
- **System B** - the volume contained inside the balloon.

  This is an open system because we have identified a region of volume as the system. In this system, the system boundary consists of the inner surface of the balloon membrane and the imaginary surface that covers the opening at the neck of the balloon. The air rushing out of the balloon passes out of this system and the control volume will shrink in size as the balloon deflates. Although we have defined the system in terms of a volume, the mass of air contained in the volume is also in the system and as the balloon deflates the amount of mass in this system decreases.
- **System C** - the balloon (just the rubber membrane).

  This is also a closed system. Why isn't it an open system? Does the mass of the system change? Does the volume of the system change?
- **System D** - the balloon plus the volume contained inside the balloon.

  The boundary of this system consists of the outer surface of the balloon and an imaginary surface that covers the opening at the neck of the balloon.

✔ Example — Test your understanding of systems

To test your understanding try to answer the following questions about System D:

- Is System D an open or closed system? Why?
- Does its mass change? Does its volume change?
- How does the system boundary of System D differ from that of System B?
- To better understand the distinctions between systems, resketch the balloon and identify (draw) all four systems below. Use dashed lines to indicate the control surfaces.

Typically a system interacts with its surroundings in various ways. We will refer to any exchange (or communication) between a system and its surroundings as an **interaction**. Interactions occur across the boundary of a system and are represented as arrows on Figure $2.1.2$. The interactions that can occur depend on both the system you have selected and the physical description of the problem. Without interactions, system behavior would be pretty boring. Much of our effort this quarter will be aimed at learning how to make **modeling assumptions** that capture the important features of the interactions in a problem.

✔ Test Yourself

Using a dashed line, sketch a system that encloses the light bulb and cuts the electric cord.

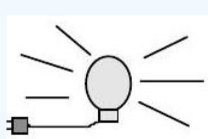


Figure 2.1.3: Sketch of a light bulb

(a) Is this an open or closed system?

(b) List all the interactions that occur between this system and its surroundings.

There is one very special type of system that has no interactions with its surroundings. We call this an **isolated system**. It is physically impossible to construct a system that is completely isolated from its surroundings; however, it is often a useful approximation (assumption) for modeling purposes. If you assume that a system is isolated, you are assuming that it has NO interactions of any kind with the surroundings. This is a very powerful assumption and places a severe restriction on the behavior of a system.

2.1.2 Property

Every system has many characteristics that we could use to describe it. We are specifically interested in characteristics that can be measured by observing the system or can be calculated directly from these observations. These are called properties. A property is any characteristic of a system that can be assigned a numerical value at a specified time without considering the history of the system.

Now consider the air inside our balloon just before it starts to zoom around. How many different properties of the system could you list? After some thought, your list might look like this:

• temperature • pressure • velocity • density	• kinetic energy • potential energy • mass • volume

All of these are properties of the air in the system. Now examine the two lists. What's different about the two lists? Any ideas?

? Test Yourself

What are the three key characteristics of a property?

1)

2)

3)

It turns out that the list on the properties on the left are all intensive properties and the properties on the right are extensive properties. An **intensive property** is independent of the extent of the system and has a value at a point. By contrast an **extensive property** depends upon the extent of the system and cannot be evaluated at a point. Some intensive properties have an extensive counterpart; these are called **specific properties**.

Again consider the air in the balloon. The mass of the air in the balloon can be thought of as the sum of all of the individual masses within the balloon. The temperature on the other hand cannot be thought of in the same way. *Extensive properties are additive, but intensive properties are not.*

2.1.3 State

The **state** of a system is a complete description of the system in terms of its properties. When we attempt to describe the condition of a physical device, we are actually trying to specify its state.

Strictly speaking, the state of a system can only be known when *all* of its properties are known. Luckily, we will find that under many conditions only a few properties must be known to specify the state sufficiently for our analysis. In addition, we will learn that many properties are related to each other. For instance, the pressure, temperature, and density of a gas at room conditions are related by something called the ideal gas model, and you need only specify two of these properties to uniquely fix the value of the third property.

2.1.4 Process

As a system interacts with its surroundings its properties may change, and when they do, we say that it has changed state or undergone a **process**. Without processes, our study of engineering systems would be pretty simple—nothing changes. During the quarter we will examine many different processes and how systems change.

There are two processes that merit special attention because they are so common in nature—a cyclic process and a steady-state process. A **cycle** is a sequence of processes that, combined, begin and end at the same state. The operation of many important engineering systems can often be modeled as a periodic cycle. Examples of this type of system include the refrigerator in your house and the engine of your car.

A **steady-state process** is a process in which the intensive properties of a system and it interactions are independent of time. Many real devices can be modeled using the steady-state assumption. Instead of talking about a steady-state process, we will often talk about a **steady-state system**. We will take these two expressions—steady-state process and steady-state system—as being synonymous. This is a very powerful assumption, and where applicable we will use it to remove time as a variable from our analysis.

2.1.5 Properties and Processes — A Test

Recall that we defined a property as any characteristic of a system that can be assigned a numerical value without knowing the history of the system. Another way of saying this is that *the value of a property only depends upon the state of the system*. If a system undergoes a process between two specified end states, the change in any property of the system is solely a function of the end states and is independent of the history of the system.

This result provides us with a useful test for determining if something is a property or not:

If the change in the value of a system characteristic between two end states is independent of the process, then the characteristic is a property.

Properties are sometimes called **point** or **state functions** because of this functional dependence on the state of the system.

We will see later that there are many physical quantities that are not properties because their value depends upon knowing the path of the process. One important class of these is system interactions. We will learn to call these quantities **path functions**.

2.2: The Accounting Concept

Experience has shown that the accounting concept is most useful when it is applied to account for extensive properties, so the "stuff" mentioned at the end of Chapter 1 should really be interpreted as extensive properties. Figure $2.2.1$ shows a system whose interactions with its surroundings are shown as transfers of an extensive property. This is a useful picture to help us visualize what is going on as we apply the accounting concept.

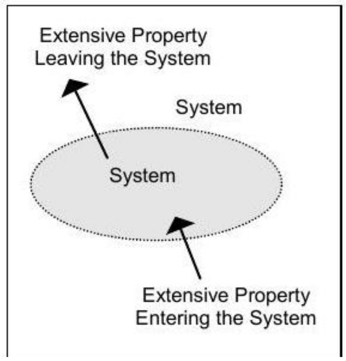


Figure $2.2.1$: System-surroundings interactions are represented as transfers of an extensive property.

2.2.1 Rate Form of the Accounting Concept

The most general form of the accounting concept we will use is the rate form. This form is valid at any instant in time. The rate-form of the accounting principle for an extensive property (EP) can be written in words for any system:

Rate-Form Accounting Principle for an Extensive Property

The rate of accumulation of EP inside the system at time t equals the rate of transport of EP into the system at time t minus the rate of transport of EP out of the system at time t plus the rate of generation (production) of EP inside the system at time t minus the rate of consumption (destruction) of EP inside the system at time t.

In an equation-like format, the words can be presented as

$$\begin{bmatrix}\text{Rate of Accumulation}\\ \text{of EP}\\ \text{inside the system}\\ \text{at time t}\end{bmatrix} = \underbrace{\begin{bmatrix}\text{Rate of Transport}\\ \text{of EP}\\ \text{into the system}\\ \text{at time t}\end{bmatrix} - \begin{bmatrix}\text{Rate of Transport}\\ \text{of EP}\\ \text{out of the system}\\ \text{at time t}\end{bmatrix}}_{\text{Transport across boundaries}} + \underbrace{\begin{bmatrix}\text{Rate of Generation}\\ \text{of EP}\\ \text{inside the system}\\ \text{at time t}\end{bmatrix} - \begin{bmatrix}\text{Rate of Consumption}\\ \text{of EP}\\ \text{inside the system}\\ \text{at time t}\end{bmatrix}}_{\text{Generation/Consumption inside the system}}$$

Figure $2.2.2$: The rate-form accounting principle for an extensive property expressed in words in the form of an equation.

If we let the symbol B represent a generic extensive property, we can write this more compactly in symbols:

$$\frac{d}{dt}\underbrace{B_{sys}}_{B\text{ inside the system}} = \underbrace{\{\dot{B}_{in} - \dot{B}_{out}\}}_{\text{Transport of }B\text{ inside the system}} + \underbrace{\{\dot{B}_{gen} - \dot{B}_{cons}\}}_{\text{Generation/Consumption of }B\text{ inside the system}}$$

Please note that all of the terms in this equation are defined independently of the accounting concept. The accounting concept gains its power from its ability to relate these various terms for any system.

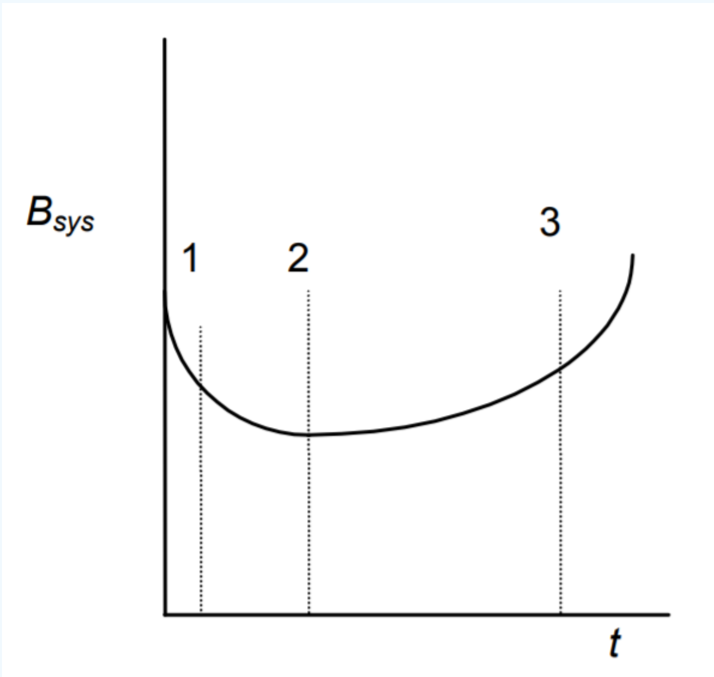
The rate-form of the accounting equation, Eq. $2.2.1$, can be written in even a more compact form if we introduce the idea of net transport and net generation:

$$\frac{d}{dt}\underbrace{B_{sys}}_{B\text{ inside the system}} = \underbrace{\{\dot{B}_{in} - \dot{B}_{out}\}}_{\text{Transport of }B\text{ inside the system}} + \underbrace{\{\dot{B}_{gen} - \dot{B}_{cons}\}}_{\text{Generation/Consumption of }B\text{ inside the system}}$$
$$= \dot{B}_{in,net} + \dot{B}_{gen,net} \tag{2.2.1}$$

In this notation, the first term on the right-hand side is positive if there is more transport into the system than out of the system. A negative value indicates that there is more leaving the system than entering.

? Test Yourself

The graph below shows B_{sys} as a function of time for a system.



(a) How you could use this graph to explain the meaning of the term $\frac{dB_{sys}}{dt}$?

(b) Where is B_{sys} increasing? Decreasing?

(c) Where is $\frac{dB_{sys}}{dt}$ positive? Negative?

(d) How are your answers to parts (b) and (c) related?

Because mathematics is the primary language for our analysis and a convenient short-hand for communicating our ideas, it is important to clearly understand what the various symbols and notation means. Starting with Eq. $2.2.1$ and $2.2.2$, it is very important to recognize that the left-hand side and right-hand side of each of these equations represent different things. The left-hand describes what is happening inside the system and describes how the amount of extensive property B is increasing or decreasing inside the system. The right-hand side describes something about how much of extensive property B is crossing the system boundary or is being consumed or generated inside the system.

Physically, the derivative on the left-hand side of Eq. $2.2.1$ and $2.2.2$ represents the rate of change or accumulation of the extensive property B_{sys} inside the system at any time t. *It is the rate of change of something that is contained inside the system,* B_{sys}. Please note that it is just an ordinary derivative. Mathematically this is correct because the amount of the extensive property B inside the system, B_{sys}, is only a function of time, i.e. $B_{sys}(t)$. The ordinary derivative of B_{sys} with respect to time, dB_{sys}/dt, represents the rate of change of B inside the system.

If I had a graph showing the amount of B inside the system as a function of time, then dB_{sys}/dt represents the slope of the line at any desired time. To evaluate this derivative, I must keep track of all the B inside the system as a function of time. To determine the rate of change of B_{sys}, I must be able to calculate the ordinary derivative or relate this term to other measurable quantities.

Mathematically, the terms on the right-hand side of Eq. $2.2.1$ and $2.2.2$ are not derivatives. As used here, the dot above a symbol does not represent differentiation. The "dot-above" notation, $\dot{B}$, represents a transport rate, an interaction, between the system and its surroundings where B flows across the system boundary or the rate at which B is produced or consumed inside the system. Let me repeat this once again for emphasis: $\dot{B}$ is not a derivative.

What is it, then? "B with a dot above it" or "B dot," $\dot{B}$, does not represent a *rate of change* of anything inside the system. It represents the rate at which the extensive property B is being transported across the boundary of a system or generated (or consumed) inside a system. In theory, knowledge about $\dot{B}$ is unrelated to the amount of B inside the system. (Yeah, yeah, yeah! I know that it can be related to what is inside the system through the accounting principle, but that's the whole point. $\dot{B}$ and dB_{sys}/dt are two different things that are defined and can be measured independently of each other. The accounting concept shows us how they can be related to each other.)

Example — Who's Eating the Crackers?

Let's take a simple example to try and understand the differences. Assume your mom buys you a box of animal crackers. (Who doesn't love animal crackers?) As soon as you get the box, you start eating the animal crackers. Your jealous younger brother and sister start keeping track of the crackers to see if they will get any.

Your brother is pretty active and has a hard time sitting still, so every five minutes your brother runs back in the room, yanks the box away from you, and counts the number of crackers in the box. Then he runs away and doesn't see you eat the crackers. To help him remember, he writes these numbers down as a table showing the number of crackers in the box at five-minute intervals, i.e. at 0, 5, 10 min, etc.

Your sister, on the other hand, is much less combative. She watches you eat the crackers and just counts how many crackers you take out of the box during each 5 minute interval, 0 min to 5 min, 5 min to 10 min, etc. Unfortunately, your siblings don't get along either so they do not share information with each other.

The results of their efforts are shown below in the table.

Brother's Tally			Sister's Tally		
Time (min)	Crackers in box B_{sys}	$\Delta B_{sys}/\Delta t$	Time Interval (min)	Crackers taken out of box, $B_{t1\text{-}t2}$	$B_{t1\text{-}t2}/\Delta t$
0	100				
		– 2	0 – 5	9	1.80
5	90				
		– 3	5 – 10	13	2.60
10	75				
		– 4	10 – 15	17	3.40
15	55				
		– 5	15 – 20	33	6.60
20	20				
		– 2	20 – 25	10	2.00
25	10				
		– 2	25 – 30	10	2.00
30	0				

If the number of crackers in the box is called B then your brother can plot a graph that shows a series of points from the function $B_{sys}(t)$ for the box. He can use this information to calculate the average rate of change for any interval using the formula $(dB_{sys}/dt)_{average} = \Delta B/\Delta t$ for any 5-minute interval. Notice that all he has to know to compute the rate of change is how many crackers are in the box as a function of time. Unfortunately, your brother has no idea where the crackers are going unless he sees you take them out.

Now your sister has no idea how many crackers are in the box, but she has sufficient information to compute the average transport rate of crackers out of the box (how many you ate per minute). She does this by dividing the number of crackers you took out in one five-minute interval, say B_{0-5}, and dividing it by the time interval, 5 minutes, i.e., $\dot{B}_{out,average} = B_{0-5}/\Delta t$.

Notice that all your sister needed to calculate the rate of transport out of the box was to focus on the boundary of they system (the box) and count how many crackers crossed it in any give time period. This is not a derivative of anything; it is a transport rate, but not a derivative. She knows the average rate at which crackers were transported out of the box, but she knows nothing about the rate of change of crackers inside the box.

Your sister and brother get so intrigued with their measurements that they totally forget to eat any crackers; however, they do decide to compare their information. Much to their surprise, they discover the following:

$$\left(\frac{dB_{sys}}{dt}\right)_{average} \neq -\dot{B}_{out,average}$$

The average rate of change of crackers inside the box did not equal the average transport rate of crackers out of the box. What happened?

Well, it seems that someone forgot to notice the small mouse in the box that was consuming crackers at the average rate of $\dot{B}_{mouse,average}$. If we account for the mouse eating (consuming) the crackers we see that

$$\left(\frac{dB_{sys}}{dt}\right)_{average} = -\dot{B}_{out,average} - \dot{B}_{mouse,average}$$

should apply to this system. This equation could be used to solve for the average consumption rate at which the mouse ate the crackers.

Several things can be learned from this example.

- All of the terms in the accounting equation above have independent definitions and, under the best circumstances, could be measured directly.
- These terms can also be related through the accounting principle applied to the appropriate system (the box) for the appropriate property (number of crackers in the box), and the accounting principle can be used to solve for one unknown term.
- The only term that can be considered a derivative is the term on the left-hand side. The transport and consumption rates on the right-hand side are not derivatives and have no direct relationship to the rate of change of B inside the system except through the accounting concept applied correctly to a system.
- The accounting equation is most useful when we can say something *a priori* about the consumption and generation terms. If we had known that the box was mouse-free, we would only need to observe the transport rate of cookies out of the system to predict the rate of change of cookies inside the system. As we will show shortly, many important physical laws have attained this status precisely because empirical evidence allows us to say something about the consumption and generation terms.

Another way to understand the distinction between dB/dt and $\dot{B}$ is to consider what happens when each is integrated over a time interval. When B with a dot over it is multiplied by dt, the result is not the standard differential dB but a new symbol δB that represents not a "change in" but a "small amount of":

$$\begin{aligned}\delta B &= \dot{B} dt \\ &= \text{small amount of B} \\ &= \text{an inexact differential because } B \text{ is not a single-valued function of } t.\end{aligned}$$

On the other hand, you will recall from your calculus that the interpretation of the derivative dB/dt is

$$dB = \frac{dB}{dt} dt \qquad \text{because } \frac{dB}{dt} = \lim_{\Delta t \to 0} \frac{B(t+\Delta t) - B(t)}{\Delta t}$$

where dB equals "a small change" in $B(t)$, a single-valued function.

Now the integral of the ordinary derivative with respect to time is the change in B between the two times:

$$\int_{t_1}^{t_2} \frac{dB}{dt} dt = \int_{B(t_1)}^{B(t_2)} dB = B(t_2) - B(t_1) = \Delta B$$

The integration of $\dot{B}$ with respect to time over the same time interval is an amount of B, not a change in B, as shown below:

$$\int_{t_1}^{t_2} \dot{B}\, dt = \int_{t_1}^{t_2} \delta B = B_{t_1 \to t_2}$$

Take care to be sure that you understand the differences between these last two equations and that you can perform these integrations.

2.2.2 Finite-time (Integrated) Form of the Accounting Concept

There are many times when we are interested in a system that has undergone a process over a specified time interval. In these cases, it is possible to integrate the rate form to obtain a finite time form. Figure $2.2.5$ shows a picture that is helpful for interpreting the finite-time form of the accounting concept. As you can see, the system exists in two different states and is connected by a process.

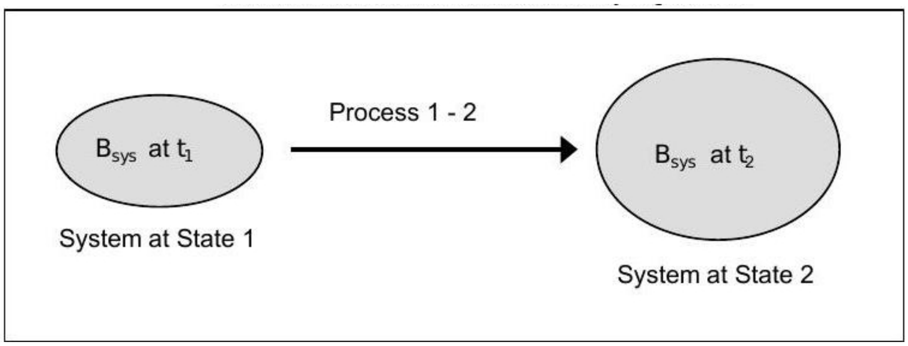


Figure $2.2.5$: System undergoing a process from State 1 to State 2

In words, the finite-time form of the accounting concept is written as follows:

$$\begin{bmatrix} \text{Change in EP} \\ \text{inside the system} \\ \text{during the} \\ \text{time period } \Delta t \end{bmatrix} = \underbrace{\begin{bmatrix} \text{Amount of EP} \\ \text{Transported} \\ \text{into the system} \\ \text{during the} \\ \text{time period } \Delta t \end{bmatrix} - \begin{bmatrix} \text{Amount of EP} \\ \text{Transported} \\ \text{out of the system} \\ \text{during the} \\ \text{time period } \Delta t \end{bmatrix}}_{\substack{\text{Transport} \\ \text{across boundaries}}} + \underbrace{\begin{bmatrix} \text{Amount of EP} \\ \text{Generated} \\ \text{inside the system} \\ \text{during the} \\ \text{time period } \Delta t \end{bmatrix} - \begin{bmatrix} \text{Amount of EP} \\ \text{Consumed} \\ \text{inside the system} \\ \text{during the} \\ \text{time period } \Delta t \end{bmatrix}}_{\substack{\text{Generation/Consumption} \\ \text{inside the system}}}$$

Figure $2.2.6$: The finite-time form of the accounting concept expressed in words in the form of an equation.

In symbols, the finite-time form is written as follows:

$$\begin{aligned} B_{sys}(t_2) - B_{sys}(t_1) &= \{B_{in} - B_{out}\} + \{B_{gen} - B_{cons}\} \\ \Delta B_{sys} &= B_{in,net} + B_{gen,net} \end{aligned} \tag{2.2.2}$$

The meaning of each of these symbols can be determined by comparison with the word statement.

Test Yourself

Now test yourself. Starting with Eq. $2.2.2$ and using the results from Eq. $2.2.3$ and $2.2.4$, show the steps to develop Eq. $2.2.5$.

2.3: Conservation

The last idea to be introduced in this chapter is the concept of conservation. In this course, a **conserved property** cannot be generated or consumed (created or destroyed). This is short and simple but represents a very, very powerful idea. When applied to the accounting concept, it means that all of the generation and consumption terms are identically equal to zero.

You should be aware of the fact that not everyone uses this definition for conservation. Most physics textbooks use the word "conserved" to indicate that there is no change in the amount of the conserved quantity inside the system. For example, conservation of momentum in most physics books is a principle that is only used to consider systems in which the momentum of the system is a constant. This approach uses "conserved" as a modeling assumption that may or may not hold for a given problem. In this course, we will always use "conserved" as a statement about a fundamental physical law. In our usage the concept of conservation relates to how the world works in general.

It turns out that most of the important fundamental laws in physics are Conservation Laws — mass, charge, linear momentum, angular momentum, and energy. The remaining law we will consider is the Second Law of Thermodynamics. It can be represented by an accounting principle where entropy can never by consumed.

We can write Accounting Statements or Accounting Equations for any extensive property. However, we can only write Conservation Laws or Conservation Equations for selected extensive properties. The validity of these laws is then based on accumulated empirical evidence that certain selected extensive properties are conserved.

Test Yourself

1. Are you sure you know what "net" means? Revisit Equation $2.2.2$ and rewrite it in terms of two new net terms — $\dot{B}_{out,net}$ and $\dot{B}_{cons,net}$.
2. Revisit Equation $2.2.1$. How would this equation simplify if you assumed that the system in question was isolated? What if it was operating under steady-state conditions?
3. Write an equation similar to Equation $2.2.1$ that would be correct for a conserved property — a conservation law.
4. Now revisit the balloon problem. What do you think would be the best system? What properties should you count? What form of the accounting concept would be the best place to start out?

CHAPTER OVERVIEW

3: Conservation of Mass

3.1: Four Questions

When developing an accounting or balance equation for a new property, there are four questions that must be answered:

1. What is it?
2. How can it be *stored* in a system?
3. How can it be *transported* across the boundary of a system?
4. How can it be *generated* or *consumed*?

In this section, we will use these questions to guide us as we develop a mass balance built on a fundamental physical principle the conservation of mass.

3.1.1 What is mass?

Mass can be described in many different ways. Newton described it as a "measure of the inertia" of a body; however, this is not very useful without defining inertia. Similarly, we could define it as the amount of a substance; however, this is also faulty precisely because it neglects the inertia idea.

This problem of definition is a common one in science and language in general. Any definition must of necessity build upon other words that are assumed to be understood. In science and engineering, we will make use of an **operational definition** to provide a precise definition of a new term or concept. An operational definition is a series of steps or operations that must be performed to define the quantity or concept in question.

Using this approach, the mass of any object could be described in terms of the mass of a standard or reference object:

Mass: In terms of gravitational mass of a reference object

The ratio of the gravitational mass of an object to the gravitational mass of a standard or reference object equals the ratio of the weight of the object of unknown mass to the weight of the standard object.

This definition assumes that the concept of *weight* is understood. An alternative definition could be developed from Newton's second law that relates the force $\vec{F}$, the system mass m and acceleration $\vec{a}$, $\vec{F}=m\vec{a}$:

Mass: In terms of inertial mass of a reference object

The ratio of the **inertial mass** of an object to the inertial mass of standard or reference object equals the ratio of the acceleration of the object to the acceleration of the reference object when both objects are subjected to the same force.

As you have learned in physics, the inertial mass and the gravitational mass of an object are equal. For our purposes, we will take the definition of mass as being an **undefined term** and rely on your background from physics.

3.1.2 How can mass be stored in a system?

Mass is an *intrinsic property* of matter. Any system that contains matter has mass. If the mass of a particle is m_i, then the mass of a system of particles m_{sys} is equal to the sum of the mass of the individual particles:

$$m_{sys}=\sum_{i=1}^{n} m_i$$

Since the mass of a system depends on the number of particles (or extent) of the system, we recognize that mass is also an *extensive property*.

More generally, the mass of a system can be found by integrating the mass density ρ over the volume of the system V_{sys}.

$$m_{sys}=\int_{V_{sys}} \rho\, dV$$

Mass density or more commonly just the density of a substance is defined as the mass per unit volume. Because it has a value at a point, density is an intensive property. The dimensions of density are $[\mathrm{M}]/[\mathrm{L}]^3$. Typical units for density are kg/m^3 in SI and

$lb \cdot m/ft^3$ in USCS.

In general, substances can be classified as either incompressible or compressible. An incompressible substance is one whose density is constant with regard to both space and time. There are no truly incompressible substances; however, many substances may be modeled as incompressible under certain conditions. For example, the hydraulic fluid in your automobile brake lines is essentially incompressible. The same is true for most liquids and solids. A compressible substance is one whose density can change significantly during a change in state. Gases and vapors fall in this category.

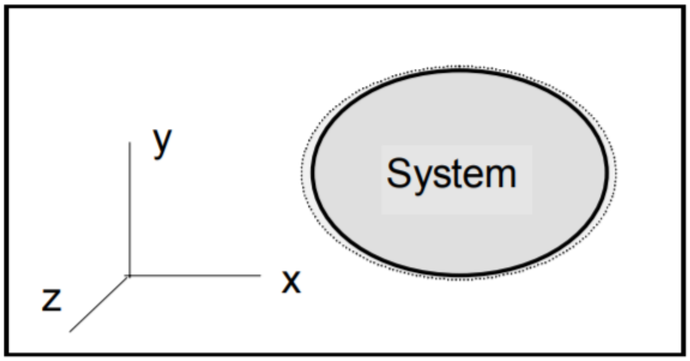


Figure $3.1.1$: System

To better understand the significance of this simple relation, consider the system and coordinate system in Figure 3-1. Depending upon whether it is an open or closed system, the mass of a system may change with time. Mathematically, we would say that the mass of the system depends upon time, $m_{sys} = m_{sys}(t)$. The mass density ρ at any point in the system may depend on both its position (x, y, z) and time (t), i.e. $\rho = \rho(x, y, z, t)$. Substituting these terms back into Eq. $3.1.2$ we have

$$m_{sys}(t) = \int_{V_{sys}} \rho(x, y, z, t)\, dV$$

Notice how the integration over the system volume removes the spatial dependence leaving only the time dependence. Typically we will not write out the spatial and time dependence as in Eq. $3.1.3$; however, you should remember that these variations may exist.

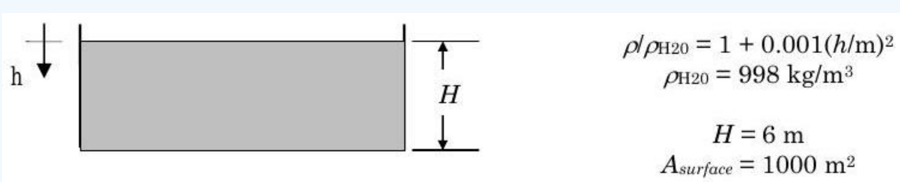
Example — Mass in a solar pond

A solar pond contains a stratified mixture of salt and water. The solar pond has a rectangular section surface area of $1000\ m^2$ and a uniform depth of $6\ m$. The density of the salt water mixture varies with the distance below the surface h according to the relation $\rho/\rho_{H2O} = 1 + 0.001(h/m)^2$ where $\rho_{H2O} = 998 kg/m^3$. Determine the mass of salt water in the pond, in kilograms.

Solution

Known: A solar pond contains a stratified mixture of salt and water.

Find: The mass of salt water in the pond, in kg.



Analysis:

We start with the mathematical relation that defines the mass inside a system in terms of its mass density and volume:

$$m_{sys(t)} = \int_{V_{sys}} \rho_{(x,y,z,t)}\, dV$$

Because the density only depends on the depth h, it makes sense to define the differential volume as $dV = A_{surface}\, dh$ and the limits of integration from $0 \to H$. Combining this with the density relationship gives

$$\begin{aligned} m_{sys} &= \int_0^H \rho_{H2O} \left[1 + 0.001 \left(\frac{h}{\mathrm{m}}\right)^2\right] \underbrace{(A_{surface}\, dh)}_{V_{sys}} \\ &= \rho_{H2O} A_{surface} \left[h + \frac{0.001}{3} \frac{h^3}{\mathrm{m}^2}\right] \Bigg|_0^H \\ &= \rho_{H2O} A_{surface} \left[H + \frac{0.001}{3} \frac{H^3}{\mathrm{m}^2}\right] = \underbrace{\rho_{H2O} \left[1 + \frac{0.001}{3} \left(\frac{H}{\mathrm{m}}\right)^2\right]}_{\rho_{average}} \underbrace{(A_{surface} H)}_{V_{sys}} \end{aligned}$$

Substituting in the numbers, we have

$$\begin{aligned} m_{pond} &= \left(998 \frac{\mathrm{kg}}{\mathrm{m}^3}\right) \times \left[1 + \frac{0.001}{3} \left(\frac{6\ \mathrm{m}}{\mathrm{m}}\right)^2\right] \times \left[\left(1000\ \mathrm{m}^2\right)(6\ \mathrm{m})\right] \\ &= [(998)(1 + 0.012)(6000)] \left[\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \mathrm{m}^2 \cdot \mathrm{m}\right] \\ &= 6.06 \times 10^6\ \mathrm{kg} \end{aligned}$$

Comments:

1. Notice the form of the equation for the density. This equation is dimensionally homogeneous because it works correctly with any set of units. If the equation had been presented without the meter unit in the expression, i.e. $\rho/\rho_{H2O} = 1 + 0.001(h)^2$ instead of $\rho/\rho_{H2O} = 1 + 0.001(h/\mathrm{m})^2$, the equation could have only been used correctly if numerical values of h were always supplied in meters.
2. Also notice how the calculations were first done symbolically. This is the preferred method for solving problems, because it allows you to keep track of the physics without getting lost in the numbers. Notice how we were able to identify the volume of the pond, V_{pond}, and the average density, ρ_{avg}, in the last step of the calculation. Constantly connecting our mathematics with our physical understanding of the problem provides a continuous check on our work.
3. When substituting in the magnitude of the physical quantities, $H\backslash(and\backslash(\rho_{H2O}$, notice how we substitute in both a number and a unit. Strictly speaking, if we were to leave off the units the expression would be mathematically incorrect. One approach to simplify the calculations, especially when you are dealing with unfamiliar units, is to segregate the numerical and the unit calculations into two parts as shown in the problem.

3.1.3 How can mass be transported across a system boundary?

There are two ways that mass can cross the boundary of a system:

- Gross motion of mass across the boundary.
- Microscopic motion of mass due to molecular diffusion.

In this course we will focus on only the first mechanism. Specifically we will define the **mass flow rate** $\dot{m}$ as the time rate at which mass crosses a boundary. The mass flow rate has dimensions of $[\mathrm{M}]/[\mathrm{T}]$. Typical units are $\mathrm{kg/s}$ in SI, and $\mathrm{lb\,m/s}$ or $\mathrm{slugs/s}$ in USCS.

3.1.4 How can mass be created or destroyed?

Based on your past experiences, how would you answer this question? If you said, "It can't!" you are recognizing a fundamental physical law known as the **Conservation of Mass**. Recognizing that mass is a property, we can state this law as follows: **Mass is a conserved property.**

Specifically this means that, excluding energy-to-mass conversion, it is impossible to generate or consume mass.

3.1.5 Putting it all together

Combining what we have learned from the first four sections about mass, we can use our accounting or balance framework to write the following **conservation of mass equation** or **mass balance**:

$$\frac{dm_{sys}}{dt} = \sum_{in} \dot{m}_i - \sum_{out} \dot{m}_e$$

where the first summation is over all the inlets (entrances) and the second summation is over all of the outlets (exits). In words, we might say that

The time rate-of-change of the mass in the system equals the sum of the mass flow rates into system minus the sum of the mass flow rates out of the system.

Another statement might be

The rate of accumulation of mass within the system equals the net mass flow rate into the system.

As with any of our balance equations, please note that all of the terms in the balance equation have independent definitions and can be calculated independently of the other expressions. The unique contribution of this mass balance and its related physical law, The Conservation of Mass, is that we now have a unique relationship between all of these quantities.

3.2: Mass Flow Rate

Before proceeding, we need to develop a better understanding of how to calculate the mass flow rate. Recall that we earlier defined the mass flow rate as the time rate at which mass crosses the boundary of a system. Note that a mass flow rate really only has meaning as it relates to the boundary for which it is defined.

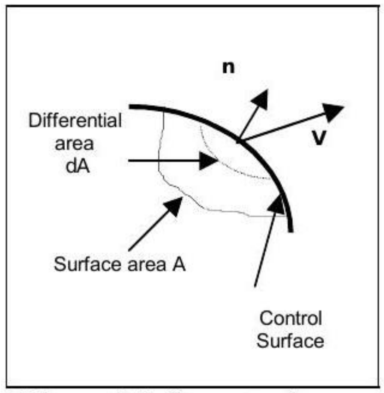


Figure $3.2.1$: Geometry for flow across a system boundary.

To calculate the mass flow rate, consider the rate at which mass flows across the system boundary with area A in Figure $3.2.1$. By definition the mass flow rate out of a system is defined by the equation

$$\dot{m}_{out} = \int\limits_{A_{sys}} \rho \left(\mathbf{V}_{rel} \cdot \mathbf{n}\right) \, dA$$

where

$$\begin{aligned} \rho &= \text{the fluid density,} \\ \mathbf{V}_{rel} &= \text{velocity (a vector) of the mass crossing the boundary measured with respect to the system boundary, and} \\ \mathbf{n} &= \text{is the unit vector normal to the differential area } dA \text{ and pointing out of the system.} \end{aligned}$$

The key to understanding the meaning of Eq. $3.2.1$ is to recall the meaning of a simple scalar or dot product of two vectors. First recall that the scalar or dot product of two vectors produces a scalar. Now applying the definition of a this operation to the unit normal vector and the velocity vector gives:

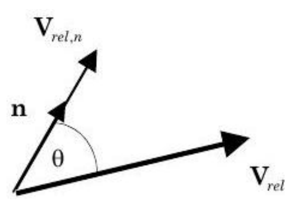


Figure $3.2.2$: Definition of the dot product of $\vec{V}_{rel}$ and $\hat{n}$.

$$\begin{aligned} V_{rel,n} &= \left|\mathbf{V}_{rel,n}\right| \\ &= \mathbf{V} \cdot \mathbf{n} = |\mathbf{V}||\mathbf{n}| \cos\theta \\ &= \text{normal velocity of the mass relative to the boundary surface} \end{aligned}$$

Using this result, the mass flow rate expression can be written as

$$\dot{m} = \int\limits_{A_{sys}} \rho V_{rel,n} \, dA$$

In general, the density ρ and the normal velocity $V_{rel,n}$ may change with position on the boundary surface. Also note that we have dropped the "out" subscript because our observation of the direction of $V_{rel,n}$ with respect to the system determines whether the mass flow rate is into or out of the system.

There are numerous modeling assumptions that are used to describe the behavior of real systems in the construction of mathematical models. If the flow has uniform density at the flow boundary, then the density is spatially uniform at the flow boundary. Under these conditions,

$$\dot{m}=\int_{A_c} \rho V_{rel,n}\, dA=\rho \underbrace{\int_{A_c} V_{rel,n}\, dA}_{=\dot{V}}=\rho \dot{V},$$

more simply expressed as

$$\dot{m}=\rho \dot{V} \quad \text{(uniform density)}$$

where the integral of relative normal velocity over the flow cross-sectional area is the **volumetric flow rate**, $\dot{V}$. The dimensions of volumetric flow rate are $[\mathrm{L}]^3/[\mathrm{T}]$ and typical units are m^3/s in SI and ft^3/s in AES. There are many other commonly used units for volumetric flow rate, including gallons per min (gpm) and cubic feet per min (cfm).

If we restrict ourselves to **planar (flat) flow boundaries**, then several additional simplifications are possible. If the flow at a boundary has uniform velocity and uniform density, both the density and relative velocity come outside the integral and the mass flow rate becomes

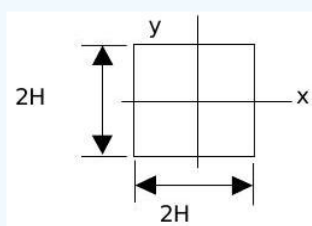
$$\dot{m}=\rho A_c V_n$$

This is one of the most commonly used forms for calculating the mass flow rate. Notice that we have dropped the "rel" subscript to simplify the notation; however, you are warned that all mass and volumetric flow rates must of necessity be calculated with respect to a flow boundary. For flow conditions where the density and normal velocity vary across the flow cross section, it is common practice to break the flow cross section into many small elements where Eq. $3.2.4$ applies and then sum the results, e.g. $\dot{m}=\sum_j \left(\rho A_c V_{rel,n}\right)_j$. In many cases, the velocity will vary across the cross section but the density will be relatively uniform. Under these conditions it is frequently useful to write the mass flow rate in terms of an average velocity $V_{n,avg}$.

$$\dot{m}=\rho A_c V_{n,avg} \quad \text{where } V_{n,avg}=\frac{\dot{m}}{\rho A_c}=\frac{\dot{V}}{A_c}$$

Example — Flow in a rectangular duct

Air flows steadily through a heating duct with a square cross section $2H \times 2H$. The measured velocity profile over the cross-sectional area can be described mathematically in terms of the position in the duct (x, y) and the centerline velocity V_o (velocity at (0,0)).



$$V_n=V_o\left[1-\left(\frac{x}{H}\right)^2\right]\left[1-\left(\frac{y}{H}\right)^2\right]$$

Determine:

(a) the volumetric flow rate, in m^3/s, if $V_o=10\mathrm{m/s}$ and $H=0.3\mathrm{m}$, and

(b) the ratio of V_{avg} to V_o.

Solution

Known: Air flows in a square duct with a specified velocity profile.

Find: (a) Volumetric flow rate if $V_o=10\mathrm{m/s}$ and $H=0.3\mathrm{m}$. (b) Ratio of V_{avg} to V_o.

Given: Sketch and velocity profile shown above.

Analysis: Strategy $\rightarrow$ Since Vn and the area are given, use the definition of volumetric flow rate. We start with the definition for volumetric flow rate

$$\begin{aligned}\dot{V} &= \int_{A_c} V_n \, dA \quad \text{where } dA = dx\ dy \\ &= \int_{-H}^{H}\int_{-H}^{H} \underbrace{V_o\left[1-\left(\frac{x}{H}\right)^2\right]\left[1-\left(\frac{y}{H}\right)^2\right]}_{V_n} \underbrace{dx\ dy}_{dA}\end{aligned}$$

Bringing V_o outside the integral and using symmetry to change the limits of integration, we have

$$\dot{V} = 4V_o \int_{0}^{H}\int_{0}^{H} \underbrace{\left[1-\left(\frac{x}{H}\right)^2\right]\left[1-\left(\frac{y}{H}\right)^2\right]}_{V_n} \underbrace{dx\ dy}_{dA}$$

Integrating first with respect to the x-axis gives

$$\int_0^H \left[1-\left(\frac{x}{H}\right)^2\right]\ dx = \left[x - \frac{1}{3}\frac{x^3}{H^2}\right]_0^H = \frac{2}{3}H$$

Substituting back into the full integral and integrating with respect to y gives

$$\begin{aligned}\dot{V} &= 4V_o\left(\frac{2}{3}H\right)\int_0^H\left[1-\left(\frac{y}{H}\right)^2\right]\ dy = 4V_o\left(\frac{2}{3}H\right)\left[y-\frac{1}{3}\frac{y^3}{H^2}\right]_0^H = 4V_o\left[\frac{2}{3}H\right]^2 \\ &= \frac{16}{9}H^2V_o\end{aligned}$$

Substituting the numbers into this expression gives the volumetric flow rate as

$$\dot{V} = \frac{16}{9}(0.3\text{ m})^2\left(10\ \frac{\text{m}}{\text{s}}\right) = 1.6\text{ m}^3/\text{s}$$

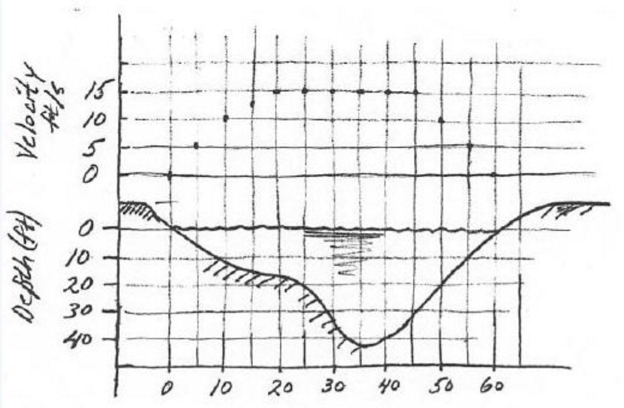
Now to find the ratio of the average to the centerline velocity, use the definition of the average velocity:

$$V_{avg} = \frac{\dot{m}}{\rho A_c} = \frac{\dot{V}}{A_c} = \frac{\frac{16}{9}H^2V_o}{[2H]^2} = \frac{4}{9}V_o$$

Thus the average velocity is $\frac{4}{9}$ of the centerline velocity.

Example — River flow rate

Water flows steadily in a river. The river cross-section is shown in the figure below. In addition, surface velocities measured by watching pieces of wood float downstream are also shown.



Compute the (a) volumetric flow rate in ft^3/s and (b) mass flow rate if the density of water is $62.4\ \mathrm{lbm}/\mathrm{ft}^3$.

Solution

Known: Velocities on the surface of a river and the depth of the river at each location.

Find: Volumetric flow rate in ft^3/s and the mass flow rate if $\rho = 62.4\mathrm{lbm}/\mathrm{ft}^3$.

Given: Channel depth and surface velocity given in figure above.

Analysis:

Strategy $\rightarrow$ Should be able to apply defining equation for volumetric and mass flow rate.

Assume $\rightarrow$ Velocity is uniform from top to bottom of channel. The defining equation for volumetric flow rate is

$$\begin{aligned} \dot{V} &= \int\limits_{A_{surface}} V_n \, dA \\ &\simeq \sum_{i=1}^{N} A_i V_{avg,\, i} \simeq \sum_{i=1}^{N} (h_i \Delta x_i) V_{avg,\, i} \end{aligned}$$

To perform the necessary integration numerically, we can set up a table:

i	x (ft)	V_n (ft/s)	Δx (ft)	h_i (ft)	$V_n(h\Delta x)$ (ft^3/s)
1	2.5	2.5	5	2.5	31.25
2	7.5	7.5	5	8.0	300.00
3	12.5	11.25	5	12.5	703.10
4	17.5	13.75	5	16.0	1100.00
5	22.5	15.0	5	18.0	1350.00
6	27.5	15.0	5	26.0	1950.00
7	32.5	15.0	5	37.0	2775.00
8	37.5	15.0	5	42.0	3150.00
9	42.5	15.0	5	36.0	2700.00
10	47.5	12.5	5	25.0	1562.50
11	52.5	7.5	5	15.0	562.50
12	57.5	2.5	5	5.0	62.50
.					16246.85

Thus the average volumetric flow rate is

$$\dot{V} = 16.2 \times 10^3 \text{ ft}^3/\text{s}$$

The mass flow rate can be found by multiplying the water density times the volumetric flow rate:

$$\dot{m} = \rho \dot{V} = \left(62.4 \frac{\text{lb}_\text{m}}{\text{ft}^3}\right)\left(16200 \frac{\text{ft}^3}{\text{s}}\right) = 1.01 \times 10^6 \frac{\text{lb}_\text{m}}{\text{s}}$$

Comment: If we recognize that the velocity is at a maximum at the free surface and zero at the bottom, a better estimate might be obtained by assuming that the average velocity is some fraction of the amount at the free surface. For example, if we assume that the average velocity is $4/5$ of the maximum velocity, then the actual volumetric flow rate is $13.0 \times 10^3 \text{ ft}^3/\text{s}$.

3.3: Conservation of Mass Equation

The recommended starting point for application of the conservation of mass equation is the rate-form of the mass balance (conservation of mass equation):

$$\frac{dm_{sys}}{dt}m = \sum_{in}\dot{m}_i - \sum_{out}\dot{m}_e$$

where $m_{sys} = \int_{V_{ms}} \rho \, dV$, the system mass, and $\dot{m}$ is the mass flow rate across the system boundary.

In applying the rate-form of the conservation of mass equation to a system, there are many common modeling assumptions that can be used to set up the mathematical model of the physical system. These are detailed in the following paragraphs.

<u>**Steady-state system:**</u> For a steady-state system, all extensive and }}\) intensive properties are independent of time. Thus

$$\underbrace{\frac{dm_{sys}}{dt}}_{=0,\text{ SS}} = \sum_{in}\dot{m}_i - \sum_{out}\dot{m}_e \;\Rightarrow\; 0 = \sum_{in}\dot{m}_i - \sum_{out}\dot{m}_e$$

<u>**Incompressible substance:**</u> An incompressible substance is one for which the density never changes. Under these conditions, the mass flow rate can be written in terms of the density and the volumetric flow rate:

$$\dot{m} = \int_{A_c} \rho V_{rel,n}\, dA = \rho \int_{A_c} V_{rel,n}\, dA \;\Rightarrow\; \dot{m} = \rho \dot{\forall}$$

$$m_{sys} = \int_{\forall_{sys}} \rho \, d\forall = \rho \int_{\forall_{sys}} \;\Rightarrow\; m_{sys} = \rho \forall_{sys}$$

Test your understanding

Although it is not a fundamental law, the following equation involving system volume and volumetric flow rates is often used to describe the behavior of systems.

$$\frac{d\forall_{sys}}{dt} = \sum_{in}\dot{\forall}_i - \sum_{out}\dot{\forall}_e :$$

What conditions (or modeling assumptions) must apply for this equation to be valid? Try starting with Eq. $3.3.1$ and deriving this equation for volume.

<u>**One-dimensional flow with uniform density at a flow boundary:**</u> Under these conditions both the density and the velocity come outside of the integral, giving

$$\dot{m} = \int_{A_c} \rho V_{rel,n}\, dA = \rho V_{rel,n} \int_{A_c} dA \;\Rightarrow\; \dot{m} = \rho A_c V$$

where velocity V used in calculating the mass flow rate is assumed to be the component of the velocity of the mass crossing the boundary that is normal to the boundary and measured relative to the boundary (regardless of whether the boundary is moving or stationary).

<u>Please</u> recognize that you should not concentrate on or memorize the final form of the equations developed above. Your focus should be on understanding the *assumptions* and learning how they impact the governing equations. As you will find later, the assumptions and their impact will be used repeatedly throughout these notes to help develop mathematical models for physical systems.

It is also sometimes required to apply the conservation of mass equation to a finite-time process when you are interested in relating what was known at some time t_1 to some later time t_2. Again, rather than memorizing a special form of the equation, just integrate the rate form of the conservation of mass equation as shown below:

$$\int_{t_1}^{t_2}\left[\frac{dm_{sys}}{dt}\right]dt = \int_{t_1}^{t_2}\left[\sum_{in}\dot{m}_i - \sum_{out}\dot{m}_e\right]dt$$

$$\int_{m_{sys,1}}^{m_{sys,2}} dm_{sys} = \int_{t_1}^{t_2}\left(\sum_{in}\dot{m}_i\right)dt - \int_{t_1}^{t_2}\left(\sum_{out}\dot{m}_e\right)dt$$

$$m_{sys,2} - m_{sys,1} = \sum_{in} m_i - \sum_{out} m_e$$

where

$m_{sys,2};\ m_{sys,1}\ =$ mass inside the system at time t_2 and t_1, respectively.

$m_i \equiv \int_{t_1}^{t_2} \dot{m}_i\, dt\ =$ the amount of mass transported into the system during the time interval t_1 to t_2. (This is NOT the change in mass!)

The following examples demonstrate various applications of the conservation of mass equation to various problems.

Example — Evaporating acetone

An open flask is filled with acetone. Initially, the flask contains $25\ \text{kg}$ of acetone. After two hours, some of the acetone evaporates and the flask contains $23\ \text{kg}$ of acetone. Taking your system to be the liquid acetone, answer the following questions by applying conservation of mass:

What is the accumulation of acetone during the two hours, in kg?

How much acetone evaporated during the two hours, in kg?

What is the average rate of evaporation during this time period, in kg/s?

Solution

Known: Acetone evaporates from an open flask

Find: Accumulation of the acetone, in kg.

Amount of acetone that evaporates, in kg.

Average rate of evaporation during this period, in kg/h.

Given:

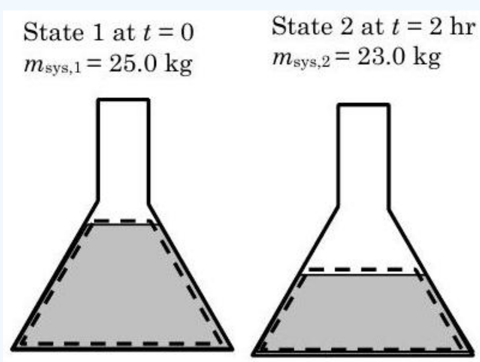


Figure 3.3.1: Acetone evaporates from the flask over the course of 2 hours, with the mass of the system decreasing from 25.0 kg to 23.0 kg.

Analysis:

Strategy $\rightarrow$ What's the system? Liquid acetone as suggested in the problem statement.

What should we count? Mass of acetone.

What's the time period? Finite time period of 2 hours.

Selecting the liquid acetone as the system as shown in the figures above, we have an open system and there is only one interaction with the surrounding, the mass flow rate of acetone out of the system due to evaporation. This occurs at the free surface of the liquid in the flask.

Writing the rate form of the conservation of mass equation for this system we have

$$\frac{dm_{sys}}{dt} = \underbrace{\overset{=0}{\cancel{\sum_{in} \dot{m}_i}}}_{\text{no inlet flows}} \underbrace{\cancel{\sum_{out} \dot{m}i_e}}_{\text{only one outlet flow}} = \dot{m}_{\text{evap}} \quad \Rightarrow \quad \frac{dm_{\text{sys}}}{dt} = -\dot{m}_{\text{evap}}$$

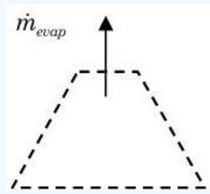


Figure $3.3.2$: Rate of mass flow out of the system.

To find the accumulation, we must understand that the accumulation of the mass in the system is the change in the system mass

$$\Delta m_{sys} = m_{sys,2} - m_{sys,1} = (23.0 \text{ kg}) - (25.0 \text{ kg}) = -2.0 \text{ kg}$$

Thus the accumulation of mass in the system is -2.0 kg of acetone.

To find the mass of acetone that has evaporated during these two hours, we must turn to the conservation of mass equation developed above. Since we want the amount of mass evaporated not the rate of evaporation, we must integrate the rate equation over the two-hour period.

$$\frac{dm_{sys}}{dt} = -\dot{m}_{evap} \quad \rightarrow \quad \int_{t_1}^{t_2} \left(\frac{dm_{sys}}{dt}\right) dt = \int_{t_1}^{t_2} (-\dot{m}_{evap})\ dt \quad \rightarrow \quad \int_{m_{sys,1}}^{m_{sys,2}} dm_{sys} = -\int_{t_1}^{t_2} \dot{m}_{evap}\ dt \quad \rightarrow$$
$$\Delta m_{sys} = -m_{evap}$$

The left hand side just equals the accumulation of mass in the system and the right side is the amount of mass transported out of the system by mass transfer during the time interval. Thus the amount of mass evaporated is

$$m_{evap} = -\Delta m_{sys} \quad \rightarrow \quad m_{evap} = -(-2.0 \text{ kg}) = 2.0 \text{ kg}$$

Note that these two quantities could, in fact, be computed independently of each other with appropriate measurements. The connection between these quantities is the conservation of mass for this system.

To determine the average rate of evaporation, we can revisit the finite-time or integrated form of the conservation of mass equation

$$\frac{dm_{sys}}{dt} = -\dot{m}_{evap} \quad \rightarrow \quad \int_{m_{sys,1}}^{m_{sys,2}} dm_{sys} = \underbrace{-\int_{t_1}^{t_2} \dot{m}_{evap}\ dt}_{\substack{\text{assume mass flow} \\ \text{rate occurs at a} \\ \text{constant rate, } \dot{m}_{sys,evap}}} \quad \rightarrow \quad \Delta m_{sys} = -\dot{m}_{evap,avg} \Delta t \quad \rightarrow$$
$$\dot{m}_{evap,sys} = -\frac{\Delta m_{sys}}{\Delta t}$$

Thus the average evaporation rate is

$$\dot{m}_{evap,avg} = -\frac{\Delta m_{sys}}{\Delta t} = -\frac{(-2.0 \ \text{ kg})}{(2.0 \text{ hr})} = 1.0 \ \frac{\text{kg}}{\text{hr}}$$

Comment:

Does the volume of the system change? If so what information would you need to be able to solve for the change in volume?

How would your solution change if you used a *closed* system that consisted of all of the acetone?

Example — Syringe oil can

A hypodermic syringe is used to oil a machine part. The inside diameter of the glass syringe tube is 1.0 cm and the plunger moves inside the tube at the rate of $0.5\ \text{cm/s}$. The syringe needle has an inside diameter of $1.0\ \text{mm}$. How long will it take to discharge 12 cc of oil? What is the velocity of the oil as it leaves the needle? (If necessary, you may assume that the oil is incompressible and the flow in the needle is one-dimensional.)

<u>**Solution**</u>

Known: A hypodermic syringe is used to oil parts.

Find: Volumetric flow rate of oil out of the needle, in cm^3/s.

Time required to discharge 12 cc of oil.

Velocity of the oil squirting out of the needle, in cm/s.

Given:

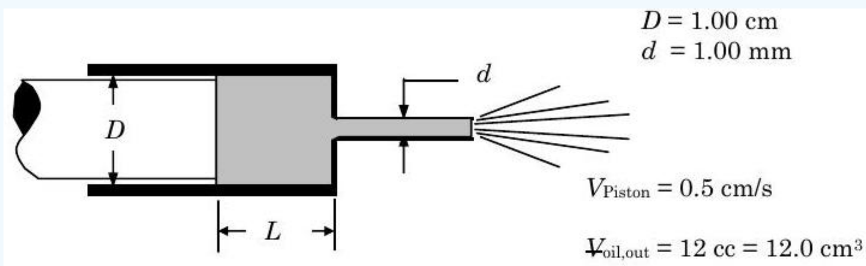


Figure $3.3.3$: Setup of a syringe of oil, with the piston moving forward at a rate of $0.5\ \text{cm/s}$.

Analysis

Strategy $\rightarrow$ System $\rightarrow$ Volume occupied by oil inside the cylinder and needle

Property $\rightarrow$ Count the mass of oil, since volume is related to mass.

Time interval $\rightarrow$ Finite-time since amount of oil extruded given net rate.

Using the volume of oil inside the cylinder *and* needle as the open system, we can sketch a system diagram showing the mass flow rates.

Writing the conservation of mass equation for this system gives

$$\frac{dm_{sys}}{dt} = \underbrace{\sum_{in} \overset{=0}{\cancel{\dot{m}_i}}}_{\substack{\text{no inlets assuming} \\ \text{boundary at piston moves}}} - \underbrace{\sum_{out} \overset{=\dot{m}_{oil,out}}{\cancel{\dot{m}_e}}}_{\substack{\text{one outlet assuming} \\ \text{no leakage at piston}}} \rightarrow \frac{dm_{sys}}{dt} = -\dot{m}_{oil,out}$$

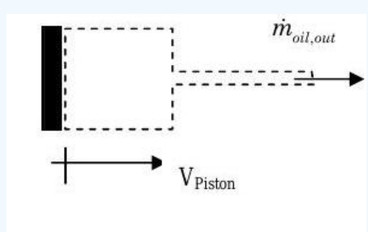


Figure $3.3.4$: Velocity of piston motion and mass flow rate of oil out of the system.

Assuming uniform density for the oil gives $m_{sys} = \rho_{oil} \forall_{sys}$ and $\dot{m}_{oil,out} = \rho_{oil} \dot{\forall}_{out}$.

Substituting these relations back into the conservation of mass equation, it becomes

$$\frac{dm_{sys}}{dt} = -\dot{m}_{oil,out} \quad \rightarrow \quad \frac{d}{dt}(\rho_{oil} \forall_{sys}) = -\rho_{oil} \dot{\forall}_{out} \quad \rightarrow \quad \cancel{\rho_{oil}} \frac{d}{dt}(\forall_{sys}) = -\cancel{\rho_{oil}} \dot{\forall}_{out} \quad \rightarrow$$

$$\frac{d\forall_{sys}}{dt} = -\dot{\forall}_{out}$$

Note that for this particular problem, it appears that volume is conserved; however, in general this is not the case, or there would be a general law of conservation of volume.

Thinking about the volume of the system and that the boundary at the piston moves with the velocity V_{Piston}, we can represent the rate of change of volume of the system in terms of the piston velocity and area as

$$\frac{d\forall_{sys}}{dt} = -A_{Piston} V_{Piston} = -\left(\frac{\pi}{4} D^2\right) V_{Piston}$$

(Why do we have a minus sign?)

Now combining this with the result from the conservation of mass gives

$$\frac{d\forall_{sys}}{dt} = -\dot{\forall}_{out}$$

$$-\left(\frac{\pi}{4} D^2\right) V_{Piston} = -\dot{\forall}_{out} \quad \rightarrow \quad \dot{\forall}_{out} = \frac{\pi}{4} D^2 V_{Piston} = \frac{\pi}{4}(1.0 \ \text{cm}^2)\left(0.5 \ \frac{\text{cm}}{\text{s}}\right) = 0.393 \ \frac{\text{cm}^3}{\text{s}}$$

Now to find the time needed to squirt $12.0 \ \text{cm}^3$, we integrate the conservation of mass result

$$\frac{d\forall_{sys}}{dt} = -\dot{\forall}_{out} \quad \rightarrow \quad \int_{\forall_{initial}}^{\forall_{final}} dV = -\int_{t_{initial}}^{t_{final}} \dot{\forall}_{out} \, dt \quad \rightarrow \quad \Delta\forall_{sys} = -\dot{\forall}_{out} \Delta t$$

$$\Delta t = \frac{\Delta\forall_{sys}}{\dot{\forall}_{sys}} = \frac{(12.0 \ \text{cm}^3)}{\left(0.393 \ \frac{\text{cm}^3}{\text{s}}\right)} = 30.5 \text{ seconds}$$

Now to find the velocity of the oil leaving the needle, we can assume one-dimensional flow and use the definition of mass flow rate

$$\dot{\forall}_{out} = A_{out} V_{out} \quad \rightarrow \quad V_{out} = \frac{\dot{\forall}_{out}}{A_{out}} = \frac{\dot{\forall}_{out}}{\left(\frac{\pi}{4} D^2\right)} = \frac{\left(0.393 \ \frac{\text{cm}^3}{\text{s}}\right)}{\left(\frac{\pi}{4}\right)(0.1 \ \text{cm})^2} = 50.0 \ \frac{\text{cm}}{\text{s}}$$

Comment

Could you solve this problem with a non-deforming control volume? [Hint: Treat the piston entering the system as a mass flow rate. Then recognize that the mass in the system is changing.]

Could you solve this problem using a deforming, closed system? [Hint: Consider a closed, deforming system that includes all of the oil originally in the cylinder volume and the needle.]

✔ Example — Draining the gas tank

Gasoline is pumped into a 1000 gallon storage tank at the rate of 10 gpm (gallons per minute). During the filling process, gasoline is being drained out at a rate of 2 gpm. The inlet and the drain are both located below the free surface of the gasoline in the tank. How long will it take to fill the tank if it initially contains 100 gallons of gasoline?

Solution

Known: Gasoline tank is being filled

Find: Time required to fill the tank.

Given:

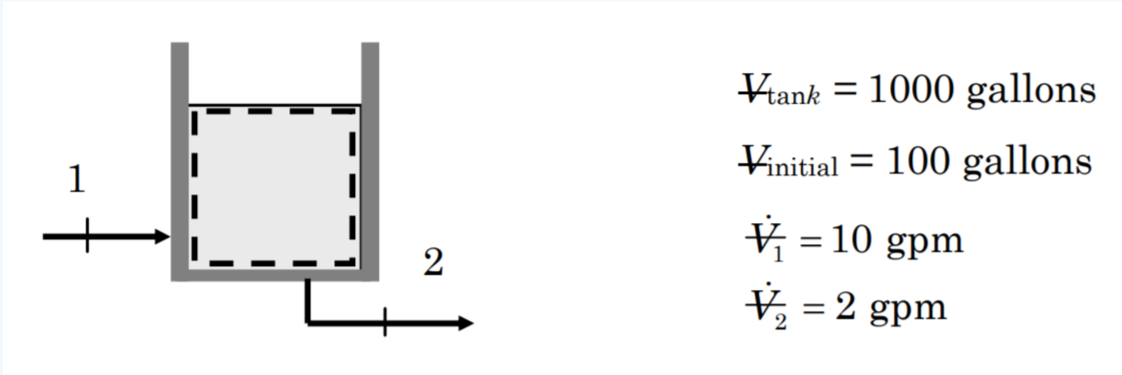


Figure 3.3.5: Defining the system and the volumetric flow rates into and out of it.

Analysis:

Strategy $\rightarrow$ System $\rightarrow$ Deforming volume of gasoline inside the tank during the entire process
Property to Count $\rightarrow$ Mass of the oil
Time period $\rightarrow$ Finite time

Writing conservation of mass for the deforming open system shown below, we have

$$\frac{dm_{sys}}{dt} = \dot{m}_1 - \dot{m}_2$$

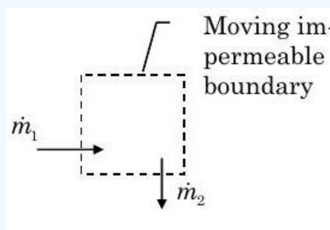


Figure 3.3.6: Gasoline in the tank forms a system with a moving impermeable boundary.

Assume uniform density and that the gasoline is incompressible, $\dot{m} = \rho\dot{\cancel{V}}$ and $m = \rho\cancel{V}$. Substituting these values back into the conservation of mass equation gives

$$\frac{dm_{sys}}{dt} = \dot{m}_1 - \dot{m}_2 \quad \rightarrow \quad \underbrace{\frac{d}{dt}\left(\cancel{\rho}\cancel{V}_{sys}\right) = \left(\cancel{\rho}\dot{\cancel{V}}_1\right) - \left(\cancel{\rho}\dot{\cancel{V}}_2\right)}_{\text{Density cancels out of all terms}} \quad \rightarrow \quad \frac{d\cancel{V}_{sys}}{dt} = \dot{\cancel{V}}_1 - \dot{\cancel{V}}_2$$

Now integrate with time to get the finite time form

$$\int_{t_i}^{t_f} \frac{d\cancel{V}_{sys}}{dt}\,dt = \int_{t_i}^{t_f} \left(\dot{\cancel{V}}_1 - \dot{\cancel{V}}_2\right) dt \quad \rightarrow \quad \Delta\cancel{V}_{sys} = \underbrace{\left(\dot{\cancel{V}}_1 - \dot{\cancel{V}}_2\right)\Delta t}_{\substack{\text{Assumes that volumetric} \\ \text{flow rates are constant}}}$$

Solving for the time interval

$$\Delta t = \frac{\Delta\cancel{V}_{sys}}{\left(\dot{\cancel{V}}_1 - \dot{\cancel{V}}_2\right)} = \frac{(1000 - 100)\ \cancel{\text{gal}}}{(10 - 2)\frac{\cancel{\text{gal}}}{\text{min}}} = 112.5 \text{ minutes}$$

Comment

How would your answer change if $\dot{\cancel{V}}_2 = \left(2\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{t}{10\text{ min}+t}\right)$?

> **Example — Flow through a reducer**

Water flows steadily through a 3-inch diameter steel pipe before it passes through a reducer fitting into a 1-inch diameter pipe. All diameters are internal diameters. The average velocity in the larger pipe is 5 ft/s. The density of water is 62.4lbm/ft^3. Determine (a) the mass flow rate and the volumetric flow rate in the larger pipe, and (b) the volumetric flow rate and the average velocity in the smaller pipe.

Solution

Known: Water flows steadily through a reducer fitting

Find: Volumetric and mass flow rate in the 3-inch pipe. Volumetric flow rate and average velocity in 1-inch pipe.

Given:

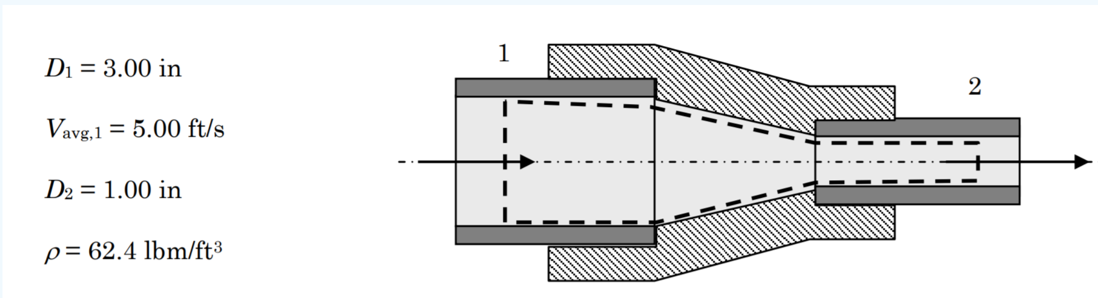


Figure 3.3.7: Defining the system as the water within the reducer.

Strategy → System -- Non-deforming open system as shown in the figure to relate flows at the inlet and outlet.
Property to count -- Mass because we have to relate flows at two locations
Time period -- Steady-state problem since given only rates.

Writing the conservation of mass equation for this open system gives

$$\underbrace{\cancel{\frac{dm_{sys}}{dt}}^{=0}}_{\substack{\text{steady-state}\\ \text{conditions}}} = \dot{m}_1 - \dot{m}_2 \quad \rightarrow \quad \dot{m}_2 = \dot{m}_1$$

From the definition of average velocity $\dot{m} = \rho A_c V_{\text{avg}}$ and $\dot{V} = A_c V_{\text{avg}}$ we can solve for the requested information at Inlet 1:

$$\dot{V}_1 = A_{c,1} V_{avg,1} = \left(\frac{\pi}{4} D_1^2\right) V_{avg,1} = \left[\frac{\pi}{4}\left(\frac{3.00}{12} \text{ ft}\right)^2\right] (5.00 \text{ ft/s}) = 0.245 \frac{\text{ft}^3}{\text{s}}$$

$$\dot{m}_1 = \rho \dot{V}_1 = \left(62.4 \frac{\text{lbm}}{\text{ft}^3}\right)\left(0.245 \frac{\text{ft}^3}{\text{s}}\right) = 15.3 \frac{\text{lbm}}{\text{s}}$$

Now at Outlet 2 we can make use of the conservation of mass results from above. Assuming that the density is uniform, then

$$\dot{m}_2 = \dot{m}_1 \quad \rightarrow \quad \cancel{\rho} \dot{V}_2 = \cancel{\rho} \dot{V}_1 \quad \rightarrow \quad \dot{V}_2 = 0.245 \frac{\text{ft}^3}{\text{s}}$$

To find the velocity

$$\dot{V}_2 = \dot{V}_1 \quad \rightarrow \quad A_{c,2} V_{avg,2} = A_{c,2} V_{c,1} \quad \rightarrow \quad V_{avg,2} = \frac{A_{c,1}}{A_{c,2}} V_{c,1}$$

$$V_{avg,2} = \frac{\left(\frac{\pi}{4} D_1^2\right)}{\left(\frac{\pi}{4} D_2^2\right)} V_{avg,1} = \left(\frac{D_1}{D_2}\right)^2 V_{avg,1} = \left(\frac{3}{1}\right)^2 \left(5 \frac{\text{ft}}{\text{s}}\right) = 45.0 \frac{\text{ft}}{\text{s}}$$

✔ Example — Fill 'er Up – A lesson in conservation of mass

Consider the water tank shown below. The tank can be filled using either the tap at the top or the inlet at the bottom of the tank. The floor of the tank has an area of $A_{\text{tank}} = 10\ \mathrm{m}^2$, and the walls of the tank are vertical.

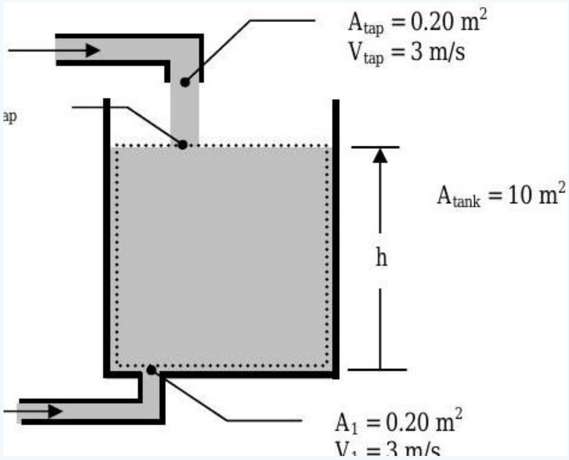


Figure 3.3.8: Defining the system and the two inlets leading into it.

The question at hand is, how long will it take to raise the level h of the water from 2 meters to 5 meters if I fill the tank with the inlet at the bottom of the tank? What if I use the tap at the top of the tank? Will it make any difference?

Filling the tank from the inlet in the floor of the tank

For purposes of this analysis, we must select a system, stuff to count, and a time period:

System $\rightarrow$ Pick the *volume* of the water inside the tank. (See the dashed lines in the figure above. Mass can flow into this volume at 1 and the top boundary of the tank, which corresponds with the free surface of the water in the tank, moves up and down with the water. Call this System I.)

Stuff to count $\rightarrow$ Mass of water inside the system

Time period $\rightarrow$ Since we are asked to find the amount of time we will eventually need a finite-time analysis. (However, my experience, which I'm sharing with you, tells me that it is easiest to start with the rate-form (infinitesimal time period) and then integrate to get the finite-time form.)

Applying the conservation of mass equation to this system gives the following:

$$\frac{dm_{sys}}{dt} = \dot{m}_1$$

Now to get the level or depth of the water into the problem, we need to consider how the mass of the system is related to the depth of the water. Applying the fundamental equation for calculating the mass inside a system gives the following result:

$$m_{sys} = \int_{V\!\!\!\!-_{sys}} \rho \, dV\!\!\!\!- = \underbrace{\rho \int_{V\!\!\!\!-_{sys}} dV\!\!\!\!-}_{\substack{\text{Assume}\\ \text{uniform}\\ \text{density}}} = \rho V\!\!\!\!-_{sys} = \underbrace{\rho A_{\text{tank}} h}_{\text{Since} V\!\!\!\!-_{sys} = A_{\text{tank}} h}$$

Similarly, we need to determine the mass flow rate at 1 in terms of the known information as follows:

$$\dot{m}_1 = \int_{A_1} \rho V_{\text{n, rel}} \, dA = \underbrace{\rho_1 \int_{A_1} V_{\text{n, rel}} \, dA =}_{\substack{\text{Uniform density}\\ \text{at the inlet}}} \underbrace{\rho_1 V_1 \int_{A_1} dA}_{\substack{\text{Uniform velocity}\\ V_1 = V_{\text{n, rel}}}} = \rho_1 V_1 A_1$$

Now we can combine all of this information back into the conservation of mass equation Eq. 3.3.2 as follows:

$$\frac{dm_{sys}}{dt} = \dot{m}_1$$
$$\frac{d}{dt}(\rho A_{\text{tank}} h) = \rho_1 A_1 V_1 \tag{3.3.1}$$

Assuming that water is incompressible, then the density of water in the system ρ and the density of the water coming into the system ρ_1 are the equal and the conservation of mass equation reduces (as developed in Eq. 3.3.5) to

$$A_{\text{tank}} \frac{dh}{dt} = A_1 V_1$$
$$\frac{dh}{dt} = \frac{A_1}{A_{\text{tank}}} V_1 \tag{3.3.2}$$

Integrating this equation to find the time it takes for h to go from 2 to 5 meters, we will use a definite integral between specified limits

$$\int dh = \int \left(\frac{A_1}{A_{\text{tank}}} V_1\right) dt = \left(\frac{A_1}{A_{\text{tank}}} V_1\right) \int dt \quad \rightarrow \quad \int_{h_1}^{h_2} dh = \int_{t_1}^{t_2} \left(\frac{A_1}{A_{\text{tank}}} V_1\right) dt = \left(\frac{A_1}{A_{\text{tank}}} V_1\right) \int_{t_1}^{t_2} dt$$
$$h_2 - h_1 = \left(\frac{A_1}{A_{\text{tank}}} V_1\right)(t_2 - t_1) \tag{3.3.3}$$

Now solving for the numerical answer we have

$$\Delta t = t_2 - t_1 = \frac{(h_2 - h_1)}{\left(\frac{A_1}{A_{\text{tank}} V_1}\right)} = \frac{(5-2)\ \text{m}}{\left(\frac{0.2\ \text{m}^2}{10\ \text{m}^2}\right)\left(3\ \frac{\text{m}}{\text{s}}\right)} = 50\ \text{s}$$

Filling the tank from the tap at the top of the tank

For purposes of this analysis, we must select a system, stuff to count, and a time period:

System $\rightarrow$ Pick the *volume* of the water inside the tank. (See the dashed lines in the figure at below.) Mass can flow into this volume at 2, which is part of the upper boundary of the system. In addition, the entire top boundary of the tank, which corresponds with the free surface of the water in the tank, moves up and down with the water. Call this System II.

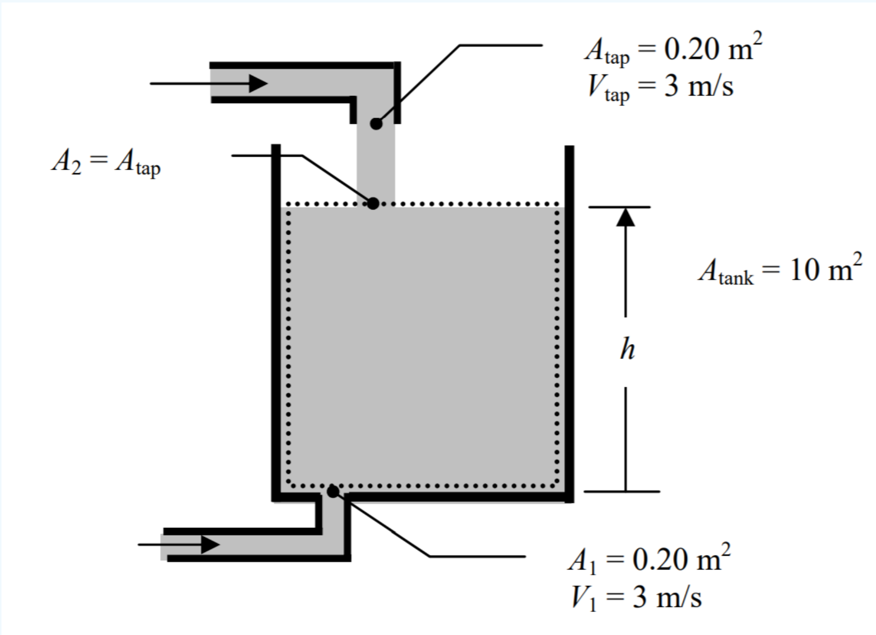


Figure 3.3.9: Defining the system and the two inlets leading into it.

Stuff to count → Mass of water inside the system

Time period → Since we are asked to find the amount of time, we will eventually need a *finite-time* analysis. (However, my experience, which I'm sharing with you, tells me that it is easiest to start with the rate-form (infinitesimal time period) and then integrate to get the finite-time form.)

Before you go on be sure you understand the difference between the system selected here (System II) and the one used previously (System I). This is very important.

Consider the questions:

- What is inside each system?
- Are they both open systems?
- Which boundaries have flow?
- Which boundaries move?
- How would you expect the *shape* of each system to change with time?

Applying the conservation of mass equation to System II gives the following:

$$\frac{dm_{sys}}{dt} = \dot{m}_2$$

Now to get the level or depth of the water into the problem, we need to consider how the mass of the system is related to the depth of the water. Applying the fundamental equation for calculating the mass inside a system gives the following result:

$$m_{sys} = \int_{V_{sys}} \rho \, dV = \underbrace{\rho \int_{V} dV}_{\text{Assume uniform density}} = \rho V = \underbrace{\rho A_{\text{tank}} h}_{\text{Since } V_{sys} = A_{\text{tank}} h}$$

Similarly we need to determine the mass flow rate at inlet 2 in terms of the known information as follows:

$$\dot{m}_2 = \int_{A_2} \rho V_{\text{n, rel}} \, dA = \underbrace{\rho_2 \int_{A_2} V_{\text{n, rel}} \, dA}_{\text{Uniform density at the inlet}} = \underbrace{\rho_2 V_{2,\text{ rel}} \int_{A_2} dA}_{\text{Uniform density } V_{2,rel} = V_{n,rel}} = \rho_2 V_{2,\text{ rel}} dA = \rho V_{2,\text{ rel}} A_2$$

At this point we must be very careful to correctly calculate the velocity of the mass crossing the system boundary. Recall that the mass flow rate is defined with respect to some boundary. For instance, the mass flow rate of the water *leaving* the tap, i.e., crossing the exit plane of the tap, is $\dot{m}_{tap} = \rho A_{tap} V_{tap}$. This assumes that the density and velocity are uniform at the flow cross-section and that **the velocity $\mathbf{V_{tap}}$ is measured with respect to the exit plane of the tap.**

Now, to calculate the mass flow rate at 2 on the boundary of our system requires that we know the velocity of the water *relative* to the system boundary *that is moving*.

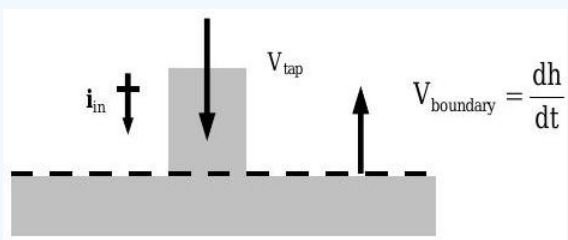


Figure 3.3.10: Rate of change of the upper system boundary.

Referring to the figure, we see that with respect to the ground (or any other stationary point) the velocity of the jet V_{tap} and the velocity of the boundary $V_{boundary}$ can be sketched on the figure. The position of the boundary (the free surface of the water) can be defined in terms of the height of the surface h measured from the bottom of the tank. The velocity of the boundary with respect to the bottom of the tank (a stationary point) can be defined as $V_{boundary} = dh/dt$.

To calculate the relative velocity of the fluid entering the system measured with respect to the moving boundary we must resort to basic physics relations for relative velocity calculations:

Given object A moving with velocity V_A measured with respect to a stationary point O and object B moving with velocity V_B measured with respect to the same stationary point O, then

the *relative velocity of* A *with respect to* B is $\mathbf{V}_{A/B} = \mathbf{V}_A - \mathbf{V}_B$ and

the *relative velocity of* B *with respect to* A is $\mathbf{V}_{B/A} = \mathbf{V}_B - \mathbf{V}_A$.

Applying this result to our problem above gives the result that

$$\begin{aligned} \mathbf{V}_{2,\text{rel}} &= \mathbf{V}_{\textbf{tap/boundary}} \\ &= \mathbf{V}_{\text{tap}} - \mathbf{V}_{\text{boundary}} \\ V_{2,\text{rel}}\mathbf{i}_{\text{in}} &= V_{\text{tap}}\mathbf{i}_{\text{in}} - \left(-V_{\text{boundary}}\mathbf{i}_{\text{in}}\right) \\ &= \left(V_{\text{tap}} + V_{\text{boundary}}\right)\mathbf{i}_{\text{in}} \\ V_{2,\text{rel}} &= V_{\text{tap}} + V_{\text{boundary}} = V_{\text{tap}} + \frac{dh}{dt} \end{aligned} \tag{3.3.4}$$

where $\mathbf{i}_{\text{in}}$ is a unit vector pointing into the system (see the figure above of the moving boundary).

Note that Eq. $3.3.12$ makes physical sense. The system boundary and the incoming water are both moving towards each other; thus, increasing the tap velocity or increasing the rate of change of h will both increase the relative velocity of the water crossing the system boundary measured with respect to the moving boundary.

Now combining these results with the conservation of mass equation gives

$$\frac{dm_{\text{sys}}}{dt} = \dot{m}_2 \quad \rightarrow \quad \frac{d}{dt}\left(\rho A_{\text{tank}} h\right) = \rho_2 A_2 V_{2,\text{rel}} \quad \rightarrow \quad \frac{d}{dt}\left(\rho A_{\text{tank}} h\right) = \rho_2 A_2 \left(V_{\text{tap}} + \frac{dh}{dt}\right)$$

Again assuming that water is an incompressible substance and also assuming that the area $A_2 = A_{\text{tap}}$ [1] gives us the following result:

$$\begin{aligned} \frac{d}{dt}\left(\rho A_{\text{tank}}\, h\right) = \rho_2 A_2 \left(V_{\text{tap}} + \frac{dh}{dt}\right) \quad &\rightarrow \quad \rho A_{\text{tank}} \frac{dh}{dt} = \rho_2 A_2 \left(V_{\text{tap}} + \frac{dh}{dt}\right) \\ \left(A_{\text{tank}} - A_2\right)\frac{dh}{dt} &= A_2 V_{\text{tap}} \\ \frac{dh}{dt} &= \left(\frac{A_2}{A_{\text{tank}} - A_2}\right) V_{\text{tap}} \end{aligned} \tag{3.3.5}$$

Integrating Eq. $3.3.14$ as before we obtain:

`h_{2} - h_{1} = \left( \frac{A_2}}{A_{\text {tank}}-A_2} \right) V_{\text {tap}} \left( t_2-t_1 \right) \nonumber`

And solving for the numerical answers, we obtain

$$\Delta t = t_2 - t_1 = \frac{(h_2 - h_1)}{\left[\left(\frac{A_2}{A_{\text{tank}} - A_2}\right) V_{\text{tap}}\right]} = \frac{(5-2)\ \text{m}}{\left[\left(\frac{0.2\ \text{m}^2}{(10-0.2)\ \text{m}^2}\right)\left(3\ \frac{\text{m}}{\text{s}}\right)\right]} = 49\ \text{ s}$$

Note: You should note that the assumption that $A_2 = A_{\text{tap}}$ is directly related to our assumption that the absolute velocity of the tap water jet at the boundary is the same as the velocity of the water leaving the tap V_{tap}

Comparing the results

Which arrangement "fills" the tank faster? Does this seem strange to you?

How would the comparison change if $A_{\text{tap}} = A_1 = 1\ \text{m}^2$? Would System II still be faster?

- System I:
- System II:

Suppose I started with an empty tank. Will it always take less time to "fill" the tank if I use a tap?

3.4: Mixture Composition

Frequently it is necessary to describe the composition of a system that contains a mixture. The composition of a mixture can be described on either a mass or a molar basis. Before proceeding we should review the relationship between the fundamental dimensions of mass and amount of substance (moles).

Molar Mass and the Amount of Substance

The quantity of material inside the system is usually specified in one of two ways: (1) the mass or (2) the **amount of substance** (the number of molecules or atoms) inside the system. The amount of substance unlike the mass of a substance specifies a unique number of particles. These particles can be atoms, molecules, electrons, etc.; thus, it is necessary to indicate exactly what particles are being counted. By definition, a **mole** is an amount of substance that contains the same number of particles as there are atoms in 0.012 kilograms of carbon-12 (6.022×10^{23} particles). The mass of one mole of substance is called its **molar (or molecular) mass**, M. If we use the symbol n for the number of moles, then we have the following relationship between the mass, moles, and molecular mass of a substance:

$$m = nM$$

In physics and chemistry, most calculations involving the amount of substance always made use of the gram-mole (mol) or just mole for short. In engineering calculations we will have need of many different moles. To understand how these are related, consider the molar mass of diatomic oxygen $\mathrm{O_2}$:

$$M_{\mathrm{O_2}} = \underbrace{32.0\,\frac{\mathrm{g}}{\mathrm{mol}}}_{\substack{\text{Form commonly}\\ \text{used in physics}\\ \text{and chemistry}}} = \underbrace{32.0\,\frac{\mathrm{kg}}{\mathrm{kmol}} = 0.0320\,\frac{\mathrm{kg}}{\mathrm{mol}}}_{\substack{\text{Other forms commonly}\\ \text{in the Si system}}} = \underbrace{32.0\,\frac{\mathrm{lbm}}{\mathrm{lbmol}} = 32.0\,\frac{\mathrm{slug}}{\mathrm{slugmol}}}_{\text{Forms used in the AES system}}$$

In addition to the gram-mole (mol), the molar mass can be described in terms of the kilogram-mole (kmol), the pound-mole ($mathrmlbmol$), and the slug-mole (slugmol). We will predominately use the kilogram-mole and the pound-mole for our calculations; however, anytime we use the word "mole" or refer to a "molar basis", the actual mole unit could be any of the units mentioned above.

The key to using these different quantities is to recognize that each one of them is a way of specifying a definite number of molecules or atoms. Consider the example below to test your understanding of this concept.

Example — Particles in a pound-mole

How many moles of diatomic oxygen $(\mathrm{O_2})$ have the same number of particles as one pound-mole of $\mathrm{O_2}$?

Solution

Starting with the molar mass of diatomic oxygen, we have $32.00\ \mathrm{lbm/lbmol} = 32.00\ \ \mathrm{g/mol}$ If we divide both sides through by 32.00 we discover that $1\ \mathrm{lbm/lbmol} = 1\ \ \mathrm{g/mol}$. Starting with this result

$$1\,\frac{\mathrm{lbm}}{\mathrm{lbmol}} = 1\,\frac{\mathrm{g}}{\mathrm{mol}} \quad \rightarrow \quad 1\mathrm{lbmol} \ = \left(\frac{\mathrm{lbm}}{\mathrm{g/mol}}\right) = \left(\frac{\mathrm{lbm}}{\mathrm{g}}\right) \times (\mathrm{mol}) \times \underbrace{\left(0.4536\,\frac{\mathrm{kg}}{\mathrm{lbm}}\right)}_{=1} \times \underbrace{\left(1000\,\frac{\mathrm{g}}{\mathrm{kg}}\right)}_{=1}$$

$$= 453.6\ \mathrm{mol}$$

Similarly,

$$\begin{aligned} 1\ \mathrm{kmol} &= 1000\ \ \mathrm{mol} \\ 1\ \mathrm{slugmol} &= 32.174\ \mathrm{lbmol} = 14,594\ \ \mathrm{mol} = 14.594\ \mathrm{kmol} \end{aligned}$$

What is going on here? Would these results hold for other substances? How are the number of particles in each "mole" related?

Example — Moles of methane

A tank contains 50 kg of methane gas. How many moles of methane (CH_4) are in the tank, in kmol?

Starting with the basic equation relating moles, mass and molar mass, we have $m = nM$. The molar mass for methane is $M_{\text{methane}} = 16.04$. Rearranging the basic relation we have

$$n = \frac{m}{M} = \frac{50 \text{ kg}}{16.04 \text{ kg/kmol}} = 3.12 \text{ kmol}$$

How many pound-moles would you have?

$$n = \frac{m}{M} = \frac{50 \text{ kg}}{16.04 \text{ lbm/lbmol}} = (3.12 \text{ lbmol}) \times \left(\frac{\text{kg}}{\text{lbm}}\right) \times \underbrace{\left(\frac{\text{lbm}}{0.4536 \text{kg}}\right)}_{=1} = 6.88 \text{ lbmol}$$

How many ton-mol if $1 \text{ ton} = 2000 \text{ lbm}$?

Now that we have reviewed the difference between mass and amount of substance, we can turn to mixture composition.

Composition on a Mass Basis

When mixture composition is specified on a **mass basis**, we are interested in knowing the composition in terms of the mass of the various components. Because of the direct relationship between the mass of mixture and its weight, you will sometimes see "analysis by weight" or "gravimetric analysis" used instead of mass basis.

To see how the mass basis is specified, consider a three component mixture formed from compounds A, B, and C. The mass of the mixture is just the sum of the mass of each component:

$$m_{mix} = m_A + m_B + m_C = \sum_{i=1}^{N} m_i$$

If we now divide by the mass of mixture, we obtain

$$1 = \frac{m_A}{m_{mix}} + \frac{m_B}{m_{mix}} + \frac{m_C}{m_{mix}} = \sum_{i=1}^{N} \frac{m_i}{m_{mix}}$$

The ratio of m_i, the mass of the i-th component, to m_{mix}, the mass of the mixture, is called the **mass fraction** mf_i of the i-th component of the mixture:

$$mf_i \equiv \frac{m_i}{m_{mix}}$$

Combining Eq. $3.4.3$ and Eq. $3.4.4$, we see that the mass fractions of a mixture sum to unity:

$$\sum_{i=1}^{N} mf_i = 1$$

When the composition of mixture is specified on a mass basis, the mass fractions are specified for the components of the mixture.

Composition on a Molar Basis

When a molar basis is used, we are interested in knowing the composition in terms of the number of moles of the various components. Again consider our three-component mixture. The number of moles of the mixture equals the sum of the number of moles of each component:

$$n_{mix} = n_A + n_B + n_C = \sum_{i=1}^{N} n_i$$

If we now divide this result by the number of moles in the mixture, we obtain

$$1=\frac{n_A}{n_{mix}}+\frac{n_B}{n_{mix}}+\frac{n_C}{n_{mix}}=\sum_{i=1}^{N}\frac{n_i}{n_{mix}}$$

The ratio of n_i, the number of moles of the i-th component, to n_{mix}, the number of moles of the mixture, is called the **mole fraction** nf_i of the i th component of the mixture:

$$nf_i \equiv \frac{n_i}{n_{mix}}$$

As with the mass fractions, combining the results of Eq. $3.4.7$ and Eq. $3.4.8$ reveals that the mole fractions for all components of mixture sum to unity:

$$\sum_{i=1}^{N} nf_i = 1$$

When the composition of a mixture is specified on a molar basis, the mole fractions of each component are specified.

Mixture Molar Mass

In addition to knowing the composition of a mixture, it is often of interest to know the molar mass for a mixture. By definition the mixture molar mass (or apparent molar mass of a mixture) is the ratio of the mixture mass to the number of moles in the mixture:

$$M_{mix}=\frac{m_{mix}}{n_{mix}}$$

The units for this quantity are the same as that for a pure quantity — $\mathrm{g/mol}$, $\mathrm{kg/kmol}$, or $\mathrm{lbm/lbmol}$.

Converting Mixture Composition Basis

Either basis can be used to specify the composition of a mixture. Experience reveals that gaseous mixtures are most commonly described on a molar basis and mixtures containing liquids and solids are described using a mass basis.

It often required to convert the mixture composition from a mole basis to an mass basis or vice versa. This is best done by taking the mixture (either the actual amount or a representative amount), setting up a table of data, and methodically working through to find the necessary information. The following examples demonstrate this technique.

✓ Example — Gas Mixture Composition

A tank contains $1500\ \mathrm{kmol}$ of a gas. The gas is a mixture of oxygen $(\mathrm{O_2})$, nitrogen $(\mathrm{N_2})$, and carbon dioxide $(\mathrm{CO_2})$. The measured mole fractions for two of the components are as follows: $\mathrm{O_2} - 20\%$ and $\mathrm{N_2} - 70\%$.

Determine (a) the number of kilograms in the tank, (b) the mixture composition in mass fractions (weight percent) and (c) the *mixture* (apparent) *molar mass*.

Solution

Known: Tank contains a gas mixture.

Find: (a) mass of the gas, m_{mix} in kg; (b) mass fractions, mf_i's; (c) mixture molar mass, M_{mix}

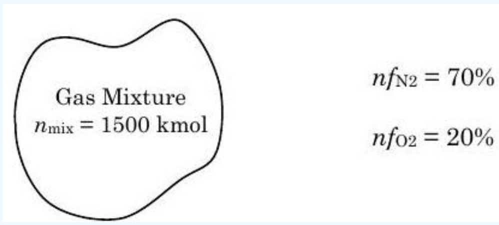


Figure $3.4.1$: Given information about a gas mixture.

Strategy $\rightarrow$ Try using a table format to simplify the calculations.

As a first step, we can complete the table for the mole fractions since the mole fraction of the mixture must sum to equal 100%. Thus $nf_{\mathrm{CO_2}} = (100-70-20)\% = 10\%$.

Now set up a table including the molar masses of the three constituent gases

	nf_j	$n_j = nf_j \cdot n_{\text{mix}}$	M_j	$m_j = n_j \cdot M_j$	$mf_j = m_j/m_{\text{mix}}$
O_2	0.20	300 kmol	32.00 kg/kmol	9600 kg	0.210
N_2	0.70	1050 kmol	28.01 kg/kmol	29410 kg	0.645
CO_2	0.10	150 kmol	44.01 kg/kmol	6602 kg	0.145
	1.00	$1500 \text{ kmol} = n_{\text{mix}}$	....	$45612 \text{ kg} = m_{\text{mix}}$	1.000

The number of kilograms of gas in the tank is found in the fifth column of the table. The mass fractions of the mixture are found in the sixth column of the table.

To find the mixture molar mass, we use the definition of mixture molar mass

$$m_{\text{mix}} = M_{\text{mix}} n_{\text{mix}} \quad \rightarrow \quad M_{\text{mix}} = \frac{m_{\text{mix}}}{n_{\text{mix}}} = \frac{45612 \text{ kg}}{1500 \text{kmol}} = 30.41 \, \frac{\text{kg}}{\text{kmol}}$$

Comments

(a) An alternate way to find the mixture molar mass M_{mix} uses the component mole fractions and molar masses directly:

$$M_{\text{mix}} = \frac{m_{\text{mix}}}{n_{\text{mix}}} = \frac{\sum_{j=1}^{N} m_j}{n_{\text{mix}}} = \frac{\sum_{j=1}^{N} (n_j M_j)}{n_{\text{mix}}} = \sum_{j=1}^{N} \left(\frac{n_j}{n_{\text{mix}}} M_j \right) = \sum_{j=1}^{N} (nf_j M_j) = M_{\text{mix}}$$

(b) Many times you are only given the compositions without any information about the actual amount of substance. Under these conditions, you can still use the table format. Simply assume an amount of substance, say 100 kmol, and then work the problem for this size mixture.

(c) If you were given the composition in terms of mass fractions (or weight percents), you would follow the same process as shown in the table, only starting with mass fractions and the known or assumed mass of substance.

Example — Flow stream composition

A stream of liquid enters a tank with a mass flow rate of 400 kg/min. The weight percents (mass fractions) of the two-components in the stream are as follows: water $(H_2O) - 90\%$ and ammonia $(NH_3) - 10\%$.

Determine the molar composition of the liquid (mole fractions) and determine the molar flow rate of the liquid in kmol/min.

Solution

Known: Mass fractions for a flow stream.

Find: Mole fractions and molar flow rate.

Given: $\dot{m} = 400 \text{ kg/min}; \quad mf_{H_20} = 0.90; \quad mf_{NH_3} = 0.10$

Analysis:

Strategy $\rightarrow$ Try the table to convert from mass fractions to mole fractions.

	mf_j	$\dot{m}_j = mf_j \cdot \dot{m}_{\text{mix}}$	M_{mix}	$\dot{n}_j = \dot{m}_j / M_{\text{mix}}$	$nf_j = \dot{n}_j / \dot{n}_{\text{mix}}$
H_2O	0.90	360.0 kg/min	18.02 kg/kmol	19.98 kmol/min	0.895
NH_3	0.10	40.0 kg/min	17.03 kg/kmol	2.35 kmol/min	0.105
	1.00	400.0 kg/min mixture mass flow rate	......	22.33 kmol/min mixture molar flow rate	1.000

3.5: Accounting of Chemical Species

For many problems, it is essential to keep track of the individual chemical species — atoms or molecules. Examples of these include

- combustion processes, e.g. burning gasoline in air
- mixing processes, e.g. mixing water and antifreeze
- any process with chemical reactions
- preparation of solid solutions, e.g. doped silicon crystal

Although the *total mass is always conserved*, no such conservation law exists for chemical species in general.

Experimenting with Conservation of Mass

To see what happens when we try to account for individual chemical species, consider the steady-state chemical reactor in Figure $3.5.1$. Compound A and compound B flow into the reactor and combine to produce compound C.

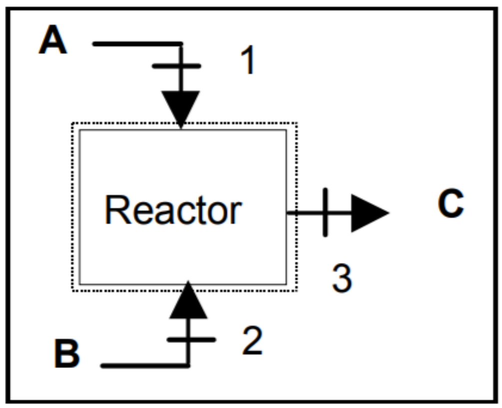


Figure $3.5.1$: Steady-state chemical reactor

Writing the rate-form of the *conservation of mass* equation gives

$$\frac{dm_{sys}}{dt} = \dot{m}_1 + \dot{m}_2 - \dot{m}_3.$$

For steady-state operation the rate of change of mass inside the system is zero. Rearranging and solving for the mass flow rate at the outlet gives

$$\dot{m}_3 = \dot{m}_1 + \dot{m}_2.$$

Now let's try writing a similar equation for just compound C. Starting with the conservation of mass equation for only compound C we have

$$\underbrace{\frac{dm_{C,sys}}{dt}}_{\substack{\text{Rate of change} \\ \text{of compound } C \\ \text{inside the system}}} = \underbrace{\dot{m}_{C,1} + \dot{m}_{C,2}}_{\substack{\text{Mass flow rate} \\ \text{of compound } C \\ \text{entering the system}}} - \underbrace{\dot{m}_{C,3}}_{\substack{\text{Mass flow rate} \\ \text{of compound } C \\ \text{leaving the system}}}$$

By definition, for a steady-state system the intensive and extensive properties of the system are independent of time. Thus, the rate-of-change term must be zero. We also know from our problem statement that only compounds A and B enter the system; thus, the mass flow rates of C at inlets 1 and 2 are identically zero. Putting this together gives the following result:

$$0 = 0 + 0 - \dot{m}_{C,3}$$

This result is inconsistent with our understanding of the physical situation. We know that compound C is leaving our steady-state reactor. Where is it coming from? A few moments of reflection might suggest that our experience tells us that compound C is being *generated* inside the system.

If we tried writing similar equations for compound A and compound B, we would come up with the same physically inconsistent result. In this case, we might conclude that compound A and compound B are being *consumed* inside this system.

To remedy this inconsistency let's attempt to modify our mass balance to account for generation and consumption of individual chemical species. For our particular chemical reactor the equation might look like this for compound i:

$$\underbrace{\frac{dm_{i,sys}}{dt}}_{\substack{\text{Rate of accumulation}\\ \text{of compound } i\\ \text{inside the system}}} = \underbrace{\dot{m}_{i,1}+\dot{m}_{i,2}-\dot{m}_{i,3}}_{\substack{\text{Transport rate of}\\ \text{compound } i\\ \text{across the}\\ \text{system boundary}}} + \underbrace{\dot{m}_{i,\text{ gen}}}_{\substack{\text{Generation rate}\\ \text{of compound } i\\ \text{inside the system}}} - \underbrace{\dot{m}_{i,\text{ cons}}}_{\substack{\text{Consumption rate}\\ \text{of compound } i\\ \text{inside the system}}}$$

Writing this equation for each of the three compounds in our chemical reactor subject to the appropriate assumptions, e.g. steady-state operation and pure compounds at the three flow boundaries, gives

$$\begin{aligned} \text{Compound } A: &\quad 0=\dot{m}_{A,1}-0+0-\dot{m}_{A,\text{ cons}} \\ \text{Compound } B: &\quad 0=\dot{m}_{B,1}-0+0-\dot{m}_{B,\text{ cons}} \\ \text{Compound } C: &\quad 0=0-\dot{m}_{C,3}+\dot{m}_{C,\text{ gen}}-0 \end{aligned}$$

If we sum these three equations we obtain the following relationship:

$$0=\underbrace{\left[\dot{m}_{A,1}+\dot{m}_{B,2}-\dot{m}_{C,3}\right]}_{\substack{=0\\ \text{From conservation}\\ \text{of total mass}}}+\left[\dot{m}_{C,\text{ gen}}-\dot{m}_{A,\text{ cons}}-\dot{m}_{B,\text{ cons}}\right]$$

Since the terms in the first brackets on the right-hand side satisfy the conservation of total mass equation written earlier, we are left with the conclusion that the terms in the second set of brackets must satisfy the following relationship:

$$\underbrace{\dot{m}_{C,\text{ gen}}}_{\substack{\text{Generation rate}\\ \text{of compound } C\\ \text{inside the system}}} - \underbrace{\dot{m}_{A,\text{ cons}}}_{\substack{\text{Consumption rate}\\ \text{of compound } A\\ \text{inside the system}}} - \underbrace{\dot{m}_{B,\text{ cons}}}_{\substack{\text{Consumption rate}\\ \text{of compound } B\\ \text{inside the system}}} = 0$$

Examining this result, it would seem that *this relationship is a direct consequence of the empirical fact that the total mass is conserved*. In the next section we will write a general accounting statement for any chemical species in terms of both the species mass and the number of moles of the species.

Accounting Equation for Chemical Species

Building on the results of the last section and our experience with the accounting framework, we can write two different accounting or balance equations for any individual chemical species.

Mass Basis for Compound i

On a mass basis, the accounting equation is written in terms of the system mass, mass flow rates, and generation and consumption of mass of the specified compound:

$$\underbrace{\frac{dm_{i,sys}}{dt}}_{\substack{\text{Rate of accumulation}\\ \text{of mass}\\ \text{of Compound } i\\ \text{inside the system}}} = \underbrace{\left[\sum_{in}\dot{m}_{i,i}-\sum_{out}\dot{m}_{i,e}\right]}_{\substack{\text{Net flow rate}\\ \text{of mass}\\ \text{of Compound } i\\ \text{into the system}}} + \underbrace{\left[\dot{m}_{i,gen}-\dot{m}_{i,cons}\right]}_{\substack{\text{Net generation rate}\\ \text{of mass}\\ \text{of Compound } i\\ \text{inside the system}}}$$

where

$$m_{i,sys} = \int_{V_{sys}} \rho_i \, dV \text{, the mass of } i \text{ inside the system (kg)}$$
$$\dot{m}_{i,i};\ \dot{m}_{i,e} = \text{mass flow rate of } i \text{ into or out of the system (kg/s)}$$
$$\dot{m}_{i,gen};\ \dot{m}_{i,cons} = \text{generation/consumption rate of } i \text{ inside the system (kg/s)}$$

Note that the generation and consumption terms are each identically zero *unless* there are chemical reactions in the system.

Molar Basis for Compound i

On a molar basis, the accounting equation is written in terms of the system moles, molar flow rates, and generation and consumption of moles of the specified compound:

$$\underbrace{\frac{dn_{i,sys}}{dt}}_{\substack{\text{Rate of Accumulation}\\ \text{of moles}\\ \text{of Compound } i\\ \text{inside the system}}} = \underbrace{\left[\sum_{in} \dot{n}_{i,i} - \sum_{out} \dot{n}_{i,e}\right]}_{\substack{\text{Net molar flow rate}\\ \text{of Compound } i\\ \text{into the system}}} + \underbrace{[\dot{n}_{i,gen} - \dot{n}_{i,cons}]}_{\substack{\text{Net generation rate}\\ \text{of moles}\\ \text{of Compound } i\\ \text{inside the system}}}$$

where

$$n_{i,sys} = \int_{V_{sys}} \bar{\rho}_i \, dV, \text{ the number of moles of} i \text{ inside the system (kmol)}$$
$$\bar{\rho}_i = \text{molar density of } i \left(\text{kmol/m}^3\right)$$
$$\dot{n}_{i,i};\ \dot{n}_{i,e} = \text{molar flow rate of } i \text{ into or out of the system (kmol/s)}$$
$$\dot{n}_{i,gen};\ \dot{n}_{i,cons} = \text{generation/consumption rate of moles of } i \text{ inside the system (kmol/s)}$$

Again, as with the mass basis equation, the generation and consumption terms are each identically zero *unless* there are chemical reactions in the system.

Application of Chemical Species Accounting

There are two broad classes of problems in which chemical species accounting are important:

- Systems without chemical reactions, and
- Systems with chemical reactions.

Systems Without Chemical Reactions

If a system has no chemical reactions, the first thing to recognize is that both the generation rate and the consumption rate terms for any compound are identically zero.

- If the chemical composition for the system is also *constant* (with time) and *uniform* (in space), the species accounting equations duplicate the conservation of mass equations and add nothing to our analysis.
- If the chemical composition is also *time dependent* and *non-uniform*, the species accounting equations *may* provide additional information that supplements the conservation of mass equations. Examples of this include problems involving mixing, separation, and distillation.

When applying both the conservation of mass and the species accounting equations, it is important to recognize how many independent equations can be written for a specific system.

For any non-reactive system, the maximum number of independent equations that can be obtained by applying the conservation of total mass and species accounting equals the number of independent species involved in the process

Test Yourself

Try your hand at deciding how many independent equations can be written for the following steady-state systems:

(a) Tee in an air duct

- Inlet-1 Air
- Outlet-2 Air

- Outlet-3 Air

(b) Air separation process

- Inlet-1 Air composed of 79% N_2 and $21\% O_2$ (molar analysis)
- Outlet-2 Pure O_2
- Outlet-3 Pure N_2

(c) Humidifier

- Inlet-1 Air
- Inlet-2 Water vapor
- Outlet-3 Air-water vapor mixture

(d) Mixing chamber

- Inlet-1 Water vapor (H_2O)
- Inlet-2 Carbon dioxide (CO_2)
- Inlet-3 Air composed of 79% N_2 and $21\% O_2$ (molar analysis)
- Outlet-4 Gaseous mixture of $H_2O,\ CO_2,\ N_2$,\) and O_2

(e) Distillation process

- Inlet-1: Water-alcohol mixture with composition 1
- Outlet-2: Water-alcohol mixture with composition 2
- Outlet-3: Water-alcohol mixture with composition 3

The next two examples demonstrate how to use the species accounting equations to solve problems that do not involve chemical reactions.

Species Accounting — Problems without Chemical Reaction

Problems with changing composition can be separated into two groups: systems with chemical reaction and systems without chemical reaction. The focus of this subsection is systems without chemical reaction.

Species Accounting Equations

For systems without chemical reaction, the species accounting equations can be written without the generation/consumption terms. In terms of the mass of component j, the species accounting equation becomes

$$\frac{dm_{j,sys}}{dt} = \sum_{in} \dot{m}_{j,i} - \sum_{out} \dot{m}_{j,e} + \cancel{\dot{m}_{j,gen}}^{=0} - \cancel{\dot{m}_{j,cons}}^{=0} \quad \rightarrow \quad \frac{dm_{j,sys}}{dt} = \sum_{in} \dot{m}_{j,i} - \sum_{out} \dot{m}_{j,e}$$

and in terms of the amount of substance j (moles of component j) the species accounting equation becomes

$$\frac{dn_{j,sys}}{dt} = \sum_{in} \dot{n}_{j,i} - \sum_{out} \dot{n}_{j,e} + \cancel{\dot{n}_{j,gen}}^{=0} - \cancel{\dot{n}_{j,cons}}^{=0} \quad \rightarrow \quad \frac{dn_{j,sys}}{dt} = \sum_{in} \dot{n}_{j,i} - \sum_{out} \dot{n}_{j,e}$$

These equations can be written to keep track of any chemical component when studying the behavior of a system with changing composition and no chemical reactions. Examples of this type of problem involve the physical processes of mixing, separation, and distillation.

Given a system with N_{comp} chemical compounds, you can write a total of $N_{\text{comp}} + 1$ equations by applying conservation of total mass m and applying species accounting for each of the N_{comp} compounds. Unfortunately, only N_{comp} of these equations are independent. (This means that any N_{comp} of the available $N_{\text{comp}} + 1$ equations above can be combined by simple algebra to recover the remaining equation.)

Composition Equations

The composition of the mixture can always be described in terms of the mass fractions and mole fractions. For any system, one composition equation can be written for each flow stream, and an additional composition equation can be written to describe the contents of the system. For a steady-state system, the system composition equation is unnecessary.

Consider an open system with three components A, B, and C. For each inlet or outlet stream, say stream 1, we can relate the mass (or molar) flow rate of each component to the total flow rate at the inlet or outlet as shown below:

$$\begin{array}{lll} \dot{m}_1 = \dot{m}_{A,1} + \dot{m}_{B,1} + \dot{m}_{C,1} & = mf_{A,1}\dot{m}_1 + mf_{B,1}\dot{m}_1 + mf_{C,1}\dot{m} & \rightarrow \quad 1 = mf_{A,1} + mf_{B,1} + mf_{C,1} \\ \dot{n}_1 = \dot{n}_{A,1} + \dot{n}_{B,1} + \dot{n}_{C,1} & = nf_{A,1}\dot{n}_1 + nf_{B,1}\dot{n}_1 + nf_{C,1}\dot{n}_1 & \rightarrow \quad 1 = nf_{A,1} + nf_{B,1} + nf_{C,1} \end{array}$$

For the contents of the system, the mass or amount of substance (moles) for each component can be written as shown below:

$$\begin{array}{lll} {}_{sys} & = m_{A,sys} + m_{B,sys} + m_{C,sys} & \\ & = mf_{A,sys}m_{sys} + mf_{B,sys}m_{sys} + mf_{C,sys}m_{sys} \quad \rightarrow & 1 = mf_{A,sys} + mf_{B,sys} + mf_{C,sys} \\ n_{sys} & = n_{A,sys} + n_{B,sys} + n_{C,sys} & \\ & = nf_{A,sys}n_{sys} + nf_{B,sys}n_{sys} + nf_{C,sys}n_{sys} \quad \rightarrow & 1 = nf_{A,sys} + nf_{B,sys} + nf_{C,sys} \end{array}$$

Example — Mixing Problem

A specific chemical process requires a mixture of methanol, ethanol, and water at a mass flow rate of 200 kg/h. The product stream is formed by using two streams each containing only two components. The known information about flow rates and composition for this steady-state mixing problem is shown in the table.

Using the information in the table and any additional assumptions, determine the unknown compositions and flow rates. [Note: No chemical reactions occur.]

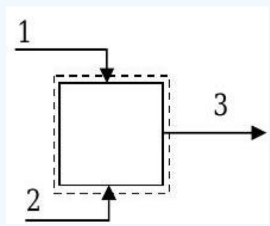


Figure $3.5.2$: System setup for the mixing problem.

Stream	**Mass Flow Rate**	**Composition — Mass %**		
		Methanol	Ethanol	Water
1			0	
2	120 kg/h	0		
3	200 kg/h	5.00	40.0	

Analysis:

System $\rightarrow$ See the picture

Time period $\rightarrow$ Rate problem (Infinitesimal time difference)

Stuff to count $\rightarrow$ Chemical species and mass.

There are 6 unknowns in this problem: one mass flow rate and five compositions. Thus we need 6 independent equations to relate these variables before we can solve the problem.

Apply conservation of mass and species accounting. Since there are 3 compounds we can obtain at most 3 independent equations by writing species accounting and conservation of mass equations for the system. Let's write equations for total mass, ethanol and methanol:

$$\text{Mass Balance:} \quad \cancel{\frac{dm_{sys}}{dt}}^{=0,SS} = \dot{m}_1 + \dot{m}_2 - \dot{m}_3$$
$$0 = \dot{m}_1 + \dot{m}_2 - \dot{m}_3$$

$$\text{Ethanol Balance:} \quad \frac{dm_{\text{eth, sys}}}{dt} = \dot{m}_{\text{eth, 1}} + \dot{m}_{\text{eth, 2}} - \dot{m}_{\text{eth, 3}} = mf_{\text{eth, 1}}\dot{m}_1 + mf_{\text{eth, 2}}\dot{m}_2 - mf_{\text{eth, 3}}\dot{m}_3$$
$$0 = mf_{\text{eth, 1}}\dot{m}_1 + mf_{\text{eth, 2}}\dot{m}_2 - mf_{\text{eth, 3}}\dot{m}_3$$

$$\text{Methanol Balance:} \quad \frac{dm_{\text{meth, sys}}}{dt} = \dot{m}_{\text{meth, 1}} + \dot{m}_{\text{meth, 2}} - \dot{m}_{\text{meth, 3}} = mf_{\text{meth, 1}}\dot{m}_1 + mf_{\text{meth, 2}}\dot{m}_2 - mf_{\text{meth, 3}}\dot{m}_3$$
$$0 = mf_{\text{meth, 1}}\dot{m}_1 + mf_{\text{meth, 2}}\dot{m}_2 - mf_{\text{meth, 3}}\dot{m}_3$$

Since we have used all of the independent species accounting or mass balance equations, we must now turn to the composition relations. There are three inlets/outlets, and so at most we can write 3 composition equations, one for each flow stream:

$$\begin{aligned} \text{Stream 1:} &\quad 1 = mf_{\text{meth, 1}} + mf_{\text{eth, 1}} + mf_{\text{w, 1}} \\ \text{Stream 2:} &\quad 1 = mf_{\text{meth, 2}} + mf_{\text{eth, 2}} + mf_{\text{w, 2}} \\ \text{Stream 3:} &\quad 1 = mf_{\text{meth, 3}} + mf_{\text{eth, 3}} + mf_{\text{w, 3}} \end{aligned}$$

Since the problem is at steady-state, there is no need for a composition equation that describes the contents of the system.

We now have six equations that involve the six unknowns. Assuming that the equations are independent, we should be able to solve them for the unknowns. If we substitute the known information into the six equations, we have the following:

$0 = \dot{m}_1 + \dot{m}_2 - \dot{m}_3$	$\rightarrow$	$0 = \dot{m}_1 + \left(120 \ \frac{\text{kg}}{\text{h}}\right) - \left(200 \ \frac{\text{kg}}{\text{h}}\right)$
$0 = mf_{\text{eth, 1}}\dot{m}_1 + mf_{\text{eth, 2}}\dot{m}_2 - mf_{\text{eth, 3}}\dot{m}_3$	$\rightarrow$	$0 = (0)\dot{m}_1 + mf_{\text{eth, 2}}\left(120 \ \frac{\text{kg}}{\text{h}}\right) - (0.40)\left(200 \ \frac{\text{kg}}{\text{h}}\right)$
$0 = mf_{\text{meth, 1}}\dot{m}_1 + mf_{\text{meth, 2}}\dot{m}_2 - mf_{\text{meth, 3}}\dot{m}_3$	$\rightarrow$	$0 = mf_{\text{meth, 1}} + (0)\left(120 \ \frac{\text{kg}}{\text{h}}\right) - (0.05)\left(200 \ \frac{\text{kg}}{\text{h}}\right)$
$1 = mf_{\text{meth, 1}} + mf_{\text{eth, 1}} + mf_{\text{w, 1}}$	$\rightarrow$	$1 = mf_{\text{meth, 1}} + 0 + mf_{\text{w, 1}}$
$1 = mf_{\text{meth, 2}} + mf_{\text{eth, 2}} + mf_{\text{w, 2}}$	$\rightarrow$	$1 = 0 + mf_{\text{eth, 2}} + mf_{\text{w, 2}}$
$1 = mf_{\text{meth, 3}} + mf_{\text{eth, 3}} + mf_{\text{w, 3}}$	$\rightarrow$	$1 = 0.05 + 0.40 + mf_{\text{w, 3}}$

Solving these equations gives the following answers:

Stream	**Mass Flow Rate**	**Composition — Mass %**		
		Methanol	Ethanol	Water
1	**80 kg/h**	**12.5**	0	**87.5**
2	120 kg/h	0	**66.7**	**33.3**
3	200 kg/h	5.00	40.0	**55.0**

This solution can be done by hand; however it is best done using a computer algebra system like MAPLE or a numerical solver like EES.

Lessons to be learned from this problem:

In general, for any problem with varying composition but no chemical reactions,

- each flow stream is specified by knowing its mass (or molar) flow rate and the mass (or mole) fractions of the components in the stream.
- the contents of each system are described by knowing the mass (or moles) and the composition of the contents.
- for each system (or subsystem), you can write one composition equation for each flow stream and one composition equation for the contents of the system. All of these equations are independent.
- for each system (or subsystem) in the problem, you can write as many independent mass and/or species accounting equations as there are chemical compounds (components) in the system (or subsystem).

For this example, there is one system with three flow streams and three compounds:

- Maximum number of variables to describe the compositions, mass, and flow rates:

$$(3 \text{ mass flow rates}) + (3 \text{ flow streams})\left(3 \ \frac{\text{compounds}}{\text{flow stream}}\right) = 12 \text{ variables}$$

- Maximum number of independent equations that can be written using composition, mass conservation, and species accounting:

$$\underbrace{(1\text{ system})(3\text{ compounds})}_{\substack{\text{Number of independent total mass}\\ \text{and/or species accounting equations}}}+\underbrace{(3\text{ flow streams})}_{\substack{\text{Number of independent}\\ \text{composition equations}}}=6\text{ independent equations}$$

- The number of **degrees of freedom** is the difference between the number of variables needed to describe the problem and the number of independent equations you can write:

$$\underbrace{\text{DOF}}_{\substack{\text{Degrees of}\\ \text{Freedom}}}=\underbrace{\text{NOV}}_{\substack{\text{Number of}\\ \text{Variables}}}-\underbrace{\text{NOE}}_{\substack{\text{Number of}\\ \text{Independent}\\ \text{Equations}}}=12-6=6=\begin{bmatrix}\text{Minimum number of variables}\\ \text{that must be specified to obtain}\\ \text{a unique solution.}\end{bmatrix}$$

- For this example, the problem statement provides information about six variables. Without this information it would have been impossible to find a unique solution.
  If only five variables were assigned values, it would have only been possible to solve for 6 of the remaining variables in terms of the seventh variable. For example, if the mass flow rate at 2 had not been specified, you could have treated it as an independent variable and solved for all of the other unknown variables as a function of $\dot{m}_2$. For a physically possible solution, $\dot{m}_2$ could only take on a limited range of values, e.g. $0<\dot{m}_2<200\text{ kg/h}$ because stream 2 is the only way that ethanol is brought into the mixer.
  With only five variables specified, a unique solution would only exist if additional constraints on the problem are provided. These could be provided in a number of ways - by specifying how some of the incoming mass or species flows must be distributed in the outlet, by giving you a range of compositions for certain variables, etc. These constraints serve as independent equations that can be combined with the independent equations developed from composition and species/mass accounting. This set can then be solved for the unknown variables.
  If more than six variables were assigned values, not all of them could take on arbitrary values for the problem to have a solution.

Example — Enriching a gas mixture

A gas cylinder initially contains 10 lbm of air. It is desired to enrich the oxygen content of the mixture by bleeding 5 lbm of air from the cylinder and then adding 3 lbm of O_2. Determine the mass and molar composition of the mixture after the enrichment process is completed. For purposes of analysis you may assume that air is a mixture of 23% oxygen and 77% nitrogen by mass.

Solution

Known: Air is bled from a gas cylinder and replaced by oxygen.

Find: Composition (mass and mole fractions) after process

Given:

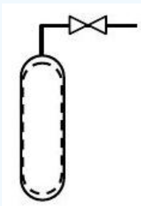


Figure 3.5.3: The system inside the tank.

$$\begin{aligned}\text{State 1:}\quad & m_1=10\text{ lbm}\\ & mf_{O_2}=0.23;\quad mf_{N_2}=0.77\end{aligned}$$

$$\text{Process } 1 \rightarrow 2: \quad \begin{array}{l} \text{Remove 5 lbm air} \\ \text{Add 3 lbm oxygen} \end{array}$$

Analysis:

Strategy $\rightarrow$ To find compositions we must know the total mass and the mass of each component in the tank at the end of the enrichment process. If we can find the total mass and the mass of each component at the final state, we can answer this question. Thus, let's assume we have three unknowns. To solve for these we will apply an accounting equation since we want to relate the contents of a system at two different times.

System $\rightarrow$ Treat the interior of the tank as a non-deforming open system.

Property $\rightarrow$ Mass and chemical species

Time period $\rightarrow$ Since only interest in initial and final states probably requires finite-time form

Writing the conservation of mass for any time during the enrichment process and then integrating over the time interval of the enrichment gives:

$$\frac{dm_{sys}}{dt} = \dot{m}_{in} - \dot{m}_{out} \quad \rightarrow \quad \int_{t_1}^{t_2} \left(\frac{dm_{sys}}{dt} \right) dt = \int_{t_1}^{t_2} (\dot{m}_{in} - \dot{m}_{out})\, dt \quad \rightarrow \quad m_{sys,2} - m_{sys,1} = m_{in} - m_{out}$$

A similar expression can also be written for each chemical species in the problem:

$$\begin{aligned} \text{O}_2: \quad & \frac{dm_{sys,\text{O}_2}}{dt} = \dot{m}_{in,\,\text{O}_2} - \dot{m}_{out,\,\text{O}_2} && \rightarrow m_{sys,\,\text{O}_2,\,2} - m_{sys,\,\text{O}_2,\,1} = m_{in,\,\text{O}_2} - m_{out,\,\text{O}_2} \\ \text{N}_2: \quad & \frac{dm_{sys,\,\text{N}_2}}{dt} = \dot{m}_{in,\,\text{N}_2} - \dot{m}_{out,\,\text{N}_2} && \rightarrow m_{sys,\,\text{N}_2,\,2} - m_{sys,\,\text{N}_2,\,1} = m_{in,\,\text{N}_2} - m_{out,\,\text{N}_2} \end{aligned}$$

This gives three equations; however, only two of them are independent, i.e. the third equation can be formed by a linear combination of the other two equations.

To obtain the third independent equation, consider the composition in State 2 . This can be written in one of two forms:

In terms of the actual mass we have $\quad m_{sys,2} = m_{sys,\text{O}_2,2} + m_{sys,\text{N}_2,2}$

In terms of the mass fractions we have $\quad 1 = mf_{sys,\text{O}_2,2} + mf_{sys,\text{N}_2,2}$

As you can easily see, these two equations are not independent since the second equation is just the first equation divided through by the mass of the system.

Solving for the final mass

$$m_{sys,\,2} = m_{sys,\,1} + m_{in} - m_{out} = (10 + 3 - 5)\text{ lbm} = 8\text{ lbm}$$

Now to find the amount of oxygen and nitrogen in the final state

m_{sys, \ \text{N}_2, \ 2} - m_{sys, \ \text{N}_2, \ 1} = \cancel{ m_{in, \ \text{N}_2} }^{=0} - m_{out, \ \text{N}_2}

m_{out, \ \text{N}_2} = m f_{out, \ \text{N}_2} \cdot m_{out} = (0.77) (5 \ \text{lbm}) = 3.85 \ \text{lbm} \quad \rightarrow \quad m_{sys, \ \text{N}_2, \ 2} = (7.70 - 3.85) \ \text{lbm} = 3.85 \ \text{lbm}

m_{sys, \ \text{N}_2, \ 1} = m f_{sys, \ \text{N}_2, \ 1} \cdot m_{sys, \ 1} = (0.77) (10 \ \text{lbm}) = 7.70 \ \text{lbm}

m_{sys, \ 2} = m_{sys, \ \text{N}_2, \ 2} + m_{sys, \ \text{O}_2, \ 2} \quad \rightarrow \quad

Now to find the compositions, use a simple table as below:

	$\frac{m_j}{\text{lbm}}$	mf_j	$\frac{M_j}{(\text{lbm/lbmol})}$	$\frac{n_j}{\text{lbmol}}$	nf_j
O_2	4.15	0.519	32.00	0.1297	0.485
N_2	3.85	0.481	28.01	0.1375	0.515
	8.00	1.000	......	0.2672	1.000

The weight percent of oxygen has increased from 23% initially to a final value of 51.9%.

Comment:

Note that the mass fractions (mf) and the mole fractions (nf) are not the same. As indicated below the mass fraction indicates the composition in terms of the amount of mass of each component while the mole fraction only considers the number of molecules of the particles. \

[

$$mf \rightarrow \frac{\text{mass of } j \text{ molecules}}{\text{total mass of molecules}}$$
$$nf \rightarrow \frac{\text{number of } j \text{ molecules}}{\text{total number of molecules}}$$

\nonumber \]

Example — Continuous flash distillation process

A liquid feed stream is fed continuously to a flash distillation chamber. As the liquid enters the chamber, its pressure is reduced and some of the entering liquid flashes to vapor. The liquid bottoms stream is drained from the floor of the chamber and the vapor distillate is removed near the top of the chamber.

The feed stream consists of water, ethanol, methanol entering at 25 lbm/h, $10 \text{lbm}/\text{h}$, and $5 \text{lbm}/\text{h}$, respectively. The bottoms stream has a mass flow rate of $24 \text{lbm}/\text{h}$ and consists of 83.3% water, 12.5% ethanol, and 4.2% methanol by weight.

Determine the composition (mass and mole fractions) and the mass flow rate of the distillate.

Solution

Known: Operating conditions for a continuous flash distillation chamber.

Find: Composition (mass and mole fractions) and mass flow rate of the distillate.

Given:

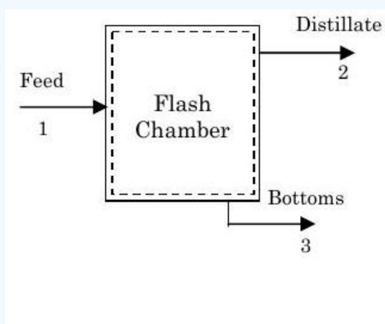


Figure $3.5.4$: The system inside the flash chamber.

Streams	
$1-\text{Feed}$	$\dot{m}_{\text{Water}} = 25 \text{lbm}/\text{h}$ $\dot{m}_{\text{Ethanol}} = 10 \text{lbm}/\text{h}$ $\dot{m}_{\text{Methanol}} = 5 \text{lbm}/\text{h}$
$2-\text{Distillate}$	
$3-\text{Bottoms}$	$mf_{\text{Water}} = 83.3\%$ $mf_{\text{Ethanol}} = 12.5\%$ $mf_{\text{Methanol}} = 4.2\%$ $\dot{m} = 24 \text{lbm}/\text{h}$

Analysis:

Strategy $\rightarrow$ Since we are interested in relating entering flows to leaving flows, this seems like a candidate for conservation of mass and species accounting.

System $\rightarrow$ Non-deforming, open system formed by the volume of the interior of the tank

Property $\rightarrow$ Mass and species since we have mixtures which change composition

Time period $\rightarrow$ Since system is fed continuously, assume steady-state

To completely specify all of the inlets and outlets of a problem like this requires knowing the composition and the total mass flow rate at each inlet/outlet OR the mass flow rate for *each* species and the total mass flow rate at each inlet/outlet. Thus the total number of unknowns is

$$\underbrace{[(\text{Number of Species})+1]}_{mf_j\text{'s and }\dot{m}} \times [\text{Number of inlets/outlets}] = [3\times 1]\times[3] = \underbrace{12 \text{ unknowns}}_{\substack{mf_{\text{Water}},\, mf_{\text{Ethanol}},\, mf_{\text{Methanol}},\, \dot{m} \\ \text{for feed, distillate, and bottoms}}}$$

$$\underbrace{[(\text{Number of Species})+1]}_{mf_j\text{'s and }\dot{m}} \times [\text{Number of inlets/outlets}] = [3\times 1]\times[3] = \underbrace{12 \text{ unknowns}}_{\substack{\dot{m}_{\text{Water}},\, \dot{m}_{\text{Ethanol}},\, \dot{m}_{\text{Methanol}} \\ \text{for feed, distillate, and bottoms}}}$$

To help visualize what we know and don't know, set up a table:

	$\dot{m}_j$ (lbm/h)	Composition		
		Water	Ethanol	Methanol
1 – Feed	Eq. 1	Eq. 2	Eq. 3	Eq. 4
2 – Distillate	Eq. 5	Eq. 6	Eq. 7	Eq. 8
3 – Bottoms	24.0	83.3	12.5	4.2

By observing the table we see we already have four pieces of information. Given the mass flow rates at the feed stream we can find the composition at the feed stream using a feed stream composition relation (Eq. 1)

$$\text{Eq. 1} \quad \rightarrow \quad \dot{m}_1 = \dot{m}_{\text{Water, 1}} + \dot{m}_{\text{Ethanol, 1}} + \dot{m}_{\text{Methanol, 2}} = (25+10+5)\ \frac{\text{lbm}}{\text{h}} = 40\ \frac{\text{lbm}}{\text{h}}$$

Then using the **definition of mass fractions** (Eq. 2, 3, and 4) for the feed stream:

$$\text{Eq. 2, 3, 4} \quad \rightarrow \quad mf_{\text{Water, 1}} = \frac{\dot{m}_{\text{water, 1}}}{\dot{m}_1} = \frac{25}{40} = 62.5\%\ ; \quad mf_{\text{Ethanol, 1}} = \frac{10}{40} = 25.0\%\ ;$$
$$mf_{\text{Methanol, 1}} = \frac{5}{40} = 12.5\%$$

Now there are only four unknowns remaining. The total mass flow rate can be solved for writing **conservation of mass for the system** (Eq. 5)

$$\text{Eq. 5} \rightarrow \underbrace{\cancel{\frac{dm_{sys}}{dt}}^{=0}}_{\substack{\text{steady-state}\\ \text{conditions}}} = \dot{m}_1 - \dot{m}_2 - \dot{m}_3 \quad \rightarrow \quad \dot{m}_2 = \dot{m}_1 - \dot{m}_3 = (40-24)\ \frac{\text{lbm}}{\text{h}} = 16\ \frac{\text{lbm}}{\text{h}}$$

The remaining unknowns can be determined by applying the species accounting equation for water, ethanol and methanol. The resulting equations look just like Eq. 5 except they are written for each species.

$$\dot{m}_{\text{water, 2}} = \dot{m}_{\text{water, 1}} - \dot{m}_{\text{water, 3}} = \dot{m}_{\text{water, 1}} - mf_{\text{water, 3}}\dot{m}_3 = [25.0-(0.833)(24.0)]\ \frac{\text{lbm}}{\text{h}} = 5.01\ \frac{\text{lbm}}{\text{h}}$$
$$mf_{\text{water, 2}} = \frac{\dot{m}_{\text{water, 2}}}{\dot{m}_2} = \frac{5.01}{16.0} = 31.3\%$$

$$\dot{m}_{\text{ethanol, 2}} = \dot{m}_{\text{ethanol, 1}} - \dot{m}_{\text{ethanol, 3}} = \dot{m}_{\text{ethanol, 1}} - mf_{\text{ethanol, 3}}\dot{m}_3 = [10.0-(0.125)(24.0)]\ \frac{\text{lbm}}{\text{h}} = 7.00\ \frac{\text{lbm}}{\text{h}}$$
$$mf_{\text{ethanol, 2}} = \frac{\dot{m}_{\text{ethanol, 2}}}{\dot{m}_2} = \frac{7.00}{16.0} = 43.7\%$$

$$\dot{m}_{\text{methanol, 2}} = \dot{m}_{\text{methanol, 1}} - \dot{m}_{\text{methanol, 3}} = \dot{m}_{\text{methanol, 1}} - mf_{\text{methanol, 3}}\dot{m}_3 = [5.0 - (0.042)(24.0)]\ \frac{\text{lbm}}{\text{h}} = 3.99\ \frac{\text{lbm}}{\text{h}}$$

$$mf_{\text{methanol, 2}} = \frac{\dot{m}_{\text{methanol, 2}}}{\dot{m}_2} = \frac{3.99}{16.0} = 25.0\%$$

The molar composition of the distillate can be computed after finding the molar flow rates:

	Mass flow rate	Molar mass	Molar flow rate	Mole fraction
Water	$5.01\ \text{lbm/h}$	$18.02\ \text{lbm/lbmol}$	$0.2780\ \text{lbmol/h}$	0.409
Ethanol	$7.00\ \text{lbm/h}$	$46.07\ \text{lbm/lbmol}$	$0.1519\ \text{lbmol/h}$	0.224
Methanol	$3.99\ \text{lbm/h}$	$32.05\ \text{lbm/lbmol}$	$0.2494\ \text{lbmol/h}$	0.367
	$16.00\ \text{lbm/h}$	$\cdots$	$0.6793\ \text{lbmol/h}$	1.000

Systems With Chemical Reactions

When chemical reactions occur in a system, the consumption and generation terms are no longer each zero. To evaluate these expressions, you must rely on the appropriate balanced chemical equations that apply to the reactions involved. Because of this, the molar form of the species accounting equation is the one commonly used when chemical reactions occur.

Again, it is important to recognize exactly how many *independent* equations can be obtained from applying conservation of mass and species accounting.

For a reactive system, the number of independent equations that can be obtained from conservation of mass and species accounting equals the number of independent species plus the number of independent chemical reactions. (The additional equations are supplied by the balanced chemical equations for each independent reaction.)

Applications for systems with chemical reactions are beyond the scope of this text. However, a brief example will illustrate how generation and consumption terms are obtained from the chemical equations. These topics are handled in greater detail in chemical engineering courses. The final example in this section demonstrates how a problem with chemical reactions could be approached.

✔ Example —Making CO_2

A tank contains $2\ \text{kmol}$ of carbon monoxide (CO) and $2\ \text{kmol}$ of oxygen (O_2). If 50% of the original amount of CO is reacted with the O_2 to form CO_2, determine the amount of CO, CO_2, and O_2 in the final mixture.

Solution

Known: A mixture of CO and O_2 is reacted in a closed system.

Find: Amount of CO, O_2, and CO_2 in mixture if 50% of original CO is reacted.

Given:

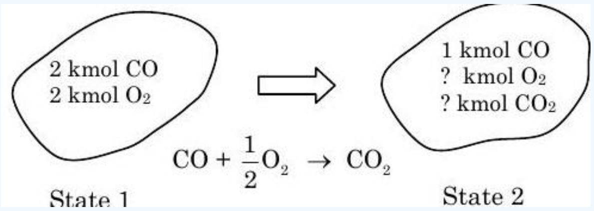


Figure $3.5.5$: States 1 and 2 of the system, and the chemical reaction responsible for the change in states.

Analysis:

Strategy $\rightarrow$ Since this involves mixtures of substances and reactions, use species accounting.

System $\rightarrow$ Consider a closed system that includes all mass initially in the system.

Property to count $\rightarrow$ Chemical species

Time period $\rightarrow$ Since interested in beginning and ending state, use finite-time form.

In general for each species j

$$\frac{dn_{sys,j}}{dt} = \underbrace{\sum_{in} \dot{n}_{j,i} - \cancelto{=0}{\sum_{out} n_{j,e}}}_{\text{closed system}} + \dot{n}_{j,\, gen} - \dot{n}_{j,\, cons} \quad \rightarrow \quad \underbrace{n_{j,2} - n_{j,1}}_{\text{dropped the system subscript}} = n_{j,\, gen} - n_{j,\, cons}$$

Now for each species

$$\begin{array}{lccc} \text{CO:} & \cancelto{=0}{n_{\text{CO},\, 2}} + \cancelto{=0}{n_{\text{CO},\, 1}} = \cancelto{=0}{n_{\text{CO},\, gen}} - n_{\text{CO},\, cons} & \rightarrow & n_{\text{CO},\, cons} = 1 \text{ kmol} \\ \text{O}_2: & n_{\text{O}_2,\, 2} - \cancelto{=0}{n_{\text{O}_2,\, 1}} = \cancelto{=0}{n_{\text{O}_2,\, gen}} - n_{\text{O}_2,\, cons} & \rightarrow & n_{\text{O}_2,\, 2} = -n_{\text{O}_2,\, cons} \\ \text{CO}_2: & n_{\text{CO}_2,\, 2} - \cancelto{=0}{n_{\text{CO}_2,\, 1}} = n_{\text{CO}_2,\, gen} - \cancelto{=0}{n_{\text{CO}_2,\, cons}} & \rightarrow & n_{\text{CO}_2,\, 2} = n_{\text{CO}_2,\, gen} \end{array}$$

This gives us four unknowns and only two equations. To get the remaining equations, look at the chemical reaction equation. In terms of consumption and generation, the chemical reaction is as follows:

$$n_{\text{CO},\, cons}\text{CO} + n_{\text{O}_2,\, cons}\text{O}_2 \rightarrow n_{\text{CO}_2,\, gen}\,\text{CO}_2$$

$$\text{CO} + \frac{n_{\text{O}_2,\, cons}}{n_{\text{CO},\, cons}}\text{O}_2 \rightarrow \frac{n_{\text{CO}_2,\, gen}}{n_{\text{CO},\, cons}}\text{CO}_2$$

If we compare this equation with the balanced chemical equation for the reaction of CO with O_2 to form CO_2, we can determine the two remaining equations between the unknowns:

$$\begin{array}{c} \text{CO} + \dfrac{n_{\text{O}_2,cons}}{n_{\text{CO},\, cons}}\text{O}_2 \rightarrow \dfrac{n_{\text{CO}_2,\, gen}}{n_{\text{CO},\, cons}}\text{CO}_2 \\ \text{CO} + \frac{1}{2}\text{O}_2 \rightarrow \text{CO}_2 \end{array} \quad \Rightarrow \quad \frac{n_{\text{O}_2,\, cons}}{n_{\text{CO},\, cons}} = \frac{1}{2} \quad \& \quad \frac{n_{\text{CO}_2,\, gen}}{n_{\text{CO},\, cons}} = 1$$

Thus the final compositions are as follows:

$$n_{\text{CO},\, 2} = 1 \text{ kmol}$$

$$n_{\text{O}_2,\, 2} = n_{\text{O}_2,\, gen} = \frac{n_{\text{O}_2,\, gen}}{n_{\text{CO},\, cons}} n_{\text{CO},\, cons} = \left(\frac{1}{2}\right)(1 \text{ kmol}) = \frac{1}{2} \text{ kmol}$$

$$n_{\text{CO}_2,\, 2} = n_{\text{CO}_2,\, gen} = \frac{n_{\text{CO}_2,\, gen}}{n_{\text{CO},\, cons}} n_{\text{CO},cons} = (1)(1 \text{ kmol}) = 1 \text{ kmol}$$

Comments:

(1) The final mixture has 3.5 kmol compared to 4.0 kmol in the initial mixture. Typically, moles are *not* conserved.

(2) An alternate approach is to write species accounting equations for the atomic species, C and O. Note that atomic species, like total mass, are conserved.

3.6: Density, Specific Volume, Specific Weight, and Specific Gravity

As we have demonstrated earlier, the mass of a system is an extensive property and as such it depends upon the extent of the system. Recall that for a specified system, the mass of the system can be calculated from this general relationship:

$$m_{sys} = \int_{V_{sys}} \rho \, dV$$

It would be impossible to tabulate a handbook of masses for all possible combinations of substances and systems, but it is feasible to tabulate or provide models for the density. Once this information is known, this equation can be used to calculate the mass of the system.

Density and Specific Volume

The density of a substance is usually found tabulated in handbooks as a function of the substance and its state (i.e. its temperature and pressure). The **density** ρ of a substance is the mass per unit volume and typically has units of $\mathrm{kg/m^3}$ or $\mathrm{lbm/ft^3}$.

Mathematically, the density at a point in space can be thought of as the limiting value of the mass per unit volume within a small cube of volume as the volume shrinks towards zero:

$$\rho = \lim_{\Delta V \to 0} \frac{m}{\Delta V}$$

Imagine, if you will, that as the cube of volume becomes smaller and smaller the density approaches some limiting value (See Figure $3.6.1$). At some point the volume of the cube becomes small enough that the number of molecules within the volume begins to fluctuate with time. Once this point is reached, the idea of a statistically significant average value (or limit) makes no sense.

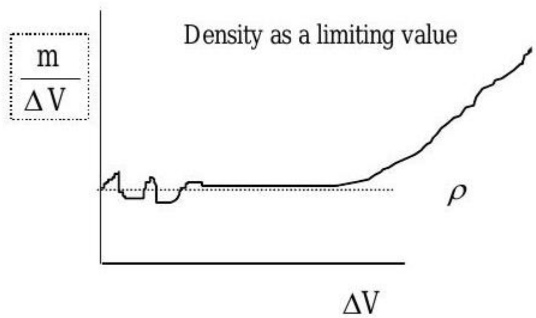


Figure $3.6.1$: Density as a limiting value.

In this course we are restricting ourselves to conditions under which this type of statistically significant average is possible. This is called the **continuum** assumption. Sometimes this viewpoint is also called a **macroscopic** viewpoint because it only addresses macroscopic variables like pressure, temperature, density, etc. One of the distinct advantages of a macroscopic or continuum viewpoint is that we do not have to explicitly consider the microscopic behavior of atoms and molecules. In this course, we will only be considering continuums and a macroscopic viewpoint. Sometimes we may find it useful to take a **microscopic** viewpoint to help us understand some phenomena. Although in most industrial situations, the continuum assumption is appropriate, there are many important situations where the continuum assumption breaks down. These are typically conditions with very low densities, e.g. vacuum conditions, conditions in space near the edge of the earth's atmosphere.

For some substances it is not the density but the specific volume that is tabulated. The **specific volume** v is the volume per unit mass. It is typically given the lower-case "v" as its symbol and can be related to the density by the equation:

$$v = \frac{1}{\rho}$$

The dimensions for specific volume are $[\mathrm{L}]^3/[\mathrm{M}]$, and typical units for specific volume are $\mathrm{m^3/kg}$ or $\mathrm{ft^3/lbm}$.

As one might expect, there is a **molar density** defined as the moles per unit volume and given the symbol $\bar{\rho}$, where *the overbar indicates that this is a mole-based quantity*. A **molar specific volume** is also defined similarly as the volume per unit mole and is given the symbol $\bar{v}$.

What would be the units for these two molar quantities in SI and in the American Engineering System?

Specific Weight

The specific weight of a substance is the weight per unit volume. Specific weight is usually given the symbol γ ("gamma") and can be calculated as the product of density times the local acceleration of gravity:

$$\gamma=\rho g$$

Any values of specific weight you find listed in a table have, of necessity, assumed a value for the local acceleration of gravity. Unless noted otherwise, tabulated specific weight values are based on the standard value for the acceleration of gravity at sea level, $9.80665 \mathrm{~m} / \mathrm{s}$or $32.174 \mathrm{ft} / \mathrm{s}$ Typical units for specific weight are $\mathrm{N} / \mathrm{m}^3$ or $\mathrm{lbf} / \mathrm{ft}^3$.

Specific Gravity

In many applications, it is important to know the density or specific weight of a substance relative to that of a reference substance.. The **specific gravity** $(\boldsymbol{SG})$ of a substance is the ratio of the density of the substance at the specified conditions to the density of a reference substance:

$$SG=\frac{\rho}{\rho_{\text{ref}}}$$

For liquids and solids, the reference substance is water at $4^{\circ} \mathrm{C}$ where the density of water is $1000 \mathrm{~kg} / \mathrm{m}^3$ or $1.940 \mathrm{~slug} / \mathrm{ft}^3$. For gases, the reference substance is air at a specified temperature and pressure, say $25^{\circ} \mathrm{C}$ and 1 atm. Under these conditions, air can be treated as an ideal gas.

3.7: Constitutive Equations

Based upon *Introduction of Chemical Engineering Analysis* by T.W.F. Russell and M. M. Denn, John Wiley & Sons, Inc, New York, 1972, p.41- 43.

As we have worked out problems involving the conservation of mass and the accounting of chemical species, we have frequently found problems where the direct application of the conservation and accounting equation was insufficient to develop a complete model, and hence a solution to our problem or required much more time and effort than we could spend. When this occurs we will often require additional relationships between important variables that do not appear to fit in the conservation and accounting framework. These equations are called constitutive relations or equations.

A **constitutive relation** is a mathematical relationship between variables that describes some physical phenomena. There are several common sources for these relationships. They may be based solely on experiments, such as Ohm's Law that relates voltage drop and electric current through a resistor. They may be based upon purely theoretical grounds, such as the predictions of statistical mechanics for the properties of gases. They may be based on a combination of experiments and theory, such the mass flow rate through an orifice as a function of the height of fluid above the orifice. All of these are constitutive relations.

Constitutive relations are by their nature *specific*. They cannot be applied in general and are only valid under a restricted set of conditions. This is in contrast to the fundamental conservation laws and the entropy accounting equation that are always valid under any circumstances. A prime example of this is the ideal gas "law" we will discuss next. Although commonly presented as a "law," it is really a model for the behavior of substances under restricted conditions of pressure and temperature and is not applicable in every situation.

Constitutive relations typically do one of two things: describe microscopic phenomena or represent empirical relationships between variables for a specific phenomena. Fourier's "law" of heat conduction, Ohm's "law", and Fick's "law" of diffusion all describe the flow of an extensive property — thermal energy, charge, and mass, respectively — in terms of a driving force — temperature, voltage, and concentration, respectively. These are all microscopic phenomena. Another example is the ideal gas "law" that relates pressure, density, and temperature of a gas. All three of these *macroscopic* variables are related to the microscopic behavior of the gas. In the other category are things like the Moody diagram that relates the pressure drop in a pipe to several variables describing the flow situation. The orifice equation $\dot{V} = C_d A_o \sqrt{2gh}$ is an example of how empirical data and theory can be used to develop a constitutive relation.

Constitutive relations play an extremely important role in modeling physical systems. You should be on the lookout for constitutive relations and how they are used in the modeling process. Although we will use several constitutive relations during the quarter, we may not provide much background on how they were developed. Later you will learn more about their specific limitations and their development. *Remember that constitutive relations by their very nature have limited ranges of applicability*. One of the major tasks of the engineer is to learn what these ranges are and how to determine the appropriate constitutive relation for a given situation.

3.8: Ideal Gas Model - A Useful Constitutive Relation

In a chapter on conservation of mass, it is frequently necessary to predict the density of a substance. For most liquids and solids, the values for density (or specific weight or specific gravity) will be found in tables in handbooks. However, for gases there is a very useful model that can accurately predict the pressure - temperature - density relationship for conditions of "low pressure" and "high temperature". The exact limitations of this model will be discussed next quarter.

The **ideal gas model** (not law) is a constitutive relationship that relates pressure, temperature, and density for a gaseous substance:

$$\begin{aligned} p &= \bar{\rho} R_u T, \\ \text{where } \bar{\rho} &= \text{molar density } \left(\mathrm{kmol/m^3}\right) \\ p &= \text{absolute pressure (kPa)} \\ T &= \text{absolute (thermodynamic) temperature (K)} \\ R_u &= \text{universal gas constant} = 8.314\ \mathrm{kJ/(kmol \cdot K)} \end{aligned} \tag{3.8.1}$$

There are many different forms of this equation. An alternative form that is also very useful is the following:

$$\begin{aligned} p &= \rho R T, \\ \text{where } \rho &= \text{density } \left(\mathrm{kg/m^3}\right) \\ p &= \text{absolute pressure (kPa)} \\ T &= \text{absolute (thermodynamic) temperature (K)} \\ R &= \frac{R_u}{M} = \text{specific gas constant } (\mathrm{kJ/(kg \cdot K)}) \end{aligned} \tag{3.8.2}$$

Please *beware*: there is much confusion between the universal gas constant and the specific gas constant. You must use the correct one in each calculation.

Many other useful forms of the ideal gas equation can be developed:

$$\begin{aligned} \text{Molar forms:} \quad pV &= nR_uT \\ p\bar{v} &= R_uT \quad \text{where } \bar{v} = V/n \\ p &= \bar{\rho} R_u T \end{aligned}$$

$$\begin{aligned} \text{Mass forms:} \quad pV &= mRT \\ pv &= RT \quad \text{where } v = V/m \end{aligned}$$

You are encouraged to learn only two or three and develop skill in converting to the other forms. The tables below give more information about the various terms in the equations and molar mass information for several substances.

Molar Mass (Molecular Weight) for some common substances

Substance	Chemical Formula	Molar Mass (g/gmol;kg/kmol;lbm/lbmol)
Air	$\cdots$	28.97
Ammonia	NH_3	17.04
Carbon dioxide	CO_2	44.01
Refrigerant 134a	$C_2F_4H_2$	102.03
Helium	He	4.003
Hydrogen	H_2	2.016
Methane	CH_4	16.04
Nitrogen	N_2	28.01
Oxygen	O_2	32.00
Water	H_2O	18.02

Molar Basis	**Mass Basis**

<table>
<tr><th>Molar Basis</th><th>Mass Basis</th></tr>
<tr><td>$$PV = nR_uT$$ $$P\bar{v} = R_uT \quad \text{and} \quad P = \bar{\rho}R_uT$$</td><td>$$PV = mRT$$ $$Pv = RT \quad \text{and } P = \rho RT$$</td></tr>
<tr><td>where $$\begin{aligned} P &= \text{absolute pressure of gas } \left[\text{kPa or lbf/ft}^2\right] \\ V &= \text{volume of gas } \left[\text{m}^3 \text{ or ft}^3\right] \\ n &= \text{number of moles of gas } [\text{kmol or lbmol}] \\ R_u &= \text{universal gas constant (the same for every gas)} \\ & \qquad [\text{kJ}/(\text{kmol}\cdot\text{K}) \text{ or } (\text{ft}\cdot\text{lbf})/(\text{lbmol}\cdot^\circ\text{R})] \\ T &= \text{absolute temperature of gas } [\text{K or }^\circ\text{R}] \\ \bar{\rho} &= \text{molar density} = 1/\bar{v} \left[\text{kmol/m}^3 \text{ or lbmol/ft}^3\right] \\ \bar{v} &= \text{molar specific volume } \left[\text{m}^3/\text{kmol or ft}^3/\text{lbmol}\right] \end{aligned}$$</td><td>where $$\begin{aligned} P &= \text{absolute pressure of gas } \left[\text{kPa or lbf/ft}^2\right] \\ V &= \text{volume of gas } \left[\text{m}^3 \text{ or ft}^3\right] \\ m &= \text{mass of gas } [\text{kg or lbm}] \\ R &= \text{specific gas constant (different for each gas)} \\ & \qquad [\text{kg}/(\text{kg}\cdot\text{K}) \text{ or } (\text{ft}\cdot\text{lbf})/(\text{lbm}\cdot^\circ\text{R})] \\ T &= \text{absolute temperature } [\text{K or }^\circ\text{R}] \\ \rho &= \text{density} = 1/v \left[\text{kg/m}^3 \text{ or lbm/ft}^3\right] \\ v &= \text{specific volume } \left[\text{m}^3/\text{kg or ft}^3/\text{lbm}\right] \end{aligned}$$</td></tr>
<tr><td>and $$\begin{aligned} R_u &= 8.314 \frac{\text{kJ}}{\text{kmol}\cdot\text{K}} = 8.314 \frac{\text{J}}{\text{mol}\cdot\text{K}} \\ &= 1545 \frac{\text{ft}\cdot\text{lbf}}{\text{lbmol}\cdot^\circ\text{R}} \end{aligned}$$</td><td>and $$R = \frac{R_u}{M}$$ where $$M = \text{molecular weight (molar mass) of a specific gas}$$</td></tr>
</table>

3.9: Problems

Problem 3.9.1

The tank shown in the figure is full of a liquid and initially contains $1000\ \mathrm{lbm}$ of liquid. (The density of the liquid is $60\ \mathrm{lbm/ft^3}$). The sides and base are rigid but the top wall of the tank can float up and down without leakage. The tank has one inlet and one outlet as shown in the figure. The following information is known about the mass flow rate at the inlet and the outlet:

$$\dot{m}_1 = \begin{cases} \left(3\ \frac{\mathrm{lbm}}{\mathrm{min}^2}\right) t \text{ when } 0 \le t \le 6\ \mathrm{min} \\ 18\ \mathrm{lbm/min} \text{ when } t > 6\ \mathrm{min} \end{cases}$$

$$\dot{m}_2 = (18\ \mathrm{lbm/min})\left(1 - e^{-t/(3\ \mathrm{min})}\right)$$

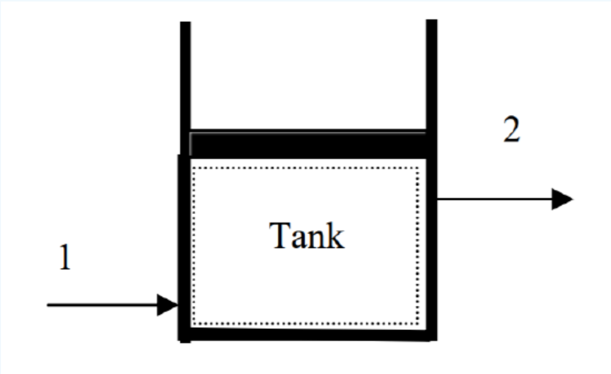


Figure 3.9.1: System consisting of the contents of a tank with a floating lid, one inlet and one outlet.

(a) Using an open system as shown in the figure that corresponds to the volume of liquid inside the tank, answer the following questions:

- Calculate and graph the *rate of change* of the mass inside the tank at one-minute intervals for $0 \le t \le 20\ \mathrm{min}$.
- Calculate and graph the *amount* of mass inside the tank at one-minute intervals for $0 \le t \le 20\ \mathrm{min}$.
- Calculate the *net* change in the volume of the system for the 20-minute time interval.
- Is the system at steady-state during this 20-minute period? Would it ever reach a steady-state condition at any time?

(b) How would your answers to part (a) change if you used an open system that included the walls of the tank inside your system? A qualitative discussion is acceptable.

Problem 3.9.2

A system of four tanks is connected as shown in the figure below.

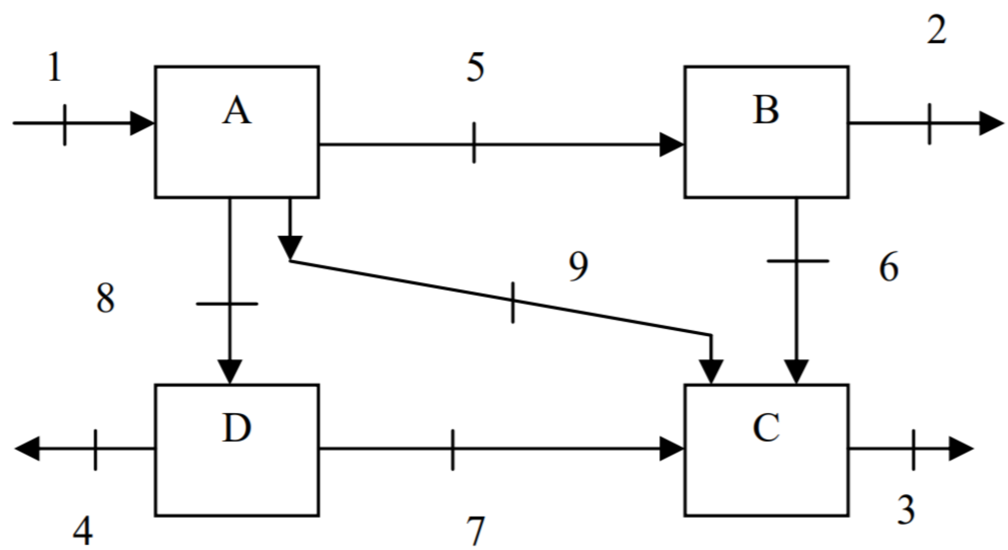


Figure $3.9.2$: System consisting of 4 tanks, interconnected by 9 pipes.

The flow net formed by the tanks and their piping operates at steady-state conditions. The known mass flow rates are indicated in the table below:

Pipe Number	$\mathrm{kg/s}$
1	50
2	30
3	
4	60
5	
6	
7	40
8	
9	80

(a) Determine the unknown mass flow rates, in $\mathrm{kg/s}$. Clearly show the system(s) you use to develop the necessary equations.

(b) The *degree of freedom* of a problem is the number of variables that must be specified before the remaining variables can be calculated. Assuming that mass flow rates are the only variables in this problem and that conservation of mass is the only pertinent physical law required for solving it, how many degrees of freedom does this problem have? Or, looking at it another way, how many of the nine mass flow rates must be specified before the remaining mass flow rates can be determined uniquely? The specified variables are sometimes referred to as the **design variables** and the unknown variables to be determined are called the **state variables**.

? Problem $3.9.3$

The air scoop on the hood of a car is sketched in the figure, along with the velocity profile of the air immediately upstream of the scoop inlet. Air flows directly into the scoop at 1 and then exits the scoop at 2 at an angle $\theta = 30^\circ$. The scoop has a rectangular cross section with a uniform depth $d = 25\ \mathrm{cm}$ into the screen. At the inlet $H = 5\ \mathrm{cm}$ and at the outlet $2H = 10\ \mathrm{cm}$. The velocity profile shows how the velocity u varies with vertical position y. For purposes of analysis, $U_0 = 27\ \mathrm{m/s}$ and $\delta = 2.5\ \mathrm{cm}$. Assume air behaves as an incompressible substance. Assume steady-state operation.

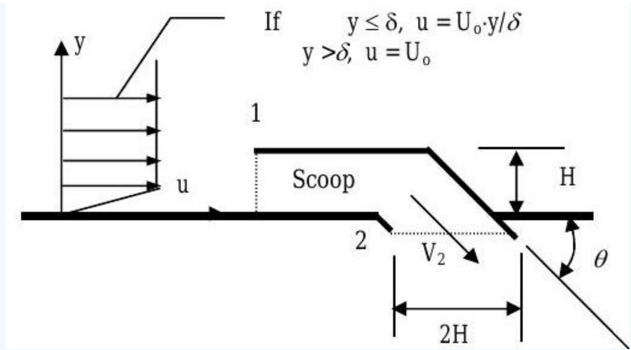


Figure 3.9.3: Profile view of the air scoop with measurements.

(a) Calculate the volumetric flow rate of air into the scoop, in $\mathrm{m^3/s}$.

(b) Determine the volumetric flow rate of air leaving the scoop at 2, in $\mathrm{m^3/s}$.

(c) Determine the magnitude of the average velocity V_2 at the scoop outlet, in $\mathrm{m/s}$.

Problem 3.9.4

The cross-sectional area of a rectangular duct is divided into 16 equal rectangular areas, as shown in the figure. The axial fluid velocity is measured at the center of each area and is reported in the figure in feet per second. Assume that atmospheric air flows in the duct.

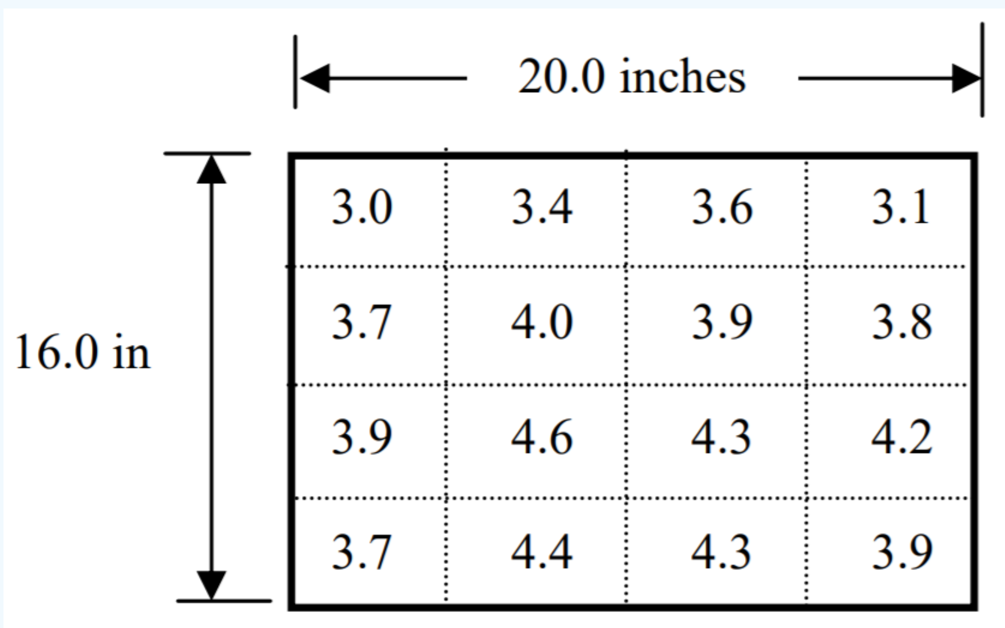


Figure 3.9.4: Divisions of the cross-section of a rectangular air duct.

(a) Estimate the volume flow rate in $\mathrm{ft^3/s}$ (cfs) and $\mathrm{ft^3/min(cfm)}$ and the average axial velocity in $\mathrm{ft/s}$.

(b) Assume that the rectangular duct is the inlet section of a transition fitting that connects rectangular to circular sheet metal ducting. If the outlet circular duct has a diameter of 12 inches, what is the volume flow rate and average velocity in the circular duct?

? Problem 3.9.5

(Adapted from White, *Fluid Mechanics*, 4th ed., WCB/McGraw-Hill.)

When a fluid (liquid or gas) flows through a surface, the velocity may vary with position. The equation or graph that describes this variation is known as the *velocity profile*. A thin layer of liquid, draining from an inclined plane as shown in the figure, will have a laminar velocity profile described by the following equation:

$$u = U_0\left[2\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right] = \left(2\frac{U_0}{h}\right)y - \left(\frac{U_0}{h^2}\right)y^2,$$

where U_0 is the surface velocity, i.e. velocity of the water at the surface of the layer, u is the velocity of the water at any y-position in the layer, and h is the thickness of the layer. A graph of this velocity profile is shown in the figure.

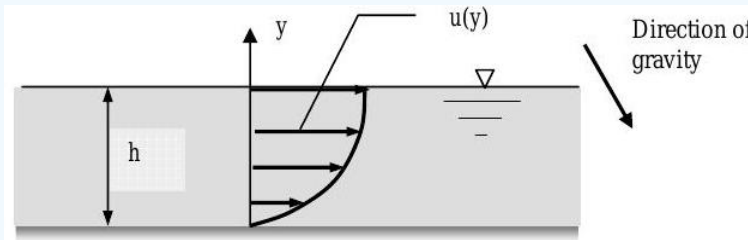


Figure 3.9.5: Velocity profile of a layer of liquid flowing down an inclined plane.

(a) If the plane has width b into the screen, develop an expression for the volumetric flow rate in the film in terms of h, U_0, and b. [Hint: Use a differential area element $dA = b\,dy$ and integrate from $y = 0$ to $y = h$.

(b) Calculate the mass flow rate of liquid down the plane, in $\mathrm{kg/s}$, if the liquid is SAE 10W motor oil with a specific gravity of 0.87 and $b = 0.3\ \mathrm{m}$, $h = 5\ \mathrm{mm}$, and $U_0 = 0.2\ \mathrm{m/s}$.

? Problem 3.9.6

Air flows over a flat plate and the flat plate produces a region of retarded flow near the plate surface. In this region, known as the boundary layer, the fluid velocity $u = 0$ at the plate surface $(y = 0)$ and u approaches U_o, the free-stream velocity, far away from the wall $(y >> \delta)$. Measurements reveal that $u/U_o = 0.99$ when $y = 23\ \mathrm{mm}$. The plate has a width (into the paper) $w = 10\ \mathrm{m}$.

Determine (a) the value of δ using the given velocity profile, (b) the volumetric flow rate of air represented by the large black arrow, i.e. flow across the dashed surface, and (c) the average velocity normal to the dashed surface.

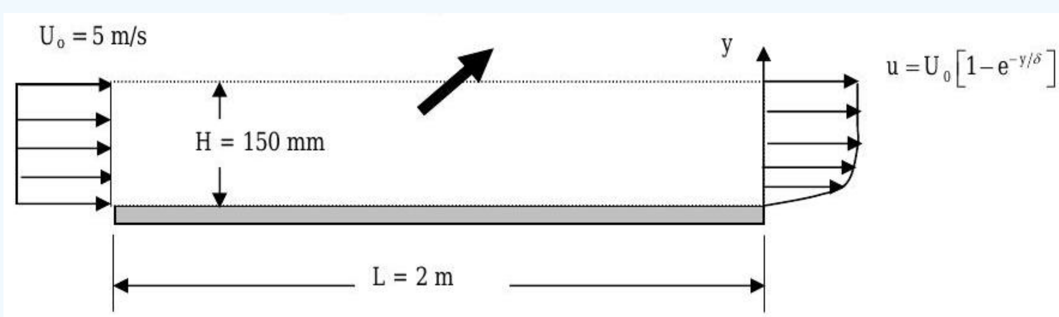


Figure 3.9.6: Air velocities at the edges of a flat plate.

? Problem 3.9.7

An air supply system has an exhaust tee as shown in the figure. The air enters at Inlet 1 with a density of $0.075\ \mathrm{lbm/ft^3}$ and a velocity of $15\ \mathrm{ft/s}$. The mass flow rates at Outlet 2 and Outlet 3 are equal; however, the outlet velocities are different. The cross-sectional area (shown by a dashed line) is $2.0\ \mathrm{ft^2}$ for Inlet 1 , $2.3\ \mathrm{ft^2}$ for Outlet 2, and $2.3\ \mathrm{ft^2}$ for Outlet 3.

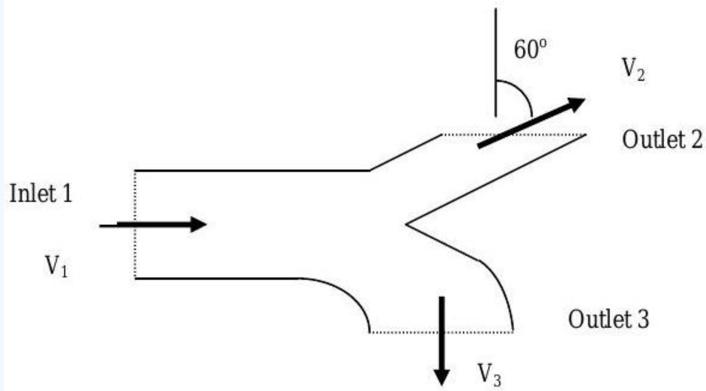


Figure $3.9.7$: Profile view of exhaust tee, showing the directions in which air enters or exits through each opening.

For purposes of analysis, you may assume that the air is incompressible and the tee operates at steady-state conditions. All flows may be assumed as one-dimensional. Except for Outlet 2, the average velocity crossing the boundary is perpendicular to the flow boundary. At Outlet 2, the flow leaves with an angle of 60° as shown on the figure.

Determine:

(a) the mass flow rate at each inlet and outlet, in $\mathrm{lbm/min}$.

(b) the average velocity V_2 at Outlet 2 and V_3 at Outlet 3, in $\mathrm{ft/s}$.

Problem $3.9.8$

(Adapted from Potter & Wiggert, *Mechanics of Fluids*, 2nd ed., Prentice Hall)

Water flows steadily through a radial outflow turbine as shown in the figure. The water enters the turbine flowing parallel to the axis-of-rotation of the turbine with a velocity of $15\ \mathrm{m/s}$ and leaves the turbine flowing radially outward with a velocity V_2 at an angle θ as shown in the figure. The exit angle depends on the operating conditions of the turbine.

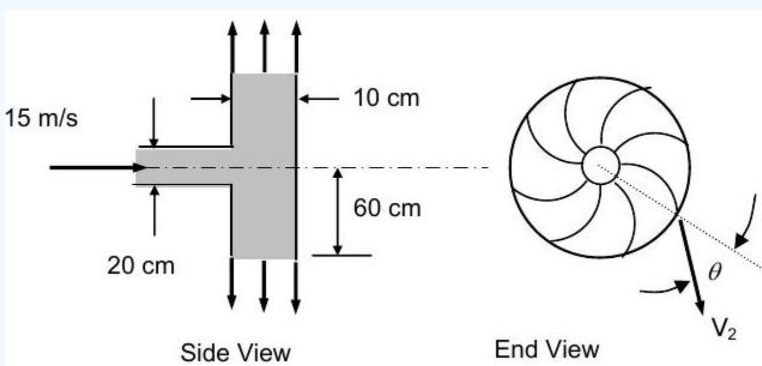


Figure $3.9.8$: Side and end views of the radial outflow turbine.

Assume that water is incompressible with a density of $1000\ \mathrm{kg/m3}$ and recognize (learn) that the phrase "flows steadily" implies that the turbine operates at steady-state conditions.

(a) Determine the mass flow rate at the turbine inlet, in $\mathrm{kg/s}$.

(b) Determine the volumetric flow rate at the turbine inlet, in $\mathrm{m^3/s}$.

(c) Determine the average velocity V_2 of the water leaving the turbine if $\theta = 0^\circ$, 30°, and 60°.

Problem 3.9.9

A hydraulic pump operates the elevators in Moench Hall by forcing fluid into a vertical cylinder with a piston that supports the elevator. As fluid is forced into the cylinder, the piston moves and pushes the elevator between floors. The hydraulic fluid is supplied to the pump from a small reservoir as shown in the figure. The elevator travels up and down at the rate of $0.6\ \mathrm{m/s}$ and travels a total distance of 6.0 meters. The diameter of the lower face of the cylindrical piston is $10\ \mathrm{cm}$. The oil reservoir is a steel tank vented to the atmosphere, is $0.6\ \mathrm{m}$ tall and has a $1.0\ \mathrm{m} \times 1.0\ \mathrm{m}$ footprint (length $\times$ width). With the elevator at its lowest elevation, the oil reservoir is $2/3$ full. For analysis purposes, you may assume that the hydraulic oil is incompressible.

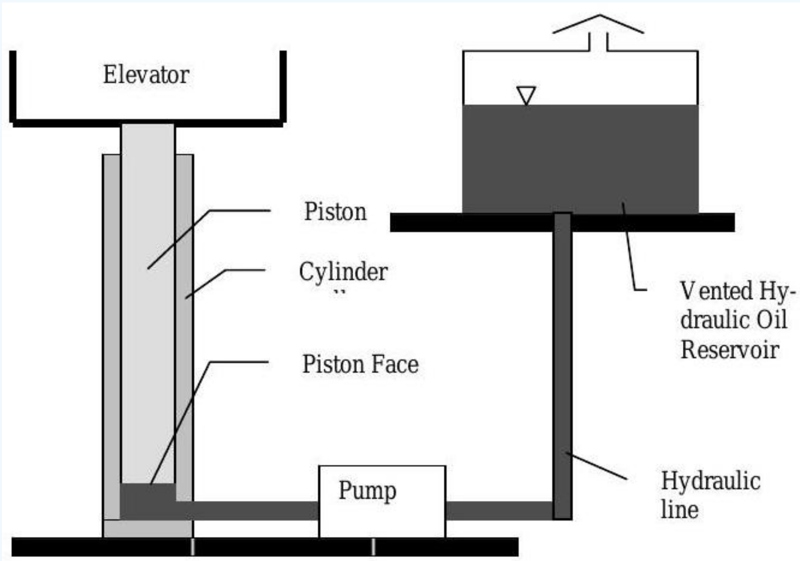


Figure $3.9.9$: Setup of the hydraulic oil reservoir used to raise an elevator.

(a) Starting with the conservation of mass equation and an open system of your choice, determine

- the required volumetric flow rate out of the pump in $\mathrm{m}^{3/s}$ and gallons per minute (gpm) to raise the elevator at the indicated velocity.
- the velocity of the oil at the pump outlet if the hydraulic lines have a diameter of $0.020\ \mathrm{m}$.

[*Hint:* Try a one-inlet, deforming open system that contains all of the oil downstream of the pump. The system inlet corresponds with the pump outlet and the moving boundary is the lower face of the piston.]

(b) Starting with the conservation of mass equation and an *open system* of your choice, determine the rate of change, in $\mathrm{m/s}$, of the oil level in the reservoir when the elevator is going up. [Hint: Try a one-inlet open system with changing volume.]

(c) Using a closed system consisting of all the oil in the cylinder, lines, pump, and reservoir, determine the rate of change, in $\mathrm{cm/s}$, of the oil level in the reservoir when the elevator is going up.

(d) Using any approach that *clearly* demonstrates its connection to the conservation of mass principle, determine the oil level in the reservoir when the elevator is at its highest elevation.

Problem 3.9.10

Gasoline (S.G. $= 0.7$) drains by gravity from the upper tank to the lower tank through a connecting pipe. The liquid depth in the upper tank is decreasing at the rate of $0.333\ \mathrm{m/min}$ The area of the base of the upper tank is Aupper $= 9\ \mathrm{m}^2$ and the area of the base of the lower tank is $A_{\text{lower}} = 50\ \mathrm{m}^2$. The walls of both tanks are vertical as shown in the figure. The pipe connecting the tanks is 50 meters long and has a diameter of $0.10\ \mathrm{m}$.

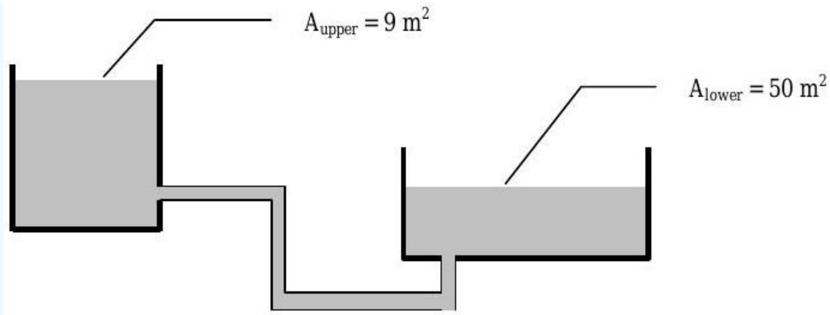


Figure 3.9.10: Setup of two pipe-connected tanks, one situated higher than the other.

(a) Starting with the rate form of the conservation of mass and an appropriate *open* system, determine the volumetric flow rate and the speed of the fluid in the connecting pipe connecting, in $\mathrm{m^3/min}$ and $\mathrm{m/s}$, respectively.

(b) Starting with the rate form of the conservation of mass and an appropriate *open* system, determine the rate at which the depth of oil in the lower tank changes, in $\mathrm{m/min}$. Clearly indicate whether the level is increasing or decreasing.

(c) Repeat Part (b), but this time use an appropriate *closed* system, e.g. all of the oil in both tanks and in the connecting pipe.

? Problem 3.9.11

A vertical cylindrical tank has a diameter $D = 5$ meters, is 10 meters deep, and contains water. The tank drains from the bottom of the tank through an opening with a diameter of 5 centimeters. The volumetric flow rate of the water out of the tank is proportional to the depth of the water in the tank h and varies according to the equation $\dot{V} = \sqrt{(18.0\ \mathrm{m^5/s^2})\,h}$. If the depth of the water is initially 6 meters, determine how long it will take the tank to drain until the water depth decreases to 2 meters. If necessary, assume that water is an incompressible substance with a density of $1000\ \mathrm{kg/m^3}$.

? Problem 3.9.12

An automobile tire having a volume of $3\ \mathrm{ft^3}$ contains air at an absolute pressure of $32\ \mathrm{psi}\ (\mathrm{lbf/in^2})$ and a temperature of $70^\circ\mathrm{F}\ (530^\circ\mathrm{R})$. Assume that air can be modeled as an ideal gas under these conditions. [If Fahrenheit ($^\circ$ \text{F}) and Rankine $(^\circ\mathrm{R})$ temperatures are new or confusing, check out the unit conversion page and possibly your physics book or a dictionary.]

(a) Determine the mass of the air inside the tire, in lbm.

(b) If the tire sits overnight and the temperature drops to $32^\circ\mathrm{F}\ (492^\circ\mathrm{R})$, determine the new pressure in the tire. Assume no air leaks out of the tire and the volume of the tire remains constant.

(Just for fun. Not required: How much would the tire footprint change?)

? Problem 3.9.13

The breathing rate of an adult is approximately 12 breaths per minute and with each breath approximately $0.18\ \mathrm{mol}$ (not kmol) of gas is exhaled. The table below shows the molar composition (mole fractions) of the gases in the gas mixture exhaled.

Species		Mole Percent
Oxygen	O_2	15.1%
Carbon dioxide	CO_2	3.7%
Nitrogen	N_2	75.0%
Water vapor	H_2O	6.2%

(a) Determine the composition of the gas mixture on a mass basis, i.e. find the mass fraction of each mixture component. [Hint: Set up a table as shown in the text and work across from known mole fractions to mass fractions.]

(b) Determine mass of exhaled gases, in grams.

(c) Determine the apparent molar mass for the gases, in $\mathrm{g/mol}$ or $\mathrm{kg/kmol}$.

(d) Calculate the volume of gas exhaled, in $\mathrm{m^3}$ and in $\mathrm{cm^3}$, if the gas exhaled has a temperature of $37^{\circ}\mathrm{C}$ $(310\ \mathrm{K})$ and a pressure of $99\ \mathrm{kPa}$. Assume that the exhaled gas may be modeled as an ideal gas.

Problem 3.9.14

The composition of a gaseous fuel has been experimentally determined as shown in the table. The fuel is flowing at a rate of $1000\ \mathrm{ft^3/min}$ at a pressure of $200\ \mathrm{psi}$ and $60^{\circ}\mathrm{F}$.

Compound		Mole %
Methane	CH_4	65%
Ethane	C_2H_6	25%
Carbon Dioxide	CO_2	5%
Nitrogen	N_2	5%

Determine the

(a) composition of the fuel on a mass basis,

(b) apparent molar mass of the fuel, and

(c) the mass flow rate of the fuel at the conditions indicated assuming the fuel behaves like an ideal gas, in $\mathrm{lbm/min}$.

Problem 3.9.15

A mixture containing 55% benzene (B) and 45% toluene (T) by mass is fed into a distillation column. The overhead stream is 95% benzene by mass. The feed rate is $3000\ \mathrm{kg/h}$. Assume steady-state operation. Determine the unknown flow rates and compositions.

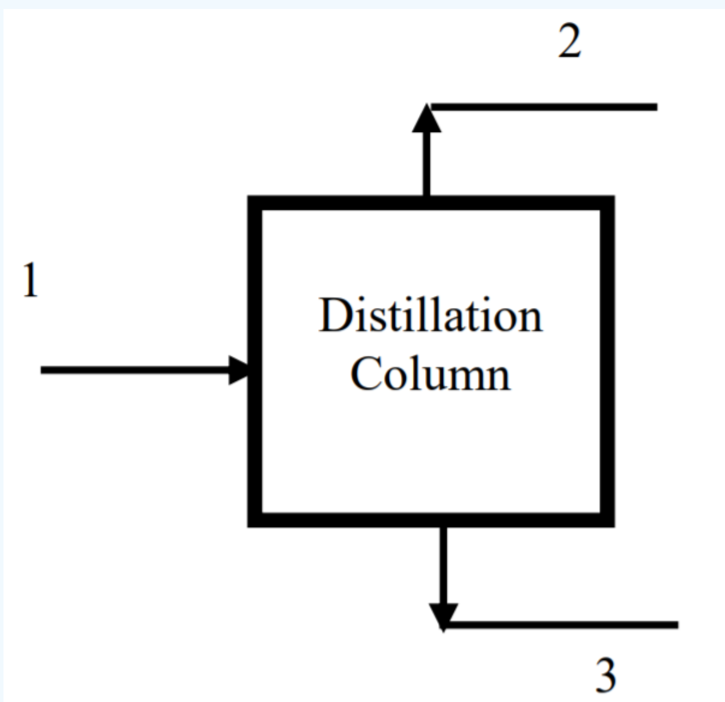


Figure $3.9.11$: The streams of material entering and leaving a distillation column.

	Stream	Mass Flow Rate	Composition	
			Benzene	Toluene
1	Feed	$3000\ \mathrm{kg/h}$	55%	45%
2	Overhead		95%	
3	Bottom	$2130\ \mathrm{kg/h}$		

Problem 3.9.16

In a process producing jam, crushed fruit containing 14% fruit solids by weight is mixed in a mixer with a sugar solution and pectin. The resultant mixture is then evaporated in a kettle to produce a jam. The known flow rates and compositions are shown in the table.

Determine the unknown flow rates and compositions, i.e. complete the table below. Assume steady-state operation.

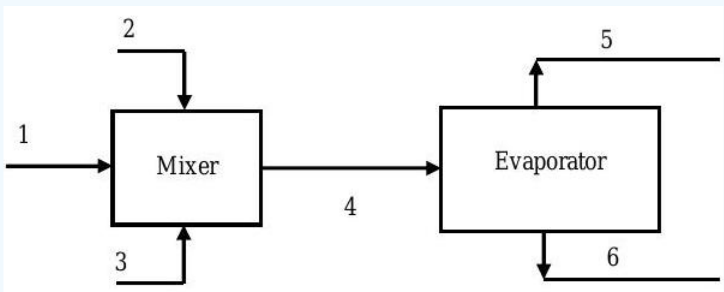


Figure $3.9.12$: Streams of material entering and leaving a system consisting of a mixer and an evaporator.

Stream	Stream Contents	Mass Flow Rates	Composition (Mass %)			
			Fruit Solids	Sugar	Pectin	Water
1	Crushed fruit	$1000\ \mathrm{kg/h}$	14%	$\cdots$	$\cdots$	86%
2	Sugar solution	$1.3\dot{m}_1$	$\cdots$	94	$\cdots$	6%
3	Pectin	$0.0025\dot{m}_1$	$\cdots$	$\cdots$	100%	$\cdots$
4	Mixture					
5	Water		$\cdots$	$\cdots$	$\cdots$	100%
6	Jam					33%

Problem 3.9.17

A solid material containing 15.0% moisture by weight is dried in a steady-state process so that it contains 7.0% water by weight. The moisture is removed by blowing fresh warm air mixed with recycled air over the solid in the dryer. Atmospheric air (moist air) used in this process can be modeled as a two-part mixture of dry air and water vapor (moisture). The known flow rates and compositions are shown in the table.

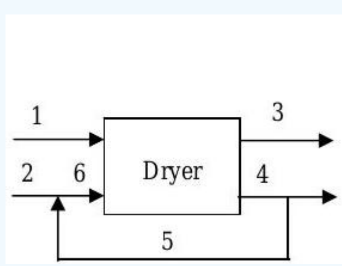


Figure 3.9.13: Streams of material entering and leaving the system of a dryer.

Stream	Stream Contents	Mass Flow Rate	Composition (Mass %)		
			Water	Dry Air	Solid
1	Moist solid	100 kg/h	15.0	...	
2	Fresh warm air		0.99		...
3	Dry solid		7.0	...	
4	Moist air out		9.09		...
5	Moist air recycled		9.09		...
6	Mixed air		2.91		

(a) Determine the unknown mass flow rates and compositions on a mass basis.

(b) The fresh warm air is supplied to the dryer at $50^\circ C$ and $120\ kPa$. Assuming as a first approximation that the air can be modeled as an ideal gas with a molar mass $M = 28.97\ kg/kmol$, calculate the volumetric flow rate of the fresh warm air in m^3/min. [Hint: Calculate the density of the air first and then the volumetric flow rate.]

Problem 3.9.18

Carbon disulfide (CS_2) is to be recovered from a gas containing $15.0\% CS_2$, $17.8\% O_2$, and $67.2\% N_2$. The gas is fed to a continuous absorption tower that operates at steady-state conditions. Inside the tower the feed gas contacts liquid benzene, which absorbs CS_2 but not O_2 or N_2. The liquid benzene is fed to the column in a $2:1$ mole ratio to the feed gas. Some of the benzene entering as liquid evaporates and mixes with other gases and leaves the top of the tower in the product gas. No chemical reactions occur inside the tower. (Adapted from *Elementary Principles of Chemical Processes* by Felder and Rousseau.)

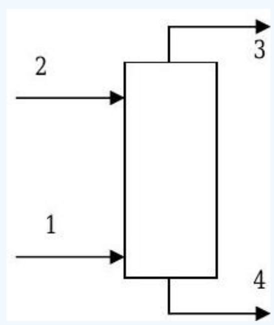


Figure 3.9.14: Inlets and outlets for a continuous absorption tower.

Inlets/Outlets	$\dot{n}_i$ (kmol/min)	Composition (mole %)			
		CS_2	Benzene	N_2	O_2
1: Feed gas	100	15.0	...	67.2	17.8
2: Feed liquid		...	100	...	...
3: Product gas		2.0	2.0		
4: Product liquid					

(a) Find the unknown mole fractions and molar flow rates indicated in the table.

(b) Find the fraction of the CS_2 fed to the column that is carried out in the product liquid.

(c) Find the fraction of benzene fed to the column that is carried out in the product gas.

? Problem 3.9.19

To make strawberry jam, a two-stage process is used as shown in the figure. First, crushed strawberries and sugar are mixed in a $45:55$ ratio by mass. Then the mixture is heated to evaporate water until the residue contains one-third water by mass. Strawberries contain about 15% solids (S) and 85% water (W) by mass.

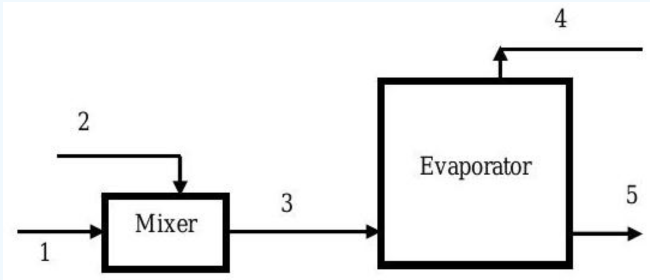


Figure $3.9.15$: Process stages and inlet/outlet streams in the process of jam making.

Assume steady-state operation and provide the missing information in the table below:

Stream		Mass Flow Rate (lbm/h)	Composition (Weight Percent)		
			Strawberry Solids	Water	Sugar
1	Strawberries	4500	15%	85%	...
2	Sugar	5500	...	...	100%
3	Evaporator Feed				
4	Water		...	100%	...
5	Strawberry Jam			33%	

? Problem 3.9.20

[Adapted from Felder & Rousseau]

In the production of a bean oil, beans containing 10% oil and 90% solids (by weight) are ground and fed to a stirred tank extractor where they are suspended in liquid n-hexane. Essentially all of the oil in the beans is extracted into the hexane in the extractor.

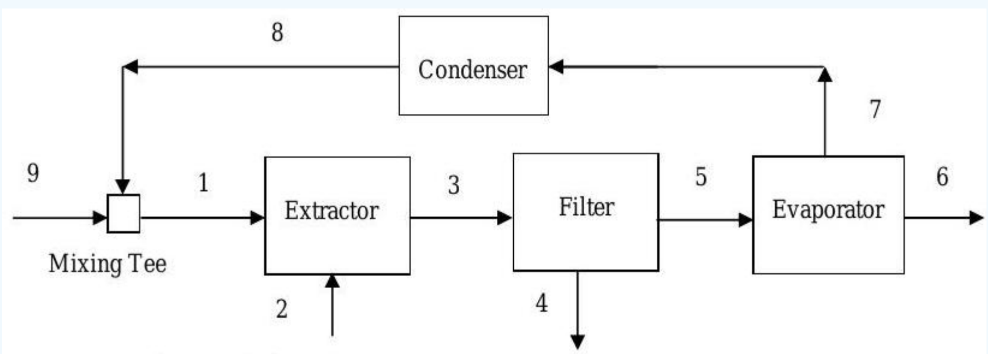


Figure $3.9.16$: Process stages and inlet/outlet streams in the process of bean oil production.

The effluent from the extractor passes through a filter as shown in the figure. The filter cake contains 75% by weight bean solids and the balance is bean oil and hexane. The ratio of the mass of bean oil to the mass of hexane in the filter cake is the same as it was in the extractor effluent, i.e. $mf_{4,\text{ oil}}/mf_{4,\text{ hexane}} = mf_{3,\text{ oil}}/mf_{3,\text{ hexane}}$.

The filter cake is discarded, and the liquid filtrate is fed to a vacuum evaporator, in which the hexane is vaporized and thereby separated from the oil. The hexane vapor is subsequently condensed and recycled to the extractor. Assume the process operates at steady-state conditions.

Determine the unknown information in the table.

Stream		Mass Flow Rate (kg/h)	Weight %		
			Hexane	Solids	Oil
1	Hexane (Recycled)	3000	100	...	...
2	Beans	1000	...	90	10
3	Effluent				
4	Filter Cake			75	
5	Liquid Filtrate			...	...
6	Oil		...	...	100
7	Hexane (Vapor)		100	...	...
8	Hexane (Liquid)		100	...	...
9	Hexane (Fresh Liquid)		100	...	...

Problem 3.9.21

An elevated water-storage tank is fed from a very large reservoir and supplies water to a community as shown in the figure. The level H of the water in the reservoir remains constant at $400\ \mathrm{ft}$ under all conditions and the level h of water in the tank changes with time. The base of the water tank has an area of $750\ \mathrm{ft}^2$ and the tank is a vertical cylinder. The following volumetric flow rates are known:

Into the tank from the reservoir: $\backslash(\dot{V}\backslash)_{reservoir} = K_{res}\sqrt{H-h} = \left(9.0\ \mathrm{ft}^{2.5}/\mathrm{min}\right)\sqrt{H-h}$

Out of the tank to the community: $\backslash(\dot{V}\backslash)_{community} = K_{comm}\sqrt{h} = \left(18.0\ \mathrm{ft}^{2.5}/\mathrm{min}\right)\sqrt{h}.$

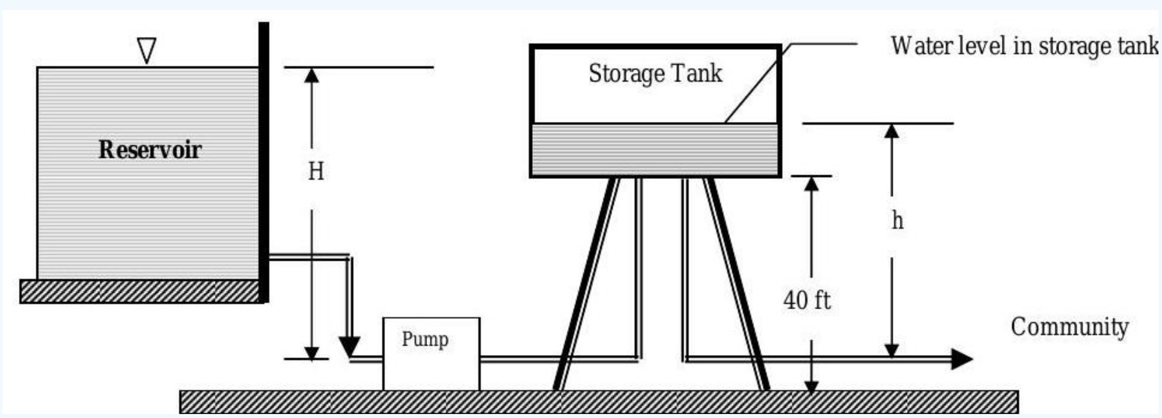


Figure 3.9.17: Connections between a reservoir, pump, elevated water storage tank, and community.

(a) Develop a differential equation, using only symbols, that describes how the elevation h of the water in the tank changes as a function of time. Assume that the tank is both receiving and discharging water.

(b) Determine the steady-state elevation of water in the tank, in feet, assuming the reservoir is charging the tank and the tank is simultaneously supplying the community.

(c) After a long period of steady-state operation, the flow from the reservoir stops unexpectedly. Determine the time it takes, in minutes, to lower the level of the water in the tank by 10 feet from the steady-state level found in part (b) under these conditions.

Problem 3.9.22

A balloon is filled with methane gas (CH_4) at $20^\circ C$ and 100 kPa until the volume is $26.4\ m^3$. Find (a) the mass of the gas (in kg) and (b) the volume (in \mathrm{m}^{3}) if the balloon rises to a height where the pressure and temperature change to 84 kPa and $0^\circ C$, respectively. Assume that the methane can be modeled as an ideal gas. [You may need to refer to your chemistry text or a thermodynamics text in order to find the molar mass of CH_4 and/or to review the ideal gas relation.]

Problem 3.9.23

A system of three tanks is connected as shown in the figure. The flow net formed by the tanks and their piping operates at steady-state conditions. The known mass flow rates are

$$\dot{m}_1 = 15\ \mathrm{lbm/s},\ \dot{m}_3 = 20\ \mathrm{lbm/s},\ \text{and}\ \dot{m}_5 = 12\mathrm{lbm/s}.$$

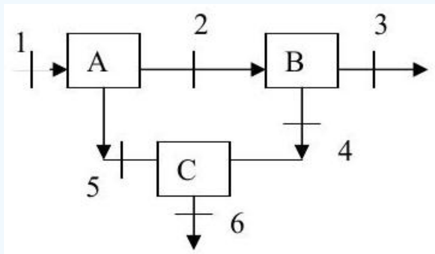


Figure 3.9.18: Streams of material entering and exiting a system of 3 tanks.

(a) Determine the unknown mass flow rates, in lbm/s. (Clearly show the system(s) you use to solve for each unknown.)

(b) If you had only been given two known flow rates could you have solved the problem? What if you were only given two of the external flow rates (1, 3, and 6)?

(c) What is the maximum number of unknown mass flow rates that you could solve for in this flow network if the only tool available is the conservation of mass, i.e. how many independent equations can you write for this system?

Problem 3.9.24

The rigid steel tank shown in the figure contains carbon dioxide gas. Initially the tank contains 300 kg of gas. The tank has an internal volume of $100\ m^3$, one inlet, and one outlet as shown in the figure. The following information is known about the mass flow rate at the inlet:

$$\dot{m}_1 = \begin{cases} \left(10.0\ \mathrm{kg/min^3}\right) t^2 \text{ if } 0 \le \mathrm{t} \le 3\ \mathrm{min} \\ 90.0\ \mathrm{kg/min} \text{ if } t \ge 3\ \mathrm{min} \end{cases}$$

The outlet mass flow rate is a constant $\dot{m}_2 = 90\ \mathrm{kg/\ min}$ for $t \ge 0$.

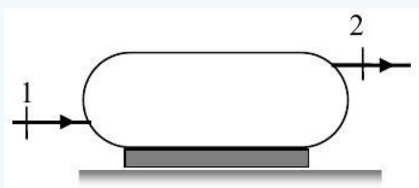


Figure 3.9.19: System consisting of a rigid steel gas tank with one inlet and one outlet.

(a) Calculate and graph the *mass* of gas and the *time rate of change of the mass* of the gas in the tank at one minute intervals for $0 \le \mathrm{t} \le 20$ min. Use an open system that corresponds with the interior volume of the tank. When, if ever, is the system at steady-state conditions?

(b) Is the density of the gas in the tank increasing or decreasing? What is the net change in the density of the gas in the tank for this 20-minute interval?

(c) How would your answer to part (a) change if you used an open system that included the walls of the tank inside your system? A qualitative discussion is acceptable.

Problem 3.9.25

The following equation is proposed as a model for the velocity profile in a river when the river channel is modeled as a rectangular channel of width $2L$ and water depth H :

$$V = V(x, y) = V_{\max} \cos\left(\frac{x}{L}\frac{\pi}{2}\right) \sin\left(\frac{y}{H}\frac{\pi}{2}\right)$$

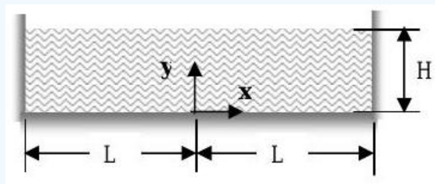


Figure 3.9.20: Rectangular cross-section of a river, to be used in calculating the river's velocity profile.

The following values have been measured for one river: $V_{\max} = 5 \text{ mph}, \; L = 50 \text{ ft}, \; H = 20 \text{ ft}$.

(a) Make a single graph that shows the velocity profile $V(x, y)$ in ft/s for all y at $x = 0, 10, 20, 30, 40,$and 50 ft. [Hint: Use a spreadsheet or MAPLE.]

(b) Calculate the volumetric flow rate by the two methods described below and report your answers in both cubic feet per second $\left(\text{ft}^3/\text{s}\right)$ and gallons per minute (gpm or gal/min).

Method 1 - Exact solution: Evaluate the appropriate two-dimensional (surface) integral for the cross-section. [Revisit your calculus notes/text for evaluating a surface integral.]

Method 2 - Approximate solution: Divide the cross-sectional area into N equal-sized rectangles, assume a uniform velocity inside each rectangle, calculate the approximate volumetric flow rate for each rectangle, and sum the results to get the total flow rate. Your answer will strongly depend on N, so you must provide some support for why your value of N gives you a "good" answer. [Hint: Use a spreadsheet.]

(c) Calculate the average velocity for the river, in mph and ft/s.

Problem 3.9.26

You have been asked to design a hydraulic system that uses a single pump to move two pistons simultaneously. As shown in the figure, the system consists of two cylinders, A and B, connected by a pump. When the pump is operating at steady-state conditions, Piston A moves at 5 cm/s.

Piston A has a diameter of 6 cm and Piston B has a diameter of 8 cm. The hydraulic line has a diameter of 2.5 cm. The hydraulic lines and cylinder walls are rigid. The hydraulic fluid is incompressible with a specific gravity of 0.8.

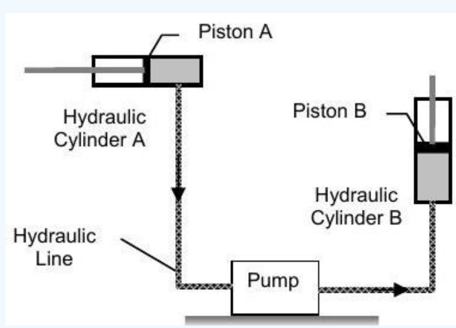


Figure 3.9.21: System consisting of two hydraulic cylinders connected by a hydraulic line passing through a pump.

(a) Starting with the conservation of mass equation and an open system of your choice, determine

- the required volumetric flow rate into the pump in $\mathrm{m^3/s}$ and gallons per minute (gpm) for Piston A to have the desired speed, and
- the velocity of the hydraulic fluid at the pump inlet under these conditions.

[*Hint:* Try a one-inlet open system with moving boundaries that contains the hydraulic fluid upstream of the pump.]

(b) Starting with the conservation of mass equation and a *closed system* of your choice, determine the speed of Piston B when Piston A is moving at the desired value. [*Hint:* Try a deforming, closed system that includes all of the hydraulic fluid.]

(c) Could you answer Part (b) using an *open system*? Briefly explain your answer, but you need not do the analysis.

? Problem 3.9.27

The rigid-walled tank contains water $\left(\rho = 1000\ \mathrm{kg/m^3}\right)$ and is open to the atmosphere on top. The tank is basically a vertical cylinder with a diameter $D_{\text{tank}} = 3\ \mathrm{m}$. Water flows into the tank at Inlet 1 with a velocity of $4\ \mathrm{m/s}$ through a pipe of diameter $D_1 = 0.2\ \mathrm{m}$. Water also enters the tank through Inlet 2. Water flows out of the tank through Outlet 3 at a mass flow rate of $80\ \mathrm{kg/s}$.

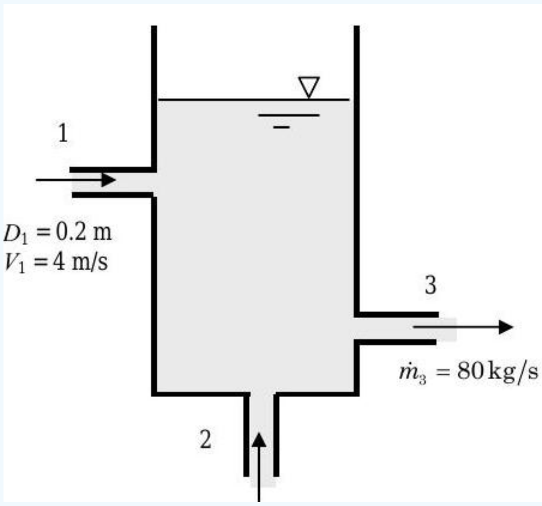


Figure $3.9.22$: System consisting of a tank of water with two inlets and one outlet.

(a) Determine the rate of change of the water level in $\mathrm{m/s}$ if water is entering the tank through Inlet 2 at $200\ \mathrm{m^3/h}$ (cubic meters per hour). Is the level increasing or decreasing?

(b) For the tank to have a steady-state operating condition with the given flow rates at Inlet 1 and Outlet 3, what volumetric flow rate is required at Inlet 2?

(c) What volumetric flow rate is required at Inlet 2 if the water level in the tank rises steadily at $10\ \mathrm{cm/min}$?

? Problem 3.9.28

A simple air duct with a side exhaust is shown below along with the direction of the velocity vectors at each inlet or outlet. The cross-sectional area at regions 1 and 3 are identical, $A_{c1} = A_{c3} = 0.5\ \mathrm{m^2}$. The velocity at 2, V_2, makes a 20° angle with the opening in the side of the duct $A_2 = 0.5\ \mathrm{m^2}$. Measurements indicate that the flow is steady with velocities $V_1 = 10\ \mathrm{m/s}$ and $V_3 = 3\ \mathrm{m/s}$.

Assume that air behaves like an incompressible substance under these conditions.

(a) Determine the ratio of the volumetric flow rate at 2 to the volumetric flow rate at 1, $\dot{V}_2/\dot{V}_1$.

(b) Determine the ratio of the velocity at 2 to the velocity at 1, V_2/V_1.

Problem 3.9.29

Water enters the steady-state system (shown in the figure) at inlet 1 with a mass flow rate of 500 kg/s and exits the system at outlets 2 and 3.

All known areas are shown as dashed lines on the figure. The velocity V_1 makes an angle of 45° with the known area A_1. Velocities V_2 and V_3 are normal to the known areas A_2 and A_3, respectively.

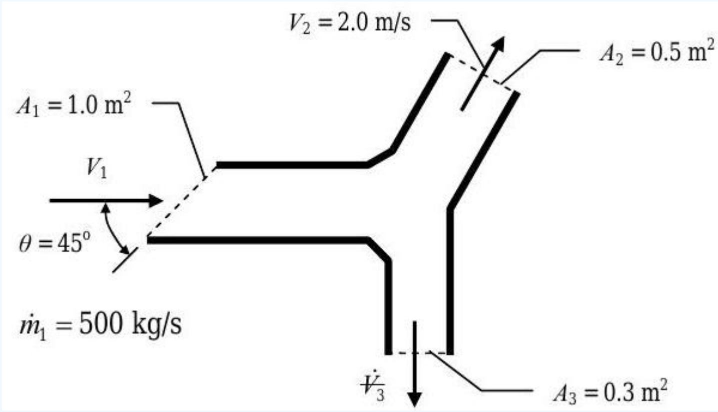


Figure 3.9.23: System consisting of a three-branched pipe, where water flows into one branch and out of the other two.

(a) Calculate the volumetric flow rate at outlet 3.

(b) Calculate the value of velocity V_1 at inlet 1.

Show *all* steps in your solution.

Problem 3.9.30

An incompressible fluid enters a rectangular channel with a uniform velocity profile with an average velocity of $V_{\text{avg}} = 3 \text{ m/s}$. The channel has a height $H = 1 \text{ m}$ and width $W = 3 \text{ m}$. As the fluid flows through the channel the velocity profile is distorted by obstructions in the flow (not shown on the figure). The distorted velocity profile is described by the equation shown on the figure. Assume all flows are steady-state.

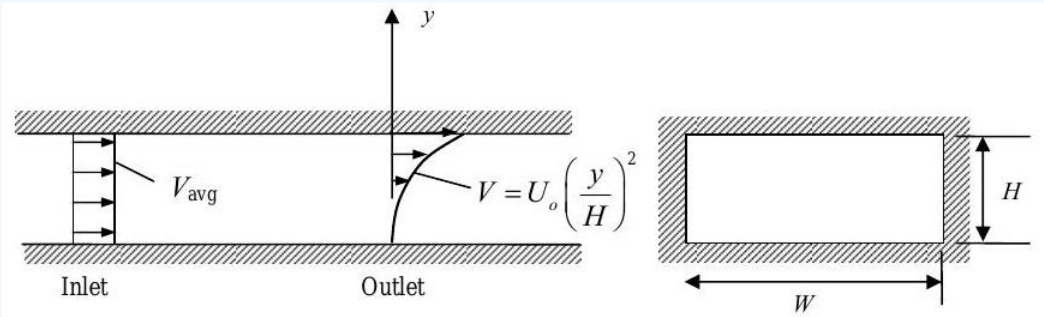


Figure 3.9.24: System consisting of a rectangular channel which water enters with a uniform velocity profile and exits with a distorted velocity profile given by $V = U_0 \left(\frac{y}{H}\right)^2$.

(a) Calculate the volumetric flow rate at the channel inlet, in m^3/s.

(b) Calculate the value of the constant U_0 in the distorted velocity profile equation, in m/s.

Problem 3.9.31

Consider the hydraulic dam shown in the figure. Water is released from the dam and flows steadily over the top of the dam into a dry riverbed at a volumetric flow rate of $200 \text{ m}^3/\text{s}$. The riverbed is 50 m wide. The riverbed absorbs water, and the velocity of the water entering the ground is $V_{\text{ground}} = 0.01 \text{ m/s}$ (see the arrows in the diagram).

Determine the extent of the riverbed, the length L, that will be affected by the released water, i.e. how far (L) will the released water flow before it is completely absorbed by the riverbed?

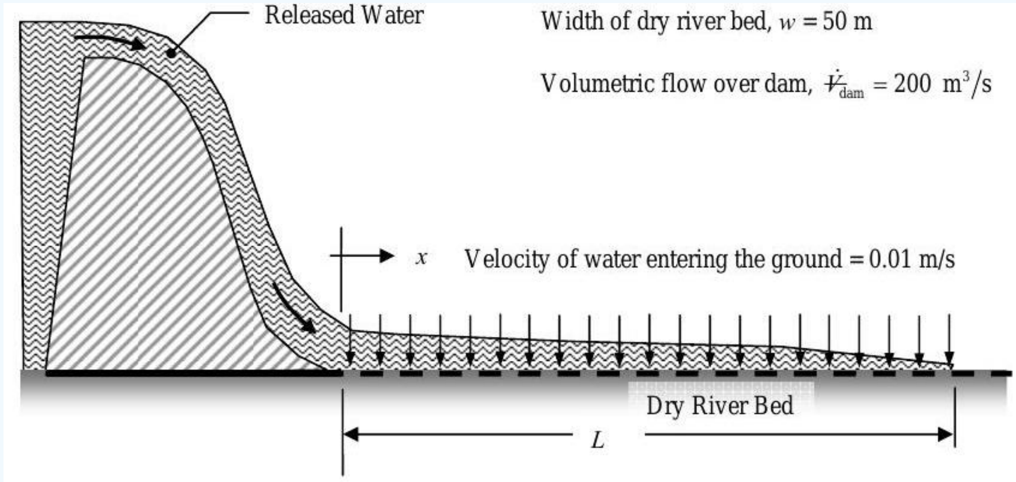


Figure 3.9.25: Water flows over the surface of a dam and onto a dry riverbed, which it flows along while also being absorbed by the ground.

? Problem 3.9.32

The tank shown below handles gasoline which has a density of $680\ \mathrm{kg/m^3}$. The tank is a vertical cylinder with a diameter D of 2 meters. Normally gasoline flows into the tank at Inlet 1 with a specified mass flow rate and leaves the tank at Outlet 2 .

A siphon is built into the side of the tank to prevent an overflow if there is a sudden change in the flow conditions for the tank. The siphon is simply a bent tube with one end inside the tank and the other outside the tank as shown. It will not start operating ("kick in") until the level of the gasoline in the tank exceeds the level of the siphon elbow by at least 0.1 meters, i.e. $h > H + 0.1\ \mathrm{m}$. Once it "kicks in", the siphon will stop working ("break") if the water level drops below the siphon inlet, i.e. $h < (H - L)$.

Under normal conditions, the mass flow rate of gasoline into the tank is $700\ \mathrm{kg/min}$, and the gasoline level in the tank is $h = 4.0$ meters. The siphon mass flow rate is $\dot{m}_{\text{siphon}} = \left(300\ \dfrac{\mathrm{kg}}{\mathrm{min \cdot m^{1/2}}}\right)\sqrt{h}$.

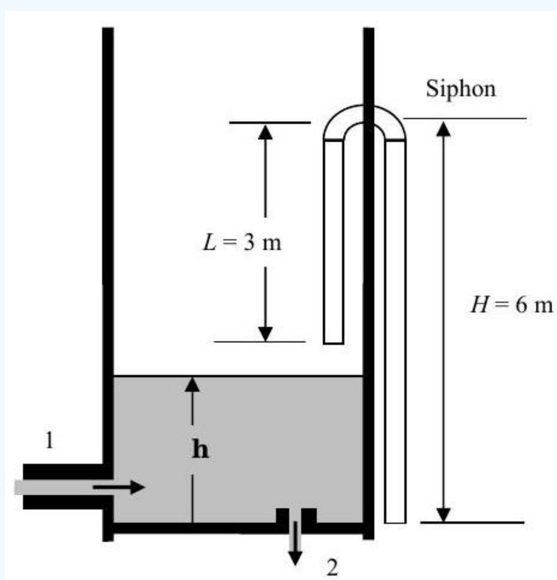


Figure 3.9.26: System consisting of a gasoline tank with one inlet, one outlet, and one siphon that operates when the liquid levels are sufficiently high.

(a) For some unknown reason, the outlet at 2 becomes totally blocked and the level begins to rise.

- Determine how long it will take the siphon to start working ("kick in").
- SET UP, but do not solve the differential equation that describes the rate of change of the gasoline level h during the transient to the new steady-state conditions after the siphon starts working.
- Determine the new steady-state height h of the gasoline in the tank under these conditions.

(b) After some time, the operator notices the siphon is discharging gasoline and stops the flow of gasoline into the tank; however, the tank will continue to drain until the siphon "breaks." How long will it take the siphon to "break" (stop working)?

Problem 3.9.33

A conically-shaped tank is partially filled with an incompressible liquid (specific gravity $SG = 0.30$). The volume of the liquid in the tank can be described by the equation $\forall_{\text{tank}} = h^3$ where h is the height of liquid in the tank (see diagram). Initially the fill valve and drain valve are both closed. (Show all your work for full credit.)

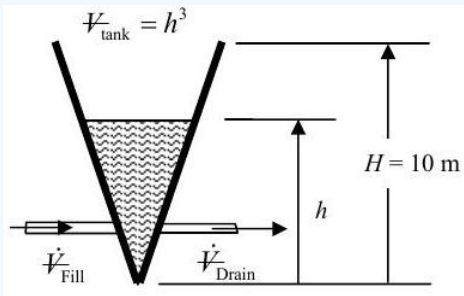


Figure 3.9.27: System consisting of a conical tank, with a fill valve and drain valve, that contains liquid.

(a) The fill valve is opened with the drain valve closed and the tank fills with more liquid. Determine a formula for the volumetric flow rate $\dot{\forall}_{\text{Fill}}$ such that dh/dt, the rate of change of the liquid level in the tank, is constant at 1 m/s, i.e. $dh/dt = 1 \text{ m/s}$.

(b) Under a different set of operating conditions, both the fill and drain valves are open with the following volumetric flow rates:

$$\dot{\forall}_{\text{Fill}} = (2/\text{s})h^3 \text{ and } \dot{\forall}_{\text{Drain}} = \left(50 \text{ m}^2/\text{s}\right) h.$$

Determine if there is a steady-state level for the liquid in the tank, or if it will just overflow.

(c) If the tank is just about to overflow, $h = H$, the fill valve is automatically shut and the drain valve opened. Develop an expression for h as a function of time as liquid drains from the tank. The draining rate is the same as in part (b): $\dot{\forall}_{\text{Drain}} = \left(50 \text{ m}^2/\text{s}\right) h$.

Problem 3.9.34

A boy starts drinking orange juice through a straw at the same time his mother is pouring the juice into his glass. The cross-sectional area of the cup, A, is 15 cm^2 and the juice is being poured into the cup at $20 \text{ cm}^3/\text{s}$.

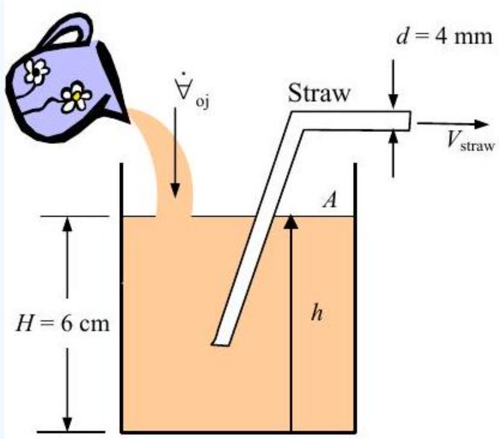


Figure 3.9.28: Orange juice is simulatneously poured into a glass and sucked out of it using a straw, at differing rates.

a) Determine the velocity of the orange juice in the straw required to keep the height of fluid at 6 cm. Comment on your answer.

b) If the mother stops pouring and the child drinks out of the straw at a time-varying rate of $10e^{-t/\tau}$ cm^3/s, where $\tau = 5$ s, determine an equation for the rate of change of the height of fluid (dh/dt) as a function of time.

c) What is the final height of fluid in the glass?

Problem 3.9.35

PVC pipe is manufactured using a steady-state extrusion process as shown in the figure. A liquid melt with density ρ_m is fed into the tank and the PVC pipe is extruded through a die in the side of the tank. As the extrusion travels to the right, the PVC material solidifies. Solid PVC has a density of ρ_s. The finished PVC pipe has inner and outer diameter of D_i and D_o, respectively, and travels to the right with a velocity of V.

(a) Determine the volumetric flow rate of liquid melt that must be supplied to the tank. Express your answer in terms of ρ_m, ρ_s, D_i, D_o and V.

(b) The extrusion passing through the die in the side of the tank is not solidified, but still in the liquid state. As a result, the pipe wall thickness is larger in this region. If the outer diameter of the pipe extrusion at the die (tank) exit is D_e as shown in the figure, find the velocity V_e of the liquid PVC material at this location. Express your answer in terms of the other problem variables as required.

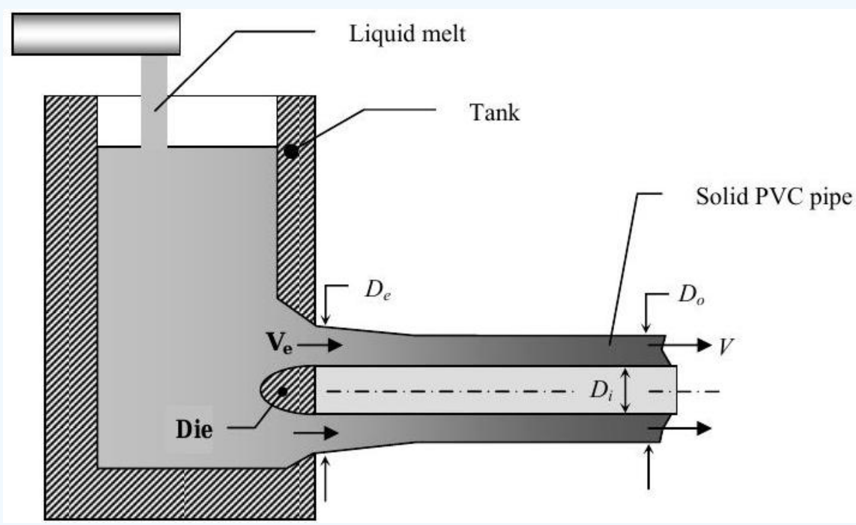


Figure 3.9.29: System to manufacture PVC pipe, consisting of liquid melt in a tank being extruded through a die in the tank side.

Problem 3.9.36

Hoosier Angels Inc. has geared up for the Christmas season and makes white plastic angels using the plastics molding system shown below.

- The feed conveyor delivers white plastic feed pellets to the feed bin. The feed has a density $\rho_{\text{feed}} = 300 \text{ kg/m}^3$.
- The feed bin is a vertical tank with a conical lower section. The cylindrical (upper) portion of the feed bin has a diameter $D_{\text{bin}} = 1.0 \text{ m}$, and a clear panel in the side of the feed bin allows the operator to observe the level h of the feed in the bin. At standard operating conditions the bin is filled so that the level of feed in the bin is $h = h_{ss} = 2.0 \text{ m}$.
- The feed flows from the feed bin to the melter where it is turned into a liquid with a density $\rho_{\text{melt}} = 200 \text{ kg/m}^3$. The liquid melt then flows to the molder through the melt transfer pipe which has a diameter $d_{\text{pipe}} = 5.0 \text{ cm}$.
- The angels are formed in the molder, drop onto a conveyor belt and are carried away for packaging. Each angel has a mass of 85 grams and the molding system produces 60 angels per minute.

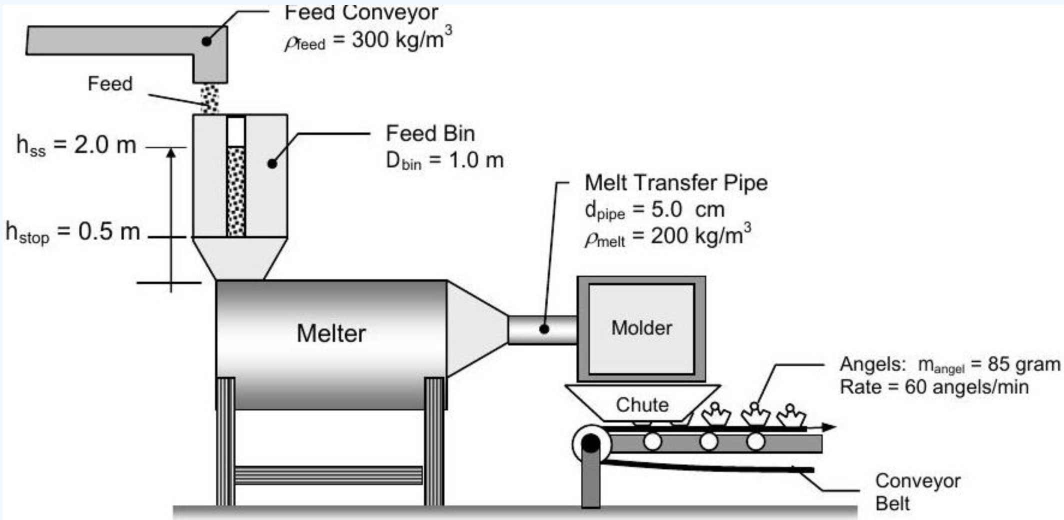


Figure 3.9.30: Diagram of a plastic molding system.

For questions, (a), (b), and (c) assume the process is operating at steady-state conditions.

(a) Determine the mass flow rate of feed supplied to the feed bin by the feed conveyor, in kg/min.

(b) Determine the volumetric flow rate of feed supplied by the feed conveyor, in m^3/min.

(c) Determine the average velocity of the liquid melt in the transfer pipe, in m/min.

If the conveyor breaks down and stops supplying feed, the melter and molder can keep functioning at steady state; however, the level h of feed in the feed bin changes.

(d) Determine the rate of change of the feed level in the bin once the feed conveyor shuts down.

(e) How long after the feed conveyor shuts down can the molder continue making angels at the current rate? Assume that the feed level drops from $h_{ss} = 2.0 \text{ m}$ to the shutoff level $h_{\text{stop}} = 0.50 \text{ m}$ during this time period.

[For full credit, you must clearly identify your system and state the pertinent assumptions in your model. Feel free to use the problem figure without redrawing it.]

Problem 3.9.37

A new waterfall is to be installed at one end of Speed Lake. The water level in Speed Lake is to be held constant by feeding it with water from Scum Pond. The volume flow rates into and out of Speed Lake are given by the equations $C_1\sqrt{h_1}$ and $C_2\sqrt{h_2}$, respectively, where $C_1 = 700 \text{ m}^{5/2}/\text{hr}$ and $C_2 = 1400 \text{ m}^{5/2}/\text{hr}$.

At the design conditions which produce a pleasing waterfall, $h_2 = 0.50 \mathrm{~m}$ and the surface area of Speed Lake is $A_{\text{surface}} = 1000 \mathrm{~m}^2$. By design, the surface area of Speed Lake, A_{surface}, is a constant for all values of $h_2 \geq 0$.

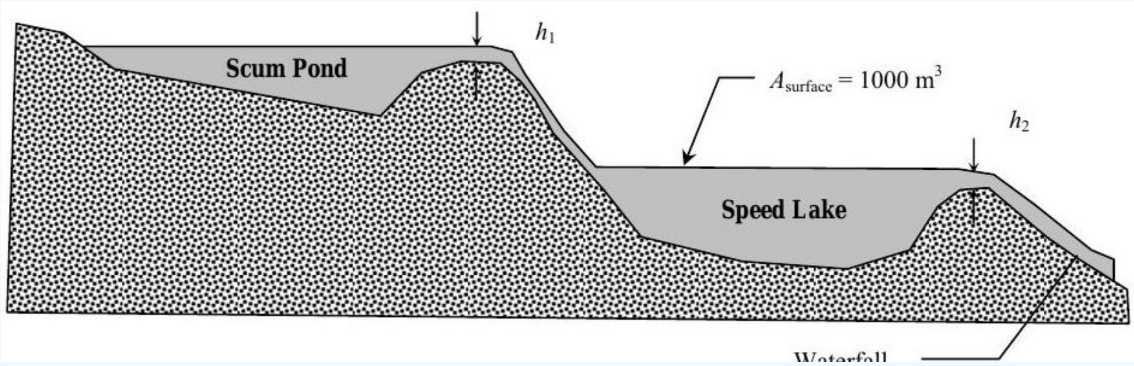


Figure 3.9.31: Diagram of a system consisting of a pond-fed lake, the pond, and a waterfall exiting the lake.

(a) Determine the required height h_1 to keep the water level in Speed Lake constant at the design conditions.

(b) During long dry periods, evaporation from the surface of Speed Lake may also be important. If $50,000 \mathrm{~kg/hr}$ of water evaporates from the surface of Speed Lake, calculate the new height h_1 required to match the design conditions.

(c) Flow into Speed Lake from Scum Pond suddenly stops. Determine the time rate of change of the water level, h_2, in Speed Lake immediately after the flow from Scum Pond is stopped and the time it takes for $h_2 \rightarrow 0$. You may neglect evaporation and assume Speed Lake was at design conditions, $h_2 = 0.50 \mathrm{~m}$, when the inlet flow stopped.

Problem 3.9.38

One of the important characteristics of paper towels is their ability to absorb liquid. The experimental setup shown in the figure is used to measure the absorption (storage) rate of liquid for paper towel material. The apparatus is designed so that any water stored in the sample holder is assumed to be absorbed by the paper towel.

To run the test, a dry towel sample is placed in the sample holder. Then water is allowed to drain from the supply tank into the sample holder. Excess liquid not absorbed by the towel material drains to the catch tank. The absorption (storage) rate data is developed by monitoring the liquid levels H and h as a function of time.

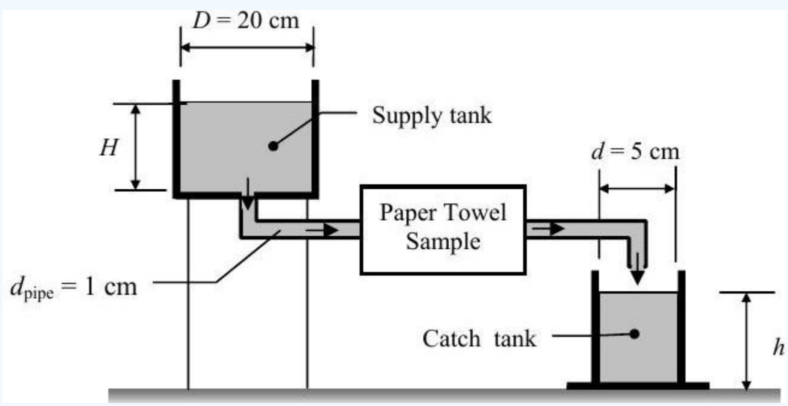


Figure 3.9.32: System to test the liquid absorbancy of a paper towel sample.

(a) Develop an expression for the rate of storage of liquid in the paper towel sample (dm/dt), in terms of the relevant information. A symbolic solution in terms of the problem variables is desired.

(b) When the towel material saturates, it will stop absorbing (storing) liquid. How will dH/dt and dh/dt be related when the towel saturates?

(c) Determine the time it takes to drain the liquid in the supply tank from $H = 10 \mathrm{~cm}$ to $5 \mathrm{~cm}$ when the sample holder is empty. Under these conditions the volumetric flow rate out of the supply tank is given by the expression

$\dot{V}_{\text{supply}} = (0.30\ \text{cm}^{2.5}/\text{s})\sqrt{H}.$

❓ Problem 3.9.39

You are given a leaky balloon (see figure) and asked to inflate it. Initially, the balloon is deflated with a volume $V = V_o$ and a pressure inside $P = P_{atm}$, the atmospheric pressure.

At time $t = 0$, you begin to inflate the balloon by blowing at a constant volumetric flow rate $\dot{V}_B$.

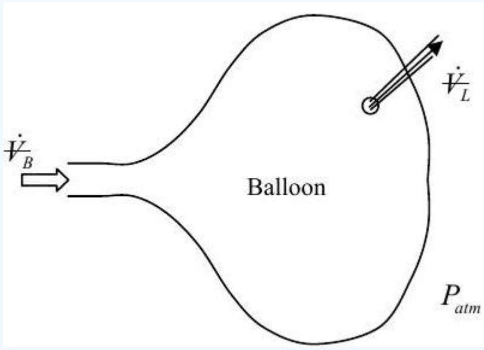


Figure 3.9.33: A balloon that leaks at a constant rate is being inflated at a constant rate.

The following information is also known:

- The balloon volume V is related to the balloon pressure P by the equation: $V = V_o + C(P - P_{atm})$ where C is a dimensional constant and P_{atm} is the atmospheric pressure.
- The leakage volumetric flow rate, $\dot{V}_L$, is proportional to the square root of the pressure difference across the wall: $\dot{V}_L = K\sqrt{P - P_{atm}}$ where K is a dimensional constant and P_{atm} is the atmospheric pressure.
- As an approximation, assume the air to be incompressible.
- The following parameters are assumed to be given (i.e. have constant, known values): P_{atm}, $\dot{V}_B$, $\dot{V}_o$, C, and K.

(a) Develop an equation for the *time rate of change* of the balloon volume, dV/dt. (Do not solve the equation.)

(b) Assuming that the balloon doesn't break and that you don't run out of air, what is the steady-state volume of the balloon? Give your answer in terms of the given parameters listed above.

Note: For full credit show *all* steps in your solution.

❓ Problem 3.9.40

A two-column distillation column operates at steady-state conditions and there are no chemical reactions. The known flow rates and compositions are shown in the table.

(a) Develop the necessary equations to calculate the unknown flow rates and compositions. DO NOT SOLVE THE EQUATIONS.

(b) Solve for the unknowns.

	Stream	Mass Flow Rate $(\mathbf{kg/h})$	Composition (Mass %)		
			A	B	C
1	Feed	1000	20.0	30.0	50
2	Overhead I	250	62.0	5.0	
3	Bottoms I				
4	Overhead II		15.0	80.0	
5	Bottoms II		0.5		

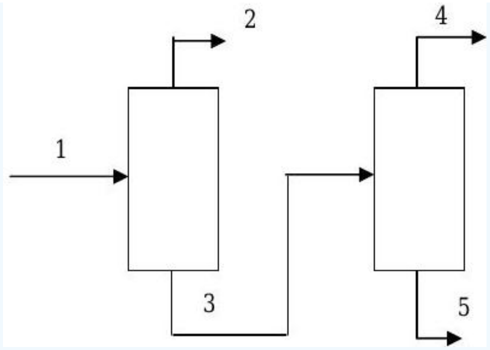


Figure $3.9.34$: System of material streams entering and exiting a two-column distillation column.

Problem $3.9.41$

The ItsaVegetable Co. makes ketchup for school lunchrooms using a two-stage process.

Raw tomatoes and tomato concentrate are fed into the crusher to make tomato slurry. The tomato slurry then flows into the mixer where spices are added to produce the ketchup. Additional flow rate and composition data is shown in the table.

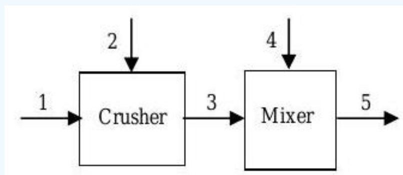


Figure $3.9.35$: Material streams entering and exiting the two stages of a ketchup production process.

(a) Develop a sufficient set of independent equations that could be used to determine the unknown information. DO NOT SOLVE THE EQUATIONS.

(b) Solve for the unknowns.

Stream	Contents	Mass Flow Rate (lbm/h)	Composition (Mass Fraction)		
			Tomato Solids	Water	Spices
1	Raw Tomato Feed	1000	0.500	0.500	0.000
2	Tomato Concentrate Feed		0.900	0.100	0.000
3	Tomato Slurry				
4	Spice Feed		0.000	0.000	1.000
5	Wet Ketchup Slurry		0.714		0.036

Problem $3.9.42$ (Revised 11 September 2006)

In the winter, a major problem in buildings is low relative humidity that occurs when dry outside air is heated. The schematic below shows an air-handling system designed to humidify outside air for ventilation and heating purposes.

The humidifier supplies conditioned air to the room. Outside air leaks into the room due to leaky windows and doors. Return air from the room is split in a splitter tee with some exhausted (exhaust air) and some recycled (recycled air). The recycled air mixes with outside air in a mixing tee. The mixed air then enters the humidifier where moisture is added.

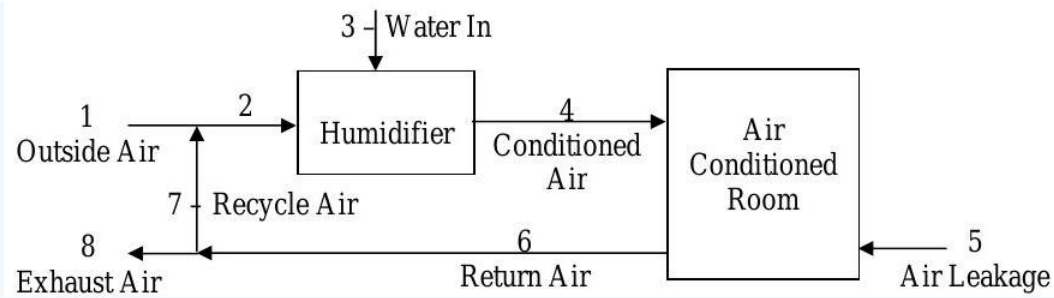


Figure 3.9.36: Streams of material entering and exiting the stages of an air-handling system.

(a) Develop a sufficient set of equation to calculate the unknown flow rates and compositions. DO NOT SOLVE THE EQUATIONS.

(b) Solve for the unknowns.

Stream		Mass Flow Rate (kg/min)	Composition	
			Water	Dry Air
1	Outside Air	$0.30 \cdot \dot{m}_4$	0.004	
2	Mixed Air			
3	Water In		1.000	
4	Conditioned Air	1000		
5	Air Leakage	200	0.004	
6	Return Air		0.011	
7	Recycle Air			
8	Exhaust			

Problem 3.9.43

A mixture of benzene, toluene, and xylene enters a two-stage distillation process where some of the components are recovered. The distillation process operates at steady-state conditions with operating information as shown in the table. In addition, the following information is known about the distillation process:

- 98.0% (by weight) of the xylene that enters Column A in the feed stream (Stream 1) leaves in the intermediate stream (Stream 3)
- 96.0% (by weight) of the benzene that enters in the feed stream (Stream 1) leaves in the bottoms of Column B (Stream 4)

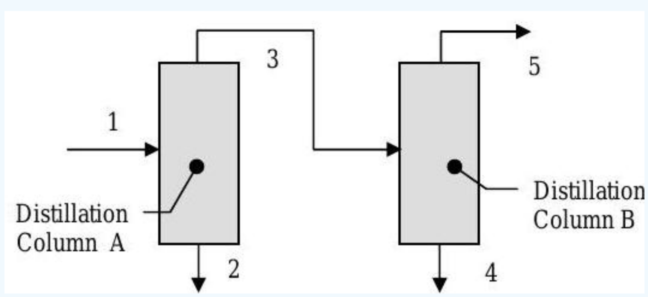


Figure 3.9.37: Streams of material entering and exiting the columns of a two-colum distillation system.

(a) Find a set of ***independent*** equations that can be solved for all the unknowns. Clearly identify the number of pertinent unknowns and number your equations. DO NOT SOLVE!

(b) Solve for the unknowns.

	Stream	Mass Flow Rate (kg/h)	Composition (Mass Percent)		
			Benzene	Toluene	Xylene
1	Feed	1275	30.0	25.0	45.0
2	Bottoms A				
3	Intermediate		0.0	1.0	99.0
4	Bottoms B		99.0	1.0	0.0
5	Tops B				

Problem 3.9.44

A paper dryer uses treated air to lower the moisture content of wet paper. The known process information is shown in the table below the figure.

(a) Find a set of ***independent*** equations that can be solved for all the unknowns. Clearly identify the number of pertinent unknowns and number your set of equations. DO NOT SOLVE!

(b) Solve for the unknowns.

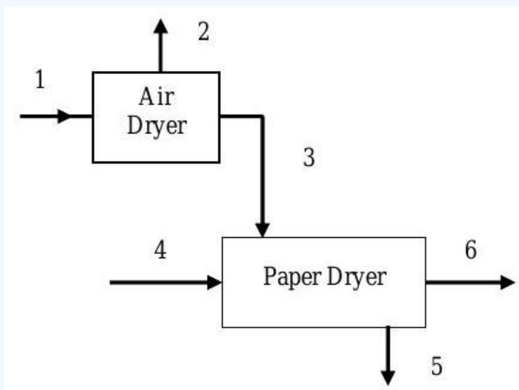


Figure 3.9.38: Streams of material entering and exiting the stages of a paper-drying system.

Stream	Description	Mass Flow Rate (lbm/h)	Composition (Mass %)		
			Paper	Water	Air
1	Atmospheric Air	10,000	0.00	0.02	99.98
2	Condensate		0.00	100.00	0.00
3	Treated Air		0.00	0.01	99.99
4	Wet Paper	1000	97.00	3.00	0.00
5	Exhaust Air		0.00	0.04	99.96
6	Dry Paper				0.00

Problem 3.9.45

A two-step distillation and mixing process is shown in the figure. The system operates at steady-state conditions and there are no chemical reactions. The known flow rates and compositions are shown in the table.

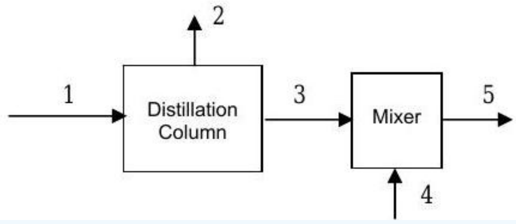


Figure 3.9.39: Streams of material entering and exiting the stages of a distillation and mixing process.

(a) Develop the necessary equations to calculate the unknown flow rates and compositions. DO NOT SOLVE THE EQUATIONS.

(b) Solve for the unknowns.

Stream		Mass Flow Rate	Composition (Mass %)	
		(kg/h)	A	B
1	Feed	100	50.0	50.0
2	Distillate		90.0	10.0
3	Concentrate			
4	Diluent		30.0	70.0
5	Product	90	26.0	

Problem 3.9.46

A glycerol plant operates at steady-state conditions and treats a glycerol solution (1) by feeding it into an extraction tower with an alcohol solvent (3). Two streams leave the extraction tower: a raffinate stream (2) and an extract stream (4). The extraction tower involves four compounds: glycerin, salt (NaCl), butyl alcohol, and water.

The extract stream (4) is fed into a distillation tower. No salt enters the distillation tower, and the distillation process involves only three compounds: glycerin, butyl alcohol, and water. A distillate stream (5) and a bottoms stream (6) leave the distillation tower. Detailed information about the known flow rates and mass compositions is shown in the table.

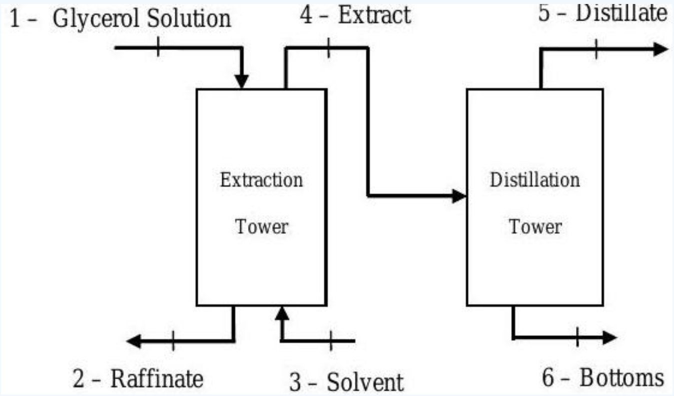


Figure 3.9.40: Streams of material entering and exiting an extraction and distillation system.

(a) Develop a *sufficient* set of equations to determine the unknown flow rates and compositions.

(b) Solve for the unknowns.

Stream		Mass Flow Rate	Composition (Mass %)			
		(lbm/h)	Glycerin	Salt (NaCl)	Butyl Alcohol	Water
1	Glycerol Solution	1000	10.00	3.00	....	87.00

Stream		Mass Flow Rate	Composition (Mass %)			
		(lbm/h)	Glycerin	Salt (NaCl)	Butyl Alcohol	Water
2	Raffinate		1.00		1.00	
3	Solvent	1000	………	………	98.00	2.00
4	Extract			0.00		
5	Distillate		………	………	95.00	5.00
6	Bottoms		25.00	………	………	75.00

? Problem 3.9.47

Salad dressing [1] is made in a two-stage mixing process as shown in the figure below. A sugar solution is mixed with pure water and crushed herbs in the first stage. The exiting stream is mixed with vinegar and olive oil in the second stage. Known flow rates and compositions are listed in the table.

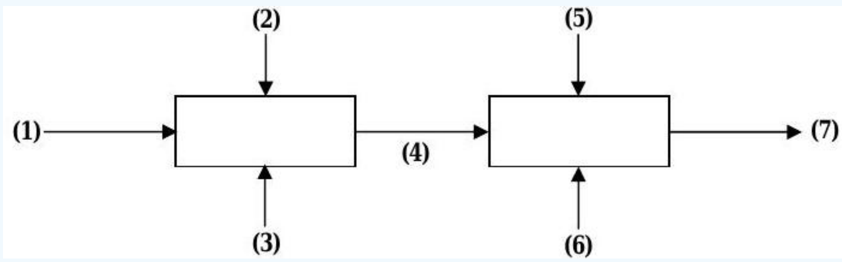


Figure 3.9.41: Streams of material entering and exiting the two stages of a mixing process.

Stream		Mass Flow Rate	Composition (Mass %)				
		(lbm/h)	Sugar	Herbs	Vinegar	Oil	Water
1	Sugar solution		0.3	⋯	⋯	⋯	0.7
2	Herbs		⋯	1.0	⋯	⋯	⋯
3	Water		⋯	⋯	⋯	⋯	1.0
4	Mixture						
5	Vinegar		⋯	⋯	1.0	⋯	⋯
6	Oil		⋯	⋯	⋯	1.0	⋯
7	Dressing	500	0.10	0.09			

a) It is desired to have equal amounts (by mass) of water, oil and vinegar in the dressing. What are the required mass fractions of vinegar, oil and water in the exit stream (7)?

b) Develop a set of equations that could be used to solve for all unknown mass flow rates as well as the mixture composition of stream (4).

c) Solve for the unknowns.

{ }^{1} Crushed herbs include basil, thyme, rosemary and just a hint of saffron. Of course, only the finest red wine vinegar and extra-virgin olive oil go into any ConAps salad dressing.

CHAPTER OVERVIEW

4: Conservation of Charge

4.1: Four Questions

Following the same pattern we used in Chapter 3 to develop the conservation of mass equations, we will once again begin our discussions by answering four questions:

- What is electric charge?
- How is electric charge stored in a system?
- How can it be transported across the boundary of a system?
- How can it be generated or consumed inside the system?

After we have answered the questions, we will then put it all together in the accounting framework.

What is electric charge?

Anyone who has walked across a carpet on a dry day and then "zapped" someone or something has at least an acquaintance with electric charge. A detailed discussion of electric charge can be found in any physics textbook[1] and is beyond the scope of the present discussion. However, certain facts are of interest as we try to understand electric charge:

- Electric charge is an attribute of matter and depends on the extent of the system, i.e. electric charge is an extensive property.
- Electric charge is granular and comes in discrete chunks.
- There are two types of electric charge. Benjamin Franklin called these positive charge and negative charge: a glass rod becomes positively charged when rubbed with a piece of silk, and a hard rubber rod becomes negatively charged when rubbed with a piece of cat's fur.
- Unlike charges attract each other, and like charges repel each other. This phenomenon is described in terms of Coulomb's Law.
- The unit of charge is the **coulomb (C)**. The coulomb can be defined operationally using Coulomb's Law in terms of the force of attraction or repulsion between two charged particles. A more practical operational definition can be developed in terms of the force of attraction or repulsion between two parallel wires in which an electric charge is flowing. Again, an undergraduate physics text is the best resource for learning more about this.
- The smallest granule of charge is given the symbol e and its magnitude in coulombs is $e = 1.602189 \times 10^{-19}\ \mathrm{C}$.

Notation for charge can sometimes be confusing. In this course we will use the following symbols and conventions:

$$\begin{aligned}
&q^{+} = \text{positive electric charge} = \mathrm{n}e \text{ where } \mathrm{n} = 0, 1, 2, 3, \ldots \\
&q^{-} = \text{negative electric charge} = \mathrm{n}e \text{ where } \mathrm{n} = 0, 1, 2, 3, \ldots \\
&q = q^{+} - q^{-} = \text{net electric charge}
\end{aligned}$$

Although electric charge comes in discrete chunks, we will assume that the amount of charge in a system can take on any amount unless our system consists of an extremely small number of subatomic particles, atoms, or molecules.

[1] For example *University Physics: Models and Applications* by W. P. Crummet and A. B. Western. Wm. C. Brown Publishers, Dubuque, IA, 1994.

How is charge stored in a system?

Because positive charge, negative charge, and net charge are all extensive properties, the amount of charge in a system can be calculated in a manner that parallels our calculation of mass in a system. To do this we first need to define a charge density. **Charge density** ρ_q is the amount of charge per unit volume with typical units of C/m^3 or C/ft^3. As with the mass density, charge density can be a function of all three spatial coordinates and time. To calculate the net charge within a system, we once again integrate the charge density over the system volume:

$$q_{sys}(t) = \int\limits_{V_{sys}} \rho_q(x, y, z, t)\, dV$$

A similar integration could be performed for positive and negative charge.

Under certain conditions, it may be more useful to work in terms of the charge per unit mass or unit mole rather than per unit volume. To do this, we need to define **mass specific charge** $\tilde{q}$ as the charge per unit mass with units of C/kg or C/lbm. Similarly a molar specific charge $\bar{q}$ could also be defined as the charge per unit mole with typical units of $\mathrm{C/kmol}$. Once the specific charge is known then the charge for the system would be calculated as

$$q_{sys}(t) = \int_{V_{sys}} \tilde{q}(x,y,z,t)\rho(x,y,z,t)\,dV.$$

Similar calculations could be performed using the molar density and the molar specific charge. Once again, notice that integrating over the system volume produces a system charge that only depends upon time. Charge calculations might be useful in areas like electrochemistry or magnetohydrodynamics, where the charge characteristics of various ions are known as is the amount of matter in the system.

How can charge be transported across the boundary of a system?

In answering this question, it is useful to consider the mechanisms for closed and open systems separately. For closed systems, experience has shown that charge can flow across the system boundary. The following symbols are used to describe the rate at which charge crosses or flows across a boundary:

$$\begin{aligned}\dot{q}^{+} &= \text{flow rate of positive charge, in amperes (A)}\\ \dot{q}^{-} &= \text{flow rate of negative charge, in amperes (A)}\\ \dot{q} = \dot{q}^{+} - \dot{q}^{-} &= \text{flow rate of net charge, in amperes (A)}\end{aligned}$$

Recall that a flow rate by definition can only be defined with respect to a boundary. The standard unit for the flow rate of charge is the **ampere** (A), which is defined as $1\ \mathrm{A} = 1\ \mathrm{C/s}$. The "q-dot" notation is consistent with our convention for describing flow rates of an extensive property; however, by long-standing convention the most commonly used symbol for the flow rate of net charge is the lower-case i, e.g. $i \equiv \dot{q}$. We will use both symbols interchangeably. Also by convention, the flow rate of net charge is referred to as the **electric current** and *the electric current is assumed to flow in the direction of the movement of positive charge*.

For an open system, there is an additional mechanism for charge to flow across the boundary — transport of charge with mass flow. An example where this mechanism is important is inside a battery where a flow of ions occurs internal to the battery to match the flow of current in the external circuit. Another application where this is important is in the design of electrostatic precipitators that remove pollutants from combustion exhaust gases. Although it is a different mechanism than the current flow into a closed system, we will not introduce a special set of transport terms. Just remember that if mass crosses the boundary of a system and the mass carries a charge, it must be included in the overall charge equation.

How can electic charge be generated or consumed inside the system?

Experience has shown that *net charge is conserved*. In equation form this can be written as

$$\begin{aligned}\dot{q}_{\text{gen}} &= \dot{q}^{+}_{\text{gen}} - \dot{q}^{-}_{\text{gen}} \equiv 0 &\quad\rightarrow\quad& \dot{q}^{+}_{\text{gen}} = \dot{q}^{-}_{\text{gen}}\\ \dot{q}_{\text{cons}} &= \dot{q}^{+}_{\text{cons}} - \dot{q}^{-}_{\text{cons}} \equiv 0 &\quad\rightarrow\quad& \dot{q}^{+}_{\text{cons}} = \dot{q}^{-}_{\text{cons}}\end{aligned}$$

Based on this empirical evidence and our definition of net charge, we can see from Equation $4.1.4$ that conservation of net charge implies that positive and negative charges can only be generated or consumed in matched pairs

4.2: Conservation of Charge

Using the accounting framework, we can now develop the following statement for positive charge and negative charge:

$$\begin{bmatrix}\text{Rate of}\\ \text{Accumulation}\\ \text{of }+/-\text{charge}\\ \text{inside the system}\\ \text{at time } t\end{bmatrix} = \underbrace{\begin{bmatrix}\text{Transport Rate}\\ \text{of }+/-\text{ charge}\\ \text{into the system}\\ \text{at time } t\end{bmatrix} - \begin{bmatrix}\text{Transport Rate}\\ \text{of }+/-\text{ charge}\\ \text{out of the system}\\ \text{at time } t\end{bmatrix}}_{\text{Net Transport Rate Into The System}}$$
$$+\underbrace{\begin{bmatrix}\text{Generation Rate}\\ \text{of }+/-\text{ charge}\\ \text{inside the system}\\ \text{at time } t\end{bmatrix} - \begin{bmatrix}\text{Consumption Rate}\\ \text{of }+/-\text{ charge}\\ \text{inside the system}\\ \text{at time } t\end{bmatrix}}_{\text{Net Generation Rate Inside the System}}$$

In symbols, the rate-form of the accounting equation for positive charge can be written as

$$\frac{d}{dt}q^+_{sys} = \sum_{in}\dot{q}^+_i - \sum_{out}\dot{q}^+_e \quad + \quad \dot{q}^+_{gen} - \dot{q}^+_{cons}$$

In symbols, the rate-form of the accounting equation for negative charge can be written as

$$\frac{d}{dt}q^-_{sys} = \sum_{in}\dot{q}^-_i - \sum_{out}\dot{q}^-_e \quad + \quad \dot{q}^-_{gen} - \dot{q}^-_{cons}$$

If we subtract Equation $4.2.3$ from Equation $4.2.2$ and apply our definition of net charge, we obtain the following results:

$$\frac{d}{dt}\left(q^+_{sys} - q^-_{sys}\right) = \sum_{in}\left(\dot{q}^+_i - \dot{q}^-_i\right) - \sum_{out}\left(\dot{q}^+_e - \dot{q}^-_e\right) \quad + \quad \left(\dot{q}^+_{gen} - \dot{q}^-_{gen}\right) - \left(\dot{q}^+_{cons} - \dot{q}^-_{cons}\right)$$
$$\frac{d}{dt}q_{sys} = \underbrace{\sum_{in}\dot{q}_i - \sum_{out}\dot{q}_e}_{\text{Transport across boundaries}} \quad + \quad \underbrace{\dot{q}_{gen} - \dot{q}_{cons}}_{\substack{\text{Generation/Consumption}\\ \text{inside the system}}}$$

But we know that net charge is conserved so the generation and consumption terms on the right-hand side of Equation $4.2.5$ must be zero. Thus Equation $4.2.5$ can be written as the *rate-form* of the **conservation of net charge equation**:

$$\frac{d}{dt}q_{sys} = \sum_{in}\dot{q}_i - \sum_{out}\dot{q}_e$$

This is the primary equation for solving problems involving charge. We will often refer to "charge" without specifically saying "net charge". Unless we explicitly refer to positive or negative charge, the term charge should be interpreted as meaning net charge. We could also write Eq. $4.2.6$ using the long established current notation as

$$\frac{d}{dt}q_{sys} = \sum_{in}i_i - \sum_{out}i_e$$

In applying the conservation of charge equation, Equations $4.2.6$ or $4.2.7$, the approach is the same as we have used before. First you must clearly identify the system and think about the time period. Finite-time forms of the conservation of charge equation can be developed by integrating these rate forms of the conservation of charge equation with respect to time as done for mass. Be careful to clearly indicate the direction of all currents on your system diagram.

4.3: Physical Circuits and The Lumped Circuit Model

Based on "From Conservation to Kirchhoff: Getting Started in Circuits With Conservation and Accounting," by Bruce A. Black, 14 February 1996, published in the Proceedings of the 1996 ASEE Frontiers In Engineering Education Conference.

One of the most important and common applications of the conservation of charge equation is in the study of electrical circuits. It is a common experience in physics lab to connect electrical components we commonly call resistors, capacitors, and inductors along with various current and voltage sources, and to then study the behavior of this **physical circuit**. To predict the behavior of these physical circuits, we must have an accurate and mathematically tractable model. A distinct engineering science called *circuit theory* has been developed to model and analyze physical circuits.

Circuit theory is the vast collection of specialized techniques and results for analyzing and designing physical circuits that satisfy the *lumped circuit model*. The key assumptions underlying the **lumped circuit model** are listed below:

- Lumped circuits are constructed from discrete lumped circuit elements that are physically distinct and are connected only by wires.
- Electrical and magnetic energy is stored or converted to other forms of energy only within the circuit elements, i.e. we assume that no electric fields exist in the space outside the elements; that is, no electric fields exist either between the elements or between the elements and ground.
- No time-varying magnetic fields intersect any of the circuit loops.
- Lumped circuits are physically "small", i.e. the time scales of interest are much greater than the time for an electrical disturbance to propagate at the speed of light or for charges and currents to redistribute inside the devices.

Typical lumped circuit elements include resistors, inductors, capacitors, and voltage and current sources connected by wires. Remember that lumped circuit elements are models for the real physical components. Although physical circuits never exactly match the corresponding lumped circuit, the power of circuit theory is its ability to accurately predict the behavior of physical circuits for a wide range of conditions. In fact, their use is so prevalent and so accepted that we often forget the assumptions inherent in the model.

When the conservation of charge equation is applied to a lumped circuit element, the key lumped circuit assumption is that "no electric fields exist in the space outside the elements." *If a system boundary is not pierced by electric field lines, then it is impossible to change the amount of net charge inside the system, and charge cannot accumulate.* This is a consequence of Gauss's Law, which studied in physics, and is always true for a lumped circuit element. This result is commonly referred to as Kirchhoff's current law (KCL):

$$\sum_{\text{in}} i_i = \sum_{\text{out}} i_e \qquad \text{[Kirchhoff's Current Law]}$$

which follows directly from Eq. $4.1.7$ when $\dfrac{dq_{sys}}{dt} = 0$.

Note that we have not discussed Kirchhoff's voltage law (KVL): *The summation of voltage drops around any circuit loop is zero.* This is an extremely powerful tool for circuit analysis and is a direct consequence of the lumped circuit modeling assumptions. Although all of the circuits studied in this course must satisfy KVL because they are lumped circuits, we will not use this tool directly in solving for currents and voltages in our circuits. A more detailed discussion of this important result will be discussed in ES 203 - Electrical Systems.

4.4: Ohm's "Law" -- A Constitutive Relation

One of the most frequently used constitutive relationships in studying electrical phenomena is referred to as Ohm's Law. This, like every constitutive relationship, is not really a law but a well-worn and useful model for a specific physical phenomenon.

Ohm's Law relates three circuit variables: current, voltage, and resistance. For purposes of our discussion here, we will define voltage at a point A with respect to a reference point N as the value read by a "voltmeter" connected between the two points. The voltage at point A will be given the symbol V_{AN} to remind us that every voltage reading is always measured with respect to some reference, e.g. point N. A typical system is shown in Figure $4.4.1$.

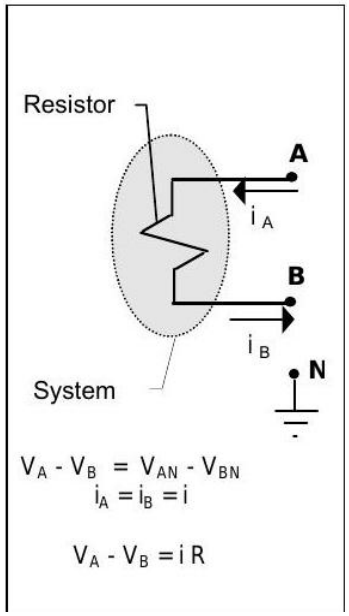


Figure $4.4.1$: A system consisting of a resistor within a physical circuit.

Empirical evidence for lumped circuit resistors shows that (1) there is no accumulation of charge in the resistor and (2) the current flowing through the resistor is related to the voltage change across a resistor by the following equation:

$$V_{\mathrm{AN}}-V_{\mathrm{BN}}=i R$$

where

- V_{AN} is the voltage at point A measured with reference to ground N, in volts
- V_{BN} is the voltage at point B measured with reference to ground N, in volts
- i is the electrical current flowing from point A $\rightarrow$ B, in amperes
- R is the electrical resistance, in ohms where 1 $\mathrm{volt}=(1\ \mathrm{A})\times(1\ \mathrm{ohm})$

Equation $4.4.1$ is known as Ohm's Law and the various terms in the equation are shown in Figure $4.4.1$. In a resistor, the magnitude of the voltage always decreases in the direction of the current flow, i.e. current flows *inside* a resistor from a high voltage to a lower voltage.

4.5: Simple Resistive Circuits

The only circuits we will study in this course are simple resistive circuits that satisfy the lumped circuit model. Subsequent courses, especially ES 203 - Electrical Systems, will introduce you to the more complicated circuits and more complete solution techniques of circuit theory.

Simple resistive circuits are constructed of resistors and current and voltage sources. A **current source** is a circuit element that maintains a specified current independent of the voltage change across its terminals. A **voltage source** is a circuit element that maintains a specified voltage difference across its terminals independent of the current flowing through the element. The junction where two or more circuit elements meet is called a **node**. The voltage at a node is called a **node voltage**.

The brute-force approach in solving a simple resistive circuit is to identify each node as a system and apply the conservation of charge to each system. In the circuit theory language, we would say "apply Kirchhoff's current law to each node." For N nodes, this should provide N independent equations to solve for N unknowns. If additional information is required, you can introduce node voltages using Ohm's Law to relate the branch current through a resistor to the resistance and the differences in node voltages across the resistor. Sometimes this process can be shortened by applying conservation of charge to larger systems that include several nodes. However, there is not a single node voltage for the system.

A simple algorithm is given here for solving simple resistive circuits that satisfy the lumped circuit model. Although it is not always the quickest approach, it will almost always lead you to an answer by brute force. In fact, you will learn other methods later. (In later courses on circuit analysis you will learn that this approach is called nodal analysis.) We will restrict our attention here to simple resistive circuits composed of current sources, voltage sources, and resistors. The following algorithm makes use of Kirchhoff's current law (conservation of charge for lumped circuits) and Ohm's law.

Solving Simple Resistive Circuits using Conservation of Charge and Ohm's law

- **Identify and label all nodes and branch currents.** Nodes have a single voltage and at least one current input and one current output. Be sure to indicate the direction of the current in each branch of the circuit. Branches connect nodes and for our purposes a branch will consist of a resistor, a voltage source, or a current source.
- **Apply conservation of charge (Kirchhoff's current law) to each node to relate the branch currents entering and leaving the node.** (Recall that by definition, a node cannot accumulate charge.) Applying Kirchhoff's current law to a circuit with N nodes results in at most N independent equations relating the branch currents in the circuit. If the entire circuit can be included in a system with no charge flowing across the system boundary, only $N-1$ independent equations can be obtained from applying KCL to the N nodes.
- **Apply the appropriate constitutive relationship to describe the behavior of each branch.**
  - **Resistors** - Relate the voltage drop across each resistance to the current flowing through the resistance using Ohm's resistance model. It is critical that your voltage drop correspond with the assumed direction of the current in a resistor. By definition, $i=\left(V_{in}-V_{out}\right)/R$ where the current flows into the resistor at a voltage V_{in} and leaves the resistor at a voltage V_{out}.
  - **Voltage Sources** - Specified voltage difference with no constraint on the direction or magnitude of the branch current
  - **Current Source** - Specified current direction and magnitude for the branch with no constraint on the voltage difference across the branch.

    (Note that even though we call these sources, they are not generating charge within our system. It would be more correct to think of current and voltage sources as being charge movers with no charge storage.)
- **Select one node as the zero point and arbitrarily set its voltage value to zero.** This is typically the grounded node. (For our purposes, we will assume that there is no current flowing into or out of ground. The existence of ground loops with non-zero current flowing often occurs in real systems; however, we will assume that our circuits are correctly grounded.) All node voltages are then defined with respect to this zero node.
- **Check to see if you have sufficient equations to handle the number of unknowns.** If you have followed this procedure a system with N nodes and B branches will have at most $N+B$ unknowns: N node voltages and B branch currents. The actual number of unknowns is reduces by the known voltages (including ground) and currents specified for the circuit.
- **Solve for the unknowns.** (Matrix algebra can be a real time saver for large problems especially where there is a simple pattern for setting up the equations. For small systems of equations, it may be quicker to just set up the equations and let MAPLE or

Mathematica solve them directly. The preferred method usually depends upon your familiarity with the software and how easily you can set up the matrices.)

- **Check your answers.** One approach is to examine a system that contains several nodes and then check to see that conservation of charge is satisfied.

The following example shows how to apply this technique to a simple resistive circuit that includes both a voltage source and a current source.

Example — Simple Circuit #1

Find the unknown branch currents and node voltages in the circuit shown in the figure.

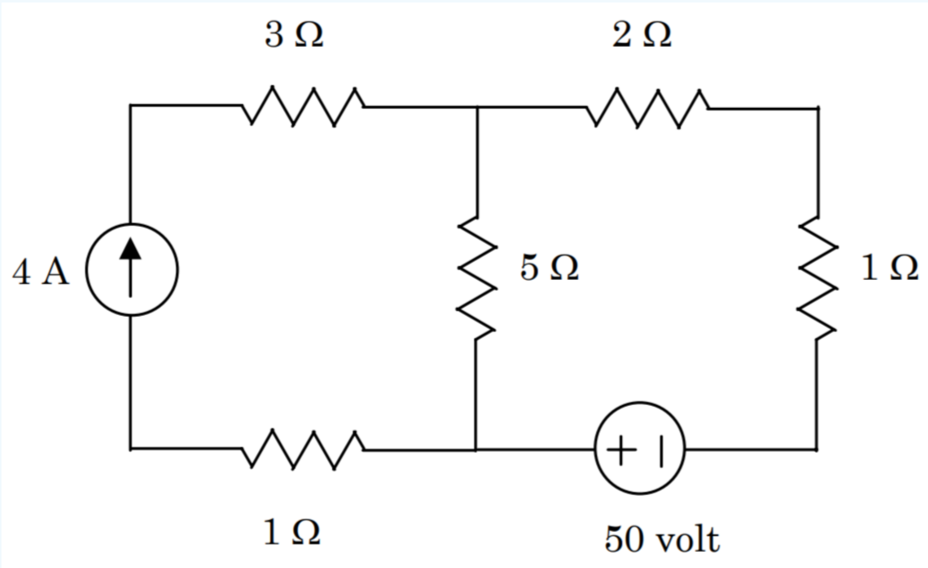


Figure $4.5.1$: Circuit diagram for a circuit with 6 nodes, 1 battery, 5 resistors, and one known branch current.

Solution

Strategy $\rightarrow$ This is clearly a circuit problem and will require application of conservation of charge along with suitable constitutive relationships.

First step is to label and locate all the nodes and branches. This gives

6 nodes: $V_1, V_2, V_3, V_4, V_{5,} V_6$

7 branches: $i_a, i_b, i_c, i_d, i_e, i_f, i_g$

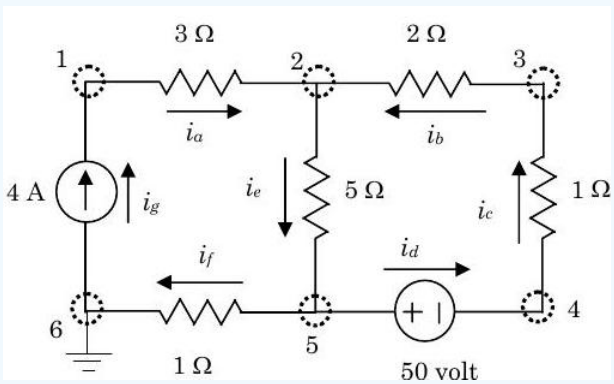


Figure $4.5.2$: Circuit diagram from above with all nodes and branch currents marked and labeled.

Writing conservation of charge for each node assuming no charge accumulation (lumped circuit assumption), we get six equations relating the currents:

$$\begin{aligned} &\text{Node 1:} \quad 0 = i_g - i_a &\qquad &\text{Node 4:} \quad 0 = i_d - i_c \\ &\text{Node 2:} \quad 0 = i_a + i_b - i_e &\qquad &\text{Node 5:} \quad 0 = i_e - i_f - i_e \\ &\text{Node 3:} \quad 0 = i_c - i_d &\qquad &\text{Node 6:} \quad 0 = i_f - i_g \end{aligned}$$

Note that we have assumed no current in the ground connection. Also note that only five of these six equations are independent equations.

Now we can *write the branch equations* by using the appropriate constitutive relations:

$$\begin{aligned} &\text{Branch a:} \quad (V_1 - V_2) = (3\ \Omega) i_a &\qquad &\text{Branch e:} \quad (V_2 - V_5) = (5\ \Omega) i_e \\ &\text{Branch b:} \quad (V_3 - V_2) = (2\ \Omega) i_b &\qquad &\text{Branch f:} \quad (V_5 - V_6) = (1\ \Omega) i_f \\ &\text{Branch c:} \quad (V_4 - V_3) = (1\ \Omega) i_c &\qquad &\text{Branch g:} \quad i_g = 4\text{ A} \\ &\text{Branch d:} \quad V_5 - V_4 = 50\text{ volts} \end{aligned}$$

All seven of the branch equations are independent.

We now have 13 unknowns and 12 equations (5 node and 7 branch). The remaining equation comes from assigning the *ground voltage* to node 6.

$$V_6 = 0$$

We should now have sufficient information to solve for the remaining voltages and currents. This is most easily done use a software package, e.g. MAPLE™ or EES.

MAPLE Solution:

There are 6 nodes in this problem (numbered 1,2,3,4,5,6). If we write conservation of charge for each node we will get 6 equations that relate the 6 unknown currents in the circuit:

```
>Node1:= 0= ig - ia;
Node2:= 0= ia + ib - ie;
Node3:= 0= ic - ib;
Node4:= 0= id - ic;
Node5:= 0= ie - iff - id;
Node6:= 0= iff - ig;
```

$$\begin{aligned} Node1 &:= 0 = ig - ia \\ Node2 &:= 0 = ia + ib - ie \\ Node3 &:= 0 = ic - ib \\ Node4 &:= 0 = id - ic \\ Node5 &:= 0 = ie - iff - id \\ Node6 &:= 0 = iff - ig \end{aligned}$$

Figure 4.5.3: Code in MAPLE to set up equations relating conservation of charge for each node to the circuit currents.

Now writing the 7 branch equations:

```
>
 Brancha:= (v1-v2)= ia*3;
Branchb:= (v3-v2)= ib*2;
Branchc:= (v4-v3)= ic*1;
Branchd:=  v5-v4 = 50;
Branche:= (v2-v5)= ie*5;
Branchf:= (v5-v6)= iff*1;
Branchg:=  ig = 4;
```

Figure 4.5.4: Code in MAPLE to set up equations for voltage drop in each branch of the circuit, based on constitutive relations.

There are 7 branch currents and 6 node voltages. Unfortunately only 5 of the 6 conservation of charge relations are independent, so we need an additional constraint. The remaining constraint is supplied by establishing a ground node with zero potential. Based on the circuit schematic this node should probably be Node 6.

```
> Ground:= v6=0;
```

$$Ground := v6 = 0$$

Figure 4.5.5: Code in MAPLE to assign a value of 0 to the ground node (node 6).

Now we can collect 12 independent equations to solve for the 6 currents and the 6 voltages.

```
>
evalf(solve({Node1,Node2,Node3,Node4,Node5,Brancha,Branchb,Branchc,Branchd,Branch
e,Branchf,Branchg,Ground},{v1,v2,v3,v4,v5,v6,ia,ib,ic,id,ie,iff,ig}),4);
```

{v6 = 0., ig = 4., v3 = -37.25, v1 = -7.750, v2 = -19.75, id = -8.750, ib = -8.750, v5 = 4.,
ia = 4., ic = -8.750, iff = 4., ie = -4.750, v4 = -46. }

Figure 4.5.6: Code to solve for the problem variables, using 5 of the conservation of charge equations, and the numerical solutions returned by the program.

To check our solutions, let's try some variations and see what happens. If any 5 of the conservation of charge equations work then we ought to be able to solve with a different set. Let's try for a couple of cases. First, let's try using Node 6 instead of Node 1:

```
>
evalf(solve({Node6,Node2,Node3,Node4,Node5,Brancha,Branchb,Branchc,Branchd,Branch
e,Branchf,Branchg,Ground},{v1,v2,v3,v4,v5,v6,ia,ib,ic,id,ie,iff,ig}),4);
```

{v6 = 0., ig = 4., v3 = -37.25, v1 = -7.750, v2 = -19.75, id = -8.750, ib = -8.750, v5 = 4.,
ia = 4., ic = -8.750, iff = 4., ie = -4.750, v4 = -46. }

Figure 4.5.7: Code to solve for the problem variables, using another set of 5 of the conservation of charge equations, and the numerical solutions returned by the program.

Now let's try using all of the nodes except Node 5:

```
>
evalf(solve({Node1,Node2,Node3,Node4,Node6,Brancha,Branchb,Branchc,Branchd,Branch
e,Branchf,Branchg,Ground},{v1,v2,v3,v4,v5,v6,ia,ib,ic,id,ie,iff,ig}),4);
```

$$\{v6 = 0., ig = 4., v3 = -37.25, v1 = -7.750, v2 = -19.75, id = -8.750, ib = -8.750, v5 = 4., ia = 4., ic = -8.750, iff = 4., ie = -4.750, v4 = -46.\}$$

Figure 4.5.8: Code to solve for the problem variables, using another set of 5 of the conservation of charge equations, and the numerical solutions returned by the program.

Looks OK. Now let's see what happens if we don't include the GROUND condition and don't put v_6 in the variable list:

```
>
evalf(solve({Node1,Node2,Node3,Node4,Node6,Brancha,Branchb,Branchc,Branchd,Branche
,Branchf,Branchg},{v1,v2,v3,v4,v5,ia,ib,ic,id,ie,iff,ig}),4);
```

$$\{v5 = v6 + 4., v2 = v6 - 19.75, v1 = -7.750 + v6, v3 = v6 - 37.25, v4 = v6 - 46., ig = 4., id = -8.750, ib = -8.750, ia = 4., ic = -8.750, iff = 4., ie = -4.750\}$$

Figure 4.5.9: Code from Figure 4.5.8 with v_6 omitted from the variable list and the GROUND condition omitted from the conditions list, and the solutions returned by the program. All voltage solutions are in terms of v_6.

Now let's try and see what happens if we just accidentally don't include the GROUND condition but we do include v6 in the variable list:

```
>
evalf(solve({Node1,Node2,Node3,Node4,Node6,Brancha,Branchb,Branchc,Branchd,Branch
e,Branchf,Branchg},{v1,v2,v3,v4,v5,v6,ia,ib,ic,id,ie,iff,ig}),4);
```

{v5 = v6 + 4., v2 = v6 – 19.75, v1 = –7.750 + v6, v3 = v6 – 37.25, v4 = v6 – 46., ig = 4.,
id = -8.750, ib = -8.750, ia = 4., ic = -8.750, iff = 4., ie = -4.750, v6 = v6 }

Figure 4.5.10: Code from Figure 4.5.8 with the GROUND condition omitted from the conditions list, and the solutions returned by the program. All voltage solutions are in terms of v_6.

Notice how all the voltages include a v_6 component. What does this mean?

Finally, what happens if we just leave out the variables list?

```
>
evalf(solve({Node1,Node2,Node3,Node4,Node6,Brancha,Branchb,Branchc,Branchd,Branch
e,Branchf,Branchg,Ground}),4);
```

$\{v6 = 0., ig = 4., v3 = -37.25, v1 = -7.750, v2 = -19.75, id = -8.750, ib = -8.750, v5 = 4.,$
$ia = 4., ic = -8.750, iff = 4., ie = -4.750, v4 = -46.\}$

Figure 4.5.11: Code from Figure 4.5.8 with the variables list removed, and the numerical solutions returned by the program.

Same answers, just in a different order.

4.6: Problems

Problem 4.6.1

The electrical device shown below is designed so that it can accumulate net charge. Initially the net charge on the device is zero.

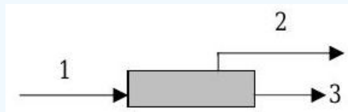


Figure 4.6.1: A device with one current flowing in and two currents flowing out of it.

The electric currents for the device are given below for the current arrows as shown on the diagram.

$$\begin{aligned}&\text{Wire 1: } i_1(\mathrm{t}) = (9\ \mathrm{mA}) \cdot \cos[t/(6\ \mathrm{s})]\\&\text{Wire 2: } i_2(\mathrm{t}) = (6\ \mathrm{mA}) \cdot \exp[-t/(6\ \mathrm{s})]\\&\text{Wire 3: } i_3(t) = (9\ \mathrm{mA}) \cdot \{1-\exp[-t/(6\ \mathrm{s})]\} \cdot \cos[(t/(6\ \mathrm{s})]\end{aligned}$$

(a) Develop an equation that describes the time rate of change of net charge in the device. Plot the rate of change versus time for $t=0$ to 60 s. Using this equation alone (and the associated plot), discuss what how you expect the net charge of the device to change for $t=0$ to $t>>0$.

(b) Develop an equation for the net charge on the device as a function of time and plot your result for 60 seconds. What is the net charge on the device at $t=6, 12, 18, 24, 30$ and 60 s ? How does this result compare with your predictions from part (a)?

[Note: Any graphs or plots must be complete plots with titles, labels., etc. Hand drawn or computer generated is acceptable.]

Problem 4.6.2

Two different circuits are shown below. Each black dot represents a node and the line connecting two nodes is a branch. The circuits are designed so that there is no accumulation of net charge anywhere in the circuit.

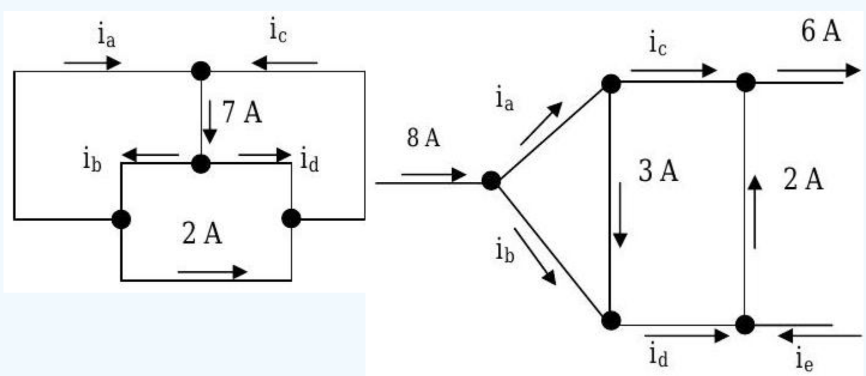


Figure 4.6.2: Circuit A (left) and Circuit B (right).

For each circuit, answer the following questions:

(a) Using only conservation of charge, what is the maximum number of independent equations that you can develop relating the currents in the circuit?

(b) How many unknown currents can you solve for in each problem?

(c) If you have sufficient information, solve for the unknown currents, in amps.

(d) Is there a *General Rule* here about number of nodes and number of independent current equation? What is it?

? Problem 4.6.3

Determine the value of the node voltages (measured with respect to ground) and the magnitude and direction of the current flowing in each resistor. Your final answer should include a diagram with labeled with arrows showing current direction and magnitudes and labeled node voltages.

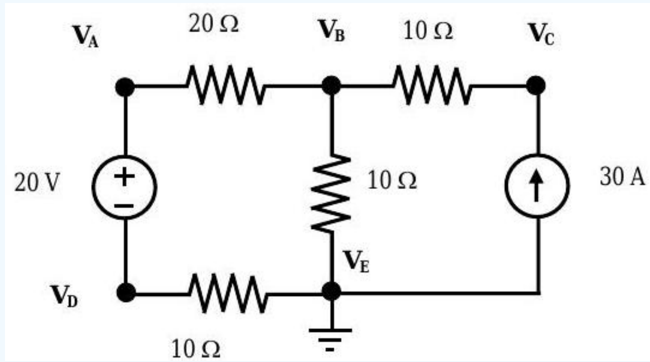


Figure 4.6.3: Circuit diagram with 5 node voltages and 5 branch currents to solve for.

? Problem 4.6.4

Determine the value of the node voltages (measured with respect to ground) and the magnitude and direction of the current flowing in each resistor. Your final answer should include a diagram with labeled with arrows showing current direction and magnitudes and labeled node voltages.

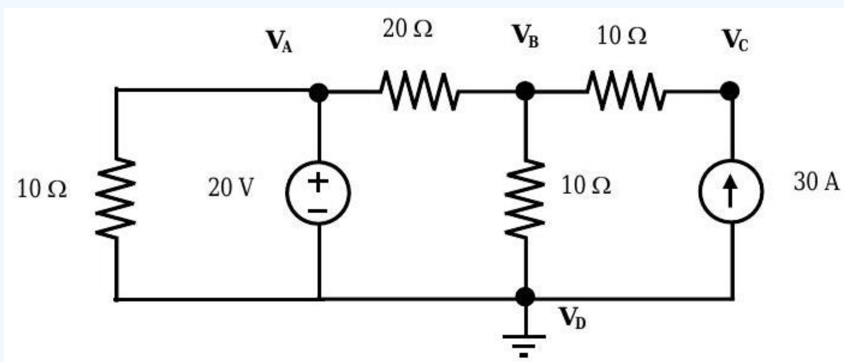


Figure 4.6.4: Circuit diagram with a total of 5 nodes and 7 branches.

? Problem 4.6.5

For the circuit shown below, solve for the unknown currents and voltages. Make the standard lumped circuit assumptions and assume that all resistances are known and have a value of 2 ohms.

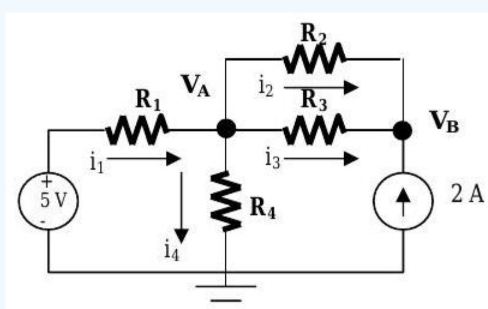


Figure 4.6.5: Circuit digram with 3 node voltages and 4 branch currents to solve for.

Problem 4.6.6

For the circuit shown, determine the currents i_1, i_2, and i_3 and the voltages at A, B, and C measured with respect to ground.

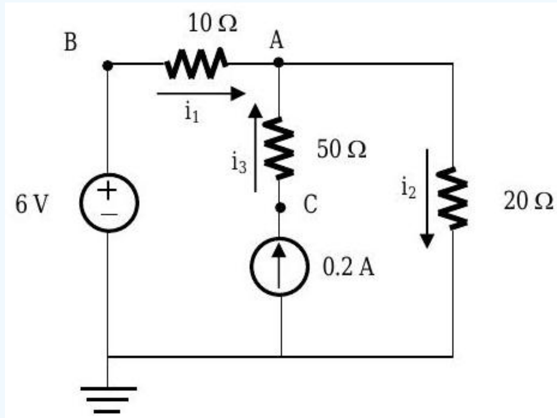


Figure 4.6.6: Circuit with 3 point voltages and 3 branch currents to solve for.

Problem 4.6.7

For the circuit shown, determine the unknown currents and node voltages.

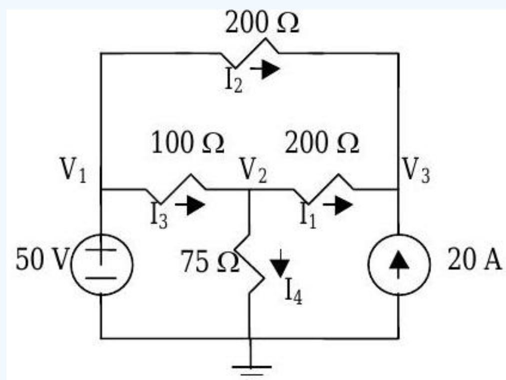


Figure 4.6.7: Circuit with 3 node voltages and 4 branch currents to solve for.

Problem 4.6.8

For the circuit shown below, determine the indicated branch currents and node voltages.

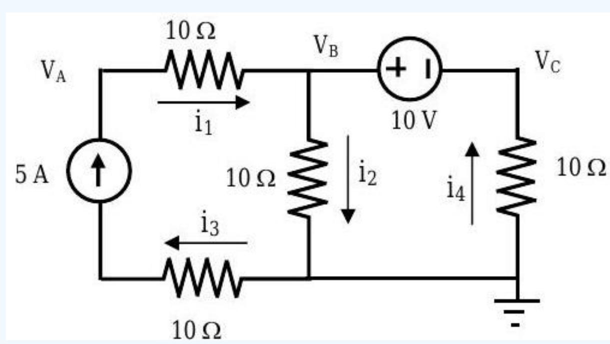


Figure 4.6.8: Circuit with 3 node voltages and 4 branch currents to solve for.

Problem 4.6.9

In the circuit shown, the source currents are $I_A = 3\text{ A}$ and $I_B = 5\text{ A}$. Determine I_1, I_2, and I_3.

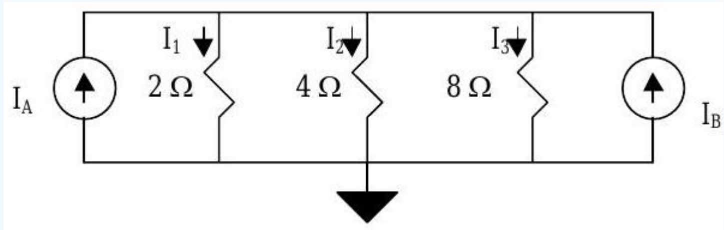


Figure 4.6.9: Circuit with 3 branch currents to solve for.

Problem 4.6.10

Using conservation of charge and Ohm's Law, find the current I_x and voltage V_x.

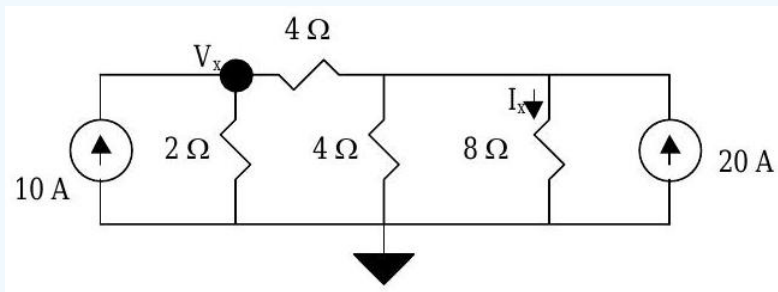


Figure 4.6.10: Circuit with one branch current and one node voltage to solve for.

Problem 4.6.11

Using conservation of charge and Ohm's Law, find the current I_x and voltage V_x.

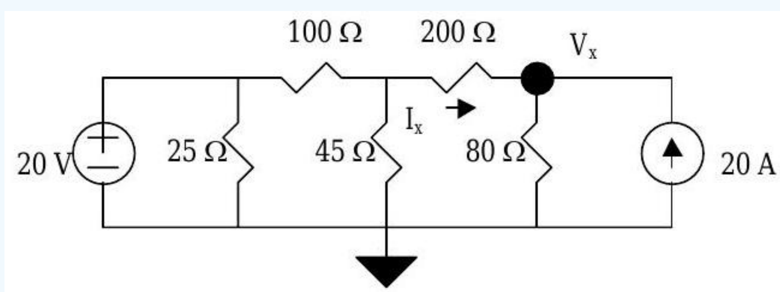


Figure 4.6.11: Circuit with one branch current and one node voltage to solve for.

Problem 4.6.12

Using conservation of charge and Ohm's Law, find the current I_x and voltage V_x.\

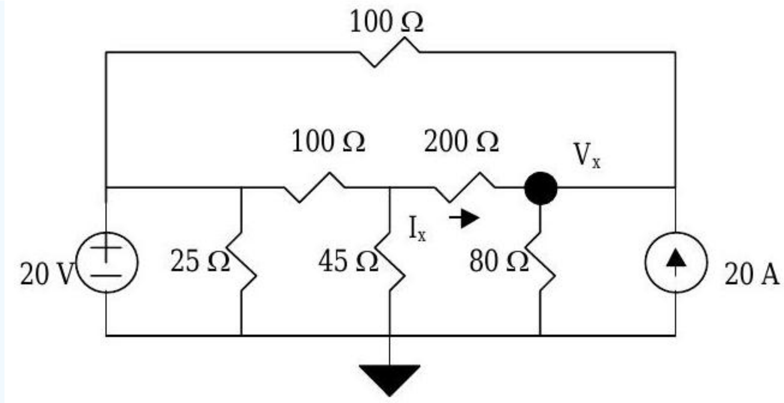


Figure 4.6.12: Circuit with one branch current and one node voltage to solve for.

Problem 4.6.13

Using conservation of charge and Ohm's Law, find the current I_x.

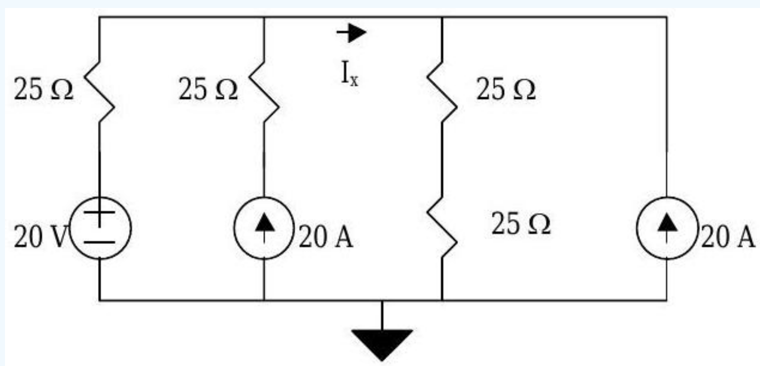


Figure 4.6.13: Circuit with one branch current to solve for.

Problem 4.6.14

Using conservation of charge and Ohm's Law, find the current I_x.

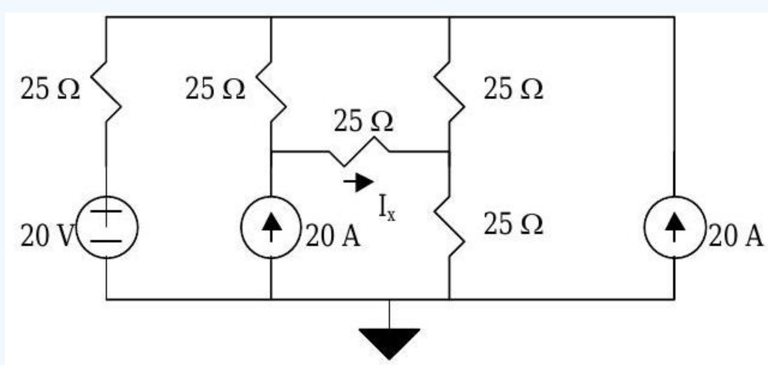


Figure 4.6.14: Circuit with one branch current to solve for.

CHAPTER OVERVIEW

5: Conservation of Linear Momentum

5.1: Four Questions

Newton's second law, commonly written as $F = ma$, is one of the most famous relations in physics. This relation, along with Newton's first and third laws, is typically the starting point for mechanics instruction in high school and undergraduate physics. Mechanics instruction is often broken down into three different areas: forces in stationary systems, motion of a particle subjected to a force, and impact problems. By focusing on each area individually, it is possible to begin to develop a physical understanding of the important physical phenomena.

When engineers apply these principles to realistic problems, they find it helpful to have a more general framework, one that provides a common basis for developing useful mathematical models. To provide this framework, we develop a physical law called the **Conservation of Linear Momentum** that, with few restrictions, is universally applicable. Then, as we did with mass and charge, we will explore the modeling assumptions that engineers commonly use when they build mathematical models of physical systems.

When developing an accounting concept for a new property, there are four questions that must be answered. When applied to linear momentum, the questions become

1. What is linear momentum?
2. How can linear momentum be *stored* in a system?
3. How can linear momentum be *transported*?
4. How can linear momentum be *created* or *destroyed*?

Once we have answered these questions we will have the appropriate accounting equation for the linear momentum of a system.

What is linear momentum?

As a starting point, let's consider the motion of a very specialized system, a particle. A **particle** is a closed system that behaves as if its volume was negligible and its mass was concentrated at a single point. The motion of a particle is completely described by its velocity, and all interactions between a particle and its surroundings occur at a single point. Later we will consider more realistic and larger extended systems for which the particle assumption is insufficient; however, we will find that the particle assumption is often a very good model.

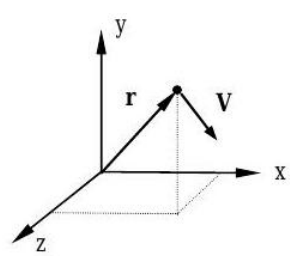


Figure $5.1.1$: Particle with mass m at position $\mathbf{r}$ with velocity $\mathbf{V}$.

For a particle with mass m and velocity $\mathbf{V}$ as shown Figure $5.1.1$, the **linear momentum** of the particle, P, is defined as the product of the particle mass and the particle velocity:

$$\mathbf{P} = m\mathbf{V}$$

The dimensions of linear momentum in terms of primary dimensions are the product of the dimensions of mass and velocity, $[\mathrm{M}][\mathrm{L}]/[\mathrm{T}]$. Typical units for linear momentum are $\mathrm{kg \cdot m/s}$ in SI and $\mathrm{slug \cdot ft/s}$ (or $\mathrm{lbm \cdot ft/s}$) in USCS.

It is important to note that the linear momentum $\mathbf{P}$, like velocity $\mathbf{V}$, is a **vector**. As shown in Figure $5.1.2$, vectors are typically represented by arrows. A vector is a mathematical expression that has a magnitude, a direction, and adds according to the parallelogram law. The **magnitude** of the vector describes the "length" of the arrow, and if it represents a physical quantity, the magnitude includes both a number and the associated units. The **direction** of a vector is specified in terms of its *line of action* and its *sense*. As shown in Figure $5.1.2$ vectors $\mathbf{P}_\mathrm{A}$ and $\mathbf{P}_\mathrm{B}$ have the same magnitude and same line of action but opposite sense. The magnitude of a vector is a positive number and is represented mathematically as $P_A = |\mathbf{P}_A|$. The complete vector is described by giving the magnitude and the direction of each vector: $\mathbf{P}_A = P_A \angle\theta = (5\ \mathrm{kg \cdot m/s}) \angle\theta$ and $\mathbf{P}_B = P_B \angle(\theta+\pi) = (5\ \mathrm{kg \cdot m/s}) \angle(\theta+\pi)$, where all angles are measured from the reference line.

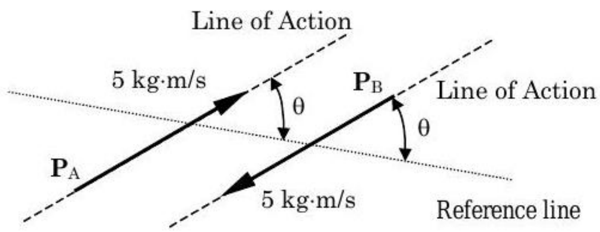


Figure 5.1.2: Magnitude and direction of a vector.

In addition to magnitude and direction, vectors that represent linear momentum and its transport also have a **point of application**. A vector with a point of application is called a bound vector. In summary, a linear momentum vector is completely specified on]ce we know its magnitude, direction, and point of application.

Because the linear momentum of a particle depends upon its mass, *linear momentum is an extensive property*. As with other extensive properties linear momentum has an intensive form. The linear momentum per unit mass, or the *specific linear momentum* $\mathbf{p}$, is defined as

$$\mathbf{p} \equiv \lim_{V_{\text{sys}} \to 0} \frac{\mathbf{P}_{\text{sys}}}{m_{\text{sys}}} = \mathbf{V}$$

Thus the velocity of a particle is really just the specific linear momentum of particle.

It is important to note that velocity, like position, is always measured relative to a reference point. When used to define the linear momentum, the velocity must be measured relative to an **inertial reference frame**. For our purposes this means a coordinate system that is not rotating or accelerating with respect to the Earth, i.e. an Earth-fixed coordinate system. In addition, any coordinate system that is moving with a constant velocity with respect to the Earth is also an inertial reference frame.

As we will show later, it is frequently useful to attach the coordinate system to a moving system *if* the system is moving with a constant velocity. Under these conditions all velocities are computed *relative* to the moving coordinate system. For motion of spacecraft you would need to use a Sun-centered, non-rotating reference frame. See any undergraduate physics textbook to learn more about the conditions of an inertial reference frame.

5.1.1 Kinematics of a particle

Because of the relationship of velocity to linear momentum, an understanding of kinematics will play a crucial role in our study of linear momentum. Kinematics is the study of the geometry of motion without regard to the cause of the motion and provides the relationship between the position r, velocity V, and acceleration a of a point. The velocity vector V is related to the position vector r through the basic kinematic relationship you learned in physics—velocity is the first derivative of position:

$$\mathbf{V} = \frac{d\mathbf{r}}{dt}$$

The dimensions of velocity are length per unit time, $[\mathrm{L}]/[\mathrm{T}]$. Typical units for velocity are $\mathrm{m/s}$ in SI or $\mathrm{ft/s}$ in USCS.

The first derivative of velocity (or the second derivative of position) is the acceleration vector $\mathbf{a}$:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)$$

The dimensions of acceleration are length per unit time squared, $[\mathrm{L}]/[\mathrm{T}]^2$. Typical units for acceleration are $\mathrm{m/s^2}$ in SI or $\mathrm{ft/s^2}$ in USCS.

Rectilinear motion

In rectilinear motion, a particle can only move along a straight line. Under these conditions only one spatial coordinate, say x, is required to describe the particle position, and the velocity and acceleration are written as

$$V = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dV}{dt} = \frac{d^2x}{dt^2}$$

These relationships are most useful when position x, velocity V, and acceleration a are all functions of time. Under conditions of rectilinear motion, a positive numerical value for the velocity implies that the particle is moving in the same direction as the positive x-axis. Similarly, a positive numerical value for the acceleration implies that the velocity of the particle is increasing as it moves in the direction of the positive x-axis.

An alternate expression for acceleration is sometimes useful if acceleration and velocity are known as functions of position, e.g. $V = V(x)$ and $a = a(x)$. Under these conditions the acceleration can be written as

$$\begin{aligned} &= f_x(t) \\ V &= f_V(x) \quad \rightarrow \quad a \equiv \tfrac{dV}{dt} = \overbrace{\left(\tfrac{dV}{dx}\right)\underbrace{\left(\tfrac{dx}{dt}\right)}_{=V}}^{\text{Chain Rule of Differentiation}} = V\tfrac{dV}{dx} \\ a &= f_a(x) \end{aligned}$$

using the chain rule of differentiation from calculus. Note that the notation "$f(t)$" is read as "a function of t."

Example — Rectilinear Kinematics

One of the common tasks of kinematics is to integrate the acceleration (time rate of change, or first derivative, of velocity) of a point with respect to time or with respect to position. Do not memorize the formulas described below. Each of the results is different but the approach is the same. Learn the process, not the result!

Case I: Acceleration is known as a function of time, e.g. $a = f(t)$. To find the velocity, integrate with respect to time as follows:

$$a = f(t) = \frac{dV}{dt} \rightarrow dV = f(t)dt \quad \rightarrow \quad V - V_o = \int_{t_o}^{t} f(t)\, dt$$

Practice: (a) Given that $a = bt$ and $V = V_o$ at $t = 0$ s, develop an equation for the velocity V. (b) Determine the velocity at $t = 5$ s, if $b = 15\ \mathrm{m/s^3}$ and $V_o = 20\ \mathrm{m/s}$.

Case II: Acceleration is known as a function of position, e.g. $a = f(x)$. The velocity can be determined by integrating with respect to position as below:

$$a = f(x) = V\frac{dV}{dx} \quad \rightarrow \quad V\,dV = f(x)dx \quad \rightarrow \quad \frac{V^2}{2} - \frac{V_o^2}{2} = \int_{x_o}^{x} f(x)\, dx$$

Practice: (a) Given that $a = cx$ and $V = V_o$ at $x = 0$, develop an equation for the velocity V. (b) Determine the velocity at $x = 5$ m, if $c = 10\ \mathrm{s^{-2}}$ and $V_o = 10\ \mathrm{m/s}$.

Case III: Acceleration is known as a function of velocity, e.g. $a = f(V)$. The velocity can be determined by integrating with respect to time or position as below:

$$\begin{aligned} a = f(V) = \frac{dV}{dt} \qquad &\rightarrow \quad dt = \frac{dV}{f(V)} \qquad &\rightarrow \quad t - t_o = \int_{V_o}^{V} \frac{dV}{f(V)} \\ a = f(V) = V\frac{dV}{dx} \qquad &\rightarrow \quad dx = \left[\frac{V}{f(V)}\right] dV \qquad &\rightarrow \quad x - x_o = \int_{V_o}^{V} \left[\frac{V}{f(V)}\right] dV \end{aligned}$$

Practice: (a) Given that $a = bV^2$ and $V = V_o$ at $x = x_o$, develop an equation for the velocity V. (b) Given that $a = bV^2$ and $V = V_0$ at $t = t_o$, develop an equation for the velocity V.

Case IV: Acceleration is a constant, e.g. $a = \text{constant}$. The velocity can be determined by integrating with respect to time or position as below:

$$a = \text{constant} \quad \rightarrow \quad \begin{array}{lllll} a = \dfrac{dV}{dt} & \rightarrow & dV = a\,dt & \rightarrow & V - V_o = a\,(t - t_o) \\ a = V\dfrac{dV}{dx} & \rightarrow & V\,dV = a\,dx & \rightarrow & \dfrac{V^2}{2} - \dfrac{V_o^2}{2} = a\,(x - x_o) \end{array}$$

Practice: Starting with the definition of velocity $V = dx/dt$, prove that the position can be described by an equation of the form $x - x_o = V_o t + (1/2)at^2$.

Plane curvilinear motion

In **plane** (or two-dimensional) **curvilinear motion**, the motion is constrained to a plane. Writing the position vector $\mathbf{r}$ (Figure $5.1.3$) for a point in the plane in terms of its rectangular components gives

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the two orthogonal unit vectors and x and y are the corresponding distances measured in the direction of each unit vector.

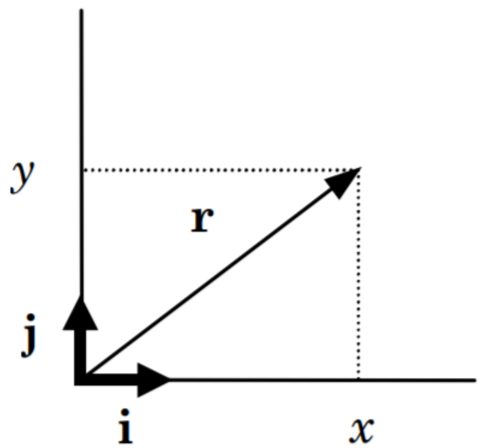


Figure $5.1.3$: Rectangular coordinates

Differentiating Eq. $5.1.7$ as required by Eq. $5.1.3$ gives the velocity vector in terms of its components

$$\begin{aligned} \mathbf{V} &= \frac{d}{dt}(x\mathbf{i} + y\mathbf{j}) \\ &= \frac{d(x\mathbf{i})}{dt} + \frac{d(y\mathbf{j})}{dt} = \left(\frac{dx}{dt}\mathbf{i} + x\,\frac{d\mathbf{i}}{\cancel{dt}}^{=0}\right) + \left(\frac{dy}{dt}\mathbf{j} + y\,\frac{d\mathbf{j}}{\cancel{dt}}^{=0}\right) \\ &= \left(\frac{dx}{dt}\right)\mathbf{i} + \left(\frac{dy}{dt}\right)\mathbf{j} \\ &= V_x\mathbf{i} + V_y\mathbf{j} \end{aligned} \tag{5.1.1}$$

where V_x and V_y are the velocity components in the x and y directions, respectively. Recall from your study of calculus that the derivative of a constant, such as the unit vector $\mathbf{i}$, is identically zero. Similarly, it can be shown that the acceleration vector has the components:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{V}}{dt} = \frac{d}{dt}(V_x\mathbf{i} + V_y\mathbf{j}) \\ &= \frac{dV_x}{dt}\mathbf{i} + \frac{dV_y}{dt}\mathbf{j} \\ &= a_x\mathbf{i} + a_y\mathbf{j} \end{aligned} \tag{5.1.2}$$

where a_x and a_y are the velocity components in the x and y directions, respectively.

Relative Motion

Recall that when you are computing the linear momentum of a system all velocities must be measured with respect to an inertial, i.e. non-accelerating, reference frame. Thus, *if the velocity is measured in an accelerating reference frame, it will be necessary to determine the absolute velocity before we can compute the system linear momentum*. Furthermore, if a system is translating with constant velocity, it will sometimes simplify a problem to compute the linear momentum with respect to this moving, inertial reference frame.

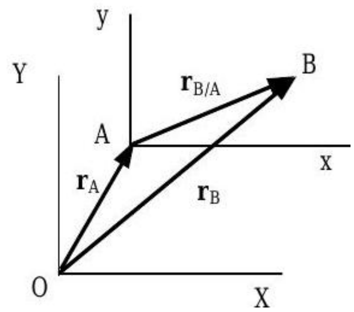


Figure 5.1.4: Relative position.

The relative position vector $\mathbf{r}_{B/A}$ of point B with respect to point A (See Figure 5.1.4) is defined by the relation:

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A$$

where the position vector $\mathbf{r}_A$ of point A and position vector $\mathbf{r}_B$ of point B are both measured with respect to point O, the origin of coordinate system XYZ. Given the relative position of point B with respect to point A and the absolute position of point A, Eq. 5.1.10 can be rewritten as below to find the absolute velocity of point B :

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Similar relations can be written for the relative and absolute velocities of points A and B :

$$\mathbf{V}_{B/A} = \mathbf{V}_B - \mathbf{V}_A \rightarrow \qquad \mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{B/A} \tag{5.1.3}$$

$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A \rightarrow \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{5.1.4}$$

Example — Linear momentum of two particles with relative motion

Imagine that a railroad flatcar is accelerating and moving to the right with an instantaneous velocity of $5 \mathrm{~m/s}$ as measured by an observer on the ground. A forklift located on the bed of the flatcar is moving with a velocity of $3 \mathrm{~m/s}$ as measured by an observer sitting on the moving flatcar. Both observers see left and right directions similarly.

Determine the absolute velocity of the forklift, i.e. the velocity of the forklift as measured by an observer on the ground,

(a) if the observer sitting on the flatcar sees the forklift moving to her left and

(b) if the observer sitting on the flatcar sees the forklift moving to her right.

Solution

The following equation defines the relative velocity of the forklift with respect to the flatcar:

$$\mathbf{V}_{\text{forklift / flat car}} = \mathbf{V}_{\text{forklift}} - \mathbf{V}_{\text{flatcar}} \quad \rightarrow \quad \mathbf{V}_{\text{forklift}} = \mathbf{V}_{\text{flatcar}} + \mathbf{V}_{\text{forklift / flatcar}}$$

Now assume that the positive axis for our coordinate system points to the right.

(a)

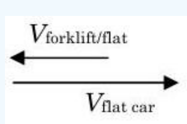


Figure 5.1.5: Forklift moves to the left as observed by an observer on the flat car.

$$\begin{aligned} \rightarrow + \quad V_{\text{forklift}} &= V_{\text{flatcar}} - V_{\text{forklift / flatcar}} \\ &= (5 \text{ m/s}) - (3 \text{ m/s}) \\ &= 2 \text{ m/s} \\ \mathbf{V}_{\text{forklift}} &= 2 \text{ m/s to the right.} \end{aligned}$$

(b)

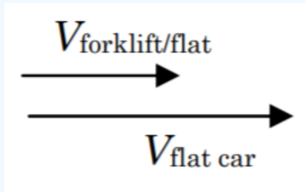


Figure 5.1.6: Forklift moves to the right as observed by an observer on the flat car.

$$\begin{aligned} \rightarrow + \quad V_{\text{forklift}} &= V_{\text{flatcar}} + V_{\text{forklift / flatcar}} \\ &= (5 \text{ m/s}) + (3 \text{ m/s}) \\ &= 8 \text{ m/s} \\ \mathbf{V}_{\text{forklift}} &= 8 \text{ m/s to the right.} \end{aligned}$$

5.1.2 How can linear momentum be stored in a system?

Linear momentum is stored in the motion of the mass of a system; thus any system that has mass has the ability to store linear momentum. Because it is an extensive property, the linear momentum of a system can be found by adding up the linear momentum of each mass inside the system. For a system consisting of N particles, the linear momentum of the system $\mathbf{P}_{\text{sys}}$ is found by summing the linear momentum of the $\mathbf{N}$ particles in the system,

$$\mathbf{P}_{\text{sys}} = \sum_{j=1}^{N} \mathbf{P}_j = \sum_{j=1}^{N} m_j \mathbf{V}_j$$

For a continuous system, this summation is performed using an integral over the system volume:

$$\mathbf{P}_{\text{sys}} = \int_{\forall_{\text{sys}}} (\mathbf{V}\rho)\, d\forall$$

where in general velocity and density can vary with both time and position.

Center of mass

To calculate the linear momentum for an extended system, i.e. something other than a particle, it is necessary to evaluate the integral in Eq. 5.1.15. Fortunately this process can be greatly simplified by introducing the concept of center of mass. The **center of mass** is the point where all of the mass for a system can be assumed to be concentrated for purposes of calculating the linear momentum. As we will show later, this makes the evaluation of the linear momentum a simpler task.

The **velocity of the center of mass**, $\mathbf{V}_G$, is defined by the following equation:

$$\mathbf{P}_{sys} = m_{sys}\mathbf{V}_G \equiv \int_{\forall_{sys}} (\mathbf{V}\rho)\, d\forall \quad \rightarrow \quad \mathbf{V}_G = \frac{1}{m_{sys}} \int_{\forall_{sys}} (\mathbf{V}\rho)\, d\forall$$

Thus the velocity of the center of mass equals the linear momentum of the system divided by the mass of the system.

The position of the center of mass, $\mathbf{r}_G$, is the *mass-weighted average* of the position of all of the mass within the system:

$$\mathbf{r}_G = \frac{1}{m_{sys}} \int_{\forall_{sys}} \mathbf{r}\rho\, d\forall$$

On the surface of the earth, the location of the center of mass is essentially the same as the location of the **center of gravity**. From experience, we know that an object suspended from its center of gravity has no tendency to rotate, i.e. it is perfectly balanced. When the density of a system is uniform, the location of the center of mass is identical with the **centroid** of the volume of the system. The location of the centroid for most common shapes has been tabulated and can be found in mathematical and physical handbooks.

Example — Linear momentum of a system of particles

A system consists of three particles with mass, velocity, and positions as indicated in the table.

Particle	m (kg)	$\mathbf{r}$ (m)	$\mathbf{V}$ (m/s)
1	15	$3\mathbf{i}+3\mathbf{j}$	$5\mathbf{i}+5\mathbf{j}$
2	10	$-4\mathbf{i}+2\mathbf{j}$	$3\mathbf{i}-5\mathbf{j}$
3	5	$1\mathbf{i}+2\mathbf{j}$	$2\mathbf{i}-2\mathbf{j}$

Determine the following for the system:

(a) Linear momentum, in $\mathrm{kg \cdot m/s}$.

(b) Velocity of the center of mass, in $\mathrm{m/s}$.

(c) Location of the center of mass, in m.

Solution

Known: Description of a system with pertinent information

Find: Calculate the linear momentum for each system.

Given: See below in each example.

Analysis

(a) Use the definition of linear momentum.

$$\begin{aligned}\mathbf{P}_{\mathrm{sys}} &= \mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3\\ &= m_1\mathbf{V}_1+m_2\mathbf{V}_2+m_3\mathbf{V}_3\\ &= (15\ \mathrm{kg})\left[\left(5\ \frac{\mathrm{m}}{\mathrm{s}}\right)\mathbf{i}+\left(5\ \frac{\mathrm{m}}{\mathrm{s}}\right)\mathbf{j}\right]+(10\ \mathrm{kg})\left[\left(3\ \frac{\mathrm{m}}{\mathrm{s}}\right)\mathbf{i}-\left(5\ \frac{\mathrm{m}}{\mathrm{s}}\right)\mathbf{j}\right]+(5\ \mathrm{kg})\left[\left(2\ \frac{\mathrm{m}}{\mathrm{s}}\right)\mathbf{i}-\left(2\ \frac{\mathrm{m}}{\mathrm{s}}\right)\mathbf{j}\right]\\ &= [(75+30+10)\mathbf{i}+(75-50-10)\mathbf{j}]\left(\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\right)\\ &= (115\mathbf{i}+15\mathbf{j})\left(\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\right)\ \text{or}\ \left(115\ \frac{\mathrm{kg\cdot m}}{\mathrm{s}}\right)\mathbf{i}+\left(15\ \frac{\mathrm{kg\cdot m}}{\mathrm{s}}\right)\mathbf{j}\end{aligned}$$

(b) Use the defining equation for velocity of the center of mass

(c) Use the defining equation for location of center of mass

Plane motion of a rigid body

Many physical systems of interest can be modeled as a rigid body. A r**igid body** is a body that does not deform, i.e., the relative distance between any two points on the body remains a constant.

In this course we will restrict most of our discussions to what is referred to as plane motion. Plane motion occurs when any point in the body travels in a plane and the planes formed by the locus of each point in the body are all parallel to a common reference plane. The plane motion of a rigid body can be classified as *translation*, *rotation about a fixed axis*, or *general plane motion*, which is a combination of translation and rotation; see Figure 5.1.7:

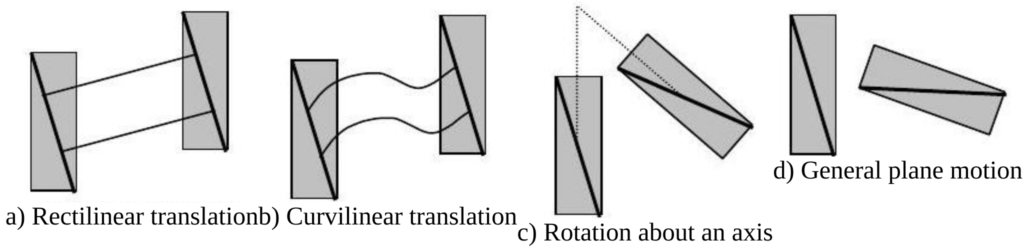


Figure 5.1.7: Plane motion of a rigid body.

1. **Translation.** If any straight line inside the body remains parallel to itself as the body moves, we say the motion of the body is translation. If a point in the body also follows a straight line, we say the motion is **rectilinear translation**. If on the other hand, a point in the body follows a curved path, we say the motion is **curvilinear translation**. The velocity for every point in a translating body is written in rectangular coordinates as $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j}$.
2. **Rotation about a fixed axis.** If all points in a rigid body move in parallel planes and follow concentric circles about the same fixed axis, we say the body is rotating about a fixed axis. We call the fixed axis the *axis of rotation*. Under these conditions the velocity of each point in the body can be described by the relation $\mathbf{V} = r\omega\mathbf{e}_\theta = r\omega(-(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j})$
3. **General plane motion.** Experience has shown that the plane motion of *any* rigid body can be described as a combination of rotation of a rigid body around a translating axis of rotation.

The kinematics of general plane motion is complicated and its study will be delayed until a subsequent course. We will spend most of our time addressing rectilinear translation and rotation about a fixed axis. Because of this, it is important for you to be able to identify the three types of planar motion.

5.1.3 How can linear momentum be transported across a system boundary?

Linear momentum can be transported across a system boundary by two different mechanisms: forces and mass flow. As with linear momentum, both forces and mass transports of linear momentum are vectors and to specify each requires that the magnitude, direction (sense and line of action) and the point of application be specified.

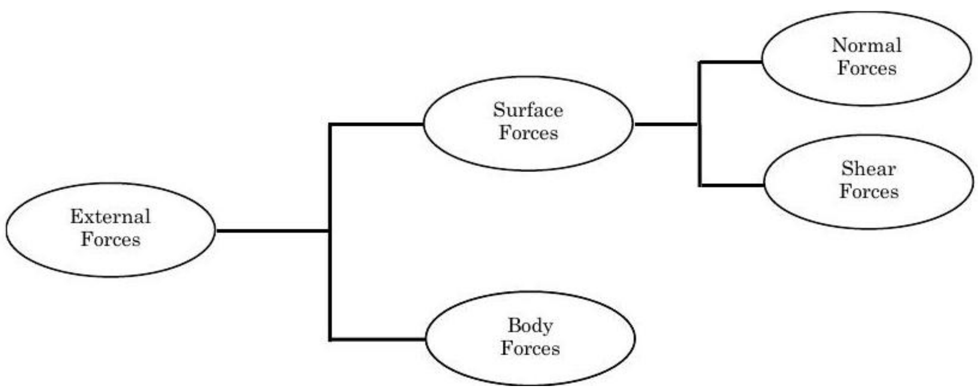


Figure 5.1.8: Classification of External Forces

Transport of linear momentum with force

Experience has shown that the linear momentum of a closed system can only be changed by the application of an external force (This is how Newton originally stated his second law). Thus an external force applied to a closed system must transfer linear momentum between the system and its surroundings. More specifically,

$$\mathbf{F}_{\text{external}} \equiv \dot{\mathbf{P}}_{\text{Force}} = \left\{ \begin{array}{c} \text{transport rate of linear momentum} \\ \text{across a boundary due to a force} \end{array} \right\}$$

where the direction of the momentum flow is in the same direction as the force.

An **external force** is a mechanism for transferring linear momentum and is a transport rate for linear momentum across the boundary between a system and its surroundings. A force is a *vector* with all of the properties discussed earlier. The primary dimensions for force are $[\mathrm{M}][\mathrm{L}]/[\mathrm{T}]^2$, which are the dimensions of [Linear Momentum]/[Time]. The typical unit for force is a newton (N) in SI and a pound-force (lbf) in USCS.

When several forces act on a system, the net transport rate of linear momentum into a system with the external forces is written mathematically as below:

$$\sum_j \mathbf{F}_{\text{ext } j} \equiv \left\{ \begin{array}{c} \textit{net} \text{ transport rate of linear momentum} \\ \text{into the system with force} \end{array} \right\}$$

External forces, as shown in Figure $5.1.8$, can be classified as either surface forces or body forces. Surface forces can be further classified as either normal or shear forces.

Body Forces

A **body force** always acts in a distributed fashion within the volume of the system and is the result of placing the system in a force field, e.g. a gravitational field, electric field, or magnetic field. Because of this a body force is sometimes called a "force at a distance." Since it acts over the volume of the system, the body force can only be calculated once the system is defined.

The most commonly occurring body force is **weight** — the force exerted by the Earth on a mass placed in the Earth's gravitational field. Although a body force acts throughout the system, its effect is frequently represented as a concentrated force that acts at a specified point *within* the system. When the field strength is spatially uniform as it is in the Earth's gravitational field, the weight of an object is assumed to act at the *center of gravity* of the system. Near the Earth's surface, when an object is suspended from its center of gravity, the object will be perfectly balanced.

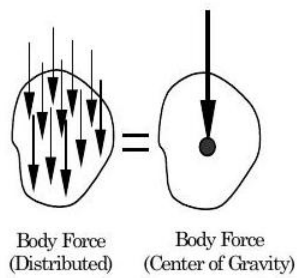


Figure $5.1.9$: Body force due to gravity

Example — Weight as a body force

Determine the magnitude, direction, and point of application of the weight vector for systems A, B, and C. Each system is made of a material with a density of 900 kg/m^3. In each case, sketch your system and locate the center of gravity. [Hint: You might try your physics book, your calculus or a good mathematics table to find the location of the centroid of these volumes.]

System A	System B	System C
Right circular cylinder with diameter $D = 20$ cm and height $H = 40$ cm.	A right circular cone with diameter $D = 20$ cm and height $H = 40$ cm.	A hemisphere of diameter $D = 20$ cm.

Surface Forces

A **surface force** is characterized by having a point or points of physical contact on the boundary surface, and for this reason it is sometimes referred to as a *contact force*. Physically, surface forces always act over a finite area, i.e. they are distributed forces.

Typically, we usually represent the surface forces by a single concentrated force (see Figure 5.1.10). When a surface force acts on a plane surface and either the area is small or the force is uniformly distributed over the area, it is reasonable to model the distributed force as a single concentrated force with a point of application at the centroid of the area.

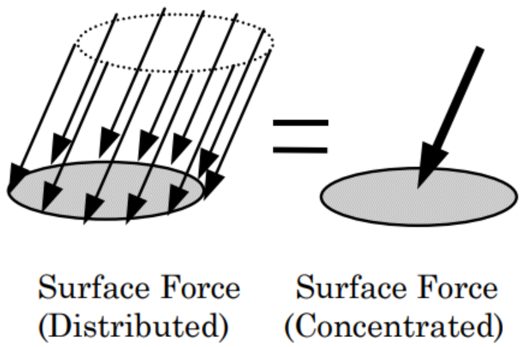


Figure 5.1.10: Surface forces

When the surface is curved or the distributed force is non-uniform, a single concentrated force can still be used to model the distributed force; however, the exact point of application must be carefully computed and is not necessarily the centroid of the area over which the distributed force acts.

Every surface force can, if necessary, be further decomposed into two components: a normal and a shear component, as shown in Figure 5.1.11. The line of action of a **normal force** is perpendicular to the surface at the point of application, and the line of action of a **shear force** is tangent to the surface at the point of application. Some external forces are by their very nature purely normal or purely shear forces. For example, the force produced by the pressure of the atmosphere on any surface is always a normal force, while the force due to friction is always a shear force.

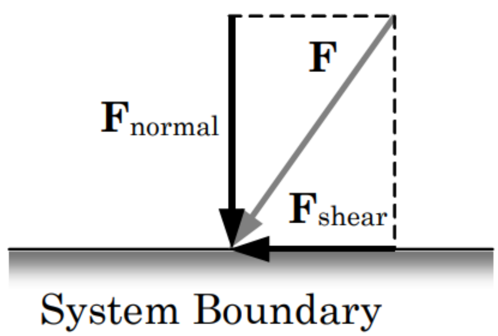


Figure 5.1.11: Normal and shear forces.

Surface Forces due to Pressure

One of the most ubiquitous forces in our lives is the force of the atmosphere. Because of this we must consider how to determine surface forces due to pressure. By definition, **pressure** is defined as the force per unit area. The dimensions for pressure are $[\mathrm{F}]/[\mathrm{L}]^2 = \{[\mathrm{M}][\mathrm{L}]/[\mathrm{T}]^2\}/[\mathrm{L}]^2 = [\mathrm{M}][\mathrm{L}]^{-1}[\mathrm{T}]^{-2}$. Typical units for pressure are newton per square meter or pascals (Pa) in SI with $1\ \mathrm{Pa} = 1\ \mathrm{N/m^2}$. In USCS, the units are pounds-force per square inch (psi or $\mathrm{lbf/in^2}$) or pounds-force per square foot (psf or $\mathrm{lbf/ft^2}$) with $144\ \mathrm{psi} = 144\ \mathrm{lbf/in^2} = 1\ \mathrm{lbf/ft^2}$.

The standard value of atmospheric pressure on the surface of the earth is $P_{\text{atm}} = 101325\ \mathrm{Pa} = 101.325\ \mathrm{kPa} = 14.696\ \mathrm{lbf/in^2}$. These values are frequently rounded off to $101.3\ \mathrm{kPa}$ and $14.7\mathrm{lbf/in^2}$.

To evaluate the force due to pressure acting on a surface we must first define the surface (see Figure 5.1.12). Mathematically, the force is calculated using the integral of the pressure over the surface area as below:

$$\mathbf{F}_{\text{pressure}} = -\int_{A_{\text{surface}}} \mathbf{n}P\, dA$$

where P is the pressure acting on the boundary, $\mathbf{n}$ is the unit normal vector that points out from the system into the surroundings, and dA is the differential area over which the pressure acts. Thus, *the direction of the pressure force is always into the system*.

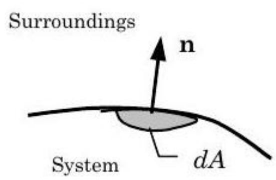


Figure $5.1.12$: Geometry for pressure force calculation.

The simplest and also the most common application is one where the pressure is uniform *and* the surface is flat. Under these conditions Eq. $5.1.20$ can be greatly simplified since both the unit normal vector and the pressure come outside the integral:

$$\begin{aligned}\mathbf{F}_{\text{pressure}} &= -\int_{A_{\text{surface}}} \mathbf{n}P\, dA \\ &= -(\mathbf{n}P)\underbrace{\int_{A_{\text{surface}}} dA}_{=A_{\text{surface}}} = -PA_{\text{surface}}\mathbf{n}\end{aligned} \tag{5.1.5}$$

In words, this says that the pressure force on a flat surface due to a uniform pressure is a vector that is perpendicular to the surface, points into the system, and has a magnitude equal to PA_{surface}, , the product of the pressure and the surface area. Furthermore, it can be shown (after we study angular momentum) that the point of application of the force is at the centroid of the surface area.

As an example, consider a triangular solid with constant width into the paper as shown in Figure $5.1.13$. A uniform pressure acts on each of the three sides as shown in the figure. Since each pressure is uniform and the surfaces are flat the equivalent pressure force vector can be calculated using Eq. $5.1.21$. For each side, the force vector points into the surface, is normal to the surface, and has a magnitude equal to the product of the pressure and the surface area over which the uniform pressure acts.

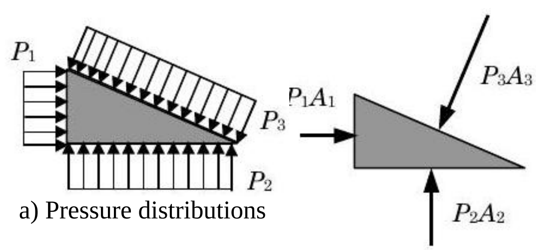


Figure 5.1.13: Forces due to uniform pressure.

Frequently, we will have systems that are subjected to atmospheric pressure over the entire system boundary. Imagine for a minute any particular object you would like, say your car or an apple hanging on a string, that is subjected to a uniform atmospheric pressure on all sides. What is the *net* force of the atmosphere on the object? Based on your experience, I assume you would say the net force due to the atmosphere pressure is zero. You can repeat this experiment and you will find that *the net force on any object surrounded by a uniform pressure will be zero regardless of the shape of the object or the value of the pressure*.

This leads us to another interesting result. Consider the irregular shaped solid shown in Figure $5.1.14$. Imagine that the object is subjected to a uniform pressure on all sides and that the resulting pressure force vectors are shown on the figure. Now we know from experience that if the only forces are due to the uniform pressure then there is no net force on the object due to the pressures. If this were *not* true, then the object would have a tendency to move (and we would patent this device and retire to the Caribbean). This must mean that the pressure force acting on the left side of the solid must equal the sum of the pressure forces acting over the right side of the solid. As we have shown the pressure force is proportional to surface area. So why doesn't the left side have more pressure force than the right side? The key is in the vector nature of force.

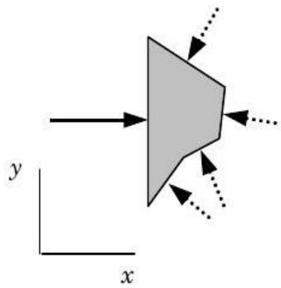


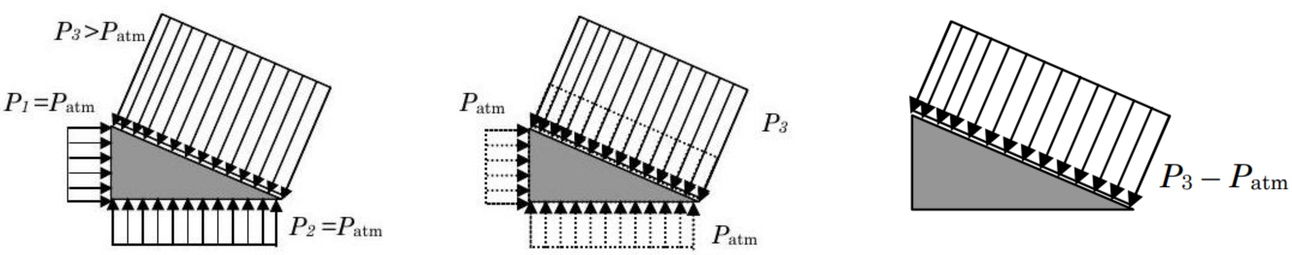
Figure 5.1.14: Pressure forces using projected areas.

To help us understand this result, let's only consider the force in the x-direction. Now we see that the pressure force acting over the left side of the solid must equal the x-components of the pressure forces acting on the right side of the solid. Without resorting to complicated mathematics, we can summarize this result as follows:

The magnitude of the pressure force in any direction is equal to the product of the pressure and the projected area in the desired direction.

The projected area is the two-dimensional area observed when the surface is viewed in the direction of interest. Consider a sphere with diameter D and a cylinder with diameter D and height H. When observed from any direction, the sphere looks like a circle with projected area $\pi D^2/4$. Now consider the cylinder. When viewed from either end the cylinder also has a projected area of $\pi D^2/4$; however, when viewed from the side (perpendicular to its axis of symmetry), the projected area of the cylinder is a rectangle of area DH.

We will also frequently have problems where only a small portion of the system boundary is subjected to non-atmospheric pressure. Under these conditions, we can greatly simplify the pressure force calculations by "subtracting off" atmospheric pressure. Consider the familiar triangular solid as shown in Figure 5.1.15.



$$\mathbf{F}_\text{net} = -\int_{A_\text{surface}} \mathbf{n} P \, dA$$

$$\mathbf{F}_\text{net} = -\int_{A_\text{surface}} \mathbf{n} P \, dA - \underbrace{\left[-\int_{A_\text{surface}} \mathbf{n} P_\text{atm} \, dA\right]}_{=0}$$

$$= -\int_{A_\text{surface}} \mathbf{n}(P - P_\text{atm}) \, dA$$

c) Resulting pressure distribution with $P_{3,\text{new}} = P_3 - P_\text{atm}$

$$\mathbf{F}_\text{net} = -(P_3 - P_\text{atm}) A_3 \mathbf{n}$$

Figure 5.1.15: Calculating net pressure force when most of the system sees atmospheric pressure.

For this example, we will assume that all surfaces except one are subjected to atmospheric pressure and this surface has a uniform pressure $P_3 > P_\text{atm}$. As shown in the figure, we can use the standard integral to evaluate the net force due to the pressures acting on the system. Since the net force due to a uniform atmospheric pressure on all surfaces is zero as shown in part (b) of Figure 5.1.15 we can subtract this zero force from the integral. Once this is done, the integral for net force can be rearranged as shown and we discover that the net force is the result of the pressure measured with respect to atmospheric pressure (this is sometimes referred to as gage pressure). Finally, as shown in part (c) the magnitude of the net force is just $(P_3 - P_\text{atm}) A_3$. Since $P_3 > P_\text{atm}$, the force arrow points into surface 3. When some portion of a system's boundary is subjected to non-atmospheric pressure, subtracting out the atmospheric pressure will greatly simplify the calculation of the net pressure force.

Example — The Plumber's Helper

To give us some quick practice with these ideas, consider the plumber's helper used to unclog toilets. The classic plumber's helper consists of a hollow rubber hemisphere attached to a wooden stick. The rubber material is flexible and will deform when pushed against a surface. If applied to a smooth surface, as shown in the figure, the air will be squeezed out of the hemisphere. This lowers the pressure inside the deformed hemisphere and causes the plumber's helper to "stick" to the surface. What exactly is the net force in the vertical direction due to pressure on the plumber's helper?

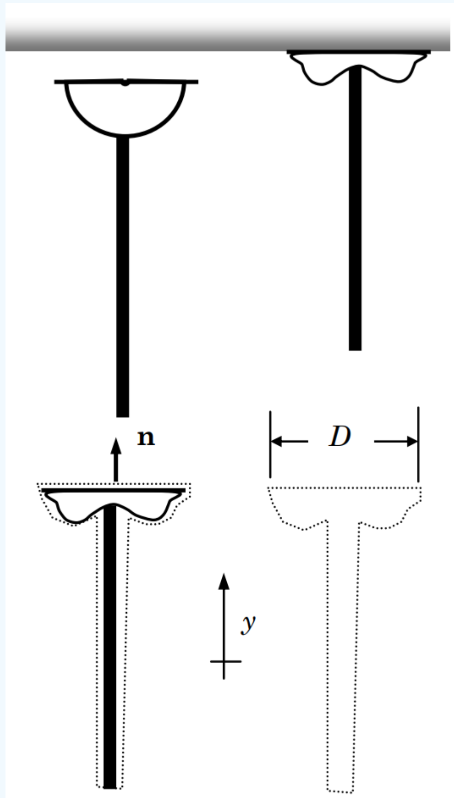


Figure $5.1.16$: The plumber's helper.

Solution

Consider the system shown above by the dashed lines. The cup in contact with the surface is of diameter D. The pressure on every surface of the system except the open end of the rubber cup (top surface) is subjected to atmospheric pressure, P_{atm}. The surface that covers the open end of the cup sees a pressure P_{cup}—the pressure of the air trapped inside the deformed hemisphere.

The projected area of the plumber's helper, when viewed from the top or bottom along the vertical direction, is just a circle with area $(\pi/4)D^2$. Also, since we want to find the net force we can just subtract out the atmospheric pressure.

Thus

$$\begin{aligned} \mathbf{F}_{\text{net pressure}} &= -\left(P_{\text{cup}} - P_{\text{atm}}\right) A_{\text{projected}} \mathbf{n} \\ \uparrow + \; F_{\text{net pressure}} &= -\left(P_{\text{cup}} - P_{\text{atm}}\right)\left(\frac{\pi}{4} D^2\right)(1) = \left(P_{\text{atm}} - P_{\text{cup}}\right)\left(\frac{\pi}{4} D^2\right) \\ \mathbf{F}_{\text{net pressure}} &= \left(P_{\text{atm}} - P_{\text{cup}}\right)\left(\frac{\pi}{4} D^2\right) \angle 90^\circ \quad \text{or} \quad \left(P_{\text{atm}} - P_{\text{cup}}\right)\left(\frac{\pi}{4} D^2\right) \uparrow \end{aligned}$$

As you might expect if \(P_{\text {cup}}<P_{\text {atm}}\), the net force will be up (in the positive y-direction). There are at least two other forces acting on this system that will govern whether it "sticks" or falls—the weight of the system and the force of the smooth surface on the system where it contacts the rubber hemisphere.

Linear momentum transport with mass flow

As was shown earlier, every lump of mass with a velocity has linear momentum. When mass is allowed to flow across the boundary of an open system, each lump of mass carries with it linear momentum. Thus the linear momentum of an open system can also be changed by mass flow carrying linear momentum across the system boundary. The rate at which linear momentum is transported across the boundary can be represented by the product of the mass flow rate and the local velocity at the boundary, assuming that the velocity is uniform:

$$\dot{\mathbf{P}}_{\text{mass}} = \dot{m}\mathbf{V} = \left\{ \begin{array}{c} \text{transport rate of linear momentum} \\ \text{with mass flow} \end{array} \right\}$$

where $\dot{m}$ is the mass flow rate at the flow boundary where the velocity $\mathbf{V}$ is uniform. The dimension for this quantity is $[\mathrm{M}][\mathrm{L}]/[\mathrm{T}]^2$.

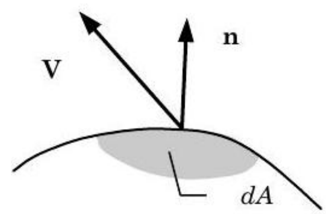


Figure $5.1.17$: Mass transport of linear momentum at a boundary.

For conditions where the velocity is not spatially uniform at the flow boundary, the linear momentum transport rate must be calculated from using an integral. At any flow boundary as shown in Figure $5.1.17$, the transport of linear momentum out of the system can be characterized by the integral:

$$\begin{aligned} \mathbf{P}_{\text{mass, out}} &= \int_{A_{\text{sys}}} (\mathbf{V})\left(\rho \mathbf{V}_{\text{rel}} \cdot \mathbf{n}\right) \, dA \\ &= \int_{A_{\text{sys}}} (\mathbf{V})\left(\rho V_{\text{n, rel}}\right) \, dA \end{aligned}$$

where $\mathbf{V}$ is the local velocity vector measured with respect to an inertial reference frame, $\mathbf{V}_{\text{rel}}$ is the local velocity vector measured with respect to the surface, $\mathbf{n}$ is the unit vector normal to the differential surface area dA and points out from the system, and $V_{\text{n, rel}} = \mathbf{V}_{\text{rel}} \cdot \mathbf{n}$, the magnitude of the normal component of the velocity $\mathbf{V}_{\text{rel}}$.

If the velocity is spatially uniform at the flow cross section, the mass transport of linear momentum can be written in terms of the mass flow rate at the boundary and the velocity as follows:

$$\begin{aligned} \dot{\mathbf{P}}_{\text{mass, out}} &= \int_{A_{\text{sys}}} (\mathbf{V})\left(\rho V_{\text{n, rel}}\right) \, dA = (\mathbf{V})\left(\rho V_{\text{n, rel}}\right) \int_{A_{\text{sys}}} dA \\ &= (\mathbf{V})\left(\rho V_{\text{n, rel}}\right) A_{\text{sys}} = \left(\rho V_{\text{n, rel}} A_{\text{sys}}\right) \mathbf{V} \\ &= \dot{m}_{\text{out}} \mathbf{V} \end{aligned} \tag{5.1.6}$$

This is often referred to as **one-dimensional flow**. If the velocity is not uniform across the flow area, the flow area must be broken into a suitable number of areas where the velocity can be assumed uniform and Eq. $5.1.24$ will become a summation. Alternatively, the non-uniform velocity distribution can be handled by performing the integral in Eq. $5.1.23$.

The **net** transport of linear momentum with mass flow rate can be obtained by summing up the transports at all flow boundaries:

$$\sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e \equiv \left\{ \begin{array}{c} \textit{net} \text{ transport rate of linear momentum} \\ \textit{into} \text{ the system with mass flow} \end{array} \right\}$$

Notice that the net transport may be positive or negative depending upon the relative transports.

5.1.4 How can linear momentum be generated or destroyed in a system?

Experience has shown that the linear momentum of a system cannot be created or destroyed. Recognizing that linear momentum is an extensive property, we can state this law as follows:

Linear momentum is a conserved property.

Once again you are reminded that the use of the word "conservation" is different in this course (ES201) than you are familiar with in physics. (C'mon, take at look at your physics book and see what kind of problems they solve in the section labeled "conservation of linear momentum." Do the problems involving conservation of energy differ from the problems involving conservation of linear momentum?) In this course, when we say that something is conserved we are making a general statement about the way the world behaves and not a problem specific modeling assumption. In fact, you could say that it is incorrect (if not redundant) to write in a problem solution "assume that linear momentum (or mass or charge) is conserved." You can't invoke basic physical laws by *assuming* their existence. It would be correct to say "using (or applying) the conservation of linear momentum (or mass or charge) we find that..." The basic physical laws apply whether we want them to or not.

5.1.5 Putting it all together!

Using the accounting framework, we can now develop the following statement for conservation of linear momentum:

$$\left[\begin{array}{c}\text{Rate of accumulation} \\ \text{of} \\ \text{linear momentum} \\ \text{inside a system} \\ \text{at time } t\end{array}\right]=\left[\begin{array}{c}\text{Net transport rate of} \\ \text{linear momentum} \\ \text{into the system} \\ \text{by external forces} \\ \text{at time } t\end{array}\right]+\left[\begin{array}{c}\text{Net transport rate of} \\ \text{linear momentum} \\ \text{into the system} \\ \text{by mass flow} \\ \text{at time } t\end{array}\right]$$

or in symbols, conservation of linear momentum (rate form) becomes

$$\frac{d\mathbf{P}_{\text{sys}}}{dt}=\sum \dot{\mathbf{P}}_{\text{force}}+\sum_{\text{in}} \dot{\mathbf{P}}_{\text{mass, in}}-\sum_{\text{out}} \dot{\mathbf{P}}_{\text{mass, out}}$$

Using the more familiar symbols of forces, mass flow rates, and specific linear momentum, **the rate form of the conservation of linear momentum** becomes

$$\frac{d\mathbf{P}_{\text{sys}}}{dt}=\sum \mathbf{F}_{\text{ext}}+\sum_{\text{in}} \dot{m}_i \mathbf{V}_i-\sum_{\text{out}} \dot{m}_e \mathbf{V}_e \tag{5.1.7}$$

In words we might say the following:

The time rate of change of the linear momentum of the system equals the net transport rate of linear momentum into the system by external forces plus the net transport rate of linear momentum into the system by mass flow.

Under some conditions, the integrated or finite-time form is useful. In words, the finite-time form of the conservation of linear momentum is

$$\left[\begin{array}{c}\text{Linear} \\ \text{momentum} \\ \text{inside the} \\ \text{system at} \\ \text{time } t_2\end{array}\right]-\left[\begin{array}{c}\text{Linear} \\ \text{momentum} \\ \text{inside the} \\ \text{system at} \\ \text{time } t_1\end{array}\right]=\left[\begin{array}{c}\text{Net transport of} \\ \text{linear momentum} \\ \text{into the system} \\ \text{by external forces} \\ \text{during the period } \Delta t\end{array}\right]+\left[\begin{array}{c}\text{Net transport of} \\ \text{linear momentum} \\ \text{into the system} \\ \text{by mass flow} \\ \text{during the period } \Delta t\end{array}\right]$$

or in equation form,

$$\Delta \mathbf{P}_{\text{sys}}=\sum_{\text{external}}\left(\int_{t_1}^{t_2} \mathbf{F}\, dt\right)+\sum_{\text{in}}\left(\int_{t_1}^{t_2}\left(\dot{m}_i \mathbf{V}_i\right) dt\right)-\sum_{\text{out}}\left(\int_{t_1}^{t_2}\left(\dot{m}_e \mathbf{V}_e\right) dt\right)$$

Although this last form of the conservation of linear momentum equation looks rather complicated, it is really much simpler to focus on the rate form and then, if necessary, perform the required integration to recover Equation $5.1.30$.

Example — Newton's Laws and Conservation of Linear Momentum

Newton's Laws are commonly written as follows:

Law I: If the resultant force acting on a *particle* is zero, the *particle* will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion)

Law II: If the resultant force acting on a *particle* is not zero, the particle will have an acceleration proportional to the magnitude of the resultant force and in the direction of this resultant force

Law III: The forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense.

Demonstrate that these three laws are satisfied by the application of the rate-form of the Conservation of Linear Momentum.

Solution

Known: Newton's three laws of mechanics

Find: Show that these laws are supported by the conservation of linear momentum

Given: Provided as needed.

Analysis:

Newton's First Law: Look at a single particle as shown in the figure. To apply the conservation of linear momentum, we select the particle as a system and sketch a linear momentum interaction (free-body) diagram for the body.

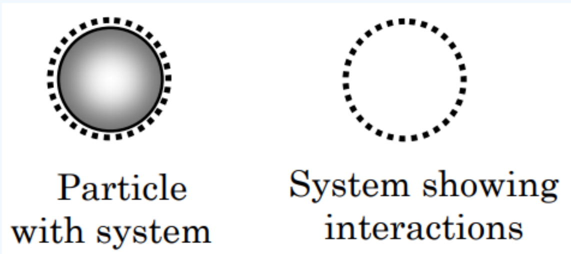


Figure 5.1.18: A system consisting of a particle. Free-body diagram of this system, which has no interactions with its surroundings.

Writing the rate-form of the conservation of linear momentum for this system we have:

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \underbrace{\cancel{\sum \mathbf{F}_{\text{ext}}}}_{\substack{=0 \text{ because there} \\ \text{are no external} \\ \text{forces}}} + \underbrace{\cancel{\sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e}}_{\substack{=0 \text{ because} \\ \text{it is a closed system}}}$$

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = 0 \quad \rightarrow \quad 0 = \frac{d\mathbf{P}_{\text{sys}}}{dt} = \frac{d(m\mathbf{V})}{dt} = m\frac{d\mathbf{V}}{dt}$$

Thus, $\frac{d\mathbf{V}}{dt} = 0$ and $\mathbf{V}$ is a constant!

This demonstrates that with no transports of linear momentum by force, i.e. with no external forces acting on the particle, the velocity of the particle remains constant.

Newton's Second Law: The system is the same as before, only this time there is a force acting on the system.

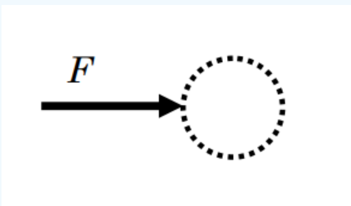


Figure 5.1.19: Free-body diagram of a system consisting of a particle with a force acting on it.

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \mathbf{F} + \underbrace{\cancel{\sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e}}_{\substack{=0 \text{ because} \\ \text{it is a closed system}}}$$

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \mathbf{F} \quad \rightarrow \quad \mathbf{F} = \frac{d\mathbf{P}_{\text{sys}}}{dt} = \frac{d(m\mathbf{V})}{dt} = m\underbrace{\frac{d\mathbf{V}}{dt}}_{=\mathbf{a}} = m\mathbf{a}$$

Thus $\mathbf{F} = m\mathbf{a}$ and $\mathbf{a} = \mathbf{F}/m$.

This demonstrates that the acceleration vector is proportional to the external force vector, so the magnitude of the acceleration is proportional to the applied force and the direction is the same as the direction of the applied force.

Newton's Third Law: This time our system is the interface or the contact surface between two particles that are in contact. This interface as shown in the drawing below has no mass inside it and is acted on by two forces - $\mathbf{F}_{\text{A}-\text{B}}$, the force of particle A on particle B, and $\mathbf{F}_{\text{B}-\text{A}}$, the force of particle B on particle A.

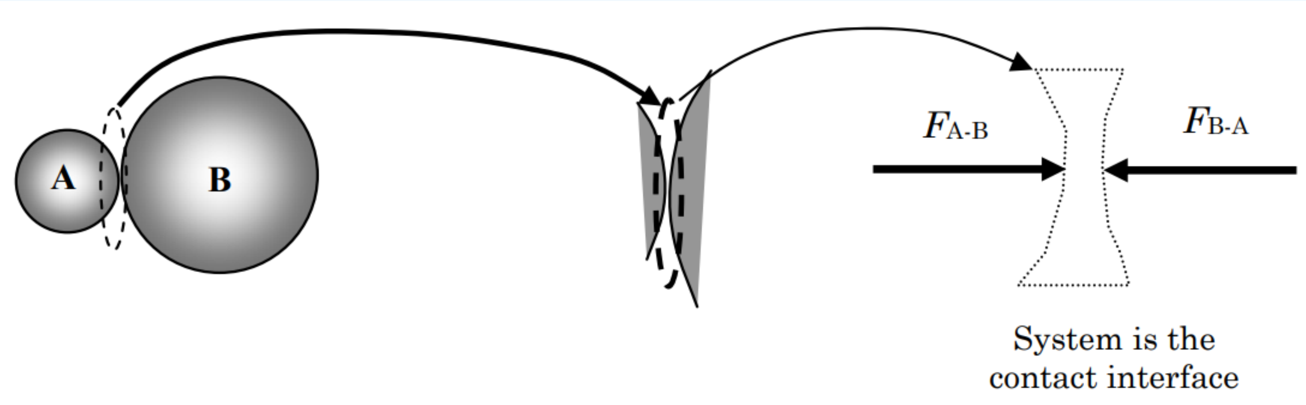


Figure 5.1.20: System consisting of the contact interface between two particles.

Writing the conservation of linear momentum equation for this system gives the following:

$$\underbrace{\cancel{\frac{d\mathbf{P}_{\text{sys}}}{dt}}}_{\substack{=0 \text{ because the system} \\ \text{has no mass, so } \mathbf{P}=0 \\ \text{for all time.}}} = \mathbf{F}_{\text{A-B}} + \mathbf{F}_{\text{B-A}} + \underbrace{\cancel{\sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e}}_{\substack{=0 \text{ because} \\ \text{it is a closed system}}}$$

$$0 = \mathbf{F}_{\text{A-B}} + \mathbf{F}_{\text{B-A}} \quad \rightarrow \quad \mathbf{F}_{\text{B-A}} = -\mathbf{F}_{\text{A-B}}$$

Thus, the forces of action and reaction at a point of contact between two systems are vectors of equal magnitude, opposite sign, and the same line of action.

Comment: The important point here is that all three of Newton's laws of mechanics are incorporated into the Conservation of Linear Momentum principle as formulated here.

5.2: Conservation of Linear Momentum Equation

The recommended starting point for any application of the conservation of linear momentum is the rate form of the linear momentum equation (previously given as Equation(5.1.28):

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \sum \mathbf{F}_{\text{ext}} + \sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e$$

where $\mathbf{P}_{\text{sys}}$ is the system linear momentum, $\mathbf{F}_{\text{ext}}$ is the transport rate of linear momentum by an external force, and $\dot{m}\mathbf{V}$ is the transport rate of linear momentum by mass flow.

In applying the rate form of the conservation linear momentum to a system, there are many modeling assumptions that are frequently used to build the mathematical model of the physical system. These are detailed in the following paragraphs. As always, you should focus on understanding what the assumptions mean physically and how they can be used to simplify the equations for a given system. *Do not just memorize the results.*

Steady-state system:

If a system is operating under steady-state conditions, all intensive properties are independent of time, i.e., the values of intensive properties may only vary with position. Thus, the linear momentum of the system is a constant, $\mathbf{P}_{\text{sys}} = \text{constant}$. When this assumption is applied to the conservation of linear momentum equation, we have

$$\underbrace{\cancelto{=0}{\frac{d\mathbf{P}_{\text{sys}}}{dt}}}_{\mathbf{P}_{\text{sys}}=\text{constant}} = \sum \mathbf{F}_{\text{ext}} + \sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e$$

$$0 = \sum \mathbf{F}_{\text{ext}} + \sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e$$

In words, this says that the net transport rate of linear momentum *into* the system by force must equal the net transport rate of linear momentum *out of* the system by mass flow.

Closed system:

A system with boundaries selected so that the mass flow rate at any boundary is identically zero is a closed system; thus the mass of the system is constant. When applied to the conservation of linear momentum, this assumption has the following effect:

$$\frac{d\,\cancelto{=m_{\text{sys}}\mathbf{V}_G}{\mathbf{P}_{\text{sys}}}}{dt} = \sum \mathbf{F}_{\text{ext}} + \sum_{\text{in}} \cancelto{=0}{\dot{m}_i} \mathbf{V}_i - \sum_{\text{out}} \cancelto{=0}{\dot{m}_e} \mathbf{V}_e$$

$$m_{\text{sys}} \frac{d\mathbf{V}_G}{dt} + \underbrace{\cancelto{=0}{\frac{dm_{\text{sys}}}{dt}}}_{\substack{\text{Closed} \\ \text{system}}} \mathbf{V}_G = \sum \mathbf{F}_{\text{ext}}$$

$$m_{\text{sys}} \frac{d\mathbf{V}_G}{dt} = \sum \mathbf{F}_{\text{ext}}$$

where $\mathbf{V}_G$ is the velocity of the center of mass of the system. Recalling that acceleration is just the time derivative of velocity, we immediately recognize that this is our old friend from physics: $F = ma$.

Closed, steady-state system:

Using what you've learned to this point, how would you expect the linear momentum equation to simplify for a closed system that operates at steady state conditions?

In applying the conservation of linear momentum to model a system, we will need to, as always, select a system and identify the transports of linear momentum across the system boundary. Guidelines for drawing a linear momentum (or free-body) diagram are found below:

Guidelines for Drawing a Linear Momentum (Free-body) Diagram

1. **Select a system.** Every system can be broken down into smaller subsystems. For a given problem, there may be several possible systems and different questions may require a different system.
2. **Sketch the physical object clearly identifying the boundaries of *your* system.** This is usually done with a dashed line to indicate the system boundary.
3. **Detach the system from its surroundings and sketch the isolated contour of the system**, i.e., the system boundaries.
4. **Identify the transports of momentum between the system and the surroundings.** When identifying external forces, only consider the forces exerted *by* the surroundings *on* the system. Remember that there are two types of external forces: contact (or surface) forces that act on the boundaries of the system and body forces produced by fields like the gravitational force commonly called *weight*.
5. **For each momentum transport, draw an arrow on the system diagram showing the direction and location of the transport.** Care should be taken to draw each arrow with the correct direction (line-of-action and sense) and position. Label all forces and mass transports of momentum with a name, number, or a symbol. Draw the vector on the diagram by placing either the tail or the head of the arrow at the point of application:
   - Contact forces should be applied at the appropriate point on the system boundary where the system was "cut away" from the surroundings.
   - The weight vector should be applied at the center of gravity of the body.
   - The mass transport of linear momentum vector should be applied at the point where the mass crosses the boundary and in the direction of the velocity.
   - If you do not know the correct direction of a momentum transfer, assume a direction. If your analysis results in a negative numerical value for the transfer, then the actual direction is opposite to the direction assumed.
6. **Draw a coordinate system and indicate all pertinent dimensions and angles on the free-body diagram.**

Example — What a serve!

A tennis player serves the ball a distance $L = 40\text{ ft}$ back from the net and a height $H = 9\text{ ft}$ above the court. The tennis ball leaves the racket with an angle $\theta = 0^\circ$ below the horizontal as shown in the figure. The net is 3ft high and the center of the tennis ball clears the net by 3 inches. Determine (a) the initial velocity V_o of the ball when it leaves the racket and (b) the distance s behind the net where the ball hits the court. Assume air drag is negligible. The mass of the tennis ball is 0.25 lbm.

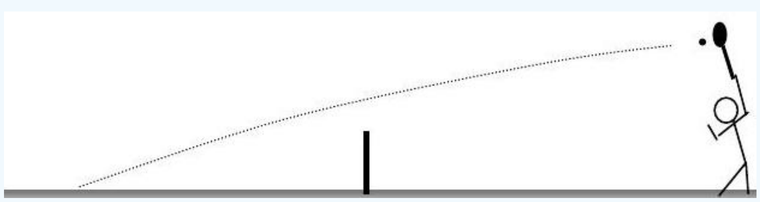


Figure $5.2.1$: A tennis player serves a ball.

<u>Solution</u>

Known: Tennis ball is served with a specified angle from a specified location.

Find: (a) The velocity of the ball when it crosses the net and (b) the distance from the net where it hits the court.

Given:

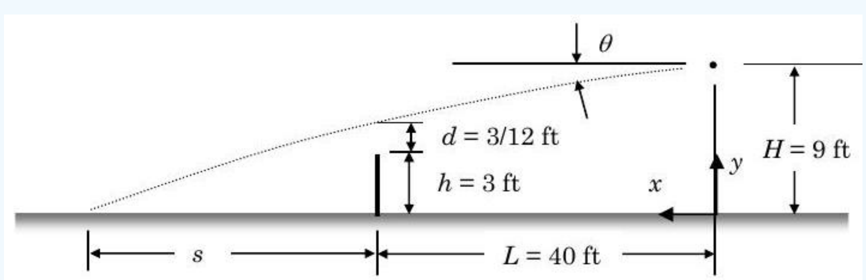


Figure 5.2.2: Diagram indicating the path traveled by the ball, assigned a coordinate system and labeled with all relevant measurements.

Analysis:

System $\rightarrow$ Select the tennis ball as the system. This is a closed system.
Property to count $\rightarrow$ Since trajectory is governed by effect of gravity, count linear momentum.
Time period $\rightarrow$ Will eventually need to integrate over finite-time to get trajectory and velocity.

First we must sketch the free-body (linear momentum) diagram for the system. The only transports of linear momentum occur due to the gravity body force-the weight. The inertial coordinate system and the other dimensions are shown on Figure 5.2.2 above.

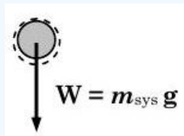


Figure 5.2.3: Free-body diagram of the system.

Writing the rate form of the conservation of linear momentum for the system gives

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \mathbf{W} + \underbrace{\left[\sum_{\text{in}} \dot{m}_i \mathbf{V}_i + \cancel{\sum_{\text{out}} \dot{m}_e \mathbf{V}_e}\right]^{=0}}_{\text{Closed system, } \dot{m}=0} \rightarrow \frac{d}{dt}\left(m_{\text{sys}} \cancel{\mathbf{V}_G}\right)^{=m_{\text{sys}}\frac{d\mathbf{V}_G}{dt}} = m_{\text{sys}}\,\mathbf{g} \rightarrow m_{\text{sys}} \frac{d\mathbf{V}_G}{dt} = m_{\text{sys}}\,\mathbf{g}$$

Now writing equation in scalar form using the coordinate system defined above and $\mathbf{V}_G = V_x\mathbf{i} + V_y\mathbf{j}$, we have the following equations:

$$\cancel{m_{\text{sys}}} \frac{d\mathbf{V}_G}{dt} = \cancel{m_{\text{sys}}}\,\mathbf{g} \rightarrow \frac{d}{dt}(V_x\mathbf{i} + V_y\mathbf{j}) = -g\mathbf{j} \rightarrow \begin{cases} x\text{- axis:} & \dfrac{dV_x}{dt} = 0 \\ y\text{-axis:} & \dfrac{dV_y}{dt} = -g \end{cases}$$

Integrating once to get velocity gives

$$\begin{array}{llll} x\text{-axis:} & \dfrac{dV_x}{dt} = 0 & \rightarrow V_x = \text{constant} = V_{x,o} = V_o(\cos\theta) & \rightarrow V_x = V_o(\cos\theta) \\ y\text{-axis:} & \dfrac{dV_y}{dt} = -g & \rightarrow \int_{V_{y,o}}^{V_y} dV_y = -\int_0^t g\,dt \rightarrow V_y - V_o(\sin\theta) = -gt & \rightarrow V_y = -gt + V_o(\sin\theta) \end{array}$$

Now integrating again to get position:

$$\begin{array}{llll} x\text{-axis:} & V_x = \dfrac{dx}{dt} = V_o\cos\theta & \rightarrow \int_0^x dx = \int_0^t \underbrace{(V_o\cos\theta)}_{\text{Constant}}\,dt & \rightarrow x = (V_o\cos\theta)\,t \\ y\text{- axis:} & V_y = \frac{dy}{dt} = -gt - V_o\sin\theta & \rightarrow \int_H^y dx = -\int_0^t (gt + V_o\sin\theta)\;dt & \rightarrow y - H = -\left(\dfrac{gt^2}{2} + (V_o\sin\theta)\,t\right) \end{array}$$

Now that we know position and velocity as a function of time, let's first solve for the initial velocity required to clear the net at $x = L$ and $y = h + d$.

For $\theta = 0$ with x and y as specified above, we have two equations for two unknowns t and V_o :

$$\begin{array}{lll} x\text{-axis:} & L = V_o t & \rightarrow t = L/V_o \\ y\text{-axis:} & (h+d) - H = -\left[\dfrac{1}{2}gt^2\right] & \rightarrow t^2 = \dfrac{2}{g}[H - (h+d)] \end{array}$$

Combining these equations and eliminating the time t gives the following result:

$$t^2 = \left(\frac{L}{V_o}\right)^2 = \frac{2}{g}[H-(h+d)] \qquad \rightarrow \qquad V_o = \sqrt{\frac{gL^2}{2[H-(h+d)]}} = \sqrt{\frac{\left(32.2\ \frac{\text{ft}}{\text{s}^2}\right)(40\text{ ft})^2}{2\left[9-\left(3+\frac{3}{12}\right)\right]\text{ ft}}} = 66.9\ \frac{\text{ft}}{\text{s}}$$

$$V_o^2 = \frac{gL^2}{2[H-(h+d)]}$$

Now to solve for s, we recognize that $s = x - 40$ when $y = 0$ and use the two displacement equations:

$$s = x - (40\text{ ft}) = V_o t - (40\text{ ft})$$

$$0 - H = -\frac{1}{2}gt^2 \quad \rightarrow \quad H = \frac{1}{2}gt^2 \qquad \rightarrow \qquad s = V_o\sqrt{\frac{2H}{g}} - (40\text{ ft})$$

Solving for the distance s: $\quad s = \left(66.9\ \frac{\text{ft}}{\text{s}}\right)\sqrt{2\frac{(9\text{ ft})}{(32.3\text{ ft/s})}} - (40\text{ ft}) = 10.0\text{ ft}$

Comments:

1. Assuming that the diameter of the tennis ball is 3 inches and the tennis still hits with the initial velocity calculated above, determine the maximum value of θ before the ball hits the net.
2. Now determine the minimum initial speed that the ball must have to clear the net if it is hit horizontally, e.g. $\theta = 0$.

Example — Forces to Restrain a Nozzle

Water flows steadily through a horizontal nozzle attached to a pipe. The nozzle is flanged and attached to the pipe with 6 bolts as shown in the figure. The pipe and the nozzle inlet has an interior diameter of 25 cm and the outlet diameter of the nozzle is 12 cm. The absolute pressure at the inlet to the nozzle is 500 kPa and the water velocity is 5 m/s. The pressure at the nozzle outlet is atmospheric pressure $(P_{\text{atm}} = 100\text{kPa})$. Assume that the density of liquid water is $1000\ \text{kg/m}^3$.

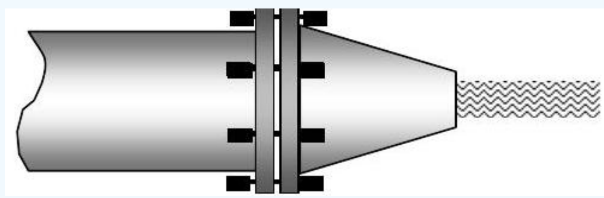


Figure 5.2.4: Water moves through a cylindrical pipe attached to a flanged nozzle.

Determine the total force, in newtons, applied by the bolts to hold the nozzle in place. Assume the bolts only support tension.

Solution

Known: Water flows steadily through a nozzle.

Find: Find the total force exerted by the bolts on the nozzle to hold it in place.

Given:

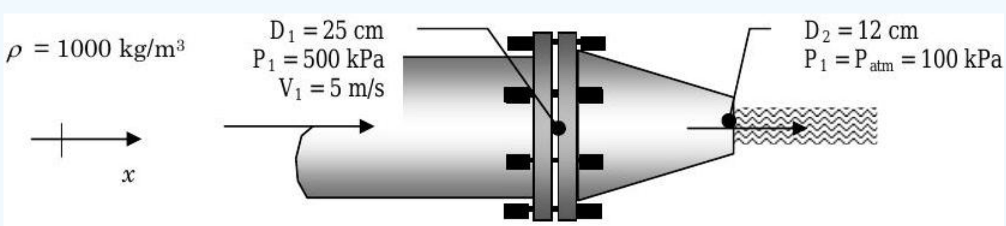


Figure 5.2.5: The pipe and nozzle, assigned with a coordinate system and labeled with all given quantities.

Analysis:

Strategy: Try linear momentum in the x-direction since the problem asks for forces (a mechanism for transferring linear momentum) and the forces are in the x-direction.

System $\rightarrow$ Pick an open, non-deforming system that cuts the pipe at the pipe nozzle connection and includes the nozzle.

Property to count → Try linear momentum and possibly mass (if we need to relate input to output flow rates.

Time period → Looks like it may be a rate (infinitesimal time interval) or steady-state problem.

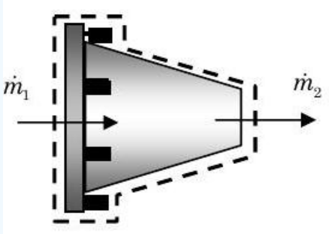


Figure 5.2.6: System choice for this problem.

Starting with the system shown by the dashed line, identify all of the momentum transfers for the system. Walking your fingers around the system boundary you would identify six bolt forces, pressure forces, and the two mass transports of momentum. Now writing these in a vector equation for linear momentum in the x-direction, we have:

$$\frac{d\mathbf{P}_{x,\text{ sys}}}{dt} = [6\mathbf{F}_{\text{bolt},\,x} + \mathbf{F}_{\text{net pressure},\,x}] + [\dot{m}\mathbf{V}_{1,\,x} - \dot{m}\mathbf{V}_{2,\,x}]$$

A more complete momentum diagram is now required to obtain the correct directions of the forces. The complete diagram is shown below.

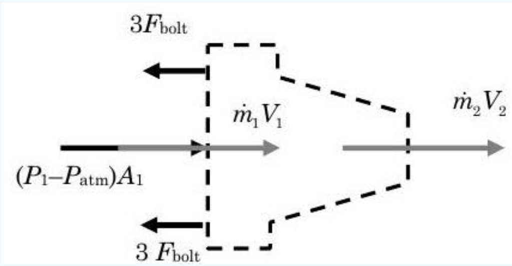


Figure 5.2.7: Free-body diagram for the nozzle system.

Before continuing, we should explain how we obtained the various terms.

- The mass transport terms are the product of the mass flow rate and the velocity at any flow boundary. The direction of the arrow is in the direction of the velocity.
- Two arrows show the bolt forces. Each arrow represents three (3) bolts. In addition, only the average bolt force can be determined. Note that since the bolts are in tension, the arrows indicate the bolts are pulling on the system.
- The net pressure forces in the x-direction must be calculated by considering the pressure distribution over the entire system boundary. As shown in the figure, atmospheric pressure P atm acts on all surfaces except the inlet flow area where the pressure is P_1. Notice that all of the pressure arrows point into the surface. The net force can be determined by subtracting off the atmospheric pressure as shown in the figure. The result is a uniform pressure of $P_{\text{net}} = P_1 - P_{\text{atm}}$ acting over the inlet area. The magnitude of the resulting force is $F_{\text{net pressure},\,x} = (P_1 - P_{\text{atm}})\,A_1$. This arrow points into the surface and is shown on the momentum diagram above.

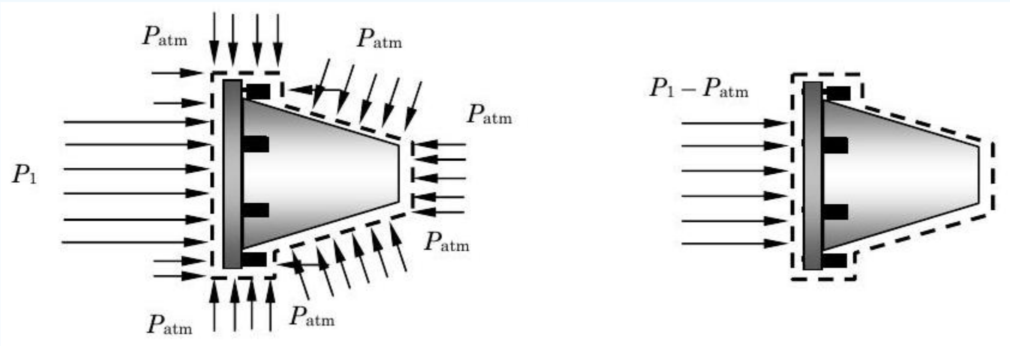


Figure 5.2.8: Equivalent methods of expressing the net pressure experienced by the nozzle system.

Now all that remains is to solve for the bolt forces. Writing the momentum equation as a scalar for the x− direction we have:

$$\underbrace{\frac{dP_{x,\,\text{sys}}}{\cancel{dt}}^{=0}}_{\text{Steady-state}} = \left[-6F_{\text{bolt},\,x} + \underbrace{F_{\text{net pressure},\,x}}_{=(P_1-P_{\text{atm}})A_1}\right] + \underbrace{\left[\dot{m}_1\left(+V_{1,\,x}\right) - \dot{m}_2\left(+V_{2,\,x}\right)\right]}_{\substack{+\text{ sign for each } V \text{ since a positive value}\\ \text{for } V_{1,\,x} \text{ and } V_{2,\,x} \text{ represents a positive}\\ \text{specific linear momentum}}}$$

$$0 = -6F_{\text{bolt},\,x} + \left(P_1 - P_{\text{atm}}\right)A_1 + \left[\dot{m}_1 V_1 - \dot{m}_2 V_2\right] \quad \rightarrow \quad 6F_{\text{bolt},\,x} = \left(P_1 - P_{\text{atm}}\right)A_1 + \left[\dot{m}_1 V_1 - \dot{m}_2 V_2\right]$$

Further simplification can be achieved by applying conservation of mass to the same system and then applying the steady-state assumption:

$$\frac{dm_{\text{sys}}}{\cancel{dt}}^{=0} = \dot{m}_1 - \dot{m}_2 \quad \rightarrow \quad \dot{m}_2 = \dot{m}_1 \quad \rightarrow \quad \underbrace{\cancel{\rho_1} A_1 V_1 = \cancel{\rho_2} A_2 V_2}_{\text{Incompressible liquid}}$$

Using this result, the equation for the bolt forces become

$$6F_{\text{bolt}} = \left(P_1 - P_{\text{atm}}\right)A_1 + \dot{m}_1\left(V_1 - V_2\right)$$

Solving for the area at the inlet: $A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}(0.25\text{ m})^2 = 4.909 \times 10^{-2}\text{ m}^2$

Solving for the mass flow rate: $\dot{m}_1 = \rho A_1 V_1 = \left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\left(4.909 \times 10^{-2}\text{ m}^2\right)\left(5\ \frac{\text{m}}{\text{s}}\right) = 245.5\ \frac{\text{kg}}{\text{s}}$

The velocity at the exit becomes: $V_2 = \frac{A_1}{A_2}V_1 = \left[\frac{(\pi/4)D_1^2}{(\pi/4)D_2^2}\right]V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{25\text{ cm}}{12\text{ cm}}\right)^2\left(5\,\frac{\text{m}}{\text{s}}\right) = 21.70\ \frac{\text{m}}{\text{s}}$

Substituting these numbers back into the general result gives

$$\begin{aligned} 6F_{\text{bolt}} &= \left(P_1 - P_{\text{atm}}\right)A_1 + \dot{m}_1\left(V_1 - V_2\right) \\ &= \left[(500 - 100)\text{ kPa}\right]\left(4.904 \times 10^{-2}\text{ m}^2\right) + \left(245.5\ \frac{\text{kg}}{\text{s}}\right)(5.00 - 21.70)\frac{\text{m}}{\text{s}} \\ &= \left(19.64\text{ kPa}\cdot\text{m}^2\right)\left(\frac{1000\text{ N/m}^2}{\text{kPa}}\right) + \left(-4100\,\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)\left(\frac{\text{N}}{\left(\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)}\right) \\ &= 19.64 \times 10^3\text{ N} + \left(-4.10 \times 10^3\text{ N}\right) \\ &= 15.5\text{ kN} \end{aligned}$$

Thus the six bolts must exert a total force of 15.5 kN acting to the left (or towards the pipe) to hold the nozzle in place.

Comments:

1. Note that the increase in the specific linear momentum, i.e. V, of the mass leaving the system actually serves to reduce the load on the bolts. To see this, consider what the bolt force would be if the nozzle was capped.
2. How would the solution change if we had picked the basic x coordinate to the left instead of to the right as shown? The vector relation would have stayed the same, but the scalar equation would have become as follows

$$0 = 6F_{\text{bolt},\,x} - \left(P_1 - P_{\text{atm}}\right)A_1 + \left[\dot{m}_1\left(-V_1\right) - \dot{m}_2\left(-V_2\right)\right] \quad \rightarrow \quad 6F_{\text{bolt},\,x} = \left(P_1 - P_{\text{atm}}\right)A_1 + \left[\dot{m}_1\left(V_1\right) - \dot{m}_2\left(V_2\right)\right]$$

Example — Cable Car

A cable car is pulled along a fixed overhead cable by a cable attached at point A. The car has a mass of 200 kg, and the tension in the cable is 2400 N. The car is supported by wheels that rest on the cable. The cable is inclined at an angle of $\theta = 22.6^\circ$ with the horizontal as shown in the figure.

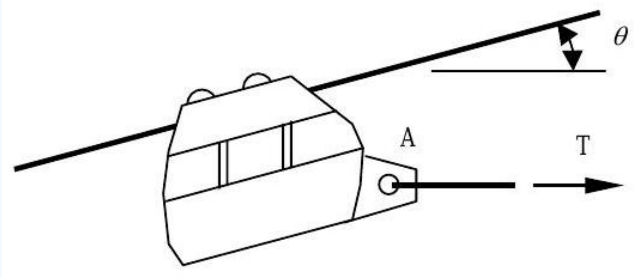


Figure 5.2.9: Cable car suspended from an angled cable is pulled to the right by a horizontal cable.

Determine:

(a) The magnitude and direction of the force R exerted by the overhead cable on the car wheels, in newtons.

(b) The magnitude and direction of the acceleration of the car in $\mathrm{m/s^2}$.

Solution

Known: A small inspection car is pulled along a fixed overhead cable by a cable attached at point A

Find: (a) The magnitude and direction of force R exerted by the overhead cable on the wheels of the car, in newtons. (b) The magnitude and direction of the acceleration of the car, in $\mathrm{m/s^2}$.

Given:

$$\mathrm{T}=2400\ \mathrm{N}$$
$$m_{\mathrm{car}}=200\ \mathrm{kg}$$
$$\theta=22.6^{\circ}$$

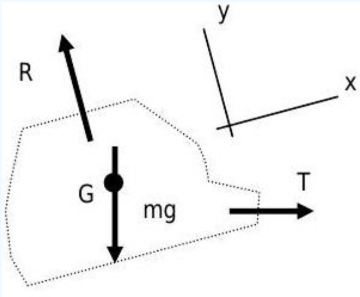


Figure 5.2.10: Free-body diagram of the isolated cable car.

Analysis:

Strategy $\rightarrow$ Since the problem involves forces, try conservation of linear momentum.
System $\rightarrow$ Closed system including only the car as shown in the momentum system diagram at right.
Property $\rightarrow$ Linear Momentum
Time Period $\rightarrow$ Instantaneous

Without selecting a coordinate system, the conservation of linear momentum equation becomes

$$\frac{d}{dt}\mathbf{P}_{\mathrm{sys}}=\sum_{\text{external}}\mathbf{F}_j+\sum_{\text{in}}\dot{m}_i\cancelto{=0}{\mathbf{V}_i}-\sum_{\text{out}}\dot{m}_e\cancelto{=0,\ \text{closed system}}{\mathbf{V}_e}\quad\rightarrow\quad\frac{d}{dt}\mathbf{P}_{\mathrm{sys}}=m_{\mathrm{sys}}\mathbf{g}+\mathbf{T}+\mathbf{R}$$

where there are only three external forces - the force on the wheels $\mathbf{R}$, the weight of the car $m_{\mathrm{sys}}\mathbf{g}$, and the cable force $\mathbf{T}$. The net pressure force is zero since atmospheric pressure surrounds the car.

Also, because this is a closed system the linear momentum of the system $\mathbf{P}_{\mathrm{sys}}=m_{\mathrm{sys}}\mathbf{V}_G$ where $\mathbf{V}_G$ is the velocity of the center of mass of the system. Thus, the conservation of linear momentum can now be written as

$$\frac{d}{dt}(m_{\mathrm{sys}}\mathbf{V}_{\mathrm{G}})=m_{\mathrm{sys}}\mathbf{g}+\mathbf{T}+\mathbf{R}\quad\rightarrow\quad m_{\mathrm{sys}}\frac{d\mathbf{V}_{\mathrm{G}}}{dt}=m_{\mathrm{sys}}\mathbf{g}+\mathbf{T}+\mathbf{R}$$

Now selecting a coordinate system that is aligned with the cable, the conservation of linear momentum can be written in two components. In the y direction, the linear momentum equation becomes

$$m_{\text{sys}} \overbrace{\frac{dV_{G,y}}{\cancel{dt}}}^{\substack{=0;\text{ no motion} \\ \text{in } y\text{-direction}}} = -(T \cdot \sin\theta) + R - (m_{\text{sys}}\, g \cdot \cos\theta) \quad \rightarrow \quad R = (T \cdot \sin\theta) + (m_{\text{sys}}\, g \cdot \cos\theta)$$

after solving for the force R.

Substituting in the numerical information gives

$$\begin{aligned} R &= (2400 \text{ N}) \cdot \sin(22.6^\circ) + (200 \text{ kg}) \left(9.81 \frac{\text{m}}{\text{s}^2}\right) \cdot \cos(22.6^\circ) \\ &= \qquad 922.3 \text{ N} \qquad + \qquad 1811 \text{ N} \qquad = 2733 \text{ N} \end{aligned}$$

In the x direction, the linear momentum equation becomes

$$\begin{aligned} m_{\text{sys}} \frac{dV_{G,x}}{dt} &= (T \cdot \cos\theta) - (m_{\text{sys}}\, g \cdot \sin\theta) \\ \frac{dV_{G,x}}{dt} &= \frac{(T \cdot \cos\theta) - (m_{\text{sys}}\, g \cdot \sin\theta)}{m_{\text{sys}}} = \left(\frac{T}{m_{\text{sys}}} \cos\theta\right) - (g \cdot \sin\theta) \end{aligned}$$

Recalling the definition of acceleration, the x-momentum equation can be solved for acceleration as

$$a_{G,x} \equiv \frac{dV_{G,x}}{dt} = \left(\frac{T}{m_{\text{sys}}} \cos\theta\right) - (g \cdot \sin\theta)$$

Substituting in the numerical information gives

$$\begin{aligned} a_{G,x} &= \left(\frac{2400 \text{ N}}{200 \text{ kg}}\right) \cos(22.6^\circ) - \left(9.81 \frac{\text{m}}{\text{s}^2}\right) \sin(22.6^\circ) = \left(12.0 \frac{\text{m}}{\text{s}^2}\right)(0.9232) - \left(9.81 \frac{\text{m}}{\text{s}^2}\right)(0.3843) \\ &= 7.31 \frac{\text{m}}{\text{s}^2} \end{aligned}$$

Comments:

1. As part of a safety study, determine how the force R and the car acceleration would change if the pulling cable suddenly broke.
2. Following a failure of the pulling cable, the maximum speed of the cable car is limited to 15 m/s by a emergency braking mechanism in the wheel carriage. The braking force is activated when the speed of a "falling" car reaches 5 m/s. What braking force must be exerted on the cable car to limit the car speed? If the car is stationary when the pulling cable breaks, how long will it take for the emergency brake to activate and when will the maximum speed be reached?

Example — Weighing Water

A scale is used to weigh the water in a tank. Water flows steadily through the tank as shown in the figure. It enters at the top of the tank with volumetric flow rate of $30 \text{ m}^3/\text{s}$ through a pipe with a diameter of 6 cm. It leaves at the side of the tank through a circular opening with a $6-\text{cm}$ diameter. The volume of the water in the tank is 0.6 m^3, and the dry weight of the tank is 500 N. Determine the scale reading, in newtons.

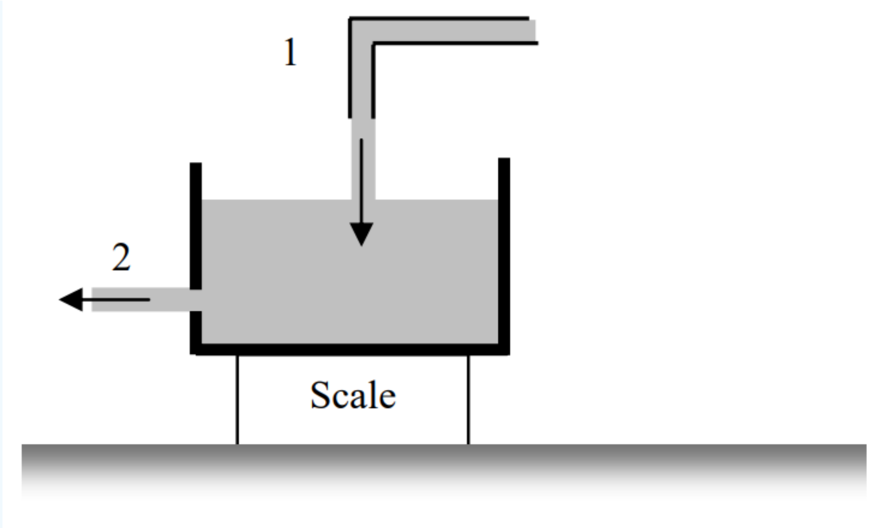


Figure 5.2.11: Water enters a tank on a scale through Opening 1 and leaves the tank through Opening 2.

Solution

Known: Water flows steadily through a tank that rests on a scale.

Find: The scale reading.

Given:

Inlet Pipe @ 1
Volumetric flow rate $\dot{V}_1 = 30 \text{ m}^3/\text{h}$
Diameter $D_1 = 6 \text{ cm}$

Outlet Opening @ 2
Diameter $D_2 = 6 \text{ cm}$

Volume of water in tank at steady-state: $V_{\text{water}} = 0.6 \text{ m}^3$

Weight of tank: $W_{\text{tank}} = 500 \text{ N}$

Analysis:

Strategy $\rightarrow$ Since the problem involves forces, try conservation of linear momentum.
System $\rightarrow$ Open system that includes all water in the tank and the tank as shown in the momentum system diagram.
Property to count $\rightarrow$ Linear momentum and mass
Time Period $\rightarrow$ Instantaneous

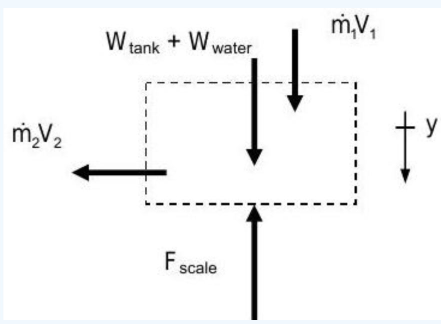


Figure 5.2.12: Free body diagram of the system consisting of the tank and the water in it.

Writing the rate-form of the conservation of linear momentum equation for this problem gives

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \sum_{\text{external}} \mathbf{F}_j + \sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e$$

$$\overbrace{\frac{d\mathbf{P}_{\text{sys}}}{\cancel{s}}}^{=0}_{\text{steady state}} = (\mathbf{W}_{\text{tank}} + \mathbf{W}_{\text{water}}) + \mathbf{F}_{\text{scale}} + \dot{m}_1 \mathbf{V}_1 - \dot{m}_2 \mathbf{V}_2 \text{ A}$$

$$0 = (\mathbf{W}_{\text{tank}} + \mathbf{W}_{\text{water}}) + \mathbf{F}_{\text{scale}} + \dot{m}_1 \mathbf{V}_1 - \dot{m}_2 \mathbf{V}_2$$

Now writing the component of this equation in the y-direction as defined in the figure above,

$$0 = [W_{\text{tank}} + W_{\text{water}}] - F_{\text{scale}} + \dot{m}_1 V_{1,y} - \dot{m}_2 \overbrace{\cancel{V_{2,y}}}^{=0}_{\text{No } y\text{-component @ 2}}$$

Solving for F_{scale} we have

$$F_{\text{scale}} = W_{\text{tank}} + W_{\text{water}} + \dot{m}_1 V_{1,y}$$

Now solving for the weight of the tank W_{tank} we have

$$W_{\text{water}} = m_{\text{water}}\, g = (\rho_{\text{water}} V_{\text{water}})\, g = \left(1000\ \frac{\text{kg}}{\text{m}^3}\right) (0.600\ \text{m}^3) \left(9.81\ \frac{\text{m}}{\text{s}^2}\right) = 5886\ \text{N}$$

The y-component of the velocity at 1 and the mass flow rate at 1 are

$$V_{1,y} = V_1 = \frac{\dot{\forall}_1}{A_1} = \frac{\dot{\forall}_1}{\left(\frac{\pi}{4} D_1^2\right)} = \frac{\left(30\ \frac{\text{m}^3}{\text{h}} \times \frac{1\ \text{h}}{3600\ \text{s}}\right)}{\frac{\pi}{4}(0.06\ \text{m})^2} = 2.95\ \frac{\text{m}}{\text{s}}$$

$$\dot{m}_1 = \rho_1 \dot{\forall}_1 = \left(1000\ \frac{\text{kg}}{\text{m}^3}\right) \left(30\ \frac{\text{m}^3}{\text{h}} \times \frac{1\ \text{h}}{3600\ \text{s}}\right) = 8.33\ \frac{\text{kg}}{\text{s}}$$

Combining this to solve for the force of the scale on the tank, F_{scale}:

$$\begin{aligned} F_{\text{scale}} &= W_{\text{tank}} + W_{\text{water}} + \dot{m}_1 V_{1,y} \\ &= (500\ \text{N}) + (5886\ \text{N}) + \left(8.33\ \frac{\text{kg}}{\text{s}}\right) \left(2.95\ \frac{\text{m}}{\text{s}}\right) \\ &= (6386\ \text{N}) + (24.6\ \text{N}) \\ &= 6411\ \text{N} \end{aligned}$$

If the operator had neglected the effect of the water flowing into the tank on the reading, he or she would have overestimated the amount of water in the tank by roughly 0.4%.

Comment:

What is the magnitude and direction of the horizontal force that the scale must exert on the tank to keep it from sliding off the platform? [Answer: $24.6\ \text{N} \leftarrow$] How could you use a side-force measurement to indicate flow rate?

5.3: Friction Forces

Whenever a solid comes into contact with another solid or a fluid, the force at the interface has both a normal and a shear component. The focus of this section is on the shear component of the force. These shear forces are produced by shear stress at the interface and are called **friction forces**.

Traditionally friction forces are classified into two categories: dry friction (Coulomb friction) or fluid friction. **Dry friction** is a shear force produced at a solid-solid interface and occurs regardless of whether or not there is relative motion (sliding) at the interface. **Fluid friction**, by contrast, is a shear force produced at a solid-fluid interface by the relative motion between the bulk fluid and the solid. The major characteristics of these friction models are presented in the table below:

Dry Friction Model	Fluid Friction Model
Magnitude of the force acting on a system assumed to depend on • the magnitude of the normal force N at the boundary and • the relative motion (sliding) between the surfaces. If the *surfaces do not slide* (no relative motion), $$F_f \leq F_{s,\max} = \mu_s N$$ $$\begin{aligned} \text{where} \quad F_f &= \text{the friction force} \\ F_{s,\max} &= \mathit{maximum} \text{ static–friction force possible,} \\ \mu_s &= \text{coefficient of static friction} \\ N &= \text{the normal force at the interface} \end{aligned}$$ If the surfaces do slide (relative motion), $$F_f = F_k = \mu_k N$$ $$\begin{aligned} \text{where} \quad F_k &= \text{the kinetic-friction force} \\ \mu_k &= \text{coefficient of kinetic friction} \end{aligned}$$ **Direction of the force** acting *on* a system is in the plane of contact and in the direction of relative motion of the external object.	**Magnitude of the force** acting on a system is assumed to depend on the relative velocity between the surface and the fluid free stream. Models for fluid friction: *Viscous friction forces* occur between close-fitting surfaces with a gas or liquid lubricant at the interface and its magnitude is modeled as being proportional to the relative velocity between the surfaces (the sliding velocity), V_{slide} : $$F_{\text{viscous}} = k_1 V_{\text{slide}}$$ *Fluid dynamic drag forces* are exerted on any object immersed in a moving fluid, and its magnitude is modeled as being proportional to the fluid velocity squared: $$F_{\text{drag}} = k_2 V_{\text{fluid}}^2$$ where V_{fluid} is the velocity of the bulk fluid measured relative to the solid surface. **Direction of the force** acting *on* a system is in the direction of the fluid velocity V_{fluid} (or sliding velocity V_{slide}) measured relative to the surface.

Note the significant differences between fluid friction and dry friction. A dry friction force is proportional to the normal force at the interface between between the two solids and occurs with or without relative motion. With no sliding, we have static friction; with sliding, we have kinetic (or sliding) friction. In contrast, a fluid friction force is independent of the normal force at the interface and is zero when there is no relative motion between the bulk fluid and the solid surface. Detailed information about the motion of the fluid is required to evaluate the proportionality constants in the fluid friction models. In this section we will concentrate on dry friction. A detailed discussion of fluid friction, especially how to find the proportionality constants, will be reserved for a later course, e.g. ES 202—Fluid and Thermal Systems.

To explore dry friction in more detail, consider a block resting on a surface as shown in Figure $5.3.1$. Under what conditions will the block move and what is the value of the friction force at the interface? For our study, select a closed system that encloses just the block. Note that the lower boundary of our system is placed at the interface between the block and the horizontal surface that it rests on. A linear momentum system diagram (free body diagram) showing all linear momentum interactions with the surroundings is also shown in the figure.

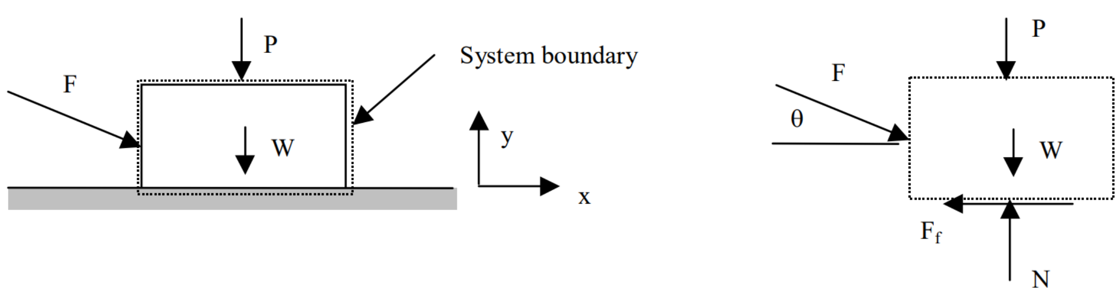


Figure 5.3.1: Block moving on a horizontal surface with friction.

Writing the rate-form of conservation of linear momentum in vector form we have for the closed system:

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \mathbf{F} + \mathbf{P} + \mathbf{F}_{\text{f}} + \mathbf{N}$$

where $\mathbf{F_f}$ is the dry friction force at the interface between the block and the surface. We can simplify this equation by recalling that for a closed system $\mathbf{P}_{\text{sys}} = m_{\text{sys}}\mathbf{V}_G$. To write this equation in terms of scalar components use the (x, y) coordinate system shown in the figure. In the x-direction we have the following:

$$\frac{d\left(mV_x\right)}{dt} = F\cos\theta - F_{\text{f}} \quad \rightarrow \quad m\,\frac{dV_x}{dt} = F\cos\theta - F_{\text{f}}$$

In the y-direction, we have the following result:

$$\begin{aligned} \frac{d\left(mV_y\right)}{dt} &= N - P - W - F\sin\theta \\ \underbrace{m\,\frac{dV_y}{\cancel{dt}}^{=0}}_{\substack{\text{No motion in} \\ y \text{ direction}}} &= N - P - W - F\sin\theta \quad \rightarrow \quad N = P + W + F\sin\theta \end{aligned}$$

Note that the normal force N does not just equal the weight of the block. Please be careful in determining the normal force. Students often *assume* that the friction force is proportional to the weight of the object. This mistake is the result of carelessly assuming that the normal force always equals the weight.

To investigate how the friction force influences the system behavior, we apply the *dry friction model*:

$$\begin{aligned} \text{No sliding:} \quad V_x = 0 \quad &\rightarrow \quad F_{\text{f}} = F\cos\theta \leq \mu_{\text{s}} N \\ \text{Surfaces sliding:} \quad V_x \neq 0 \quad &\rightarrow \quad F_{\text{f}} = \mu_{\text{k}} N \end{aligned}$$

where V_x is the relative (sliding) velocity at the interface between the surfaces in contact. Typically the coefficient of kinetic friction is approximately 75% of the coefficient of static friction.

If there is no sliding, i.e. relative velocity between the surfaces is zero, the friction force can assume a range of values less than or equal to the value of the maximum possible static-friction force. Students frequently assume that the static-friction force is single valued and always equals the maximum possible static-friction force. This is incorrect and a major cause of errors. Only when sliding is impending does the value of the friction force equal that of the maximum static-friction force.

By contrast, if the surfaces are sliding, the friction force has a single value and equals the kinetic-friction force. Note that the kinetic-friction force only depends on the coefficient of kinetic friction and the normal force.

In many problems with dry friction it is unclear whether or not the system slides due to the applied loads. In these problems, the system (block) can behave in one of three ways:

Case	Relative motion at the interface	Friction Force
I	No motion — The block is not sliding.	$F_{\text{f}} < \mu_{\text{s}} N$
II	Impending motion — The block is not sliding but is on the verge of sliding.	$F_{\text{f}} = \mu_{\text{s}} N$
III	Motion — The block is sliding.	$F_{\text{f}} = \mu_{\text{k}} N$

When the motion is indeterminate do the following:

- First assume there is no sliding and determine the *required friction force* to maintain this condition.
- Next, compare the required friction force against the maximum possible static friction force, $F_{\text{s,max}} = \mu_{\text{s}} N$.
- If the required friction force is less than or equals the maximum possible static friction force, your assumption of no sliding was correct and the actual friction force at the interface equals the required friction force calculated previously.

- If the required friction force exceeds the maximum possible static friction force, your assumption of no sliding was incorrect. Under these conditions, the system slides and the actual friction force equals the kinetic-friction force.

Alternatively, instead of assuming something about the motion and then solving for the friction force required, you can determine the value of the external forces that are required for impending motion. To do this, assume that sliding is impending and the friction force equals the maximum possible static-friction force. Then determine the external forces required for this condition. Actual external forces with values that exceed this required value will produce sliding, and the friction force will equal the kinetic-friction force value. Smaller values will result in no motion, and the actual friction force can be predicted using the actual external forces.

? Test Your Knowledge

To test your understanding, reconsider the problem discussed above. Using the graph shown below plot a graph showing how the friction force F_{f} generated at the block interface changes as the applied force F increases from zero with $\theta = 0$.

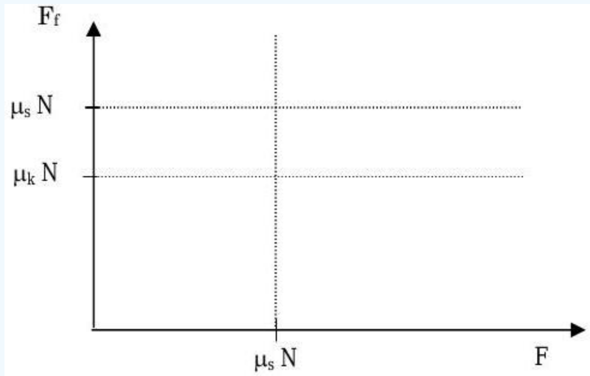


Figure $5.3.2$: Coordinate system for graphing F_{f} vs F.

5.4: Linear Impulse, Linear Momentum, and Impulsive Forces

As we have already demonstrated, it is sometimes necessary to integrate the rate form of the conservation of linear momentum equation over a specified time interval. This gives a relationship between the *change of linear momentum within the system* and the *amount of linear momentum transported into the system*. Historically, these types of calculations have been done through the introduction of a quantity known as the *linear impulse*. Under certain conditions, a system will be subjected to a relatively *large* force over a very *short* time interval, such as the contact between two billiard balls or the impact between two cars as they collide. These large, short-duration forces are known as *impulsive forces* and the interaction is referred to as an *impact*. In this section we will demonstrate that impact calculations for impulse and impulsive forces follow naturally from our understanding of the basic conservation of linear momentum equation.

5.4.1 Linear Impulse

When a force acts on a system boundary, linear momentum flows across the boundary at a specified rate — the greater the magnitude of the force, the greater the transport rate of linear momentum. If we consider a simple particle with a single force $\mathbf{F}$ acting on it, we know that the rate of change of the linear momentum of the system is

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \mathbf{F}$$

By definition, the **linear impulse** of the force $\mathbf{F}$ is the integral of the force with respect to time:

$$\mathbf{Impulse}_{t_1 \to t_2} = \int_{t_1}^{t_2} \mathbf{F}\, dt$$

As can be seen in Figure $5.4.1$, this represents the area under a $\mathbf{F}$-t curve.

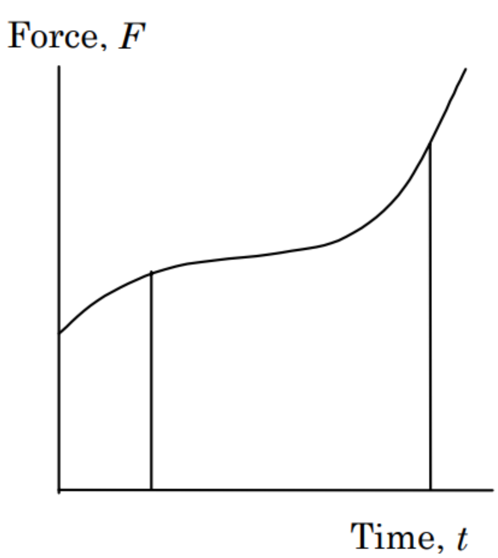


Figure $5.4.1$: $\mathbf{F}$-t curve.

For the particle, we see that the impulse of force $\mathbf{F}$ just equals the change in linear momentum of the particle:

$$\mathbf{Impulse}\big|_{\mathbf{F},\, t_1 \to t_2} = \int_{t_1}^{t_2} \mathbf{F}\, dt = \int_{t_1}^{t_2} \left(\frac{d\mathbf{P}_{\text{sys}}}{dt} \right) dt = \mathbf{P}_{\text{sys},2} - \mathbf{P}_{\text{sys},1}$$

For any system, the impulse of a given force equals the amount of linear momentum transferred across the system boundary in the specified time interval. It does not always equal the change in linear momentum of the system. Frequently, it is impossible to directly measure the magnitude of the force as a function of time. In this case, it is common practice to talk about the impulse of the force and, if possible, to determine the impulse by measuring the change in linear momentum of the system.

5.4.2 Impulsive Forces

The inability to measure the details of a given force as a function of time is especially true during impacts. An **impulsive force** is a relatively *large* force that acts over a very *short* time period — for instance, when a bowling ball hits a bowling pin. When this occurs, the bowling ball transfers linear momentum to the pin by a short-duration contact force. The same thing would be true of the linear momentum imparted to a homerun pitch as the baseball impacts the slugger's bat for a very short period of time. For a very small interval of time, it is possible to *estimate* the magnitude of the *average impulsive force* by assuming that the force is constant over the small time interval:

$$\mathbf{F}_{\text{avg}}\,\Delta t = \int_{t}^{t+\Delta t} \mathbf{F}\,dt = \mathbf{Impulse}\Big|_{\mathbf{F},\,t_1\to t_2} \quad \to \quad \mathbf{F}_{\text{avg}} = \frac{1}{\Delta t}\int_{t}^{t+\Delta t} \mathbf{F}\,dt$$

As shown in Figure $5.4.2$, the average impulsive force equals the constant force that acts over the same time interval and transfers the same amount of linear momentum as the original time-varying force.

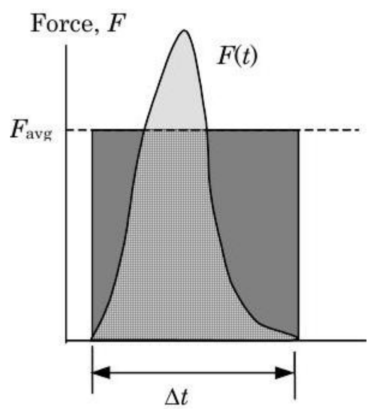


Figure $5.4.2$: Impulsive force.

Test Your Understanding — Estimating the peak force during impulse loading

In calculating the average impulsive force above we have assumed that the shape and the area of the $\mathbf{F}$-t curve can best be approximated by a rectangular box of height F_{avg} and width Δt. There are other possible shapes we could use to approximate the $\mathbf{F}$-t curve. Suppose we approximated the area under the $\mathbf{F}$-t curve using an isosceles triangle of height F_{peak} and base Δt.

What is the relationship between F_{avg} and F_{peak} for a impulsive loading situation of duration Δt ? Which value gives a better estimate of the maximum force experienced by the system during the impact and why is it better?

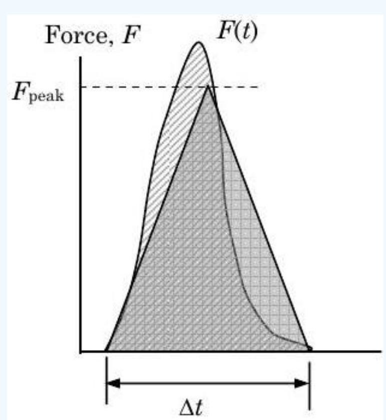


Figure $5.4.3$: Impulsive force approximation using isosceles triangle.

Example — Colliding boxcars

Two boxcars are to be coupled together in a railroad switchyard. Boxcar A is moving to the right with a velocity of $3 \mathrm{~km/h}$ and is to be coupled to boxcar B, which is initially stationary. Boxcar A has a mass of $45,000 \mathrm{~kg}$ and boxcar B has a mass of $25,000 \mathrm{~kg}$.

Determine (a) the final velocity of the coupled cars, and (b) the average impulsive force acting on each car during the coupling if the coupling process takes $0.3 \mathrm{~s}$.

Solution

Known: Two boxcars collide and are coupled together.

Find: (a) final velocity of the coupled cars

(b) average impulsive force on cars if coupling process takes $0.3 \mathrm{~s}$.

Given:

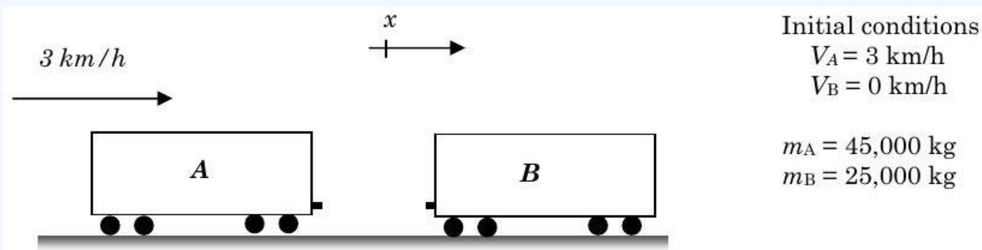


Figure $5.4.4$: Initial conditions and relative positions of two boxcars.

Analysis:

Strategy $\rightarrow$ Since this problem involves two objects colliding with an impact, try looking at conservation of linear momentum.

System $\rightarrow$ I'm not sure yet, I'll decide this in a minute.

Property to count $\rightarrow$ Linear momentum in the x-direction.

Time interval $\rightarrow$ Finite time since interested in behavior before and after.

Now consider what system to pick. Let's try using a moving closed system that includes only the boxcars.

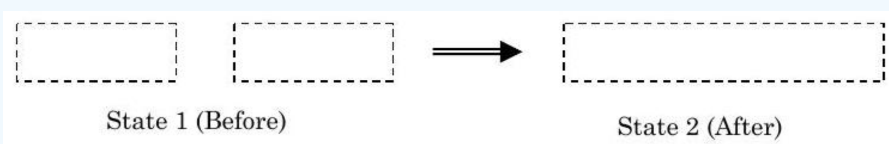


Figure $5.4.5$: Choice of system, before and after the collision.

Notice that if we only consider x-momentum, there are no forces acting on the system with a component in the x-direction. So the conservation of linear momentum equation becomes

$$\frac{dP_{x,\,\text{sys}}}{dt} = \sum \underbrace{F_{x\,\text{external}}}_{\text{No } x\text{-forces}} + \underbrace{\sum_i \dot{m} \overbrace{\cancel{V_{x,i}}}^{=0} - \sum_e \dot{m} \overbrace{\cancel{V_{x,e}}}^{=0}}_{\text{Closed System}} \rightarrow P_{x,\,\text{sys}} = \text{constant}$$

Using this result to relate the linear momentum of the system before and after the impact gives the following:

$$\begin{aligned} P_{x,\,1} &= m_A V_{A,\,1} + m_B \overbrace{\cancel{V_{B,1}}}^{=0} \\ P_{x,2} &= m_A \overbrace{\cancel{V_{A,2}}}^{=V_2} + m_B \overbrace{\cancel{V_{B,2}}}^{=V_2} \end{aligned} \rightarrow \quad \rightarrow \quad \begin{aligned} P_{x,\,1} &= P_{x,\,2} \\ m_A V_{A,\,1} &= (m_A + m_B)\, V_2 \end{aligned} \quad \rightarrow \quad V_2 = \frac{m_A}{(m_A + m_B)} V_{A,\,1}$$

Solving for the final velocity we have $V_2 = \dfrac{m_A}{(m_A + m_B)} V_{A,\,1} = \dfrac{45}{(45+25)} \left(3 \ \dfrac{\mathrm{km}}{\mathrm{h}}\right) = 1.93 \mathrm{~km/h}$.

And since this is a positive number, the coupled trains will continue to move to the right (positive x-direction). This is the answer to Part (a).

Now to determine the average impulsive force, we must place a boundary where the force occurs. Consider a closed system that includes only boxcar B. Starting with the conservation of linear momentum for this closed system gives $\frac{dP_{x,\text{ sys}}}{dt} = F_{\text{coupling}}$.

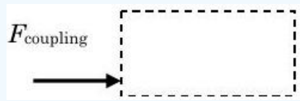


Figure 5.4.6: Closed system consisting of Boxcar B.

But to find the average coupling force on car B, we must integrate this expression over the time interval:

$$\int_{t_1}^{t_2} \left(\frac{dP_{x,\text{ sys}}}{dt} \right) dt = \int_{t_1}^{t_2} F_{\text{coupling}}\, dt \quad \rightarrow \quad P_{x,\,2} - P_{x,\,1} = F_{\text{coupling, avg}}\, \Delta t$$

Solving for the force we have

$$\begin{aligned} F_{\text{coupling, avg}} &= \frac{P_{x,\,2} - P_{x,\,1}}{\Delta t} = m_B \frac{V_{B,\,2} - V_{B,\,1}}{\Delta t} \\ &= (25{,}000 \text{ kg}) \left[\frac{(1.93 - 0)\ \frac{\text{m}}{\text{h}}}{(0.3 \text{ s})} \right] \left(\frac{\text{h}}{3600 \text{ s}} \right) \left(\frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \right) = 44.7 \times 10^3 \text{ N} = 44.7 \text{ kN} \end{aligned}$$

Since the value is positive, the direction of the coupling force on car B is as shown in the figure. The coupling force acting on car A is of the same magnitude and opposite direction.

Comment

(1) Consider an alternate system for solving Part (a). This time assume an open system that initially includes car B and finally includes both cars.

Starting with the linear momentum equation:

$$\frac{dP_{x,\text{ sys}}}{dt} = \underbrace{\sum \cancel{F_x}^{=0}}_{\text{No forces in } x\text{-direction}} + \dot{m}_i V_{x,\,i} - \underbrace{\dot{m}_e \cancel{V_e}^{=0}}_{\text{No flow out of system}} \quad \rightarrow \quad \frac{dP_{x,\text{ sys}}}{dt} = \underbrace{\dot{m}_i V_{x,\,i}}_{\substack{x\text{-momentum carried} \\ \text{in with boxcar A}}}$$

To find the velocity (or linear momentum) of the system after the coupling we must integrate this equation over the time interval:

$$\int_{t_1}^{t_2} \left(\frac{dP_{x,\text{ sys}}}{dt} \right) dt = \underbrace{\int_{t_1}^{t_2} (\dot{m}_i V_i)\, dt}_{\substack{= \text{all of the momentum} \\ \text{carried into the system} \\ \text{by boxcar A}}} \quad \rightarrow \quad \underbrace{P_{x,\,2}}_{\substack{\text{Includes} \\ \text{both cars}}} - \underbrace{P_{x,\,1}}_{\substack{\text{Only} \\ \text{car B}}} = m_A V_A \quad \rightarrow \quad (m_A + m_B) V_2 = m_A V_A$$

The solution continues on from this point as above. Notice how starting with a different system, even an open system, we can recover the same equation using consistent assumptions. This illustrates that is often possible to solve the same problem using several different systems.

(2) Select an open system that has mass flowing in and out during the impact and solve for the final velocity of the coupled cars. [Hint: This system would have no linear momentum in either the initial or final states.]

(3) How long would it take the coupled cars to come to a complete stop if the brakes on car B were locked at the time of impact and the coefficient of kinetic friction between the wheels and the rail is 0.10? What would the velocity of the coupled cars be immediately after the impact? Would it be reasonable to neglect the friction force during the impact? [Answer: 1.53 seconds].

5.5: Linear Momentum Revisited

This section considers additional aspects of calculating linear momentum. The first subsection considers the use of relative velocities and the second subsection addresses the use of cylindrical coordinates to describe curvilinear plane motion.

Using Relative Velocities

The following example illustrates how conservation of linear momentum can be used to solve for the motion of a rocket that is accelerating. This example makes use of relative velocities.

Example — An Accelerating Rocket

A small rocket fully loaded with fuel and oxidizer has a mass of $500\ \mathrm{Mg}$, and 80% of the total mass is fuel and oxidizer. The rocket is pointed vertically up and is placed on a launching pad on the surface of Earth. After ignition, the mass flow rate of the exhaust from the rocket engine is $2300\ \mathrm{kg/s}$ with an exit velocity of $2100\ \mathrm{m/s}$ relative to the rocket. Neglecting pressure forces and fluid drag, determine (a) how long after ignition does burnout occur (i.e. the rocket runs out of fuel and oxidizer)? (b) how long after ignition does liftoff occur? and (c) what is the velocity of the rocket at burnout?

<u>**Solution**</u>

Known: A small rocket takes off from the surface of the earth.

Find: (a) time after ignition until burnout; (b) time after ignition until liftoff; (c) velocity of the rocket at burnout.

Given:

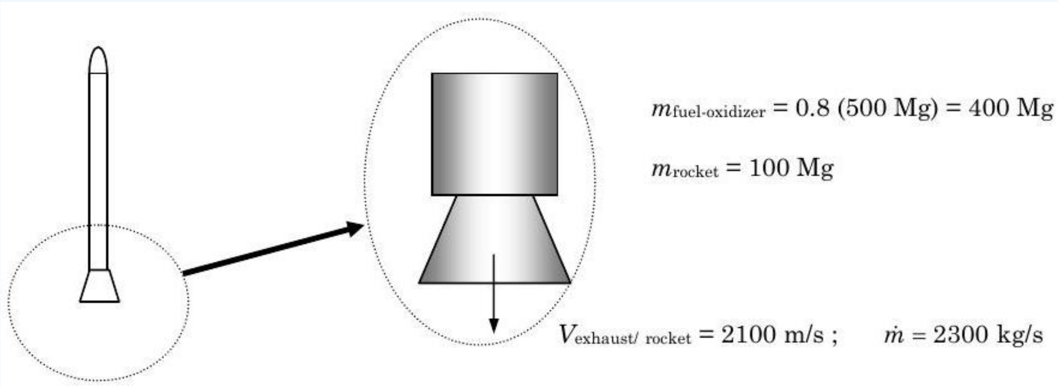


Figure $5.5.1$: Given information.

Analysis:

Strategy $\rightarrow$ Since we are concerned about velocity of an accelerating object and about the time needed to run out of the mass of fuel and oxidizer, let's try using mass and linear momentum.
System $\rightarrow$ Take an open, non-deforming system which includes the rocket with mass crossing the boundary at the exhaust.
Property to count $\rightarrow$ As stated above, try mass and linear momentum.
Time interval $\rightarrow$ Start with rate form and then see what's needed.

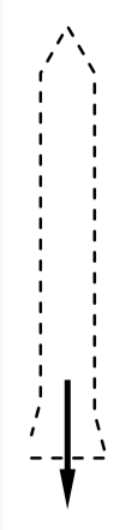


Figure 5.5.2: Choice of system and direction of mass flow across boundary.

For the system above, the mass flows out of the bottom of the rocket as indicated by the arrow. To solve part (a), write the conservation of mass for the system; then by integrating with time, we have:

$$\frac{dm_{\text{sys}}}{dt} = -\dot{m}_e \quad \rightarrow \quad \int_{t_1}^{t_2} \left(\frac{dm_{\text{sys}}}{dt}\right) dt = \int_{t_1}^{t_2} (-\dot{m}_e)\, dt \quad \rightarrow \quad m - m_o = -\dot{m}_e t$$

$$m = m_o - \dot{m}_e t \quad \rightarrow \quad m = (500,000 \text{ kg}) - \left(2300 \ \frac{\text{kg}}{\text{s}}\right) t$$

Notice that the mass flow rate is calculated with respect to the rocket exhaust. At burnout, the mass of the system equals just the mass of the empty rocket:

$$t_{\text{burnout}} = \frac{(500,000 \text{ kg} - 100,000 \text{ kg})}{\left(2300 \ \frac{\text{kg}}{\text{s}}\right)} = 173.9 \text{ seconds}$$

Now to solve for the liftoff time, we need to apply the conservation of linear momentum:

$$\boxed{\uparrow +} \qquad \frac{dP_{y,\text{ sys}}}{dt} = -mg - \dot{m}_e V_{y,\,e}$$

$$\frac{d\,(mV_G)}{dt} = -mg - \dot{m}_e \underbrace{\left(V_G - V_{\text{exhaust / rocket}}\right)}_{\substack{\text{Absolute velocity of the exhaust} \\ \text{with respect to the ground}}}$$

$$m\frac{dV_G}{dt} + V_G \underbrace{\overset{=-\dot{m}_e}{\frac{dm}{\cancel{dt}}}}_{\substack{\text{Using conservation} \\ \text{of mass results}}} = -mg - \dot{m}_e \left(V_G - V_{\text{exhaust / rocket}}\right)$$

$$m\frac{dV_G}{dt} + V_G(-\dot{m}_e) = -mg - \dot{m}_e \left(V_G - V_{\text{exhaust / rocket}}\right)$$

$$m\frac{dV_G}{dt} = -mg - \dot{m}_e \left(V_G - V_{\text{exhaust / rocket}}\right) + \dot{m}_e V_G$$

$$m\frac{dV_G}{dt} = -mg - \cancel{\dot{m}_e V_G} + \dot{m}_e V_{\text{exhaust / rocket}} + \cancel{\dot{m}_e V_G}$$

$$m\frac{dV_G}{dt} = -mg + \dot{m}_e V_{\text{exhaust / rocket}}$$

Carefully study the steps above. Note that you must be careful to use the absolute velocity of the exhaust leaving the exhaust. Also note that we make use of the conservation of mass equation in simplifying the conservation of linear momentum.

To solve for part (b), the liftoff time, we can use the equation above. At liftoff the acceleration of the system is zero, as the thrust just balances the weight of gravity:

$$m \overbrace{\cancel{\frac{dV_G}{dt}}}^{=0} = -mg + \dot{m}_e V_{\text{exhaust / rocket}} \quad \rightarrow \quad \begin{aligned} mg &= \dot{m}_e V_{\text{exhaust / rocket}} \\ (m_o - \dot{m}_e t)\, g &= \dot{m}_e V_{\text{exhaust / rocket}} \\ m_o g &= \dot{m}_e \left(gt + V_{\text{exhaust / rocket}}\right) \end{aligned} \quad \rightarrow$$

$$t_{\text{liftoff}} = \frac{m_o}{\dot{m}_e} - \frac{V_{\text{exhaust / rocket}}}{g}$$

Then solving for the liftoff time:

$$t_{\text{liftoff}} = \frac{m_o}{\dot{m}_e} - \frac{V_{\text{exhaust / rocket}}}{g} = \left(\frac{500,000 \text{ kg}}{2300 \text{ kg/s}}\right) - \left(\frac{2100 \text{ m/s}}{9.81 \text{ m/s}^2}\right) = 3.32 \text{ seconds}$$

Finally to solve for the velocity at burnout, we must calculate the velocity of the rocket as a function of time. We do this by integrating the acceleration equation for the rocket. First rearranging the conservation of linear momentum result, we have an expression for the acceleration of the rocket:

$$\frac{dV_G}{dt} = -g + \frac{\dot{m}_e}{m} V_{\text{exhaust / rocket}}$$

Integrating this with time from the point of liftoff gives the following:

$$\begin{aligned} \int_{t_{\text{liftoff}}}^{t} \left(\frac{dV_G}{dt}\right) dt &= \int_{t_{\text{liftoff}}}^{t} \left[-g + \frac{\dot{m}_e V_{\text{exhaust / rocket}}}{(m_o - \dot{m}_e t)}\right] dt \\ V_G - V_G\big|_{t_{\text{liftoff}}} &= \int_{t_{\text{liftoff}}}^{t} [-g]\, dt + \int_{t_{\text{liftoff}}}^{t} \left[\frac{\dot{m}_e V_{\text{exhaust / rocket}}}{(m_o - \dot{m}_e t)}\right] dt \\ V_G - \overbrace{\cancel{V_G\big|_{t_{\text{liftoff}}}}}^{=0} &= -g\,(t - t_{\text{liftoff}}) + \dot{m}_e V_{\text{exhaust / rocket}} \left[\frac{1}{\dot{m}_e} \int_{t_{\text{liftoff}}}^{t} \frac{\dot{m}_e\, dt}{(m_o - \dot{m}_e t)}\right] \\ V_G &= -g\,(t - t_{\text{liftoff}}) + \dot{m}_e V_{\text{exhaust / rocket}} \left[-\frac{1}{\dot{m}_e} \ln\left(\frac{m_o - \dot{m}_e t}{m_o - \dot{m}_e t_{\text{liftoff}}}\right)\right] \\ V_G &= -g\,(t - t_{\text{liftoff}}) - V_{\text{exhaust / rocket}} \left[\ln\left(\frac{m_o - \dot{m}_e t}{m_o - \dot{m}_e t_{\text{liftoff}}}\right)\right] \end{aligned}$$

Note that this equation is only valid for $t_{\text{liftoff}} \leq t \leq t_{\text{burnout}}$. Solving for the velocity at burnout gives

$$\begin{aligned} V_G &= -g\,(t_{\text{burnout}} - t_{\text{liftoff}}) - V_{\text{exhaust / rocket}} \left[\ln\left(\frac{m_o - \dot{m}_e t_{\text{burnout}}}{m_o - \dot{m}_e t_{\text{liftoff}}}\right)\right] \\ &= -\left(9.81 \, \frac{\text{kg}}{\text{s}}\right)(173.9 - 3.32)\text{s} - \left(2100 \, \frac{\text{m}}{\text{s}}\right)\left[\ln \frac{(100 \text{ Mg})}{\left(500 \text{ Mg} - 2.3 \, \frac{\text{Mg}}{\text{s}}(3.32 \text{ s})\right)}\right] \\ &= \left(-1673 \, \frac{\text{m}}{\text{s}}\right) - \left(-3348 \, \frac{\text{m}}{\text{s}}\right) \\ &= 1675 \text{ m/s} \end{aligned}$$

Comments

How far above the earth would the rocket be at burnout? How could you calculate this? Do you think our model is accurate all the way to burnout? What possible problems would you see?

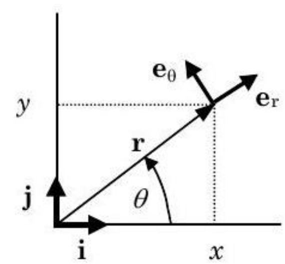
If the diameter of the rocket nozzle exhaust is $5\ \mathrm{m}$ and the exhaust pressure is $10\ \mathrm{kPa}$, determine the net pressure force on the system. Assume atmospheric pressure is $100\ \mathrm{kPa}$. Was neglecting the pressure force reasonable?

Motion with Cylindrical Coordinates

The following application shows how cylindrical coordinates can be used to describe plane curvilinear motion.

Appication — Plane curvilinear motion in cylindrical coordinates

Sometimes it is advantageous to describe the motion of a point in a plane in terms of the distance r from the origin and the angle θ measured from the positive x-axis, i.e. cylindrical coordinates. Using the definitions of velocity and acceleration as first and second derivatives of position with respect to time and the trigonometric relations between $x, y, r,$ and θ the following relationships can be developed:



$x = r(\cos\theta)$

$y = r(\sin\theta)$

	Rectangular Coordinates	Cylindrical Coordinates
Position	$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$	$\mathbf{r} = r\,\mathbf{e}_r$
Velocity	$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j}$	$\mathbf{V} = \frac{dr}{dt}\mathbf{e}_r + r\omega\,\mathbf{e}_\theta$
Acceleration	$a = a_x\mathbf{i} + a_y\mathbf{j}$	$\mathbf{a} = \left(\frac{d^2r}{dt} - r\omega^2\right)\mathbf{e}_r + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\mathbf{e}_\theta$

where $\omega = \frac{d\theta}{dt}$, the **angular velocity** in radians per second (rad/s)

$\alpha = \frac{d\omega}{dt}$, the **angular acceleration** in radians per second squared (rad/s²)

Unit vector relations:

$$\mathbf{e}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \qquad \frac{d}{dt}\mathbf{e}_r = \omega\,\mathbf{e}_\theta$$

$$\mathbf{e}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} \qquad \frac{d}{dt}\mathbf{e}_\theta = -\omega\,\mathbf{e}_r\ ;$$

Note that the direction of the unit vectors **i** and **j** are constants; however, both of the cylindrical coordinate unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ change with θ.

Figure 5.5.3: Table of unit vector and position, velocity, and acceleration equation relationships between rectangular and cylindrical coordinates.

(a) Starting with the position vector $\mathbf{r}$, show that $\mathbf{e_r} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$.

(b) Now differentiate $\mathbf{e_r}$ with respect to time and prove to yourself that $\frac{d}{dt}\mathbf{e_r} = \omega\mathbf{e}_\theta$.

(c) How do the relations for $\mathbf{r}$, $\mathbf{V}$, and $\mathbf{a}$ in cylindrical coordinates simplify if $r = R$, a constant?

(d) How do the relations for $\mathbf{r}$, $\mathbf{V}$, and $\mathbf{a}$ in cylindrical coordinates simplify if $\omega = \text{constant}$?

Example — Linear momentum of a circular disk

A circular disk of radius R and thickness t is made of a material with density ρ. Determine the linear momentum for the disk under the following conditions:

(a) The disk is rotating with a rotational velocity ω about its centroidal axis, G. Even though this disk appears to touch the ground, it is slipping and is *not* rolling.

(b) The disk is sliding to the right without rotating, i.e. $\omega = 0$. The disk is translating with velocity $\mathbf{V}$.

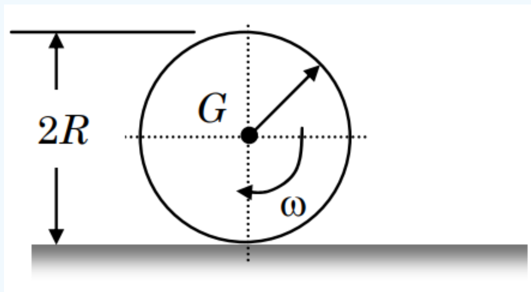


Figure 5.5.4: Given information about disk.

Solution:

Known: A circular disk moves with specified motion.

Find: (a) Linear momentum when the disk is rotating about its centroidal axis.
(b) Linear momentum when the disk is translating with velocity $\mathbf{V}$.

Given: See figure above.

Analysis: Strategy $\rightarrow$ Use the definition of linear momentum.

(a) The velocity of the rotating disk measured with respect to the center of mass G is $\mathbf{V} = r\omega\mathbf{e}_\theta = r\omega\left(-(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}\right)$. Using the definition for linear momentum of a system we have

$$\begin{aligned}
\mathbf{P}_{\text{sys}} &= \int_{\forall_{\text{sys}}} (\mathbf{V}\rho)d\forall = \int_0^{2\pi}\int_0^R \underbrace{(r\omega\mathbf{e}_\theta)}_{\mathbf{V}}\rho\underbrace{(r\,d\theta\,dr)}_{d\forall} \\
&= (\rho\omega)\int_0^{2\pi} \underbrace{((\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j})}_{\mathbf{e}_\theta}\left(\int_0^R r^2\,dr\right)d\theta = (\rho\omega)\int_0^{2\pi}\underbrace{((\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j})}_{\mathbf{e}_\theta}\left(\frac{R^3}{3}\right)d\theta \\
&= \left(\frac{\rho\omega R^3}{3}\right)\int_0^{2\pi}((\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j})\;d\theta \\
&= \left(\frac{\rho\omega R^3}{3}\right)\left[\mathbf{i}\int_0^{2\pi}(\sin\theta)\;d\theta - \mathbf{j}\int_0^{2\pi}(\cos\theta)\;d\theta\right] = \left(\frac{\rho\omega R^3}{3}\right)(0) = 0
\end{aligned}$$

Thus, as you might have guessed the linear momentum of a disk rotating about its centroidal axis is identically zero. In general, the linear momentum of a system rotating about a fixed axis is always zero if (1) the axis of rotation passes through the center of mass, i.e. its centroidal axis, and (2) the system has a plane of symmetry that is perpendicular to the axis of rotation. If either of these two conditions is violated, the linear momentum will not be zero.

Note that this integration could have been simplified greatly if we had used the integral for velocity of the center of mass, Eq. (5.1.16). The center of mass G of the disk is on the axis of rotation and the axis of rotation is not translating; thus,

$$\mathbf{P}_{\text{sys}} = \int\limits_{\forall_{\text{sys}}} (\mathbf{V}\rho) d\forall = m_{sys}\ V_{\cancel{G}}^{=0} = 0$$

and the linear momentum of the system is zero. This is much easier than doing the full integration!

(b) Starting with the definition of linear momentum and the fact that the disk is translating with velocity $\mathbf{V}$, we have

$$\begin{aligned}
\mathbf{P}_{\text{sys}} &= \int\limits_{\forall_{\text{sys}}} (\mathbf{V}\rho) d\forall = \int\limits_0^{2\pi} \int\limits_0^R (\mathbf{V}\rho) \underbrace{(rt\ d\theta\ dr)}_{d\forall} = (\mathbf{V}\rho t) \int\limits_0^{2\pi} \int\limits_0^R r\ dr\ d\theta \\
&= (\mathbf{V}\rho t) \int\limits_0^{2\pi} \left(\int\limits_0^R r\ dr \right) d\theta = (\mathbf{V}\rho t) \int\limits_0^{2\pi} \left(\frac{R^2}{2} \right) d\theta \\
&= (\mathbf{V}\rho t) \left(\frac{R^2}{2} \right) \int\limits_0^{2\pi} d\theta = (\mathbf{V}\rho t) \left(\frac{R^2}{2} \right) (2\pi) \\
&= \underbrace{\left(\rho\left(\pi R^2 t\right)\right)}_{\rho\forall_{\text{sys}}} \mathbf{V} = m_{sys}\mathbf{V}
\end{aligned}$$

Thus the linear momentum is just the product of the translational velocity and the system mass. Again, this calculation can be greatly simplified using the equation for the velocity of the center of mass:

$$\mathbf{P}_{\text{sys}} = \int_{\forall_{\text{sys}}} (\mathbf{V}\rho) d\forall = m_{sys}\ \mathbf{V}_{\cancel{G}}^{=\text{V}} = m_{sys}\mathbf{V}$$

5.6: Problems

Problem 5.1

The motion of a particle is described by the relation

$$x=\left(3.0\ \frac{\mathrm{m}}{\mathrm{s}^3}\right)t^3-\left(6.0\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2-\left(12.0\ \mathrm{ft}\frac{\mathrm{m}}{\mathrm{s}}\right)t+(5.0\ \mathrm{ft})$$

The particle has a mass of 5 kilograms and all motion is in a horizontal plane, i.e. gravity has no effect. Answer the following questions:

(a) Determine when the velocity is zero.

(b) Determine the position and the total distance traveled when the acceleration is zero.

(c) Determine the *linear momentum* and the *rate of change of the linear momentum* of the particle for $t=0,1,2,3,$ and 4 seconds.

(d) Graph the *linear momentum* and the *rate of change of the linear momentum* of the particle as a function of time for $0 \leq t \leq 4$ seconds. [Note: you may need more than 5 points to accurately plot this function.]

- What is the maximum value of the linear momentum of the particle during this time interval and when does it occur?
- What is the maximum value of the *rate of change* of the linear momentum of the particle and when does it occur?

(e) Using conservation of linear momentum for this particle, determine the *net external force* acting on the particle for $t=0,1,2,3,$ and 4 seconds.

(f) Graph the *net external force* acting on the particle as a function of time for $0 \leq t \leq 4$ seconds. [Note: you may need more than 5 points to accurately plot this function.]

- What is the maximum value of the net external force acting on the particle during this time interval and when does it occur?
- Does the direction of the net external force change during this time interval? If so, when does it occur?

(g) Compare your results for (c), (d), (e), and (f). Comment on similarities and differences.

Problem 5.2

A rocket sled weighing $3,220\ \mathrm{lbf}$ reaches a constant speed of $700\ \mathrm{ft/s}$ under a thrust force of $8,000\ \mathrm{lbf}$. The primary resistance to motion is a fluid drag force with a magnitude of $F_{\text{drag}}=kV^2$ and a direction that opposes the motion.

(a) Determine the value of the constant k in the drag force model.

(b) If the engine is shut off, determine how long it takes and how far the sled travels as it slows down to $70\ \mathrm{ft/s}$.

Problem 5.3

Two cables are tied together at C and are loaded as shown. The entire system is stationary. You *must* sketch a complete linear momentum interaction diagram (also called a free-body diagram) for each problem clearly showing the important transports of linear momentum.

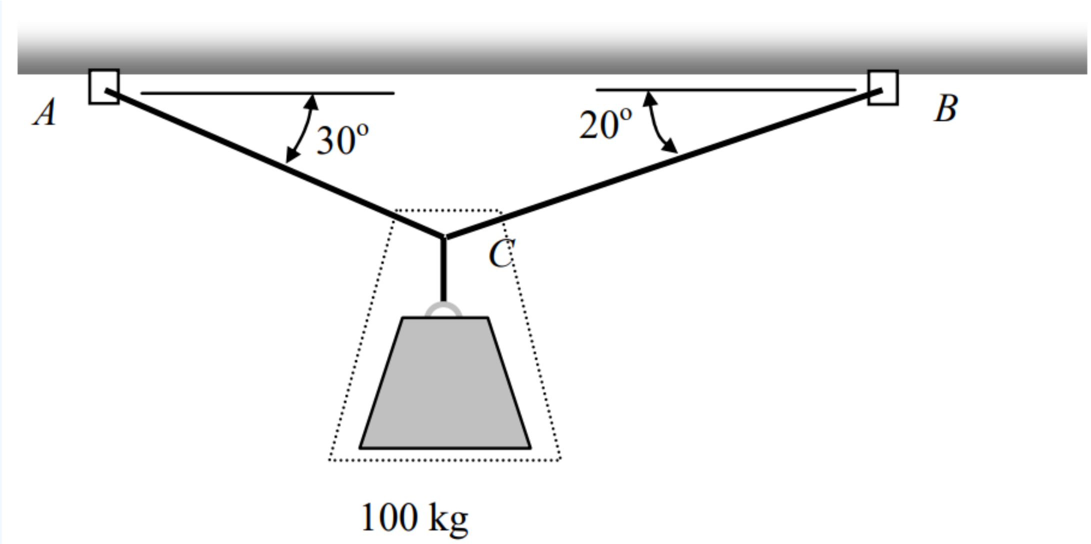


Figure 5.6.1: A loaded cable system.

(a) Starting with the rate-form of the conservation of linear momentum, use the *closed* system shown by the dashed line to solve for the tension in the cables AC and BC.

(b) How would your analysis change if you used a closed system that only included the joint at C? How would you handle the force applied by the hanging object?

(c) What would the tension be in both cables if the angle for both AC and BC was 5° ? What is the ratio of the tension in each cable to the weight of the mass? This ratio is sometimes referred to as the *mechanical advantage* you learned about when you studied levers. Does this give you any ideas about what would be the best way to rig up a cable to pull a car up an embankment? Any problems with your idea?

? Problem 5.4

A tractor-trailer is traveling at $60 \mathrm{~mi/h}$ when the driver applies her brakes. The tractor has a mass of $15,000 \mathrm{~lbm}$, and the trailer has a mass of $17,400 \mathrm{~lbm}$. Knowing that the braking forces of the tractor and the trailer are $3600 \mathrm{~lbf}$ and $13,700 \mathrm{~lbf}$, respectively, determine (a) the distance traveled by the tractor trailer before it comes to a stop, and (b) the horizontal component of the force in the hitch between the tractor and the trailer while they are slowing down.

Remember to start with the appropriate conservation equations and carefully select your system and identify the important interactions between the system and the surroundings. Also, remember to draw a free body diagram (linear momentum interaction diagram). [Hint: Use two different systems to answer the questions. For part (a) take the entire truck and trailer as the system. For part (b) use either the truck or the trailer for your analysis. Note that since the truck and trailer are hooked together, information like acceleration and velocity you calculate in part (a) can be used directly in part (b) without redeveloping the information. Why do you need to use this system for part (b)?]

Why does atmospheric pressure cancel out in this problem — or does it?

? Problem 5.5

(Adapted from *Dynamics* by Beer & Johnson, 6th edition)

A man hosing down his driveway hits the back of his mailbox by mistake. The velocity and cross-sectional area of the water stream as it hits the back of the mailbox are $25 \mathrm{~m/s}$ and $300 \mathrm{~mm}^2$, respectively. The water stream is horizontal as it hits the vertical surface of the mailbox. Assume all systems are steady-state and answer the following questions. Carefully show all of your work, especially how you create an appropriate model for this system starting with the rate form of the linear momentum equation. If needed, assume atmospheric pressure to be $100 \mathrm{~kPa}$.

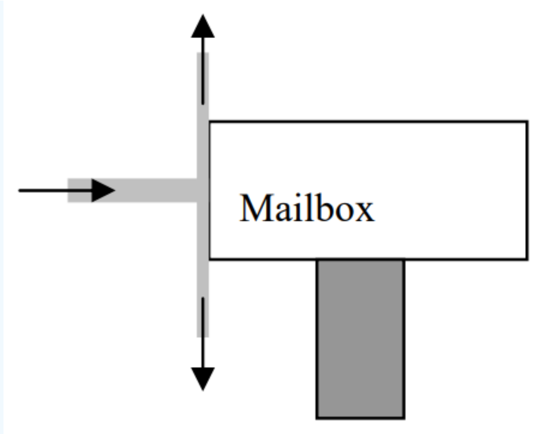


Figure 5.6.2: Behavior of a water stream as it hits a vertical side of a mailbox.

(a) Determine the direction and magnitude, in Newtons, of the horizontal shear force that is applied by the post to the mailbox. [Hint: Pick an open system that cuts the wooden post and the water stream. For this system, there is only *one* contact force with a horizontal component. Assume all water enters the system with a horizontal velocity of $25 \mathrm{~m/s}$ and that it leaves the system with no horizontal component of velocity.]

(b) Now determine the direction and magnitude, in Newtons, of the force of the water acting on the mailbox. Is this answer different than your answer above? [Hint: This time pick a *closed system*, the mailbox. For this system, the water force appears as a contact (or surface) force acting on the boundary of the system. Be careful to consider the pressure forces acting on the system.]

Answer

a) 187.5 Nacting to the left.

b) Yes, it is greater than the answer in part (a).

Problem 5.6

Water flows steadily through a 180° reducing pipe bend as shown in the figure. The atmospheric pressure outside the piping system is $P_{\mathrm{atm}}=100 \mathrm{~kPa}$. The pipe bend is connected to the two pipes by flanges. The flanges are held together by flange bolts. Assume the density of water is $1000 \mathrm{~kg/m}^{3}$.

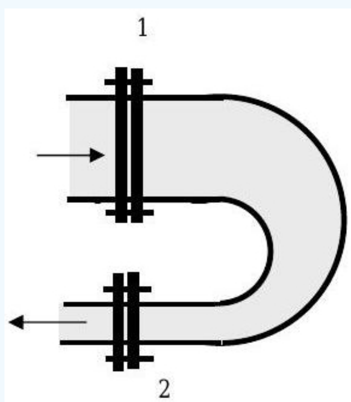


Figure 5.6.3: Water moves through a reducing pipe bend shaped in a semicircle.

At the inlet to the bend, the pressure is $P_{1}=350 \mathrm{~kPa}$, the pipe diameter is $D_{1}=25 \mathrm{~cm}$, and the water velocity is $V_{1}=2.2 \mathrm{~m/s}$. At the outlet of the bend, the pressure is $P_{2}=120 \mathrm{~kPa}$ and the pipe diameter is $D_{2}=8 \mathrm{~cm}$.

Neglecting the weight of the pipe bend and the water in the bend, determine the magnitude and direction of the total force (not force in the x-direction) of the flange bolts on the pipe bend. Would it have been "OK" to neglect the pressure effects?

Problem 5.7

(Adapted from *Dynamics* by Beer & Johnson, 6th edition)

A light train made of two cars travels at $45 \mathrm{~mi/h}$. Car A, which is in the front of the train, weighs 18 tons and car B weighs 13 tons. A constant breaking force of $4300 \mathrm{~lbf}$ is applied to car B but the brakes on car A are not applied. Determine (a) the time required for the train to stop after the brakes are applied, and (b) the force in the coupling between the cars which the train is slowing down.

Answer

(a) $29.6 \mathrm{~s}$ and (b) $2500 \mathrm{~lbf}$ in tension

Problem 5.8

A block of mass m_2 is placed on a wedge-shaped block of mass m_1. The angle of the wedge is θ. The inclined surface is frictionless and the small block slides freely on the incline. In addition, the small casters under the wedge are also frictionless, so the wedge glides freely on the horizontal surface.

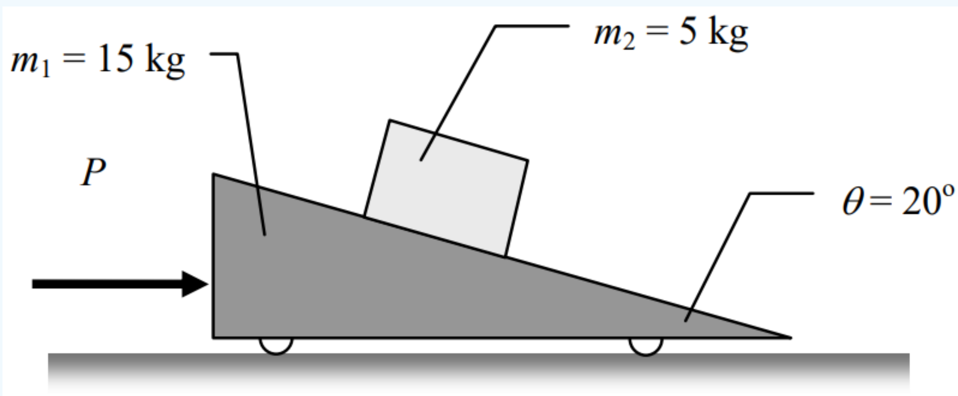


Figure $5.6.4$: A system where a force is exerted on a wedge on which a block rests.

(a) Determine the value of applied force P, in Newtons, that must be applied so that the smaller block will not slip on the wedge-shaped block, i.e. is no relative motion between the two blocks.

(b) How fast will the blocks be accelerating, in $\mathrm{m/s^2}$?

Note: It may not be possible to solve this problem using only one system

Problem 5.9

A 20-kg package is at rest on an incline when a force $\mathbf{P}$ is applied to it. The coefficients of static and kinetic friction between the package and the incline are 0.4 and 0.3, respectively.

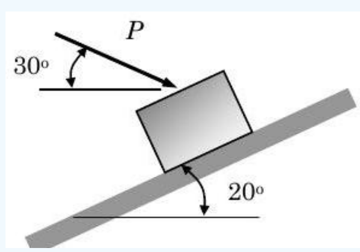


Figure $5.6.5$: A system where a force is exerted on a package on an incline.

(a) Determine the minimum force $\mathbf{P}$ required to move the package.

(b) Determine the magnitude of $\mathbf{P}$ if 10 s is required for the package to travel $5 \mathrm{~m}$ up the incline.

Problem 5.10

(Adapted from *Dynamics* by Beer & Johnson, 6th edition)

The triple jump is a track-and-field event in which an athlete gets a running start and tries to leap as far as he can with a hop, step, and jump. The runner approaches the takeoff line from the left with a horizontal velocity of $10 \mathrm{~m/s}$, remains in contact with the ground for 0.18 s, and takes off with a velocity of $12 \mathrm{m/s}$ at an angle of 50° from the horizontal.

Determine the vertical component of the average impulsive force exerted by the ground on his foot. Carefully identify your system and the associated transfers of linear momentum. Give your answer in terms of the weight W of the athlete.

Problem 5.11

(Adapted from *Dynamics* by Beer & Johnson, 6th edition)

Two swimmers A and B of weight 190 lbf and 125 lbf, respectivey, are at diagonally opposite corners of a floating raft that is $20 \mathrm{~ft}$ wide and $10 \mathrm{~ft}$ long. Suddenly they realize that the raft has broken away from its moorings and is floating free. Swimmer A immediately starts walking toward B at a speed of $2 \mathrm{~ft/s}$ relative to the raft. Knowing that the raft weighs 300 lbf, determine (a) the speed of the raft if B does not move, and (b) the speed with which B must walk toward A if the raft is not to move.

Problem 5.12

(Adapted from *Dynamics* by Beer & Johnson, 6th edition)

A block slides up an inclined plane from point A where its velocity is $30 \mathrm{~ft/s}$ to a point where its velocity is zero. The plane makes an angle of 20° with the horizontal. The coefficient of kinetic friction between the block and the plane is 0.30.

(a) Determine the time it takes for the block to travel from A to B.

(b) Determine the minimum value of the static-friction coefficient that is required to keep the block from sliding back down the slope after it reaches point B.

Problem 5.13

(taken from *Fundamentals of Fluid Mechanics* by Munson, Young and Okiishi)

A vertical, circular cross-sectional jet of air strikes a conical deflector as indicated in the figure. A vertical anchoring force of $0.1 \mathrm{~N}$ is required to hold the deflector in place. Determine the mass, in kg, of the deflector. The magnitude of the air velocity remains constant and the density of air is $1.23 \mathrm{~kg/m}^3$.

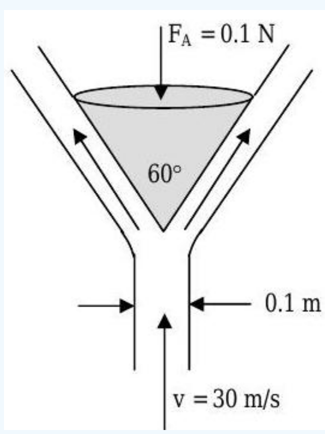


Figure 5.6.6: Vertical air stream splits around a tip-down conical deflector.

Problem 5.14

The conveyor belt shown below moves a constant speed of $v_o = 24\ \mathrm{ft/s}$. The length of the belt is $L = 20\ \mathrm{ft}$.

(a) Determine the angle α for which the sand is deposited on the stockpile at B.

(b) If the sand falls with practically zero velocity onto the conveyor at a constant rate $\dot{m} = 100\ \mathrm{lbm/s}$ as shown below, determine the magnitude of the net force P required to maintain a constant belt speed v (use the smaller of the two angles from part a). Note: P would not act at the location shown below - it is placed there for convenience only. Assume you can neglect the mass of the belt, but not the mass of the sand on the belt.

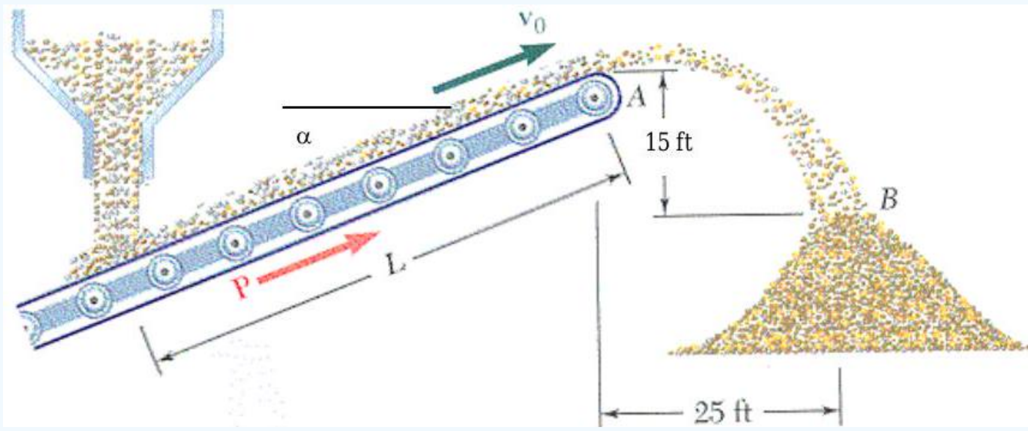


Figure 5.6.7: Sand falls onto an inclined conveyor belt and arcs off the end of the belt.

Problem 5.15

The resistance R to penetration of a $0.25\ \mathrm{kg}$ projectile fired with a velocity of $600\ \mathrm{m/s}$ into a certain block of fibrous material is shown in the graph below. Represent this resistance by the dashed line and compute the velocity of the projectile for the instant when $x = 25\ \mathrm{mm}$ if the projectile is brought to rest after a total penetration of $75\ \mathrm{mm}$.

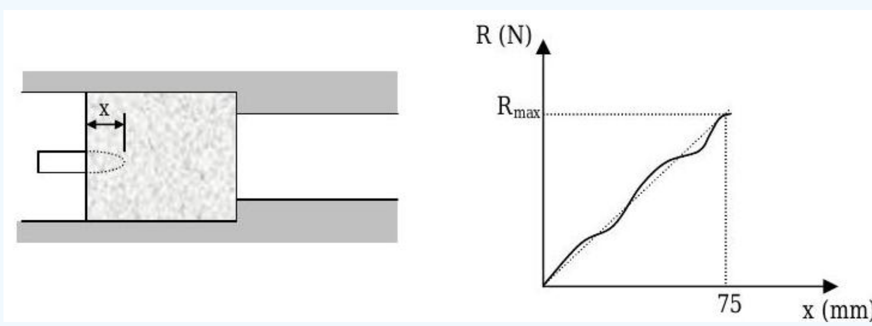


Figure 5.6.8: Graph of a block's penetration resistance vs distance penetrated into block by a projectile.

Problem 5.16}

A $125\ \mathrm{lb}$ block initially at rest is acted upon by a force $\mathbf{P}$ which varies as shown. Knowing that the coefficients of friction between the block and the horizontal surface are $\mu_s = 0.50$ and $\mu_k = 0.40$, determine

(a) the time at which the block will start moving

(b) the maximum velocity reached by the block

(c) the time at which the block will stop moving.

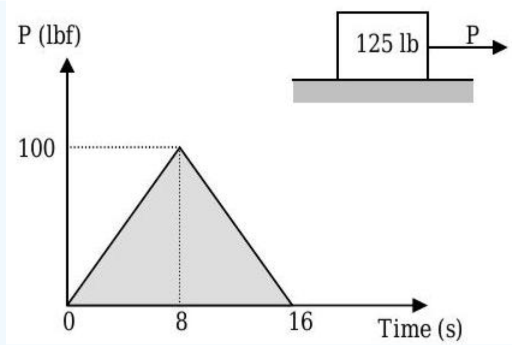


Figure 5.6.9: Graph showing time variation of a force exerted on a block.

Problem 5.17

Boxes A and B are at rest on a conveyor belt that is initially at rest. The belt suddenly started in an upward direction so that slipping occurs between the belt and the boxes. The coefficients of kinetic friction between the belt and the boxes are $(\mu_k)_A = 0.30$ and $(\mu_k)_B = 0.32$.

(a) Determine the initial acceleration of each box.

(b) If the two blocks remain in contact, how long will it take them to travel 3 ft? If they separate, what will the distance between the two blocks be after A has traveled up the incline 3 ft?

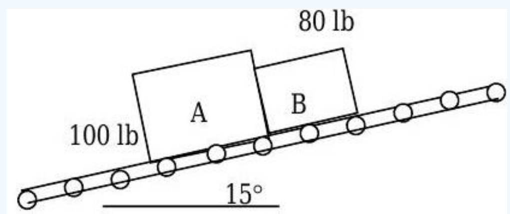


Figure 5.6.10: A system where two boxes in contact rest on an inclined conveyor belt.

Problem 5.18

If the coefficient of kinetic friction between the 20−kg block A and the 100−kg cart B is 0.50, and the coefficient of static friction is 0.55, determine the acceleration of each block when:

a) $P = 40$ N,

b) $P = 60$ N,

c) $P = 100$ N.

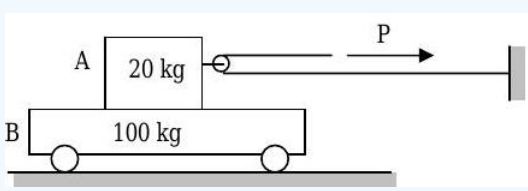


Figure 5.6.11: A force is applied to a box stacked on another box that can roll.

(Hint: To determine if A slides on B, assume that it doesn't and then solve for the friction force and compare to $\mu_s N$)

Problem 5.19

(modified version of a problem taken from V*ector Mechanics for Engineers* by Beer and Johnson)

A 10-kg package drops from a chute into a 25-kg cart with a velocity of $3 \mathrm{~m/s}$. Knowing that the cart is initially at rest and can roll freely, determine

(a) the final velocity of the cart

(b) the impulse exerted by the cart on the package.

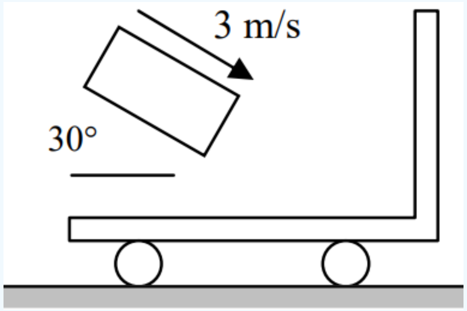


Figure 5.6.12: A box falls at an angle towards a open cart.

? Problem 5.20

(modified version of a problem taken from *Vector Mechanics for Engineers* by Beer and Johnson)

At an intersection car B was traveling south and car A was traveling 30 degrees north of east when they slammed into each other. Upon investigation it was found that after the crash the two cars got stuck and skidded off at an angle of 10 degrees north of east. Each driver claimed that he was going at the speed limit of $50 \mathrm{~km/h}$ and that he tried to slow down but couldn't avoid the crash because the other driver was going a lot faster. Knowing the mass of the two cars A and B were $1500 \mathrm{~kg}$ and $2000 \mathrm{~kg}$, respectively, determine

(a) which car was going faster

(b) the speed of the faster car if the slower car was traveling at the speed limit.

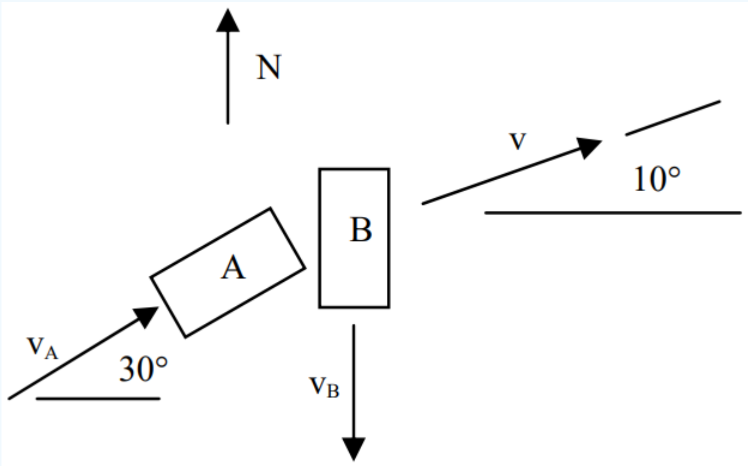


Figure 5.6.13: Directions of travel of two cars, before and after they collide.

? Problem 5.21

(modified version of a problem taken from *Vector Mechanics for Engineers* by Beer and Johnson)

A mother and her child are skiing together, with the mother holding the end of a rope tied to the child's waist. They are moving at a speed of $7.2 \mathrm{~km/h}$ on a flat portion of the ski trail when the mother observes that they are approaching a steep descent. She decides to pull on the rope to decrease her child's speed. Knowing that this maneuver causes the child's speed to be cut in half in $3 \mathrm{~s}$ and neglecting friction, determine

(a) the mother's speed at the end of the 3-s interval

(b) the average value of the tension in the rope during that time interval.

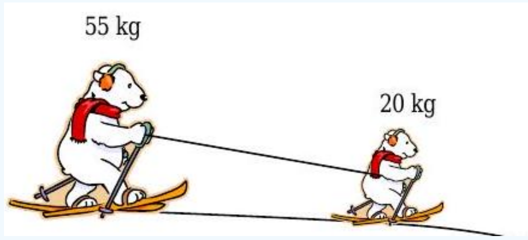


Figure 5.6.14: A mother and child skiing single file, connected by a rope

Problem 5.22

(taken from *Fundamentals of Fluid Mechanics* by Gerhart, Gross and Hochstein)

A carbon steel, 14-in outer diameter (13.25-in inner diameter), schedule 30, 90° elbow is to be butt-welded to a pipe carrying water at $Q = 4000\ \mathrm{gal/min}$

Find the force required in the weld to support the elbow. The elbow weighs 150 lbf. The density of water is $1.94\ \mathrm{slug/ft^3}$.

(Hint: The mass of water is approximately equal to $\mathrm{m_w} \approx \left(\pi D^2/4\right) L\rho$ where D is the inside diameter of the pipe and L is the arc length associated with Re.)

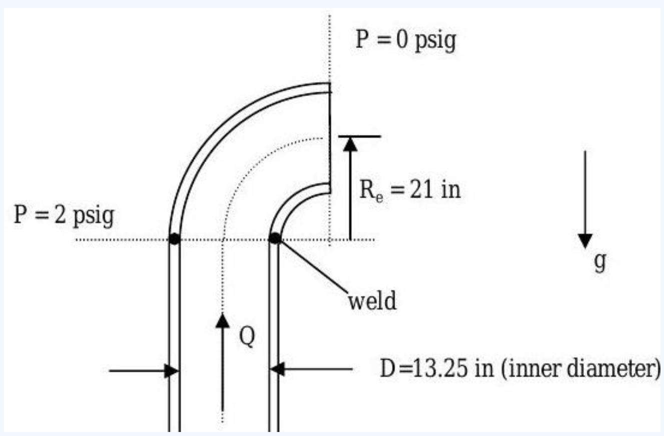


Figure 5.6.15: Water travels up a vertical pipe welded to a 90° elbow.

Problem 5.23

A 0.02 kg bullet strikes block A with a velocity of $v_0 = 150\ \mathrm{m/s}$ as shown below. The masses of the blocks are given in the figure. Assume the bullet becomes lodged in A and the time of impact is very small.

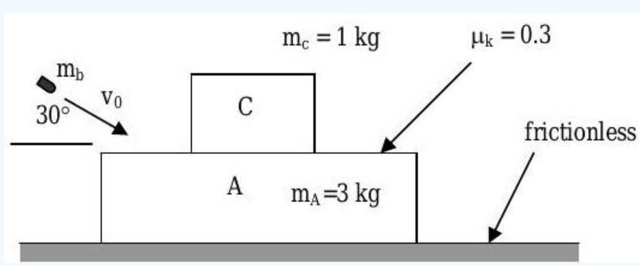


Figure 5.6.16: Bullet moves towards one of two stacked blocks.

Determine:

(a) the velocity of A after the impact,

(b) the acceleration of A and C after the impact, and

(c) the time it will take for block C to stop sliding on block A. (Assume block A is large enough so that C will not fall off of it.)

Hints: The friction force between the blocks is not an impulse force. There will be sliding between the blocks.

Problem 5.24

A particle is traveling in a straight line in a horizontal plane. The mass of the particle is 0.10 kg. The acceleration of the particle is described by the following equation:

$$a=\frac{dv}{dt}=A-Bt^2$$

where $A=\text{constant}$ and $B=6\ \mathrm{m/s^4}$. The following additional information is known about the motion of the particle:

$$\text{At } t=0: \quad x=x_0=8\ \mathrm{m},\ V=V_0=0 \quad \text{and} \quad \text{At } t=1 \text{ second:} \quad V=30\ \mathrm{m/s}$$

(a) Determine the time(s) at which the velocity is zero.

(b) Determine the total distance traveled when $\mathrm{t}=5\ \mathrm{sec}$. (Note that the change in position x is not the same as the distance traveled.)

(c) Plot the acceleration, velocity, and position of the particle for $0 \leq \mathrm{t} \leq 5$ seconds. Determine the time(s) when the motion of the particle changes direction.

(d) For the same time interval as in Part (c), plot the linear momentum of the particle, the rate of change of linear momentum of the particle, and the net external force on the particle in the horizontal plane. Determine the time(s) when the net external force changes direction.

(e) Discuss how the time(s) you calculated in Part (c) when the particle changed its direction of motion relate to the time(s) you calculated in Part (d) when the net external force changes direction. Is there any relation?

Problem 5.25

(Adapted from *Engineering Mechanics: Statics* by Bedford & Fowler, Addison-Wesley)

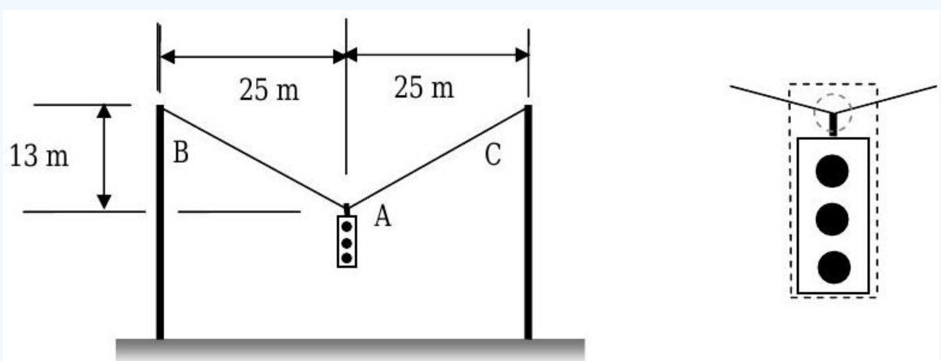


Figure 5.6.17: A traffic light hangs from cables connected to two posts.

A 140-kg traffic light is suspended above the street by two cables as shown in the figure above. On a windy day the wind blows from left to right on the figure and creates a horizontal force of 200 N on the traffic light. Assume that the deflection of the traffic light from a vertical orientation due to the crosswind is negligible.

(a) Starting with the rate-form of the conservation of linear momentum, use the closed system shown by the dashed lines in the figure to solve for the tension in the cables AB and AC with and without the crosswind force.

(b) How would your analysis change for (a) if your system only included the intersection of the two cables and the traffic light as shown inside the light gray dashed circle? How would you handle the force applied by the hanging traffic light?

Problem 5.26

A 30-kg package is placed on an incline when a force $\mathbf{P}$ is applied to it as shown on the figure. The coefficients of static and kinetic friction between the package and the incline are 0.2 and 0.1, respectively. The motion of the package depends on the magnitude of the force $\mathbf{P}$.

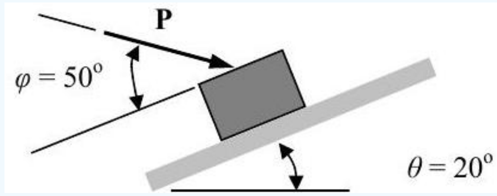


Figure $5.6.18$: A package resting on a ramp experiences a force $\mathbf{P}$ applied to it.

(a) Determine the range of values for $\mathbf{P}$ for which the package will remain stationary on the inclined plane.

(b) Assuming the package is initially stationary, determine its velocity and position 10 seconds after the force $\mathbf{P}$ is reduced to zero, i.e. $\mathbf{P}=0$.

Problem 5.27

The mass of Block A is 30 kg and the mass of Sphere B is 5 kg. Block A slides on the surface, and as it slides Sphere B is free to move as shown in the figure. The coefficient of kinetic friction between Block A and the surface is 0.24. Assume that the link connecting the block and the sphere has negligible mass.

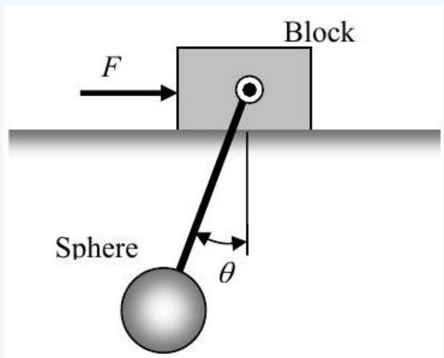


Figure $5.6.19$: A sphere hangs from a block sliding along a surface.

(a) Determine the numerical value of the force F, if the angle $\theta=20^{\circ}$ is constant.

(b) Is the linear momentum of the block increasing, decreasing, or constant? Is the block accelerating, decelerating, or moving at constant velocity?

(c) Determine the magnitude and direction of the force of the link acting on the sphere. Is there tension or compression in the link connecting Block A and Sphere B ?

Problem \(5.28)

In a cathode-ray tube, an electron with mass m enters the gap between two charged plates at Point O with a velocity $\mathbf{V}=V_o\mathbf{i}$. While it is between the charged plates, the electric field generated by the plates subjects the electron to a force $\mathbf{F}=-eE\mathbf{j}$ where e is the charge of the electron and E is the electric field strength. The plate spacing is $2h$, the length of the charged plates is l, and the distance to the screen is L. (See figure below.) Assume that gravitational forces can be neglected and that external forces on the electron are negligible when it is not between the charge plates.

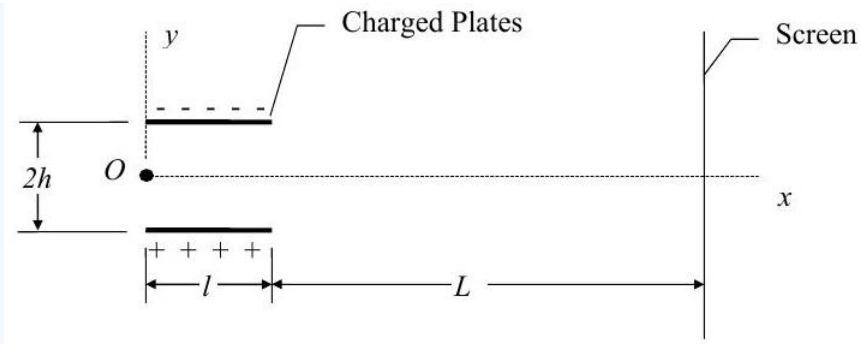


Figure 5.6.20: An electron passes between oppositely charged plates to impact a screen.

(a) Starting with the appropriate conservation and accounting relations, develop an expression for the location of the point of impact of the electron with the screen. Your answer should be presented in terms of the initial speed of the electron (V_o), the mass of the electron (m), the charge of the electron (e), the electric field strength (E), and the dimensions h, l, and L. Clearly identify your system and show how you use the material in the problem and any additional assumptions to develop your answer.

(b) Sketch the path of the electron on the figure.

(c) Calculate the deflection in terms of h if the following numerical values are known:

$$V_o = 2.2 \times 10^7 \text{ m/s}; \quad m = 9.11 \times 10^{-31} \text{ kg}; \quad e = 1.6 \times 10^{-19} \text{C (coulombs)}; \quad E = 15 \text{ kN/C}$$
$$L = 100 \text{ mm}; \quad l = 30 \text{ mm}$$

? Problem 5.29

A block with a mass $m = 200$ kg rests on an inclined plane with an applied load $\mathbf{P}$. Depending upon the magnitude of $\mathbf{P}$, the block may move up the incline, down the incline, or remain stationary. Between the surfaces of contact, the coefficient of static friction is $\mu_s = 0.25$ and the coefficient of kinetic friction is $\mu_k = 0.20$.

Determine the range of the magnitude of the horizontal force $\mathbf{P}$, in Newtons, that will keep the block in equilibrium, i.e. not moving.

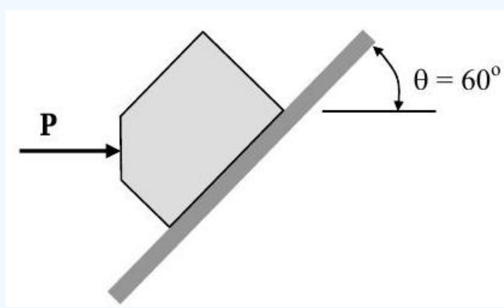


Figure 5.6.21: An irregularly shaped block resting on an incline experiences a force $\mathbf{P}$.

? Problem 5.30

The ducted fan unit shown in the figure has mass $m = 100$ kg and is supported in the vertical position on its flange at A. The unit draws in air with a density $\rho = 1.200 \text{ kg/m}^3$ and a velocity $V_1 = 5$ m/s through an inlet with diameter $D_1 = 1.00$ m. It discharges air through two outlets at the bottom of the fan. The mass flow rate through each outlet is $1/2$ of the entering mass flow rate, and the velocity at each outlet is $V_2 = V_3 = 15$ m/s. Both inlet and outlet pressures are atmospheric.

Determine the vertical force R, in Newtons, applied to the flange of the fan unit by the supports.

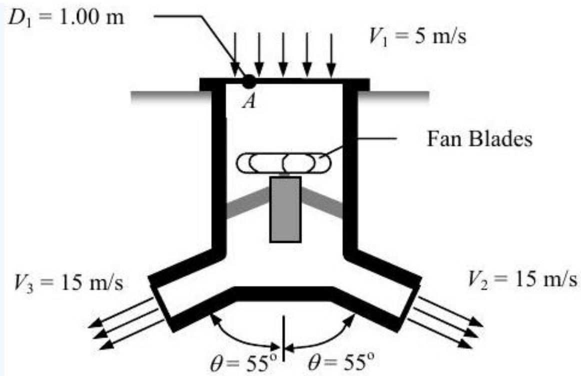


Figure $5.6.22$: A fan unit is held in a vertical position by a flanged upper rim.

Problem 5.31

A crate with mass $m = 500 \text{ lbm}$ is attached to a motorized winch and positioned on a sloping dock with inclination angle $\theta = 30^\circ$ as shown in the figure. The winch exerts a tensions force T on the crate through the winch cable. The friction coefficients between the crate and the dock surface are $\mu_s = 0.30$ for static friction and $\mu_k = 0.25$ for kinetic friction.

Initially, the crate is stationary and the winch brake is on to prevent any motion. Suddenly at $t = 0$, the brake is released and the winch exerts a constant tension force $T = 100 \text{ lbf}$ on the crate. At $t > 2$ seconds, the tension force exerted by the winch on the crate suddenly increases to a constant value $T = 400 \text{ lbf}$.

A ramp slants upwards and to the right at 30 degrees above the horizontal. A crate rests on the ramp and is connected by a cable to a motorized winch that sits at the top of the ramp. At t<0, the winch is locked and the crate is stationary. When t is greater than or equal to 0 and less than or equal to 2 seconds, the winch operates with the cable tension force being 100 lbf. When t

2 seconds, the winch operates with the cable tension force being 400 lbf." src="/@api/deki/files/54938/Screenshot_(79).png">

Figure $5.6.23$: A crate is pulled up a ramp by a cable attached to a winch. Cable tension varies as a function of time.

(a) Consider the forces on the *stationary* crate when the winch brake is on and find the value or range of values for the tension force T under these conditions.

(b) Consider the crate's motion for $0 \leq t \leq 2 \text{ s}$ when the tension force is $T = 100 \text{ lbf}$. Find the crate's *acceleration, velocity* and *position* at $t = 2$ seconds.

(c) Consider the crate's motion for the period $t > 2 \text{ s}$ when the tension force is $T = 400 \text{ lbf}$. *Qualitatively* describe how the crate moves during this period (Be concise; use words and not numbers!)

(d) Calculate the *impulse* for the tension force T over the time interval $t = 0$ to 4 s. Be sure to indicate *both* the direction and magnitude.

Problem 5.32

You have been asked to investigate the performance of a jet-propelled boat using a water channel where the water velocity V_{water} can be varied as required. The boat is placed in the channel and tethered so that it is stationary. The boat is jet-propelled by a pump that develops a constant volumetric flow rate of water, $\dot{V}_{\text{pump}}$. Water enters the aft (front) of the boat through an area of A_1 and leaves at the stern (rear) through an area A_2.

Water flowing over the hull of the boat exerts a drag force on the boat in the direction the water is flowing. This horizontal drag force which includes the net pressure forces on the hull is given by the following equation:

$$F_{\text{drag}} = kV_{\text{water}}^2$$

where k is a constant.

Assume that the angle θ and water density ρ are both known.

a) Find expressions for the water velocities V_1 and V_2 in terms of the pump flow rate, $\dot{V}_{\text{pump}}$.

b) Find an expression for the volumetric flow rate through the pump $\dot{V}_{\text{pump}}$ as a function of the water velocity in the channel, V_{water}, when the tension in the tether is zero, i.e. $\dot{V}_{\text{pump}} = f\left(V_{\text{water}}\right)$.

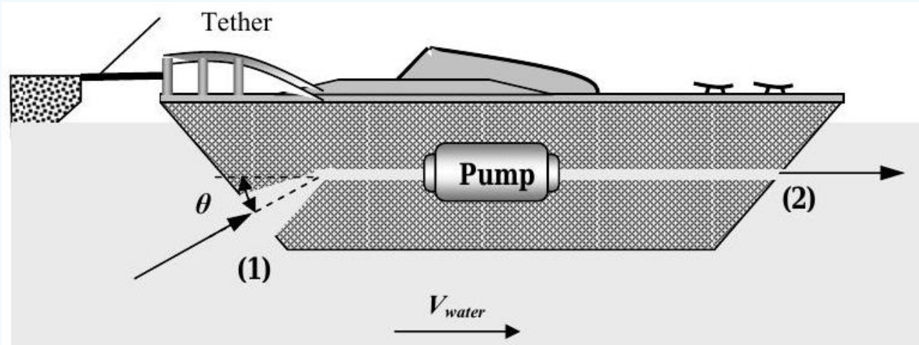


Figure 5.6.24: Water enters a ship to be pumped out the other side.

Problem 5.33

A $10-\text{kg}$ steel sphere is suspended from a $15-\text{kg}$ frame, and the sphere-frame combination slides down a 20° incline as shown in the figure. The coefficient of kinetic friction between the frame and the incline is $\mu_{\text{k}} = 0.15$.

Determine the tension in each of the supporting wires, in Newtons.

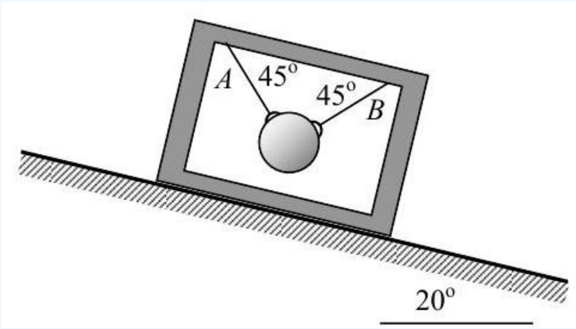


Figure 5.6.25: A sphere suspended in a rectangular frame by two cables slides down an incline.

Problem 5.34

You have been hired by NASCAR to analyze vehicle impacts into the wall. For the crash below, solve for the average reactions ($R_{x,\,\text{avg}}$ and $R_{y,\,\text{avg}}$) of the wall on the car in terms of the mass of the vehicle m, the initial velocity V_1, the angle θ, the distances h and d, and the time interval $\Delta t = t_2 - t_l$ assuming the vehicle comes to a complete stop during the time interval.

Feel free to assume that the car remains a rectangle during the impact.

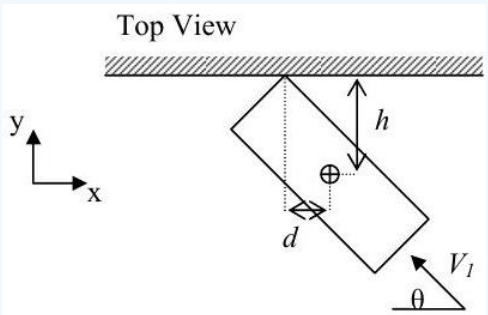


Figure 5.6.26: Top-down view of a car moving towards a wall.

> ❓ Problem 5.35

Blocks A and B are identical and each have a mass of 10 kg. Block B is at rest when it is hit by block A, which is moving with velocity $V_A = 6\ \mathrm{m/s}$ just before impact. Blocks A and B stick together after the impact, and you may neglect friction during the impact.

After the impact, the velocity of blocks A and B decreases due to friction. The coefficient of kinetic friction between all surfaces is $\mu_k = 0.20$.

(a) Determine the velocity of block A and B immediately after A hits B.

(b) Determine the impulse of the force of block A on block B during the impact.

(c) Determine the time required for the velocity of the blocks to drop to $1\ \mathrm{m/s}$.

(d) Determine the distance traveled by the blocks during this time interval.

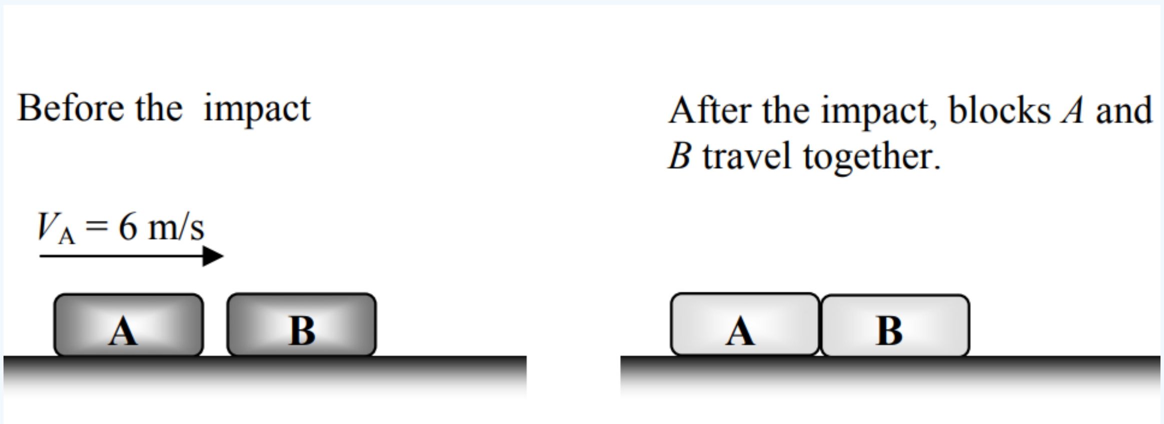


Figure 5.6.27: Behavior of blocks before and after impact.

> ❓ Problem 5.36

A piece of wood is held at rest on the smooth (frictionless) inclined plane by the stop block at A. A bullet is traveling as shown in the figure with velocity V when it becomes embedded in the block. The embedding takes a short time, Δt. The mass of the wood is m_w, the mass of the bullet is $m_{\mathbf{B}}$, the angle of the inclined plane is θ, and the acceleration due to gravity is g.

Provide symbolic solutions to answer the following questions:

(a) What is the velocity V_a of the bullet/wood immediately after the bullet becomes embedded? SET UP BUT DO NOT SOLVE. Clearly show how you would use your equations to solve for V_a

For the remaining questions, you may assume that V_a is known.

(b) What is the average impulsive force acting on the bullet during the impact?

(c) What is the equation for the rate of change of velocity of the bullet/wood after the impact?

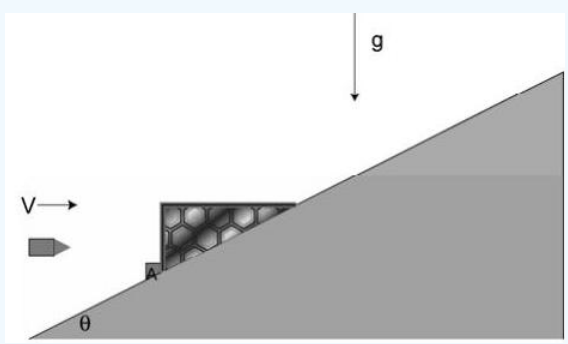


Figure 5.6.28: Bullet approaches a wooden wedge held in place on a ramp.

Problem 5.37

Salt water $\left(\rho = 1025\ \mathrm{kg/m^3}\right)$ enters a vertical pipe at a volumetric flow rate of $0.5\ \mathrm{m^3/s}$ and is discharged into the atmosphere from the two 30° outlets as shown in the figure. The flow divides equally between the two outlets.

Each of the discharge nozzles has an outlet diameter of $10.0\ \mathrm{cm}$ and the inside diameter of the pipe at section A-A is $25\ \mathrm{cm}$. The pressure of the water at section A-A is $550\ \mathrm{kPa}$ and the atmospheric pressure is $100\ \mathrm{kPa}$. The pipe above the flange and the water within it has a mass of 60 kg.

Determine the total force exerted by the lower pipe on the section of pipe above the flange A-A. Indicate both its magnitude, in Newtons, and its direction.

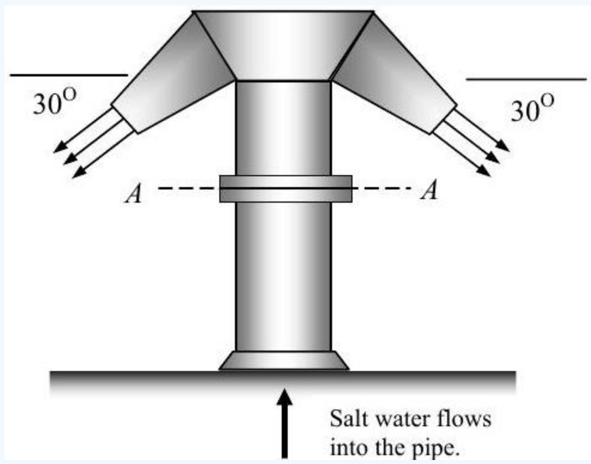


Figure $5.6.29$: Salt water flows up a vertical pipe and out of two angled nozzles.

Problem 5.38

A jet engine with exhaust nozzle is mounted on a test stand as shown in the figure. The engine is mounted as shown on two hangars and a diagonal brace. All connections are with frictionless pinned joints.

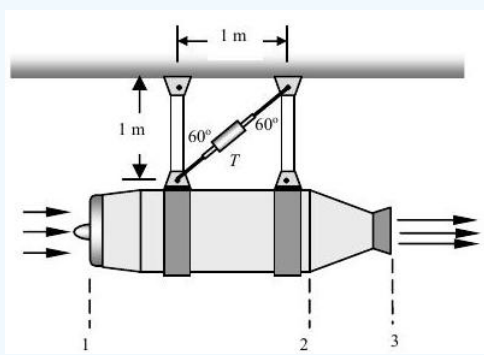


Figure $5.6.30$: A jet engine hangs from a test stand composed of three struts.

Information about the flow area, pressure, and air velocity at three locations along the engine are given in the table. At steady-state operation, air is sucked into the inlet at the rate of $30\ \mathrm{kg/s}$.

Determine the direction and the magnitude of the force T in the diagonal brace. Is the brace in tension or compression?

		Sec. 1	Sec. 2	Sec. 3
Flow area	$\mathrm{m^2}$	0.15	0.16	0.06
Pressure	kPa	84	240	114
Air velocity	$\mathrm{m/s}$	120	315	600

Problem 5.39

A bullet strikes and glances off a flat plate as shown in the figure. The plate is resting on a frictionless horizontal surface. Initially, the bullet has a velocity of $800\ \mathrm{ft/s}$ and the plate is stationary. After striking the plate, the bullet velocity is $600\ \mathrm{ft/s}$.

Determine the final velocity of the plate.

$$m_{\text{Bullet}} = 0.05\ \text{lbm}$$
$$m_{\text{Plate}} = 3\ \text{lbm}$$

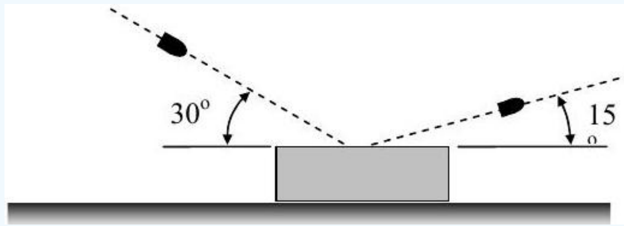


Figure $5.6.31$: A bullet strikes a flat plate at an angle and glances off.

Problem 5.40

A truck is traveling down a long steady grade $(\theta = 15^\circ)$ as shown in the figure. The crate has mass $m_{\text{Crate}} = 500\ \text{kg}$ with height $H = 1\ \text{m}$ and length $L = 2\ \text{m}$. The crate rests on the trailer bed and the surface has a static friction coefficient $\mu_S = 0.4$ and a kinetic friction coefficient $\mu_K = 0.3$. To prevent the load from shifting, the driver must limit his braking.

Determine the maximum possible truck deceleration if the crate does not slide on the trailer bed.

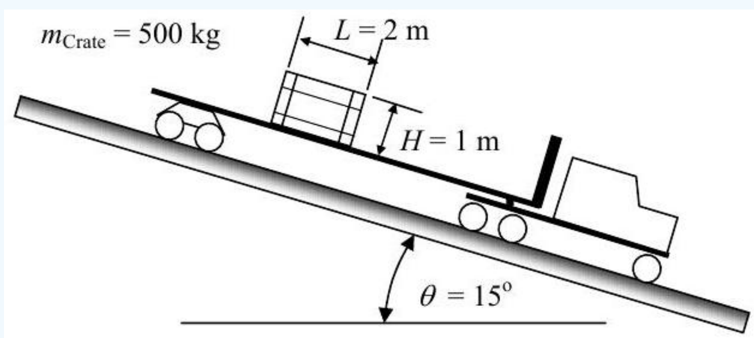


Figure $5.6.32$: A truck carrying a crate on its bed travels down a steady grade.

Problem 5.41

Two blocks rest on an inclined plane with $\theta = 35^\circ$ as shown in the figure. Block A has mass $m_A = 13.5\ \text{kg}$ and block B has mass $m_B = 40\ \text{kg}$. The coefficients of static and kinetic friction between all surfaces are $\mu_S = 0.3$ and $\mu_k = 0.2$, respectively. Initially the blocks are stationary and are supported by a stop block and fixed-length wire, as shown on the figure.

When the stop block is removed the block B immediately starts to move because the angle θ is large enough to produce motion.

Find the acceleration of block B and the tension in the wire immediately after the stop block is removed.

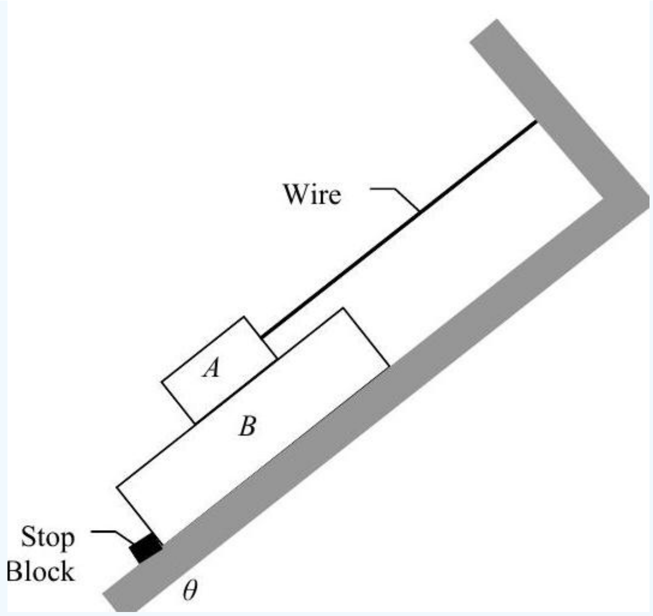


Figure 5.6.33: Blocks stacked on an incline are held stationary by a stop block and a wire attached to a support.

? Problem 5.42

The convertible shown is moving at a constant velocity, v_c, in the direction shown. At the instant shown a passenger in the car throws a ball from the car. The magnitude and direction of the initial velocity of the ball with respect to the car, v_0, is shown in the figure. The ball hits the ground $30\ \mathrm{ft}$ to the right of the point from which it was released.

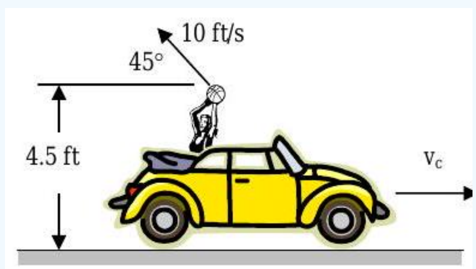


Figure 5.6.34: A car passenger throws a ball upwards and opposite the direction's car of travel.

a) How long does it take the ball to hit the ground from the moment it is released?

b) What is the velocity of the car?

? Problem 5.43

The cart has mass M and holds water that has a mass m_0. If a pump ejects water through a nozzle having a cross-sectional area A at a constant rate of v_0 relative to the cart, determine the velocity of the cart as a function of time. What is the maximum speed developed by the cart assuming all the water can be pumped out? Assume the frictional resistance to forward motion is F and the density of water is ρ.

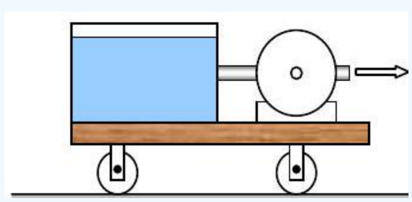


Figure 5.6.35: Water on a moving cart is steadily pumped off the cart.

Problem 5.44

Two swimmers A and B, of mass 75 kg and 50 kg, respectively, dive off the end of a 200-kg boat. Each swimmer has a relative horizontal velocity of $3\ \mathrm{m/s}$ when leaving the boat. If the boat is initially at rest, determine its final velocity, assuming that (a) the two swimmers dive simultaneously, (b) swimmer A dives first, (c) swimmer B dives first.

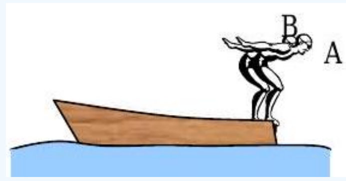


Figure $5.6.36$: Two swimmers line up to dive off the end of a boat.

CHAPTER OVERVIEW

6: Conservation of Angular Momentum

6.1: Why is this thing turning?

Anyone who has changed a flat tire, used a wrench, thrown a Frisbee, hung on a tree limb, or pushed a large box knows something about the tendency for objects to rotate or resist rotation. Angular momentum is the extensive physical property related to this phenomenon. Before we can develop the conservation of angular momentum relation, we must introduce several concepts for describing angular motion and the moment of a force.

To refresh your memory, imagine that your car has a flat tire and you must change it (See Figure $6.1.1$). To change the tire, you first raise the tire off the ground using a car jack. Next, you place the tire iron on a lug nut and try to loosen it. In the sketch, the force $\mathbf{F}$ acts on the tire iron at point P, the axle of the tire is at point O, and the nut is located at point N.

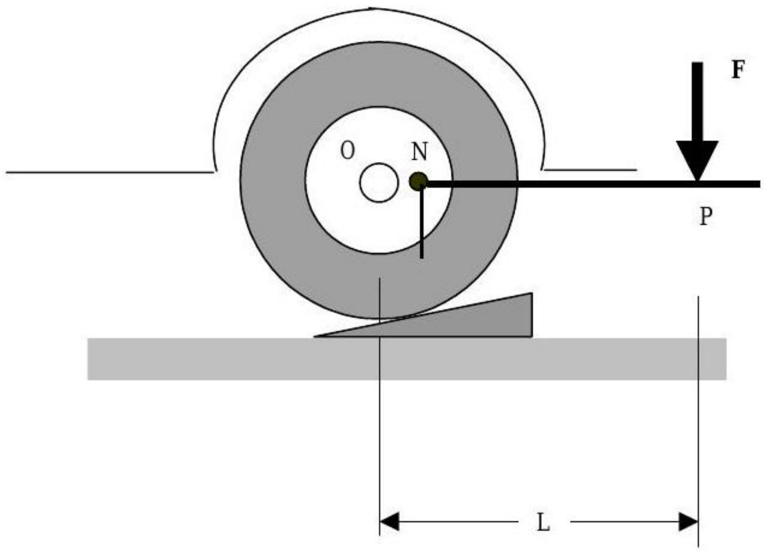


Figure $6.1.1$: Changing a tire.

If you have been lucky in life, you may be a novice at tire changing, and will discover that the tire rotates when you push on the tire iron. A driver with less luck and more experience changing tires might only raise the tire partially or insert a wedge to prevent the tire from rotating. Experienced tire changers know that our ability to loosen the nut depends on both the force $\mathbf{F}$ and its point of application (point P). They also know that the effectiveness of the applied force $\mathbf{F}$ is less if it is applied at an angle to the tire iron.

What would happen if the force was applied at point N? Would you be able to loosen the nut? Would the tire turn?

6.1.1 Moment of a force about a point

To quantify the ability of the force to rotate the tire we need a physical quantity that accounts for the magnitude and direction of force $\mathbf{F}$ as well as its point of application. The quantity we desire is the moment of a force about a point.

The **moment of a force F about a point O** is the vector (or cross) product of the position vector $\mathbf{r}$ and the force $\mathbf{F}$:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ is the position vector that extends from the point O to the point of application of the force (See Figure $6.1.2$).

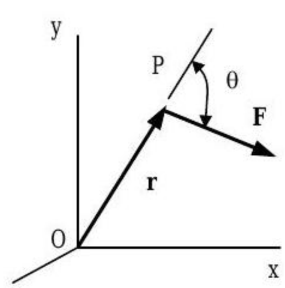


Figure 6.1.2: Calculating moment of a force $\mathbf{F}$ about a point O.

When the position vector $\mathbf{r}$ and the force vector $\mathbf{F}$ are written in terms of their components as

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \quad \text{and} \quad \mathbf{F}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}$$

Eq. $(6.1.1)$ is evaluated as the follows using standard cross-product operations:

$$\begin{aligned}\mathbf{M}_o &= \mathbf{r}\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ F_x & F_y & F_z\end{vmatrix}\\ &=\mathbf{i}\begin{vmatrix}y & z\\ F_y & F_z\end{vmatrix}-\mathbf{j}\begin{vmatrix}x & z\\ F_x & F_z\end{vmatrix}+\mathbf{k}\begin{vmatrix}x & y\\ F_x & F_y\end{vmatrix}\\ &=(yF_z-zF_y)\,\mathbf{i}-(xF_z-zF_x)\,\mathbf{j}+(xF_y-yF_x)\,\mathbf{k}\end{aligned} \tag{6.1.1}$$

If we restrict ourselves to motion in the x-y plane (plane motion), the only component of $\mathbf{M}_O$ with a non-zero value is the z-component — the last term on the right-hand side of Eq. $(6.1.3)$.

From our knowledge of vectors, we can say several things about this result:

- The position vector $\mathbf{r}$ and the force vector $\mathbf{F}$ lie in a plane.
- The moment $\mathbf{M}_O$ of the force $\mathbf{F}$ about point O is a vector (see Eq. $(6.1.3)$).
- The *line of action* of the moment vector $\mathbf{M}_O$ is normal (perpendicular) to the plane that contains both $\mathbf{r}$ and $\mathbf{F}$.
- The *point of application* of the moment vector $\mathbf{M}_O$ is at point O, the point about which we are taking the moment.
- The *sense* of the *direction* of the moment $\mathbf{M}_O$ and the sense of the rotation it could impart can be described by the **right-hand rule** in several ways (See Figure 6.1.3):

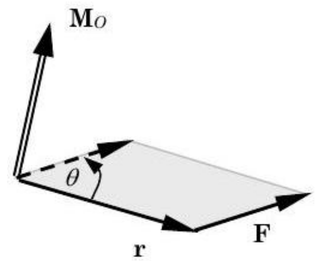


Figure 6.1.3: Sense of direction of $\mathbf{M}_O$ using the right-hand rule.

  - Align the fingers of your right hand in the direction of the position vector $\mathbf{r}$ and curl them in the direction of the force $\mathbf{F}$. Your thumb now points in the direction of the moment $\mathbf{M}_O$ and your fingers curl to show the sense of rotation for the moment $\mathbf{M}_O$.
  - Imagine sliding vector $\mathbf{r}$ or $\mathbf{F}$ until they are tail-to-tail. Align the fingers of your right hand in the direction of the position vector $\mathbf{r}$ and curl your fingers in the direction of force $\mathbf{F}$. Your curled fingers now indicate the sense of rotation for the moment $\mathbf{M}_O$ and your thumb points in the direction of moment $\mathbf{M}_O$.
  - Curl the fingers of your right hand and point the thumb of your right hand in the direction of moment $\mathbf{M}_O$. The direction your fingers curl is the sense of rotation for the moment $\mathbf{M}_O$.
- The *magnitude* of the moment vector $\mathbf{M}_O$ is the square root of the dot product of $\mathbf{M}_O$ with itself and is calculated as follows:

$$\begin{aligned}M_O &= |\mathbf{M}_o|=(\mathbf{M}_O\cdot\mathbf{M}_O)^{1/2}\\ &=\left[(yF_z-zF_y)^2+(xF_z-zF_x)^2+(xF_y-yF_x)^2\right]^{1/2}\end{aligned}$$

  It can also be calculated using the relationship

$$M_o=|\mathbf{M}_o|=|\mathbf{r}||\mathbf{F}|\sin\theta=rF(\sin\theta)$$

  where θ is the angle between the lines-of-action of vectors $\mathbf{r}$ and $\mathbf{F}$ and is measured as though $\mathbf{r}$ and $\mathbf{F}$ were placed tail-to-tail. A positive angle satisfies the right-hand rule as described above. Eq. $(6.1.5)$ can be interpreted as the area of a parallelogram formed by the vectors $\mathbf{r}$ and $\mathbf{F}$ (See Figure 6.1.4).

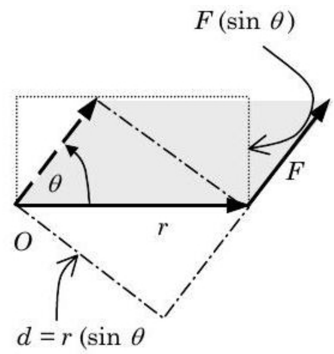


Figure 6.1.4: Interpreting the magnitude of $\mathbf{M}_O$ as an area.

Two additional rectangles shown in Figure 6.1.4 have areas equal in magnitude to the shaded area. The rectangle of length $d_\perp$ and width F has an area of

$$\begin{aligned} M_O &= (r\sin\theta)F \\ &= d_\perp \times F \quad = \underbrace{\begin{bmatrix} \text{Shortest distance} \\ \text{between point } O \\ \text{and the line-of-action} \\ \text{of force } \mathbf{F} \end{bmatrix}}_{\text{Often called the "lever arm"}} \times \begin{bmatrix} \text{Magnitude} \\ \text{of} \\ \text{force } \mathbf{F} \end{bmatrix} \end{aligned} \tag{6.1.2}$$

The rectangle of length r and width $F\sin\theta$ has an area of

$$\begin{aligned} M_O &= r(F\sin\theta) \\ &= r \times F_\perp = \begin{bmatrix} \text{Magnitude} \\ \text{of the} \\ \text{position vector } \mathbf{r} \end{bmatrix} \times \begin{bmatrix} \text{Component of force } \mathbf{F} \\ \text{that is } \perp \text{ to} \\ \text{the line-of-action of } \mathbf{r} \end{bmatrix} \end{aligned} \tag{6.1.3}$$

For most cases with two-dimensional (plane) motion, you will find that using Eqs. (6.1.6) or (6.1.7) is much simpler than using the formal cross product mathematics.

? Test your understanding

For each of the following two-dimensional cases, determine the magnitude and direction [Clockwise (CW) or counter-clockwise (CCW)] of the moment of force $\mathbf{F}$ about point O:

a)

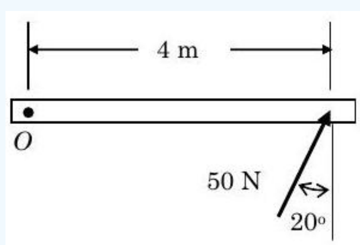


Figure 6.1.5: A force is applied at an angle to one end of a bar.

b)

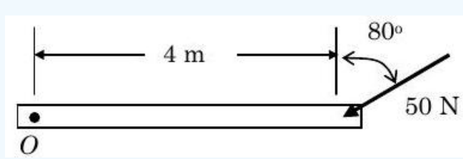


Figure 6.1.6: A force is applied at an angle to one end of a bar.

c)

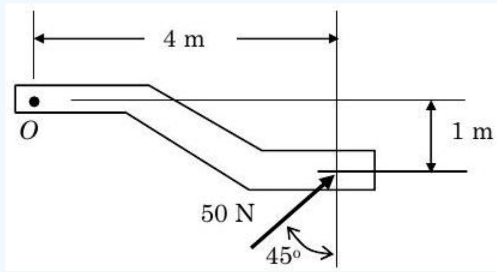


Figure 6.1.7: A force is applied at an angle to one end of a rigid body.

Answer

a) $187.9\ \mathrm{N \cdot m CCW}$

b) $34.73\ \mathrm{N \cdot m CW}$

c) $176.8\ \mathrm{N \cdot m CCW}$

6.1.2 Moment of a force couple

There is a special type of external force system called a force couple. A **force couple** consists of two external forces that have equal magnitude, parallel lines-of-action, and opposite sense (See Figure 6.1.8). Thus, a *force couple results in a zero net force on a system*. Or stating this another way, *a force couple transfers zero linear momentum to a system*. However, a quick examination of the figure shows that the force couple does in fact attempt to turn or rotate the system.

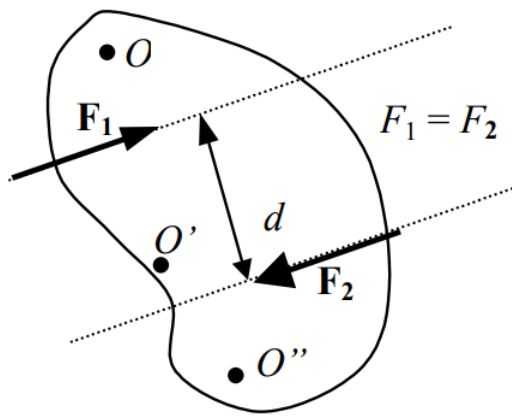


Figure 6.1.8: A force couple.

The **moment of a force couple** about point O can be calculated with reference to Figure 6.1.8 as follows:

$$\begin{aligned}\mathbf{M}_{O,\text{ couple}} &= \mathbf{M}_{O,\,F_1} + \mathbf{M}_{O,\,F_2} \\ &= (\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) \qquad \text{where } |\mathbf{F}_1| = |\mathbf{F}_2| = F \\ &= [(d_{\perp,1} \times F_1)\ \text{ CCW}] + [(d_{\perp,2} \times F_2)\ \text{ CW}] \\ &= [-(d_{\perp,1} \times F_1) \quad + \quad (d_{\perp,2} \times F_2)] \quad \text{ CW} \\ &= d \times F \quad \text{ CW} \qquad \text{where } d = |d_{\perp,1} - d_{\perp,2}| \end{aligned} \tag{6.1.4}$$

$$\mathbf{M}_{O,\text{ couple}} = d \times F \text{ in a CW direction}$$

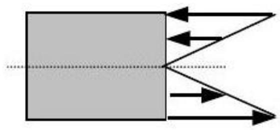
Note that if we had calculated the moment of the force couple shown in Figure 6.1.8 about point O' or point O'' we would have found exactly the same result. Thus, the moment of a couple does not seem to be tied to a particular point, unlike what we found with the moment of a force about a point.

We can say the following about a force couple with force $\mathbf{F}$ and a distance d separating the forces' lines of action:

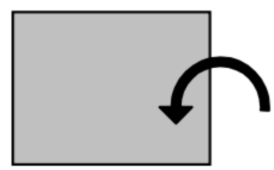
- The lines of action of the two forces in a force couple lie in the same plane.
- A force couple transfers no net linear momentum to a system.
- The moment produced by a force couple is a vector $\mathbf{M}_{\text{couple}}$.
- The *line of action* of the moment vector $\mathbf{M}_{\text{couple}}$ is normal (perpendicular) to the plane that contains the force couple.
- The *point of application* of the moment vector $\mathbf{M}_{\text{couple}}$ can be at any point in the plane of the couple. Because the point of application is free to move, this type of vector is sometimes called a *free* vector.
- The *sense* of the moment vector $\mathbf{M}_{\text{couple can be obtained from ob.}}$ serving the direction of rotation the force couple could impart. The *direction* of the moment vector can be found by curling the fingers of your right hand to match the sense of the force couple and aligning your thumb along the line of action. Your thumb now points in direction for the moment vector $\mathbf{M}_{\text{couple.}}$.
- The *magnitude* of the moment vector $\mathbf{M}_{\text{couple is equal to the product}}$ of the magnitude of one of the force vectors and the shortest distance between the lines of action, d:

$$M_{\text{couple}} = |\mathbf{M}_{\text{couple}}| = d \times |\mathbf{F}| = d \times F$$

We will frequently encounter systems where the force distribution on some portion of the boundary looks like that shown in Part (a) of Figure $6.1.9$. On the upper half of the boundary, the forces are compressive (they push on the system), and on the lower half of the boundary, the forces are tensile (they pull on the system). If every force on the upper half has a mirror image of opposite sense on the lower half, this distribution represents a couple and the net force on the boundary is zero; however there is a net moment due to the couple as shown in Part (b) of Figure $6.1.9$. There will also be occasions where the force distribution on the boundary is actually the sum of a couple and a net force.



a) Distributed couple acting on the boundary



b) Equivalent moment due to distributed couple

Figure $6.1.9$: Moment of a distributed couple.

? Test your understanding

Calculate the magnitude and the direction (CW or CCW) for the net moment about the point O :

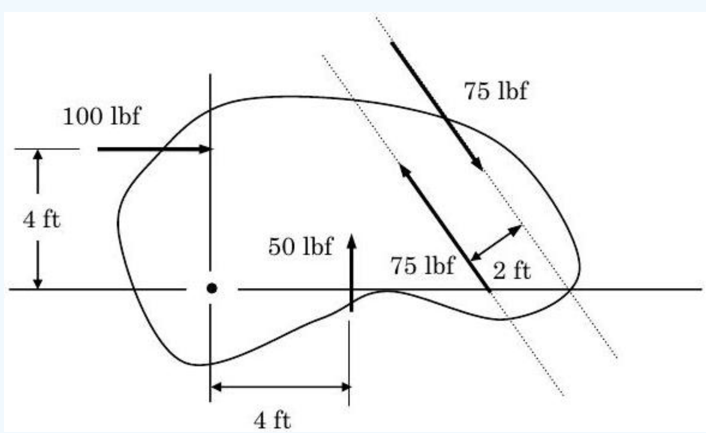


Figure $6.1.10$: Point forces and a force couple are applied to a curved shape.

6.2: Four Questions

6.2.1 What is angular momentum?

In Chapter 5, we showed that the linear momentum $\mathbf{P}$ of a particle with mass m and velocity $\mathbf{V}$ is the product of the particle mass and particle velocity, e.g. $\mathbf{P}=\mathrm{mV}$. The **angular momentum of a particle with respect to point O** is the vector (cross) product of the position vector $\mathbf{r}$ of the particle with respect to point O and the linear momentum of the particle $\mathbf{P}$ (See Figure $6.2.1$):

$$\mathbf{L}_0=\mathbf{r}\times\mathbf{P}=\mathbf{r}\times(m\mathbf{V})$$

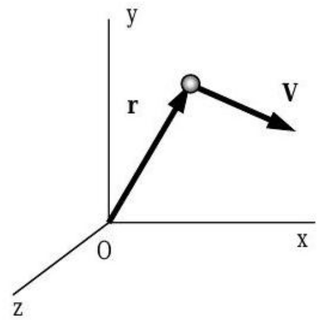


Figure $6.2.1$: Calculating linear momentum of a particle.

It is important to recognize that angular momentum, like linear momentum, is a vector—it has both magnitude and direction. The dimensions of angular momentum are $[\mathrm{L}]^2[\mathrm{M}][\mathrm{T}]^{-1}$. Typical units in the SI system are $\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$ or $\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}$ and in the USCS system are $\mathrm{lbm}\cdot\mathrm{ft}^2/\mathrm{s}$ or $\mathrm{lbf}\cdot\mathrm{ft}\cdot\mathrm{s}$.

The specific angular momentum for the particle can be calculated as the angular momentum per unit mass:

$$\mathbf{l}_0=\frac{\mathbf{L}_0}{m}=\mathbf{r}\times\mathbf{V}$$

The dimensions of specific angular momentum are $[\mathrm{L}]^2[\mathrm{~T}]^{-1}$.

6.2.2 How can angular momentum be stored in a system?

For a system of particles, the angular momentum of the system with respect to the point O is the sum of the angular momentum of each particle in the system:

$$\begin{aligned}\mathbf{L}_{O,\,\mathrm{sys}}&=\sum_{j=1}^{n}\mathbf{L}_{O,\,j}=\sum_{j=1}^{n}(\mathbf{l}_o m)_j\\&=\sum_{j=1}^{n}(\mathbf{r}_j\times\mathbf{V}_j)\,m_j\end{aligned}\tag{6.2.1}$$

For a continuous system, the summation is replaced by an integral over the system volume:

$$\mathbf{L}_{O,\,\mathrm{sys}}=\int_{M_{\mathrm{sys}}}\mathbf{l}_o\,dm=\int_{\unicode{x0194}_{\mathrm{sys}}}(\mathbf{r}\times\mathbf{V})\rho\,d\unicode{x0194}$$

Because of the kinematics describing the motion of a system, the evaluation of this integral can be very difficult. In this chapter, we will only consider two types of motion for which this integral can be easily evaluated — translation and rotation about a fixed, centroidal axis. A more detailed investigation of angular momentum for general motion will be delayed until a later course, i.e. ES 204–Analysis of Mechanical Systems.

Plane motion of a plane system

In our applications of angular momentum to a system, we will restrict ourselves to plane motion of a plane system. A **plane system** is a system that has a plane of symmetry that contains the center of mass of the system. Plane motion as discussed in Section 5.1.2 requires that the plane of symmetry of the system and the x-y plane remain parallel at all times.

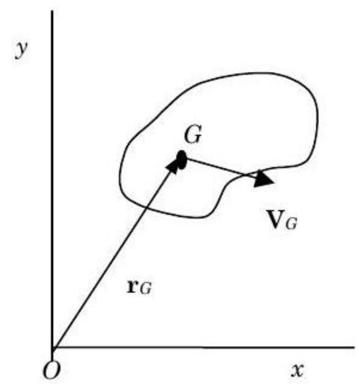


Figure 6.2.2: Plane system with translation.

A plane system with translation

When a system is translating every point in the system has the same velocity, $\mathbf{V}=\mathbf{V_x}+\mathbf{V_y}$. To calculate the angular momentum of the system about point O, we must evaluate Eq. 6.2.4 as follows:

$$\begin{aligned}
\mathbf{L}_{0,\text{ sys}} &= \int_{\forall_{\text{sys}}} (\mathbf{r}\times\mathbf{V})\rho\, d\forall && \left|\begin{array}{l}\mathbf{V}\text{ comes outside the}\\ \text{integral because}\\ \text{it is a constant.}\end{array}\right.\\
&= \left(\int_{\forall_{\text{sys}}} \mathbf{r}\rho\, d\forall\right)\times \underbrace{\mathbf{V}}_{\substack{\mathbf{V}_G=\mathbf{V}\\ \text{since}\\ \text{translation}}} && \left|\begin{array}{l}\text{The integral within () can be}\\ \text{rewritten in terms of the}\\ \text{center of mass (Section 5.1.2)}\end{array}\right.\\
&= (m_{\text{sys}}\,\mathbf{r}_G)\times\mathbf{V}_G && \left|\begin{array}{l}\mathbf{r}_G\text{ is the position vector of the}\\ \text{center of mass with respect to}\\ \text{point } O.\end{array}\right.\\
&= m_{\text{sys}}\,(\mathbf{r}_G\times\mathbf{V}_G)
\end{aligned}$$

Thus the angular momentum of a translating plane system equals the product of the system mass and the cross product of $\mathbf{r}_G$ and $\mathbf{V}_G$, the position and velocity vector, respectively, of the system center of mass.

? Test your knowledge

(a) Starting with the result in Eq. 6.2.5, prove that the following expression is correct for an *open* system:

$$\frac{d\mathbf{L}_{0,\text{ sys}}}{dt}=\frac{d}{dt}[m_{\text{sys}}\,(\mathbf{r}_G\times\mathbf{V}_G)]=(\mathbf{r}_G\times\mathbf{V}_G)\,\frac{dm_{\text{sys}}}{dt}+m_{\text{sys}}\left(\mathbf{r}_G\times\frac{d\mathbf{V}_G}{dt}\right)$$

What happened to $m_{\text{sys}}\left(\frac{d\mathbf{r}_G}{dt}\times\mathbf{V}_G\right)$?

(b) The block shown in the figure has a mass of 50 kg and is translating to the right with a velocity of 10 m/s. Determine the magnitude and direction of (a) the linear momentum of the system; (b) the angular momentum of the system with respect to point A, (c) the angular momentum of the system with respect to point B, and (d) the angular momentum of the system with respect to point C.

If a plane system is undergoing linear translation, i.e. it is traveling in a straight line, can it have angular momentum?

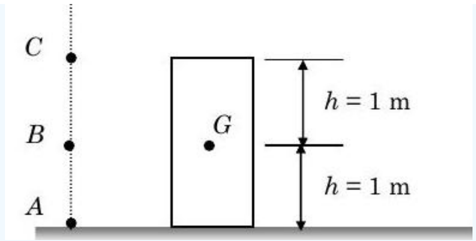


Figure 6.2.3: Block traveling along a horizontal surface, with three vertically aligned reference points.

A plane, rigid system rotating about a fixed, centroidal axis

Many systems exhibit this type of motion: the rotation of a ceiling fan, the rotation of the rotor in a motor, and the rotation of a merry-go-round. A centroidal axis is an axis of rotation that intersects the center of mass of the system. The rotation of your windshield wiper blade and arm on a car does not fit this category because it is not rotating about a centroidal axis of the wiper blade-arm combination. The same would be true of the rotation of a simple pendulum about a non-centroidal pivot point.

For a rotating plane system we will make use, without proof, of a kinematic relation for the velocity of any point in a rigid system rotating in the x-y plane about a fixed axis:

$$\begin{aligned} \mathbf{V} &= \vec{\omega} \times \mathbf{r} \\ &= (\omega_z \mathbf{k}) \times (x\mathbf{i} + y\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_z \\ x & y & 0 \end{vmatrix} \\ &= -y\omega_z \mathbf{i} + x\omega_z \mathbf{j} \\ &= r\omega_z \left(\frac{-y\mathbf{i} + x\mathbf{j}}{r} \right) = (r\omega_z)\left[-(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}\right] \end{aligned}$$

Note that the angular velocity vector has a magnitude ω_z and points in the direction of the positive z-axis.

When you think about using the right-hand rule to cross the vector $\omega = \omega_z \mathbf{k}$ with the $\mathbf{r}$ vector, align your fingers with the positive z-axis and curl them towards the radial vector $\mathbf{r}$. Your thumb should now point in a direction *normal* to the $\mathbf{r}$ vector.

When we further restrict ourselves to plane systems which are rotating around a centroidal axis, an axis that passes through the center of mass, we calculate the angular momentum as follows:

$$\begin{aligned} L_{G,\,\text{sys}} &= \int_{\forall_{\text{sys}}} (\mathbf{r} \times \mathbf{V}) \rho \, d\forall && \left| \begin{array}{l} \text{where } \mathbf{V} = \omega_z \mathbf{k} \times \mathbf{r} \text{ since only} \\ \text{rotation about z-axis is possible} \end{array} \right. \\ &= \int_{\forall_{\text{sys}}} [\mathbf{r} \times (\omega_z \mathbf{k} \times \mathbf{r})] \, \rho \, d\forall && \left| \begin{array}{l} \text{The rotational velocity } \omega_z \\ \text{is a constant.} \end{array} \right. \\ &= \omega_z \int_{\forall_{\text{sys}}} \underbrace{[\mathbf{r} \times (\mathbf{k} \times \mathbf{r})]}_{=|\mathbf{r}\cdot\mathbf{r}|\mathbf{k} = r^2\mathbf{k}} \rho \, d\forall && \left| \begin{array}{l} \text{The double cross product can be} \\ \text{simplified using cylindrical coordinates} \\ \text{where } \mathbf{r} = r\,\mathbf{e_r},\ \mathbf{k} \times \mathbf{e_r} = \mathbf{e}_\theta, \text{ and} \\ \mathbf{e_r} \times \mathbf{e}_\theta = \mathbf{k}. \text{ (See Section 5.5)} \end{array} \right. \\ &= \omega_z \int_{\forall_{\text{sys}}} \left[r^2 \mathbf{k}\right] \rho \, d\forall \\ &= \omega_z \mathbf{k} \underbrace{\left[\int_{\forall_{\text{sys}}} r^2 \rho \, d\forall \right]}_{=\text{Mass moment of inertia } I_G} && \left| \begin{array}{l} \text{The quantity in the brackets is} \\ \text{called the mass moment of inertia} \\ \text{about the } z\text{-axis that contains} \\ \text{the center of mass } G. \end{array} \right. \\ &= \omega_z I_G \mathbf{k} \end{aligned}$$

Thus the magnitude of the angular momentum of a plane, rigid system rotating about a fixed, centroidal axis equals the product of the rotational velocity ω_z about the z-axis and the mass moment of inertia I_G of the system about the z-axis. As shown in Figure $6.2.4$, a mass can have a mass moment of inertia about any axis—x, y, or z.

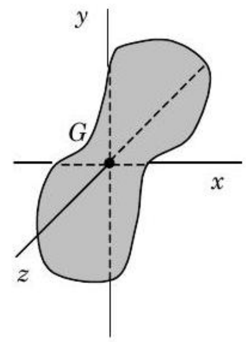


Figure $6.2.4$: Moment of inertia of a mass.

The mass moment of inertia about the z-axis would be evaluated using the integral

$$I_{G,\,z} = \int_{V_{\text{sys}}} r^2 \rho \, dV = \begin{cases} \displaystyle\int_{V_{\text{sys}}} \left(x^2 + y^2\right) \rho \, dx \, dy & \text{where } dV = dx \, dy \\ \qquad \text{or} & \\ \displaystyle\int_{V_{\text{sys}}} r^2 \rho \, r \, d\theta \, dr & \text{where } dV = r \, d\theta \, dr \end{cases}$$

Depending upon the shape of the object, the x-y or r-θ formulation may be preferable. The dimensions of the mass moment of inertia are $[\mathrm{M}][\mathrm{L}]^2$. Typical units are $\mathrm{kg} \cdot \mathrm{m}^2$ in SI and $\mathrm{lbm} \cdot \mathrm{ft}^2$ in USCS. Values for a rectangular solid, a solid cylinder, and a solid sphere can be found in Figure $6.2.5$.

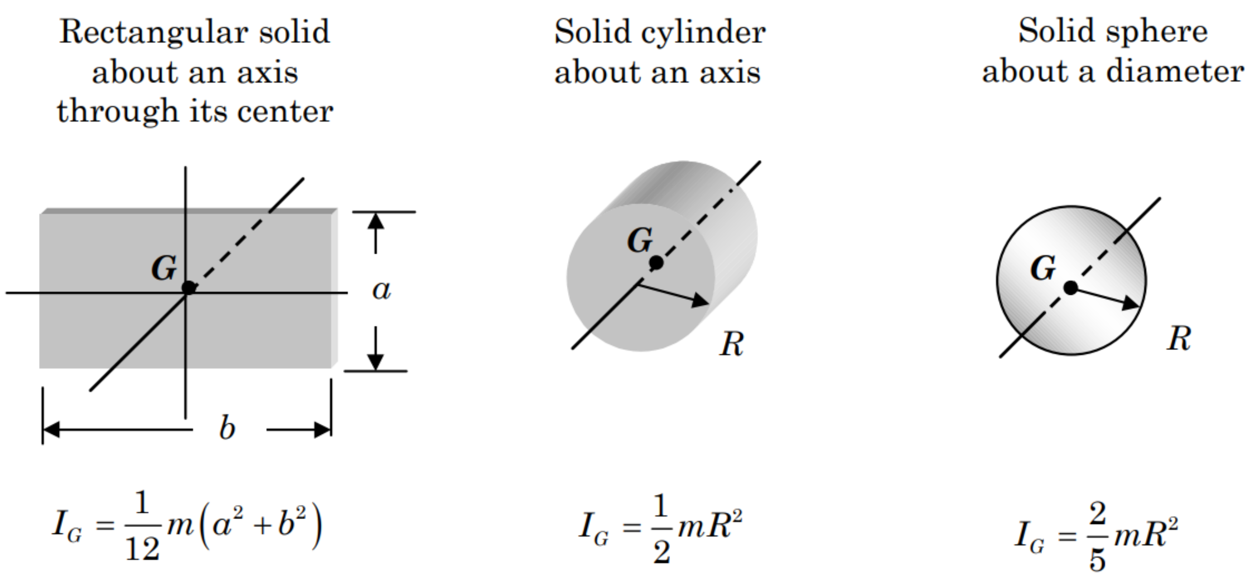


Figure $6.2.5$: Centroidal mass moment of inertia for some common shapes.

6.2.3 How is angular momentum transported across a system boundary?

Angular momentum about point O is transported across the boundary of a system by two mechanisms: angular momentum transport with external forces and angular momentum transport with mass flow.

Angular momentum transport with forces

When an external force is applied to a system as discussed earlier it produces a moment with respect to point O. This moment represents a transport rate of angular momentum. More specifically, the net transport rate of angular momentum about point O into a system with due to external force is written mathematically as the sum of the individual moments:

$$\dot{\mathbf{L}}_{O,\text{ forces}}=\sum_j \mathbf{M}_{O,\,j}=\sum_j\left(\mathbf{r}_j\times\mathbf{F}_j\right)=\begin{bmatrix}\text{Transport rate of}\\ \text{angular momentum}\\ \text{with external forces}\end{bmatrix}$$

The moment about point O due to the distributed gravitational force or weight can be shown to equal the moment produced by the force of the weight acting at the center of mass (or center of gravity) of the system. Force couples acting on the system produce a moment and also transport angular momentum across the system boundary. Thinking of the moment of a force as a transport rate of linear momentum, the dimensions of a moment are $[\text{Force}][\text{L}]=\left\{[\text{M}][\text{L}]^2/[\text{T}]\right\}/[\text{T}]=[\text{M}][\text{L}]^2[\text{T}]^{-2}$.

Angular momentum transport with mass flow

As was shown earlier, every lump of mass with a velocity has angular momentum about a point O. When mass is allowed to flow across the boundary of an open system, each lump of mass carries with it linear momentum. Thus the angular momentum of an open system can also be changed by mass flow carrying angular momentum across the system boundary.

The rate at which angular momentum about point O is transported across the boundary can be represented by the product of the mass flow rate and the local velocity at the boundary assuming that the velocity is uniform:

$$\dot{\mathbf{L}}_{O,\text{ mass}}=\dot{m}(\mathbf{r}\times\mathbf{V})=\begin{bmatrix}\text{Transport rate of}\\ \text{angular momentum}\\ \text{with mass flow}\end{bmatrix}$$

where $\dot{m}$ is the mass flow rate at the flow boundary, $\mathbf{r}$ is the position of the flow boundary with respect to point O, and $\mathbf{V}$ is the local velocity of the flow. Now combining this for all the flow boundaries, the net transport rate into the system of angular momentum about point O can be obtained by summing up the transports at all flow boundaries:

$$\mathbf{L}_{O,\text{ mass, net}}=\sum_{in}\left(\mathbf{r}_i\times\mathbf{V}_i\right)\dot{m}_i-\sum_{out}\left(\mathbf{r}_e\times\mathbf{V}_e\right)\dot{m}_e$$

6.2.4 How can angular momentum be generated or consumed in a system?

Experience has shown that the angular momentum of a system cannot be created or destroyed; thus *angular momentum is conserved*.

6.2.5 Putting It All Together

Using the accounting framework, we can develop the following statement for conservation of angular momentum about point O:

$$\begin{bmatrix}\text{Rate of accumulation}\\ \text{of}\\ \text{angular momentum}\\ \text{inside a system}\\ \text{at time } t\end{bmatrix}=\begin{bmatrix}\text{Net transport rate of}\\ \text{angular momentum}\\ \text{into the system}\\ \text{by external forces}\\ \text{at time } t\end{bmatrix}+\begin{bmatrix}\text{Net transport rate of}\\ \text{angular momentum}\\ \text{into the system}\\ \text{by mass flow}\\ \text{at time } t\end{bmatrix}$$

In symbols the **rate-form of the conservation of angular momentum** about point O becomes

$$\frac{d\mathbf{L}_{O,\text{ sys}}}{dt}=\sum_j \mathbf{M}_{O,\,j}+\sum_{in}\left(\mathbf{r}_i\times\mathbf{V}_i\right)\dot{m}_i-\sum_{out}\left(\mathbf{r}_e\times\mathbf{V}_e\right)\dot{m}_e$$

Again, as with linear momentum and mass, the judicious use of modeling assumptions will often simplify this equation as we model systems.

6.3: Conservation of Angular Momentum Equation

The recommended starting point for the application of the conservation of angular momentum principle is the rate form of the equation:

$$\frac{d\mathbf{L}_{O,\text{ sys}}}{dt}=\sum_j \mathbf{M}_{O,\,j}+\sum_{in}(\mathbf{r}_i\times\mathbf{V}_i)\,\dot{m}_i-\sum_{out}(\mathbf{r}_e\times\mathbf{V}_e)\,\dot{m}_e$$

where $\mathbf{L}_{O,\text{ sys}}$ is the system angular momentum about point O, $\mathbf{M}_O$ is the transport rate of angular momentum about point O with moments, and $\dot{m}(\mathbf{r}\times\mathbf{V})$ is the transport rate of angular momentum about point O with mass flow across the boundary.

In applying the rate form of the conservation of angular momentum equation, there are many modeling assumptions that will be used to simplify the basic equation for specific systems. As always, you should focus on understanding the physical meaning of the assumptions and how they simplify the equations for a given system. Do not just memorize the results.

Steady-state system: If a system is operating under steady-state conditions, all intensive properties and interactions are independent of time. When this assumption is applied to the conservation of angular momentum equation, we have

$$\underbrace{\cancelto{=0}{\frac{d\mathbf{L}_{O,\text{ sys}}}{dt}}}_{\mathbf{L}_{O,\text{ sys}}\text{ is constant}}=\sum_j \mathbf{M}_{O,\,j}+\sum_{in}(\mathbf{r}_i\times\mathbf{V}_i)\,\dot{m}_i-\sum_{out}(\mathbf{r}_e\times\mathbf{V}_e)\,\dot{m}_e$$

$$0=\sum_j \mathbf{M}_{O,\,j}+\sum_{in}(\mathbf{r}_i\times\mathbf{V}_i)\,\dot{m}_i-\sum_{out}(\mathbf{r}_e\times\mathbf{V}_e)\,\dot{m}_e$$

Closed system: A closed system has no flow at the boundary so the conservation of angular momentum equation reduces to the following:

$$\frac{d\mathbf{L}_{O,\text{ sys}}}{dt}=\sum_j \mathbf{M}_{O,\,j}+\sum_{in}(\mathbf{r}_i\times\mathbf{V}_i)\,\cancelto{=0}{\dot{m}_i}-\sum_{out}(\mathbf{r}_e\times\mathbf{V}_e)\,\cancelto{=0}{\dot{m}_e}$$

$$\frac{d\mathbf{L}_{O,\text{ sys}}}{dt}=\sum_j \mathbf{M}_{O,\,j}$$

To make any further simplications, it is necessary to have additional information or make assumptions about the kinematics of the moving system. If we can assume that *a plane system is translating* then

$$\left.\begin{array}{r}\mathbf{L}_{O,\text{ sys}}=m_{\text{sys}}(\mathbf{r}_G\times\mathbf{V}_G)\\ \frac{d\mathbf{L}_{O,\text{ sys}}}{dt}=\sum_j\mathbf{M}_{O,\,j}\end{array}\right\}\quad\rightarrow\quad m_{\text{sys}}\frac{d(\mathbf{r}_G\times\mathbf{V}_G)}{dt}=\sum_j\mathbf{M}_{O,\,j}$$

If we can assume that a plane, rigid system is rotating about an axis (the z-axis) that passes through the system center of mass then

$$\left.\begin{array}{r}\mathbf{L}_{O,\text{ sys}}=\omega_z I_G\mathbf{k}\\ \frac{d\mathbf{L}_{O,\text{ sys}}}{dt}=\sum_j\mathbf{M}_{O,\,j}\end{array}\right\}\quad\rightarrow\quad \frac{d(\omega_z I_G\mathbf{k})}{dt}=\sum_j\mathbf{M}_{O,\,j}$$

where $\mathbf{k}$ is the unit vector the points in the positive direction of the z-axis.

Modeling Reactions at Supports and Connections:

One of the significant problems in selecting a system for momentum transfer is the identification of the interactions between the system and its surroundings. What exactly are the constraints on how momentum is being transmitted across the boundary at a specific interactions? Contact forces on the boundary of a system are sometimes referred to as reactions, especially at places where the boundary cuts a support or connection. According to Beer and Johnston[1], reactions can be classified in three broad categories:

(a) Force with a known line of action. Many connections satisfy this condition among these are *rollers, rockers, frictionless surfaces, short links and cables, collars on frictionless rods*, and *frictionless pins in slots*. Each of these reactions is characterized by the ability to only transport linear momentum by a force in a single, clearly identifiable direction. (See Figure 6.3.1).

(b) Force of unknown direction and magnitude. Hardware or connections that provide this type of reaction include *frictionless pins in fitted holes, frictionless ball-and-socket joints, hinges, and rough surfaces* (Although frictionless surfaces only transmit a force normal to the surface, friction introduces shear forces at the surface.)

(c) Force of unknown direction and magnitude *and* a couple. This last type of reaction is usually referred to as a *fixed* support. The reaction of the ground on a telephone pole stuck vertically in the ground is an excellent example. When you push of the telephone pole in any direction, the ground exerts three forces on the pole—a horizontal shear force, a vertical normal force, and a distributed force that is a force couple. The couple, as shown earlier, only transmits angular momentum.

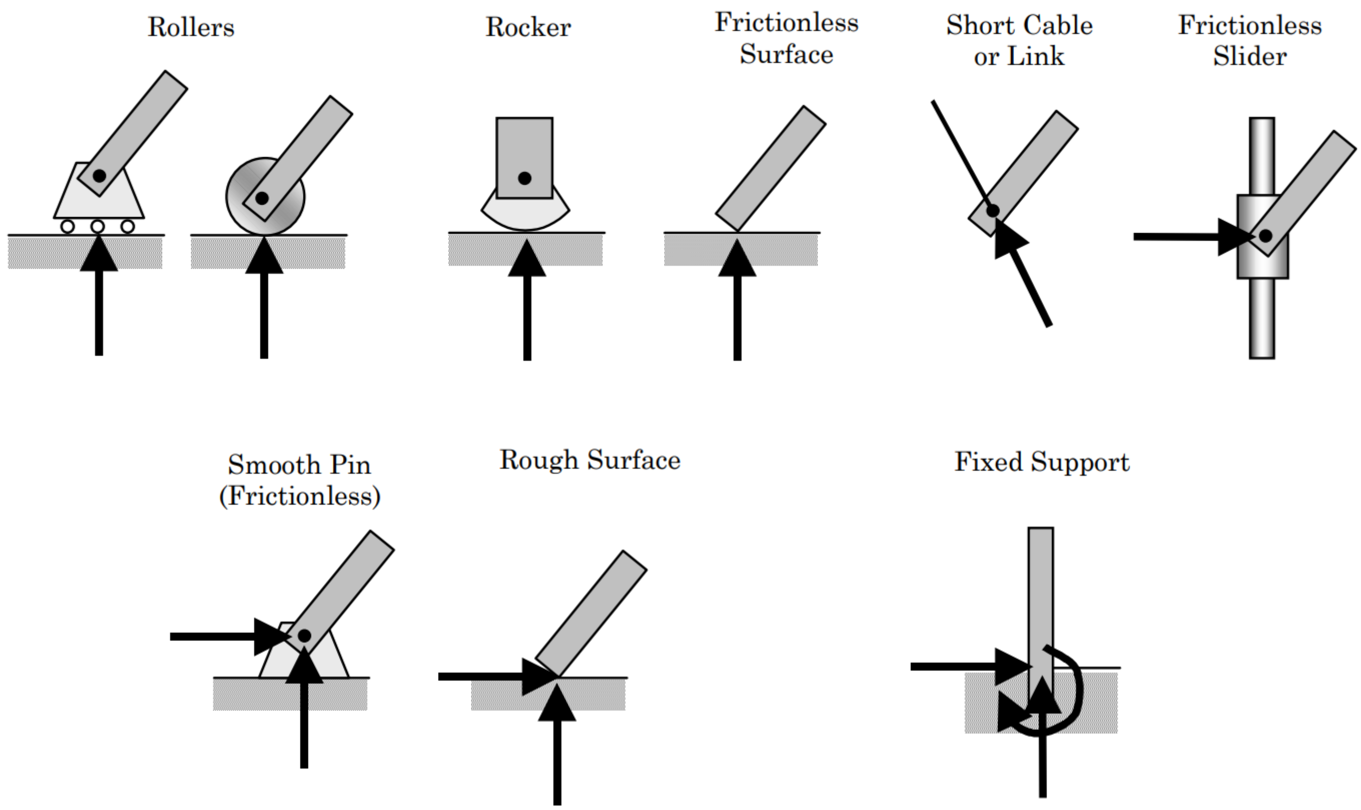


Figure $6.3.1$: Surface force reactions.

As one might expect, these reactions can be seen in everyday life if one is observant. Look around you right now and see if you can find a physical connection that falls in each of these categories. For example, look at the caster or wheel on your desk chair or the hinge on the door. Next time you drive under an overpass on the highway, look at how the bridge roadbed is supported on the columns. You very likely will see a roller.

[1] F. P. Beer and E. R. Johnston, Jr., *Vector Mechanics for Engineers: Statics*, 6th ed., McGraw-Hill, New York, 1996.

Example — What a reaction!

Determine the reactions at A and C. Assume the weight of the mechanism is negligible and that all joints are frictionless.

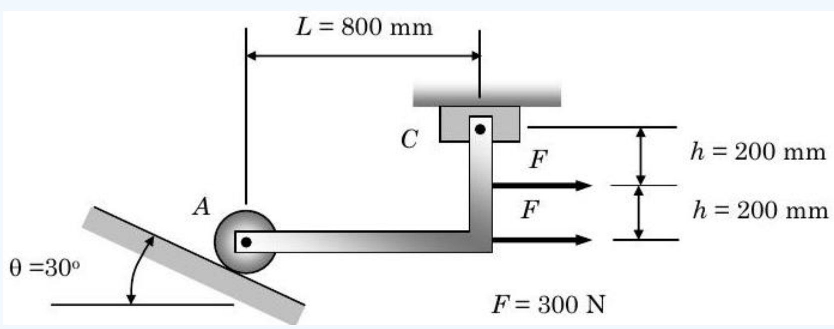


Figure $6.3.2$: An L-shaped bar is supported at two ends and has point loads applied at two points.

Solution

Known: A specific device with given loads.

Find: The reactions (magnitude and direction) at points A and C.

Given: See figure drawn above. (Notice how all physical quantities have been given symbols.)

Analysis:

Strategy $\rightarrow$ Since this problem concerns forces, at a minimum it will require linear momentum and possibly angular momentum. (The last part is a giveaway since this problem is in the angular momentum chapter. Unfortunately, problems in real life never come with the chapter number.)

System $\rightarrow$ Closed, non-deforming system including the pinned joint C, the roller at A and the connecting bar.
Property to count $\rightarrow$ Linear and angular momentum
Time interval $\rightarrow$ Try rate form. (Momentum is flowing through this system.)

The system drawn below shows the reactions at A and C and all pertinent dimensions on the figure. Assuming that this is a closed and steady-state system, the linear and angular momentum equations become the following:

$$0=\sum_j \mathbf{F}_j \quad \text{and} \quad 0=\sum_j \mathbf{M}_{0,\,j}$$

(Could you explicitly show how these two equations were obtained given the assumptions above?)

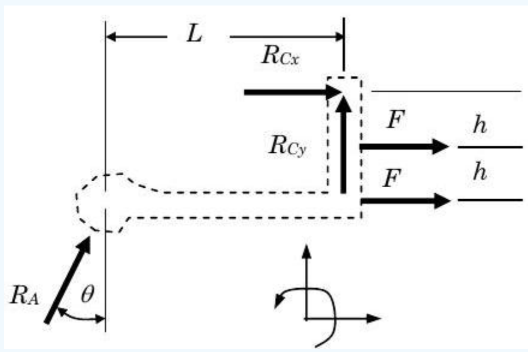


Figure $6.3.3$: Free-body diagram.

Before writing the component equations in scalar form we must clearly indicate a coordinate system and what we are assuming as positive directions. This is done with the small coordinate system in the figure. Note that we are assuming counterclockwise (CCW) as the positive direction for angular momentum.

Now looking at the x-momentum:

$$\boxed{\rightarrow +} \qquad 0=R_A(\sin\theta)+2F+R_{Cx} \quad \rightarrow \quad -2F=R_A(\sin\theta)+R_{Cx} \quad \rightarrow \quad -(600\text{ N})=R_A\left(\sin 30^\circ\right)+R_{Cx}$$

For the y-momentum:

$$\boxed{\uparrow +} \qquad 0=R_A(\cos\theta)+R_{Cy} \quad \rightarrow \quad R_{Cy}=-R_A\left(\cos 30^\circ\right)$$

This gives two equations for three unknowns — R_A, R_{Cx}, and R_{Cy}. To get the remaining equation we must apply conservation of angular momentum.

Applying conservation of angular momentum about Point A we eliminate any moment due to one of the applied forces F and the reaction R_A :

$$\boxed{\text{CCW} +} \qquad 0=[+LR_{Cy}]+[-2hR_{Cx}]+[-hF] \quad \rightarrow \quad F=\left(\frac{L}{h}\right)R_{Cy}-2R_{Cx} \quad \rightarrow$$
$$(300\text{ N})=(4)R_{Cy}-2R_{Cx}$$

Solving these three equations simultaneously, we get the following:

$$\left.\begin{aligned} -(600\text{ N}) &= R_A\sin 30^\circ - 2R_{Cx} \\ R_{Cy} &= -R_A\cos 30^\circ \\ (300\text{ N}) &= R_A\sin 30^\circ + R_{Cx} \end{aligned}\right\} \rightarrow \begin{aligned} R_A &= 365.2\text{ N} \\ R_{Cx} &= -782.6\text{ N} \\ R_{Cy} &= -316.3\text{ N} \end{aligned}$$

Notice that the reaction at C occurs in a direction opposite to that assumed on the drawing.

Example — Going to the wall for a reaction

A water jet steadily impacts a blade that is attached to a cantilever beam as shown in the figure. The cantilever beam is fixed to the wall at A and holds the blade stationary. The water hits the blade at B and leaves the blade at C.

The mass of the blade/beam combination is 100 lbm with a known center of mass at point G located at a distance $L/2$ from the wall. The water has a mass flow rate of 150lbm/s, and the speed of the water is the same entering and leaving the blade, $V_1 = V_2 = 20\text{ ft/s}$. The water leaves the blade at an angle of 30° below the horizontal.

Other information is as follows: $L = 5\text{ ft}$, $d = 7\text{ ft}$, and $h = 2\text{ ft}$.

Determine the reactions on the cantilever beam at the wall (A).

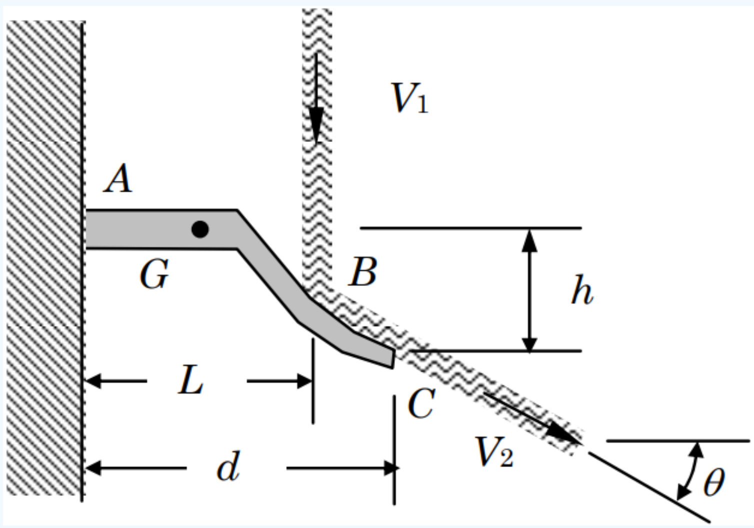


Figure $6.3.4$: Water falls onto a curved blade attached to a cantilever beam and runs off the blade's lower end.

Solution

Known: Water is deflected by a blade supported by a cantilever beam.

Find: The reactions on the cantilever beam at the wall.

Given: See the figure above. Again, note how all physical quantities have symbols.

Analysis:

Strategy $\rightarrow$ Since we are asked for reactions (forces and moments) we should try using conservation of linear and angular momentum.

System $\rightarrow$ Open, non-deforming system that includes the blade-beam-water with water flowing in and out as shown below.
Property to count $\rightarrow$ Angular and linear momentum (and possibly mass).
Time interval $\rightarrow$ Probably an infinitesimal time interval, i.e. the rate form.

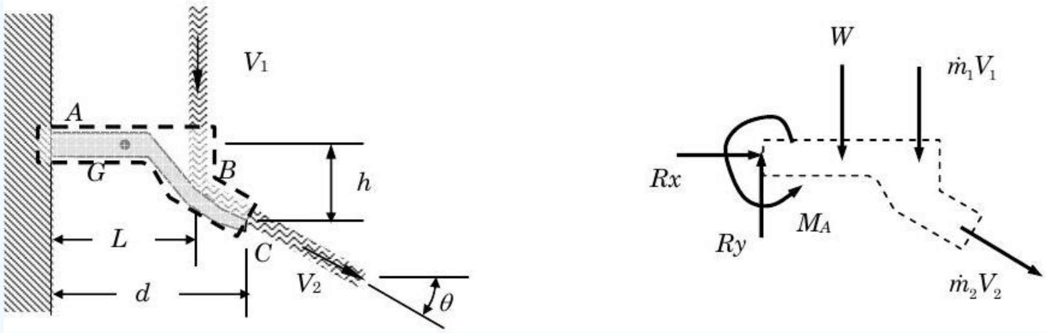


Figure 6.3.5: Free-body diagram of the beam, blade, and water system.

The figure above shows the free-body (or system interaction) diagram. Notice that the connection at the wall is replaced by two forces and a moment because the wall-beam connection is a fixed support. Also note that the mass transports of linear momentum are shown on the figure along with the weight acting at point G.

As a starting point, let's write the conservation of mass and see what it tells us:

$$\underbrace{\cancelto{=0}{\frac{dm_{\text{sys}}}{dt}}}_{\text{steady state}} = \dot{m}_1 - \dot{m}_2 \quad \rightarrow \quad \dot{m}_1 = \dot{m}_2 = \dot{m}$$

This may seem like a trivial result, but if you didn't see it immediately it isn't trivial. Furthermore, this just confirms your intuition.

Now let's write the conservation of linear momentum:

$$\cancelto{=0}{\frac{d\mathbf{P}_{\text{sys}}}{dt}} = \mathbf{Rx} + \mathbf{Ry} + \mathbf{W} + \dot{m}_1 \mathbf{V}_1 - \dot{m} \mathbf{V}_2 \quad \rightarrow \quad 0 = \mathbf{Rx} + \mathbf{Ry} + \mathbf{W} + \dot{m}_1 \mathbf{V}_1 - \dot{m} \mathbf{V}_2$$

Now writing the scalar form of this equation, assume positive x is to the right and positive y is up.

$$\boxed{\rightarrow +} \qquad 0 = Rx - \dot{m}\left(V_2 \cos\theta\right)$$

$$Rx = \dot{m}\left(V_2 \cos\theta\right) \quad \rightarrow \quad Rx = \left(150\ \frac{\text{lbm}}{\text{s}}\right)\left(20\ \frac{\text{ft}}{\text{s}}\right)\left(\cos 30^\circ\right)\left(\frac{1\ \text{lbf}}{32.174\ \frac{\text{lbm} \cdot \text{ft}}{\text{s}^2}}\right) = 80.8\ \text{lbf}$$

$$\begin{aligned}
\boxed{\uparrow +} \qquad 0 &= Ry - W + \dot{m}\left(-V_1\right) - \dot{m}\left(-V_2 \sin 30^\circ\right) \\
Ry &= W + \dot{m}\left(V_1 - V_2 \sin 30^\circ\right) \\
&= W + \dot{m} V\left(1 - \sin 30^\circ\right) \\
Ry &= \left[(100\ \text{lbm})\left(\frac{1\ \text{slug}}{32.174\ \text{lbm}}\right)\left(32.174\ \frac{\text{ft}}{\text{s}^2}\right)\right] \\
&= [100 + 46.6]\ \text{lbf} \\
&= 146.6\ \text{lbf}
\end{aligned}$$

Be very careful about the signs on the terms, especially the mass flow rate terms. Note that the $+/-$ in front of the mass flow rates depend upon whether the flow is into or out of the system; the signs inside of the parenthesis on the velocity (specific linear momentum) terms are the result of translating the velocity vector to this specific coordinate system.

With one remaining unknown M_A, we need to apply conservation of angular momentum. Although we can sum moments about *any* point, let's use point C. This has the benefit of eliminating the angular momentum carried out in the water jet at C.

$$\frac{d\mathbf{L}_{C,\text{ sys}}}{dt} = M_A + (\mathbf{r}_A \times \mathbf{Rx}) + (\mathbf{r}_A \times \mathbf{Ry}) + (\mathbf{r}_G \times \mathbf{W}) + \dot{m}\,(\mathbf{r}_B \times \mathbf{V}_1)$$

$$\boxed{\text{CW}+} \quad 0 = (-M_A) + (+h \cdot Rx) + (d \cdot Ry) + \left[-\left(d - \frac{L}{2}\right) W\right] + \dot{m}\,[-(d-L)V_1]$$

$$M_A = (h \cdot Rx) + (d \cdot Ry) + \left(-\left(d - \frac{L}{2}\right) W\right) + \dot{m}\,[-(d-L)V_1]$$

$$M_A = [(2\text{ ft})(80.8\text{ lbf})] + [(7\text{ ft})(146.6\text{ lbf})] + \left[-\left(7 - \frac{5}{2}\right)(\text{ft})(100\text{ lbf})\right] + \left(150\ \frac{\text{lbm}}{\text{s}}\right)\left[-(2\text{ ft})\left(20\ \frac{\text{ft}}{\text{s}}\right)\right]$$

$$= (161.6\text{ ft}\cdot\text{lbf}) + (1026.2\text{ ft}\cdot\text{lbf}) + (-450\text{ ft}\cdot\text{lbf}) + \underbrace{\left(-6000\ \frac{\text{lbm}\cdot\text{ft}}{\text{s}^2}\right)\left(\frac{1\text{ lbf}}{32.174\ \frac{\text{lbm}\cdot\text{ft}}{\text{s}^2}}\right)}_{=-186.5\text{ ft}\cdot\text{lbm}}$$

$$= [161.6 + 1026.2 - 450 - 186.5]\text{ ft}\cdot\text{lbf}$$

$$M_A = 551.3\text{ ft}\cdot\text{lbf}$$

Similar comments about the signs apply to angular momentum as given for linear momentum.

Thus the reactions at A are $Rx = 80.8\text{ lbf}$, $Ry = 146.6\text{ lbf}$, and $M_A = 551.3\text{ ft}\cdot\text{lbf}$ in the directions as indicated on the diagram.

Example — Tipping a box

You have been asked to help move a filing cabinet by pushing it on the floor. Luckily for you the floor has very low friction coefficients ($\mu_{\text{static}} = 0.3$ and $\mu_{\text{kinetic}} = 0.1$). The filing cabinet is 0.3 m wide and 1.6 m high and has a mass of 100 kg. You will push it at a position above the floor equal to $3/4$ of its height.

(a) Determine the force F required to get the cabinet moving. Will it begin to tip before it slips?

(b) If it tips before it slips, what force F is required to begin tipping the stationary cabinet?

(c) If it slips before it tips, what is the maximum force F that can be applied before the slipping cabinet tips?

Solution

Known: A filing cabinet is being pushed along the floor.

Find:

(a) The force required to slip the cabinet

(b) If it tips before it slips, the force required to begin tipping

(c) If it slips before it tips, the maximum force that can be applied before the slipping cabinet tips

Given:

$$\begin{aligned}&h = 1.6\text{ m} \quad m = 100\text{ kg}\\&L = 0.3\text{ m}\\&d = 1.2\text{ m}\\&G\text{ in center of rectangle}\\&\\&\text{Positive directions on coordinate system}\end{aligned}$$

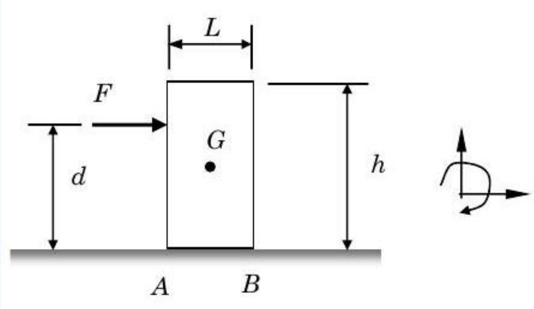


Figure 6.3.6: Labeled diagram showing all given information.

Analysis

Strategy $\rightarrow$ Try conservation of linear and angular momentum since we are considering friction forces and motion and tipping.

System $\rightarrow$ Closed, non-deforming system that includes the cabinet only.
Property to count $\rightarrow$ Linear momentum and angular momentum.
Time interval $\rightarrow$ Probably the rate form since we are interested in forces.

First we need to draw the free-body diagram. Notice that the direction of the friction force along the bottom of the cabinet opposes the motion of the cabinet. You should also notice that the normal force F_{N} on the bottom of the cabinet can act anywhere between A and B depending on the value of the applied force F. Clearly when F is zero the normal force is applied immediately under the center of mass.

First, let's determine what force will cause the cabinet to slip. To do this we apply conservation of linear momentum to a *stationary*, *closed* system.

$$\underbrace{\cancel{\frac{d\mathbf{P}_{\text{sys}}}{dt}}^{=0}}_{\substack{\mathbf{P}=0 \\ \text{a constant}}} = \sum_j \mathbf{F}_j + \underbrace{\cancel{\sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e}^{=0}}_{\text{closed system, no mass flow}} \quad \rightarrow \quad 0 = \mathbf{F} + \mathbf{F}_f + \mathbf{F}_{\mathrm{N}} + m\mathbf{g}$$

Where the friction force must equal the maximum static friction force possible

$$\boxed{\rightarrow +} \qquad 0 = -F_f + F \quad \rightarrow \quad F = F_f \leq \mu_{\text{static}} F_N$$

$$\boxed{\uparrow +} \qquad 0 = -mg + F_N \quad \rightarrow \quad FN = mg$$

$$F = F_f \leq \mu_{\text{static}} mg = (0.3)\left[(100 \text{ kg})\left(9.81 \ \tfrac{\text{m}}{\text{s}^2}\right)\right] = 294.3 \text{ N}$$

So the minimum force that must be applied to get the cabinet to slide is 294.3 N.

Now to determine if it tips, apply the conservation of angular momentum to the same system. To do this we must recognize that just at it starts to tip (tipping impending) the normal force is acting at point B (the right side of the cabinet). So let's calculate angular momentum about point B and solve for the force that must be applied to tip the cabinet assuming it is stationary, F_{tip}.

$$\underbrace{\cancel{\frac{d\mathbf{L}_{B,\text{ sys}}}{dt}}^{=0}}_{\substack{\mathbf{L}=0 \text{ since} \\ \text{stationary}}} = \sum_j \mathbf{M}_{B,\,j} + \underbrace{\cancel{\sum - \sum}^{=0}}_{\substack{\text{Closed system} \\ \text{No mass flow terms}}} \quad \rightarrow \quad 0 = (\mathbf{r}_F \times \mathbf{F}_{\text{tip}}) + (\mathbf{r}_G \times m\mathbf{g})$$

$$\boxed{\text{CW} +} \qquad 0 = (d\,F_{\text{tip}}) + \left(\frac{L}{2} mg\right)$$

$$F_{\text{tip}} = \frac{1}{2}\frac{L}{d} mg \quad \rightarrow \quad F_{\text{tip}} = \frac{1}{2}\left(\frac{0.3 \text{ m}}{1.2 \text{ m}}\right)(981.0 \text{ N}) = 122.6 \text{ N}$$

Comparing the force to slip against the force to tip, we find that the force to tip is less than the force to slip the cabinet—*the cabinet tips before it slips*. The maximum force that can be applied to the cabinet before it tips is 122.6 N and the cabinet will never slip.

The answer to Part (c) is academic since the cabinet never slips under these conditions.

Comment:

But wait, we need to move the cabinet! Suppose we place the cabinet on some very small roller bearings so it essentially glides over the floor with no static or kinetic friction. How much force can I apply now before it tips?

The y-momentum results will be identical $\rightarrow$ $F_{\mathrm{N}} = mg = 981 \text{ N}$.

The x-momentum results must consider motion in the x-direction so we have $\rightarrow m\frac{dV_G}{dt} = F$.

The angular momentum equation is trickier. First we must recognize that the cabinet is *translating*, and that when tipping is impending the normal force F_{N} will act at the right corner:

$$\frac{d\mathbf{L}_B}{dt} = (\mathbf{r}_F \times \mathbf{F}) + (\mathbf{r}_G \times m\mathbf{g}) \quad \text{and} \quad \mathbf{L}_B = m\left(\mathbf{r}_G \times \mathbf{V}_G\right)$$

$$\boxed{\text{CW}+} \qquad \underbrace{\frac{d}{dt}\left[m\left(\frac{h}{2}V_G\right)\right]}_{\substack{\text{Angular momentum}\\ \text{of system about}\\ \text{point } B}} = \underbrace{(d\,F)}_{\substack{\text{Moment of}\\ \text{force } F \text{ about}\\ \text{point } B}} + \underbrace{\left(-\frac{L}{2}mg\right)}_{\substack{\text{Moment of}\\ \text{the weight about}\\ \text{point } B}}$$

$$m\frac{d}{dt}\left(\frac{h}{2}V_G\right) = dF - \frac{L}{2}mg$$

$$m\frac{h}{2}\frac{dV_G}{dt} = dF - \frac{L}{2}mg \qquad \rightarrow \qquad F = \frac{m}{2}\frac{h}{d}\frac{dV_G}{dt} + \frac{m}{2}\frac{L}{2}g$$

$$F = \frac{m}{2}\left(\frac{h}{d}\right)\left[\frac{dV_G}{dt} + g\left(\frac{L}{h}\right)\right]$$

Now we need to use our results from the x-momentum to eliminate the acceleration term

$$F = \frac{m}{2}\left(\frac{h}{d}\right)\left[\frac{dV_G}{dt} + g\left(\frac{L}{h}\right)\right] = \frac{m}{2}\left(\frac{h}{d}\right)\underbrace{\left[\left(\frac{F}{m}\right) + g\left(\frac{L}{h}\right)\right]}_{\substack{\text{Here's where we}\\ \text{made the substitution}}}$$

$$F - \frac{m}{2}\left(\frac{h}{d}\right)\left(\frac{F}{m}\right) = \frac{m}{2}\left(\frac{h}{d}\right)\left(\frac{L}{h}\right)g$$

$$F\left[1 - \frac{1}{2}\left(\frac{h}{d}\right)\right] = \frac{m}{2}\left(\frac{L}{d}\right)g \qquad \rightarrow \qquad F = \frac{\frac{m}{2}\left(\frac{L}{d}\right)g}{\left[1 - \frac{1}{2}\left(\frac{h}{d}\right)\right]} = (mg)\frac{\frac{1}{2}\left(\frac{L}{d}\right)}{\left[1 - \frac{1}{2}\left(\frac{h}{d}\right)\right]}$$

Now substituting the numerical values we have

$$F = (mg)\frac{\frac{1}{2}\left(\frac{L}{d}\right)}{\left[1 - \frac{1}{2}\left(\frac{h}{d}\right)\right]} = (981 \text{ N})\frac{\frac{1}{2}\left(\frac{0.3}{1.2}\right)}{\left[1 - \frac{1}{2}\left(\frac{1.6}{1.2}\right)\right]} = 367.9 \text{ N}$$

Notice that the value of F depends on the weight of the object, the ratio h/d, and the ratio L/d. What happens when $d = h/2$? What does this mean physically? What about when $L = 0$? What does this mean?

Example — It's Turning!

Two metal cylinders A and B are suspended from a frictionless pulley. Cylinder A has a mass of 30 kg and cylinder B has a mass of 60 kg. The pulley is essentially a flat disk with a diameter of 0.5 m and a mass of 10 kg. The cables have negligible mass.

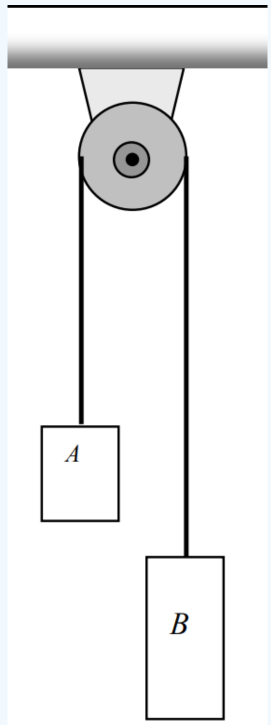


Figure $6.3.7$: Two cylinders suspended from a frictionless pulley.

Initially the cylinders are stationary. If they are suddenly released, calculate the acceleration of the cylinders.

Solution

Known: Two masses suspended from a pulley suddenly start moving.

Find: The acceleration of the cylinders after they are released.

Given: See the figure above.

$$\begin{aligned}
&\text{Diameter of the pulley} = D = 0.5 \text{ m} \\
&\text{Mass of the pulley} = m_{\text{Pulley}} = 10 \text{ kg} \\
&\text{Mass of cylinder } A = m_{\text{A}} = 30 \text{ kg} \\
&\text{Mass of cylinder } B = m_{\text{B}} = 60 \text{ kg}
\end{aligned}$$

Analysis:

Strategy $\rightarrow$ Try angular momentum since pulley is turning *and* we have not been told to ignore the mass of the pulley.
System $\rightarrow$ Closed system containing the cylinders, the cable, and the pulley.
Property to count $\rightarrow$ Angular momentum
Time interval $\rightarrow$ Start with rate equation

The free-body diagram is shown below, indicating the known forces on the system. Since this is a closed system there are no mass transfers of momentum. See the arrows on the diagram to indicate the positive directions for x_A, x_B, and ω.

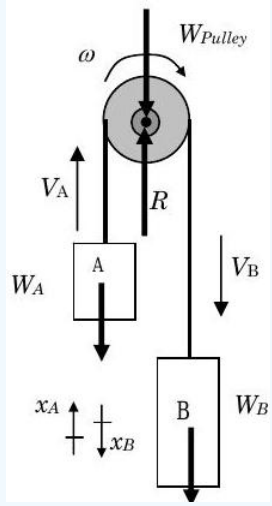


Figure 6.3.8: Free-body diagram of the system consisting of the pulley and its attached blocks.

Writing angular momentum about the pulley axis of rotation for the system and assuming clockwise rotation is positive, we have the following result:

$$\begin{aligned} & \frac{d\mathbf{L}_{0,\text{ sys}}}{dt} = (\mathbf{r}_A \times \mathbf{W}_A) + (\mathbf{r}_B \times \mathbf{W}_B) \\ \text{where} \quad & \mathbf{L}_{0,\text{ sys}} = m_A(\mathbf{r}_A \times \mathbf{V}_A) + m_B(\mathbf{r}_B \times \mathbf{V}_B) + \omega I_G \end{aligned}$$

$$\boxed{\text{CW} +} \quad \frac{dL_{0,\text{ sys}}}{dt} = \left(\frac{D}{2}W_B\right) + \left(-\frac{D}{2}W_A\right)$$

$$\text{where} \quad L_{0,\text{sys}} = \underbrace{\left(\frac{D}{2}m_A V_A\right)}_{\substack{\text{Angular momentum of} \\ \text{Cylinder A about axle with} \\ \text{positive velocity } V_A \text{ upward}}} + \underbrace{\left(\frac{D}{2}m_A V_A\right)}_{\substack{\text{Angular momentum of} \\ \text{Cylinder B about axle with} \\ \text{positive velocity } V_B \text{ downward}}} + \underbrace{\omega I_G}_{\substack{\text{Angular momentum of} \\ \text{the pulley assuming positive} \\ \omega \text{ is in clockwise direction}}}$$

Now substituting the system angular momentum into the momentum balance we have

$$\boxed{\text{CW} +} \quad \frac{d}{dt}\left[\left(\frac{D}{2}m_A V_A\right) + \left(\frac{D}{2}m_A V_A\right) + \omega I_G\right] = \left(\frac{D}{2}W_B\right) + \left(-\frac{D}{2}W_A\right)$$

This equation can be further simplified by recognizing that only the velocities are constants, so bringing the constants outside the derivative and dividing through by $D/2$ gives the following

$$\boxed{\text{CW} +} \quad m_A\frac{dV_A}{dt} + m_B\frac{dV_B}{dt} + \left(\frac{2}{D}\right) I_G \frac{d\omega}{dt} = -W_A + W_B$$

To go further, we must relate the translational and the rotational velocities. Assuming that a positive rotational direction is clockwise which corresponds with the velocity arrows on the diagram, we have the following:

$$V_A = V_B = \omega(D/2)$$

Using this result to replace V_B and ω in the angular momentum balance gives the following:

$$\begin{aligned} \boxed{\text{CW} +} \quad & m_A\frac{dV_A}{dt} + m_B\frac{dV_B}{dt} + \left(\frac{2}{D}\right) I_G \frac{d}{dt}\left(\frac{2}{D}V_A\right) = W_B - W_A \\ & m_A\frac{dV_A}{dt} + m_B\frac{dV_A}{dt} + \left(\frac{2}{D}\right)^2 I_G \frac{dV_A}{dt} = W_B - W_A \end{aligned}$$

To finish, we must write the weights and the mass moment of inertia of the pulley in terms of their mass as follows:

$$m_A \frac{dV_A}{dt} + m_B \frac{dV_A}{dt} + \left(\frac{2}{D}\right)^2 I_G \frac{dV_A}{dt} = W_B - W_A$$

$$m_A \frac{dV_A}{dt} + m_B \frac{dV_B}{dt} + \left(\frac{2}{D}\right)^2 \underbrace{\left[\frac{1}{2} m_{\text{Pulley}} \left(\frac{D}{2}\right)^2\right]}_{\substack{I_G \text{ for the pulley treated} \\ \text{as a disk}}} \frac{dV_A}{dt} = m_B\, g - m_A\, g$$

$$m_A \frac{dV_A}{dt} + m_B \frac{dV_A}{dt} + \left(\frac{m_{\text{Pulley}}}{2}\right) \left[\frac{dV_A}{dt}\right] = (m_B - m_A)\, g$$

$$\frac{dV_A}{dt} = \frac{(m_B - m_A)}{\left(m_A + m_B + \frac{m_{\text{Pulley}}}{2}\right)} g$$

Substituting in the numbers gives

$$\frac{dV_A}{dt} = \frac{(m_B - m_A)}{\left(m_A + m_B + \frac{m_{\text{Pulley}}}{2}\right)} g = \frac{(60 - 30)}{\left(30 + 60 + \frac{10}{2}\right)} g = \left(\frac{30}{95}\right) g = 3.098 \, \frac{\text{m}}{\text{s}^2}$$

Thus cylinder A accelerates up and cylinder B accelerates down at the rate of 3.098 m/s^2.

Comment:

Determine the *error* if we had neglected the mass of the pulley: i.e., what would the acceleration be if we had neglected the pulley mass?

6.4: Problems

Problem 6.1

The bracket BCD is hinged at C and attached to a control cable at B. For the loading shown, determine (a) the tension in the cable, (b) the reaction at C. [Remember to explicitly select a system, apply the basic equations, and state your simplifying assumptions.]

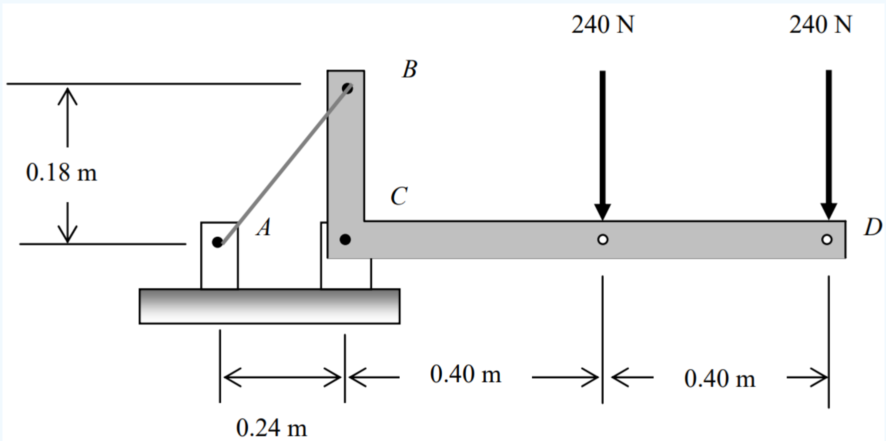


Figure 6.4.1: A bracket is loaded with two point forces.

Problem 6.2

(Adapted from Beer & Johnston, *Dynamics*, 6th ed., McGraw-Hill)

Grain falls from a hopper onto a chute CB at the rate of 240 $\mathrm{lbm/s}$. It hits the chute at A with a velocity of 20 $\mathrm{ft/s}$ and leaves at B with a velocity of 15 $\mathrm{ft/s}$, forming an angle of 10° with the horizontal. Knowing that the combined weight of the chute and of the grain it supports is 600 lbf and acts at G, determine the reaction of the roller support B and the components of the reaction at the hinge C.

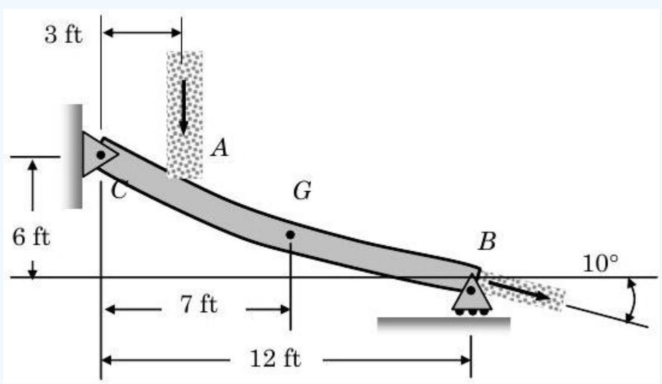


Figure 6.4.2: Side view of grain falling onto a curved chute and exiting in an angled stream.

[Be sure to sketch your system carefully so that you can see what is happening. Do not assume that you know any dimensions other than the ones given to you in the problem statement and figure. Also note that you can calculate the angular momentum about any point inside or on the boundary of the system. It is usually best to pick the point that minimizes the calculations or where the transports of angular momentum are easiest to see.]

Problem 6.3

(Adapted from Pestel & Thomson, *Statics*, McGraw-Hill)

For the clamping device shown in the figure, determine the forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

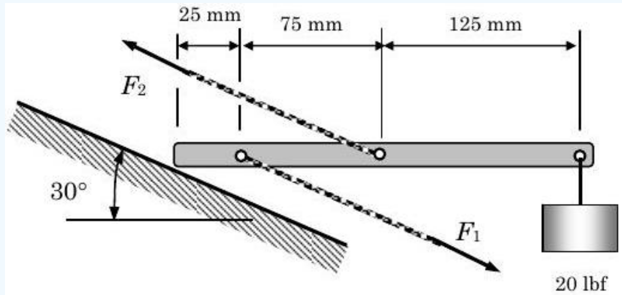


Figure 6.4.3: A clamping device experiences two unknown tension forces.

Problem 6.4

(Adapted from Pestel & Thomson, *Statics*, McGraw-Hill)

A monorail car with the dimensions shown in the figure is driven by only the front wheel. If the coefficient of static friction between the wheel and the rail is 0.60, determine the maximum acceleration possible for the car.

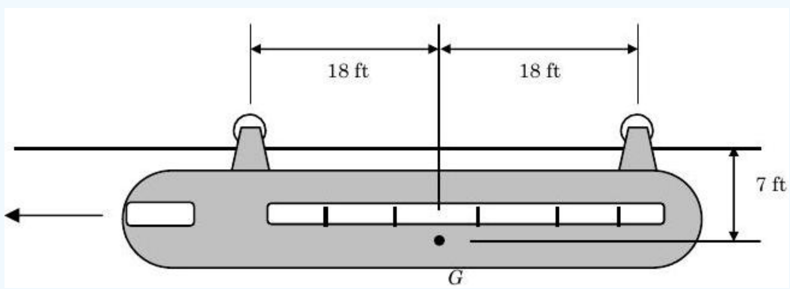


Figure 6.4.4: A monorail car hangs below a track, supported by its wheels.

Answer

0.340 g

Problem 6.5

The loaded trailer shown in the figure has a mass of 900 kg with a center of mass at G and is attached at A to a rear-bumper hitch. Pertinent dimensions are given on the diagram.

(a) Determine the vertical component of the hitch force acting on the trailer at A when the trailer is stationary. Give both the magnitude, in newtons, and the direction of the force.

(b) If the car accelerates to the right at the rate of $4.5 \mathrm{~m/s^2}$, determine the vertical component of the hitch force acting on the trailer at A. Neglect the small friction force exerted on the relatively light wheels. Give both the magnitude, in newtons, and the direction of the force.

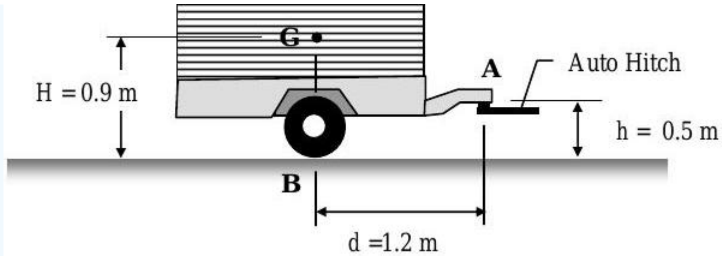


Figure 6.4.5: Side view of a loaded trailer.

Problem 6.6

You have been challenged by your buddies to balance a broomstick in your hand. So that you don't embarrass yourself, you've decided to do some analysis before you put on a show.

As a first approximation, you assume that the broomstick can only move in the plane of the paper. The broom has a mass $m = 2.0\ \mathrm{lbm}$ and an overall length $L = 5\ \mathrm{ft}$. The center of mass of the broom, including the bristles, is located a distance $0.6L$ from the end of the broom handle

If you start to balance the broom when it leans an angle $\theta = 30^\circ$ from the vertical, your challenge is to move your hand horizontally so that the broom maintains this orientation, i.e. it undergoes linear translation. Since you cannot grab the broom, assume that your hand can only resist forces in the x - and y-directions.

For these conditions, determine (a) the magnitude and direction of the reaction of your hand on the broom, and (b) the direction and magnitude of the horizontal acceleration of the broom.

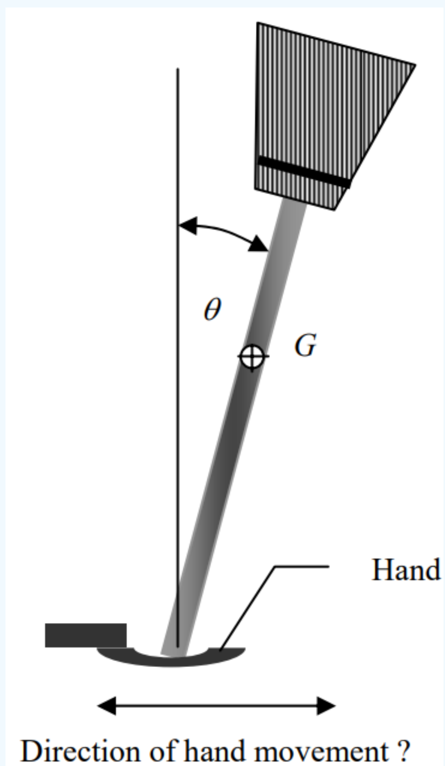


Figure 6.4.6: A broom is balanced on a hand via the end of its broomstick.

Problem 6.7

A Pelton wheel turbine is used extract energy from a stream of flowing water. When it is operating at steady-state conditions, the water stream enters the rotating turbine wheel as shown with velocity V_1 and leaves the turbine with velocity V_2 at an angle θ. Under these conditions, the turbine wheel rotates about the axis through point O. The black dot at O represents the turbine shaft. For steady-state operation, two reaction forces R_x and R_y and a torque (or moment) M_O must be applied to the shaft at point O.

If the mass flow rate is $50\ \mathrm{kg/s}$ and $V_1 = V_2 = 30\ \mathrm{m/s}$, determine the two reaction forces and torque at O if $\theta =$ (a) 0°, (b) 30°, (c) 60°, (d) 90°, (d) 120°, (e) 150°, and (f) 180°.

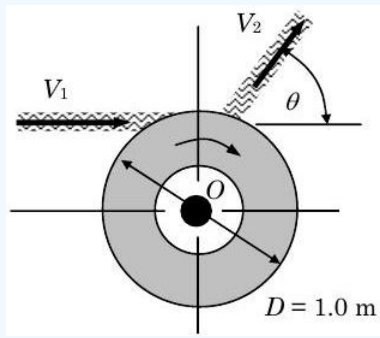


Figure $6.4.8$: Water runs along a section of a wheel turbine, turning it clockwise.

Problem 6.8

(Adapted from Bedford & Wallace, *Dynamics*, 2nd ed., Addison-Wesley)

The slender bar weighs $10\ \mathrm{lbf}$ and the disk weighs $20\ \mathrm{lbf}$. The coefficient of kinetic friction between the disk and the horizontal surface is 0.1. If the disk has an initial counterclockwise angular velocity of $10\ \mathrm{rad/s}$, how long does it take for the disk to stop spinning.

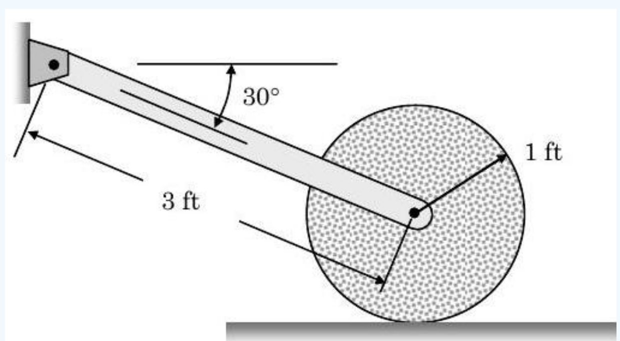


Figure $6.4.9$: An angled bar has one end pinned to a wall and the other pinned to a disk resting on the ground.

Problem 6.9

Water flows out of a fire hydrant with a velocity of $50\ \mathrm{ft/s}$ and a volumetric flow rate of $1000\ \mathrm{gpm}$. The water pressure at the inlet to the hydrant is $200\ \mathrm{psia}$ and the atmospheric pressure is $14.7\ \mathrm{psia}$. At the base of the hydrant the bolts must resist a normal force holding the hydrant down, a shear force parallel to the ground, and a couple trying to rotate the hydrant off its base. Calculate these reactions assuming steady-state conditions. Assume the density of water is $62.4\ \mathrm{lbm/ft^3}$.

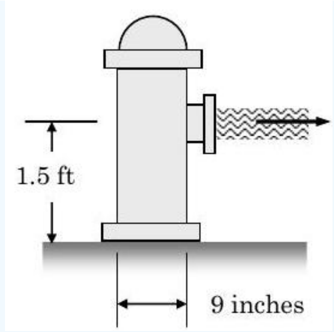


Figure 6.4.9: Side view of a fire hydrant with water exiting its nozzle in a horizontal stream.

? Problem 6.10

The frame shown supports part of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end E (forces and moment).

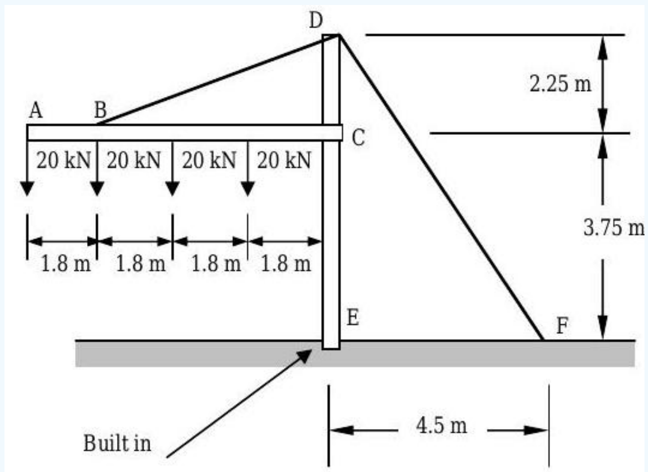


Figure 6.4.10: A frame consisting of two beams and two support cables is loaded with several point loads.

? Problem 6.11

A high-speed jet of air issues from the nozzle A, which has a diameter of 40 mm, with a velocity of 240 m/s and mass flow rate of 0.36 kg/s and impinges on the vane OB, shown on its edge view. The vane and its right angle extension have negligible mass compared to the attached $6-$kg cylinder, and are freely pivoted about a horizontal axis through O. The air density under the prevailing condition is 1.206 kg/m^3.

Determine:

(a) the steady state angle θ assumed by the vane with the horizontal, and

(b) the reaction forces at O.

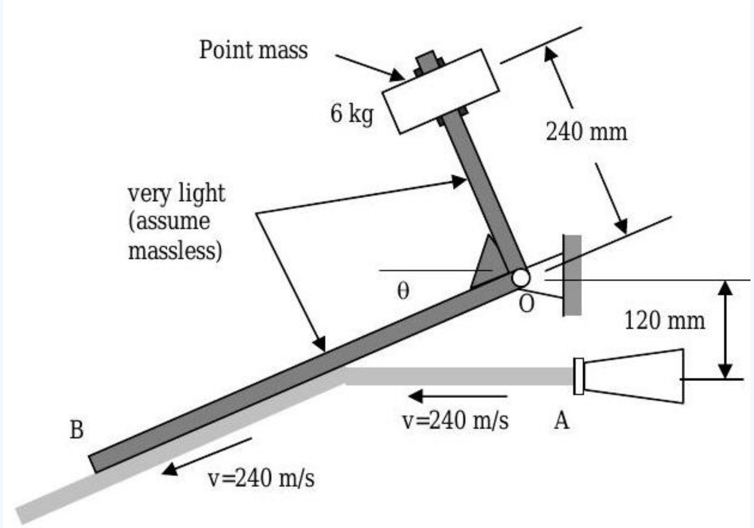


Figure 6.4.11: A jet of air travels horizontally until it impacts a slanted surface, moving at constant speed throughout.

Problem 6.12

A ball with mass $m = 5 \text{ lbm}$ is mounted on a horizontal rod that is free to rotate about a vertical shaft as shown in the figure. In the position shown (position A), the rod rotates and the ball is held by a cord attached to the shaft. In this state, the speed of the ball is $V_1 = 24 \text{ in/s}$. The cord is suddenly cut and the ball moves to the position B as the rod continues to rotate. Neglecting the mass of the rod, determine the speed of the ball after it has reached the stop B. Be careful to show all of your work.

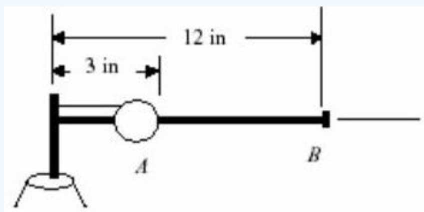


Figure 6.4.12: Two instantaneous positions of a ball in relation to a rotating rod on a shaft.

Problem 6.13

(From *Dynamics* by Beer and Johnson)

Coal is being discharged from a horizontal conveyor belt at the rate of 120 kg/s. It is received at A by a second belt which discharges it again at B. Knowing that $v_1 = 3 \text{ m/s}$ and $\text{v}_2 = 4.25 \text{ m/s}$, and that the second belt assembly and the coal it supports have a total mass of 472 kg, determine horizontal and vertical components of the reactions at C and D.

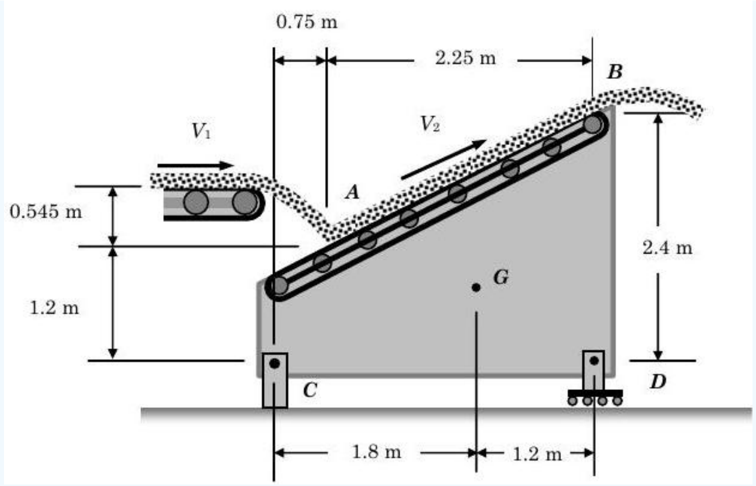


Figure 6.4.13: Coal falls off a horizontal conveyor belt onto a diagonal one where it is again discharged.

? Problem 6.14

A conveyor system is fitted with vertical panels, and a 300 mm rod AB of mass 2.5 kg is lodged between the panel as shown. Assume all the surfaces are smooth. Knowing the acceleration of the panel and the rod is $1.5\ \mathrm{m/s^2}$ to the left, determine the reactions of the carrier on the rod at C and B.

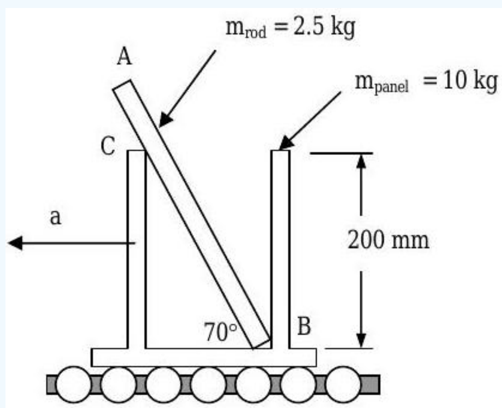


Figure 6.4.14: A rod rests against two vertical panels traveling along a conveyor belt.

? Problem 6.15

(Modified from *Dynamics* by Beer and Johnson)

The forklift shown weighs 2250 lbf and is used to lift a crate of weight $W = 2500$ lbf. The coefficient of static friction between the crate and the fork lift is 0.3.

Determine:

(a) the maximum deceleration the forklift can have for the crate not to slip, and

(b) the maximum deceleration the forklift can have for the forklift not to tip.

(c) If the truck is moving to the left at a speed of 10 ft/s when the brakes are applied, determine the smallest distance in which the truck can be brought to a stop if the crate is not to slide and if the truck is not to tip forward.

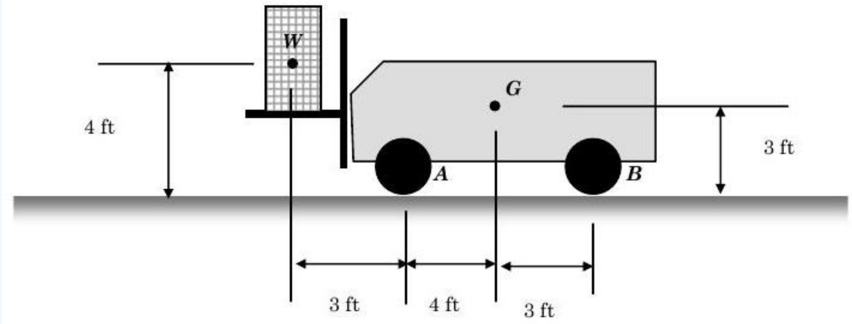


Figure 6.4.15: Side view of a two-axle forklift lifting a crate.

❓ Problem 6.16

Part (a) Three forces are applied to the L-shaped plate shown in the figure. All forces are applied in the plane of the paper.

(i) Determine the moment of each force about Point O and the *sum* of the moments about Point O, in lbf-ft.

(ii) Determine the moment of each force about Point P and the *sum* of the moments about Point P, in lbf-ft.

(iii) Do any of the forces applied to the plate form a couple? If the answer is yes, which ones?

[Note: Remember to indicate both the direction (label CW or CCW, or use arrows) and magnitude of all vector quantities.]

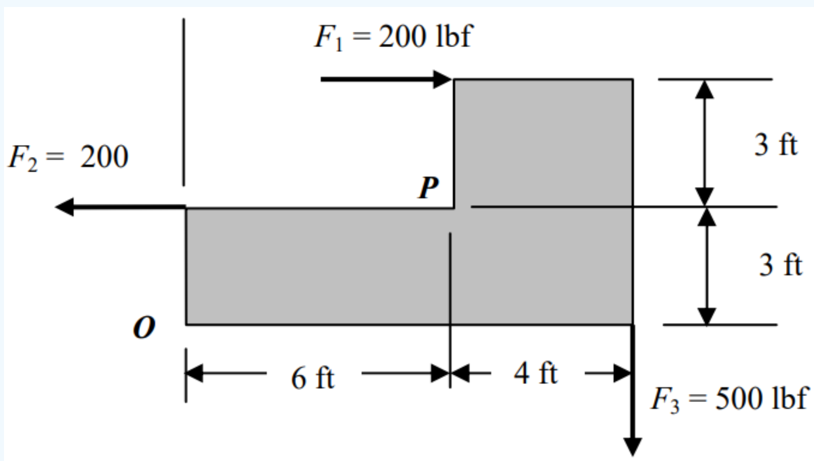


Figure 6.4.16a: Point forces are applied at different locations on an L-shaped plate.

Part (b) Two forces act on the planar, rigid body shown in the figure.

Determine the individual moment of each force about the point O and the *sum* of the moments about point O in $\mathrm{N-m}$. Remember to indicate the direction and magnitude of all vector quantities.

Figure 6.4.16b: Point forces are applied at different locations on an irregularly shaped plate.

❓ Problem 6.17

Consider the pulley-mass system shown in the figure. The diameters of the large and small pulley are $D = 0.5\ \mathrm{m}$ and $d = 0.25\ \mathrm{m}$, respectively. Both pulleys turn together around the same axle, point P.

(a) If the pulleys are locked by a brake and cannot turn, the weight of each block produces a moment about point P. Determine the net moment about point P due to the stationary blocks, in N-m.

(b) If the pulleys turn together at the rate of 2.0 radians per second in the direction shown, i.e. $\omega = 2.0\mathrm{rad/s}$ in the direction shown, each of the blocks has angular momentum and linear momentum. Determine the following for each block:

- the velocity, in $\mathrm{m/s}$,

- the linear momentum, in $\mathrm{kg}-\mathrm{m}/\mathrm{s}$, and
- the angular momentum of each mass with respect to axle point P, in $\mathrm{kg}-\mathrm{m}^2/\mathrm{s}$.

Remember to indicate both the magnitude and direction of all vector quantities.

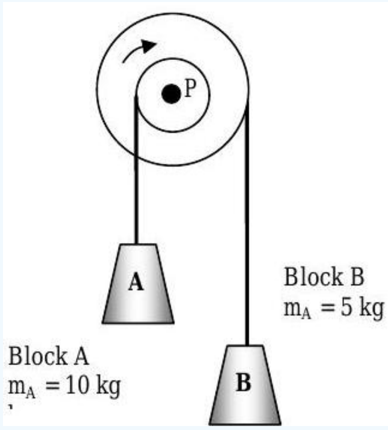


Figure $6.4.17$: Two weights are suspended from two pulleys through which a single axle passes.

? Problem 6.18

A horizontal force $\mathbf{P}$ acts on a cabinet that rests on a floor as shown. The cabinet weighs $120\,\mathrm{lbf}$. It is known that the coefficient of static friction is $\mu_s = 0.30$ and the coefficient of kinetic friction is $\mu_k = 0.24$.

(a) If slipping impends, what is the magnitude of $\mathbf{P}$?

(b) If tipping impends,

(i) what is the magnitude of $\mathbf{P}$, and

(ii) at what point will the resultant floor reaction act?

(c) What is the *smallest* magnitude of $\mathbf{P}$ that will cause the cabinet to move, i.e. either tip or slip?

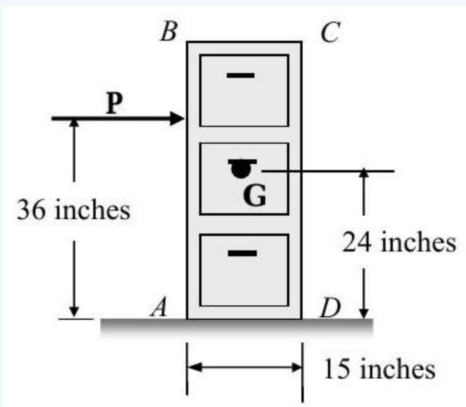


Figure $6.4.18$: A horizontal force is applied to a tall cabinet resting on the floor.

? Problem 6.19

Water flows steadily through the elbow-nozzle assembly shown in the figure. The assembly is in the vertical plane (the plane of the paper) and gravity acts as indicated. The assembly is supported entirely by the flange bolts which must resist forces in the x-and y-directions as well as a moment. For purposes of analysis, you may assume that all flange reactions are concentrated at the dark "dot" at the centerline of the flange. The available information about the geometry and operating conditions of the assembly are shown in the figure.

Determine the forces and moment (the reactions) at the flange to support the elbow-nozzle assembly.

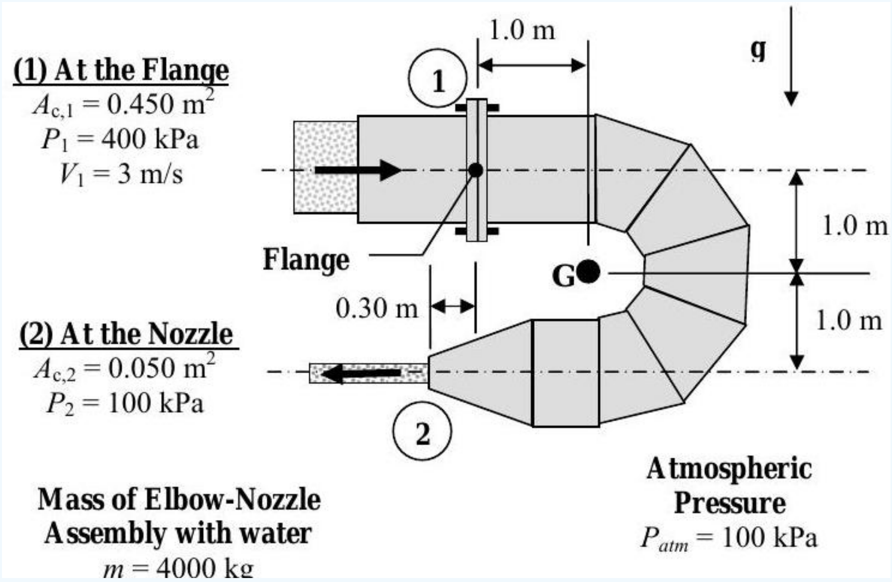


Figure $6.4.19$: Water flows from a horizontal pipe into an elbow-nozzle assembly, connected to the pipe by a flange.

Problem \(6.20)

A cyclist is traveling on a level road at a speed $V = 15\ \mathrm{m/s}$. The combined mass of the person and bicycle shown in the figure is $m = 77\ \mathrm{kg}$. The location of the combined center of mass relative to the wheels is also shown.

Suddenly the cyclist uses the handlebar brakes to stop the bike. If she only applies the front brakes, determine

(a) the maximum horizontal force of the ground (the braking force) on the bike at Point A that can be applied without the bike flipping (neglect any horizontal force exerted by the ground on the rear wheel), and

(b) the corresponding deceleration of the cyclist and bike measured in g's, e.g. $8.3g$, and

(c) the maximum allowable value of the *kinetic friction coefficient* between the front wheel and the ground if the front wheel locks under these conditions and slides on the ground.

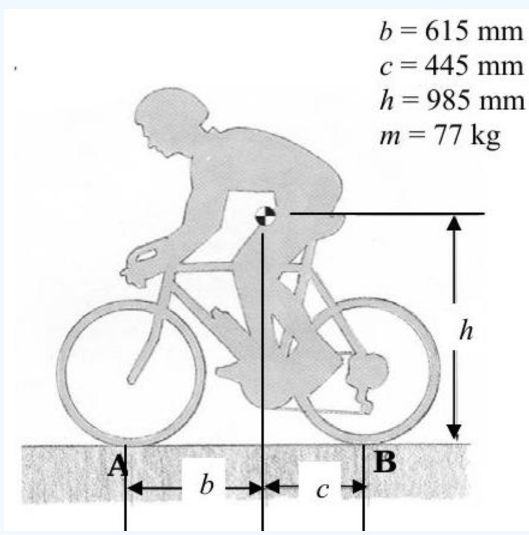


Figure $6.4.20$: System consisting of a cyclist and bike, traveling at a constant velocity.

Problem 6.21

As part of a school bus safety test program, school buses are being tested for potential roll-over hazards. To test the bus, it is placed on a 1000-pound moveable concrete pad that rolls freely without friction. The horizontal motion of the pad is produced by a hydraulic ram which pulls the pad to left. A 5000-pound school bus is placed on the pad as shown in the figure

(a) Assuming the bus does not slip on the pad, determine the minimum value of the horizontal acceleration (dV/dt) of the pad in the direction indicated that will cause the bus to tip, in ft/s^2.

(b) Determine the force, in lbf, that the ram must apply to the platform to produce the acceleration found in part (a).

(c) Determine the minimum static coefficient of friction between the tires and the concrete pad that is required to keep the bus from slipping on the moving pad.

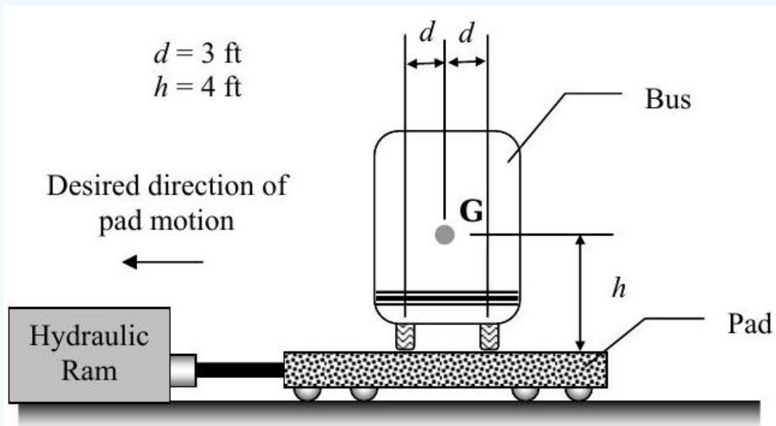


Figure $6.4.21$: End-on view of a school bus, placed on a rolling pad that is pulled to the left.

Problem 6.22

A manufacturer of handheld showerheads uses the setup shown in the figure to test the "handling" characteristics of their showerheads. For the tests, the showerhead is hung from a vertical water-supply pipe using a pinned-joint connection that can only resist horizontal and vertical forces.

Water enters the showerhead at B with a purely vertical velocity of $1.00\ \text{ft/s}$ and a volumetric flow rate of $0.0050\ \text{ft}^3/\text{s}$ The water pressure in the water supply line is $35\ \text{psia}$.

At the spray outlet, the water velocity is $25\ \text{ft/s}$ and the pressure is atmospheric ($P_{\text{atm}} = 14.7\ \text{psia}$).

The test showerhead weighs $1.3\ \text{lbf}$. The density of water can be assumed to be $62.4\ \text{lbm/ft}^3$

For these steady-state test conditions, determine

(a) the angle θ the showerhead makes with the vertical, and

(b) the horizontal reaction force at the pinned joint.

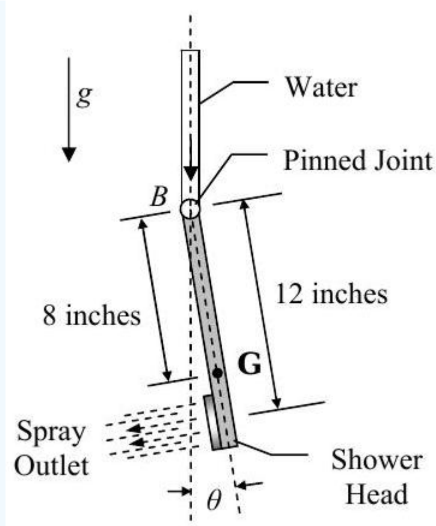


Figure $6.4.22$: Water moves through a system consisting of a shower head pinned to a vertical pipe.

? Problem 6.23

A steel pipe BC, of length L_p with a mass m_p, is attached to the rear bumper of a truck using a lightweight rope AB of length L_r. The coefficient of kinetic friction at point C is μ_k. The angles that the rope and the steel pipe make with the horizontal are constant and shown on the figure.

Determine the constant acceleration of the truck a_{truck} and the tension in the rope T required to maintain these conditions.

SET UP BUT DO NOT SOLVE. Clearly identify your unknowns and the set of equations you would use to solve for the unknowns.

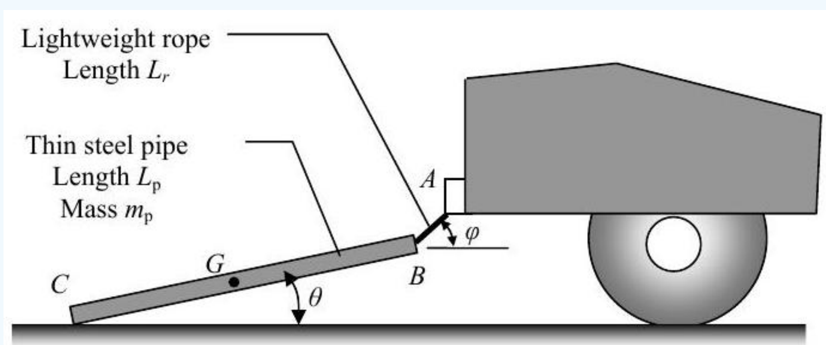


Figure $6.4.23$: A pipe with one end resting on the ground is connected to a truck bumper by a taut rope.

? Problem 6.24

The air-handling unit (AHU) shown in the figure is attached to a fixed support that ultimately rests on the roof. The fixed support corresponds with the center of mass G of the AHU. The AHU weighs 500 lbf.

The expected steady-state operating conditions are shown on the figure. Note that the pressure around the AHU is everywhere atmospheric, $P_{\text{atm}} = 14.7$ psia, except at the air inlet (state 1) where $P_1 = 14.6$ psia. You may assume that the air density is constant and uniform at $\rho_{\text{air}} = 0.075 \text{ lbm/ft}^3$.

Determine the reactions at point G required to support the AHU.

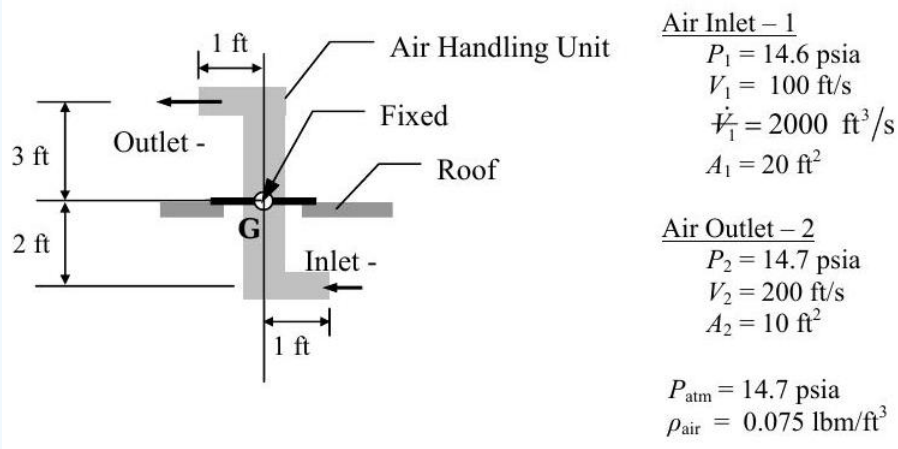


Figure 6.4.24: An air handling unit with one inlet and one outlet is suspended via a fixed support through its center of mass.

? Problem 6.25

The dragster has a mass of 1200 kg and a center of mass at G. A braking parachute is attached at C and, when released, provides a horizontal braking force of $F = k_O V^2$ where $k_O = 1.6\ \mathrm{N \cdot s^2/m^2}$ and V is the dragster velocity. If the parachute is deployed when the dragster is traveling too fast, there is a danger that the dragster will flip.

You may neglect the mass of the wheels and assume the engine is disengaged so that the wheels are freely rolling (so there is no horizontal force between the wheels and the ground) when the parachute is released.

a) Determine the critical speed (the maximum safe speed) the dragster can have such that the wheels at B are on the verge of leaving the ground when the parachute is released

b) If the dragster is traveling at the critical speed when the parachute is deployed, determine the distance it will travel before it stops. Does your answer make sense? If not, what do you think is the problem?

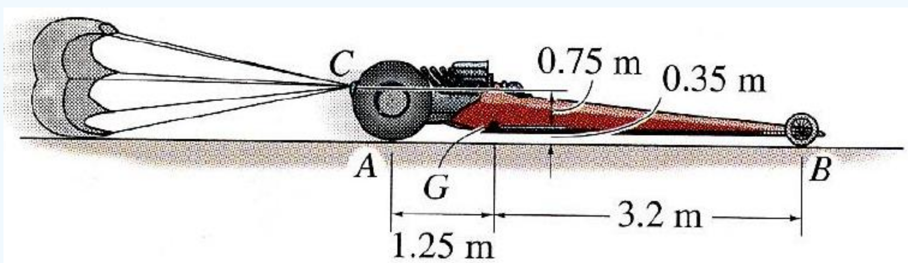


Figure 6.4.25: A dragster traveling to the right deploys a parachute from its rear.

? Problem 6.26

The girder shown in the figure weighs 4000 N and the motor weighs 1200 N. The motor is hoisting a load that weighs 8000 N. (Assume mass moment of inertia of the motor I_{motor} is negligible.)

Determine the reactions at A and B if the motor is *raising* the load and the load has an acceleration of $1.5\ \mathrm{m/s}$ upward.

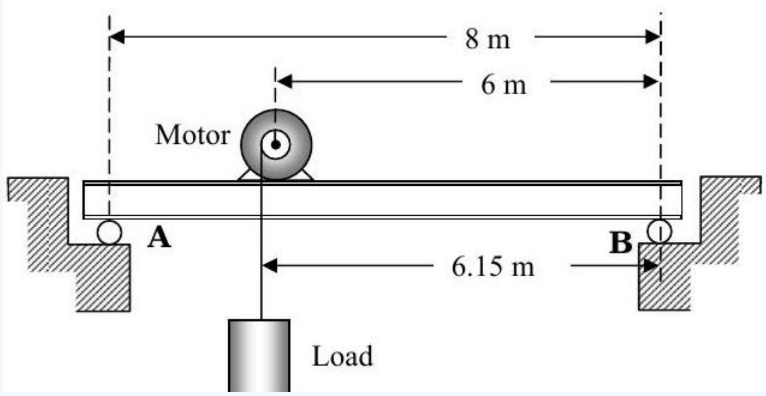


Figure 6.4.26: A motor resting on a girder turns to raise a load.

Problem 6.27

A jet of water with density ρ hits a hinged flap with a mass m as shown in the figure. The velocity of the water of both the incoming and outgoing jet is V_{jet}. The incoming water jet is circular with a diameter d. Known dimensions are given in the figure.

a) Find the angle θ that the stationary flap makes with the horizontal. Express your answer in terms of the known quantities.

b) Find the horizontal and vertical reaction forces at the pin connection A. You may assume that θ is known from part (a).

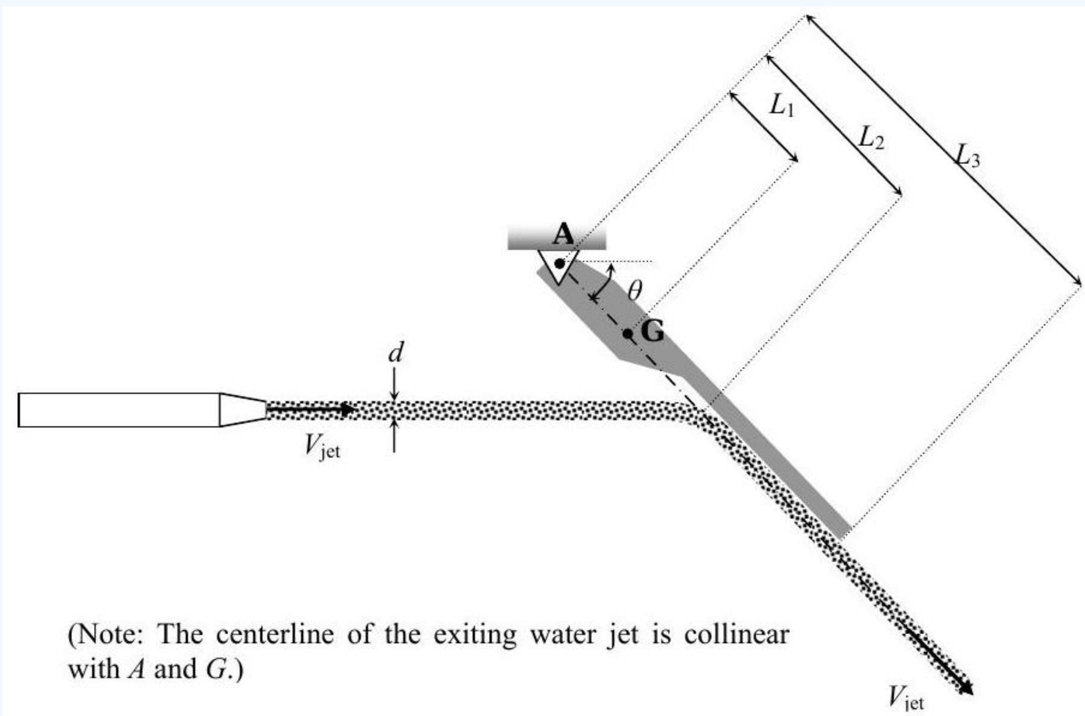


Figure 6.4.27: Water strikes a hinged, angled flap and runs down it.

Problem 6.28

Moo's Dairy has entered the annual Dairy Drag Race at the State Fair. His drag racer is a fully-loaded milk truck shown in the figure. When fully loaded, the milk truck weighs 5000 lbf. The truck is a *rear-wheel drive* vehicle, and the front tires provide negligible frictional drag when rolling.

The maximum traction between the tires and the road occurs when there is no slip between the tires and the road, i.e. the force between the road and the tires is due to static friction. The static coefficient of friction between the rubber tires and the concrete pavement is $\mu_s = 0.80$.

(a) Determine the reactions between the tires and the road at points A and B, in lbf, when the truck is stationary.

(b) Determine the maximum acceleration possible for the fully-loaded, rear-wheel drive milk truck, in $\mathrm{ft/s^2}$ *or* in g's. Also determine the corresponding reactions at points A and B, in lbf. Is there any danger of the truck tipping under these conditions?

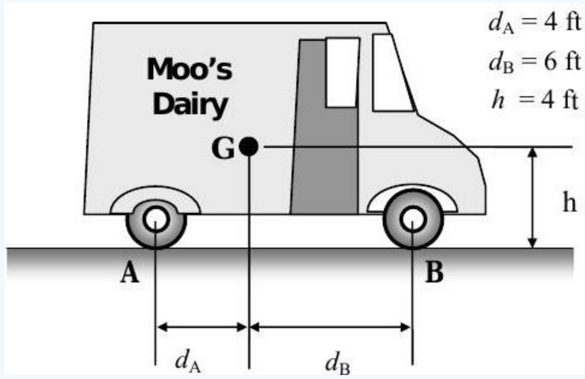


Figure 6.4.28: Side view of a milk truck on level ground.

CHAPTER OVERVIEW

7: Conservation of Energy

7.1: Mechanics and the Mechanical Energy Balance

Historically, the general conservation of energy principle was one of the last principles identified by the scientific community. However, restricted forms of the principle were derived earlier from other fundamental principles. One of the most useful of these is the mechanical energy balance. In this section, we will demonstrate how this balance can be developed from the conservation of linear momentum. Along the way, we will introduce several new concepts — such as mechanical work, mechanical power, and mechanical energy — that are rooted in the study of mechanics.

7.1.1 Mechanical Work and Mechanical Power

Early in our study of conservation of linear momentum we examined different ways to write the time rate of change of linear momentum for a particle:

$$\frac{d}{dt}(mV) = m\frac{dV}{dt} = m\left(\frac{dV}{dx}\right)\left(\frac{dx}{dt}\right) = m\left[V\frac{dV}{dx}\right]$$

The motivation for this exercise was the need to integrate the linear momentum equation when information was only given as a function of only position x. This same concern leads us to consider the integral of a force with position.

Mechanical Work

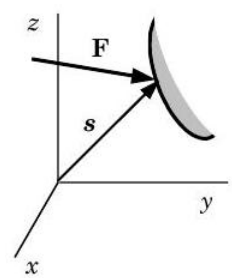


Figure $7.1.1$: Surface force acting on the boundary of a system.

The mechanical work done by a surface force $\mathbf{F}$ acting on the boundary of a system (see Figure $7.1.1$) is the integral

$$W_{\text{mech}} \equiv \int_{\mathbf{s}_1}^{\mathbf{s}_2} \mathbf{F}\cdot d\mathbf{s} = \int_1^2 \delta W_{\text{mech}} \quad \text{where } \delta W_{\text{mech}} \equiv \mathbf{F}\cdot d\mathbf{s}$$

where $d\mathbf{s}$ is the differential displacement vector of the point of application of the force on the boundary and $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively, are the initial and final positions of the boundary.

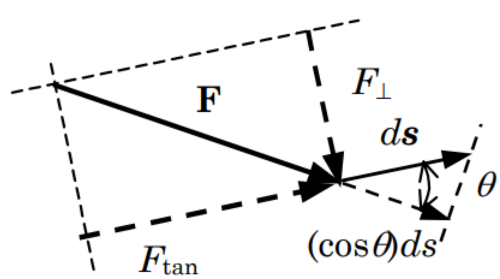


Figure $7.1.2$: Evaluating differential mechanical work, δW_{mech}.

To better understand this integral, it is useful to examine the differential amount of work δW_{mech}. To help, Figure $7.1.2$ shows both the force $\mathbf{F}$ and the displacement $d\mathbf{s}$. Note that force $\mathbf{F}$ can be decomposed into two components: $\mathbf{F}_\perp$ normal to $d\mathbf{s}$ and $\mathbf{F}_{\text{tan}}$ tangent to $d\mathbf{s}$. Using the definition of a dot product between two vectors, we have the following result for δW_{mech}:

$$\begin{aligned}\delta W_{\text{mech}} &= \mathbf{F}\cdot d\mathbf{s} = |\mathbf{F}||d\mathbf{s}|\cos\theta = F\cdot ds\cdot\cos\theta \\ &= F\cdot \underbrace{[ds\cdot\cos\theta]}_{\substack{\text{Component of displacement}\\ \text{that is parallel to the}\\ \text{force's line of action}}} = \underbrace{[F\cdot\cos\theta]}_{\substack{\mathbf{F}_{\tan}=\text{Component of the force}\\ \text{that is parallel to the}\\ \text{displacement's line of action}}}\end{aligned} \tag{7.1.1}$$

From Eq. $7.1.2$ above, we see that there are two different interpretations for the differential amount of work. δW_{mech} can interpreted as (1) the product of the force and the component of the displacement that is parallel to the force or (2) the product of the displacement and the component of the force that is parallel to the displacement.

Several additional points about mechanical work can be discovered by a close examination of the integral, Eq. $7.1.1$:

- Mechanical work is a *scalar* quantity; however, it is a signed quantity because the dot product $\mathbf{F}\cdot d\mathbf{s}$ can be positive or negative depending on the angle θ between the vectors (See Figure $7.1.2$)
- The dimensions of mechanical work are $[\text{Force}][\text{L}]$ or $[\text{M}][\text{L}]^2[\text{T}]^{-2}$
- The units for work are $\text{N}\cdot\text{m}$ in SI and $\text{ft}\cdot\text{lbf}$ in the USCS system. The fact that one set of units is force-length and the other is length-force is arbitrary and probably the result of what sounded good to the ear.
- The integral and hence mechanical work can only be evaluated if one knows the *end states* and the *path of the process* connecting these states. Mathematically, you must know $\mathbf{F}$ as a function of position $\mathbf{s}$ to evaluate the integral.
- Mechanical work for a system cannot be evaluated in a specific state or at a specific time. It can only be evaluated for a process — a change in state. Mechanical work is an example of a **path function** because its defining integral, Eq. $7.1.1$, can only be evaluated when the path of the process is known. This is in contrast to, say, the problem of finding the change in volume for a process. To find the change in volume of a system, we need only know the system volume at the each end state without regard to the path of the process. Thus, *volume, like all properties of a system, is a* **state (or point) function**.

Mechanical Power

The *time rate* at which surface force $\mathbf{F}$ does work on the system is called the **mechanical power** and is defined by the equation

$$\dot{W}_{\text{mech}} \equiv \mathbf{F}\cdot\mathbf{V}$$

where $\mathbf{V}$ is the velocity of the point of application of the force $\mathbf{F}$ on the boundary of the system.

By integrating the mechanical power with respect to time over a finite-time interval, we can demonstrate the relationship between mechanical power and mechanical work:

$$\int_{t_1}^{t_2}\dot{W}_{\text{mech}}\,dt = \begin{cases}\int_{t_1}^{t_2}(\mathbf{F}\cdot\mathbf{V})\,dt = \int_{t_1}^{t_2}\left(\mathbf{F}\cdot\frac{d\mathbf{s}}{dt}\right)\,dt = \int_{t_1}^{t_2}\underbrace{\mathbf{F}\cdot d\mathbf{s}}_{\delta W_{\text{mech}}} = W_{\text{mech}} \\ \int_1^2 \delta W_{\text{mech}} = W_{\text{mech}} \quad \text{where} \quad \delta W_{\text{mech}} = \dot{W}_{\text{mech}}\,dt\end{cases}$$

Note that the integral of δW_{mech} does not equal ΔW_{mech}.

Several additional points should be made about mechanical power and its relation to mechanical work:

- Mechanical power, unlike mechanical work, can be evaluated at a specific time because it is the instantaneous *rate* at which work is being done.
- The dimensions for mechanical power are $[\text{Force}][\text{L}]/[\text{T}]$ or $[\text{M}][\text{L}]^2[\text{T}]^{-3}$.
- The units for mechanical power are $\text{N}\cdot\text{m/s}$ in SI and $\text{ft}\cdot\text{lbf/s}$ or hp (horsepower) in USCS. ($1\text{hp} = 550\text{ft}\cdot\text{lbf/s}$, approximately).

Before leaving the discussion of mechanical work and mechanical power, please note that they have very precise definitions and can, given sufficient information, be evaluated from their defining equations. Now we investigate how our knowledge of mechanical power and work can help us solve some problems of mechanics using scalars instead of vectors.

7.1.2 The Work-Energy Principle

The work-energy principle is the direct link between the conservation of linear momentum and the mechanical energy balance. It is developed here first for a particle and then expanded to include a system of particles.

Work-Energy Principle for a Particle

Consider the motion of a particle of mass m in a gravitational field. (Recall that a particle is a system with its mass concentrated at a point that may only translate as it moves through space, i.e. its mass moment of inertia is zero.) As shown in Figure 7.1.3, the position vector $\mathbf{s}$ describes the position of the particle. The only external forces acting on the particle are the weight $\mathbf{W}$ and the net surface (or contact) forces $\mathbf{R}$. Note that the gravity vector acts in the direction of the negative z-axis.

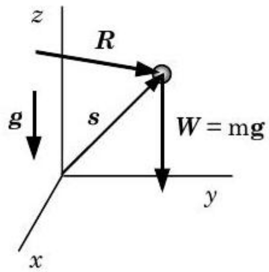


Figure 7.1.3: Motion of a particle in a gravitational field..

If we apply the conservation of linear momentum to the particle we have

$$\frac{d\mathbf{P}_{\text{sys}}}{dt} = \mathbf{R} + \mathbf{W} \quad \rightarrow \quad m\frac{d\mathbf{V}_G}{dt} = \mathbf{R} + m\mathbf{g}$$

where we have recognized that a particle is a closed system, its linear momentum is $\mathbf{P}_{\text{sys}} = m\mathbf{V}_G$, and its weight is $m\mathbf{g}$.

Now let us evaluate the mechanical power supplied to the particle by the contact force $\mathbf{R}$. Since the particle is a point, the force $\mathbf{R}$ moves with the velocity $\mathbf{V}_G$ and the mechanical work is

$$\dot{W}_{\text{mech}} = \mathbf{R}\cdot\mathbf{V}_G$$

To eliminate the surface force $\mathbf{R}$, we use the results of the conservation of linear momentum, Eq. 7.1.5, as follows:

$$\begin{aligned}\dot{W}_{\text{mech}} &= \mathbf{R}\cdot\mathbf{V}_G = \left[m\frac{d\mathbf{V}_G}{dt} - m\mathbf{g}\right]\cdot\mathbf{V}_G \\ &= \underbrace{m\frac{d\mathbf{V}_G}{dt}\cdot\mathbf{V}_G}_{\text{Inertia term}} - \underbrace{m\mathbf{g}\cdot\mathbf{V}_G}_{\text{Gravitation Term}}\end{aligned}$$

We have identified an "Inertia term" and a "Gravitation term" on the right hand side of Eq. 7.1.7. Before continuing, we need to investigate these two terms.

The inertia term in Eq. 7.1.7 is the dot product of the particle velocity with the rate of change of the particle linear momentum. Starting with the inertia term and performing the dot product, we have

$$\begin{aligned}\begin{bmatrix}\text{Inertia}\\ \text{term}\end{bmatrix} &= m\frac{d\mathbf{V}_G}{dt}\cdot\mathbf{V}_G = \frac{m}{2}\frac{d}{dt}(\mathbf{V}_G\cdot\mathbf{V}_G) \\ &= \frac{m}{2}\frac{d}{dt}\left(V_G{}^2\right) = \frac{d}{dt}\left(m\frac{V_G{}^2}{2}\right) = \frac{dE_K}{dt} \\ \text{where } E_K &\equiv m\frac{V_G{}^2}{2} = \text{ the kinetic energy of a particle}\end{aligned}$$

Thus the inertia term is just the ordinary derivative of a *scalar* quantity we typically refer to as the **kinetic energy** of the particle. Because the kinetic energy only depends on properties of the system and also depends on the mass of the system, *kinetic energy is an extensive property*.

Similarly, we can massage the gravitation term. To do this, we refer back to Figure 7.1.3 and first write the velocity and the weight term using the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then we perform the dot product as follows:

$$\begin{aligned}
\begin{bmatrix}\text{Gravitation}\\ \text{term}\end{bmatrix} &= m\mathbf{g}\cdot\mathbf{V}_G = m(-g\mathbf{k})\cdot\left(V_{G,\,x}\mathbf{i}+V_{G,\,y}\mathbf{j}+V_{G,\,z}\mathbf{k}\right)\\
&= -mg\left[V_{G,\,x}\underbrace{(\mathbf{k}\cdot\mathbf{i})}_{=0}+V_{G,\,y}\underbrace{(\mathbf{k}\cdot\mathbf{j})}_{=0}+V_{G,\,z}\underbrace{(\mathbf{k}\cdot\mathbf{k})}_{=1}\right]\\
&= -mgV_{G,\,z} = -mg\left(\frac{dz}{dt}\right) \quad \text{because } V_{G,\,z}=\frac{dz}{dt}\\
&= -\frac{d}{dt}(mgz) = -\frac{dE_{GP}}{dt}\\
\text{where } E_{GP} &\equiv mgz = \text{the gravitational potential energy of a particle.}
\end{aligned}$$

Thus the inertia term is proportional to the ordinary derivative of another scalar typically referred to as the **gravitational potential energy** of the particle. Frequently this is abbreviated to "potential energy." Based on an argument analogous to the one for kinetic energy, *gravitational potential energy is an extensive property*.

Using these new energy terms, we can rewrite the equation for the mechanical power supplied to the particle by the contact force $\mathbf{R}$, Eq. $7.1.7$, as follows:

$$\begin{aligned}
\dot{W}_{\text{mech}} = \mathbf{R}\cdot\mathbf{V}_G &= \underbrace{m\frac{d\mathbf{V}_G}{dt}\cdot\mathbf{V}_G}_{\text{Inertia term}} - \underbrace{m\mathbf{g}\cdot\mathbf{V}_G}_{\text{Gravitational term}}\\
&= \left[\frac{d}{dt}\left(m\frac{V_G^2}{2}\right)\right] - \left[-\frac{d}{dt}(mgz)\right]\\
\dot{W}_{\text{mech}} &= \frac{dE_K}{dt}+\frac{dE_{GP}}{dt}
\end{aligned} \tag{7.1.2}$$

This is the *rate form* of the **work-energy principle** for a particle. It is a very powerful and useful equation when applied correctly. In words this equation says,

> The mechanical power supplied by the *net* surface forces to a particle moving in a gravitational field equals the time rate of change of the kinetic energy and the gravitational potential energy of the particle.

Frequently, the finite-time form is useful. To obtain this form, we integrate both sides of Eq. $7.1.10$ over a finite-time interval as follows:

$$\begin{aligned}
\int_{t_1}^{t_2}\dot{\mathbf{W}}_{\text{mech}}\,dt &= \int_{t_1}^{t_2}\left[\frac{dE_K}{dt}+\frac{dE_{GP}}{dt}\right]dt\\
\int_{t_1}^{t_2}\delta W_{\text{mech}} &= \int_{1}^{2}dE_K+\int_{t_1}^{t_2}dE_{GP}\\
W_{\text{mech}} &= \underbrace{\left(E_{K,\,2}-E_{K,\,1}\right)}_{=\Delta E_K}+\underbrace{\left(E_{GP,\,2}-E_{GP,\,1}\right)}_{=\Delta E_{GP}}
\end{aligned} \tag{7.1.3}$$

This equation is the *finite-time form* of the **work-energy principle** for a particle. In words, this equation says

> The mechanical work done by the *net* surface forces on a particle moving in a gravitational field equals the change of the kinetic energy and gravitational potential energy of the particle.

Note how both of these equations involve only scalar terms, as compared to the conservation of linear momentum equation with which we started. Before we continue, let's recap the steps we used to derive the work-energy principle:

- We wrote the rate form of the conservation of linear momentum for a particle moving in a gravitational field and separated the external forces into two groups: the force of gravity (a body force) and the surface forces.
- We wrote the mechanical power supplied by the surface forces to the particle.
- We used the results of the conservation of linear momentum to replace the surface force term in the mechanical power expression.

- We evaluated the inertia and the gravitation terms (the dot product terms) separately and identified two new extensive properties — kinetic energy and gravitational potential energy.
- Finally, we rewrote the equation for mechanical power in terms of the time rate of change of the kinetic and the gravitational potential energy of the particle.

Note that this was in fact a derivation starting with conservation of linear momentum. Because of this, the principle of work-energy contributes nothing new to our analysis over what we learn from applying conservation of linear momentum. However, its scalar nature often makes it a useful alternative to the conservation of linear momentum, which by its very nature is a vector relationship.

Work-Energy Principle for a System of Particles

Because our derivation of the work-energy principle was only for a single particle, one might ask if it is applicable to systems composed of several particles. To investigate this question, consider the following example of two blocks sliding on a plane.

Example — Applying the work-energy principle to a two-particle system

Consider a system consisting of two blocks, Blocks A and B, stacked initially as shown in the figure. Block A rests on block B. The contact surface between A and B is rough, with kinetic friction coefficient μ_k and static friction coefficient μ_s. Block B rests on a smooth horizontal surface and slides freely on the surface with no friction. A force $\mathbf{F}$ is suddenly applied to block A and both blocks begin to move.

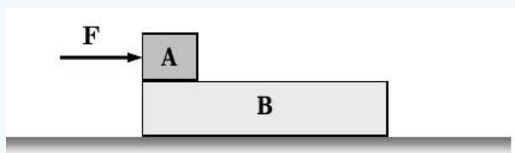


Figure 7.1.4: A system consisting of two blocks held together by friction.

To study the limitations of the work-energy principle, determine the rate of change of the kinetic energy of the blocks after force $\mathbf{F}$ is applied. Compare your results using two single-particle systems and using a combined system that includes *both* particles.

To make use of the work-energy principle, we must first clearly identify all of the *external* forces acting on the system. This is done below for the System AB, System A, and System B.

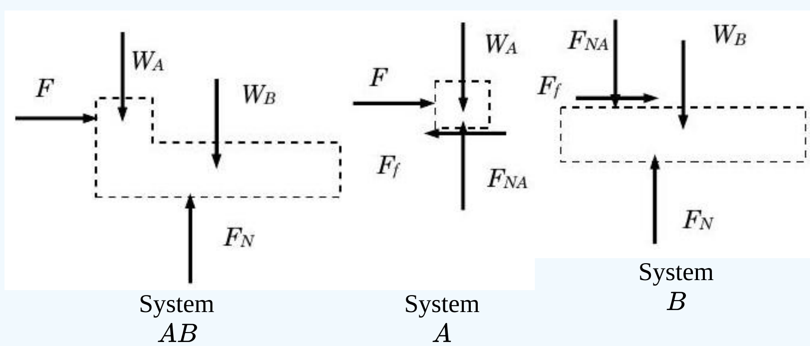


Figure 7.1.5: Free-body diagrams for systems consisting of blocks AB, of block A only, and of block B only.

For system A, the work-energy principle can be written as

$$\frac{d}{dt}\left(E_K+E_{GP}\right)_A=\dot{W}_{\text{mech}}$$

$$\frac{dE_{K,\,A}}{dt}+\underbrace{\cancel{\frac{dE_{GP,\,A}}{dt}}^{=0}}_{\text{No change in elevation}}=(\mathbf{F}\cdot\mathbf{V}_A)+\underbrace{\cancel{(\mathbf{F}_{NA}\cdot\mathbf{V}_A)}^{=0}}_{\substack{\text{Force and velocity}\\ \text{are perpendicular}}}+\underbrace{(\mathbf{F}_f\cdot\mathbf{V}_A)}_{\text{Friction}}$$

$$\frac{dE_{K,\,A}}{dt}=(F\cdot V_A)+\underbrace{(-F_f\cdot V_A)}_{\substack{\text{Friction force opposes}\\ \text{motion with velocity } V_A}}\quad\rightarrow\quad\boxed{\frac{d}{dt}\left(m_A\frac{V_A{}^2}{2}\right)=(F-F_f)\cdot V_A}\,*$$

Note how for system B, the work-energy principle can be written as

$$\frac{d}{dt}\left(E_K+E_{GP}\right)_B=\dot{W}_{\text{mech}}$$

$$\frac{dE_{K,\,B}}{dt}+\underbrace{\cancel{\frac{dE_{GP,\,B}}{dt}}^{=0}}_{\text{No change in elevation}}=\underbrace{\cancel{(\mathbf{F}_N\cdot\mathbf{V}_B)}^{=0}}_{\substack{\text{Force and velocity}\\ \text{are perpendicular}}}+\underbrace{\cancel{(\mathbf{F}_{NA}\cdot\mathbf{V}_B)}^{=0}}_{\substack{\text{Force and velocity}\\ \text{are perpendicular}}}+\underbrace{(\mathbf{F}_f\cdot\mathbf{V}_B)}_{\text{Friction}}$$

$$\frac{dE_{K,\,B}}{dt}=(F_f\cdot V_B)\quad\rightarrow\quad\boxed{\frac{d}{dt}\left(m_B\frac{V_B{}^2}{2}\right)=(F_f\cdot V_B)}\,**$$

where the velocity used in calculating the mechanical power must be the velocity of the block B based on our derivation of the work-energy principle for a particle.

Now adding these two equations together, we have

$$\left.\begin{array}{c}\dfrac{d}{dt}\left(m_A\dfrac{V_A{}^2}{2}\right)=(F-F_f)\cdot V_A\\ +\\ \dfrac{d}{dt}\left(m_B\dfrac{V_B{}^2}{2}\right)=F_f\cdot V_B\end{array}\right\}\rightarrow\quad\frac{d}{dt}\left(m_A\frac{V_A{}^2}{2}\right)+\frac{d}{dt}\left(m_B\frac{V_B{}^2}{2}\right)=(F-F_f)\cdot V_A+F_f\cdot V_B$$

$$\boxed{\underbrace{\frac{d}{dt}\left(m_A\frac{V_A{}^2}{2}\right)+\frac{d}{dt}\left(m_B\frac{V_B{}^2}{2}\right)}_{\substack{\text{Rate of change of the kinetic energy}\\ \text{of blocks } A \text{ and } B}}=F\cdot V_A-(F_f)\cdot(V_A-V_B)}\,***$$

For System AB, the *only* contact forces are $\mathbf{F}_N$, the normal force exerted by the ground on block B, and $\mathbf{F}$, the horizontal force applied to block A. *Assuming it is valid to write the work energy principle for two particles as a single system,* we write

$$\frac{d\left(E_K+E_{GP}\right)_{sys}}{dt}=\dot{W}_{\text{mech}}$$

$$\frac{d}{dt}\left[\left(E_K+E_{GP}\right)_A+\left(E_K+E_{GP}\right)_B\right]=(\mathbf{F}\cdot\mathbf{V}_A)+\underbrace{\cancel{(\mathbf{F}_B\cdot\mathbf{V}_B)}^{=0}}_{\substack{\text{Force normal}\\ \text{to velocity}}}$$

$$\frac{dE_{K,\,A}}{dt}+\cancel{\frac{dE_{GP,\,A}}{dt}}^{=0}+\frac{dE_{K,\,B}}{dt}+\cancel{\frac{dE_{GP,\,B}}{dt}}^{=0}=F\cdot V_A\quad\rightarrow\quad\boxed{\frac{d}{dt}\left(m_A\frac{V_A{}^2}{2}\right)+\frac{d}{dt}\left(m_B\frac{V_B{}^2}{2}\right)=F\cdot V_A}\,****$$

If the work-energy principle applies to systems with two particles, then Eq. $***$ and Eq. **** should give us the same information. Under what conditions are these two equations identical?

Condition 1: The friction force is zero.

Condition 2: There is no relative velocity between the particles, i.e. $V_A = V_B$. Thus it would appear that the work-energy principle accurately describes the behavior of a system with multiple particles only under a limited set of conditions.

Thus it would appear that the work-energy principle accurately describes the behavior of a system with multiple particles *only under a limited set of conditions*.

Fortunately, experiences like the one above allow us to establish *general guidelines for the use of the work-energy principle:*

> The work-energy principle can be applied to any *closed* system if (1) the system can be modeled as a collection of particles, i.e. moment of inertia of each subsystem is negligible, and (2) there is no friction inside the system.

As we will show shortly, these limitations are related to the fact that the work-energy principle is a restricted application of the general conservation of energy principle. Now we will turn to an application of the work-energy principle:

Example — Movin' boxes

Two boxes rest on individual inclined planes and are connected by a pulley as shown in the figure. The mass of each box is 12 kg and the boxes rest on frictionless surfaces. If the system is released from rest, determine the magnitude of the velocity of the weights when they have moved a distance $L = 1 \text{ m}$ along the planes.

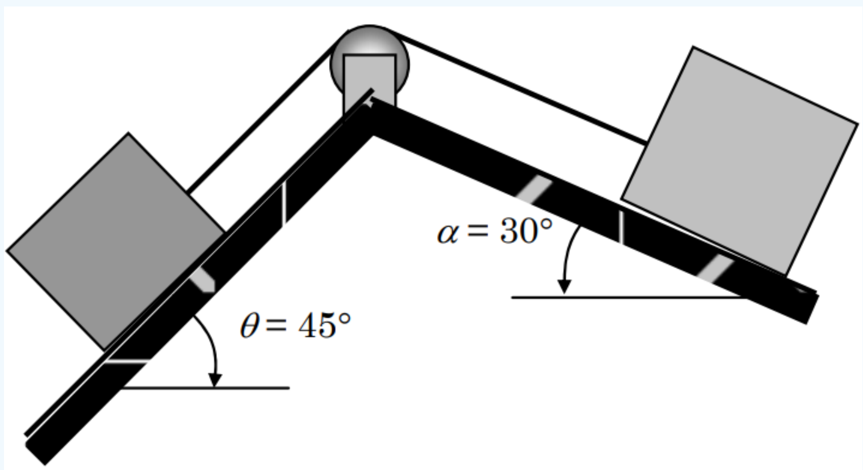


Figure $7.1.6$: A pulley at the intersection of two inclined planes supports a string connected to boxes on each plane.

<u>**Solution**</u>

Known: Two boxes connected by a pulley and cable slide on inclined surfaces.

Find: The velocity of the boxes when they have traveled a distance of 1 m along the planes.

Given: See the figure above.

Strategy $\rightarrow$ There are at least three possible ways to attempt this problem. One is to apply conservation of linear momentum. A second is to apply conservation of angular momentum. A third is to try the work-energy principle. (This is, in reality, just an alternate way to apply conservation of linear momentum.)

System $\rightarrow$ Both blocks, the pulley, the cable, and the pulley support.
Property to count $\rightarrow$ Lets try mechanical energy, i.e. work-energy principle.
Time interval $\rightarrow$ Start with the finite-time form since we have a finite displacement.

First we should sketch the free-body diagram and identify all of the external forces acting on our system. If you look carefully you will see the weights of the two boxes, the normal forces on the two boxes, two forces where we cut the pulley support, and the weight of the pulley and its support. (Look at the figure and find each of these.) You will also see the positive z-coordinate which points opposite to the direction of gravity.

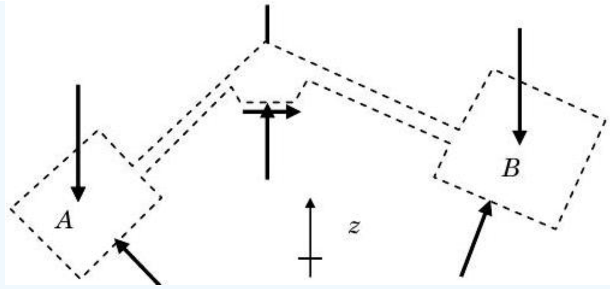


Figure 7.1.7: Free-body diagram of the system consisting of the boxes, their connecting cable, and the pulley the cable wraps around.

Now to apply the principle of work-energy to this system, we must model the closed system as a collection of particles with no internal friction.

$$\Delta\left(E_K+E_{GP}\right)_{sys}=W_{\text{mech}}$$

$$\Delta E_K+\Delta E_{GP}=\underbrace{\cancel{W_{\text{mech}}}^{=0}}_{\substack{\mathbf{F}\cdot d\mathbf{s}=0 \text{ for all} \\ \text{contact forces}}}$$

$$\Delta\left(m_A\frac{V_A{}^2}{2}+m_B\frac{V_B{}^2}{2}\right)+\Delta\left(m_A g z_A+m_B g z_B\right)=0$$

$$\underbrace{\left(m_A\frac{V_A{}^2}{2}+m_B\frac{V_B{}^2}{2}\right)_2}_{\substack{V_A=V_B=V \text{ since connected} \\ \text{by inextensible cable}}}+\left(m_A g z_A+m_B g z_B\right)_2=\underbrace{\cancel{\left(m_A\frac{V_A{}^2}{2}+m_B\frac{V_B{}^2}{2}\right)}^{=0}}_{\text{Initially stationary, no velocity}}+\left(m_A g z_A+m_B g z_B\right)_1$$

$$\left(m_A+m_B\right)\frac{V_2{}^2}{2}=m_A g\left(z_1-z_2\right)_A+m_B g\left(z_1-z_2\right)_B$$

In developing this equation we have also assumed that there is *no* friction inside the system and the rotation of the pulley is insignificant, i.e. it has a negligibly small mass moment of inertia about its axis of rotation.

Carefully notice that the force of gravity, or the weight of the masses, does not do any work on the system. Including weight as a force in the work term is a common mistake and results in a double counting of the effect of gravity. Ignoring the effect of gravity in the work term is a direct consequence of our definition of mechanical work in terms of *surface (contact) forces*. We can handle the effect of gravity in two different ways. If we include the force of gravity in the work term, there is no gravitational potential energy. If we handle the effect of gravity through the gravitational potential energy term, then only surface forces do work. To be consistent with the general conservation of energy principle, we have elected to use the latter approach.

To go further, we must indicate how the change in elevation is related to the displacement of the boxes along the planes. *Assume that box A moves up the plane and box B moves down the plane.*

Therefore

$$\left(z_2-z_1\right)_A=L\sin\theta \quad \text{and} \quad \left(z_2-z_1\right)_B=-L\sin\alpha$$

Substituting this back into the result from the work-energy principle, we have

$$\begin{aligned}(m_A+m_B)\frac{V_2{}^2}{2} &= m_A g(z_1-z_2)_A + m_B g(z_1-z_2)_B \\ &= -m_A g(z_2-z_1)_A - m_B g(z_2-z_1)_B \\ &= -m_A g(L\sin\theta) - m_B g(-L\sin\alpha) \quad \text{but} \quad m_A = m_B = m_{Box} \\ (2m_{box})\frac{V_2{}^2}{2} &= m_{box} gL(\sin\alpha - \sin\theta) \\ V^2 &= gL(\sin\alpha-\sin\theta) = \left(9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)(1\ \mathrm{m})\left(\sin 30^\circ - \sin 45^\circ\right) = -2.032\ \frac{\mathrm{m}^2}{\mathrm{s}^2}\end{aligned}$$

What happened? How can we take the square root of a negative number? Does this mean we have a *complex* velocity? Be very careful when you find yourself faced with taking the square root of a negative number. It almost always means that something is in error, as a *complex* velocity has no physical meaning in this problem.

What happened is that we guessed wrong about the original direction of motion! If we had assumed that *box A moved down the plane and box B moved up the hill* our equations for Δz in terms of L would have had opposite signs and

$$\begin{aligned}(m_A+m_B)\frac{V_2{}^2}{2} &= -m_A g(z_2-z_1)_A - m_B g(z_2-z_1)_B \\ &= -m_A g(-L\sin\theta) - m_B g(L\sin\alpha) \quad \text{but} \quad m_A = m_B = m_{Box} \\ (2m_{box})\frac{V_2{}^2}{2} &= m_{box} gL(-\sin\alpha + \sin\theta) \\ V^2 &= gL(\sin\theta-\sin\alpha) = \left(9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)(1\ \mathrm{m})\left(\sin 45^\circ - \sin 30^\circ\right) = 2.032\ \frac{\mathrm{m}^2}{\mathrm{s}^2} \\ V &= 1.43\ \frac{\mathrm{m}}{\mathrm{s}} \text{ where the elevation of } B \text{ increases and the elevation of } A \text{ decreases.}\end{aligned}$$

Form an energy standpoint what this means is that the increased kinetic energy comes about from a decrease in the total gravitational potential energy of the system. The decrease in gravitational potential energy of box A equals the increase in gravitational potential energy of box B plus the increase in kinetic energy of the entire system.

Comment:

(a) To check this answer, try using one of the other methods.

(b) How would your answer change if the kinetic friction coefficient for surface B was 0.1? [Ans: $V = 1.09\ \mathrm{m/s}$]

7.1.3 The Mechanical Energy Balance

It is useful at this point to apply what we know about the accounting concept to better understand the work-energy principle. If we start with the rate form, Eq. $7.1.10$ and rearrange it so that the derivatives are on the left-hand side, we have the following equation:

$$\frac{d}{dt}(E_K + E_G) = \dot{W}_{\text{mech}}$$

Because the kinetic energy and gravitational energy are both extensive properties, we may increase our understanding of Eq. $7.1.12$ by examining it in light of the accounting framework.

For this equation, the extensive property is the mechanical energy of the system. Because of their roots in mechanics, kinetic energy and gravitational potential energy are collectively referred to as mechanical energy. If the left-hand side of Eq. $7.1.12$ represents the rate of change of an extensive property of the system, then the right-hand side must represent either a transport rate or generation rate. Because mechanical power is defined in terms of a force and velocity on the boundary, we will refer to mechanical power as a transport rate of mechanical energy.

In terms of mechanical energy, the rate form of the work-energy principle, Eq. $7.1.12$, can be interpreted as follows:

> The time rate of change (or accumulation) of mechanical energy of a system equals the net transport rate of mechanical energy into the system by mechanical work.

Unfortunately, there is no general physical principle that always satisfies this statement. However, if we restrict ourselves to mechanical energy and only consider those cases where there is no destruction or generation of mechanical energy, then Eq. $7.1.12$ is the **mechanical energy balance** *with no generation or destruction of mechanical energy*.

A more general analysis would demonstrate that under most conditions, mechanical energy can only be *destroyed* within a system. Think about what happens when you drop a golf ball on the ground and let it bounce until it stops moving. Now assume you did this experiment in a vacuum to remove air friction. Initially, the golf ball has gravitational potential energy and no kinetic energy. When it finishes bouncing and rests on the ground it has less gravitational potential energy and no kinetic energy. Where did the initial mechanical energy go? We'll see later that it is irreversibly converted into internal energy — the energy of the motion of the atoms and molecules of the system.

Sometimes Eq. $7.1.12$ is called the "conservation of mechanical energy" for a closed system. Please be careful if you want to think in these terms. In this course, we typically reserve the word "conservation" for a general usage — describing how the world always works. As we will demonstrate later using the general conservation of energy principle, Eq. $7.1.12$ is restricted to an *adiabatic, closed system* where *mechanical work is the only work mechanism to transport energy*, and *energy can only be stored as mechanical energy*. See how many qualifiers have to be added just to get to "conservation of mechanical energy" from the general principle of conservation of energy.

7.1.4 An Additional Mechanical Energy — Spring Energy

If you think back to your physics class, you studied *ideal* springs. For our purposes an ideal spring is one that follows the same force-displacement curve regardless of the direction of displacement of the end of the spring, i.e. a spring with no hysteresis. One of the first things you learned is that the force required to compress or extend a spring is described by the equation:

$$F_{\text{spring}} = kx$$

where k is the spring constant with units $\mathrm{N/m}$ or $\mathrm{lbf/in}$ or $\mathrm{lbf/ft}$ and x is the displacement of the end of the spring from its unloaded or *free length* position.

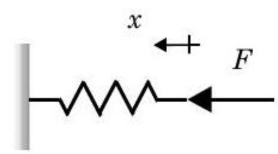


Figure $7.1.8$: An ideal spring.

Now consider a block of mass m on a horizontal, frictionless surface as shown in Figure $7.1.9$. Initially the block is held against the spring, and the spring is compressed to a length L. The free length of the spring is L_o. Suddenly the block is released and the block moves to the right. How could we use the mechanical energy balance to solve for the velocity of the block?

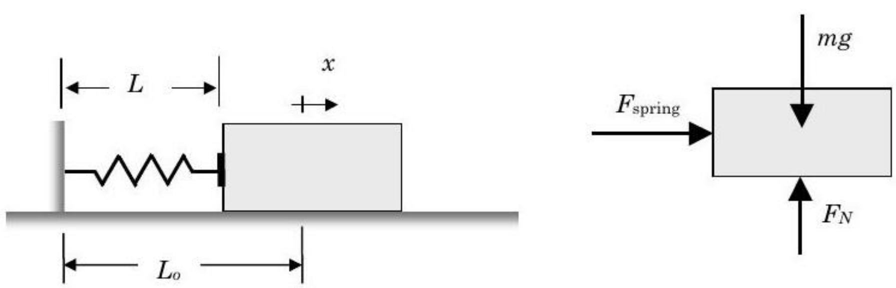


Figure $7.1.9$: Forces acting on a block due to an expanding spring.

First we select the block as our system and draw a free body diagram showing all of the external forces acting on our system (see Figure $7.1.9$). Pay particular attention to the location of the x-coordinate system. Next we will assume that the block can be modeled as a particle and we can apply the finite-time form of the mechanical energy balance:

$$
\begin{aligned}
\Delta E_K \quad + \quad \underbrace{\Delta \cancel{E_{GP}}^{=0}}_{\text{No change in elevation}} \quad & = W_{\text{mech}} = \int_{\mathbf{s}_1}^{\mathbf{s}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{x_1}^{x_2} F_{\text{spring}} \cdot dx \\
\tfrac{m}{2}\left(V_2{}^2 - V_1{}^2\right) \quad & = \int_{x_1}^{x_2} \left[k\left(L_o - L\right)\right] \cdot dx \quad \text{where} \quad L = L_o + x \\
& = \int_{x_1}^{x_2} \left[k\left(L_o - \left(L_o + x\right)\right)\right] \cdot dx \\
& = \int_{x_1}^{x_2} \left[-kx\right] \cdot dx = -\tfrac{k}{2}\left(x_2{}^2 - x_1{}^2\right)
\end{aligned}
$$

With the initial compressed length of the spring L_1, the velocity at any location x_2 can be developed from the result above as follows:

$$
\begin{gathered}
\frac{m}{2}\left(V_2{}^2 - \underbrace{V_1{}^2}_{=0}\right) = -\frac{k}{2}\left[x_2{}^2 - x_1{}^2\right] = -\frac{k}{2}\left[x_2{}^2 - \left(L_1 - L_o\right)^2\right] \\
m\frac{V_2{}^2}{2} = \frac{k}{2}\left[\left(L_1 - L_o\right)^2 - x_2{}^2\right] \\
V_2 = \sqrt{\frac{k}{m}\left[\left(L_1 - L_o\right)^2 - x_2{}^2\right]}
\end{gathered}
$$

Now let's return to Eq. $7.1.14$ and rearrange that result:

$$
\begin{gathered}
\tfrac{m}{2}\left(V_2{}^2 - V_1{}^2\right) = -\tfrac{k}{2}\left(x_2{}^2 - x_1{}^2\right) \\
\underbrace{\tfrac{m}{2}\left(V_2{}^2 - V_1{}^2\right)}_{=\Delta E_K} + \underbrace{\tfrac{k}{2}\left(x_2{}^2 - x_1{}^2\right)}_{=\Delta E_{\text{spring}}} = 0
\end{gathered}
$$

Notice that the term on the left-hand side only depends upon the end states of the system. This is characteristic of a property and in fact both quantities on the left-hand side are changes in extensive properties of the system. The first one is our old friend the change in kinetic energy. The second term is new and is called the change in spring (elastic) energy of the system. We define the **spring (elastic) energy** of the system as:

$$
E_{\text{spring}} \equiv \frac{1}{2} k x^2
$$

The ideal spring stores energy in the elastic deformation of the spring with the amount of energy directly related to the compression or expansion of the spring from its unloaded or free length.

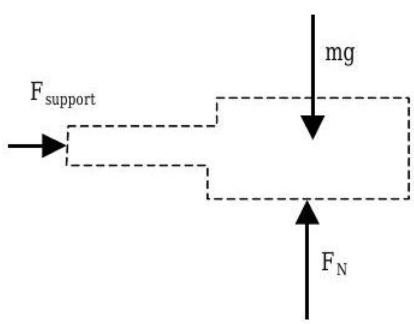


Figure $7.1.10$: Analyzing a system with the spring *inside* the system.

If we treated spring energy as a form of mechanical energy and we had placed the spring inside our system (see Figure $7.1.10$), the mechanical energy balance with no losses would look like the following for the spring-block system:

$$
\Delta E_K + \Delta E_{GP} + \Delta E_{\text{spring}} = W_{\text{mech}}
$$

To solve the problem, we would again calculate the W_{mech} for the system; however, none of the external forces contribute to the mechanical work. The normal force fails to contribute because it is perpendicular to the motion; the weight fails to contribute because it is a body force and not a surface force; and the support force holding the spring to the wall does not move. Thus the energy equation would reduce as follows:

$$\underbrace{\Delta E_K}_{V_1=0} + \underbrace{\Delta \cancel{E_{GP}}^{=0}}_{\text{No change in elevation}} + \Delta E_{\text{spring}} = \cancel{W_{\text{mech}}}^{=0}$$

$$\Delta E_K + \Delta E_{\text{spring}} = 0$$

$$m\frac{V_2^2}{2} + \frac{k}{2}\left({x_2}^2 - {x_1}^2\right) = 0$$

This result is identical to Eq. 7.1.14, the result obtained when the spring was outside the system; however, the current approach is simpler because there was no work done on the system. From a mechanical energy perspective, what is happening here is that the initial energy stored in the spring is converted to the kinetic energy of the block repeatedly as the block oscillates back and forth. This oscillation will continue forever unless there is friction internal to the system or friction between the block and the horizontal surface.

In the current section, we have extended our definition of mechanical energy to include kinetic energy, gravitational potential energy, and now spring (or elastic) energy. This by itself would be of little consequence, except we have also claimed that this new energy can be included within our existing mechanical energy balance. This experience of labeling or identifying a new form of energy and then fitting it into an existing framework is characteristic of the historical development of the conservation of energy principle. Now we will turn to this more general principle, the conservation of energy — one of the most powerful and pervasive principles of physics and engineering science.

7.2: Four Questions

When developing an accounting concept for a new property, there are four questions that must be answered. When applied to energy, the questions become

1. What is energy?
2. How can it be stored in a system?
3. How can it be transported?
4. How can it be created or destroyed?

Once we have answered these questions we will have the appropriate balance equation for energy.

7.2.1 What is energy?

From your study of basic mechanics, you have already been introduced to three types of energy — kinetic energy, gravitational potential energy, and spring (elastic) energy. Each of these mechanical energies followed naturally from a calculation of the mechanical work required to change the state of a system:

Mechanical Energy	Property Changed
Kinetic energy, E_K	Translational velocity of a particle.
Gravitational potential energy, E_{GP}	Elevation of a particle in a gravitational field.
Spring energy, E_{spring}	Length of an elastic material.

It also seemed reasonable to apply the accounting principle and combine mechanical energy and mechanical power in a mechanical energy balance:

$$\frac{d}{dt}\left(E_K+E_G+E_{\text{Spring}}\right)=\dot{W}_{\text{mech}}$$

Unfortunately, this mechanical energy balance is only valid under certain restrictions.

Historically, the study of what we now call energy was divided into the study of *heat* and the study of *work*. The remnants of this division are still with us today. Common usage of these terms implies that *heat* is something a system has and can transfer, and *work* is something that a system does. One of the crowning achievements of 19th century physics was the recognition that these two concepts could be unified through the First Law of Thermodynamics, and that both represent mechanisms for transferring energy.

In our study of linear momentum, we had a very precise definition for the linear momentum of a particle and subsequently for a system. Unfortunately for the student, there is no single expression for energy because energy takes many different forms. Thus our understanding of energy will be based on a set of *operational definitions* that help us recognize it. (Although this may sound complicated, do not fret. In most applications, only a few easily recognizable forms of energy are involved. The operational definitions provide the foundation for understanding the concept, but are rarely required in daily practice.)

Thermodynamic Work

We have already explored mechanical work and learned how it may change a system. Many other interactions between a system and its surrounding can also affect a system in ways that mimic mechanical work. To investigate these effects requires a broader concept that includes mechanical work as a subset. This broader concept is called *thermodynamic work*. The operational definition for thermodynamic work is as follows:

> **Thermodynamic work** is an interaction between a system and its surroundings that occurs in such a fashion that the *only* change in either the system *or* the surroundings *could* have been an increase in the gravitational potential energy of the system *or* the surroundings. The magnitude of the work equals the increase in the gravitational potential energy that could have occurred.

Note that this definition does not say that the gravitational potential energy of the system or surroundings did in fact increase. It says that it *could* have increased. Typically, we will just use the term **work** as a synonym for thermodynamic work. (We will decide shortly that this interaction is a transfer of energy; however, to state this now is logically premature.) **Mechanical work** is a subset of thermodynamic work. In general, we will not apply this definition directly once we have identified the common forms of work. However, it does serve a purpose when new interactions are studied.

Adiabatic Process, Boundary, and System

An **adiabatic process** is any process that involves only work interactions with the surroundings. An **adiabatic boundary** is a boundary that only allows work interactions with the surroundings. A system that only has adiabatic boundaries is called an **adiabatic system**.

First Law of Thermodynamics

After much experimentation, the **First Law of Thermodynamics** became an established fundamental principle in the latter half of the 19th century. One form of it says the following:

> When any closed system undergoes an adiabatic process, the net work associated with the change of state is the same for all possible adiabatic processes that connect the same two equilibrium end states.

Because the amount of adiabatic work for a closed system only depends on knowing the two end states of the system, *the adiabatic work for a closed system defines the change in a property for the system*. This property is called the **energy** of the system and defined by the relation:

$$\Delta E_{sys} = E_{sys,\,2} - E_{sys,\,1} = W_{\text{1-2, adiabatic}} \quad \text{for a closed system.}$$

Experience has shown that the adiabatic work required to change the state of a closed system depends on the amount of mass of the system; thus, *energy is an extensive property*.

The first law of thermodynamics can also be stated as two postulates:

1. There exists an extensive property called energy, E.
2. The change in energy for a closed system between any two states is defined as the work done on the system during an adiabatic process connecting the two states, $\Delta E = E_2 - E_1 = W_{\text{1-2 adiabatic}}$.

This now provides us a way to calculate the change in energy for a closed system.

The dimensions of energy are the same as for work: $[\text{Force}][\text{Length}]$. Although the dimensions are the same, there are several units for energy that are commonly used by engineers. In the SI system, the standard unit is the joule (J) where $1\ \text{J} = 1\ \text{N} \cdot \text{m}$. In the USCS system, there are two units that commonly occur: the foot-pound-force $(\text{ft} \cdot \text{lbf})$ and the British thermal unit (Btu). The foot-pound-force had its birth in the study of mechanical work and the British thermal unit was used in the study of heat. One of the most famous results of physics was the experimental determination of the "mechanical equivalent of heat" by Joule. Today we know that the correct relationship between these two units of energy is approximately

$$1\ \text{Btu} = 778.17\ \text{ft} \cdot \text{lbf}$$

One British thermal unit approximately equals the amount of energy required to raise the temperature of one pound-mass of liquid water one degree Fahrenheit at room temperature.

The first law of thermodynamics only tells us how to calculate the change in energy of a system. This is also a characteristic of the property energy — we can only calculate energy differences.

But what about kinetic energy and gravitational potential energy and spring energy? Can't we evaluate absolute values for these energies using our familiar equations? The answer is no. For example, the numerical value of the kinetic energy of a particle, $mV^2/2$, changes depending upon which inertial reference frame you choose when evaluating the velocity. Imagine you are throwing a baseball on a steadily moving train. If the velocity is measured with respect to the train the kinetic energy has one value; if the velocity is measured with respect to the ground, the kinetic energy will have a different value. Similar arguments can be made for gravitational potential energy and spring energy.

Following the same line of reasoning we used earlier with mechanical energy and mechanical work, it seems reasonable that we should interpret work as a mechanism for transporting energy across the boundary of a system. We will show shortly that work alone is insufficient to explain how the energy of a system can change.

7.2.2 How can energy be stored in a system?

Based on the first law of thermodynamics and the concept of thermodynamic work, we have a way to investigate changes in the energy of a closed system (see Eq. $7.2.1$). Fortunately, engineers and scientists have developed a fairly comprehensive picture of the various forms in which energy can be stored in a system.

Types of Energy

Energy can be stored in many forms. In addition to kinetic energy, gravitational potential energy, and elastic energy, there are several other types of energy. We will find it useful to classify the energy stored in a system E_{sys} into four groups:

$$E_{sys} = \underbrace{U}_{\substack{\text{Internal}\\ \text{energy}}} + \underbrace{E_K}_{\substack{\text{Translational}\\ \text{kinetic energy}}} + \underbrace{E_{GP}}_{\substack{\text{Gravitational}\\ \text{potential energy}}} + E_{other}$$

where U is the internal energy of the system, E_K is the translational kinetic energy of the system, E_{GP} is the gravitational potential energy of the system, and E_{other} represents all other forms of energy. Two of these are familiar and two are not.

First, let's consider the familiar terms. The translational kinetic energy and the gravitational potential energy are the same ones we have used previously in this course and require no additional discussion. Notice that we have now been more explicit about the type of kinetic energy our system has. A rotating system with a stationary center of mass may still store energy as rotational kinetic energy. (Any piece of rotating machinery can store a significant amount of energy in its rotation. Containing this energy during a catastrophic failure of the device, e.g. the failure of a turbine blade or the failure of a tire rim, is often a major safety factor in the design of the device.)

Of the two new energies, the most important is the internal energy. The identification of internal energy was a direct consequence of the development of the first law of thermodynamics.

> The **internal energy** of a substance is the extensive property that represents the microscopic kinetic and potential energy of the molecules and atoms that make up a substance.

Any object with mass has internal energy as a result of the "motions and configurations of its internal particles" (E. F. Obert and R. A. Gaggioli, *Thermodynamics*, 2nd Ed., McGraw-Hill, New York, 1963, pg. 18). Changes in the internal energy of a system are manifest by changes in other properties such as temperature, pressure, and density.

Internal Energy — Where did it come from?

To give us some insight into how internal energy was discovered *and* what it is physically, consider the device shown in the figure. The device consists of a rigid tank that is heavily insulated. The tank contains air and a set of fan blades attached to a shaft. The other end of the shaft is attached to a pulley and suspended from the pulley by a cable is a mass.

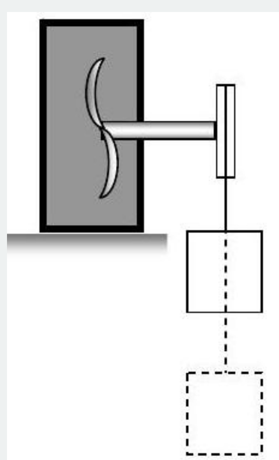


Figure $7.2.1$: System consisting of fan blades inside a rigid container attached to a shaft and pulley, from which a mass is suspended outside the container.

Initially the fan-shaft-pulley-mass system is stationary. If the mass is allowed to fall a distance L and once again become stationary, what happens to the gas inside the tank?

You might suggest that the temperature of the gas might increase, and also its pressure. If you wait long enough the gas would also become stationary once again.

Since the energy of the closed system cannot change, the decrease in gravitational potential energy must be balanced by an increase in another form of energy.

If we examined a system that includes everything in the picture, we would discover two things:

(1) There is no interaction on the boundary of the system that qualifies as thermodynamic work. This is a direct consequence of the insulation. Thus, the energy of the system is constant.

(2) The gravitational potential energy of the system decreases.

For this system, the energy that increases is the internal energy of the tank walls, the fan blade and the portion of the shaft inside the tank. On a macroscopic level, this change in internal energy might manifest itself as an increase in the temperature of the system. On a microscopic level, we would observe changes in the motion of the atoms and molecules that form these objects; hence the name internal energy.

The interaction that occurs between the portion of the shaft outside the tank and the shaft inside the tank does in fact represent thermodynamic work (shortly we will identify it as *shaft work*). To justify this, imagine that the tank and its contents were replaced with a system containing a frictionless pulley and mass. Now the falling of the weight in the original system results in the raising of the imagined mass in a gravitational field with only an increase of the gravitational potential energy of the imagined system. This satisfies our definition of thermodynamic work. Note that in identifying thermodynamic work, it is not what actually happens to the energy transferred but what could happen

The remaining term E_{other} in our energy expression, Eq. $7.2.2$, includes all other forms of energy. These may or may not be important depending upon the system being considered. Some of the most common forms of energy that can be stored in a system are listed below:

Energy	Example
Elastic	Energy stored in the deformation of an elastic material, e.g. a spring.
Kinetic, Rotational	Energy stored in an object due to its rotation about an axis, e.g. a flywheel
Surface tension	Energy stored in the stretching of a liquid film, e.g. formation of a raindrop.
Electric field	Energy stored in an electric field, e.g. a capacitor
Magnetic field	Energy stored in a magnetic field, e.g. an inductor

One of the major challenges in problem solving is deciding which forms of energy are important for a particular system. This list is not all inclusive. As you continue your education, you may discover other forms of energy that play a prominent role in specific physical phenomena.

Specific energy and the energy of a system

For our purposes, we will assume that the energy of the system is associated with the mass contained inside the system; thus,

$$E_{\text{sys}} = \int_{V_{sys}} e\,\rho dV$$

where e is the energy per unit mass, or the **specific energy**, of the system. The dimensions of specific energy are $[\text{Energy}]/[\text{Mass}]$ and typical units are kJ/kg in SI and $\text{ft}\cdot\text{lbf}/\text{lbm}$ or Btu/lbm in USCS. Since energy is an extensive property, the energy of a system is equal to the sum of the energy of its subsystems. For some types of energy, the energy is actually stored in an electric or magnetic field that may or may not correspond with the physical mass of the system. This will require a different approach to assigning the energy of the system. Typically, we will assume that the energy is located within a specific physical device, such as a capacitor or inductor.

7.2.3 How can it be transported?

Energy can be transported across the boundary of a system by three different mechanisms — work, heat transfer, and mass transfer of energy.

Energy transport by work

Energy transport by work can occur at both flow and non-flow boundaries. Any interaction between a system and its surroundings that satisfies the definition of thermodynamic work is a work transfer of energy. The work transfer rate of energy into the system is also known as the power and has the same characteristics as mechanical power discussed earlier. Thinking of power as an energy transfer rate, we can write

$$\dot{E}_{\text{work}} = \dot{W}$$

Subscripts "in" or "out" are often used to indicate the direction of the transfer. When we sum up the work transfer rates of energy for a system we will frequently report this as a net rate:

$$\sum \dot{W}_{\text{in}} - \sum \dot{W}_{\text{out}} = \dot{W}_{\text{net, in}} = -\dot{W}_{\text{net, out}}$$

Notice how the subscripts are connected with the signs. This becomes especially important when you copy an equation from the text, because you must understand the sign convention observed to correctly use the derived equation.

There are many different units for work and power. The most common ones used in this course are presented below:

	SI System	USCS System
Work	newton-meter $(\mathrm{N} \cdot \mathrm{m})$ joule (J) kilojoule (kJ)	foot-pound-force $(\mathrm{ft} \cdot \mathrm{lbf})$ British thermal unit (Btu)
Power	newton-meter/second $(\mathrm{N} \cdot \mathrm{m/s})$ joule/second $(\mathrm{J/s})$ watt (W) kilowatt (kW)	foot-pound-force/second (ft-lbf/s) British-thermal-unit/second $(\mathrm{Btu/s})$ horsepower (hp)

There are many ways that energy can be transported across a non-flow boundary of a system by work. Consider the device shown in Figure \\($7.2.2$ consisting of a battery connected to a DC motor that turns a paddle wheel that stirs a gas contained in a piston-cylinder device.

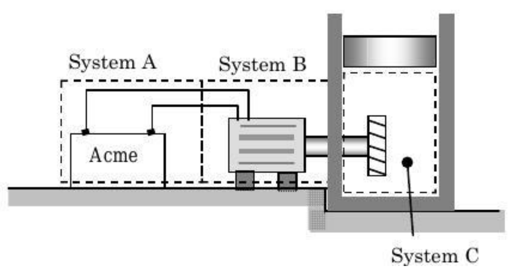


Figure $7.2.2$: Examples of work transfer of energy.

Initially, everything in the device is stationary and the pressure of the gas in the piston-cylinder device is sufficient to "float" the piston. If we close a switch between the battery and the motor, what would happen? We would expect the motor to turn the paddle wheel and with time the piston would probably rise. (Take a minute and think about this. Would you expect the piston to rise?) Now what are the work transfers of energy for this system? Since work is defined in terms of transfers at a boundary, we will identify three systems A, B, and C. From a work perspective, the battery (System A) does *electrical work* on the motor (System B). The motor does *shaft work* on System C, the paddle wheel, shaft, and gas inside the cylinder. Finally, the gas does expansion work to move the piston. In the following sections, we will examine each of these work modes in more detail.

Compression-Expansion (P dV) Work

One of the most important and pervasive work modes is compression-expansion work. This is the work mode that occurs in your car engine as hot, high-pressure gas expands against the piston. As they expand, the gas does work on the piston. This work is then transferred through the connecting rod to the crankshaft to the transmission to the drive shaft through the differential to the axle and finally to the wheels of the car. (Whew, what a path!)

But let's get back to what is happening to the hot gas. Consider the simple piston-cylinder device shown in Figure $7.2.3a$. The hot, high-pressure gas is contained in the closed volume formed by the piston and the cylinder walls. For clarity, let's consider the case of expansion where the piston is moving to the right. We are interested in finding the work done *by* the piston on the gas:

$$W_{in} = \int_{\mathrm{x}_1}^{\mathrm{x}_2} \mathbf{F}_{\text{surface}} \cdot d\mathbf{x} = \int_{\mathrm{x}_1}^{\mathrm{x}_2} (-F_{\text{piston}}\mathbf{i}) \cdot (dx\mathbf{i}) = -\int_{x_1}^{x_2} F_{\text{piston}} \cdot dx$$

where the only surface force that moves is F_{piston}, the force that the piston exerts on the gas (See Figure \(\PageIndex{3b\}). Note that the minus sign in the equation comes about because the force acts opposite to the displacement direction.

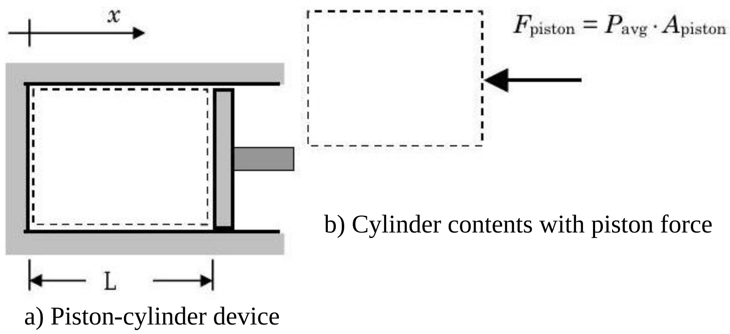


Figure $7.2.3$: Compression-expansion work for a piston-cylinder device.

This is a perfectly good expression; however it would be more useful if it could be expressed in terms of the properties the gas inside the system. We can do this by rewriting Eq $7.2.6$ in terms the average pressure at the piston-gas interface and the system volume:

$$\begin{aligned} W_{\text{in}} &= -\int_{x_1}^{x_2} F_{\text{piston}} \cdot dx = -\int_{x_1}^{x_2} \underbrace{\left(\frac{F_{\text{piston}}}{A_{\text{piston}}}\right)}_{=P_{\text{avg}}} \cdot \underbrace{A_{\text{piston}}\, dx}_{=dV} \\ &= -\int_{V_{sys,\,1}}^{V_{sys,\,2}} P_{\text{avg}} \cdot dV \end{aligned}$$

Again note that to evaluate this equation only requires that we know the average pressure on the moving boundary of the system as a function of the system volume.

If we further assume that the expansion (or compression) process occurs slowly enough that the pressure within the system is uniform, then $P_{\text{avg}} = P$, and the work *on* the gas during a compression-expansion process can be described by the equation:

$$W_{\text{in}} = -\int_{V_{sys,\,1}}^{V_{sys,\,2}} P \cdot dV \qquad \begin{array}{l}\text{Compression-Expansion (PdV) Work} \\ \text{(Assumes spatially uniform pressure)}\end{array}$$

This work mode is commonly referred to as **compression-expansion work** or PdV work. When the system is compressed $(dV < 0)$, the piston does work on the system and $W_{\text{in}} > 0$. When the system expands $(dV > 0)$, the system does work on the piston and $W_{\text{in}} < 0$ because energy is being transferred out of the system. This result is valid for both liquids and gases.

Example — PdV Work for a Closed System

In practice, the basic equation for PdV work of a closed system must be integrated to determine the work for a process. Experience indicates that there are several processes that occur frequently and will be investigated here. As always you are warned against memorizing formulas. Focus on how the process assumption is used to integrate the basic equation for PdV work.

Constant-Volume Process: $\cancel{V} = \cancel{V}_1 = \cancel{V}_2 = C,$ a constant

$$W_{PdV,\text{ in}} = -\int_1^2 P\, d\cancel{V} = -\int_1^2 P\, \cancelto{=0}{d\cancel{V}} = 0$$

Constant-Pressure (Isobaric) Process: $P = P_1 = P_2 = C,$ a constant

$$W_{PdV,\text{ in}} = -\int_1^2 P\, d\cancel{V} = -\int_1^2 C\, d\cancel{V} = -C\,(\cancel{V}_2 - \cancel{V}_1) = -P\,(\cancel{V}_2 - \cancel{V}_1)$$

Polytropic Process: $P\cancel{V}^n = P_1\cancel{V}_1{}^n = P_2\cancel{V}_2{}^n = C,$ a constant

$$W_{PdV,\text{ in}} = -\int_1^2 P\, d\cancel{V} = -\int_1^2 \left(\frac{C}{\cancel{V}^n}\right) d\cancel{V}$$

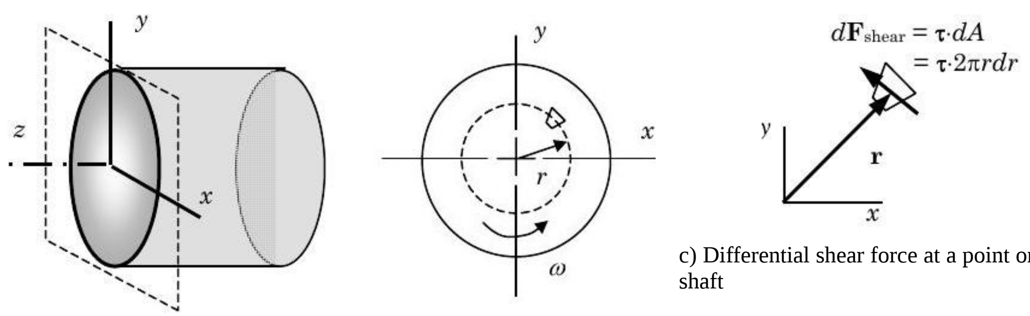
If $n = 1$:

$$W_{PdV,\text{ in}} = -\int_1^2 P\, d\cancel{V} = -\int_1^2 \left(\frac{C}{\cancel{V}}\right) d\cancel{V} = -C\int_1^2 \frac{d\cancel{V}}{\cancel{V}} = -C\ln\left(\frac{\cancel{V}_2}{\cancel{V}_1}\right) = -P\cancel{V}\ln\left(\frac{\cancel{V}_2}{\cancel{V}_1}\right)$$

If $n \neq 1$:

$$\begin{aligned} W_{PdV,\text{ in}} &= -\int_1^2 P\, d\cancel{V} = -\int_1^2 \left(\frac{C}{\cancel{V}^n}\right) d\cancel{V} = -C\int_1^2 \cancel{V}^{-n}\, d\cancel{V} \\ &= -C\left[\frac{\cancel{V}^{-n+1}}{-n+1}\right]_{\cancel{V}_1}^{\cancel{V}_2} = -C\left[\frac{\cancel{V}_2{}^{1-n} - \cancel{V}_1{}^{1-n}}{1-n}\right] = C\left[\frac{\cancel{V}_2{}^{1-n} - \cancel{V}_1{}^{1-n}}{n-1}\right] \\ &= \left[\frac{C\cancel{V}_2{}^{1-n} - C\cancel{V}_1{}^{1-n}}{n-1}\right] = \left[\frac{(P_2\cancel{V}_2{}^n)\cancel{V}_2{}^{1-n} - (P_1\cancel{V}_1{}^n)\cancel{V}_1{}^{1-n}}{n-1}\right] = \frac{(P_2\cancel{V}_2 - P_1\cancel{V}_1)}{n-1} \end{aligned}$$

Starting at the same initial state 1, plot a constant volume process, a constant pressure process, and a polytropic process with $n = 1$ on a graph of P vs. V. How is W_{PdV} related to the area under the PV curve for each process?



a) Looking into the system at a cut shaft b) End view of cut shaft c) Differential shear force at a point on the cut shaft

Figure 7.2.4: View into a system containing a rotating shaft.

Shaft Work

When a portion of a system boundary is rotating, mechanical work is done on the system by shear forces at the rotating surface. The most common case of this is when a system boundary intersects (cuts) a rotating shaft (See Figure $7.2.4$). To calculate the mechanical work, we must identify the surface forces and their motion. Figure $7.2.4a$ shows a system boundary that intersects a shaft. Figure $7.2.4b$ shows the cross-section of the shaft and the direction of rotation as viewed looking into the system. Figure $7.2.4c$ shows the differential force shearing force $d\mathbf{F}_{shear}$ acting on the cut shaft and the position vector $\mathbf{r}$ where it is applied. The mechanical work done on the rotating shaft inside the system is evaluated as

$$W_{\text{shaft, in}} = \int_{\theta_1}^{\theta_2} \int_{Ac} \underbrace{(\tau\, dA)}_{dF_{shear}} \underbrace{(r\, d\theta)}_{ds} = \int_{\theta_1}^{\theta_2} \underbrace{\left[\int_0^R \tau(2\pi r) r\, dr\right]}_{M_0 = \text{moment about axis } 0} d\theta = \int_{\theta_1}^{\theta_2} M_0 \cdot d\theta$$

where M_0 is the moment of the couple formed by all the shear forces acting on the cut shaft and θ is the angular rotation of the shaft expressed in radians. If the shaft rotation (looking into the system) has the same sense as the moment applied to the system, $W_{\text{shaft, in}} > 0$. If the sense of the moment and shaft rotation are opposite, $W_{\text{shaft, in}} < 0$.

The **shaft power** is calculated using the equation:

$$\dot{W}_{\text{shaft, in}} = M_0 \cdot \omega$$

where ω is the rotational speed in radians per second (rad/s) at the boundary. For most systems with rotating shafts, the shaft power is of more interest than the shaft work.

Electrical work and power

Energy can be transferred to a system by electric charge flowing through a system. When a charge is moved within an electric field a force is exerted on the charge (Figure $7.2.5$). When the charge is moved within the field, work is done on the charge. This situation is analogous to our earlier experience with a moving mass in a gravitational field.

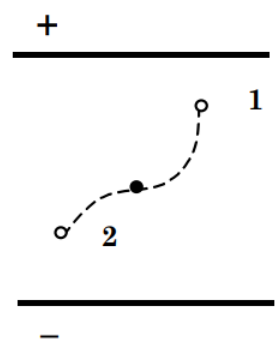


Figure $7.2.5$: A charged particle moving in an electric field.

For a particle moving in a gravitational field we found that

$$W_{\text{mech}} = \Delta E_K + \Delta E_{GP} \quad \text{and} \quad \Delta E_{GP} = m\Delta e_{GP} = mg\Delta z$$

In an analogous manner, the work done to move a charged particle with charge q in an electric field equals

$$W_{\text{electric}} = \int_1^2 q\, dV = q\left(V_{2-0} - V_{1-0}\right)$$

where V_{i-0} is the electric potential at point i. The **electric potential** is the mechanical work per unit charge required to move a charged particle in a stationary electric field between a reference point O and arbitrary point i. The standard unit for electric potential is the volt (V), where 1 volt = 1 joule/coulomb $(1\text{ V} = 1\text{ J/C})$. As with any mechanical work, the electric work may be positive or negative.

In many applications, the problem is a system where an electric current is flowing across the boundary. In a typical configuration, electric current flows through the system at a known rate with no accumulation inside the system, see Figure $7.2.6$. The electric potential on the boundary where the current crosses the system boundary is known with respect to a common ground. Under these conditions, the instantaneous **electric power** into the system is

$$\dot{W}_{\text{electric, in}} = i \cdot (V_{\text{in-o}=} - V_{\text{out-o}}) \qquad \text{Electric Power}$$

where i is the electric current crossing the boundary and the electric potential at the points on the boundary where the electric current enters and leaves the system are $V_{\text{in-o}}$ and $V_{\text{out-o}}$, respectively. When the electric potential *decreases* in the direction of the current flow, the electric power in is positive. When the electric potential *increases* in the direction of current flow, the electric power in is negative.

An example showing how the direction current flow influences the electric power for a simple dc battery is shown in Figure \($7.2.6$ When the battery is charging, energy is being added to the battery and the electric power in is positive. When the battery is discharging through a resistor, energy is leaving the battery and the electric power into the battery is negative. What would be the direction and the magnitude for the electric power for the resistor?

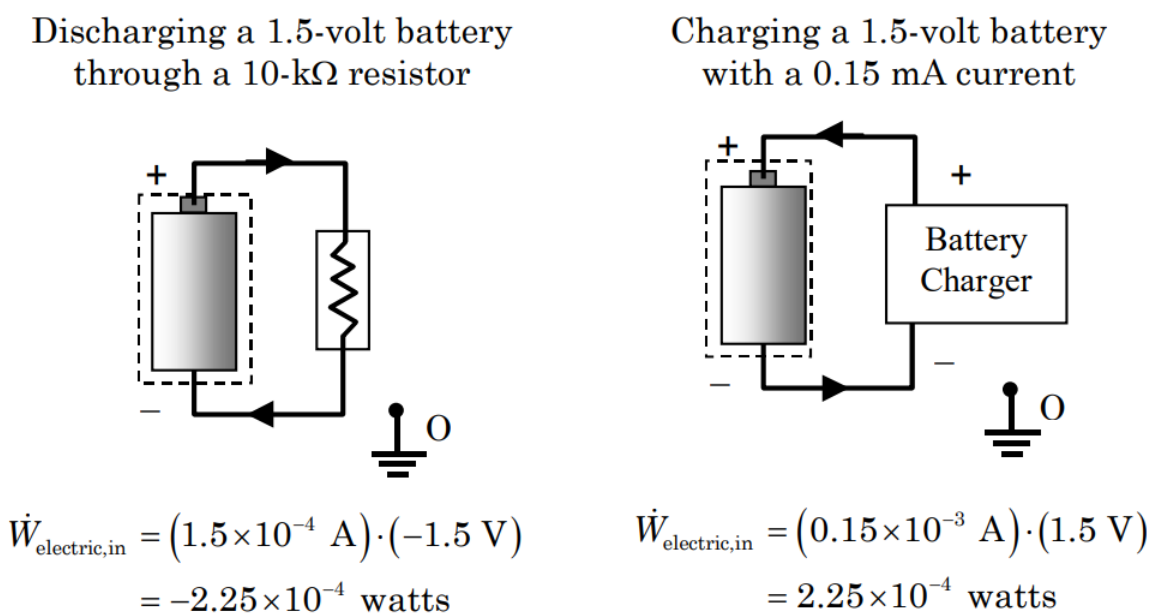


Figure $7.2.6$: Electric power for a 1.5-volt battery.

Flow work and power

At flow boundaries, the mass flowing into the system serves to compress the mass already inside the open system. This type of work is known as **flow work** and the rate of doing flow work is known as **flow power**. The flow power for a system is written as

$$\begin{aligned} \dot{W}_{\text{flow, net in}} &= \sum_{\text{in}} \dot{W}_{\text{flow, in}} - \sum_{\text{out}} \dot{W}_{\text{flow, out}} \\ &= \sum_{\text{in}} \dot{m}_i \left(P_i \nu_i\right) - \sum_{\text{out}} \dot{m}_e \left(P_e \nu_e\right) \end{aligned}$$

where $P\nu$, the product of pressure and specific volume, is the specific flow work evaluated at the boundary where the mass flow occurs. A detailed development of this term will be delayed until later. Note that for a closed system, there is no mass flow rate and thus no flow power. Flow work is a mechanism for transferring energy that only occurs in open systems.

Work and power for a system

When we combine the expressions for the work transfer rates, the power, for a system we have the following expression:

$$\begin{aligned}\dot{E}_{\text{work, net in}} &= \underbrace{\left[\sum \dot{W}_{\text{in}} - \sum \dot{W}_{\text{out}}\right]}_{\substack{\text{Rate of energy transfer by work} \\ \text{(excluding flow work)} \\ \text{(Power)}}} + \underbrace{\left[\sum_{\text{in}} \dot{m}_i \left(P_i \nu_i\right) - \sum_{\text{out}} \dot{m}_e \left(P_e \nu_e\right)\right]}_{\substack{\text{Rate of energy transfer by flow work} \\ \text{(Flow Power)}}} \\ &= \dot{W}_{\text{net, in}} + \left[\sum_{\text{in}} \dot{m}_i \left(P_i v_i\right) - \sum_{\text{out}} \dot{m}_e \left(P_e v_e\right)\right]\end{aligned}$$

where P is the pressure and ν is the specific volume at the boundary where the flow occurs. You should recognize that there may be several different types of work (or power) that are included in this term, e.g. an electric motor has both shaft work and electrical work.

Energy transport with mass flow

At flow boundaries, every lump of mass that enters or leaves the system carries with it some amount of energy. The rate at which energy is carried across a boundary is the product of the mass flow rate and the specific energy, e, of the mass at the boundary:

$$\dot{E}_{\text{mass flow}} = \dot{m}e$$

The net rate at which energy is carried into a system by mass flow is

$$\dot{E}_{\text{mass flow, net in}} = \sum_{\text{in}} \dot{m}_i e_i - \sum_{\text{out}} \dot{m}_e e_e$$

Energy transport by heat transfer

Heat transfer is the third and final mechanism for transferring energy across the boundary of a system. Heat transfer is any transfer of energy that cannot be classified as an energy transfer with mass flow or an energy transfer by work. Using the first law of thermodynamics and thermodynamic work, heat transfer for a closed system is precisely defined by the equation:

$$Q_{1-2,\text{net in}} \equiv \left(E_{\text{sys, 2}} - E_{\text{sys, 1}}\right) - W_{1-2,\text{net in}}$$

This says that when a closed system undergoes a process, any process, between states 1 and 2, the difference between the change in energy of the system and the net work into the system equals the heat transfer for the system.

Just like work, heat transfer has both a rate form and a finite time form:

$$Q_{1-2} = \int_{t_1}^{t_2} \dot{Q}\, dt = \int_{1}^{2} \delta Q$$

Also like work, heat transfer is a path function and has no value at a specified state. The heat transfer Q_{1-2}, like work W_{1-2}, can only be calculated once you know the path, the process, and the end states. We will use the same sign convention as used previously with other energy transfers. The use of subscripts, Q_{in} or Q_{out}, is encouraged to avoid confusion.

Our experience tells us that heat transfer is closely related to the idea of temperature. For example, imagine that I have two rectangular blocks of copper. One has been sitting in an ice bath and the other has been sitting in a pot of boiling water. You could easily recognize which block is hot and which block is cold. Now suppose I place these two blocks in contact and then heavily insulate the touching blocks with some fiberglass insulation. Given sufficient time, you would discover that neither block is hot or cold-they are the same. We would commonly say that "the two blocks are at the same temperature." Alternatively, we might say the blocks had reached *thermal equilibrium*. What happened to the blocks? How did they interact? Again, you would probably say "the cold one heated up" and "the hot one cooled down."

This may be a correct layman's interpretation, but how does this fit into our energy picture? If you carefully consider how the two blocks interacted during the process and how they could interact with the surroundings, you will discover that there was no work transfer of energy or mass transfer of energy. All that remains to transfer energy is heat transfer, and that is what happened. The two blocks reached thermal equilibrium — a state where the temperature is spatially uniform and unchanging with time — by exchanging energy by heat transfer. This highlights another important characteristic of heat transfer — *heat transfer will not occur between two systems in thermal equilibrium.*

Our experience also tells us that *heat transfer only occurs spontaneously from a region of high temperature to a region of low temperature*. (Would you have expected the hot block of copper to get warmer and the cold block of copper to get cooler?)

The idea of temperature is so closely involved with heat transfer that the following definition is often given for heat transfer:

> **Heat transfer** is a mechanism for transferring energy across the boundary of a system due to a temperature difference

There are three physical mechanisms for transporting energy by heat transfer — conduction heat transfer, convection heat transfer, and radiation heat transfer. We will revisit these three later, but an entire course on heat transfer is a required component of many curricula. E.g., mechanical engineering requires a complete course, and electrical and computer engineers will learn about specific applications to the cooling of electrical systems.

Heat transfers of energy may occur at any system boundary, although heat transfer at flow boundaries is usually neglected and is insignificant when compared to the transfer at non-flow boundaries. Symbolically, this can be written as

$$\dot{E}_{\text{heat transfer, net in}} = \dot{Q}_{\text{net, in}}$$

Although only a single heat transfer term appears, the total heat transfer rate can be found by summing the heat transfer rates at all boundaries of the system.

How can energy be created or destroyed?

The First Law of Thermodynamics, one of the bedrock principles of physics, is really a statement that energy cannot be created or destroyed. Thus, *energy is conserved!*

As with species accounting earlier, if you only count one type of energy, such as mechanical energy, it appears that a specific energy can be created and destroyed. However, this only occurs because you are counting one only type of energy. Experience has repeatedly shown that when all forms of energy are accounted for, energy is conserved. In fact, our belief in this law is so complete that physicists have discovered new particles when they tried to understand why their experiments did not conserve energy.

7.2.5 Putting it all together — Conservation of Energy

Applying the accounting framework to energy, we know that

$$\frac{dE_{\text{sys}}}{dt} = \dot{E}_{\text{work, net in}} + \dot{E}_{\text{heat transfer, net in}} + \dot{E}_{\text{mass flow, net in}}$$

Now, collecting all of the results developed above, we have an initial version of the rate form of the conservation of energy:

$$dE_{\text{sys}} = \underbrace{\dot{W}_{\text{net, in}} + \left[\sum_{\text{in}} \dot{m}_i \left(P_i \nu_i\right) - \sum_{\text{out}} \dot{m}_e \left(P_e \nu_e\right)\right]}_{\substack{\text{Net rate of} \\ \text{energy transport} \\ \text{by work}}} + \underbrace{\dot{Q}_{\text{net, in}}}_{\substack{\text{Net rate of} \\ \text{energy transport} \\ \text{by heat transfer}}} + \underbrace{\left[\sum_{\text{in}} \dot{m}_i e_i - \sum_{\text{out}} \dot{m}_e e_e\right]}_{\substack{\text{Net rate of} \\ \text{energy transport} \\ \text{by mass flow}}}$$

Although this result is perfectly acceptable, we will find it advantageous to combine all of the terms that involve mass flow before we write a final equation.

Energy, enthalpy and the mass flow terms

If we group the mass flow terms together, we have the following:

$$\frac{dE_{\text{sys}}}{dt} = \dot{W}_{\text{non-flow, net in}} + \dot{Q}_{\text{net, in}} + \underbrace{\left[\sum_{\text{in}} \dot{m}_i \left(e_i + P_i v_i\right) - \sum_{\text{out}} \dot{m}_e \left(e_e + P_e v_e\right)\right]}_{\text{Mass flow terms}}$$

For many problems, we discover that there are only three types of energy that are important to consider — internal energy, translational kinetic energy, and gravitational potential energy. Writing this in terms of specific energy we have

$$e = \frac{E}{m} = u + e_K + e_G = u + \frac{V^2}{2} + gz$$

where u is the specific internal energy, $V^2/2$ is the specific translational kinetic energy, and gz is the specific gravitational potential energy.

If we now substitute this back into the mass flow terms we can regroup the terms as

$$\begin{aligned} \dot{m}(e+Pv) &= \dot{m}\left[\left(u+\frac{V^2}{2}+gz\right)+Pv\right] \\ &= \dot{m}\left[(u+Pv)+\frac{V^2}{2}+gz\right] \end{aligned}$$

Because the group of terms $u+Pv$ always shows up together in the energy balance, it is useful to give them a name. To do this we define a new extensive property called enthalpy (pronounced en-thal'-py), defined as the sum of the internal energy and the product of pressure and volume:

$$H = U + PV$$

The specific enthalpy is defined similarly as follows:

$$h = \frac{H}{m} = u + Pv$$

As you might surmise, the units for enthalpy are the same as for energy, kJ or Btu or $\mathrm{ft}\cdot\mathrm{lbf}$, and the units for specific enthalpy are the same as for specific internal energy, kJ/kg or $\mathrm{Btu}/\mathrm{lbm}$ or $\mathrm{ft}\cdot\mathrm{lbf}/\mathrm{lbm}$.

Using these results, the mass flow terms can be written more concisely as follows in terms of the specific enthalpy, the specific kinetic energy, and the specific gravitational potential energy:

$$\dot{m}\left[(u+Pv)+\frac{V^2}{2}+gz\right] = \dot{m}\left(h+\frac{V^2}{2}+gz\right)$$

Rate form of the conservation of energy

We are now ready to write the rate-form of the conservation of energy equation in the form we find most useful:

$$\frac{dE_{\text{sys}}}{dt} = \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} + \left[\sum_{\text{in}} \dot{m}_i\left(h_i+\frac{V_i^2}{2}+gz_i\right) - \sum_{\text{out}} \dot{m}_e\left(h_e+\frac{V_e^2}{2}+gz_e\right)\right]$$

where the power term only excludes the flow power that is included in the mass flow rate terms. This is the most general form of the conservation of energy equation that we will be using and is the place to start all problems involving energy. Note that we have made some important assumptions in the types of energy that can be transferred with mass; however, experience indicates that these are sufficient for most problems. Also recognize that for a closed system, the mass flow rates are zero and the term in brackets disappears.

Conservation of Energy and the Work-Energy Principle

Conservation of Energy and the Work-Energy Principle for a Particle

The work-energy principle for a particle provides the same information as writing conservation of linear momentum for a particle. In developing the work-energy principle for a particle, we considered a particle moving in a gravitational field subject to a net surface force $\mathbf{R}$. After a series of well-defined mathematical operations, we obtained the ***work-energy principle* for a *particle*:**

$$\text{Rate form:}\quad \frac{\mathrm{d}}{\mathrm{dt}}\left(\underbrace{\mathrm{m}\frac{\mathrm{V}^2}{2}}_{\substack{\text{Kinetic}\\ \text{energy}}} + \underbrace{mgz}_{\substack{\text{Gravitational}\\ \text{potential}\\ \text{energy}}}\right) = \underbrace{\mathbf{R}\cdot\mathbf{V}}_{\substack{\text{mechanical}\\ \text{power into}\\ \text{the system}}} \quad \text{or} \quad \boxed{\frac{d}{dt}(E_K + E_G) = \dot{W}_{\text{mech, in}}}$$

$$\text{Finite-time form:}\quad \Delta\left(m\frac{V^2}{2}\right) + \Delta(mgz) = \int_{t_1}^{t_2} \mathbf{R}\cdot\mathbf{V}\,dt = \int_1^2 \mathbf{R}\cdot d\mathbf{s} \quad \text{or} \quad \boxed{\Delta E_K + \Delta E_G = W_{\text{mech, in}}}$$

The work-energy principle states that the mechanical work done by the net surface forces on a particle equals the change in the kinetic energy and gravitational potential energy of the particle. Writing the ***conservation of energy for a particle*** we obtain the following:

$$\text{Rate form:}\quad \frac{d}{dt}(U_{\text{sys}} + E_{K,\,\text{sys}} + E_{G,\,\text{sys}}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$$
$$\text{Finite-time form:}\quad \Delta U_{\text{sys}} + \Delta E_{K,\,\text{sys}} + \Delta E_{G,\,\text{sys}} = Q_{\text{net, in}} + W_{\text{net, in}}$$

Examining the two sets of equations, we see that the conservation of energy equation reduces to the work-energy principle for a particle under two different conditions:

- Condition (1): The rate of change of the internal energy of the particle equals the net transport rate of energy into the particle by heat transfer, $dU_{\text{sys}}/dt = Q_{\text{net, in}}$
- Condition (2): The internal energy of the particle is constant, $du_{\text{sys}}/dt = 0$, and the particle is adiabatic, $\dot{Q}_{\text{net, in}} = 0$.

To determine if condition (1) is satisfied requires information about the material substance that makes up the particle. Typically, condition (2) would be satisfied under conditions with no intentional heat transfer and no way to dissipate mechanical energy—that is, no friction—inside the system.

Conservation of Energy and the Work-Energy Principle for a System of Particles

If we had a system that contained n particles and we applied the work-energy principle for a particle individually to each particle, we would have for particle i

$$\frac{d}{dt}(E_{K,\,i} + E_{G,\,i}) = \mathbf{R}_i \cdot \mathbf{V}_i$$

If we summed this equation up over the n particles in our system, we would obtain the following:

$$\sum_{i=1}^{n}\left[\frac{d}{dt}(E_{K,\,i} + E_{G,\,i})\right] = \frac{d}{dt}\underbrace{\left[\sum_{i=1}^{n}(E_{K,\,i} + E_{G,\,i})\right]}_{E_{K,\,sys}+E_{G,\,sys}} = \sum_{i=1}^{n}\underbrace{[\mathbf{R}_i\cdot\mathbf{V}_i]}_{\dot{W}_{mech,\,in,\,i}}$$

$$\boxed{\frac{d}{dt}(E_{K,\,sys} + E_{G,\,sys}) = \sum_{i=1}^{n}\dot{W}_{mech,\,in,\,i}}$$

$$\text{where}\ \sum_{i=1}^{n}\dot{W}_{mech,\,in,i} \neq \underbrace{\left(\sum_{i=1}^{n}\mathbf{R}_i\right)}_{\substack{\text{Net surface force}\\ \text{acting on the}\\ \text{system of particles}}} \cdot \underbrace{\left(\frac{1}{m_{sys}}\sum_{i=1}^{n}m_i\mathbf{V}_i\right)}_{\substack{\text{Velocity of the center of mass}\\ \text{of the system of particles}}}$$

\

The boxed equation is the ***work-energy principle for a system of particles***. It looks much like the original work-energy principle for a particle; however, the mechanical power term on the right is the summation of the mechanical work done by surface forces on each particle. It is not equal to the dot product of the net surface force on the system of particles and the velocity of the center of mass of the system. This is a significant difference and failure to recognize this can lead to a serious misuse of the work-energy principle.

If we now write the ***conservation of energy equation for a closed system of particles*** and assume only kinetic, gravitational potential, and internal energies are important, we have the following:

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,\, in} + \dot{W}_{net,\, in} + \overbrace{\sum_{in} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i \right) - \sum_{in} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e \right)}^{=0}$$

$$\boxed{\frac{d}{dt}(U_{sys} + E_{K,\, sys} + E_{G,\, sys}) = \dot{Q}_{net,\, in} + \dot{W}_{net,\, in}}$$

Now if we subtract the *work-energy principle for a system of particles* from the *conservation of energy principle for the system of particles* we have the following:

$$\left[\frac{d}{dt}(U_{sys} + E_{K,\, sys} + E_{G,\, sys}) = \dot{Q}_{net,\, in} + \dot{W}_{net,\, in}\right] - \left[\frac{d}{dt}(E_{K,\, sys} + E_{G,\, sys}) = \sum_{i=1}^{n} \dot{W}_{mech,\, in,\, i}\right] =$$

$$\underbrace{\frac{dU_{sys}}{dt}}_{\substack{\text{Time rate of change}\\ \text{of the internal energy}\\ \text{within our system}}} = \underbrace{\dot{Q}_{net,\, in}}_{\substack{\text{Heat transfer rate}\\ \text{for the system}}} + \underbrace{\dot{W}_{net,\, in}}_{\substack{\text{Rate that work}\\ \text{is done by surface}\\ \text{forces acting on the}\\ \text{boundary of our system}}} - \underbrace{\sum_{i=1}^{n} \dot{W}_{mech,\, in,\, i}}_{\substack{\text{Rate that work}\\ \text{is done by surface}\\ \text{forces acting on each}\\ \text{particle inside our system}}}$$

As with a single particle, the net work done on our system of particles by the surface forces acting on the system boundary only equals the sum of the mechanical work done by the surface forces acting on each of the particles under two conditions:

- Condition (1): The rate of change of the internal energy of the system of particles equals the net transport rate of energy into the system of particles by heat transfer, $dU_{\text{sys}}/dt = \dot{Q}_{\text{net, in}}$, or
- Condition (2): The internal energy of the system of particles is constant, $dU_{\text{sys}}/dt = 0$, and the particle is adiabatic, $\dot{Q}_{\text{net, in}} = 0$.

To validate condition (1) requires additional knowledge about the material substance of the particles. Typically, condition (2) occurs when the closed system of particles has three characteristics: (1) each particle in the system is incompressible, (2) there is no purposeful heat transfer, and (3) there is no friction of any kind within the system, e.g. particles can only deform elastically and there is no friction between particles. Friction on the system boundary is not prohibited; however, there can be no friction inside the system.

Counting Mechanical Energy for a Closed System

Starting with the rate-form of the conservation of energy for a closed system,

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,\, in} + \dot{W}_{net,\, in}$$

Expanding the system energy and identifying mechanical forms of energy, we get

$$\frac{d}{dt}\left(U + \underbrace{E_k + E_G + E_{spring}}_{\text{mechanical}}\right) = \dot{Q}_{\text{net, \textbackslash in}} + \dot{W}_{\text{net, \textbackslash in}}$$

Because, the mechanical forms are extensive properties, we may rearrange this equation in light of an accounting principle for mechanical energy:

$$\frac{d}{dt}\underbrace{\left(E_k+E_G+E_{spring}\right)}_{\substack{\text{Rate of change of the}\\ \text{mechanical energy in}\\ \text{the system}}} = \underbrace{\dot{W}_{net,\ in}}_{\substack{\text{Transport rate}\\ \text{of energy}\\ \text{by work}}} + \underbrace{\left[\dot{Q}_{net,\ in}-\frac{dU}{dt}\right]}_{\substack{\text{Rate of production}\\ \text{of mechanical energy}\\ \text{within the system}}}$$

The left-hand side can now be interpreted as the rate of change of the mechanical energy in the system. The right-hand side contains two terms: the net work-transport rate of energy into the system and the production rate of mechanical energy within the system. This does not violate the general conservation of energy principle because we are only counting one form of energy.

Experience has shown that the term in brackets on the right-hand side, the production term, can be either positive or negative. This term is identically equal to zero if our closed system is (1) composed of incompressible substances, (2) operates without intentional heat transfer, and (3) has no friction internal to our system and we have the *mechanical energy balance for a closed system:*

$$\boxed{\frac{d}{dt}\left(E_k+E_G+E_{spring}\right)=\dot{W}_{net,\ in}}$$

If we allow internal friction, mechanical energy can be destroyed, and it can be shown that the rate of production of mechanical energy is *always* less than or equal to zero. This means that internal friction always results in the destruction of mechanical energy.

7.3: Conservation of Energy

The recommended starting point for any application of the conservation of energy is the rate form of the conservation of energy equation:

$$\frac{dE_{\text{sys}}}{dt} = \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} + \left[\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e\right)\right]$$

Remember, the only restriction built into this equation is that the mass crossing the boundary of the system can only carry with it internal energy, translational kinetic energy, and gravitational potential energy. If a more general form is required, we need only add " $+e_{\text{other}}$ " after the specific gravitational potential energy term, gz.

In applying this equation to describe the behavior of a system, there are several modeling assumptions that are commonly used. These are described in detail in the following paragraphs. As always, you should focus on understanding the physics underlying the assumption and how they are used. Do not just memorize the simplified equations.

Steady-state system

If a system is operating under steady-state conditions, all intensive properties and interactions are independent of time. Thus, the energy of the system is a constant, $E_{\text{sys}} = \text{constant}$. When applied to the conservation of energy equation you have

$$\begin{aligned} \underbrace{\cancelto{0}{\frac{dE_{\text{sys}}}{dt}}}_{E_{\text{sys}}=\text{constant}} &= \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} + \left[\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e\right)\right] \\ 0 &= \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} + \left[\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e\right)\right] \end{aligned}$$

In words, the sum of the net transport rates of energy by work, heat transfer, and mass must equal zero.

Closed system

A closed system has no mass flow across its boundary. With this constraint, the conservation of energy equation simplifies as follows:

$$\begin{aligned} \frac{dE_{\text{sys}}}{dt} &= \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} + \underbrace{\cancelto{0}{\left[\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e\right)\right]}}_{\text{No mass flow rates}} \\ \frac{dE_{\text{sys}}}{dt} &= \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} \end{aligned}$$

Finite-time, closed system

For a closed system over a finite-time interval, you first apply the closed system assumption and then integrate the equation over the specified time interval:

$$\frac{dE_{\text{sys}}}{dt} = \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} + \cancelto{=0}{\left[\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e\right)\right]}$$

$$\frac{dE_{\text{sys}}}{dt} = \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}}$$

$$\int_{t_1}^{t_2} \left(\frac{dE_{\text{sys}}}{dt}\right) dt = \int_{t_1}^{t_2} \left(\dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}}\right) dt$$

$$\int_1^2 dE_{\text{sys}} = \int_1^2 \left(\delta W_{\text{net, in}} + \delta Q_{\text{net, in}}\right) \quad \rightarrow \quad \Delta E_{\text{sys}} = W_{\text{net, in}} + Q_{\text{net, in}}$$

In words, this says the change in the energy of the system equals the net transport of energy into the system by work and by heat transfer.

Assumptions about heat transfer and work:

One of the aspects of applying the conservation of energy that frequently puzzles students is the need to say something about the heat transfer and work interactions for the system we are modeling.

Heat transfer:

In this course, we will usually make one of three assumptions about the heat transfer of energy for a system:

1. There is no heat transfer. This is called an *adiabatic process* or *adiabatic system*. Physically, applying thermal insulation to the surface approximates an adiabatic surface. Unfortunately, there are no perfect thermal insulators; however, if the time scale of the process is small relative to the time it takes for the heat transfer of energy to occur, then the adiabatic assumption is usually good.
2. The heat transfer Q or heat transfer rate $\dot{Q}$ is the unknown we are solving for in the problem.
3. The heat transfer Q or heat transfer rate $\dot{Q}$ is given in the problem statement.

Please remember that heat transfer can only be defined with respect to a boundary. If you move the boundary, you change the heat transfer. Without clearly indicating your system boundary, it is impossible to apply any of these assumptions.

The study of heat transfer as a discipline attempts to relate the heat transfer rate at a boundary to other characteristics of the system, such as the thermal conductivity, the convection heat transfer coefficient, and the temperature difference across the boundary. In some problems, you may be given a specific constitutive relation that allows you to calculate the heat transfer rate without using the conservation of energy, similar to our work equations. In all other cases, assume that heat transfer can be modeled using one of the three assumptions listed above.

Work:

In this course we will focus most of our attention on four of the possible work mechanisms. The key to making the correct assumption is to carefully examine the system you select and identify any interactions that look like work. (Physically, imagine walking around the system and looking for any clues that would lead you to believe one of these mechanisms is present.) Remember work is only defined with respect to a boundary. No boundary, no work! Here are some clues for each of the four work mechanisms:

1. Compression-expansion (PdV) work — see if any system boundary is moving in a direction normal to the surface, e.g. a boundary next to a piston.
2. Electric work — see if your system boundary cuts any electrical wires.
3. Shaft work — look to see if your system boundary cuts any rotating shafts.
4. Mechanical work and power — look to see if there are any other forces that move on the boundary of the system.

When we revisit work later we may identify a few more mechanisms; however, this list will suffice for a wide range of important problems.

Assumptions about the substance:: Another new and sometimes puzzling problem students encounter when applying the conservation of energy equation is the need to evaluate thermophysical properties — u, h, s, T, P, ρ, and v. This requires empirical knowledge about the behavior of the material within the system. This knowledge represents a constitutive equation that allows us to predict the values of the properties once we have identified the state of the substance. We will delay this complication for a while, but shortly we will introduce two substance models that accurately describe the behavior of gases, liquids, and solids under certain conditions.

Example — Motor startup

(adapted from Moran & Shapiro, *Thermodynamics*)

Under steady-state operating conditions, the shaft of a motor rotates at a constant speed of $955\ \mathrm{rpm}\ (100\ \mathrm{rad/s})$ and applies a constant torque of $18\ \mathrm{N\cdot m}$ to an external load, and the 110-volt motor draws a constant electric current of 18.2 amps.

Figure $7.3.1$: An electrical motor turns a shaft.

(a) Determine the magnitude and direction of the steady-state heat transfer rate for the motor, in kW.

(b) During the start-up transient, the rate of heat transfer between the electric motor and its surrounding varies with time as follows:

$$\dot{Q}_{\text{out}} = \dot{Q}_{\text{out, ss}} \left[1 - e^{-t/(20\ \mathrm{s})}\right]$$

where $Q_{\text{out, ss}}$ is the steady-state heat transfer rate from the motor. Obtain an expression for the time rate of change of the energy of the motor using your result from Part (a) and the heat transfer rate equation above.

Solution

Known: A motor operates with known operating conditions

Find: (a) Steady-state heat transfer rate from the motor, in kW.
(b) Time-rate of change of the motor energy during the startup transient.

Given: During the transient startup: $\dot{Q}_{\text{out}} = \dot{Q}_{\text{out, ss}} \left[1 - e^{-t/(20\ \mathrm{s})}\right]$

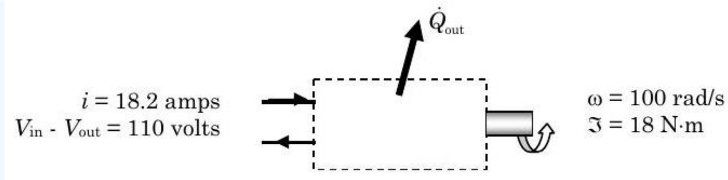


Figure 7.3.2: Known transfers of heat and work between the system of the motor and its surroundings.

Analysis:

Strategy $\rightarrow$ Because the question involves energy and heat transfer, try the conservation of energy.
System $\rightarrow$ Take the motor as a closed system
Property to count $\rightarrow$ Energy
Time period $\rightarrow$ Both parts appear to require the rate form (infinitesimal time interval)

Sketching a system and identifying the energy transports, we have the figure below.

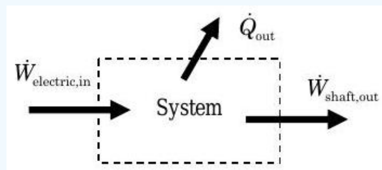


Figure 7.3.3: System with identified energy transports in and out.

Now applying the closed-system form of the rate form of the conservation of energy equation, we have:

$$\frac{dE_{sys}}{dt} = \dot{W}_{\text{net, in}} + \dot{Q}_{\text{net, in}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} - \dot{Q}_{\text{out}}$$

where we have used the subscripts to indicate the positive directions corresponding to our diagram

For part (a) we are asked to consider the steady state heat transfer rate, therefore

$$\underbrace{\frac{dE_{sys}}{dt}}_{\text{Steady state}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} - \dot{Q}_{\text{out}} \quad \rightarrow \quad \dot{Q}_{\text{out}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}}$$

Now we have to determine the electric power and the shaft power using their defining equations:

$$\begin{aligned} \dot{W}_{\text{electric, in}} &= i \cdot (V_{\text{in}-\text{o}} - V_{\text{out}-\text{o}}) \\ &= (18.2\text{ A}) \cdot (110\text{ V}) \\ &= 2.00 \times 10^3\text{ W} \\ &= 2.00\text{ kW} \end{aligned} \qquad \begin{aligned} \dot{W}_{\text{shaft, out}} &= \tau \cdot \omega \\ &= (18.0\text{ N} \cdot \text{m}) \cdot (100\text{ rad/s}) \\ &= 1.80 \times 10^3\text{ W} \\ &= 1.80\text{ kW} \end{aligned}$$

Substituting these results back into the steady-state energy balance, we have

$$\dot{Q}_{\text{out}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} = (2.00 - 1.80)\text{ kW} = 0.20\text{ kW}$$

Now for part (b), we are interested in rate of energy storage in the motor, not the amount of energy stored in the motor. Starting with the rate form of the energy balance we have

$$\frac{dE_{sys}}{dt} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} - \dot{Q}_{\text{out}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} - \dot{Q}_{\text{out, ss}}\left[1 - e^{-t/(20\text{ s})}\right]$$

after substituting in the given information for the heat transfer rate.

Assuming that the shaft and electric powers are constant during this transient will give the following result:

$$\begin{aligned} \frac{dE_{sys}}{dt} &= \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} - \dot{Q}_{\text{out, ss}}\left[1 - e^{-t/(20\text{ s})}\right] \\ &= (2.00\text{ kW}) - (1.800\text{ kW}) - (0.20\text{ kW})\left[1 - e^{-t/(20\text{ s})}\right] = (0.20\text{ kW})e^{-t/(20\text{ s})} \end{aligned}$$

Thus the time rate of change is greatest at $t = 0$ and then decreases exponentially.

Comments:

(1) The assumption that the electric power and the shaft power are constant during this startup transient is probably incorrect. A more accurate model of the motor would require a motor performance curve that shows torque and electric current as a function of rotational speed. Typically, when a motor starts up the current surges to provide the necessary startup torque. A portion of the system energy would be stored in the rotational kinetic energy of the rotor as it spins up.

(2) How does the energy inside the system as a function of time? To do this you must integrate the rate of change with time. What's the maximum change? Is there one? Try it and see what you get. Answer: $\Delta E = (4.0 \text{ kJ}) \left[1 - e^{-t/(20 \text{ s})}\right]$

Example — Refrigerant Compressor

A refrigeration system includes a compressor that takes in refrigerant at State 1 and discharges the refrigerant at State 2. Available state information is shown in the figure. The power input to the compressor is 2.2 kW. The mass flow rate of refrigerant is 0.014 kg/s.

Determine (a) the direction and magnitude of the heat transfer rate and (b) the shaft torque if the compressor power is supplied as shaft work and the compressor operates at 600rpm.

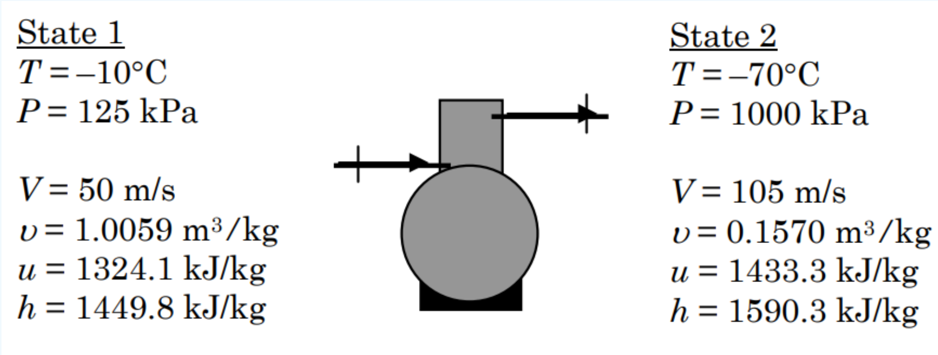


Figure $7.3.4$: System of refrigerant compressor, with all known information about the refrigerant's entering and exiting states.

Solution

Known: A compressor operates at steady-state conditions.

Find: Determine the steady-state heat transfer rate and shaft torque for the compressor.

Given: State, power, and shaft information as included in the problem statement above. [Students, this information is not repeated here because it is so clearly stated above; however, if you were preparing an engineering solution for the record, you should use this space to document all of the information and symbols gleaned from the problem.]

Analysis:

Strategy $\rightarrow$ Again, since we are talking about energy, try the conservation of energy and mass.
System $\rightarrow$ Treat the compressor as a non-deforming open system with one inlet and one outlet
Property $\rightarrow$ Energy and mass (since it is an open system)
Time interval $\rightarrow$ Infinitesimal time interval, rate form

Starting with a system diagram shown below, we have a heat transfer rate in, a shaft power in, and two places where mass crosses the boundary of the system.

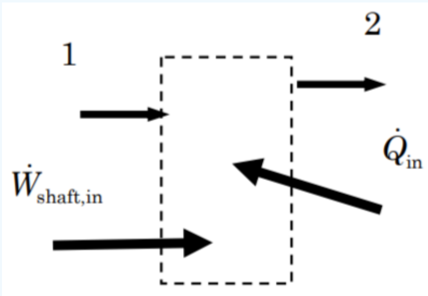


Figure $7.3.5$: Mass and energy transfers into/out of the compressor system.

Writing the rate form of the energy balance and the mass balance for this system we have the following:

$$\text{Energy:} \quad \underbrace{\cancelto{=0}{\frac{dE_{\text{sys}}}{dt}}}_{\text{Steady state}} = \dot{W}_{\text{shaft, in}} + \dot{Q}_{\text{in}} + \dot{m}_1\left(h_1 + \frac{V_1{}^2}{2} + gz_1\right) - \dot{m}_2\left(h_2 + \frac{V_2{}^2}{2} + gz_2\right)$$

$$\text{Mass:} \quad \underbrace{\cancelto{=0}{\frac{dm_{\text{sys}}}{dt}}}_{\text{Steady state}} = \dot{m}_1 - \dot{m}_2 \quad \rightarrow \quad \dot{m}_1 = \dot{m}_2 = \dot{m}$$

Combining these results and solving for the heat transfer rate we have the following:

$$0 = \dot{W}_{\text{shaft, in}} + \dot{Q}_{\text{in}} + \dot{m}_{\text{in}}\left[(h_1 - h_2) + \left(\frac{V_1{}^2}{2} - \frac{V_2{}^2}{2}\right) + \underbrace{\cancelto{=0}{g\,(z_1 - z_2)}}_{\substack{\text{No information about}\\ \text{change in elevation given.}\\ \text{Assume this is negligible.}}}\right]$$

$$\dot{Q}_{\text{in}} = -\dot{W}_{\text{shaft, in}} - \dot{m}_{\text{in}}\left[(h_1 - h_2) + \left(\frac{V_1{}^2}{2} - \frac{V_2{}^2}{2}\right)\right]$$

where we have explicitly recognized that we know nothing about the change in elevation and thus have assumed it will be insignificant. (We have not forgotten it. We have consciously made a modeling assumption.)

(a) Now to solve for the heat transfer we must substitute information back into the energy balance:

$$\begin{aligned}
\dot{Q}_{\text{in}} &= -(2.2\ \text{kW}) - \left(0.014\,\frac{\text{kg}}{\text{s}}\right)\left[(1449.8 - 1590.3)\,\frac{\text{kJ}}{\text{kg}} + \left(\frac{50^2 - 105^2}{2}\right)\left(\frac{\text{m}}{\text{s}}\right)^2\right] \\
&= -(2.2\ \text{kW}) - \left(0.014\ \frac{\text{kg}}{\text{s}}\right)\left[(-140.5)\,\frac{\text{kJ}}{\text{kg}} + (-4262.5)\,\frac{\text{m}^2}{\text{s}^2} \times \left(\frac{1\ \text{kJ}}{1000\ \text{N}\cdot\text{m}}\right) \times \left(\frac{1\ \text{N}}{\frac{\text{kg}\cdot\text{m}}{\text{s}^2}}\right)\right] \\
&= -(2.2\ \text{kW}) - \underbrace{\left(0.014\ \frac{\text{kg}}{\text{s}}\right)\left[(-140.5)\,\frac{\text{kJ}}{\text{kg}} + (-4.2625)\,\frac{\text{kJ}}{\text{kg}}\right]}_{-2.027\ \text{kW}} \\
&= -0.173\ \text{kW}
\end{aligned}$$

Thus the heat transfer rate for the compressor is $0.173\ \text{kW}$ out of the system. Typically, you might say "the compressor loses energy by heat transfer at the rate of $0.173\ \text{kW}$."

(b) Now to find the shaft torque we have to look at the shaft power and apply the definition of shaft power:

$$\begin{aligned}
\dot{W}_{\text{shaft, in}} = \tau\cdot\omega \quad \rightarrow \quad \tau &= \frac{\dot{W}_{\text{shaft, in}}}{\omega} = \frac{2.2\ \text{kW}}{600\ \text{rpm}} \times \frac{\left(\frac{1\ \text{kN}\cdot\text{m}}{\text{s}\cdot\text{kJ}}\right)}{\left(\frac{\text{rev/min}}{\text{rpm}}\right)\times\left(\frac{2\pi\ \text{rad}}{\text{rev}}\right)} \\
&= \frac{\left(2.2\ \frac{\text{kN}\cdot\text{m}}{\text{s}}\right)}{\left(1200\pi\ \frac{\text{rad}}{\text{min}}\right)} \times \left(\frac{60\ \text{s}}{\text{min}}\right) = 0.0350\ \text{kN}\cdot\text{m} = 35.0\ \text{N}\cdot\text{m}
\end{aligned}$$

The shaft torque applied to the compressor will have the same sense as the direction of shaft rotation.

Comment

(1) Notice how we have explicitly indicated our assumptions as we moved from the most general form of the conservation equations to the specific form of the equation used to model this system.

(2) Typically, the only properties at state 1 and 2 that we could have measured would be P, T, and V. All of the other properties would have been found from tables or equations that relate u, h, and v to P and T.

Example — Steam separator

Most steam power plants have a device that separates steam (gaseous water) from liquid water before it enters the steam turbine. The figure below shows one example. Experience has shown that droplets of liquid water, even small ones, can significantly erode the blades in a steam turbine.

A mixture of liquid water and steam enters a separator vessel at 1 with a mass flow rate of $10,000 \text{ lbm/h}$. Steam exits the vessel at 2 and liquid water exits the vessel at 3. The separator operates adiabatically at steady-state conditions with negligible changes in kinetic and gravitational potential energy. Measurements on the vessel indicate that when the system is operating at 2000 psia, the specific enthalpies and specific volumes of the three streams are as follows:

$$\begin{aligned} &h_1 = 787.7 \text{ Btu/lbm}; \quad h_2 = 1136.1 \text{ Btu/lbm}; \quad h_3 = 671.6 \text{ Btu/lbm}; \\ &v_1 = 0.0662 \text{ ft}^3/\text{lbm}; \quad \text{v}_2 = 0.1881 \text{ ft}^3/\text{lbm}; \quad \text{v}_3 = 0.02563 \text{ ft}^3/\text{lbm} \end{aligned}$$

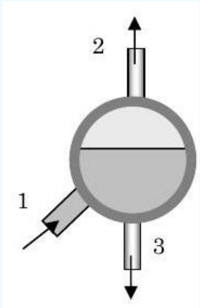


Figure $7.3.6$: Steam separator with three openings.

Determine (a) the mass flow rates at 2 and 3, and (b) the flow areas at 1 and 3 assuming the velocity is 15 ft/s.

Solution

Known: A steam separator operates at adiabatic, steady-state conditions.

Find: (a) Mass flow rates leaving the vessel
(b) Fluid velocities at all flow cross-sections assuming a 15 ft/s velocity.

Given: See figure above.

Analysis:

Strategy $\rightarrow$ Try conservation of mass.
System $\rightarrow$ Take the vessel as a non-deforming open system.
Property to count $\rightarrow$ Mass
Time interval $\rightarrow$ Infinitesimal interval, rate form

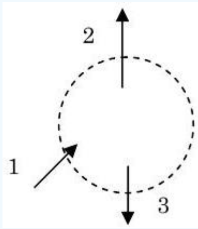


Figure $7.3.7$: System boundary and the mass flows across it.

Sketching the system diagram, we have three mass flows crossing the system boundary as shown in the sketch. Now writing the equations for conservation of mass, we have the following:

$$\text{Mass:} \quad \underbrace{\cancel{\frac{dm_{\text{sys}}}{dt}}^{=0}}_{\text{Steady-state}} = \dot{m}_1 - \dot{m}_2 - \dot{m}_3 \quad \rightarrow \quad \dot{m}_3 = \dot{m}_1 - \dot{m}_2$$

Unfortunately there are two unknowns, so we need another equation. (Notice that our strategy is being revised as we work, since we didn't notice that conservation of mass would give us one equation with two unknowns.) To get another equation, apply the conservation of energy to this system:

$$\text{Energy:} \quad \underbrace{\cancelto{=0}{\frac{dE_{\text{sys}}}{dt}}}_{\text{Steady state}} = \underbrace{\cancelto{=0}{\dot{Q}_{\text{net, in}}}}_{\text{Adiabatic}} + \underbrace{\cancelto{=0}{\dot{W}_{\text{net, in}}}}_{\substack{\text{Nothing on boundary} \\ \text{looks like power}}} + \underbrace{\dot{m}_1 h_1 - \dot{m}_2 h_2 - \dot{m}_3 h_3}_{\substack{\text{Neglecting kinetic and potential} \\ \text{energy per the problem statement}}} \rightarrow$$

$$0 = \dot{m}_1 h_1 - \dot{m}_2 h_2 - \dot{m}_3 h_3$$

The energy equation has the same two unknowns — the mass flow rates at 2 and 3. Substituting in the values from the problem statement and solving simultaneously gives the following:

$$\left. \begin{aligned} \dot{m}_3 &= \left(10000 \, \frac{\text{lbm}}{\text{h}}\right) - \dot{m}_2 \\ 0 &= \left(10000 \, \frac{\text{lbm}}{\text{h}}\right)\left(778.7 \, \frac{\text{Btu}}{\text{lbm}}\right) - \dot{m}_2 \left(1136.1 \, \frac{\text{Btu}}{\text{lbm}}\right) - \dot{m}_3 \left(671.6 \, \frac{\text{Btu}}{\text{lbm}}\right) \end{aligned} \right\} \rightarrow$$

$$\begin{cases} \dot{m}_2 = 2.50 \times 10^3 \text{ lbm/h} \\ \dot{m}_3 = 7.50 \times 10^3 \text{ lbm/h} \end{cases}$$

So roughly 25% of the entering water leaves the vessel as steam and 75% leaves the vessel as liquid water.

Now to find the cross-sectional areas if the velocity is 15 ft/s, we can make use of the definition of mass flow rate as follows:

$$\dot{m} = \rho V A_c = \frac{V A_c}{v} \rightarrow A_c = \frac{\dot{m} v}{V} = \frac{\dot{m} v}{\left(15 \, \frac{\text{ft}}{\text{s}} \times \frac{3600 \text{ s}}{\text{h}}\right)} = \frac{\dot{m} v}{\left(54000 \, \frac{\text{ft}}{\text{h}}\right)}$$

$$\text{At 1:} \quad A_{c,1} = \frac{\left(10,000 \, \frac{\text{lbm}}{\text{h}}\right)\left(0.0662 \, \frac{\text{ft}^3}{\text{lbm}}\right)}{\left(54000 \, \frac{\text{ft}}{\text{h}}\right)} = 12.3 \times 10^{-3} \text{ ft}^2 = 1.77 \text{ in}^2$$

$$\text{At 3:} \quad A_{c,3} = \frac{\left(7500 \, \frac{\text{lbm}}{\text{h}}\right)\left(0.02563 \, \frac{\text{ft}^3}{\text{lbm}}\right)}{\left(54000 \, \frac{\text{ft}}{\text{h}}\right)} = 3.56 \times 10^{-3} \text{ ft}^2 = 0.513 \text{ in}^2$$

Notice how the area at 3 is approximately 30% of the area at 1, even though the mass flow rate is only 77% of the mass flow rate at 1. This is the result of the changes in fluid specific volume.

Comments

(1) Notice how in this problem we were forced to use both the energy and the mass balances to get sufficient equations for solving the problem. Often our initial strategy will be incorrect. A hallmark of a good problem solved is the ability to not get locked into a single approach. Be flexible.

(2) If the entering mass flow rate had not been given, we could still have solved for the mass flow split in the separator. To do this, we divide both the mass and energy equations through by one of the three unknown flow rates, say the mass flow rate at 1 :

$$\begin{aligned} 0 &= \dot{m}_1 - \dot{m}_2 - \dot{m}_3 \\ 0 &= \dot{m}_1 h_1 - \dot{m}_2 h_2 - \dot{m}_3 h_3 \end{aligned} \quad \rightarrow \quad \begin{aligned} 0 &= 1 - \frac{\dot{m}_2}{\dot{m}_1} - \frac{\dot{m}_3}{\dot{m}_1} \\ 0 &= h_1 - \left(\frac{\dot{m}_2}{\dot{m}_1}\right) h_2 - \left(\frac{\dot{m}_3}{\dot{m}_1}\right) h_3 \end{aligned}$$

In this way, we've gone from three unknowns to two unknowns and now have sufficient equations to solve for the split. Open system problems are often solved on a "per unit mass" basis by dividing everything in the equation by a mass flow rate and eliminating one unknown.

✔ Example — Making the rounds

A gas is contained inside of a simple piston cylinder device and initially occupies a volume of 0.020 m and at a pressure of 1.0 MPa. The gas executes three processes in series and returns to its initial state as described below:

State 1: $P_1 = 1.0 \text{ MPa}, \forall_1 = 0.020 \text{ m}^3$

Process 1 $\rightarrow$ 2: Polytropic expansion with $P\forall^{1.4} = C$

State 2: $\forall_2 = 2\forall_1$

Process 2 $\rightarrow$ 3: Constant volume heating

State 3: $P_3 = P_1, \quad V_3 = V_2$

Process 3: $\rightarrow$ 1: Constant pressure compression

(This is called a *thermodynamic cycle* because the system executes a series of processes and then returns to its initial state.) Assume that changes in kinetic and gravitational potential energy are negligible for all processes.

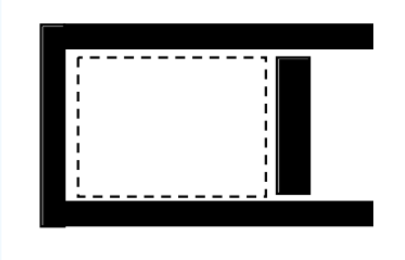


Figure $7.3.8$: System consisting of the gas inside a cylinder piston device.

Determine (a) the work done on the gas inside the piston for each process; (b) the net work for the entire cycle, i.e. sum of the work for each process in the cycle; (c) the heat transfer for the entire cycle.

Solution

Known: A gas contained in a piston-cylinder device executes a three-process cycle.

Find: (a) work done on the gas for each process
(b) net work done on the gas for the cycle.
(c) net heat transfer for the entire cycle.

Given: See figure and state/process information above.

Analysis:

Strategy $\rightarrow$ Should require the use of conservation of energy and may be able to use the definition of PdV work to evaluate the work for at least some of the processes.
System $\rightarrow$ Closed, deforming system consisting of the gas in the cylinder (see dashed line)
Property to count $\rightarrow$ Energy
Time interval $\rightarrow$ Should be finite time since each process has a definite beginning and ending.

(a) The only type of work that is possible for this system is compression-expansion (PdV) work. Assuming that each process occurs slow enough so that the pressure is uniform inside the gas throughout the process we have the following equation:

$$W_{\mathrm{PdV,\,in}} = -\int_1^2 P\, dV$$

The trick then is to evaluate it for each process as required.

Process $1 \rightarrow 2$: Polytropic process with $PV^{1.4} = C$. Integrating subject to this constraint, we have

$$\begin{aligned} W_{1-2,\,\mathrm{in}} &= -\int_1^2 P\, dV = -\int_{V_1}^{V_2} \frac{C}{V^{1.4}}\, dV = -C \int_{V_1}^{V_2} V^{1.4}\, dV = -C\left[\frac{1}{-1.4+1} V^{(-1.4+1)}\right]_{V_1}^{V_2} = \frac{C}{1.4-1}\left[V_2{}^{-0.4} - V_1{}^{-0.4}\right] \\ &= \frac{\left(P_1 V_1{}^{1.4}\right)}{0.4}\left(V_1{}^{-0.4}\right)\left[\left(\frac{V_2}{V_1}\right)^{-0.4} - 1\right] = 2.5\left(P_1 V_1\right)\left[\left(\frac{V_2}{V_1}\right)^{-0.4} - 1\right] \\ &= 2.5\left[(1000\ \mathrm{kPa})\left(0.020\ \mathrm{m}^3\right)\right]\left[\left(\frac{2}{1}\right)^{-0.4} - 1\right] = \left(-12.1\ \mathrm{kPa}\cdot\mathrm{m}^3\right) \times \left(\frac{\mathrm{N/m}^3}{\mathrm{Pa}}\right) = -12.1\ \mathrm{kN}\cdot\mathrm{m} \end{aligned}$$

Process $2 \rightarrow 3$: Constant volume heating

Since there is no volume change and PdV work is the only kind possible here, $W_{2-3,\,\mathrm{in}} = 0$.

Process $3 \rightarrow$ 1: Constant pressure cooling

$$W_{3-1,\,\mathrm{in}} = -\int_3^1 P\, dV = -P_3\left(V_1 - V_3\right) = -P_3\left(V_1 - V_2\right) \quad \text{because } V_3 = V_2$$

But we already know that for State 2 $V_2 = 2V_1$. Combining this give us the work for process $3 \rightarrow 1$:

$$W_{3-1,\text{ in}} = -P_3\left(V_1 - V_2\right) = -(1000\text{ kPa})(0.020 - 0.040)\text{ m}^3 = 20.0\text{ kN}\cdot\text{m}$$

(b) To find the net work for the cycle we just add up the three work terms:

$$W_{\text{cycle, net in}} = W_{1-2,\text{ in}} + W_{2-3,\text{ in}} + W_{3-1,\text{ in}} = [(-12.1) + 0 + 20.0]\text{ kN}\cdot\text{m} = 7.9\text{ kN}\cdot\text{m} = 7.9\text{ kJ}$$

(c) To find the net heat transfer of energy, we will resort to the finite-time energy balance for a closed system

$$\begin{array}{lll} & \text{Process } 1 \rightarrow 2: & \Delta E = E_2 - E_1 = Q_{1-2,\text{ in}} + W_{1-2,\text{ in}} \\ & \text{Process } 2 \rightarrow 3: & \Delta E = E_3 - E_2 = Q_{2-3,\text{ in}} + W_{2-3,\text{ in}} \\ + & \text{Process } 3 \rightarrow 1: & \Delta E = E_1 - E_3 = Q_{3-1,\text{ in}} + W_{3-1,\text{ in}} \\ \hline \end{array}$$
$$\underbrace{(E_2 - E_1) + (E_3 - E_2) + (E_1 - E_3)}_{=0} = \underbrace{(Q_{1-2,\text{ in}} + Q_{2-3,\text{ in}} + Q_{3-1,\text{ in}})}_{Q_{\text{cycle, net in}}} + \underbrace{(W_{1-2,\text{ in}} + W_{2-3,\text{ in}} + W_{3-1,\text{ in}})}_{W_{\text{cycle, net in}}}$$

Thus we have $0 = Q_{\text{cycle, net in}} + W_{\text{cycle, net in}}$

And for the net heat transfer rate for the cycle we have $Q_{\text{cycle, net in}} = -W_{\text{cycle, net in}} = 7.9\text{ kJ}$

Comment

The figure below shows the various areas that correspond to the work for the various processes. The rectangle under line 3-1 represents the work for Process $3 \rightarrow 1$ and the area under the curve 1-2 represents the work for Process $1 \rightarrow 2$. The area enclosed inside the closed curve 1-2-3 represents the net work for the cycle. If we had reversed the direction of the cycle, how would the values of Q and W change for the cycle?

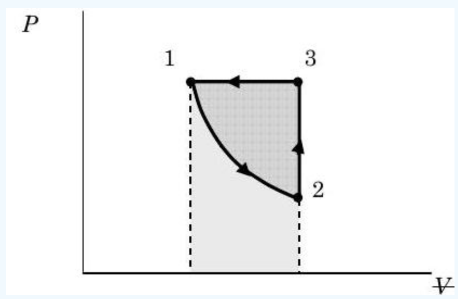


Figure $7.3.9$: Graph of pressure against volume for the gas as it undergoes the three processes.

Example — They say there's resistance

A 12-volt auto battery is connected to a $100\text{ k}\Omega$ (100 kilo-ohm) resistor. Assume changes in kinetic and gravitational potential energy are negligible and that the battery voltage and current do not change with time for the period of this problem. Measurements indicate that the heat transfer rate from the battery is approximately 2% of the electric power it delivers.

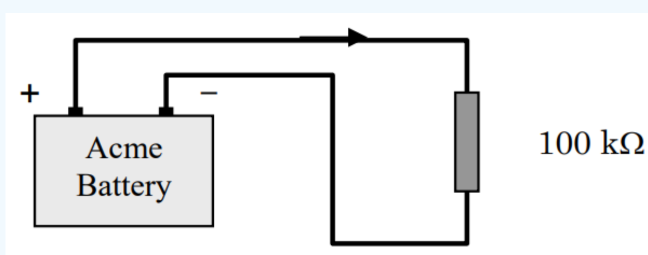


Figure $7.3.10$: Battery connected to a 100 kilo-Ohm resistor.

Determine: (a) the rate of change of the internal energy of the battery, in J/s; (b) the heat transfer rate for the resistor, in watts.

Solution

Known: A 12 -volt car battery is connected to a $100 - \text{k}\Omega$ resistor

Find: (a) Rate of change of internal energy of the battery, in kJ/s.
(b) Heat transfer rate for the resistor, in kW.

For the battery:

$$\dot{Q}_{\text{battery, out}} = (0.02) \cdot \dot{W}_{\text{battery, out}}$$

$$V^{+} - V^{-} = 12 \text{ volts}$$

For the resistor: $R = 100 \text{ k}\Omega$

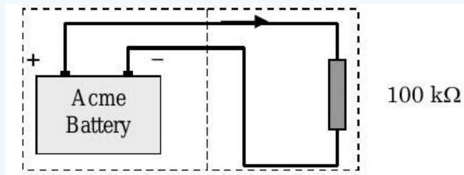


Figure 7.3.11: Separating the circuit into systems for analysis.

Analysis:

Strategy $\rightarrow$ Because we are interested in internal energy changes and heat transfer, try conservation of energy.
System $\rightarrow$ May need to use both battery and resistor.
Property to count $\rightarrow$ Energy.
Time interval $\rightarrow$ Infinitesimal time interval, rate form of equations.

Before we can solve for any other information, we will need to know the electric current flowing in the system. Assuming that the resistor obeys Ohm's Law, then

$$\Delta V = iR \quad \rightarrow \quad i = \frac{\Delta V}{R} = \frac{(12 \text{ V})}{(100 \times 10^{3} \ \Omega)} \times \left(\frac{1 \text{ A} \cdot \Omega}{\text{V}}\right) = 12 \times 10^{-5} \text{ A} = 0.12 \text{ mA}$$

Now to answer part (a), let's consider a system that includes the battery and some of the wires as shown above. Applying the energy balance to this closed system we have

$$\underbrace{\cancel{\frac{dE_{\text{sys}}}{dt}}^{=U}}_{\substack{\text{Neglecting all} \\ \text{but internal energy}}} = -\dot{W}_{\text{out}} - \cancel{\dot{Q}_{\text{out}}}^{=0.02\dot{W}_{\text{out}}} \quad \rightarrow \quad \frac{dU_{\text{battery}}}{dt} = -\dot{W}_{\text{out}} - 0.02\dot{W}_{\text{out}} = -1.02 \cdot \dot{W}_{\text{out}}$$

To continue requires using our definition for electric power as follows:

$$\begin{aligned} \frac{dU_{\text{battery}}}{dt} &= (-1.02) \cdot \dot{W}_{\text{electric, out}} = (-1.02) \cdot (i\Delta V) \\ &= (-1.02) \cdot [(0.12 \text{ mA}) \cdot (12 \text{ volts})] = (-1.02) \cdot \underbrace{\left[1.44 \times 10^{-3} \text{ W}\right]}_{\dot{W}_{\text{battery, out}}} = -1.47 \times 10^{-3} \ \frac{\text{J}}{\text{s}} \end{aligned}$$

Notice that even though the battery has a constant voltage difference and a constant current, it is not a steady-state system. This should make physical sense as the battery is supplying energy to another system; thus, its internal energy should be decreasing.

For part (b) we have two choices at this point. We can either use a system surrounding the resistor or one that encompasses the battery, wires and resistor. Let's show both for comparison of two alternate approaches:

System = Resistor	System = Battery + Resistor + Wires

System = Resistor	System = Battery + Resistor + Wires
$\begin{aligned} \underbrace{\overset{=0}{\cancel{\frac{dE_{\text{sys}}}{dt}}}}_{\substack{\text{Assume} \\ \text{steady-state}}} &= \dot{Q}_{\text{Resistor, in}} + \dot{W}_{\text{Resistor, in}} \\ 0 &= \dot{Q}_{\text{Resistor, in}} + \dot{W}_{\text{Resistor, in}} \\ -\dot{Q}_{\text{Resistor, in}} &= \dot{W}_{\text{Resistor, in}} \\ &= \dot{W}_{\text{Battery, out}} = 1.44\ \text{mW} \\ \dot{Q}_{\text{Resistor, out}} &= -\dot{Q}_{\text{Resistor, in}} = 1.44\ \text{mW} \end{aligned}$	$\begin{aligned} \frac{dE_{\text{sys}}}{dt} &= \dot{Q}_{\text{net, in}} + \underbrace{\overset{=0}{\cancel{\dot{W}_{\text{net, in}}}}}_{\substack{\text{No work found} \\ \text{in this system}}} \\ \overset{=U}{\cancel{\frac{dE_{\text{Battery}}}{dt}}} + \underbrace{\overset{=0}{\cancel{\frac{dE_{\text{Wires}}}{dt}}} + \overset{=0}{\cancel{\frac{dE_{\text{Resistor}}}{dt}}}}_{\substack{\text{Assume no change in } E \\ \text{or steady state}}} &= \dot{Q}_{\text{Battery, in}} + \underbrace{\overset{=0}{\cancel{\dot{Q}_{\text{Wires, in}}}}}_{\substack{\text{Assumed} \\ \text{negligible}}} + \dot{Q}_{\text{R}} \\ \frac{dU_{\text{Battery}}}{dt} &= \dot{Q}_{\text{Battery, in}} + \dot{Q}_{\text{Resistor, in}} \\ \underbrace{\frac{dU_{\text{Battery}}}{dt} - \dot{Q}_{\text{Battery, in}}}_{\dot{W}_{\text{Battery, in}}} &= \dot{Q}_{\text{Resistor, in}} \\ \dot{Q}_{\text{Resistor, in}} &= \dot{W}_{\text{Battery, in}} = -1.44\ \text{mW} \end{aligned}$

Regardless of the system we select for our analysis, we should get the same answer if we are making consistent assumptions for both systems.

Comments:

(1) In this problem, we had to pick a couple of different systems to develop all of the information we needed to solve the problem. A hallmark of a good problem solver is the ability to look and use different systems as appropriate to solve a problem. Often you will discover that selecting one particular system leads to a very difficult solution or a solution with "risky" assumptions. In this case, you should look around and see if you can find a better system.

(2) Imagine that we had hooked the battery up to the resistor backwards, i.e. with current flowing in the opposite direction. How would the answers change? The flow of current through the resistor is an example of an *internally irreversible* process. We will discover shortly that irreversible and reversible processes play a key role in establishing important limits in our ability to transfer and convert energy.

7.4: Substance Models

In applying the conservation of energy equation to model a physical device, we must frequently evaluate changes in internal energy or enthalpy (en - thal '- py). These in turn are related to the directly measurable properties such as pressure, temperature, and specific volume or density. The study of how thermophysical properties are related is one of the major aims of the study of *thermodynamics*.

For our present study we will consider only two substance models. These models will be presented without significant development or explanation. This will be covered in a later course. Each substance model provides us with an *equation of state* for an important class of substances. The major results of both models are summarized in Section 7.4.3, and you are encouraged to skip ahead and get an overall picture before you begin the more detailed descriptions of each model.

7.4.1 Ideal Gas Model with Room-Temperature Specific Heats

The ideal gas model is familiar to most students from physics and chemistry; however, we will use it more extensively than most of you have done previously. The ideal gas model is just that — a set of constitutive equations that accurately models the behavior of gases and vapors under certain restricted conditions.

Basic Assumptions

The ideal gas model is built upon three assumptions:

1. Pressure, volume and temperature obey the ideal gas equation:

$$PV = NR_uT = mRT \quad \text{where } R = \frac{R_u}{M}$$

2. The specific internal energy only depends on temperature:

$$u = u(T)$$

3. The molar mass of an ideal gas is identical with the actual molar mass of the substance:

$$M_{\text{ideal gas}} = M_{\text{real stuff}}$$

In addition, we will add one additional assumption that will greatly simplify our introduction to using ideal gas properties.

4. The specific heats are independent of temperature. (In general, the specific heats do in fact change with temperature, and this variation will be considered later to increase the accuracy of the ideal gas model.) However, as a first approximation and a pretty accurate assumption for small temperature changes near room temperature, we will assume that the specific heats are constant and equal to the values at room temperature.

We'll refer to our ideal gas model that is based on all four assumptions as the "ideal gas model with *room-temperature* specific heats."

P-v-T Relationship

An ideal gas by definition satisfies the ideal gas equation, Eq. $7.4.1$. Typically we find it most useful to rearrange this equation so it only includes intensive properties:

$$P = \rho RT \quad \text{or} \quad Pv = RT$$

where R is the specific gas constant $(= R_u/M)$ with units of $\mathrm{kJ}/(\mathrm{kg}\cdot\mathrm{K})$ or $(\mathrm{ft}\cdot\mathrm{lbf})/(\mathrm{lbm}\cdot{}^\circ\mathrm{R})$. This equation was discussed extensively in Chapter 3 and you are encouraged to review this material.

Changes in u and h

The differential change in specific internal energy u for an ideal gas is calculated using the relation:

$$du = c_v\, dT \quad [\text{Ideal gas}]$$

where c_v is called "see-sub-vee" or the **specific heat at constant volume** (see note). To calculate the finite change in u for an ideal gas with constant specific heats, we integrate Eq. $7.4.5$ between state 1 and state 2 and obtain the equation:

$$\Delta u = u_2 - u_1 = c_v\,(T_2 - T_1) \quad \left[\begin{array}{c}\text{Ideal gas,}\\ \text{constant } c_v\end{array}\right]$$

This equation can be used to calculate Δu for any process. *It is not restricted to constant-volume processes*.

The differential change in specific enthalpy h for an ideal gas is calculated using the relation:

$$dh = c_p\, dT \quad [\text{Ideal gas}]$$

where c_p is called "see-sub-pee" or the **specific heat at constant pressure** (see note). To calculate the finite change in h for an ideal gas with constant specific heats, we integrate Eq. $7.4.7$ between state 1 and state 2 and obtain the equation:

$$\Delta h = h_2 - h_1 = c_P\left(T_2 - T_1\right) \quad \left[\begin{array}{c}\text{Ideal gas,}\\ \text{constant } c_p\end{array}\right]$$

This equation can be used to calculate Δh for any process. *It is not restricted to constant-pressure processes.*

Note—Specific heats

The specific heats c_P and c_v are defined mathematically as the following partial derivatives:

$$c_v \equiv \left(\frac{\partial u}{\partial T}\right)_v \quad \text{and} \quad c_P \equiv \left(\frac{\partial h}{\partial T}\right)_P$$

where the phrase "constant volume" or "constant pressure" refers specifically to what is held constant during the differentiation.

Specific heat relationships

The specific heats for an ideal gas are related to the specific gas constant by the following equation:

$$c_P - c_v = R \quad \text{[Ideal gas]}$$

Sometimes all we know (or remember for a gas is its specific heat ratio. The **specific heat ratio** is defined as

$$k \equiv \frac{c_P}{c_v}$$

Combining Eqs. $7.4.9$ and $7.4.10$ we have the following equations that relate the specific heats, the specific gas constant, and the ratio of specific heats.

$$c_v = \frac{1}{k-1}R \quad \text{and} \quad c_P = \frac{k}{k-1}R \quad \text{[Ideal gas]}$$

These relations are frequently useful because the ratio of specific heats is a strong function of the molecular structure of the gas, i.e. whether it is monatomic, etc., and a weaker function of its temperature.

7.4.2 Incompressible Substance Model with Room-Temperature Specific Heats

The incompressible substance model is not as familiar as the ideal gas model. This model is based on observations of how liquids and solids behave. We will assume that this is an accurate model for most liquids and solids. It does not incorporate energy changes due to elastic strain of solids.

Basic Assumptions

The incompressible substance model is built upon three assumptions:

1. Specific volume of an incompressible substance is a constant:

$$v = \text{ constant}$$

   Values of specific volume will be evaluated at room temperature.
2. The specific internal energy only depends on temperature:

$$u = u(T)$$

3. The incompressible substance molar mass for a substance is identical with the actual molar mass of the substance:

$$M_{\text{incompressible substance}} = M_{\text{real stuff}}$$

   In addition, we will add one additional assumption that will greatly simplify our introduction to the incompressible substance model.
4. The specific heats are independent of temperature. (In general, the specific heats do in fact change with temperature; however, the changes with temperature for most solids and liquids are small and this is an accurate assumption.) We will assume that the specific heats are constant and equal to the values at room temperature.

We'll refer to our incompressible substance model that is based on all four assumptions as the "incompressible substance model with *room-temperature* specific heats."

P-v-T Relationships

For an incompressible substance there is no relationship between the pressure, the specific volume, and the temperature of the substance. The value of the specific heat depends on the substance.

The densities of solids and liquids do in fact change slightly with temperature. Anyone who has overfilled an ice cube tray or dealt with frozen pipe understands this effect. For our purposes, we will assume that the value of the specific volume (or density) can be evaluated at room temperature conditions.

Changes in u and h

The differential change in specific internal energy u for an incompressible substance is calculated using the relation:

$$du = c_v\, dT \quad [\text{Incompressible substance}]$$

where c_v is called "see-sub-vee" or the **specific heat at constant volume** (See note above). To calculate the finite change in u for an incompressible substance with constant specific heats, we integrate Eq. $7.4.15$between state 1 and state 2 and obtain the equation:

$$\Delta u = u_2 - u_1 = c_v\,(T_2 - T_1) \quad \begin{bmatrix} \text{Incompressible substance,} \\ \text{constant } c_v \end{bmatrix}$$

This equation can be used to calculate Δu for any process. *It is not restricted to constant volume processes.*

The differential change in specific enthalpy h for an incompressible substance is calculated using the relation:

$$dh = d(u + Pv) = du + v\, dP \quad [\text{Incompressible substance}]$$

where c_p is called "see-sub-pee" or the **specific heat at constant pressure**. To calculate the finite change in h for an incompressible substance with constant specific heats, we integrate Eq. $7.4.17$between state 1 and state 2 and obtain the equation:

$$\begin{aligned} \Delta h \quad &= u_2 - u_1 + v\,(P_2 - P_1) \\ &= c_v\,(T_2 - T_1) + v\,(P_2 - P_1) \quad \begin{bmatrix} \text{Incompressible substance,} \\ \text{constant } c_v \end{bmatrix} \end{aligned}$$

This equation can be used to calculate Δh for any process.

Specific heat relationships

Using the basic assumptions for the incompressible substance, it can be shown that two specific heat at constant pressure and constant volume are equal:

$$c_P = c_v = c \quad [\text{Incompressible substance}]$$

where c is sometimes just called the specific heat. Typically, tables of data for solids and liquids will only list values for c_p because it is the easiest to measure.

7.4.3 Summary of our substance models

The basic assumptions and key equations for each of our substance models are summarized in the following table. Once you have selected a substance model to apply in a given problem, this table gives *all* of the necessary equations for applying the model. Room-temperature values for the thermophysical properties for several gases, liquids, and solids are given in next two tables. One table gives values in SI and one gives values in USCS.

Substance models are required to relate properties in the energy balance like h and u to properties that are easy to measure like pressure, temperature, and specific volume.

Two Substance Models (Constitutive Relations)		
	Equation of State	
	Ideal Gas Model with room-temperature specific heats	**Incompressible Substance Model** with room-temperature specific heats
Used to model behavior of	gases and vapors	liquids and solids
Basic Model Assumptions	1. Pressure, volume, and temperature obey the ideal gas relation: $PV = NR_uT$ 2. Specific internal energy depends only on temperature, $u = u(T)$. 3. Molar mass of an ideal gas equals the molar mass of the real substance: $M_{\text{ideal gas}} = M_{\text{real stuff}}$. 4. The specific heats are independent of temperature, i.e. they are constants.	1. The density of the substance is a constant. 2. Specific internal energy depends only on temperature, $u = u(T)$. 3. Molar mass of an incompressible substance equals the molar mass of the real substance: $M_{\text{incomp substance}} = M_{\text{real stuff}}$. 4. The specific heats are independent of temperature, i.e. they are constants.

Two Substance Models (Constitutive Relations)		
P-T-ρ and P-T-v relations	$P=\rho RT$ and $Pv=RT$ where $R=R_u/M$	$v=1/\rho=$ constant Evaluated at room temperature
Specific heat relations	$c_P=c_v=R; \quad k=c_P/c_v$	$c_p=c_v=c$, a constant
c_P and c_V values	Evaluated at room temperature	Evaluated at room temperature
Δu — specific internal energy	$\Delta u=u_2-u_1=c_v(T_2-T_1)$	$\Delta u=u_2-u_1=c(T_2-T_1)$
Δh — specific enthalpy	$\Delta h=h_2-h_1=c_P(T_2-T_1)$	$\begin{aligned}\Delta h &= h_2-h_1\\ &=(u_2+P_2v)-(u_1+P_1v)\\ &=(u_2-u_1)+v(P_2-P_1)\end{aligned}$ thus $\Delta h=\Delta u+v\Delta P=c\Delta T+v\Delta T$
Δs — specific entropy Note: All temperatures are absolute values, i.e. K or $^\circ$R, in the entropy relations	$\begin{aligned}\Delta s &= s_2-s_1\\ &=c_P\ln(T_2-T_1)-R\ln(P_2-P_1)\\ &=c_v\ln(T_2-T_1)+R\ln(v_2-v_1)\end{aligned}$	$\begin{aligned}\Delta s &= s_2-s_1\\ &=c\ln(T_2-T_1)\end{aligned}$

Thermophysical Property Data for Some Common Substances (SI Units)

Gases (at 25°C and 1 atm)

Substance		Molar Mass	R $\left[\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}}\right]$	c_v $\left[\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}}\right]$	c_P $\left[\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}}\right]$	k	$\frac{T_c}{\mathrm{K}}$	$\frac{P_c}{\mathrm{bar}}$
Acetylene	C_2H_2	26.04	0.3193	1.37	1.69	1.23	309	62.4
Air	---	28.97	0.2870	0.718	1.005	1.40	133	37.7
Ammonia	NH_3	17.04	0.4879	1.66	2.15	1.30	406	112.8
Carbon dioxide	CO_2	44.01	0.1889	0.657	0.846	1.29	304.2	73.9
Carbon monoxide	CO	28.01	0.2968	0.744	1.04	1.40	133	35.0
Ethane	C_2H_6	30.07	0.2765	1.48	1.75	1.18	305.4	48.8
Ethylene	C_2H_4	28.05	0.2964	1.23	1.53	1.24	283	51.2
Helium	He	4.003	2.077	3.12	5.19	1.67	5.2	2.3
Hydrogen	H_2	2.016	4.124	10.2	14.3	1.40	33.2	13.0
Methane	CH_4	16.04	0.5183	1.70	2.22	1.31	190.7	46.4
Nitrogen	N_2	28.01	0.2968	0.743	1.04	1.40	126.2	33.9
Oxygen	O_2	32.00	0.2598	0.658	0.918	1.40	154.4	50.5
Propane	C_3H_8	44.09	0.1886	1.48	1.67	1.13	370	42.5
Refrigerant 134a	$C_2F_4H_2$	102.03	0.08149	0.76	0.85	1.12	374.3	40.6
Water (Steam)	H_2O	18.02	0.4614	1.40	1.86	1.33	647.3	220.9

Liquids				**Solids** *		
Substance	Temp (°C)	ρ $\left[\frac{\mathrm{kg}}{\mathrm{m^3}}\right]$	c_P $\left[\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}}\right]$	Substance	ρ $\left[\frac{\mathrm{kg}}{\mathrm{m^3}}\right]$	c_P $\left[\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}}\right]$
Ammonia	25	602	4.80	Aluminum	2,700	0.902
Benzene	20	879	1.72	Brass, yellow	8,310	0.400
Brine (20/	20	1,150	3.11	Brick (common)	1,922	0.79
Ethanol	25	783	2.46	Concrete	2,300	0.653
Ethyl alcohol	20	789	2.84	Copper	8,900	0.386
Ethylene glycol	20	1,109	2.84	Glass, window	2,700	0.800

Liquids				Solids *		
Kerosene	20	820	2.00	Iron	7,840	0.45
Mercury	25	13,560	0.139	Lead	11,310	0.128
Oil (light)	25	910	1.80	Silver	10,470	0.235
Refrigerant 134a	25	1,206	1.42	Steel (mild)	7,830	0.500
Water	25	997	4.18	* Evaluated at room temperature.		

Values adapted from K. Wark, Jr. and D. E. Richards, *Thermodynamics*, 6th ed. (McGraw-Hill, New York, 1999) and Y. A. Cengul and M. A. Boles, *Thermodynamics*, 4th ed. (McGraw-Hill, New York, 2002).

Thermophysical Property Data for Some Common Substances (USCS Units)

Gases (at 77°F and 1 atm)								
Substance		Molar Mass	$\frac{R}{\left[\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot{}^\circ\text{R}}\right]}$	$\frac{c_v}{\left[\frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{R}}\right]}$	$\frac{c_P}{\left[\frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{R}}\right]}$	k	$\frac{T_c}{{}^\circ\text{R}}$	$\frac{P_c}{\text{atm}}$
Acetylene	C_2H_2	26.04	59.33	0.328	0.404	1.23	556	61.6
Air	---	28.97	59.33	0.171	0.240	1.40	239	37.2
Ammonia	NH_3	17.04	90.67	0.397	0.514	1.30	730	111.3
Carbon dioxide	CO_2	44.01	35.11	0.156	0.202	1.29	548	72.9
Carbon monoxide	CO	28.01	55.16	0.178	0.249	1.40	239	34.5
Ethane	C_2H_6	30.07	51.38	0.353	0.419	1.19	549	48.2
Ethylene	C_2H_4	28.05	55.08	0.294	0.365	1.24	510	50.5
Helium	He	4.003	386.0	0.744	1.24	1.67	9.3	2.26
Hydrogen	H_2	2.016	766.4	2.43	3.42	1.40	59.8	12.8
Methane	CH_4	16.04	96.32	0.407	0.531	1.30	344	45.8
Nitrogen	N_2	28.01	55.16	0.178	0.248	1.39	227	33.5
Oxygen	O_2	32.00	48.28	0.157	0.219	1.40	278	49.8
Propane	C_3H_8	44.09	35.04	0.355	0.400	1.13	666	42.1
Refrigerant 134a	$C_2F_4H_2$	102.03	15.14	0.184	0.203	1.10	672.8	40.1
Water (Steam)	H_2O	18.02	87.74	0.335	0.445	1.33	1165	218.0

Liquids				**Solids ***		
Substance	Temp (°F)	$\frac{\rho}{\left[\frac{\text{lbm}}{\text{ft}^3}\right]}$	$\frac{c_P}{\left[\frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{R}}\right]}$	Substance	$\frac{\rho}{\left[\frac{\text{lbm}}{\text{ft}^3}\right]}$	$\frac{c_P}{\left[\frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{R}}\right]}$
Ammonia	80	37.5	1.135	Aluminum	170	0.215
Benzene	68	54.9	0.411	Brass, yellow	519	0.0955
Brine (20/	68	71.8	0.743	Brick (common)	120	0.189
Ethanol	77	48.9	0.588	Concrete	144	0.156
Ethyl alcohol	68	49.3	0.678	Copper	555	0.0917
Ethylene glycol	68	69.2	0.678	Glass, window	169	0.191
Kerosene	68	51.2	0.478	Iron	\(490)	0.107
Mercury	77	847	0.033	Lead	705	0.030
Oil (light)	77	56.8	0.430	Silver	655	0.056
Refrigerant 134a	32	80.9	0.318	Steel (mild)	489	0.119
Water	68	62.2	1.00	* Evaluated at room temperature.		

Values adapted from K. Wark, Jr. and D. E. Richards, *Thermodynamics*, 6th ed. (McGraw-Hill, New York, 1999) and Y. A. Cengul and M. A. Boles, *Thermodynamics*, 4th ed. (McGraw-Hill, New York, 2002).

Example — Heating Things Up

A rigid tank contains 0.80 g of air initially at 295 K and 1.5 bars and an electric resistor wire. The electric resistor within the tank has a mass of 0.05 g and is energized by passing a current of 0.6 A for 30 s from a 12.0V source. During the same time interval, 156 J of energy is lost through the walls of the tank by heat transfer. Assume that the air can be modeled as an ideal gas with room temperature specific heats and the mild steel resistance wire can be modeled as an incompressible substance with room temperature specific heats.

Determine (a) the final temperature of the gas, in kelvins, and (b) the final pressure of the gas, in bars.

Solution

Known: An electric resistor is used to heat air contained in a rigid tank.

Find: (a) The final temperature of the gas, in K.
(b) The final pressure of the gas, in bars.

Given:

Gas: $m_{\text{gas}} = 0.80\text{ g}$ (air)
$T_1 = 295\text{ K}$
$P_1 = 1.5\text{ bars}$

Resistor: $m_{\text{resistor}} = 0.05\text{ g}$ (mild steel)
$\Delta V = 12\text{ volts}$; $i = 0.60\text{ amps}$; $\Delta t = 30\text{ s}$

Heat transfer through tank walls: $Q_{1\text{-}2,\text{out}} = 156\text{ J}$

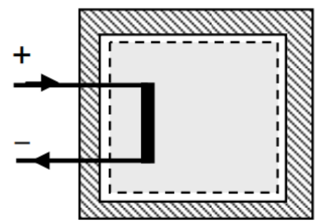


Figure 7.4.1: An electrical wire passes into a rigid container of air, containing a resistor.

Analysis:

Strategy $\rightarrow$ Because we are dealing with heat transfer and electrical work and we are asked to find the final temperature, use conservation of energy. (Questions involving changes in pressure and temperature of a substance frequently require the use of conservation of energy and some equation of state to describe how the thermophysical properties of the substance are related.)
System $\rightarrow$ Because they told us about the heat transfer from the gas to the tank, let's treat everything inside the tank, gas and resistor, as a closed system. (See dashed line drawn in figure above.)
Property to count $\rightarrow$ We want to know temperature but energy is the property we have a conservation principle for and we know energy and temperature are related, so lets count energy.
Time interval $\rightarrow$ Finite-time form since they give us the 30-second time interval.

Because we may not recall the finite-time form for a closed system, let's quickly redevelop it:

$$\frac{dE_{\text{sys}}}{dt} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \underbrace{\cancel{\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz\right)}^{=0}}_{\text{Closed system}}$$

$$\int_{t_1}^{t_2} \left(\frac{dE_{\text{sys}}}{dt}\right) dt = \int_{t_1}^{t_2} \left(\dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}\right) dt \quad \rightarrow \quad \boxed{\Delta E_{\text{sys}} = Q_{\text{net, in}} + W_{\text{net, in}}}$$

Now that we have the correct form of the energy balance, we need to evaluate the various terms as below: \[\underbrace{ \cancel{\Delta E_{\text{sys}}}^{= \Delta U_{\text{sys}}} }_{\begin{array}{c} \text{No changes in kinetic} \\ \text{and gravitational \\ \text{potential energy} \end{array}} = \underbrace{ \cancel{Q_{\text {net, in}}}^{=-Q_{\text {out}}} }_{

$$\begin{array}{c}\text{Use given information}\\ \text{about heat transfer out}\\ \text{of the system}\end{array} \tag{7.4.1}$$

} +\underbrace{ \cancel{W_{\text {net, in}}}^{=W_{\text {electric, in}}} }_{

$$\left.\begin{array}{c}\text{Only one type of work}\\ \text{because the system}\\ \text{is contained in a rigid tank.}\end{array}\right. \tag{7.4.2}$$

} \quad\quad \rightarrow \quad\quad \Delta U_{\text {sys}} = -Q_{\text {out}} + W_{\text {electric, in}} \nonumber \]

First, let's examine the change in internal energy of the system. We must recognize that the change in internal energy for the whole system can be calculated as the sum of the change in internal energy for each of its subsystems. (Recall this is a key feature of an *extensive* property like energy.)

$$\Delta U_{\text{sys}} = \Delta U_{\text{gas}} + \Delta U_{\text{resistor}} = m_{\text{gas}} \Delta u_{\text{gas}} + m_{\text{resistor}} \Delta u_{\text{resistor}}$$

$$\begin{array}{lll}\text{but} & \Delta u_{\text{gas}} = c_{v,\text{ gas}} (T_2 - T_1)_{\text{gas}} & \mid \text{Ideal gas with room temperature specific heats} \\ \text{and} & \Delta u_{\text{resistor}} = c_{\text{resistor}} (T_2 - T_1)_{\text{resistor}} & \mid \text{Incomp. substance with room temperature specific heats}\end{array}$$

$$\text{So,} \qquad \Delta U_{\text{sys}} = [m c_v (T_2 - T_1)]_{\text{gas}} + [m c (T_2 - T_1)]_{\text{resistor}}$$

Second, we must evaluate the electric work. We can recover the necessary equation from the definition of electrical power:

$$\begin{aligned} W_{\text{electric, in}} &= \int_{t_1}^{t_2} \dot{W}_{\text{electric, in}}\, dt = \int_{t_1}^{t_2} (i \Delta V) dt = (i \Delta V) \int_{t_1}^{t_2} dt = (i \Delta V) \Delta t \\ &= (0.60 \text{ A}) \cdot (12.0 \text{ V}) \cdot (30 \text{ s}) = 216 \text{ W} \cdot \text{s} = 216 \text{ J} \end{aligned}$$

Combining all of this information we have the following result:

$$\Delta U_{\text{sys}} = -Q_{\text{out}} + W_{\text{electric, in}} \quad \rightarrow \quad [m c_v (T_2 - T_1)]_{\text{gas}} + [m c (T_2 - T_1)]_{\text{resistor}} = [-156 + 216] \text{ J}$$

Before we can solve for the temperatures, we must make an assumption about the temperatures of the gas and the resistor. It would seem reasonable that the resistor and the gas have the same temperatures if the system is in thermal equilibrium at beginning and end. In addition, we must also find room temperature values for the specific heats by consulting the appropriate tables: $c_v,\text{ gas} = 0.718 \text{ kJ}/(\text{kg} \cdot \text{K})$ and $c_{\text{resistor}} = 0.500 \text{ kJ}/(\text{kg} \cdot \text{K})$.

Using this in the energy balance we can now solve for the final temperature as follows:

$$[m c_v (T_2 - T_1)]_{\text{gas}} + [m c (T_2 - T_1)]_{\text{resistor}} = [-156 + 216] \text{ J}$$

$$(m_{\text{gas}} c_{v,\text{ gas}} + m_{\text{resistor}} c_{\text{resistor}}) (T_2 - T_1) = 60 \text{ J} \quad \rightarrow \quad \text{T}_2 - T_1 = \frac{60 \text{ J}}{(m_{\text{gas}} c_{v,\text{ gas}} + m_{\text{resistor}} c_{\text{resistor}})}$$

$$T_2 - T_1 = \frac{60 \text{ J}}{\left[(0.80 \text{ g}) \cdot \left(0.718 \dfrac{\text{J}}{\text{g} \cdot \text{K}}\right) + (0.05 \text{ g}) \cdot \left(0.500 \dfrac{\text{J}}{\text{g} \cdot \text{K}}\right)\right]} = \frac{60 \text{ J}}{\left[(0.574 + 0.025) \dfrac{\text{J}}{\text{K}}\right]} = 100 \text{ K}$$

$$T_2 = T_1 + 100 \text{ K} = 295 \text{ K} + 100 \text{ K} = 395 \text{ K}$$

Solving for the final pressure is done by applying the ideal gas equation as follows:

$$\left.\begin{array}{l} P_1 V_1 = m_1 R_{\text{air}} T_1 \\ P_2 V_2 = m_2 R_{\text{air}} T_2 \end{array}\right\} \quad \rightarrow \quad \frac{P_1 V_1}{P_2 V_2} = \frac{m_1 R_{\text{air}} T_1}{m_2 R_{\text{air}} T_2} \quad \rightarrow \quad \left(\frac{P_1}{P_2}\right) \overbrace{\left(\frac{V_1}{\cancel{V_2}}\right)}^{=1} = \overbrace{\left(\frac{m_1}{\cancel{m_2}}\right)}^{=1} \cdot \overbrace{\left(\frac{R_{\text{air}}}{\cancel{R_{\text{air}}}}\right)}^{=1} \cdot \left(\frac{T_1}{T_2}\right)$$

$$4pt P_2 = P_1 \left(\frac{T_2}{T_1}\right) = (1.5 \text{ bars}) \left(\frac{395 \text{ K}}{295 \text{ K}}\right) = 2.01 \text{ bars}$$

Comment:

(1) In applying the energy balance, we went from ΔU to $m \Delta u$. This is necessary because our substance models only allow us to calculate the change in specific internal energy: Δu, not ΔU. Furthermore, this is how temperature enters the picture.

(2) When applying the ideal gas equation to find the pressure notice how we made use of ratios. This greatly simplifies the calculations and allows us to easily handle different units. Instead of solving symbolically and recognizing the ratios the final pressure could have been obtained as follows:

Step 1: Solve for V_1 $\quad V_1 = \dfrac{m_1 R_{\text{air}} T_1}{P_1}$

Step 2: Recognize that $V_2 = V_1$ and that $m_2 = m_1$

Step 3: Solve for P_2 $\quad P_2 = \dfrac{m_2 R_{\text{air}} T_2}{V_2} = \dfrac{m_1 R_{\text{air}} T_2}{V_1}$

Ratios can greatly speed-up and simplify calculations. In addition, it reduces errors, e.g. avoiding the need to find R_{gas} in units that work with bars plus the extra number punching on your calculator.

(3) If we had not made the assumption that the resistor and the gas had equal temperatures it would have been impossible to solve the problem without additional assumptions or information.

(4) What heat transfer and work interactions occurred between the resistor and its surroundings during this process? Between the gas and its surroundings?

Example — Pumping kerosene

A pump is used to move kerosene between two points in a piping system. The pump is located between the two points. Kerosene enters the piping system at an elevation of 5 ft, a pressure of 15 psia and a temperature of $70^{\circ}\mathrm{F}$ and leaves the piping system at an elevation of 20 ft and a pressure of 60 psia. During the adiabatic pumping process, the kerosene experiences a $0.5^{\circ}\mathrm{F}$ rise in temperature. Determine the power required to operate the pump in $\mathrm{ft} \cdot \mathrm{lbf} / \mathrm{lbm}$.

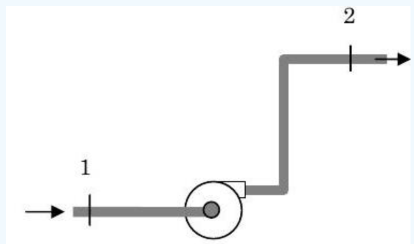


Figure $7.4.2$: Kerosene is pumped to a higher elevation.

Solution

Known: Kerosene is pumped steadily in a piping system

Find: The power required to operate the pump in $\mathrm{ft} \cdot \mathrm{lbf} / \mathrm{lbm}$.

Given:

Kerosene State 1: $z_1 = 5\text{ft};\ P_1 = 15 \text{ psia};\ T_1 = 70^{\circ}\mathrm{F}$
Kerosene State 2: $z_2 = 20\text{ft};\ P_2 = 60 \text{ psia}$
Process 1-2: Adiabatic, steady-state
$T_2 - T_1 = 0.5^{\circ}\mathrm{F}$

Analysis:

Strategy $\rightarrow$ Use conservation of energy
System $\rightarrow$ Non-deforming open system that includes pipe, pump, and contents.
Property $\rightarrow$ Energy (and possibly mass)
Time interval $\rightarrow$ Since steady-state, infinitesimal time period.

Writing the energy balance for the open system gives the following:

$$\underbrace{\cancel{\frac{dE_{\text{sys}}}{dt}}^{=0}}_{\text{Steady-state conditions}} = \underbrace{\cancel{\dot{Q}_{\text{net, in}}}^{=0}}_{\text{Adiabatic}} + \dot{W}_{\text{net, in}} + \dot{m}_1 \left(h_1 \frac{V_1^{\,2}}{2} + g z_1\right) - \dot{m}_2 \left(h_2 \frac{V_2^{\,2}}{2} + g z_2\right)$$

$$0 = \dot{W}_{\text{pump, in}} \dot{m}_1 \left(h_1 \frac{V_1^{\,2}}{2} + g z_1\right) - \dot{m}_2 \left(h_2 \frac{V_2^{\,2}}{2} + g z_2\right)$$

To go further we must say something about the mass flow rates. If we apply the conservation of mass to this steady-state, one-inlet/one-outlet system, we find that the mass flow rates are equal. Using this result, the equation above becomes:

$$\frac{\dot{W}_{\text{pump, in}}}{\dot{m}} = (h_2 - h_1) + \left(\frac{V_2^{\,2}}{2} - \frac{V_1^{\,2}}{2}\right) + g(z_2 - z_1)$$

The term on the left-hand side is the quantity we are looking for. It is the power per unit mass flow rate or the work per unit mass. (As stated earlier, it is very common to solve open system problems on a per-unit-mass basis. Then if we only change the mass flow rate we do not have to resolve the problem, unless something else changes.)

The change in specific enthalpy can be handled by assuming that kerosene can be modeled as an incompressible substance with room temperature specific heats; thus,

$$h_2-h_1=u_2-u_1+v\left(P_2-P_1\right)=c\left(T_2-T_1\right)+v\left(P_2-P_1\right)$$

where $c=0.478\ \mathrm{Btu}/\left(\mathrm{lbm}^{\circ}\mathrm{R}\right)$ and $v=1/\rho=1/\left(51.2\ \mathrm{lbm}/\mathrm{ft}^3\right)$

We have *no* information about the velocity of the kerosene at either the inlet or outlet. Without making some assumption about the velocities we cannot solve for the power. We need not assume absolute values for the velocity but only that its change—actually, the change in kinetic energy—is negligible.

Using these results we can now solve for the power: \[\begin{aligned} \frac{\dot{W}_{\text {pump, in}}}{\dot{m}} &= \left(h_{2}-h_{1}\right) + \underbrace{ \cancel{\left{\left{\frac{V_{2}{ }^{2}}{2} - \frac{V_{1}{ }^{2}}{2}\right)}^{=0} }_{

Assume change is
negligible (7.4.3)

} + g\left(z_{2}-z_{1}\right) \\ &=c\left(T_{2}-T_{1}\right)+v\left(P_{2}-P_{1}\right)+g\left(z_{2}-z_{1}\right)=c\left(T_{2}-T_{1}\right)+\frac{\left(P_{2}-P_{1}\right)}{\rho}+g\left(z_{2}-z_{1}\right) \end{aligned} \nonumber \]

Now we solve for the individual terms in the energy balance as follows:

$$c\left(T_2-T_1\right)=\left(0.478\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^{\circ}\mathrm{R}}\right)\left(0.5^{\circ}\mathrm{F}\right)=0.239\ \frac{\mathrm{Btu}}{\mathrm{lbm}}=\left(0.239\ \frac{\mathrm{Btu}}{\mathrm{lbm}}\right)\times\left(778\ \frac{\mathrm{ft}\cdot\mathrm{lbf}}{\mathrm{Btu}}\right)=185.9\ \frac{\mathrm{ft}\cdot\mathrm{lbf}}{\mathrm{lbm}}$$

where we recognize the ${}^{\circ}\mathrm{R}$ and the ${}^{\circ}\mathrm{F}$ cancel, because they both represent temperature differences and not temperatures:

$$\frac{\left(P_2-P_1\right)}{\rho}=\frac{(60-15)\ \mathrm{psia}}{51.2\ \dfrac{\mathrm{lbm}}{\mathrm{ft}^3}}=0.8789\ \frac{\mathrm{psia}\cdot\mathrm{ft}^3}{\mathrm{lbm}}=\left(0.8789\ \frac{\mathrm{psia}\cdot\mathrm{ft}^3}{\mathrm{lbm}}\right)\times\left(\frac{\mathrm{lbf}/\mathrm{in}^2}{\mathrm{psia}}\right)\times\left(\frac{144\ \mathrm{in}^2}{1\ \mathrm{ft}^2}\right)$$

$$=126.6\ \frac{\mathrm{ft}\cdot\mathrm{lbf}}{\mathrm{lbm}}$$

$$g\left(z_2-z_1\right)=\left(32.174\ \frac{\mathrm{ft}}{\mathrm{s}^2}\right)\cdot[(20-5)\ \mathrm{ft}]=482.6\ \frac{\mathrm{ft}^2}{\mathrm{s}^2}=\left(482.6\ \frac{\mathrm{ft}^2}{\mathrm{s}^2}\right)\times\left(\frac{\mathrm{lbm}}{\mathrm{lbm}}\right)\times\left(\frac{1\ \mathrm{lbf}}{32.174\ \dfrac{\mathrm{lbm}\cdot\mathrm{ft}}{\mathrm{s}^2}}\right)$$

$$=15.0\ \frac{\mathrm{ft}\cdot\mathrm{lbf}}{\mathrm{lbm}}$$

Combining these results give

$$\frac{\dot{W}_{\text{pump, in}}}{\dot{m}}=\underbrace{c\left(T_2-T_1\right)}_{=57\%}+\underbrace{\frac{1}{\rho}\left(P_2-P_1\right)}_{=39\%}+\underbrace{g\left(z_2-z_1\right)}_{=4\%}=(185.9+126.6+15.0)\ \frac{\mathrm{ft}\cdot\mathrm{lbf}}{\mathrm{lbm}}=328\ \frac{\mathrm{ft}\cdot\mathrm{lbf}}{\mathrm{lbm}}$$

Comments:

(1) Note how the power breaks down. Assuming that increasing the pressure and elevation are the desired effects for a pump, approximately 57% of the energy is "wasted" in the $0.5^{\circ}\mathrm{F}$ temperature change in the liquid. What would the power be if the process had been isothermal?

(2) What if the only change in the problem had been that the temperature of the kerosene decreased by $0.5^{\circ}\mathrm{F}$? What would the power input have been?

[This is pretty impressive. First, you hook a pipe up to some kerosene at room conditions. Then it increases the elevation and pressure of the kerosene, "cools" the kerosene in an adiabatic process, and gives you some power out. You could probably get the high-pressure, elevated kerosene to flow through a hydraulic turbine and get even more work out. As appealing as this might be, we will soon discover that it would be impossible! If you figure out a way to do this, please don't tell anyone but contact your instructor and you can both retire to the Bahamas!]

(3) Now, if the pump had really been a compressor and instead of kerosene we were compressing air, what would the power have been? (Assume changes in kinetic energy are negligible.) Does the pressure even fit into your calculations? [Answer: $108\ \mathrm{ft}\cdot\mathrm{lbf}/\mathrm{lbm}$]

[We will shortly discover that this process is also impossible. Under the most ideal conditions, the air temperature would increase approximately $258^{\circ}\mathrm{F}$ if it was compressed adiabatically between the given inlet state and final pressure.]

Example — Mixing things up

Tank A contains $1.0\ \mathrm{kg}$ of nitrogen $\left(\mathrm{N}_2\right)$ gas initally at $150\ \mathrm{kPa}$ and $300\ \mathrm{K}$. Attached to this tank through a suitable valve is a second tank B that contains $2.0\ \mathrm{kg}$ of the same gas at $300\ \mathrm{kPa}$ and $400\ \mathrm{K}$. Both tanks are rigid and insulated. Assume that nitrogen gas can be modeled as an ideal gas with room temperature specific heats.

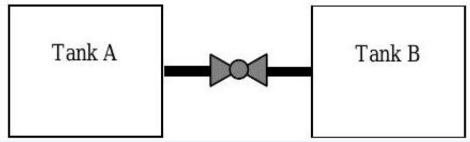


Figure 7.4.3: Setup of two connected tanks.

If the valve is opened and equilibrium is reached, determine

(a) the final mixture temperature, in K, and

(b) the final pressure of the mixture, in kPa.

Solution

Known: The contents of two tanks each containing nitrogen are mixed together.

Find: (a) final temperature of the mixture, in K, and (b) final pressure of the mixture in kPa.

Given:

Initial State (1)			Final State (2)	
Tank A	**Tank B**		**Tank A**	**Tank B**
$1.0\ \mathrm{kg}$ $150\ \mathrm{kPa}$ $300\ \mathrm{K}$	$2.0\ \mathrm{kg}$ $300\ \mathrm{kPa}$ $400\ \mathrm{K}$		T_2 and P_2	T_2 and P_2

Analysis:

Strategy $\rightarrow$ Conservation of energy
System $\rightarrow$ Treat the contents of both tanks as a single closed system.
Property to count $\rightarrow$ Energy and mass
Time interval $\rightarrow$ Finite-time since beginning information given and ending information desired.

Writing the finite-time form of the conservation of energy for a closed system we have the following:

$$\underbrace{\Delta \cancel{E_{\text{sys}}}^{=\Delta U_{\text{sys}}}}_{\substack{\text{Changes in } E_K \text{ and} \\ E_{GP} \text{ are negligible}}} = \underbrace{\cancel{Q_{\text{net, in}}}^{=0}}_{\substack{\text{Tanks specified} \\ \text{as adiabatic}}} + \underbrace{\cancel{W_{\text{net, in}}}^{=0}}_{\substack{\text{Rigid tanks and no other} \\ \text{work interactions identified}}} \quad \rightarrow \quad \Delta U_{\text{sys}} = U_{\text{sys, }2} - U_{\text{sys, }1} = 0$$

Now to evaluate the internal energies of the closed system gives the following, assuming the air can be modeled as an ideal gas with room temperature specific heats:

$$\begin{aligned} 0 &= U_{\text{sys},2} - U_{\text{sys},1} \\ &= (m_{\text{sys}} u)_2 - (m_A u_A + m_B u_B)_1 \quad \text{but} \quad m_{\text{sys}} = m_{A,2} + m_{B,2} = m_{A,1} + m_{B,1} \\ &= [(m_{A,1} + m_{B,1})\, u_2] - [m_{A,1} u_{A,1} + m_{B,1} u_{B,1}] \\ &= m_{A,1}(u_2 - u_{A,1}) + m_{B,1}(u_2 - u_{B,1}) \qquad \mid \text{ collecting the } m_A \text{ and } m_B \text{ terms to get } \Delta u \\ &= m_{A,1} c_v (T_2 - T_{A,1}) + m_{B,1} c_v (T_2 - T_{B,1}) \qquad \mid \text{applying the ideal gas model} \end{aligned}$$

Because our ideal gas model assumes that the specific heats are all evaluated at the room temperature, they are constant and will cancel out when we solve for the final temperature:

$$0 = m_{A,1} c_v (T_2 - T_{A,1}) + m_{B,1} c_v (T_2 - T_{B,1}) \quad \rightarrow \quad T_2 = \frac{m_{A,1} c_v T_{A,1} + m_{B,1} c_v T_{B,1}}{m_{A,1} c_v + m_{B,1} c_v} = \frac{m_{A,1} T_{A,1} + m_{B,1} T_{B,1}}{m_{A,1} + m_{B,1}}$$

$$T_2 = \frac{(1.0\ \mathrm{kg})(300\ \mathrm{K}) + (2.0\ \mathrm{kg})(400\ \mathrm{K})}{(1.0 + 2.0)\ \mathrm{kg}} = 367\ \mathrm{K}$$

Now to solve for the final pressure, we apply the ideal gas model:

$$\left.\begin{array}{l} P_2 = \dfrac{m_2 R_{\text{air}} T_2}{\forall_2} \\ \forall_2 = \forall_A + \forall_B \\ \forall_A = \dfrac{m_{A,1} R_{\text{air}} T_{A,1}}{P_{A,1}} \\ \forall_B = \dfrac{m_{B,1} R_{\text{air}} T_{B,1}}{P_{B,1}} \end{array}\right\} \rightarrow \quad P_2 = \frac{m_2 R_{\text{air}} T_2}{\forall_2} = \frac{m_1 R_{\text{air}} T_2}{\forall_A + \forall_B} = \frac{(m_{A,1} + m_{B,1}) R_{\text{air}} T_2}{\dfrac{m_{A,1} R_{\text{air}} T_{A,1}}{P_{A,1}} + \dfrac{m_{B,1} R_{\text{air}} T_{B,1}}{P_{B,1}}} = \frac{(m_{A,1} + m_{B,1}) T_2}{\dfrac{m_{A,1} T_{A,1}}{P_{A,1}} + \dfrac{m_{B,1} T_{B,1}}{P_{B,1}}}$$

$$P_2 = \frac{(1.0 + 2.0)(367)}{(1.0)\left(\dfrac{300}{150 \text{ kPa}}\right) + (2.0)\left(\dfrac{400}{300 \text{ kPa}}\right)} = 236 \text{ kPa}$$

Comment:

(1) Notice how we once again have used the idea of ratios to solve for the final pressure without ever calculating the tank volumes. Obviously, as a check you could find the volume of each tank, add them to find the total volume, and then use the ideal gas equation. This would be a good check.

(2) Under our model for ideal gases, the specific heat value cancels out. With a more accurate model to handle larger temperature changes, the specific heat values would not drop out of the process. Note that the final temperature is not just the average of the initial temperatures.

The substance models introduced in this section have only limited validity. Under certain conditions, the kerosene that we pumped might in fact start to vaporize or boil. Similarly under certain conditions when a gas is compressed, liquid droplets might condense or solid particles might precipitate out. Changes in phase occur suddenly and nothing about our model allows us to predict when this will occur. To learn more about these important phenomena, we will study the behavior of real substances in a later course (ES202 — Fluid & Thermal Systems). Once we have done this we will have a better basis for deciding when our substance models are accurate.

As an indication of these limitations, the ideal gas model is typically assumed to be valid at "high" temperatures and "low" pressures. But how high is "high" and how low is "low"? Well, the last two columns of the thermophysical property data for gases give us some guidance. The points of reference for gases are the critical pressure P_c and the critical temperature T_c. As we will learn later, the ideal gas equation $Pv = RT$ is in error by less than 5% when $P/P_\text{c} < 0.05$ and $T/T_\text{c} > 0.75$. To check this, compare these ratios for water, nitrogen, and helium at room conditions, say 300 K and 1 bar. Which do you usually think of as gases at room conditions?

7.5: Flow Work and Flow Power Revisited

When the volume of a *closed* system is decreased, PdV-work is done on the gas inside the system. Similarly, when mass flows into an *open* system, the mass inside the system is compressed by the entering mass.

As we count all the possible work mechanisms for transporting energy across a system boundary, we must pay particular attention to the work done at an open (or flow) boundary, especially the PdV-work done on the mass inside the open system. The purpose of this section is to provide a more detailed development and explanation of flow work and flow power.

Definitions

Earlier we defined flow work and flow power as follows:

> **Flow work** is the PdV-work (or compression work) done on the mass in an open system as a result of mass flowing into the system. (Although defined in terms of mass flowing into a system, mass flowing out of a system has a similar effect.)
>
> **Flow power** is the time rate of doing flow work.

As shown previously, the concepts of flow work and flow power are important in developing a useful form of the conservation of energy equation for open systems. Only under extremely special cases will we have any interest in computing flow work or flow power.

Mean Value Theorem:

If a function $V(t)$ is continuous for values of $t_1 \leq t \leq t_2$ then there is a value of V such that

$$\int_{t_1}^{t_2} V(t)\, dt = V\left(t^*\right)\left[t_2 - t_1\right]$$

where $t_1 \leq t^* \leq t_2$

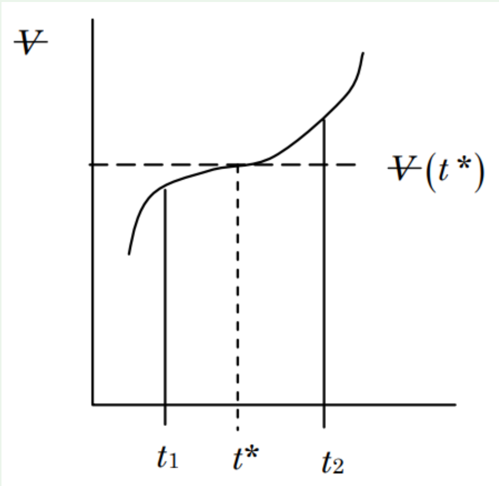
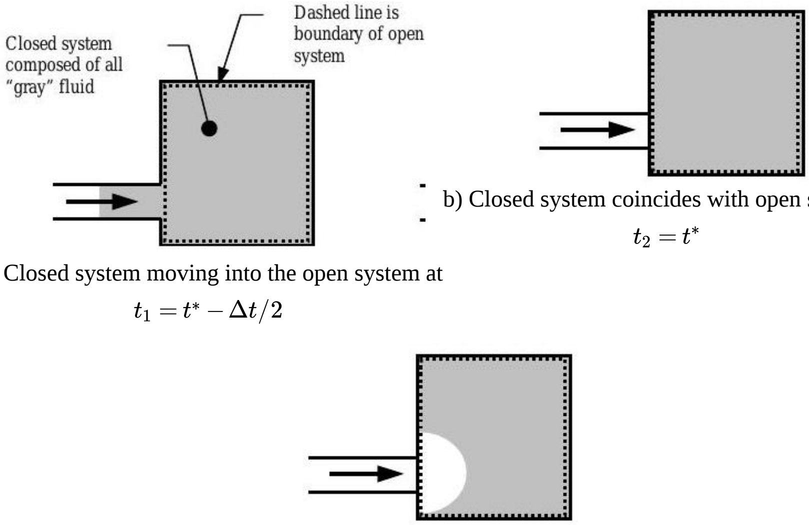


Figure 7.5.1: Mean Value Theorem.

Systems with mass flow in

Consider a rigid tank with one inlet as shown in Figure 7.5.2. The interior of the tank is an open system with a constant volume (see volume inside the dashed lines). Figure 7.5.2 shows three snapshots of a closed system (the "gray" substance) flowing into the tank. Imagine that the closed system is the mass that has been dyed "gray." At time t_1, the closed system is partially outside the tank [see Figure 7.5.2*a*]. A short time later at time t_2, the closed system occupies the same volume as the open system [see Figure 7.5.2*b*]. Still later at time t_3, the closed system occupies less volume than the open system [see Figure 7.5.2*c*].



a) Closed system moving into the open system at $t_1 = t^* - \Delta t/2$

b) Closed system coincides with open system at $t_2 = t^*$

c) Closed system has moved inside open system at $t_3 = t^* + \Delta t/2$

Figure 7.5.2: Flow of a closed system into an open system.

The compression (PdV) work done on the mass inside the closed system during the time interval t_1 to t_3 is calculated using the basic equation for PdV work:

$$W_{\text{in, 1-3}} = -\int_{\mathcal{V}_1}^{\mathcal{V}_3} P\, d\mathcal{V}$$

where P is the pressure at the moving boundary of the closed system. Using the *mean value theorem* from calculus, we can rewrite Eq. $7.5.2$ as

$$W_{\text{in, 1-3}} = -\int_{\mathcal{V}_1}^{\mathcal{V}_3} P\, d\mathcal{V} = -\tilde{P}\left(\mathcal{V}_3 - \mathcal{V}_1\right)$$

where P is a function of the closed system volume $\mathcal{V}_3$, $\tilde{P} = P\left(\mathcal{V}\right)$ and $\mathcal{V}_1 \leq \tilde{\mathcal{V}} \leq \mathcal{V}_3$.

The change in the volume of the closed system can be described in terms of the volumetric flow rate into the control volume as

$$\mathcal{V}_3 - \mathcal{V}_1 = -\int_{t_1}^{t_3} \dot{\mathcal{V}}_{\text{in}}\, dt = -\tilde{\dot{\mathcal{V}}}\left(t_3 - t_1\right)$$

where $\tilde{\dot{\mathcal{V}}}_{\text{in}} = \dot{\mathcal{V}}(\tilde{t}\,)$ and $t_1 \leq \tilde{t} \leq t_3$.

Substituting Eq. $7.5.3$ back into Eq. $7.5.2$ gives an expression for the work done on the closed system as it moves into the tank:

$$W_{\text{in, 1-3}} = -\tilde{P}\left(\mathcal{V}_3 - \mathcal{V}_1\right) = -\tilde{P}\left[-\tilde{\dot{\mathcal{V}}}_{\text{in}}\left(t_3 - t_1\right)\right] = \tilde{P}\tilde{\dot{\mathcal{V}}}_{\text{in}}\left(t_3 - t_1\right)$$

where $\tilde{P}$ and $\tilde{\dot{V}}$ are evaluated at $\tilde{t}$ and $t_1 \leq \tilde{t} \leq t_3$.

The average work per unit time (average power) over the time interval t_1 to t_3 can now be written by rearranging Eq. $7.5.4$ as follows:

$$\frac{W_{\text{in, 1-3}}}{t_3 - t_1} = \frac{W_{\text{in, 1-3}}}{\Delta t} = \tilde{P}\tilde{\dot{\mathcal{V}}}_{\text{in}}$$

where $t_3 - t_1 = \left(t^* + \Delta t/2\right) - \left(t^* - \Delta t/2\right) = \Delta t$.

The instantaneous power, as opposed to the average power, is found by taking the limit of Eq. $7.5.5$ as $\Delta t \to 0$ gives

$$\dot{W}_{\text{flow, in}} = \lim_{\Delta t \to 0} \frac{W_{\text{in,1-3}}}{\Delta t} = \lim_{\Delta t \to 0} \tilde{P}\tilde{\dot{\mathcal{V}}}_{\text{in}} = P\dot{\mathcal{V}}_{\text{in}}$$

where $\dot{W}_{\text{in, flow}}$

$=$ the rate of energy transferred by flow work (flow power) into the closed system that occupies the open system at time t

$P =$ the pressure at the boundary where the mass flows into the open system at time t.

$\dot{V}_{\text{in}} =$ the volumetric flow rate at the flow boundary at time t.

Although Eq. $7.5.6$ is correct, it is standard practice to write the flow power (rate of doing flow work) in terms of the mass flow rate. Recall that the mass flow rate is the product of the density and the volumetric flow rate. Using this result Eq. $7.5.6$ can be rewritten as follows:

$$\dot{W}_{\text{flow, in}} = P\dot{V}_{\text{in}} = P\left(\frac{\dot{m}_{\text{in}}}{\rho}\right) = \dot{m}_{\text{in}}\left(\frac{P}{\rho}\right)$$

where $\dot{m}_{\text{in}} =$ the mass flow rate at the inlet. $[\text{kg/s and lbm/s}]$

$\rho =$ the density of the substance at the inlet. $[\text{kg/m}^3 \text{ and lbm/ft}^3]$

$P =$ the pressure at the inlet. $[\text{N/m}^2, \text{kPa}, \text{ bar, and lbf/ft}^2]$

One final change of variable is traditionally made by substituting the specific volume v for the density ρ :

$$v = \frac{1}{\rho}$$

where specific volume has units of m^3/kg or ft^3/lbm. Making this substitution, we obtain the traditional form for flow power:

$$\dot{W}_{\text{flow, in}} = \dot{m}_{\text{in}}\, Pv$$

Flow power has the dimensions of $[\text{Energy}]/[\text{Time}]$

Systems with mass flow out

Using an analogous development to the one done for open systems with mass flowing in, we can show that

$$\dot{W}_{\text{flow,out}} = \dot{m}_{\text{out}}\, Pv$$

Systems with multiple inlets and outlets

A generic open system is shown in Figure $7.5.3$. The rate of energy transfer into the system by flow work (flow power in) is

$$\dot{W}_{\text{flow, net in}} = \sum_{\text{inlets}} \dot{m}_i (Pv)_i - \sum_{\text{outlets}} \dot{m}_e (Pv)_e$$

Note that the flow power for a closed system is zero since the flow rates are identically zero.

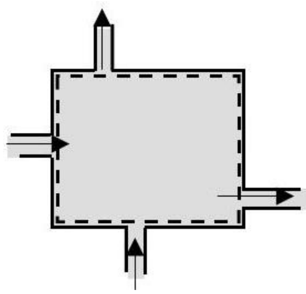


Figure $7.5.3$: Open system with multiple inlets and outlets.

7.6: Work and Power Revisited

In this chapter we have considered several different types of work or power. Specifically, we have discussed mechanical work, shaft work, PdV work, and electrical work. Each of these work transfers of energy satisfies the definition of thermodynamic work. In this section, we want to introduce another characteristic of work and power, the concept of a *quasiequilibrium* work transfer of energy.

To explore this concept in depth, we will consider electrical work. Specifically, we will examine electrical work for two situations: charging a battery and energizing a resistor. First, we will consider the electrical work required to charge a 12-volt battery. From electrochemistry, we know that the 12-volt designation is an indication of what the electrochemist might call the cell potential, ΔV_{cell}. This is an intensive property of the battery and depends solely on the chemical makeup of the battery. To recharge the battery, we must run a current backwards through the battery to reverse the chemical reaction. The differential amount of electrical work to recharge the battery is

$$\delta W_{\text{electric, in}} = i \cdot \left(V^{+} - V^{-}\right) \cdot dt = i \cdot \Delta V_{\text{terminals}} \cdot dt$$

where $\left(V^{+} - V^{-}\right) = \Delta V_{\text{terminals}}$ is the voltage difference between the battery terminals and i is the current. (Note that $i \cdot dt$ is dq, the differential amount of charge entering the system.) When the battery is being recharged, the voltage difference between the battery terminals must be at least equal to the cell potential, i.e. $\Delta V_{\text{terminals}} \geq \Delta V_{\text{cell.}}$.

If the battery is recharged in such a fashion that $\Delta V_{\text{terminals}} = \Delta V_{\text{cell}} + \varepsilon$, where ε is very, very small, we would say that this is quasiequilibrium electrical work and

$$\delta W_{\text{electric, in}} = \underbrace{\Delta V_{\text{cell}}}_{\substack{\text{An intensive} \\ \text{property}}} \cdot \underbrace{i \cdot dt}_{\substack{=dq \\ \text{Change in an} \\ \text{extensive} \\ \text{property}}} \quad \Bigg| \quad \begin{array}{c}\text{Quasiequilibrium} \\ \text{electric work}\end{array}$$

The word *quasiequilibrium* implies that the transfer of energy occurs in such a fashion that the system, as far as this work is concerned, is passing through a series of equilibrium states. For quasiequilibrium electric work, this implies that the electric current flows through the battery in such a fashion that the terminal voltage just equals the cell potential. Note that quasiequilibrium electric work is the product of an intensive property (cell potential) of the system and the change in an extensive property (charge) of the system. If ε is still very, very small but negative, the direction of the current would reverse as would the direction of the work transfer of energy. The ability to reverse the direction of the energy transfer with only an infinitesimal change in the terminal voltage difference is also a characteristic of quasiequilibrium work.

To see the significance of this, consider the case of the electric work done on a resistor:

$$\delta W_{\text{electric resistor}} = i \cdot \Delta V_{\text{terminals}} \cdot dt = i \cdot (iR) \cdot dt = i^2 R \cdot dt \geq 0$$

where R is the electrical resistance of the resistor. Unlike the case for Eq. $7.6.1$, making an infinitesimal change in the terminal voltage does not change the direction of the current or the direction of the energy transfer. In fact, electrical work transfer of energy to a resistor is always positive regardless of the direction of the current. This is an example of a *non-quasiequilibrium* work transfer of energy.

Similarly, if we examine the work done to compress a substance in a piston-cylinder device, we could write the differential amount of work as follows:

$$\delta W_{\text{piston, in}} = F_{\text{piston}}\, dx$$

where F_{piston} is the force that the piston exerts on the system and dx is the differential displacement of the boundary in the direction the external force acts. Under conditions where the piston moves slowly enough so that the force exerted by the piston just equals the uniform pressure in the system times the piston area, $F_{\text{piston}} = PA_{\text{piston}}$, we have quasiequilibrium compression-expansion work:

$$\delta W_{PdV} = -\underbrace{P}_{\substack{\text{Intensive} \\ \text{property}}} \cdot \underbrace{d\forall}_{\substack{\text{Change in an} \\ \text{extensive property}}} \quad \left|\quad \begin{array}{c}\text{Quasiequilibrium} \\ \text{compression-expansion work}\end{array}\right.$$

where P is the uniform pressure in the system and $d\forall$ is the differential change in the system volume. For this to be accurate, the system pressure (an intensive property) must be spatially uniform during the process. Again, a change in the direction of the volume change causes the sign of the work transfer of energy to change.

For comparison, consider the case where work is done on a system by a frictional force, say a block sliding on a plane.

$$\delta W_{\text{friction, in}} = -F_{\text{friction}} \cdot dx = -\mu F_{\text{normal}} dx$$

where \(F_{\text {normal}}\) is the normal force at the surface and μ is the appropriate coefficient of friction. Unlike PdV work, the transfer of energy into a system by frictional work is always the same sign and cannot be reversed by changing the direction of motion on the boundary.

So what conclusions can we draw from this discussion? First, there is a distinction between quasiequilibrium and non-quasiequilibrium work transfers of energy. Second, there are several distinctive characteristics that distinguish between these two types of work transfers of energy. These are summarized in the table below.

Characteristics of Quasiequilibrium Work Transfers of Energy

	Quasiequilibrium (Reversible) Work	Non-quasiequilibrium (Irreversible) Work
External force	Related to an intensive property of the system.	Related to the transport rate of an extensive property measured on boundary of the system.
Displacement	Related to a change of an extensive property of the system.	Related to a change of an extensive property of the system.
Constraints on system	Pertinent intensive property is spatially uniform.	No constraint on intensive properties of the system.
Generalized form	Can be written in terms of a generalized external force F_{K} (an intensive property) and a generalized displacement \(X_{\mathrm{K}}\) (an extensive property) $$\delta F_{\mathrm{K}} = \pm \underbrace{F_{\mathrm{K}}}_{\substack{\text{Intensive} \\ \text{Property}}} \cdot \underbrace{dX_{\mathrm{K}}}_{\substack{\text{Extensive} \\ \text{Property}}}$$	Frequently, most easily written as an energy transfer rate, i.e. power.
Direction of work transfer	Can change directions (reversible).	Cannot change directions (irreversible).

Third, as we will discuss in the next chapter these reversible work transfers of energy have much to do with the limits on the performance of real systems. In a later course, (ES202—Fluid & Thermal Systems), you will learn how these processes play a key role in our study of the thermophysical properties of substances. In this course, our main goal is to introduce the differences between quasiequilibrium and non-quasiequilibrium work transfers so that you better understand the consequences of modeling a work transfer using these assumptions.

Test your understanding

In the discussion above, you have seen how a work transfer of energy may or not be quasiequilibrium depending on how the work transfer of energy is related to the properties of the system. Consider shaft work under two situations:

- Case I - Shaft work drives a paddlewheel inside a tank of water.
- Case II - Shaft work is used to twist a thin wire within its elastic range. (Assume the wire acts as a torsional spring.)

Which of these cases involves a quasiequilibrium work transfer of energy? Why?

7.7: Heat Transfer Revisited

As defined earlier, heat transfer is a mechanism for transferring energy across the boundary of a system due to a temperature difference. We further distinguish between the heat transfer Q, an amount of energy (kJ and Btu), and the heat transfer rate $\dot{Q}$, an energy transfer rate (kW and $\mathrm{Btu/s}$). In this chapter, we will give you a very brief introduction to the subject of heat transfer. The goal is to introduce the basic mechanisms of heat transfer, not to make you an expert. Heat transfer is a mature discipline with applications in many fields. Key technologies, such as power generation or electronics and computer hardware, often face severe constraints because of limitations imposed by heat transfer processes.

7.7.1 Key concepts

There are three physical mechanisms for transferring energy by heat transfer - conduction, convection, and thermal radiation. Each of these can occur singly or in combination with the others. (Some people will argue that convection is not really a separate mechanism, but we will stick with three.)

Name	Physical Mechanism	Examples
Conduction	Energy transfer within a solid, liquid, or gas due to the microscopic motion of atoms and molecules. (Diffusion)	Styrofoam cooler; thermos bottle; clothing; fiberglass wall insulation.
Convection	Energy transfer within a fluid due to a combination of conduction and gross fluid motion.	Heat transfer between a surface and a fluid; heating water in a pan; car radiator;
Thermal Radiation	Energy transfer by electromagnetic radiation	Hot sand on a beach; solar collector; thermos bottle; electric toaster.

For use in the energy balance, we are interested in knowing the heat transfer rate; however, the heat transfer engineer often presents results in terms of the heat flux. The **heat flux** q'' is defined as the heat transfer rate per unit area with units of $\mathrm{W/m^2}$ or $\mathrm{Btu}/\left(\mathrm{h}\cdot\mathrm{ft^2}\right)$. The heat flux and the heat transfer rate are related by the following equation:

$$\dot{Q}_{\text{surface}} = \int\limits_{A_{\text{surface}}} q''\,dA$$

When the heat flux is uniform, we have the much simpler result that the heat transfer rate equals the product of the surface area and the heat flux:

$$\dot{Q}_{\text{surface}} = q'' A_{\text{surface}} \quad | \text{ Uniform heat flux}$$

This equation does not depend on the specific mechanism; however, the heat flux will change with the mechanism.

7.7.2 Conduction Heat Transfer

Conduction heat transfer occurs in solids, liquids, and gases. It is the diffusion of energy due to the microscopic motion of molecules and atoms. The diffusion process is limited by the ability of the atoms and molecules to transport the energy. The process of thermal energy diffusion is analogous to the diffusion of an ink drop placed in a glass of clear water.

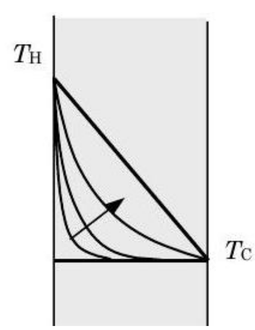


Figure 7.7.1: Temperature profile for transient conduction in a plane wall.

Consider the wall shown in Figure $7.7.1$. The wall is of thickness L with a surface area A and a thermal conductivity k. Initially, the wall is at a uniform temperature T_c. Suddenly the temperature of the left-hand side of the wall is changed to T_H as shown in the figure. Once this change in the boundary temperature occurs, thermal energy starts to diffuse through the wall and causes the temperature profile to change with time. With time, as indicated by the arrow, the temperature profile reaches a steady-state distribution that is represented by the straight line from T_H down to T_C. It takes a finite-time for the thermal energy to diffuse through the wall. At steady-state conditions, the conduction heat transfer rate through the wall is described by the equation:

$$\dot{Q}_{\text{conduction}} = k\frac{A}{L}(T_H - T_C)$$

Thus the steady-state conduction heat transfer through a plane wall is proportional to the thermal conductivity, the surface area, and the temperature difference, and it is inversely proportional to the thickness of the wall. The direction of the heat transfer of energy is always from the high temperature T_H to the lower temperature T_C.

Conduction is a strong function of the geometry of the heat flow path, with a plane wall being one of the simplest geometries. Frequently, conduction heat transfer information is contained in something called the **conduction heat transfer resistance** that is defined as follows:

$$R_{\text{conduction}} = \frac{\Delta T}{\dot{Q}_{\text{conduction}}} \geq 0$$

where all of the geometry and conductivity information is lumped into the value of the heat transfer resistance. The heat transfer resistance is always greater than or equal to zero and it typically has units of $^{\circ}\mathrm{C}/\mathrm{kW}$ or $^{\circ}\mathrm{F}/(\mathrm{Btu}/\mathrm{h})$. Given the heat transfer resistance and a temperature difference across the resistance, the conduction heat transfer rate is easily calculated as

$$\dot{Q}_{\text{conduction}} = \frac{\Delta T}{R_{\text{conduction}}}$$

The use of a heat transfer resistance is purposely meant to draw an analogy with electrical resistance. Eq. $7.7.5$ might be considered the "Ohm's Law" of heat transfer: the temperature difference (voltage difference) equals the product of the heat transfer rate (electric current) and the heat transfer resistance (electric resistance). This is a powerful analogy and forms the basis for problem solving in more advanced heat transfer courses.

7.7.3 Convection Heat Transfer

Convection is the transfer of thermal energy within a fluid due to a combination of conduction and gross fluid motion. More specifically we will focus on the convection heat transfer between a solid surface and an adjacent fluid as shown in Figure $7.7.2$.

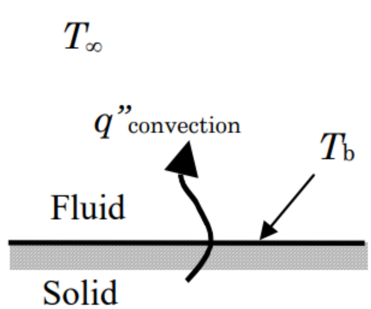


Figure $7.7.2$: Convection heat transfer from a surface at T_b to a fluid at T_∞

The equation used to describe the convection heat flux from the surface to the fluid is attributed to Newton and usually referred to as "Newton's Law of Cooling:"

$$q''_{\text{convection}} = h_{\text{convection}}\left(T_b - T_\infty\right)$$

where T_b is the surface (or boundary) temperature of the solid and T_∞ is the temperature of the fluid away from the surface, sometimes called the ambient temperature or the fluid temperature. The convection heat flux depends on the temperature difference and $h_{\text{convection}}$, the **convection heat transfer coefficient**. The convection heat transfer coefficient is typically developed empirically and depends on the fluid properties and the motion of the fluid. Fluid motion driven entirely by the naturally occurring density differences in the fluid is called *free* or *natural convection*. Fluid motion sustained by other forces is called *forced convection*. On a hot day, the convection heat transfer from the leather interior of your car to the air in the car is most likely due to natural convection, while the convection heat transfer within your car's radiator is forced convection.

For conditions where the convection heat flux is uniform, the convection heat transfer rate from a surface to a fluid is described by the following equation:

$$\begin{aligned} \dot{Q}_{\text{convection}} &= h_{\text{convection}} A_{\text{surface}} \left(T_b - T_\infty\right) \\ &= \frac{\left(T_b - T_\infty\right)}{R_{\text{convection}}} \quad \text{where } R_{\text{convection}} = \frac{1}{h_{\text{convection}} A_{\text{surface}}} \end{aligned}$$

where we have defined a convection heat transfer resistance. This is especially useful in conjunction with the energy balance to determine the temperature of a surface exposed to a fluid. If you can use the energy balance to solve for the heat transfer rate at the system boundary and you know the heat transfer resistance and the ambient fluid temperature, you can calculate the surface temperature at the boundary. In this course, when necessary we will give you the appropriate heat transfer coefficient values; however, in practice, determining the correct values for any given situation is one of the major challenges for a heat transfer engineer.

7.7.4 Thermal Radiation Heat Transfer

Last but not least, we come to thermal radiation heat transfer. We add the modifier "thermal" to distinguish this from nuclear radiation. Thermal radiation is the transfer of thermal energy between surfaces at different temperatures by electromagnetic radiation. It is the only heat transfer mechanism that can occur across a vacuum.

We will only consider one specialized case-thermal radiation heat transfer between a small convex object and its surroundings (See Figure $7.7.3$). Because this is a specialized case, we must be careful to explain the geometric limitations. By "small" we mean an object with a surface area much, much smaller than the surroundings it sees. By "convex body" we mean an object that cannot see itself. A baseball (or any sphere for that matter) is a convex surface; however, a baseball glove is not. The outer surface of a cylinder is a convex surface. The outside surface of a foam coffee cup is a convex surface. Thermal radiation proceeds between surfaces just like visible light. A convex surface is one where a person standing on the surface (a very small person) can scan the sky from horizon to horizon and not see the surface they are standing on.

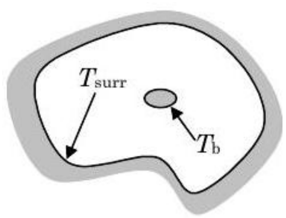


Figure $7.7.3$: Thermal radiation between a small convex object and its surroundings.

The thermal radiation heat transfer rate between a small convex object at temperature T_{b} and the surroundings at temperature T_{surr} is given by the equation:

$$\dot{Q}_{\text{radiation}} = A_{\text{surface}} \left[\varepsilon\sigma\left(T_b^4 - T_{\text{surr}}^4\right)\right] \tag{7.7.1}$$

where

$$\sigma = \text{Stefan-Boltzmann constant} \begin{cases} = 5.669 \times 10^{-8}\ \mathrm{W}/\left(\mathrm{m}^2 \cdot K^4\right) \\ = 0.1714 \times 10^{-8} \mathrm{Btu}/\left(\mathrm{hr} \cdot \mathrm{ft}^2 \cdot {}^\circ\mathrm{R}^4\right) \end{cases}$$

$$\begin{aligned} &\varepsilon = \text{emissivity of the surface of the convex body} \leq 1 \\ &A_{\text{surface}} = \text{surface area of the convex body} \\ &T_b = \text{surface temperature of the convex body (in K or } {}^\circ\mathrm{R}) \\ &T_{\text{surr}} = \text{temperature of the surroundings (in K or } {}^\circ\mathrm{R}) \end{aligned}$$

You should note that this equation has some significant differences from the previous two heat transfer rate equations we have discussed. First, note that the heat transfer rate is proportional to the difference between the absolute temperature of the body raised to the fourth power and the absolute temperature of the surroundings raised to the fourth power. Radiation heat transfer is not proportional to the simple temperature difference we saw with convection and conduction.

Second, the radiation heat transfer rate is directly proportional to the condition of the surface as indicated by the value of the emissivity of the surface. Emissivity values range from 0 to 1. The shinier the surface, the lower the emissivity value. A sheet of polished metal will typically have emissivity values on the order of 0.05, whereas oxidized metals will have values greater than 0.5. Red building brick with a rough surface has an emissivity in the range of 0.9. Radiation heat transfer depends strongly on the surface characteristics and finding an accurate emissivity value for a specific situation is a significant challenge.

Third, the temperature of the surroundings is not necessarily the temperature of the fluid adjacent to the object, but it is the temperature of the surrounding surfaces that the object sees. In our special case, we are assuming that the material in the void between the object and the surroundings does not participate in the heat transfer process. This is not always true. In many cases, the radiation proceeds from the surface of the object to the participating media in the void and finally to the surroundings.

Finally, we have totally ignored the wavelength dependence of thermal radiation. This is the phenomenon that causes a greenhouse to heat up. Solar radiation with one wavelength distribution passes easily through a glass window, while thermal radiation emitted from surfaces near room temperature with a significantly different wavelength distribution cannot pass back through the window and is trapped inside the greenhouse.

The last point to make about thermal radiation heat transfer is that it can occur in parallel with both of the other mechanisms we have discussed. Our main interest in this course will be cases where convection and thermal radiation occur in parallel from a surface. Under these conditions, the heat transfer rate from the surface is found by adding the contributions of convection and thermal radiation as follows:

$$\begin{aligned}\dot{Q}_{\text{surface}} &= \dot{Q}_{\text{convection}} + \dot{Q}_{\text{radiation}} \\ &= h_{\text{convection}}\, A_{\text{surface}} \left(T_b - T_\infty\right) + A_{\text{surface}} \left[\varepsilon\sigma \left({T_b}^4 - {T_{\text{surr}}}^4\right)\right]\end{aligned}$$

Note that there are three temperatures in this formulation: the temperature of the boundary of the object T_b, the temperature of the fluid adjacent to the body T_∞, and the temperature of the surrounding that an object sees T_{surr}. Each of these may be different. Often the fluid and surroundings temperatures are assumed to be equal if surroundings temperature does not differ significantly from the fluid temperature.

An interesting application — Skin temperature in a warm room with cold walls

This problem is for those of you who have sat in a warm room on a cold day and felt chilled.

Assume that we know you are comfortable when you sit in a room where the air temperature and the temperature of the surrounding walls are both $20^\circ\mathrm{C}\ (293\ \mathrm{K})$. Under these conditions, your skin temperature is $25^\circ\mathrm{C}\ (298\ \mathrm{K})$. Assume that the surface area of your arm is $0.120\ \mathrm{m}^2$, that your skin has an emissivity of 0.8, and that the heat transfer coefficient from your skin is $10\ \mathrm{W}/\left(\mathrm{m}^2 \cdot{}^\circ\mathrm{C}\right)$ and your skin loses no energy by heat transfer.

(1) Calculate the combined heat transfer rate from your arm by radiation and convection. This energy must be supplied to your skin by the blood supply to your arm. [Hint: Use Eq. $7.7.9$. Remember to use absolute temperatures (K) in your calculations.] What percentage of the heat transfer is by convection and by radiation?

(2) Now you are sitting in a room where the walls (the surroundings) are at $5^\circ\mathrm{C}(278\ \mathrm{K})$ because of faulty insulation in the walls. Because of the heating system, the air temperature is the same as before. Assuming the heat transfer rate to your skin that you calculated in (1) remains the same when you move to the new room, what is the temperature of your skin in the new conditions. Does it increase or decrease or stay the same? Does this explain why you feel chilled? [Hint: This may require an iterative solution since T_b appears in two places.]

It is very common to neglect thermal radiation heat transfer; however, when the convection heat transfer coefficient is low or the temperature difference is large, thermal radiation can be very important

7.8: Electrical Energy Storage and Transfer

Because of its importance and its uniqueness, we need to take a closer look at the transfer and storage of electrical energy. As a start, what exactly do we mean by electrical energy? For our purposes, we will define electrical energy as the energy that is stored in an electric or a magnetic field. Our emphasis here will be to consider how the conservation of energy principle applies to devices and systems commonly found in electrical and electronic devices. We will limit ourselves to systems that can be modeled using lumped circuit elements (as discussed in Section 4.3 of Chapter 4).

7.8.1 Instantaneous and Average Power

Earlier in this chapter, we developed an equation for the electric power in terms of the flow of an electric current through the system and the electric potential difference at the terminals where the current enters and leaves the system. In our earlier development, we tacitly assumed that the current and voltages were independent of time. In reality, we know that almost all of our electric power is supplied in the form of a time-varying alternating current. The transient behavior of electrical circuits is also of interest in the design of everything from power systems to control systems to computers. In this section, we will address the problem of calculating AC power.

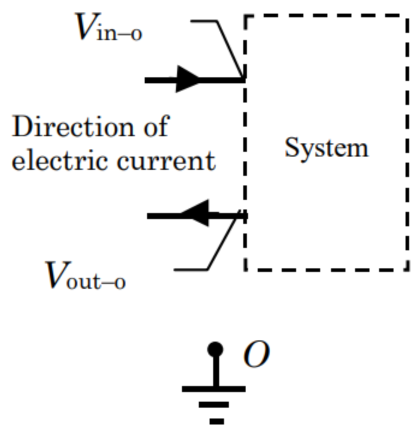


Figure 7.8.1: Two-terminal system for electric work and power.

Instantaneous Electric Power

For purposes of our discussion, we will restrict ourselves to a two-terminal system as shown in Figure 7.8.1. The electric current i flows into the system at a terminal with voltage $V_{\text{in-o}}$ and leaves the system at a terminal with voltage $V_{\text{out-o}}$. Both voltages are measured with respect to the same ground point (point O). In addition, we will assume that the voltages and current may change with time. Under these conditions, our earlier expression for electric power is still valid; but it represents the **instantaneous electric power** for a system:

$$\begin{aligned} \dot{W}_{\text{electric, in}} &= i \cdot (V_{\text{in-o}} - V_{\text{out-o}}) \\ &= i \cdot \Delta V \qquad\qquad \text{Instantaneous Electric Power} \end{aligned}$$

By convention, the instantaneous electric power is positive when the current enters the system at the terminal with the higher voltage and negative when it leaves the system at the terminal with the higher voltage.

In a DC (direct-current) system, the instantaneous power is a constant and independent of time; thus, the instantaneous electric power is also constant and independent of time.

In an AC (alternating-current) system, the current i and the voltage difference ΔV for the terminals can be described as sinusoidal functions of time:

$$\Delta V = V_{\max} \cos(\omega t) \quad \text{and} \quad i = i_{\max} \cos(\omega t + \theta)$$

where ω is the frequency (rad/s), t is the time (s), and θ is the phase angle (radians) that describes how current leads or lags the voltage $[-\pi/2 \leq \theta \leq \pi/2]$. The period for one cycle depends on the frequency as follows:

$$2\pi = \omega \cdot t_{\text{period}} \quad \rightarrow \quad t_{\text{period}} = \frac{2\pi}{\omega}$$

Under these conditions, the instantaneous power is calculated as follows:

$$\begin{aligned}
\dot{W}_{\text{electric, in}} &= i \cdot \Delta V = [i_{\max}\cos(\omega t)] \cdot [V_{\max}\cos(\omega t + \theta)] \\
&= i_{\max} \cdot V_{\max} \cdot \cos(\omega t) \cdot \cos(\omega t + \theta) \\
&= i_{\max} \cdot V_{\max} \cdot \frac{1}{2} \cdot [\cos(2\omega t + \theta) + \cos(\theta)]
\end{aligned}$$

where the last line is obtained by applying a standard trigonometric relationship for multiplying cosines. Before leaving this expression it is helpful to separate the instantaneous power, Eq. $7.8.4$, into two parts:

$$\begin{aligned}
\dot{W}_{\text{electric, in}} &= i_{\max} \cdot V_{\max} \cdot \frac{1}{2} \cdot [\cos(2\omega t + \theta) + \cos(\theta)] \\
&= \underbrace{\frac{i_{\max} \cdot V_{\max}}{2}\cos(\theta)}_{\substack{\text{Time-independent} \\ \text{component}}} + \underbrace{\frac{i_{\max} \cdot V_{\max}}{2}\cos(2\omega t + \theta)}_{\substack{\text{Time-varying periodic} \\ \text{component}}}
\end{aligned}$$

The first part is independent of time and only depends on the phase angle, θ, while the second component varies periodically with time. A careful examination of Eq. $7.8.5$ shows that the instantaneous AC power is a sinusoidal function that oscillates between zero and a maximum (or peak) value of $i_{\max} \cdot V_{\max} \cdot \cos(\theta)$.

Average Electric Power

The **average electric power** is defined as the amount of electric energy transferred across a boundary divided by the time interval over which the transfer occurs. Mathematically, the average electric power for a time interval t_{obs} can be calculated from the equation

$$\dot{W}_{\text{avg, in}} = \frac{1}{t_{\text{obs}}} \int_0^{t_{\text{obs}}} \dot{W}_{\text{electric, in}}\, dt$$

If the voltage and current are constants as they would be in a DC system, the average power and the instantaneous power are identical. In an AC system, the average power would be calculated over the time period for one cycle of the instantaneous power (or two cycles of the voltage and current) as follows:

$$\begin{aligned}
\dot{W}_{\text{avg, in}} &= \frac{1}{t_{\text{obs}}} \int_0^{t_{\text{obs}}} \dot{W}_{\text{electric, in}}\, dt \quad \text{where} \qquad t_{\text{obs}} = t_{\text{period}} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega} \\
&= \frac{1}{t_{\text{period}}} \int_0^{t_{\text{period}}} \left[\frac{i_{\max} \cdot V_{\max}}{2}\cos(\theta)\right] dt \qquad \left| \begin{array}{l} \text{What happened to the} \\ \text{time-varying component?} \end{array} \right. \\
&= \frac{1}{t_{\text{period}}} \left[\frac{i_{\max} \cdot V_{\max}}{2}\cos(\theta)\right] t_{\text{period}} = \frac{i_{\max} \cdot V_{\max}}{2}\cos(\theta)
\end{aligned}$$

This finally gives

$$\dot{W}_{\text{avg, in}} = \frac{1}{2} i_{\max} \cdot V_{\max} \cdot \cos(\theta) \qquad \begin{array}{c} \text{Average} \\ \text{AC electric power} \end{array}$$

Thus the average power can be calculated knowing the maximum (or peak) values for current voltage and the angle θ. Note, that the numerical constant of $1/2$ depends on the shape of the voltage and current signals and not the specific frequency. If the voltage and current signals had a different shape, such as a sawtooth wave or a square wave, the numerical constant would change.

7.8.2 AC Power and Effective Voltage and Current

What exactly do we mean when we say 110 ac volts? Is this the maximum (or peak) voltage? Is it the average? But the average wouldn't make any sense because the average of a sinusoid that oscillates around zero is zero. It may seem strange to you, but there once was a time when electric power was distributed using DC voltages and currents. Unfortunately for Thomas Edison, who promoted DC distribution, AC systems won out. However, the ghost of DC still lives in the way we talk about AC voltages and currents. It turns out that AC voltage and current are reported in terms of their effective values that are related to a DC equivalent that would deliver the same average power to a resistor.

The **effective value of an AC current** i_{eff} equals the value of a DC current that would deliver the same average power to a load resistor. The relationship between the maximum AC current and the effective value of the AC current is developed below.

$$\begin{aligned} \dot{W}_{\text{electric, in}}\Big|_{average,\ DC} &= \dot{W}_{\text{electric, in}}\Big|_{average,\ AC} \\ i_{\text{eff}}{}^2\ \cancel{R_{\text{load}}} = \frac{1}{t_{\text{period}}} \int_0^{t_{\text{period}}} i^2\ \cancel{R_{\text{load}}}\ dt &= \frac{1}{t_{\text{period}}} \int_0^{t_{\text{period}}} \left[i_{\max}\cos(\omega t+\theta)\right]^2\ \cancel{R_{\text{load}}}\ dt \\ i_{\text{eff}}{}^2 = i_{\max}{}^2 &\left\{\frac{1}{t_{\text{period}}} \int_0^{t_{\text{period}}} \left[\cos(\omega t+\theta)\right]^2 dt\right\} \\ = i_{\max}{}^2 &\left\{\frac{1}{t_{\text{period}}} \int_0^{t_{\text{period}}} \left[\frac{1}{2}+\frac{1}{2}\cos(2\omega t+2\theta)\right] dt\right\} \\ = &\left\{\frac{1}{t_{\text{period}}}\left[\frac{1}{2}(t_{\text{period}}-0)+0\right]\right\}^2 = \frac{i_{\max}{}^2}{2} \end{aligned}$$

Thus we have the final result:

$$i_{\text{eff}} = \frac{i_{\max}}{\sqrt{2}} \qquad \text{Effective AC current}$$

If the time-varying current had a different waveform than a sinusoid, say a square wave or a sawtooth wave, the constant in the numerator would take on a different value.

Similarly, the effective value of an AC voltage V_{eff} equals the value of a DC voltage that would deliver the same average power to a load resistor. Using a development similar to the one for effective current, we discover that:

$$V_{\text{eff}} = \frac{V_{\max}}{\sqrt{2}} \qquad \text{Effective AC voltage}$$

And this is the answer to the question we started out with — what is a $110\,\text{AC}$ voltage? The 110 volts is the effective value of the AC voltage. So on an oscilloscope, a 110-AC-volt signal would be a sinusoid with a maximum value $V_{\max} = 110\cdot\sqrt{2} = 155.6$ volts. A 220-AC-volt signal would have a maximum value of 311.1 volts.

Now if we revisit the equation for the average AC power, we can rewrite it in terms of the effective values:

$$\begin{aligned} \dot{W}_{\text{avg, in}} &= \frac{i_{\max}\cdot V_{\max}}{2}\cos(\theta) = \left(\frac{i_{\max}}{\sqrt{2}}\right)\left(\frac{V_{\max}}{\sqrt{2}}\right)\cos(\theta) \\ &= \underbrace{i_{\text{eff}}\cdot V_{\text{eff}}}_{\substack{\text{Apparent}\\ \text{Power}}}\cdot\underbrace{\cos(\theta)}_{\substack{\text{Power}\\ \text{Factor}}} \qquad \text{Average AC Power} \end{aligned}$$

Thus the average AC power transferred at a system boundary is the product of the effective current, the effective voltage, and $\cos(\theta)$.

A closer examination of the average AC power relation, Eq. $7.8.10$, shows that it is the product of the apparent power and the power factor. Because the apparent power is not really a power, it is frequently given the units of volt-amps. (If you look on the nameplate on heavy duty electrical equipment you will frequently see values reported in $\text{kV}\cdot\text{A}$, kilovolt-amps.) The power factor can have a value between 0 and 1 and depends on the behavior of the load, specifically the load impedance. Determining the phase angle and power factor for a specific electrical system is the subject of courses in circuit analysis, e.g. ES203 — Electrical Systems, which most of you will be taking later. For systems with a *purely resistive load* the power factor is one $[\theta = 0 \text{ and } \cos(\theta) = 1]$. In this text, unless you are told differently, you may assume that the power factor is unity. (Note that this assumption always gives you the maximum possible electric power for a given situation.)

7.8.3 Storage of Electrical Energy

When energy is transferred to or from a system by the flow of electrical current, what happens to this energy inside the system? The answer to this question depends on what is inside the system. In this section, we will limit ourselves to devices commonly found in electrical circuits: resistors, capacitors, inductors, and batteries.

For our discussion, we will assume that our system can store energy in six different forms:

$$E_{\text{system}} = U + \underbrace{E_{MF} + E_{EF}}_{\text{Electrical Energy}} + \underbrace{E_{K,\text{ trans}} + E_{K,\text{ rot}} + E_{GP}}_{\text{Mechanical Energy}}$$

$$\begin{aligned} &\text{where} \\ U &= \text{internal energy} \\ E_{EF} &= \text{energy stored in an electric field} \\ E_{MF} &= \text{energy stored in a magnetic field} \\ E_{K,\text{ trans}} &= \text{translational kinetic energy} \\ E_{K,\text{ rot}} &= \text{rotational kinetic energy} \\ E_{GP} &= \text{gravitational potential energy} \end{aligned}$$

The new players in this discussion are the energy stored in an electric field and the energy stored in a magnetic field. In writing Eq. $7.8.11$ with six discrete energy terms, we have assumed that the energy terms are independent of each other. This will be sufficient for our discussion here; however, a more detailed presentation of thermodynamics would reveal that there are many situations where the electrical and internal energy terms are not independent and must be treated as a single term (J. Kestin, *A Course in Thermodynamics*, Blaisdell Publishing Co, Waltham, Massachusetts, 1966, Chpt 8.).

If we restrict ourselves to a closed system, then the energy balance becomes

$$\frac{d}{dt}\left[U + E_{K,\text{ trans}} + E_{K,\text{ rot}} + E_{GP} + E_{EF} + E_{MF}\right] = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$$

Now let's consider several different components that are often found in an electrical system.

Resistor

One of the most basic components of an electric circuit is a resistor. For our purposes, we will assume that an *ideal* resistor is one that satisfies Ohm's law $V_R = iR$ as illustrated in Figure $7.8.2$ and cannot store energy in electric and magnetic fields.

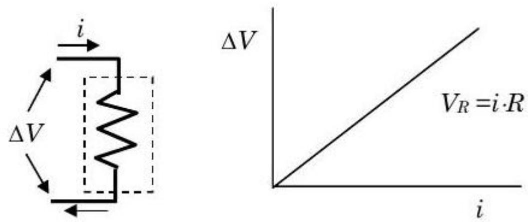


Figure $7.8.2$: Voltage-current relationship for an ideal resistor.

If we apply the conservation of energy to an adiabatic, ideal resistor we find the following:

$$\begin{aligned} \frac{d}{dt} E_{\text{resistor}} &= \dot{W}_{\text{net, in}} + \cancelto{=0}{\dot{Q}_{\text{net, in}}} \\ \frac{d}{dt}\left[U + \cancel{E_{K,\text{ trans}}} + \cancel{E_{K,\text{ rot}}} + \cancel{E_{GP}} + \cancel{E_{MF}} + \cancel{E_{EF}}\right] &= i \cdot \Delta V \quad \text{ where } \Delta V = V_R \\ \frac{dU}{dt} &= i \cdot V_R = i \cdot (iR) \end{aligned}$$

Finally we have

$$\frac{dE_{\text{resistor}}}{dt} = \frac{dU}{d} = i^2 \cdot R$$

So electric power supplied to an adiabatic, ideal resistor results in an increase in the internal energy of the system. For a finite time period, the change in energy of the resistor is

$$\Delta E_{\text{resistor}} = \Delta U = \int_{t_1}^{t_2} \left(i^2 \cdot R\right) dt \quad \geq 0$$

Note that this is an irreversible transfer of energy because changing the direction of the current will not decrease the internal energy of the system.

Capacitor

The second basic circuit component we will examine is the capacitor. A capacitor consists of two charged surfaces separated by a dielectric. For our purposes, an ideal capacitor will be one that can only store energy in an electric field within the capacitor and that satisfies the voltage-current relationship embodied in Figure $7.8.3$.

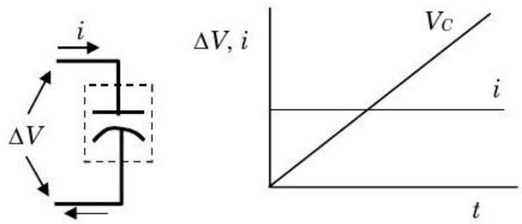


Figure $7.8.3$: Voltage-current relationship for an ideal capacitor.

Analysis of this figure shows that the voltage across the capacitor and the current are related by the expression:

$$i = C\frac{dV_C}{dt}$$

where C is the capacitance and is measured in farads (F) and $1\ \mathrm{F} = (1\ \mathrm{A})(1\ \mathrm{s})/(1\ \mathrm{V})$.

Applying the conservation of energy to an adiabatic, ideal capacitor gives the following:

$$\begin{aligned} \frac{d}{dt}E_{\text{capacitor}} &= \dot{W}_{\text{net, in}} + \overbrace{\cancel{\dot{Q}_{\text{net, in}}}}^{=0} \\ \frac{d}{dt}\left[\cancel{U} + \cancel{E_{K,\text{ trans}}} + \cancel{E_{K,\text{ rot}}} + \cancel{E_{GP}} + E_{EF} + \cancel{E_{MF}}\right] &= i \cdot \Delta V \quad \text{where } \Delta V = V_C \\ \frac{dE_{EF}}{dt} &= i \cdot V_C = \left(C\frac{dV_C}{dt}\right) \cdot V_C \end{aligned}$$

Finally we have

$$\frac{dE_{\text{capacitor}}}{dt} = \frac{dE_{EF}}{dt} = \frac{d}{dt}\left(C\frac{{V_C}^2}{2}\right) \quad \text{where } E_{\text{capacitor}} \equiv C\frac{{V_C}^2}{2}$$

So the instantaneous electric power supplied to an adiabatic, ideal capacitor results in a change in the energy stored in the electric field within the capacitor. If the capacitor is subjected to an AC voltage, the time-averaged energy stored in the capacitor is calculated by substituting the effective voltage as follows.

$$E_{\text{capacitor}}\Big|_{\text{average AC}} = C\frac{{V_{C,\text{ eff}}}^2}{2} \qquad \begin{array}{c}\text{Average energy stored}\\ \text{in a capacitor driven by}\\ \text{an AC voltage.}\end{array}$$

For a finite-time period, the change in the energy of the capacitor is just the change in the energy of the capacitor:

$$\Delta E_{\text{capacitor}} = \Delta E_{EF} = \frac{C}{2}\left({V_{C,2}}^2 - {V_{C,1}}^2\right)$$

Notice that unlike the energy storage in the resistor, the energy stored in an adiabatic capacitor can both increase and decrease.

Inductor

The third basic circuit component we will examine is the inductor. An inductor consists of cylindrical coil of wire. For our purposes, an ideal inductor will be one that can only store energy in a magnetic field within the inductor and that satisfies the voltage-current relationship embodied in Figure $7.8.4$.

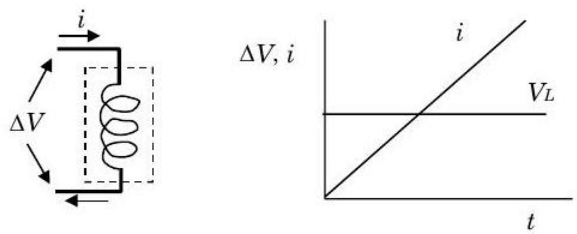


Figure $7.8.4$: Voltage-current relationship for an ideal inductor.

Analysis of this figure shows that the voltage across the inductor and the current are related by the expression:

$$V_L = L\frac{di}{dt}$$

where L is the inductance and is measured in henrys (H) and $1\mathrm{H} = (1\ \mathrm{V})(1\ \mathrm{s})/(1\ \mathrm{A})$.

Applying the conservation of energy to an adiabatic, ideal inductor gives the following results:

$$\begin{aligned} \frac{d}{dt}E_{\text{inductor}} &= \dot{W}_{\text{net, in}} + \cancelto{=0}{\dot{Q}_{\text{net, in}}} \\ \frac{d}{dt}\left[\cancel{U} + \cancel{E_{\text{K, trans}}} + \cancel{E_{K,\text{rot}}} + \cancel{E_{GP}} + \cancel{E_{EF}} + E_{MF}\right] &= i\cdot\Delta V \quad \text{ where } \Delta V = V_L \\ \frac{dE_{MF}}{dt} &= i\cdot V_L = i\cdot\left(L\frac{di}{dt}\right) \end{aligned}$$

Finally we have

$$\frac{dE_{\text{Inductor}}}{dt} = \frac{dE_{MF}}{dt} = \frac{d}{dt}\left(L\frac{i^2}{2}\right) \quad \text{ where } \quad E_{\text{Inductor}} \equiv L\frac{i^2}{2}$$

So the electric power supplied to an adiabatic, ideal inductor results in a change in the energy stored in the magnetic field within the inductor. If the inductor is subjected to an AC current, the time-averaged energy stored in the energy is calculated by substituting the effective current as follows:

$$E_{\text{inductor}}\big|_{AC} = L\frac{{i_{\text{eff}}}^2}{2} \qquad \begin{array}{c}\text{Average energy stored}\\ \text{in an inductor driven}\\ \text{by an AC current}\end{array}$$

For a finite-time period, the change in the energy of the inductor is just the change in the energy of the inductor:

$$\Delta E_{\text{inductor}} = \Delta E_{MF} = \frac{L}{2}\left({i_2}^2 - {i_1}^2\right)$$

Notice that unlike the energy stored in the resistor, the energy stored in the adiabatic inductor can both increase and decrease.

Battery

The last component we will consider is the battery. An ideal battery will satisfy the voltage-current relationship shown in Figure $7.8.5$ and cannot store energy in electric and magnetic fields.

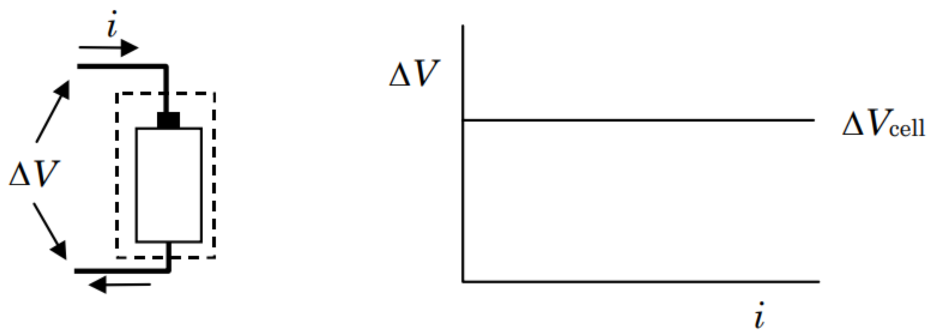


Figure 7.8.5: Voltage-current relationship for an ideal battery.

If we apply conservation of energy to the adiabatic, ideal battery we have the result that

$$\begin{aligned} \frac{dE_{\text{battery}}}{dt} &= \dot{W}_{\text{net, in}} + \overset{=0}{\cancel{\dot{Q}_{\text{net, in}}}} \\ \frac{d}{dt}\left[U + \cancel{E_{K,\text{trans}}} + \cancel{E_{K,\text{rot}}} + \cancel{E_{GP}} + \cancel{E_{UP}} + \cancel{E_{EF}}\right] &= i \cdot \Delta V \qquad \text{where } \Delta V = \Delta V_{\text{cell}} \\ \frac{dU}{dt} &= i \cdot \Delta V_{\text{cell}} \end{aligned}$$

So finally, we have for the adiabatic, ideal battery,

$$\frac{dE_{\text{battery}}}{dt} = \frac{dU}{dt} = i \cdot \Delta V_{\text{cell}}$$

Thus the electric power supplied to a battery goes into a change in the internal energy of the battery.

For a finite-time interval the change in energy of the battery is written as follows:

$$\Delta E_{\text{battery}} = \Delta U = \int_{t_1}^{t_2} (i \cdot \Delta V_{\text{cell}})\, dt$$

Note that like both the capacitor and the inductor and unlike the resistor, the internal energy of an ideal, adiabatic battery can both increase and decrease.

7.8.4 AC Power and Steady-state Systems

When a system is supplied with AC power, the instantaneous power and thus the energy transfer rate on the boundary changes with time in a periodic fashion. Our steady-state assumption requires that nothing within or on the boundary of the system change with time. This would seem to prevent us from ever assuming steady-state behavior when AC power is supplied to a system. However, our world is full of systems driven by AC power that for all appearances would seem to be operating at "steady-state" condition.

To handle this apparent conflict, we can draw on our experience with the average electric power that we developed earlier in Section 7.8.1. If we time-average the rate-form of the conservation of energy equation as we did in finding the average electric power, we end up with an equation that looks exactly like the original rate-form of the conservation of energy equation. The only difference is that each term has been time-averaged. If we are only interested in system behavior on time scales much larger than 1 period of an AC signal ($1/60$ s) the time-averaged equation will perform just like the original equation.

Now when we say that a system with AC power is operating at steady-state conditions, what we are really saying is that on average the system is not changing with time. This means the average AC power is not changing with time. It also means that anything else about the system that was varying periodically with time, e.g. energy storage in capacitors and inductors, does not change on average with time.

This phenomenon is not really unique to electrical power. If you monitor the drive shaft torque coming off your car engine, you will probably discover that although the shaft rotational speed is constant, the torque will vary with shaft angle as the shaft rotates. This gives a shaft torque that varies periodically with time. Thus shaft power may actually fluctuate, but we just report an average value. So without even knowing it, we are sometimes invoking the "time-averaged" steady-state assumptions. Clearly, if we needed to analyze the behavior of the system on a time scale equal to the period of one rotation of the shaft, it would be incorrect to average out the fluctuations

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7.9: Thermodynamic Cycles

As we study the behavior of devices, we will discover that one of the most important processes from a theoretical and a technological viewpoint is the *thermodynamic cycle*. More specifically we will be concerned with the performance of three types of thermodynamic cycles — power cycles, refrigeration cycles and heat pump cycles. In this section, we will discuss the basic features of a thermodynamic cycle, then we will discuss three ways to classify these devices, and finally we will consider how to measure their performance.

7.9.1 Key Features and Examples

The industrial revolution was driven in part by the development of the steam engine — a device that takes in "heat" from a fire, delivers "work" to the surroundings, and operates as a cycle. Our modern society is populated with machines and devices that either operate as a true thermodynamic cycle or can be modeled as one for purposes of analysis. The modern day internal combustion engine started out as, and is still modeled as, either an *Otto cycle*, a *Diesel cycle*, or a *combined cycle*. The modern-day fossil-fueled or nuclear-fueled steam power plant is modeled as a *Rankine cycle*. The modern gas-turbine engine, whether used in a jet engine to propel an aircraft or as part of a natural-gas-fired electrical energy peaking station, has its roots in the *Brayton cycle*. Some of the innovative external combustion engines currently being considered for automobiles and remote power-generation stations in remote regions are based on the *Stirling cycle*. One of the most famous theoretical cycles discussed in physics because of its relationship to the second law of thermodynamics is the *Carnot cycle*. All of these cycles are examples of power cycles.

If you are now tired of thinking of power generation but truly like to relax in your air-conditioned room, you are not finished with cycles. The guts of your window air-conditioner are modeled as a *mechanical vapor-compression cycle*. In fact, most refrigeration systems, whether used to cool the air in your house, maintain the food in your refrigerator, or keep those freezer display cases in the grocery cold, are also mechanical vapor-compression cycles. If you travel by air and enjoy a cool cabin you have benefited from a *reversed-Brayton cycle*. Sometimes you will run across a refrigerator or air conditioner that requires little or no electricity but needs a natural gas or propane flame to operate. These are examples of *absorption cycles*. (A company called Arkla used to manufacture these types of systems in Evansville. They were especially popular before rural electrification and are making a comeback today.) And last but not least, if your home has a heat pump, guess what? This is typically a mechanical vapor-compression cycle.

Now that you've been exposed to how much our society depends on this thing called a thermodynamic cycle, what is it? A **thermodynamic cycle** is a closed system that executes a series of processes that periodically return the system to its initial state.

This seems simple especially when you recognize that it is the basis for studying all of the essential devices described earlier. Since we are limited to closed systems, the rate-form of the conservation of energy equation becomes the following:

$$\frac{dE_{sys}}{dt} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$$

which is valid at any time during the cycle. We will return to this balance shortly after we discuss the physical structure of cycles.

7.9.2 Classifying Cycles

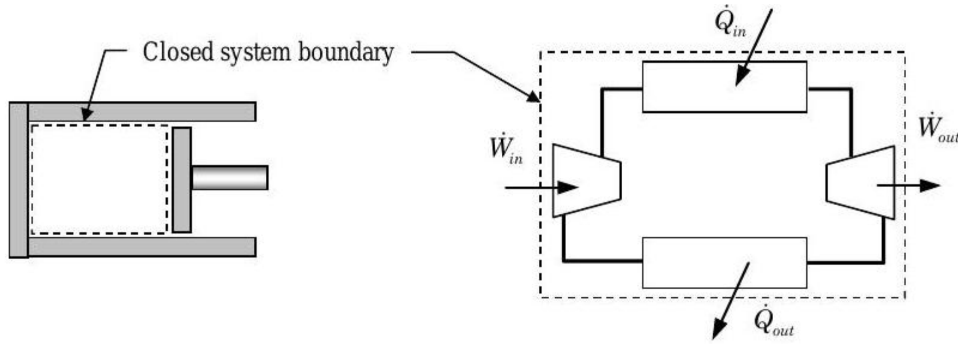
In any discussion of thermodynamic cycles it is useful to be able to classify them. We will introduce three different ways to classify cycles — working fluids, physical structure, and purpose.

Classification by Working Fluid

If you examine the various devices mentioned earlier that operate as thermodynamic cycles, you discover that they all operate by changing the properties of a substance inside the closed system. This substance is called a *working fluid*. All of the examples presented earlier fall into one of two categories. In the cycles for the automobile engine or the jet engine, the working fluid remains a gas throughout the entire cycle. These are examples of cycles that operate with a *single-phase working fluid*. The cycles that form the basis for most refrigerators and for the fossil-fueled steam power plants change the phase of the working fluid from a liquid to a vapor and then back to a liquid in the cycle. These are examples of cycles that operate with a *two-phase working fluid*. These distinctions will not be very significant to us this quarter; however, to the designer they have a great significance in determining the size, weight, cost, and performance of a specific cycle.

Classification by Physical Structure

Most devices that operate as or can be modeled as a thermodynamic cycle have a physical structure that fits into one of two categories — a closed, periodic cycle or a closed-loop, steady-state cycle. These are illustrated in Figure 7.9.1.



(a) Closed, periodic cycle (left) (b) Closed-loop, steady-state cycle (right)
Figure 7.9.1: Classification of thermodynamic cycles by physical structure

The *closed, periodic cycle* is modeled as a fixed quantity of matter contained inside of a simple piston cylinder device [see Figure 7.9.1(a)]. This cycle is characterized by spatially uniform intensive properties that vary periodically with time. This is the classic cycle that has been studied by engineers for years. It is the model for the early steam engines and is still the model for the modern internal combustion engine where a gas is compressed and expanded within a piston engine.

The *closed-loop, steady-state cycle* is modeled as a collection of steady-state devices that are connected together to form a closed loop of fluid as shown in Figure 7.9.1(b). The closed loop of fluid forms a closed system. Steady-state devices commonly used in these cycles are pumps, turbines, compressors, heat exchangers, and valves. This cycle is characterized by spatially-nonuniform intensive properties that depend on position in the fluid loop but do not change with time. This cycle is the model for the modern gas turbine engine, the modern steam power plant, and the modern refrigerator and air-conditioner.

To investigate what conservation of energy can tell us about these cycles, we apply the rate form of the closed system energy balance to each type of cycle and manipulate appropriately:

$$\frac{dE_{sys}}{dt} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$$

Closed-periodic cycle	Closed-loop, steady-state cycle
$\underbrace{\int_t^{t+\Delta t_{\text{cycle}}} \left(\frac{dE_{sys}}{dt}\right) dt}_{\substack{\text{Integrated over one period}\\ \text{of the cyle}}} = \int_t^{t+\Delta t_{\text{cycle}}} \dot{Q}_{\text{net, in}} + \int_t^{t+\Delta t_{\text{cycle}}} \dot{W}_{\text{net, in}}$ $\underbrace{E_{sys}\big\|_{t+\Delta t_{\text{cycle}}} - E_{sys}\big\|_t}_{=0.\text{ Why?}} = Q_{\text{net, in}} + W_{\text{net, in}}$ $0 = Q_{\text{net, in}} + W_{\text{net, in}}$ $Q_{\text{net, in}} = -W_{\text{net, in}} = W_{\text{net, out}}$	$\underbrace{\cancelto{=0}{\frac{dE_{sys}}{dt}}}_{\substack{\text{Steady-state}\\ \text{system}}} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$ $0 = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$ $\dot{Q}_{\text{net, in}} = -\dot{W}_{\text{net, in}} = \dot{W}_{\text{net, out}}$

In both cycles, we discover something similar — The net *heat* transfer (or transfer rate) of energy into the system equals the net *work* transfer (or transfer rate) of energy out of the system for thermodynamic cycle.

This result leads to the common interpretation of a thermodynamic cycle as an *energy conversion device* for converting heat transfer of energy into work transfers of energy.

Classification by Purpose

Experience has shown that work transfers of energy are more valuable than heat transfers of energy. This means that we can do more things with a work transfer of energy than we can with a heat transfer of energy. Because of this, we will choose to define the purpose of a device in terms of the work transfer of energy for the cycle.

If the net power or work *out* of the cycle is positive, then we call the device a **power cycle** or **heat engine**:

$$\dot{W}_{\text{net, out}} \text{ or } W_{\text{net, out}}\big|_{\text{cycle}} > 0 \quad \rightarrow \quad \text{Power cycle}$$

The purpose of a power cycle is to take a net amount of energy by heat transfer from the surroundings and transfer back to the surroundings a net amount of energy as work.

If the net power or work *into* the cycle is positive, then we call the device a refrigeration or heat pump cycle (sometimes this is called a *reversed* power cycle):

$$\dot{W}_{\text{net, in}} \text{ or } W_{\text{net, in}}\big|_{\text{cycle}} > 0 \quad \rightarrow \quad \begin{array}{c}\text{Refrigeration or}\\ \text{heat pump cycle}\end{array}$$

The purpose of a refrigeration or heat pump cycle is to take a net amount of energy by work from the surroundings and transfer a net amount of energy by heat transfer back to the surroundings. More specifically, these cycles take in an amount of energy by heat transfer at a low temperature and reject a larger amount of energy by heat transfer at a higher temperature. A **refrigeration cycle** is built to maximize the amount of energy that can be transferred into the cycle by heat transfer at the low temperature. A **heat pump cycle** is built to maximize the amount of energy that can be transferred out of the system at the high temperature.

7.9.3 Quantifying Cycle Performance

Given a specific cycle, it is helpful to be able to quantify its performance so that we can compare it with other cycles that do the same thing. As a working engineer, you may be faced with buying a piece of equipment from one of several manufacturers or vendors. Although each vendor's product does the same task, it will undoubtedly have a different performance. So we need some way to compare performance between vendors.

Measure of Performance (MOP)

Based on our discussion of the purpose of cycles, it would seem that one way to evaluate the performance of a cycle is to compare two things: *what it costs* vs. *the desired product or output*.

Using this idea, we can define a **measure of performance (MOP)** for a cycle as follows:

$$MOP = \frac{(\text{Desired product or output})}{(\text{What it costs to operate the cycle})}$$

If you think of a cycle as an energy conversion device, an equally good name for the measure of performance would be an **energy conversion ratio (ECR)**. These terms will be used interchangeably in this course. To go further, we must examine each cycle and determine what constitutes the desired output and the cost to operate the cycle.

MOP for a Power Cycle

If you now examine a power cycle, you can identify three interactions with the surroundings: heat transfer into the system, heat transfer out of the system, and a net work transfer of energy out of the system (see Figure $7.9.2$). Now what is the desired output and what is the cost to operate a power cycle?

- Desired output? $\rightarrow$ Net work transfer of energy *out* of the system.
- Cost? $\rightarrow$ Heat transfer of energy *into* the system.

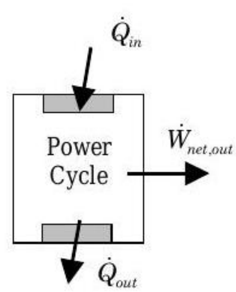


Figure $7.9.2$: Power cycle or heat engine.

Given this information, the MOP for a power cycle is called the **cycle thermal efficiency** η and is defined as follows:

$$\eta=\frac{\dot{W}_{\text{net, out}}}{\dot{Q}_{\text{in}}}\leq 1 \qquad \begin{array}{c}\text{Cycle}\\ \text{Thermal Efficiency}\end{array}$$

As indicated above, the thermal efficiency can take on values between 0 and 1. The worst power cycle would be one that takes in and rejects an equal amount of energy by heat transfer and produces no power out. On the other hand, the best power cycle would appear to be one that exchanges energy by heat transfer with only a single source and turns this energy completely into a power output. (We will show in the next chapter that it is *impossible* to build a power cycle with a thermal efficiency of one.) Some of the best fossil-fueled steam power plants only have efficiencies of approximately 30%.

MOP for a Refrigeration Cycle

If you now examine a refrigeration cycle, you can identify three interactions with the surroundings: heat transfer into the system, heat transfer out of the system, and a net work transfer of energy into the system (see Figure $7.9.3$). Now what is the desired output, and what is the cost to operate a refrigeration cycle?

- Desired output? $\rightarrow$ Heat transfer of energy *into* the system.
- Cost? $\rightarrow$ Net work transfer of energy *into* the system

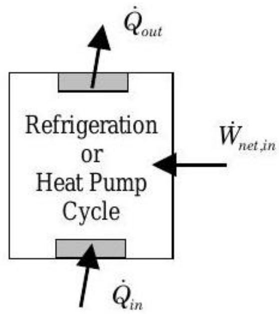


Figure $7.9.3$: Refrigeration or heat pump cycle.

Given this information the MOP for a refrigeration cycle is called the **Coefficient of Performance** $\mathrm{COP}_{\text{ref}}$ and is defined as follows:

$$0\leq COP_{\text{ref}}=\frac{\dot{Q}_{\text{in}}}{\dot{W}_{\text{net, in}}} \qquad \begin{array}{c}\text{Coefficient of Performance}\\ \text{(Refrigeration Cycle)}\end{array}$$

As indicated above, the COP for a refrigeration cycle can take on values greater than zero. The worst refrigeration cycle would be one that takes in no energy by heat transfer. On the other hand, there appears to be no upper limit on the COP. (We will show in the next chapter that there is in fact an upper limit on the COP value for a refrigeration cycle.)

MOP for a Heat Pump

If you examine a heat pump cycle, you can identify three interactions with the surroundings: heat transfer into the system, heat transfer out of the system, and a net work transfer of energy into the system (see Figure $7.9.3$). Now what is the desired output, and what is the cost to operate a heat pump cycle?

- Desired output? $\rightarrow$ Heat transfer of energy *out* of the system.
- Cost? $\rightarrow$ Net work transfer of energy *into* the system

Given this information, the MOP for a heat pump cycle is called the **Coefficient of Performance** COP_{HP} and is defined as follows:

$$1\leq COP_{\text{hp}}=\frac{\dot{Q}_{\text{out}}}{\dot{W}_{\text{net, in}}} \qquad \begin{array}{c}\text{Coefficient of Performance}\\ \text{(Heat Pump Cycle)}\end{array}$$

As indicated above the COP for a heat pump cycle can take on values greater than one. The worst heat pump cycle would be one that takes in no energy by heat transfer and converts all of the work transfer into the system into heat transfer out of the system. On the other hand, there appears to be no upper limit on the COP. (We will show in the next chapter that there is in fact an upper limit on the COP value for any heat pump cycle.)

An application of MOP's

Measures of performance are typically used in one of two ways:

- you are asked to compute the MOP for a specific cycle given all of the necessary information.
- you are given the MOP and another piece of information, e.g. refrigeration capacity (heat transfer into a refrigeration cycle), and then asked to find the other heat transfer and work transfers.

Here are examples of each type of question:

Example 7.9.1

The measured performance of a power cycle indicates that the heat transfer into the cycle is $800 \mathrm{~kJ} / \mathrm{cycle}$ and the heat transfer out of the cycle is $600 \mathrm{~kJ} / \mathrm{cycle}$. Determine the thermal efficiency of this power cycle.

Solution

$$\left.\begin{array}{l}Q_{\text {in }}=800 \mathrm{~kJ} / \text { cycle } \\ \mathrm{Q}_{\text {out }}=600 \mathrm{~kJ} / \text { cycle }\end{array}\right\} \rightarrow\left\{\begin{array}{c}0=Q_{\text {in }}-Q_{\text {out }}-W_{\text {net, out }} \\ W_{\text {net, out }}=Q_{\text {in }}-Q_{\text {out }} \\ W_{\text {net, out }}=(800-600) \dfrac{\mathrm{kJ}}{\text { cycle }}=200 \dfrac{\mathrm{kJ}}{\text { cycle }}\end{array}\right\} \rightarrow \eta=\frac{W_{\text {net, out }}}{Q_{\text {in }}}=\frac{(200)}{(800)}=0.25$$

Example 7.9.2

A heat pump is designed to deliver $10,000 \mathrm{~Btu} / \mathrm{~h}$ of heat transfer with a COP of 4. What is the power required to operate this heat pump?

Solution

$$\left.\begin{array}{l}Q_{\text {out }}=10,000 \dfrac{\mathrm{Btu}}{\mathrm{h}} \\ COP_{HP}=\dfrac{\dot{Q}_{\text {out }}}{\dot{W}}\end{array}\right\} \rightarrow \quad \dot{W}_{\text {net, in }}=\frac{\dot{Q}_{\text {out }}}{COP_{HP}}=\frac{\left(10,000 \dfrac{\mathrm{Btu}}{\mathrm{h}}\right)}{4}=2,500 \frac{\mathrm{Btu}}{\mathrm{h}}$$

How would you calculate the heat transfer rate into the system?

Example 7.9.3

Air is contained in a simple piston-cylinder device and executes a three-step cycle described in the table:

State 1	$P_1=200 \mathrm{~kPa} ; \quad T_1=27^{\circ} \mathrm{C} ; \quad V_1=0.5 \mathrm{~m}^3$
$1 \rightarrow 2$	Constant volume heating
State 2	$P_2=400 \mathrm{~kPa}$
$2 \rightarrow 3$	Constant temperature (isothermal) expansion
State 3	$P_3=P_1$
$3 \rightarrow 1$	Constant pressure (isobaric) compression

Assuming that air can be treated as an ideal gas with room temperature specific heats, determine the following:

(a) the work and heat transfer per unit mass for each process, in $\mathrm{kJ} / \mathrm{kg}$,

(b) the net work and net heat transfer per unit mass for the cycle, in $\mathrm{kJ} / \mathrm{kg}$,

(c) whether the device is a power cycle (heat engine) or a refrigerator,

(d) calculate the appropriate Measure of Performance based on your answer to (b).

Solution

$$\text{Closed system: Air inside piston;} \quad \Delta KE = 0; \quad \Delta PE = 0$$

$$\Rightarrow \Delta U = Q_{\text{in}} + W_{\text{in}}$$

$$\text{Divide by } m \text{ to get:} \quad \boxed{\Delta u = q + w}$$

$$\text{Since there is only } PdV \text{ work,} \quad W_{\text{in}} = -\int_1^2 P\, d\cancel{V}$$

$$\text{Divide by } m \text{ to get:} \quad \boxed{w_{\text{in}} = -\int P\, dv}$$

$1 \rightarrow 2:$ Constant Volume

$$\begin{aligned} \Delta u &= q_{1\text{-}2} - \cancel{w_{1\text{-}2}}^{=0} \\ w_{1\text{-}2} &= 0 \quad \text{since } v = \text{a constant} \end{aligned}$$

$$\text{Ideal gas model:} \quad \frac{P_2\, \cancel{V_2}}{P_1\, \cancel{V_1}} = \frac{\cancel{m}\, \cancel{R}\, T_2}{\cancel{m}\, \cancel{R}\, T_1}$$

$$\Rightarrow T_2 = \frac{P_2}{P_1} T_1 = (2)(600 \text{ K}) = 600 \text{ K}$$

$$\begin{aligned} q_{1\text{-}2} &= u_2 - u_1 \\ &= C_v\,(T_2 - T_1) \\ &= (0.718)(600 - 300)\frac{\text{kJ}}{\text{kg}} \end{aligned}$$

$$\boxed{q_{1\text{-}2} = 215.40 \frac{\text{kJ}}{\text{kg}}}$$

$2 \rightarrow 3:$ Isothermal

$$\begin{aligned} \Delta u &= C_v\ \cancel{(T_2 - T_1)}^{=0} = 0 \\ \cancel{\Delta u}^{=0} &= q + w \\ q_{2\text{-}3} &= -w_{2\text{-}3} \\ w_{2\text{-}3} &= -\int_2^3 P\, dv \quad \text{but } P = \frac{RT}{v} \\ w_{2\text{-}3} &= -\int_2^3 \frac{RT}{v}\, dv \\ w_{2\text{-}3} &= -RT_2 \ln\left(\frac{v_3}{v_2}\right) \end{aligned}$$

$$P_3 v_3 = P_2 v_2 \rightarrow \frac{v_3}{v_2} = \frac{P_2}{P_3} \quad \text{where } P_3 = P_1$$

$$\Rightarrow \frac{v_3}{v_2} = \frac{P_2}{P_1} = \frac{400}{200} = 2$$

$$w_{2\text{-}3} = -(0.287)(600)\ln(2) \text{ kJ/kg}$$

$$\boxed{w_{2\text{-}3} = -119.36 \frac{\text{kJ}}{\text{kg}}}$$

$$\boxed{q_{2\text{-}3} = 119.36 \frac{\text{kJ}}{\text{kg}}}$$

$3 \rightarrow 1:$ Isobaric Compression

$$\begin{aligned}
\Delta u &= q + w \\
w_{3\text{-}1} &= -\int_3^1 P\,dv \\
&= -\underbrace{P_3}_{=P_1}(v_1 - v_3) \\
&= -P_1 v_1\left(1 - \frac{v_3}{v_1}\right) \\
&= -RT_1\left(1 - \frac{P_2}{P_1}\right) \\
\frac{v_3}{v_1} &= \frac{P_2}{P_1} \text{ since } P_3 v_3 = P_2 v_2; \quad P_3 = P_1; \quad v_2 = v_1 \\
&\Rightarrow P_1 v_3 = P_2 v_1
\end{aligned}$$

$$w_{3\text{-}1} = -(0.287)(300)(1-2)\frac{\text{kJ}}{\text{kg}}$$

$$\boxed{w_{3\text{-}1} = 86.10\ \frac{\text{kJ}}{\text{kg}}}$$

$$\begin{aligned}
q_{3\text{-}1} &= (u_1 - u_3) - w_{3\text{-}1} \\
&= C_v(T_1 - T_3) - w_{3\text{-}1} \\
&= \left(0.718\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(300 - 600) - w_{3\text{-}1} \\
&= (-215.40 - 86.10)\frac{\text{kJ}}{\text{kg}}
\end{aligned}$$

$$\boxed{q_{3\text{-}1} = -301.50\ \frac{\text{kJ}}{\text{kg}}}$$

a) Summarizing these results:

	q_{in} (kJ/kg)	w_{in} (kJ/kg)
$1 \rightarrow 2$	215.40	0
$2 \rightarrow 3$	119.36	-119.36
$3 \rightarrow 1$	-301.50	86.10
$\sum$	$\mathbf{33.26}$	$\mathbf{-33.26}$

b) $q_{\text{net, in}} + w_{\text{net, in}} = 0.$ This *must* happen for a cycle!

c) Type of cycle:

$$\begin{gathered}
w_{\text{net, in}} = -33.26\ \frac{\text{kJ}}{\text{kg}} \\
\Rightarrow w_{\text{net, out}} = 33.26\ \frac{\text{kJ}}{\text{kg}} > 0 \\
\text{Power cycle}
\end{gathered}$$

d) Based on (c):

$$\begin{aligned}\eta &= \frac{w_{\text{net, out}}}{q_{\text{gross, in}}} = \frac{33.26}{215.40+119.36} \\ &= 0.09935 \Rightarrow 9.935\%\end{aligned}$$

Only $\simeq 10$ of energy brought in by heat transfer is converted to heat!

Comments:

1) The table is an essential feature of this analysis. It allows us to easily check our work.

2) The entire analysis is done on a per-mass basis.

3) This cycle is presented as a closed, periodic cycle. In theory, if we could build steady-state devices to perform these three processes, we could have a closed-loop, steady-state cycle.

If we reversed the direction of this cycle, i.e. $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$, then

	$q_{\text{in}}\ (\text{kJ}/\text{kg})$	$w_{\text{in}}\ (\text{kJ}/\text{kg})$
$1 \rightarrow 3$	301.50	-86.10
$3 \rightarrow 2$	-119.36	119.36
$2 \rightarrow 1$	-215.40	0
$\sum$	-33.26	33.26

$$w_{\text{net, in}} = 33.26\ \text{kJ}/\text{kg} > 0 \Rightarrow \text{Refrigeration cycle}$$

$$\begin{aligned}COP &= \frac{q_{\text{in, gross}}}{w_{\text{net, in}}} = \frac{119.36+215.40}{33.26} \\ &= 10.06\end{aligned}$$

7.10: Problems

Problem 7.1

The lunar module is designed to make a safe landing on the moon if its vertical velocity at impact is $V_{\max} \leq 5\ \mathrm{m/s}$. The acceleration of gravity on the moon is $1/6$ the value on Earth, $g_{\text{moon}} = g_{\text{Earth}}/6$.

(a) Using the work-energy principle (conservation of energy), determine the maximum height h above the surface of the moon at which the pilot can safely shut off the engine when the velocity of the lunar module relative to the moon's surface is (i) zero; (ii) $3\ \mathrm{m/s}$ up; and (iii) $3\ \mathrm{m/s}$ down.

(b) Repeat Part (a), only this time using conservation of linear momentum. Is one approach easier than the other?

Problem 7.2

The collar A slides on a smooth horizontal bar and is attached to the hanging mass B by a cord as shown in the figure. The collar A has mass $m_A = 30\ \mathrm{kg}$, and the hanging mass B has mass $m_B = 60\ \mathrm{kg}$. Assume that the mass of the cord and the mass of the frictionless pulley are negligible. If the collar is initially stationary, determine the velocity of collar A after it has moved 0.5 meters to the right.

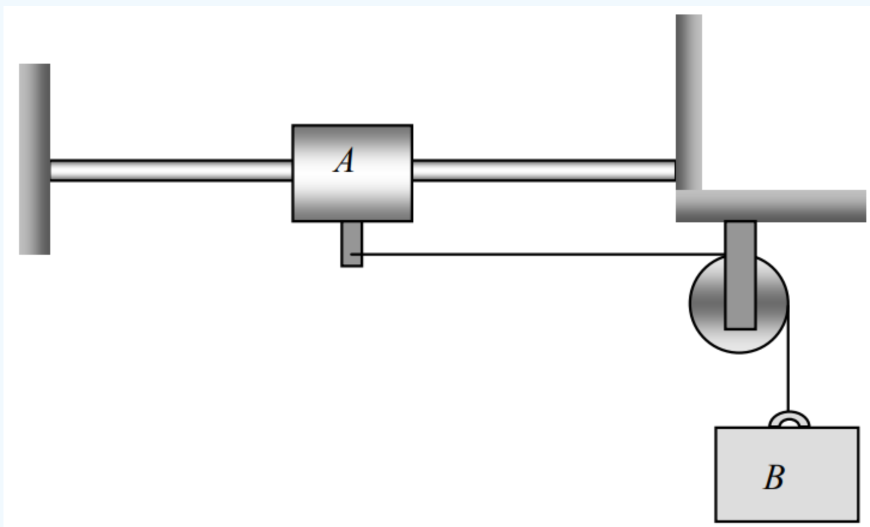


Figure $7.10.1$: A sliding collar is connected to a hanging mass by a cord and frictionless pulley.

Problem 7.3

Repeat Problem 7.2 assuming that the coefficients of static and sliding friction between collar A and the horizontal bar are 0.4 and 0.3, respectively.

A careful examination of how the cable is connected to collar A indicates that there are extra normal forces between the collar and the bar. These extra forces keep the collar from rotating and form a *force couple*. For our analysis, you may neglect the normal forces due to this couple. However, consider how you could estimate these forces and how they would change your answer.

Problem 7.4

Repeat Problem 8.2 assuming that the mass B is replaced by a constant force $T = 600\ \mathrm{N}$.

Problem 7.5

The collar A slides on a smooth horizontal bar and is attached to the hanging mass B by a cord as shown in the figure. The collar A has mass $m_A = 14\ \mathrm{kg}$, and the hanging mass B has mass $m_B = 18\ \mathrm{kg}$. The spring constant is $k = 700\ \mathrm{N/m}$. Assume that the mass of the cord and the mass of the frictionless pulley are negligible.

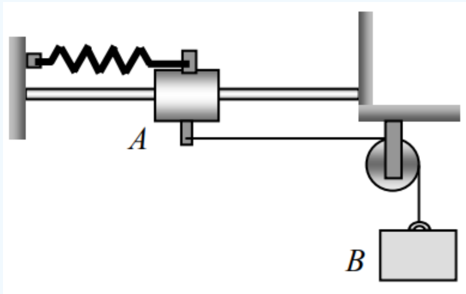


Figure 7.10.2: A sliding collar is connected to a hanging mass by a cable and to a support by a spring.

(a) If the collar is initially stationary and the spring unstretched, determine the velocity of collar A after it has moved 0.2 meters to the right.

(b) If the mass B is replaced by a constant force F_B with a magnitude equal to the weight of mass B, e.g. $F_{\mathrm{B}} = m_{\mathrm{B}} g$, and the collar is initially stationary and the spring unstretched, determine the velocity of collar A after it has moved 0.2 meters to the right. Is the value larger or smaller than the result from Part (a)? Explain

Problem7.6

A collar is attached to a linear spring as shown in the figure and moves freely on the vertical rod. The mass of the collar is $m = 2.0\ \mathrm{kg}$, and the spring constant for the spring is $k = 30\ \mathrm{N/m}$. The unstretched length of the spring is $1.5\ \mathrm{m}$. The collar is released from rest at A and slides up the smooth rod under the action of a constant force $F = 50\ \mathrm{N}$ applied at $30°$ from the vertical as shown in the figure.

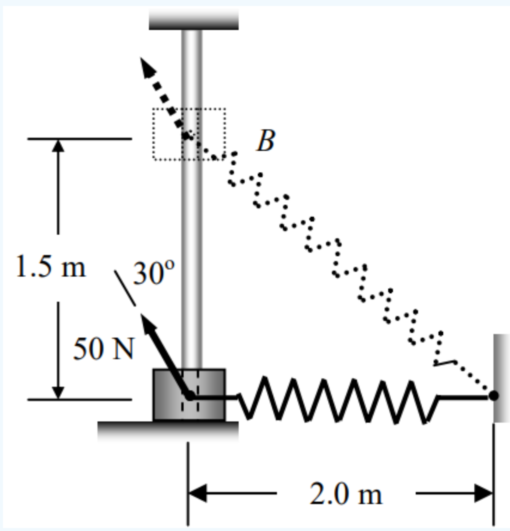


Figure 7.10.3: A spring connects a support and a sliding collar.

(a) Determine the work done on the collar by the force F in moving it from point A to point B, in $\mathrm{N \cdot m}$.

(b) Determine the velocity V of the collar as it passes point B, in $\mathrm{m/s}$.

Problem 7.7

In a preliminary design for a mail-sorting machine, parcels move down an inclined, smooth ramp and are brought to rest by a linear spring. The ramp is inclined at 30° with the horizontal. A typical packages weighs 10 lbf. The initial package velocity on the ramp is 2 ft/s and the package travels a distance of 10 ft down the ramp before it contacts the spring. The spring is designed to bring the packages to a full stop in a distance of 8 inches.

(a) Determine the spring constant k, in lbf/in, required to stop the package in the distance desired. Does your answer depend on the mass of the package?

(b) Determine the maximum deceleration for the typical package in units of standard gravitational acceleration, e.g. $3\ g$'s.

Problem 7.8

In an alternate design to the one in Problem 7.7, the spring is eliminated and the package is brought to rest solely due to a friction coating over the last 5 feet of the ramp. All variables are the same as in Problem 7.7 except the spring has been replaced by a 5-foot-long friction strip.

(a) Determine the minimum value of the coefficient of kinetic friction between the package and the friction strip material that will bring the package to rest on the friction strip. Does your answer depend on the mass of the package?

(b) Determine the maximum deceleration for the typical package in units of standard gravitational acceleration, e.g. $3g$'s.

Problem 7.9

The work done by an internal combustion engine can be modeled by considering the work done by a closed system containing a gas. The gas in the system executes a three-stage process that returns the gas to its initial state.

State 1:	$p_1 = 100 \text{ kPa};\ V_1 = 0.80 \text{ m}^3$
Process 1-2:	Compressed along a process where pV = constant.
State 2:	$V_2 = 0.2V_1$
Process 2-3:	Heated and expanded at constant pressure, e.g. p = constant
State 3:	$V_3 = V_1$
Process 3-1:	Cooled at constant volume until $p = p_1$.

(a) Sketch a generic piston-cylinder device and identify the closed system used for your analysis.

(b) Calculate the work done on the gas during each process.

(c) Sketch the three processes and their end states on a p-V diagram.

Label your axes and end states. Accurately show the path of each process on the diagram. Indicate the area on the diagram that represents the magnitude of the work done on the system during each process.

(d) Calculate the net work done on the system during this cycle and indicate the corresponding area on the p-V diagram.

Problem 7.10

A closed system of mass 5 lbm undergoes a process in which there is a heat transfer of $200 \text{ ft} \cdot \text{lbf}$ from the system to the surrounding. There is no work during the process. The velocity of the system increases from 10 ft/s to 50 ft/s, and the elevation decreases by 150 ft. The acceleration of gravity at this particular geographical location is 32.0 ft/s^2.

Please sketch your system and show all of your work to determine the change in (a) kinetic energy of the system, in $\text{ft} \cdot \text{lbf}$,

(b) gravitational potential energy of the system, in $\text{ft} \cdot \text{lbf}$, and

(c) internal energy of the system, in $\text{ft} \cdot \text{lbf}$ and Btu.

Problem 7.11

A closed system undergoes a process during which there is heat transfer to the system at a constant rate of $5\ \mathrm{kW}$, and the power out of the system varies with time according to

$$\dot{W}_{\text{out}}=\begin{cases}+2.5t & 0<t\leq 2.0\ \mathrm{h}\\ +5.0 & t>2.0\ \mathrm{h}\end{cases}$$

where t is in hours and W_{out} is in kW.

(a) Sketch the system and label the energy flows.

(b) Determine the time rate of change of system energy at $t=1.2\ \mathrm{h}$ and $2.4\ \mathrm{h}$, in kW

(c) Determine the change in system energy after $3\ \mathrm{h}$, in $\mathrm{kW}\cdot\mathrm{h}$ and in kJ.

Problem 7.12

A piston-cylinder assemby is equipped with a paddle wheel driven by an external motor and is filled with $30\ \mathrm{g}$ of a gas. The walls of the cylinder are well insulated, and the friction between the piston and the cylinder wall is negligible. Initially the gas is in state 1 (see table). The paddle wheel is then operated, but the piston is allowed to move to keep the pressure in the gas constant. When the paddle wheel is stopped, the system is in state 2 . Determine the work transfer, in joules, along the paddle-wheel shaft.

State	P, bars	v, cm^3/g	u, J/g
1	15	7.11	22.75
2	15	19.16	97.63

Problem 7.13

An insulated piston-cylinder assembly containing a fluid has a stirring device operated externally. The piston is frictionless, and the force holding it against the fluid is due to standard atmospheric pressure (\(101.3 \mathrm{~kPa}\)) and a coil spring with a spring constant of $7200\ \mathrm{N}/\mathrm{m}$. The stirring device is turned 100 revolutions with an average torque of $0.68\ \mathrm{N}\cdot\mathrm{m}$. As a result, the gas expands and the piston moves outward $0.10\ \mathrm{m}$. The diameter of the piston is $0.10\ \mathrm{m}$. Assume changes in kinetic energy and potential energy are negligible.

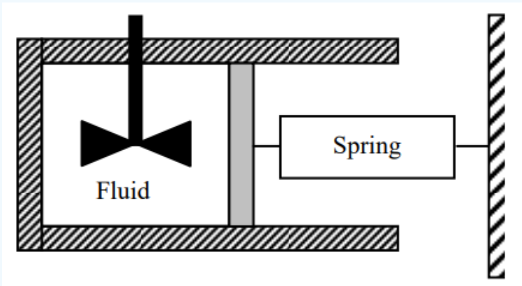


Figure $7.10.4$: Fluid is stirred in a piston-cylinder device whose piston is held in place by a spring.

(a) Determine the work done *by* the gas during its expansion, in kJ. In addition, determine how much of the work done by the gas is against the force of the atmosphere and how much is done against the spring. Assume that the spring initially exerts no force on the piston.

(b) Determine the work done *by* the stirring device on the gas, in kJ, during

(c) Determine the change in internal energy, in kJ, of the fluid.

❓ Problem 7.14

An electric motor drives an air compressor that delivers air with inlet and outlet conditions as shown on the figure. The motor-compressor set operates at steady-state conditions. The electric power supplied to the motor is 25 kilowatts at 220 volts ac, and the measured heat transfer rate from the combined motor and compressor is 4.4 kilowatts. Experience shows that changes in kinetic and potential energy are negligible for this system.

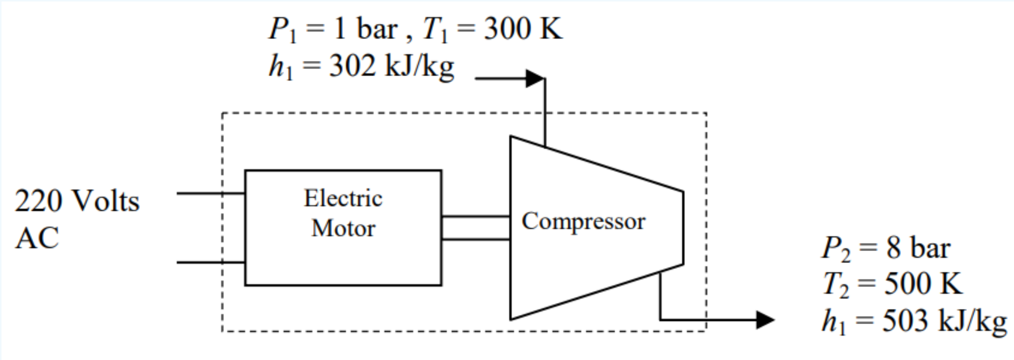


Figure $7.10.5$: An air compressor and the electric motor driving it form a system.

a) Determine the mass flow rate through the compressor in kilograms per second.

(b) Assuming a power factor of 1 (purely resistive circuit), calculate the electric current AC drawn by the motor, in amps

❓ Problem 7.15

Air enters a compressor at $100\ \mathrm{kPa}$ and $280\ \mathrm{K}$ and leaves the compressor at $600\ \mathrm{kPa}$ and $400\ \mathrm{K}$. The compressor operates at steady-state conditions with an entering mass flow rate of $0.02\ \mathrm{kg/s}$. In addition, the heat transfer rate from the compressor is $0.32\ \mathrm{kW}$. The specific enthalpy of the entering and leaving air streams are found to be $h_1 = 280.13\ \mathrm{kJ/kg}$ and $h_2 = 400.98\ \mathrm{kJ/kg}$, respectively. Assume that changes in kinetic and potential energy for the streams flowing through the system are negligible.

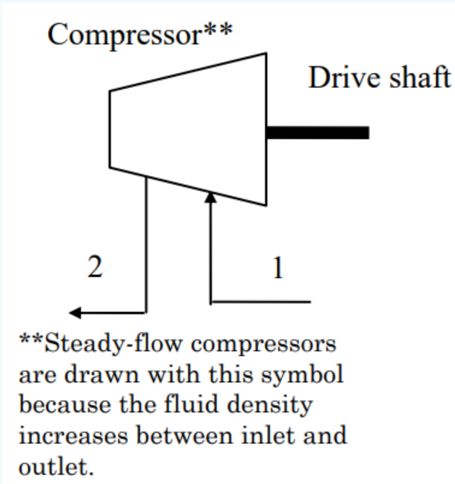


Figure $7.10.6$: A steady-state air compressor.

(a) Determine the mass flow rate of the air leaving the compressor, in $\mathrm{kg/s}$. Show your reasoning.

(b) Determine the ratio of the entering volumetric flow rate to the leaving volumetric flow rate, i.e. $\dot{V}_1/\dot{V}_2$. Assume ideal gas behavior for air. (Hint: Solve for the volumetric flow rate in terms of the mass flow rate and the density.)

(c) Determine the power required to operate the compressor, in kW and in hp.

Answer

Problem 7.16

Methane gas (CH_4) is burned with the stoichiometric amount of oxygen (O_2) in a steady-state combustion process. The methane gas enters the burner with a mass flow rate of $16.0\ \mathrm{kg/h}$ and a specific enthalpy $h = 4,778\ \mathrm{kJ/kg}$ at $1\ \mathrm{atm}$ and $25^{\circ}\mathrm{C}$. The oxygen enters the burner with a mass flow rate of $64.0\ \mathrm{kg/h}$ and a specific enthalpy $h = 100\ \mathrm{kJ/kg}$ at $1\ \mathrm{atm}$ and $25^{\circ}\mathrm{C}$. The combustion products exit the burner at $1\ \mathrm{atm}$ and $25^{\circ}\mathrm{C}$ and a specific enthalpy $h = -10,865\ \mathrm{kJ/kg}$. The burner operates with negligible work at non-flow boundaries and negligible changes in kinetic and gravitational potential energy.

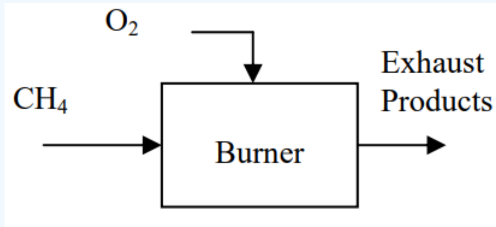


Figure $7.10.7$: Steady-state burner for methane gas.

(a) Determine the mass flow rate of combustion products leaving the burner, in $\mathrm{kg/s}$.

(b) Determine the heat transfer rate for the process, in kW. Be sure and indicate both the magnitude and the direction of the heat transfer.

(c) If the velocity of the oxygen entering the burner is approximately $1\ \mathrm{m/s}$, determine the cross-sectional area of the inlet duct. Assume that oxygen can be modeled as an ideal gas.

Problem 7.17

An dc-electric generator is attached directly to a steam turbine as shown in the figure. Steam flows into the turbine with a mass flow rate of $360\ \mathrm{lbm/h}$ and the turbine-generator set operates at steady-state conditions. The specific enthalpy of the entering and leaving steam is shown on the figure. The heat transfer rate from the turbine is $3000\ \mathrm{Btu/h}$ and from the generator is $1000\ \mathrm{Btu/h}$. Assume changes in kinetic and gravitational potential energy for the steam flowing through the generator are negligible.

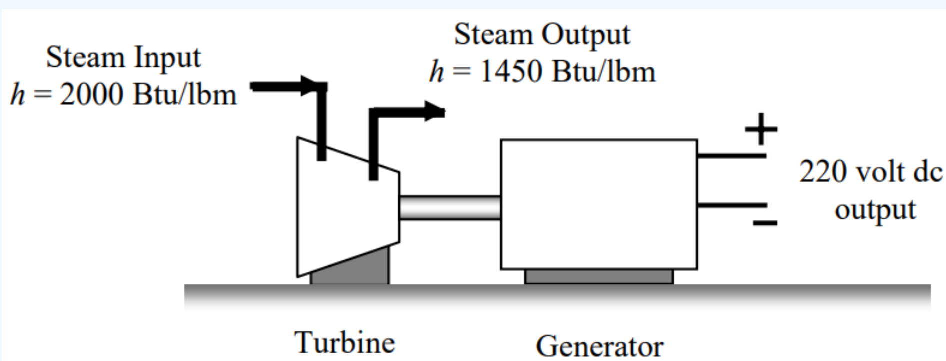


Figure $7.10.8$: System consisting of a d electric generator and steam turbine.

(a) Determine the electrical power output from the generator, in $\mathrm{Btu/h}$.

(b) Determine the de current supplied by the generator, in amps.

(c) Determine the torque, in $\mathrm{lbf \cdot ft}$, in the 220-volt dc shaft connecting the turbine to the output generator assuming it rotates at $3600\ \mathrm{rpm}$.

Problem 7.18

Steam flows through a heat exchanger and then through a steam turbine as shown in the figure. Steam leaves the turbine at two different points, State 3 and State 4. The shaft power output from the turbine is $675 \mathrm{~MW}$. All known state information is shown in the table. All devices operate at steady-state conditions and changes in kinetic and gravitational potential energy are negligible. By design, heat exchangers have no work transfer of energy. In addition, the steam turbine operates adiabatically.

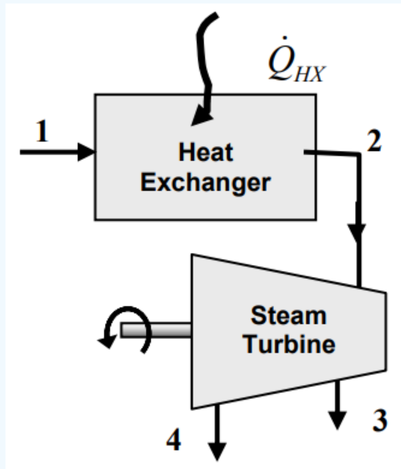


Figure $7.10.9$: Steam flows through a heat exchanger and steam turbine in steady-state conditions.

State	h (kJ/kg)	P (kPa)	T (°C)	v (m^3/kg)	$\dot{m}$ (kg/s)
1	562.0	300	133	0.001	1000
2	$\cdots$	300	400	1032	–
3	3114	150	320	1819	200
4	3034	100	280	2546	–
Note: You need not complete this table to work the problem.					

(a) Determine the mass flow rate leaving the turbine at State 4.

(b) Determine the heat transfer rate to the heat exchanger.

(c) Determine the torque transferred by the turbine shaft if the turbine rotates at $3600 \mathrm{~rpm}$.

Problem 7.19

One means for generating electricity is to use a gas-turbine engine connected to an electric generator. Although the gas-turbine engine, as shown in the figure, consists of three components—a compressor, a heat exchanger, and a turbine—connected together with air flowing through each sequentially, the engine can be analyzed using the open system indicated. The compressor and turbine have a common shaft and are directly connected to the generator. Operating information is shown on the figure. The mass flow rate of air through the gas turbine is $2.0 \mathrm{~lbm/s}$. For purposes of analysis you may assume that all systems operate at steady-state conditions and that changes in kinetic and gravitational potential energy are negligible.

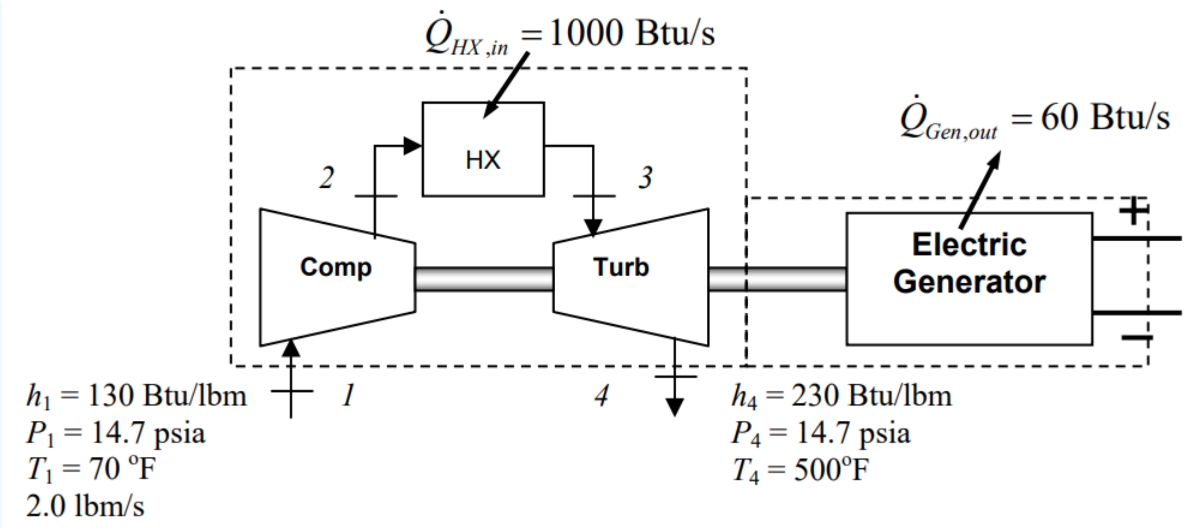


Figure 7.10.10: Electric generator is powered by a system consisting of a compressor, heat exchanger, and turbine.

(a) Determine the shaft power delivered to the electric generator, in $\mathrm{Btu/s}$, and the electrical power output of the generator, in kilowatts.

(b) Determine the shaft torque in the shaft supplying power to the electric generator, in $\mathrm{ft \cdot lbf}$, if the shaft speed is 1800 rpm.

Problem 7.20

A small steam turbine is connected to an air compressor through a gear reducer as shown in the figure. A gear reducer is a device used to change the shaft rotation speed when two devices must be connected but operate at different speeds.

Steam enters the turbine at $110^\circ\mathrm{C}$ with a specific enthalpy $h_1 = 2691.5\ \mathrm{kJ/kg}$ and exits the turbine at a pressure of $100\ \mathrm{kPa}$ and a specific enthalpy $h_2 = 2675.5\ \mathrm{kJ/kg}$. The turbine shaft rotates at 2000 rpm.

Air enters the compressor at a mass flow rate of $70\ \mathrm{kg/min}$ at $100\ \mathrm{kPa}$ and $300\ \mathrm{K}$ and exits the compressor at $500\ \mathrm{kPa}$ and $460\ \mathrm{K}$. The compressor shaft rotates at 600 rpm. Assume that air can be modeled as an ideal gas with room temperature specific heats.

Assume all devices shown in the figure—turbine, compressor and gear reducer—operate adiabatically at steady-state conditions with negligible changes in kinetic and gravitational potential energy.

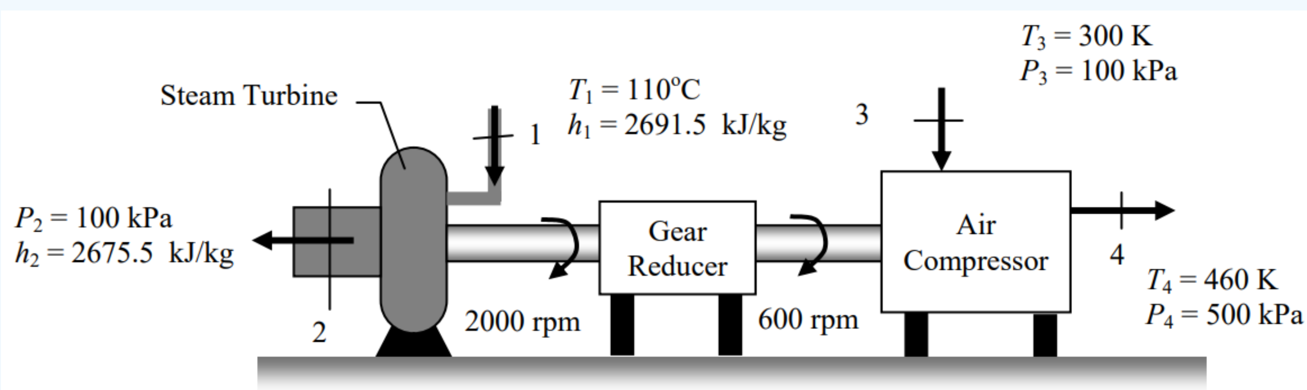


Figure 7.10.11: A steam turbine and air compressor connected by a common shaft that passes through a gear reducer.

(a) Determine the mass flow rate of steam into the turbine, in $\mathrm{kg/min}$.

(b) Determine the shaft power required by the air compressor, in kW.

(c) Determine the torque, in $\mathrm{N \cdot m}$, transmitted by the air compressor shaft.

Problem 7.21

The compression stroke of an air compressor can be modeled as a closed system. Initially the air inside the chamber occupies a volume of $100\ \mathrm{cm}^3$ and has a pressure of $100\ \mathrm{kPa}$ and a temperature of $25^{\circ}\mathrm{C}$. During the compression process the gas follows a process where the product of pressure and volume remain constant, e.g. $P\forall = C$. After the compression process, the gas occupies a volume of $12.5\ \mathrm{cm}^3$. Assume air can be modeled as an ideal gas with room-temperature specific heats.

(a) Determine the mass of air inside the piston in kg.

(b) Determine the work done on the gas by the piston during this compression process, in kilojoules.

(c) Determine the heat transfer for the process, in kilojoules.

Problem 7.22

Repeat Problem 8.21 and this time assume that the air follows a process where $P\forall^{1.3} = C$.

Problem 7.23

The water faucet in your bathroom produces a stream of warm water by mixing a hot stream and a cold stream. You adjust the temperature by adjusting the flow rates of the two entering streams. On a cold day, city water enters the house at $50^{\circ}\mathrm{F}\ (510^{\circ}\mathrm{R})$. Some of this water is diverted to the water heater and heated to a temperature of $140^{\circ}\mathrm{F}\ (600^{\circ}\mathrm{R})$.

Assume that the faucet can be modeled as a mixing tee as shown in the figure. In addition, you may assume that water can be modeled as an incompressible substance with room-temperature specific heats. Experience has also shown that for this type of problem, changes in pressure, kinetic energy, and potential energy have a negligible effect on the answer; thus, you may neglect them.

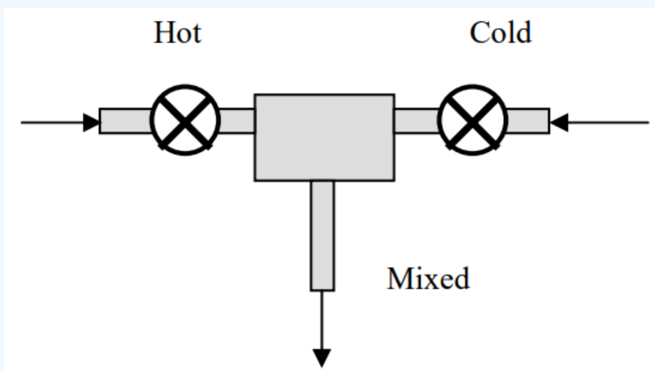


Figure 7.10.12 Mixing tee for hot and cold streams of water.

(a) Determine the steady-state volumetric flow rate of cold water that produces a warm water temperature of $100^{\circ}\mathrm{F}$ if the hot water flows at 0.100 gallons per minute. [Hint: When applying the energy balance, replace the specific enthalpy h in the energy balance with its definition, $h = u + pv = u + p/\rho$]

(b) Revisit your analysis for part (a) and rewrite the results as a function of the ratio of the flow rates, $\dot{m}_{\text{cold}}/\dot{m}_{\text{hot}}$. Plot your results in two ways:

- Plot A: T_{warm} vs. $\dot{m}_{\text{cold}}/\dot{m}_{\text{hot}}$
- Plot B: $(T_{\text{warm}} - T_{\text{cold}}) / (T_{\text{hot}} - T_{\text{cold}})$ vs. $\dot{m}_{\text{cold}}/\dot{m}_{\text{hot}}$

What, if any, is the advantage of Plot B over Plot A ?

Problem 7.24

An on-demand water heater is designed to supply hot water almost instantaneously without having to keep a tank of hot water available at all times. One design for an electrically powered on-demand water heater is shown in the figure below. For purposes of analysis, we can assume that the water heater operates at steady-state conditions immediately after the power is switched on.

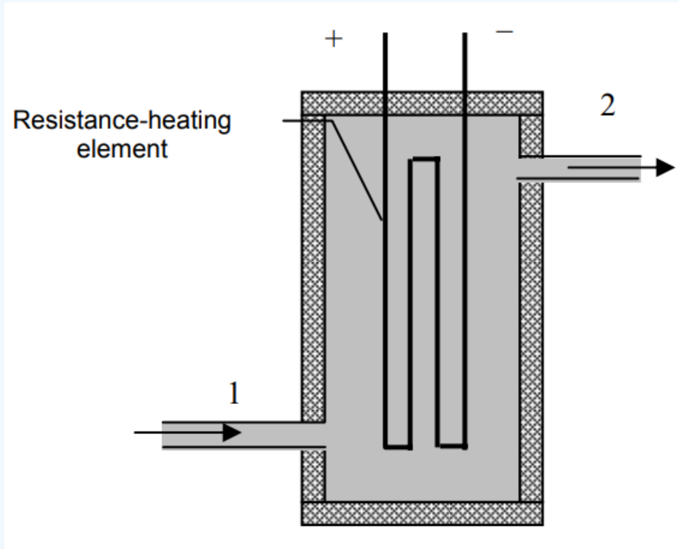


Figure 7.10.13: Water flows through a tank containing a resistance-heating element.

This design is supposed to supply 3 gallons of water per minute on a steady basis. It is also designed to operate at 220-volt ac power. The entering water temperature is $60^{\circ}\mathrm{F}$ and the leaving water temperature is $140^{\circ}\mathrm{F}$. Water can be modeled as an incompressible substance. Changes in kinetic and potential energy are negligible.

(a) Determine the mass flow rate of the water, in $\mathrm{lbm/s}$.

(b) Determine the electric power requirements to operate the water heater as designed, in kW.

(c) Determine the resistance of the resistance-heating element, in ohms.

Problem 7.25

An electric motor is used to power a mixer in a chemical process plant. The electric motor operates at $1800\ \mathrm{rpm}$ (revolutions per minute). To reduce the speed, the motor shaft is connected to a speed reducer followed by a 90° drive. The following information is known about all of the components:

- Electric Motor: Adiabatic, steady-state operation; $1800\ \mathrm{rpm}$ shaft speed.
- Speed Reducer: $15:1$ speed reduction; steady-state operation; heat transfer rate from the system is 10% of the power supplied by the input shaft.
- 90°-Drive: Adiabatic, steady-state operation; $6:5$ speed reduction.
- Mixer Shaft: Required torque is $500\ \mathrm{ft}\cdot\mathrm{lbf}$.

In addition, the tank contains $1000\ \mathrm{ft}^3$ of a liquid with a density of $70\ \mathrm{lbm/ft}^3$. The specific internal energy of the liquid in the tank can be calculated using the equation $u=c\cdot T$ where temperature is in degrees Rankine $(^{\circ}\mathrm{R})$ and \(c=1.5 \mathrm{Btu} / left(\mathrm{lbm} \cdot { }^{\circ} \mathrm{R}\right)\).

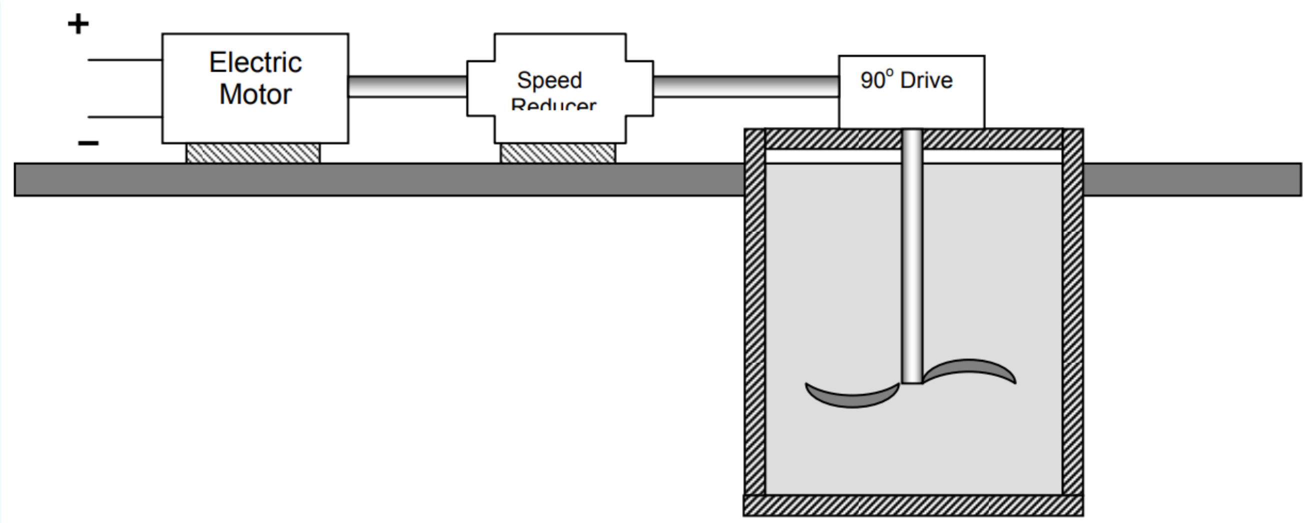


Figure 7.10.14 Mixer powered by electric motor equipped with speed reducer.

(a) Determine the rotational speed of the speed reducer output shaft and the mixer shaft, in rpm.

(b) Determine the shaft power required to turn the mixer shaft, in $\mathrm{ft \cdot lbf/s}$ and hp.

(c) Determine the power supplied by the input shaft to the 90°-drive, in $\mathrm{ft \cdot lbf/s}$ and hp.

(d) Determine the heat transfer rate from the speed reducer and the power supplied to the speed reducer by the motor shaft, in $\mathrm{ft \cdot lbf/s}$ and hp.

(e) Determine the electric power that must be supplied to the motor, in $\mathrm{ft \cdot lbf}$, hp, and kW.

(f) Determine the torque, in $\mathrm{ft \cdot lbf}$, supplied by the motor.

(g) If the mixer runs for one hour (60 minutes) and the tank is essentially adiabatic, determine the increase in temperature of the liquid in the tank. (You may neglect changes in kinetic and potential energy.)

? Problem 7.26

Two streams of air are mixed in a steady-flow, mixing process. Stream one enters at a temperature of $36^{\circ}\mathrm{C}$ and a mass flow rate of $5\ \mathrm{kg/min}$. Stream two enters at $10^{\circ}\mathrm{C}$ and $15\ \mathrm{kg/min}$. The entire mixing process occurs in a heavily insulated sheet-metal mixing tee and occurs at a pressure of $110\ \mathrm{kPa}$. An electric resistance heater element is built into the tee, so that the outlet temperature can be controlled. Assume that changes in kinetic and gravitational potential energy are negligible and that air can be modeled as an ideal gas with room temperature specific heats.

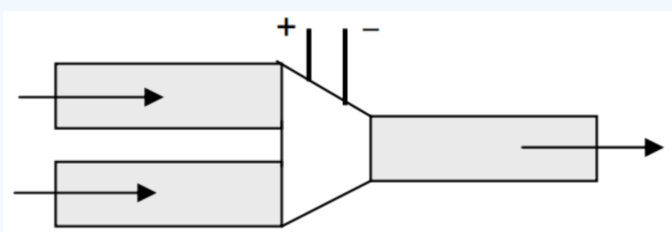


Figure 7.10.15 Streams of air are mixed in a mixing tee that contains a heating element.

(a) If the electric resistance heater is turned off, determine the temperature of the air leaving the mixing tee, in $^{\circ}\mathrm{C}$, and the volumetric flow rate of the air leaving the mixing tee, in $\mathrm{m^3/s}$.

(b) If the temperature of the low temperature stream drops from $10^{\circ}\mathrm{C}$ to $3^{\circ}\mathrm{C}$, how much power must be supplied by the electric heater to maintain the same outlet temperature as you found in Part (a)?

Problem 7.27

A centrifugal pump is driven by an electric motor as shown in the figure. Water flows steadily through the pump with the inlet and outlet conditions shown in the figure. The 440-ac-volt electric motor receives $42\ \mathrm{kW}$ of electrical power and delivers $40\ \mathrm{kW}$ of shaft power to the pump under steady-state conditions. The motor rotates at $1750\ \mathrm{rpm}$ and has a power factor of unity. Assume that water can be modeled as an incompressible substance with constant specific heats, and assume changes in gravitational potential energy are negligible.

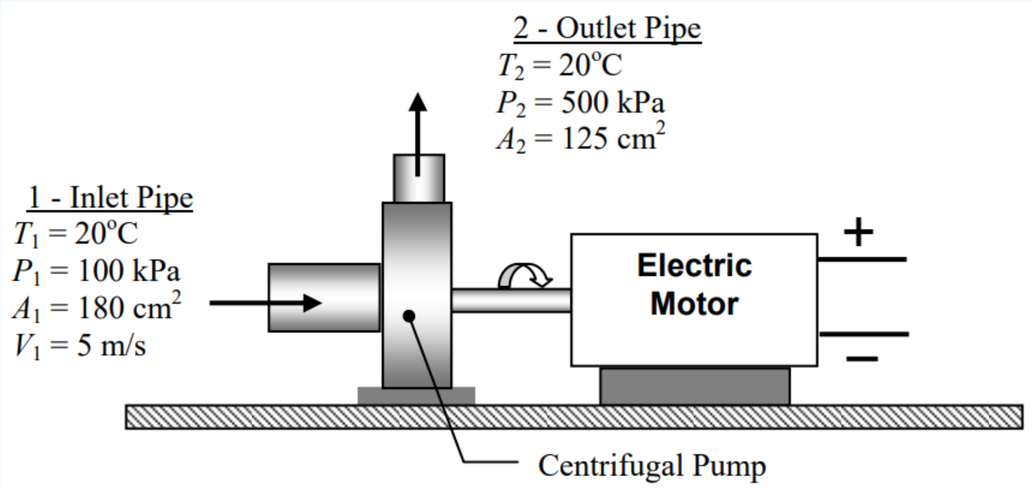


Figure $7.10.16$: An electric motor drives a centrifugal water pump.

(a) Determine the direction and magnitude of the heat transfer rate for the pump, in kilowatts.

(b) Determine the torque transmitted by the motor shaft to the pump, in $\mathrm{N \cdot m}$.

(c) Determine the electric current supplied to the motor, in amps.

Problem 7.28

A piston-cylinder device as shown in the figure contains helium gas. Initially, the gas has a pressure of $70\ \mathrm{psia}$, a temperature of $600^{\circ}\mathrm{R}$, and a volume of $7\ \mathrm{ft}^3$. During a process where $PV = C$, a constant, the helium is expanded to a final volume of $28\ \mathrm{ft}^3$. Assume that helium gas can be modeled as an ideal gas with constant specific heats and assume that changes in kinetic and potential energy are negligible.

Determine the direction and the magnitude of the work and the heat transfer for the helium gas, in $\mathrm{ft \cdot lbf}$.

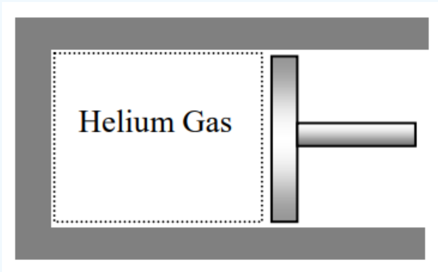


Figure $7.10.17$: A piston-cylinder device contains helium gas.

Problem 7.29

A piston-cylinder device contains carbon dioxide (CO_2) gas initially occupying the volume V_1 at the pressure P_1 and temperature T_1 indicated below. The gas undergoes the process described below:

$$\text{State 1}: P_1 = 150\ \mathrm{kPa}; \quad T_1 = 400\ \mathrm{K}; \quad V_1 = 0.5\ \mathrm{m}^3$$
$$\text{Process 1-2: Quasistatic process where } P = \left(300\ \mathrm{kPa/m^3}\right) V$$
$$\text{State 2}: V_2 = 1.0\ \mathrm{m}^3$$

Assume that carbon dioxide can be modeled as an ideal gas with constant specific heats and that changes in kinetic and gravitational potential energy are negligible for the process.

Determine the work and heat transfer of energy for process 1-2, in kJ. Be sure to also clearly indicate the direction of each energy transfer.

? Problem 7.30

An energy-recovery system reclaims energy from hot oil using a liquid-air heat exchanger to heat the air. Light oil flows through the heat exchanger at a flow rate of $100\ \mathrm{lbm/min}$. The oil enters at $80\ \mathrm{psia}$ and $200^\circ\mathrm{F}$, and leaves at $70\ \mathrm{psia}$ and $100^\circ\mathrm{F}$. The air enters the heat exchanger at $14.8\ \mathrm{psia}$ and $80\ \mathrm{F}$ and a volumetric flow rate of $17,550\ \mathrm{ft}^3/\mathrm{min}$. The exit pressure of the air is $14.6\ \mathrm{psia}$. Assume changes in kinetic and gravitational potential energy are negligible. Also assume that air can be modeled as an ideal gas with constant specific heats and liquid water can be modeled as an incompressible substance with constant specific heats. The density of light oil is $62.4\ \mathrm{lbm/ft^3}$.

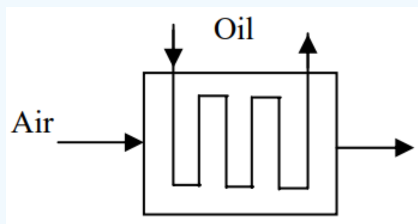


Figure $7.10.18$: A heat exchanger for hot oil and cool air.

(a) Determine the outlet temperature of the air, in $^\circ\mathrm{F}$.

(b) Determine the inlet flow area for the air, if the inlet velocity is $50\ \mathrm{ft/s}$.

(c) Determine the inlet flow area for the oil, if the inlet velocity is $10\ \mathrm{ft/s}$

? Problem 7.31

A rigid storage tank for water in a home has a volume of $0.40\ \mathrm{m}^3$. The tank initially contains $0.30\ \mathrm{m}^3$ of water at $20^\circ\mathrm{C}$ and $240\ \mathrm{kPa}$. The space above the water contains air at the same temperature and pressure as the water. An additional $0.05\ \mathrm{m}^3$ of water is slowly pumped into the tank so that the temperature of the air remains constant during the entire filling process. Assume that air behaves as an ideal gas with constant specific heats and changes in the kinetic and potential energies of the gas trapped above the water are negligible.

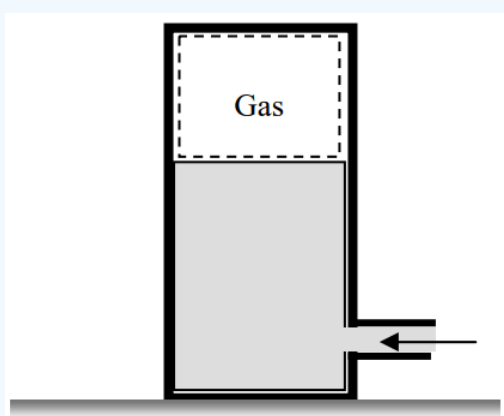


Figure $7.10.19$: Water is pumped into a tall tank from the bottom.

(a) Determine the final pressure of the air in the tank, in kPa.

(b) Determine the work and heat transfer for the gas during this isothermal compression process. Report both the direction and the magnitude of these energy transfers in kilojoules.

Problem 7.32

Liquid water enters a steady-state pump at \(1.0 \mathrm{~bar}\) and $20^{\circ}\mathrm{C}$ with a velocity of $2.6\ \mathrm{m/s}$ through an opening of $22.0\ \mathrm{cm}^2$. The water leaves the pump at \(6.0 \mathrm{~bars}\) and $7.8\ \mathrm{m/s}$. The elevation of the pump exit is 0.5 meters above the elevation of the pump inlet. Assume water can be modeled as an incompressible substance with constant specific heats and a density of $1000\ \mathrm{kg/m^3}$.

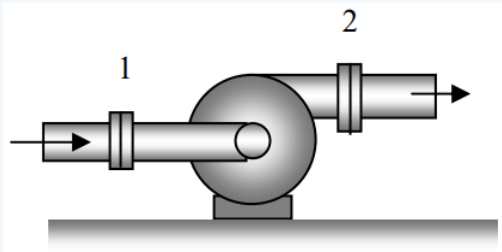


Figure $7.10.20$: Water is raised in pressure and elevation by a steady-state pump.

(a) Determine the ratio of the diameter of the outlet pipe to the diameter of the inlet pipe, e.g. D_2/D_1.

(b) Assuming that the pump operates adiabatically and the temperature of the water does not change, i.e. $T_1 = T_2$, determine the shaft power required to run the pump in kilowatts and horsepower.

(c) Repeat part (b), only this time assume that the water temperature increases by $0.10^{\circ}\mathrm{C}$. How does this change the shaft power input to the pump?

Problem 7.33

Air flows through a simple nozzle as shown in the figure. A nozzle is a rigid hollow tube whose sides are contoured as shown in the figure. If operating correctly, the velocity of the fluid leaving the device is greater than the velocity of the fluid entering the device. The walls of the nozzle are rigid and impermeable, except for the inlet and outlet areas. Typically, heat transfer from the nozzle is negligible.

Air enters the steady-state nozzle at State 1 at $200\ \mathrm{kPa}$ and $300\ \mathrm{K}$ and a velocity of $48\ \mathrm{m/s}$, and leaves the nozzle at State 2 with a pressure of $100\ \mathrm{kPa}$ and the temperature of $246\ \mathrm{K}$. Changes in gravitational potential energy are negligible. Assume that air acts like an ideal gas with room-temperature specific heats.

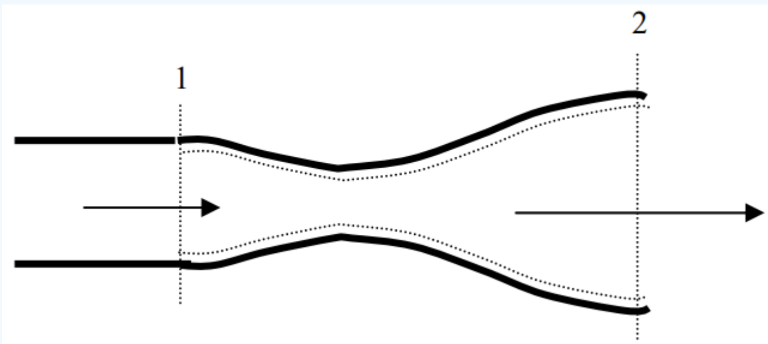


Figure $7.10.21$: Air flows through a steady-state nozzle.

(a) Determine the velocity of the air leaving the nozzle, in $\mathrm{m/s}$. Use the system shown in the figure.

(b) Determine the ratio of the exit area to the inlet area, A_2/A_1.

(c) How would your analysis change if the system was enlarged so that it included the sidewall of the nozzle? An explanation is all that is required. No numbers are required.

Problem 7.34

Liquid water flows steadily through an adiabatic nozzle. The water enters the nozzle with negligible velocity at $600\ \mathrm{kPa}$ and $100^{\circ}\mathrm{C}$ and leaves the nozzle at a pressure of $400\ \mathrm{kPa}$. Assume that liquid water can be modeled as an incompressible substance with room-temperature specific heats.

(a) Determine the exit velocity of the liquid water, in $\mathrm{m/s}$, it the exit temperature of the liquid water is $100^{\circ}\mathrm{C}$.

(b) Repeat Part (a) only this time assume the outlet temperature is $100.01^{\circ}\mathrm{C}$.

Problem 7.35

A $10,000\Omega$ resistor is connected across the terminals of a 12-volt car battery. Heat transfer occurs from the surface of the resistor by natural convection heat transfer according to the relationship

$$\dot{Q}_{\text{out}} = h_{\text{conv}}\, A_{\text{surface}}\left(T_{\text{surface}} - T_{\text{amb}}\right)$$

where

$h_{\text{conv}} = 5\ \mathrm{W}/\left(\mathrm{m}^2 \cdot \mathrm{K}\right)$, the convection heat transfer coefficient.

$A_{\text{surface}} = 1.8\ \mathrm{cm}^2$, the surface area of the resistor.

$T_{\text{surface}} =$ the surface temperature of the resistor, in degrees kelvin.

$T_{\text{amb}} =$ the temperature of the ambient air surrounding the resistor, say $300\ \mathrm{K}$.

(a) Sketch the system showing the resistor and the battery.

(b) Calculate the steady-state current through the resistor, in amps, and the electric power supplied to the resistor, in watts.

(c) If the battery is connected for 30 minutes and heat transfer from the battery is negligible, determine the change in the internal energy of the battery.

(d) Determine the surface temperature of the resistor assuming steady-state conditions, in K.

(e) If a second and identical resistor was placed in parallel across the terminals of the battery and new steady-state conditions established, would the surface temperature of the resistors increase, decrease, or remain unchanged? Explain and support your answer.

Problem 7.36

An ac transformer is supplied with electric power at 230 watts with an input voltage of 220 volts ac and a power factor of unity. The transformer output is $1.9\ \mathrm{A}$ at $110\ \mathrm{V}$ ac with a power factor of unity. The heat transfer surface area for the transformer can be modeled as a $10\ \mathrm{cm} \times 10\ \mathrm{cm} \times 10\ \mathrm{cm}$ cube. The convection heat transfer coefficient for the transformer is $\mathrm{h}_{\text{conv}} = 6\ \mathrm{W}/\mathrm{m}^2 \cdot \mathrm{K}$.

(a) Determine the input ac current to the transformer, in amps.

(b) Determine the rate of heat transfer from the transformer at steady-state operation conditions, in W.

(c) Determine the steady-state surface temperature of the transformer if the ambient air temperature is $25^{\circ}\mathrm{C}$.

Problem 7.37

A simple piston-cylinder device contains helium. The closed system formed by the helium in the device executes a four-process cycle with negligible changes in kinetic and potential energy. Assume helium can be modeled as an ideal gas with room-temperature specific heats. The available process and state information is detailed in the table below:

	T	P	V	Q_{in} (kJ)	W_{in} (kJ)	ΔU (kJ)

	T	P	V	Q_{in} (kJ)	W_{in} (kJ)	ΔU (kJ)
State 1	400 K	500 kPa	0.5 m^3			
Process 1-2	Constant Pressure Compression					
State 2	160 K	500kPa	0.2 m^3			
Process 2-3	Expansion with pV = constant					
State 3	160 K	100kPa	1.0 m^3			
Process 3-4	Constant Volume Heating					
State 4	252 K	158kPa	1.0 m^3			
Process 4-1	Adiabatic Compression					

(a) Calculate the work and the heat transfer for each process in the cycle. Clearly show your work

(b) Based on your analysis in Part (a), is the cycle a power cycle (heat engine) or a refrigerator? Clearly indicate your reasoning.

(c) Based on your answer to Part (b), calculate the appropriate measure of performance, e.g. power cycle thermal efficiency or the refrigerator coefficient of performance.

? Problem 7.38

A gas is contained in a simple piston-cylinder device. Changes in kinetic and potential energy are negligible for this closed system. The following information is known about the four processes that make up the cycle:

Process	Description	Q_{in}/m (kJ/kg)	W_{in}/m (kJ/kg)	Δu (kJ/kg)
$1\cdots2$	Adiabatic compression	\(0)	458.73	
$2\cdots3$	Isobaric heating	+1038.12		811.20
$3\cdots4$	Adiabatic expansion	0		−819.65
$4\cdots1$	Constant volume cooling	−450.23	0	
Total	······			

(a) Complete the table by providing numerical values for the unknown heat transfers and work transfers.

(b) Is this a power cycle or a refrigerator? Explain how you made this decision

(c) Based upon your answer for Part (b), calculate the appropriate measure of performance: thermal efficiency for a power cycle or coefficient of performance for a refrigerator.

? Problem 7.39

A piston-cylinder assembly contains a gas that undergoes a series of processes that make up a cycle. Assume that changes in kinetic and gravitational potential energy are negligible. The following state and process information is known about the cycle:

State/Process	P (kPa)	V (m^3)	T (°C)	U (kJ)
1	95	0.00570	20	1.47
$1 \rightarrow 2$	Adiabatic Compression			
2	2390	0.00057	465	3.67
$2 \rightarrow 3$	Constant Pressure			
3	2390	0.00171	1940	11.02

State/Process	P (kPa)	V (m^3)	T (°C)	U (kJ)
$3 \to 4$	Adiabatic Expansion			
4	445	0.00570	1095	6.79
$4 \to 1$	Constant Volume			

(a) Determine the **heat transfer** and **work** for each process of the cycle.

(b) Is this a power cycle or a refrigerator? Explain how you made your decision!

(c) Calculate the appropriate Measure of Performance (MOP) for this cycle based on Part (b).

? Problem 7.40

A heat pump cycle delivers energy by heat transfer to a dwelling at a rate of $60,000 \ \mathrm{Btu/h}$. The power input to the cycle is $7.8 \ \mathrm{hp}$.

(a) Determine the coefficient of performance for the cycle.

(b) If the heat pumps runs for 12 hours a day, how much electrical energy is required to run the heat pump for one day?

(c) If electricity costs $\$0.08$ per kilowatt-hour, how much does it cost per month to run the heat pump?

(d) If all of the heat transfer to the house is supplied by an electric-resistance furnace, the furnace would require an electrical power input of $60,000 \ \mathrm{Btu/\ h}$. How much would it cost per month to heat the house with an electric resistance furnace?

(e) Which system would you recommend to a home buyer based on your operating cost information - heat pump or electric furnace?

? Problem 7.41

A power cycle generates electricity and has a thermal efficiency of 33%. The electricity from the power cycle is used to run a refrigeration cycle which has a COP of 4. The refrigeration cycle receives $4,000 \ \mathrm{Btu/h}$ of energy by heat transfer at a low temperature. [Hint: It may help you to sketch both systems showing all pertinent heat transfer and work transfers of energy.]

(a) Determine the electrical power required to run the refrigeration cycle, in $\mathrm{Btu/s}$ and hp.

(b) Determine the total heat transfer rate of energy into the power cycle if all of its power output is going to drive the refrigeration cycle, in $\mathrm{Btu/s}$ and hp.

? Problem 7.42

In an iron ore mixing operation, a bucket full of ore is suspended from a traveling crane which moves along a stationary bridge. The bucket is to swing no more than $4 \ \mathrm{m}$ horizontally when the crane is brought to a sudden stop. Determine the maximum allowable speed v of the crane.

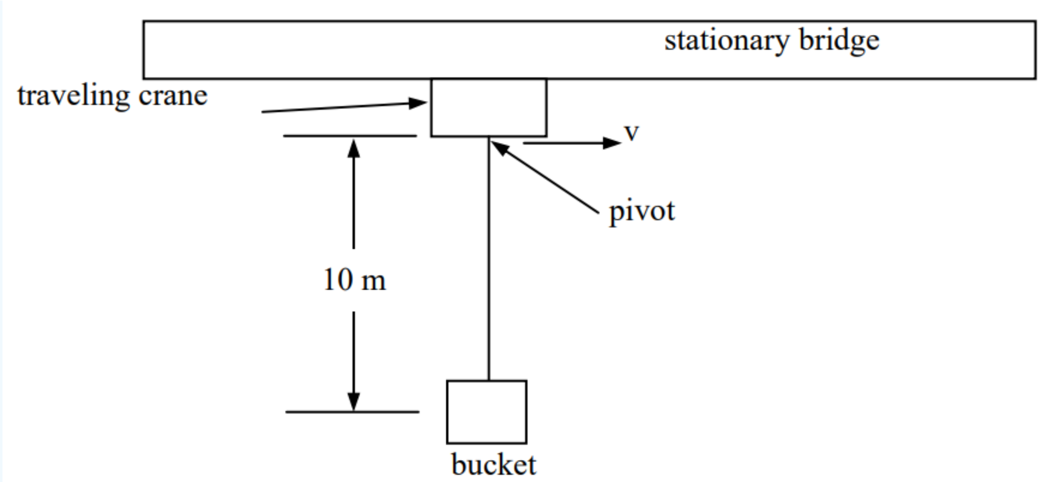


Figure 7.10.22 A bucket hangs from a traveling crane via a 10-meter cable.

? Problem 7.43

Packages are thrown down an incline at A with a velocity of 4 ft/s. The packages slide along the surface ABC to a conveyor belt which moves with a velocity of \8mathrm ft/s. Knowing that $\mu_k = 0.25$ between the packages and the surface \(\\) determine the distance d if the packages are to arrive at C with a velocity of 8 ft/s.

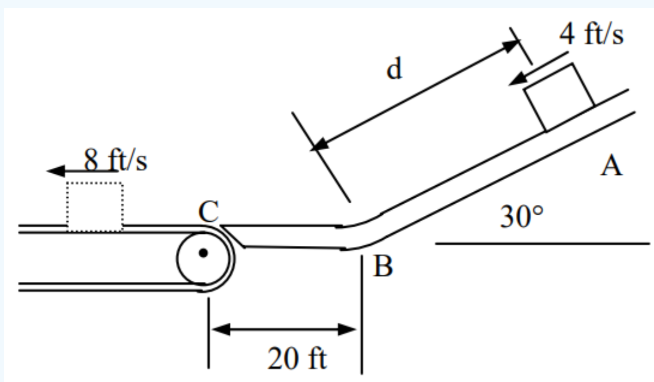


Figure 7.10.23 A package slides down an incline whose base connects to a horizontal platform.

? Problem 7.44

An elastic cable is to be designed for bungee jumping from a tower 130 ft high. The specifications call for the cable to be 85 ft long when unstretched, and to stretch to a total length of 100 ft when a 600-lb weight is attached to it and dropped from the tower. Determine:

(a) the required spring constant k of the cable,

(b) how close to the ground a 185-lb man will come if he uses this cable to jump from the tower, and

(c) the maximum acceleration experienced by the man.

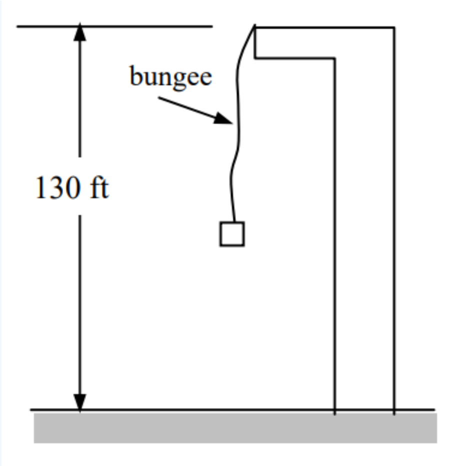


Figure 7.10.24 A bungee cable is attached to the top of a 130-ft tower.

Problem 7.45

The 4 lbf object is dropped 5 feet onto the 20 lbf block that is initially at rest on two springs, each with a stiffness $k = 5\ \mathrm{lb/in}$. Calculate the maximum deflection of the springs assuming the two objects stick together after the impact.

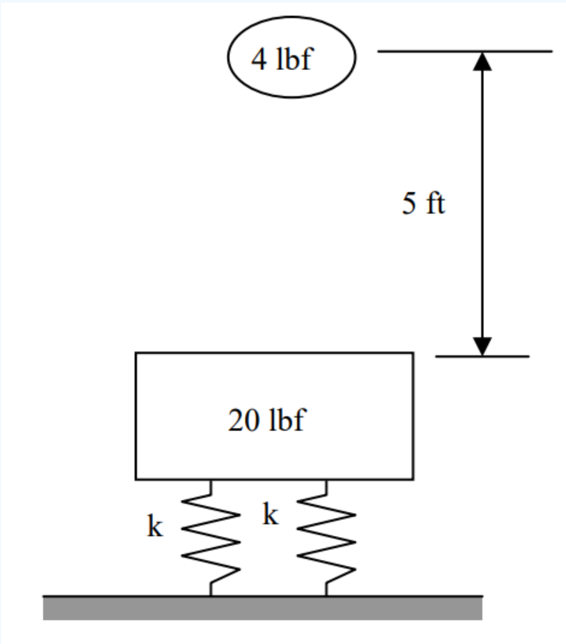


Figure 7.10.25 One object is dropped onto another supported by springs.

Problem 7.46

The system is at rest in the position shown, with the 10 kg collar A resting on the spring $(k = 500\ \mathrm{N/m})$, when a constant 0.5 kN force is applied to the cable. What is the velocity of the collar when it has risen 0.2 m ? Assume there is no friction between the vertical shaft and A.

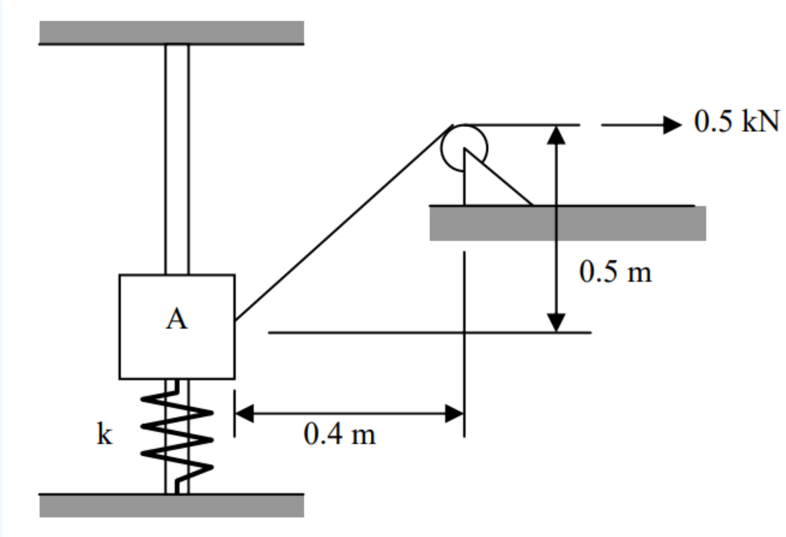


Figure 7.10.26: Sliding collar with an applied tension rests on top of a spring.

Problem 7.47

Two kg of air (assume ideal gas) in a piston-cylinder device undergoes a thermodynamic cycle consisting of three processes, each with negligible kinetic and gravitational potential energy.

Process:

$1 \rightarrow 2$ Constant Volume

$2 \rightarrow 3$: Compression with $P = -\left(250\ \mathrm{kPa/m^3}\right) V + 550\ \mathrm{kPa}$

$3 \rightarrow 1$: Adiabatic Expansion

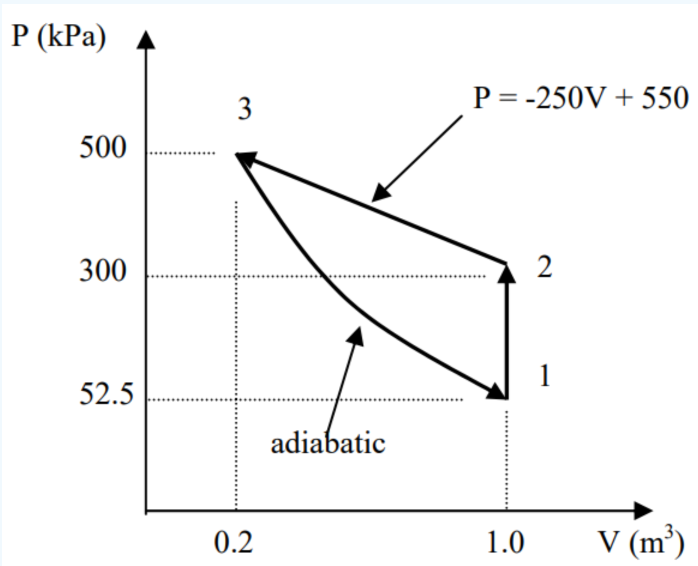


Figure 7.10.27: Pressure-volume relationships between three states of an ideal gas.

a) Complete the table given (Show work for full credit)

b) Is it a power cycle or a refrigeration cycle? Be sure to explain your answer.

Process	Q_{in} (kJ)	W_{in} (kJ)	$U_{\text{final}} - U_{\text{initial}}$ (kJ)
$1- > 2$		0	

Process	Q_{in} (kJ)	W_{in} (kJ)	$U_{\text{final}} - U_{\text{initial}}$ (kJ)
$2->3$			
$3->1$			
NET			

Problem 7.48

Block A with mass $m_A = 10$ kg slides to the right on a horizontal frictionless surface with an initial velocity of 10 m/s until it hits a Bumper B with mass m_B. The Bumper B is attached to a linear, massless spring with a spring constant $k = 1000$ N/m. The spring is initially unloaded and uncompressed. The bumper is designed so that the spring can only be compressed. Consider the distance required to stop Block A for two different cases.

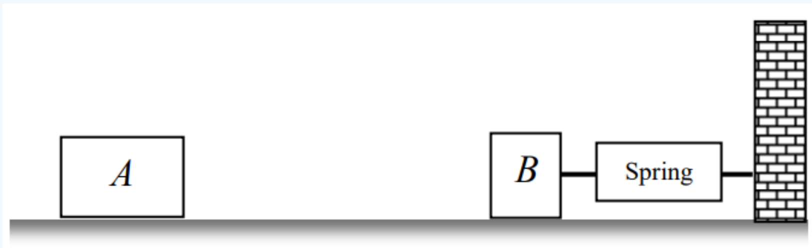


Figure 7.10.28 Two blocks on a horizontal surface, one connected to a support via spring.

(a) Case I: Determine how far Block A travels after it impacts the bumper if the impact is purely plastic (perfectly inelastic), i.e. Block A and Bumper B stick together after they contact, and the bumper is massless, i.e. $m_B = 0$ kg.

(b) Case II: Repeat Part (a), only this time assume that the bumper has mass $m_B = 5$ kg. For this part assume that the motion of the Bumper B is negligible, i.e. spring force is negligible, during the impact between Block A and the Bumper B.

Problem 7.49

A light rod with a fixed collar of mass $m = 10$ kg is initially at rest in the inclined position shown in the figure. It is then rotated counter-clockwise about the pivot B from rest by the constant force $\mathbf{P}$ until it is brought to rest in the horizontal position by the linear spring which has a spring constant of $k = 40$ kN/m. The spring is compressed from its free (uncompressed) length by 50 mm when the rod comes to rest. Assume the mass of the rod is negligible.

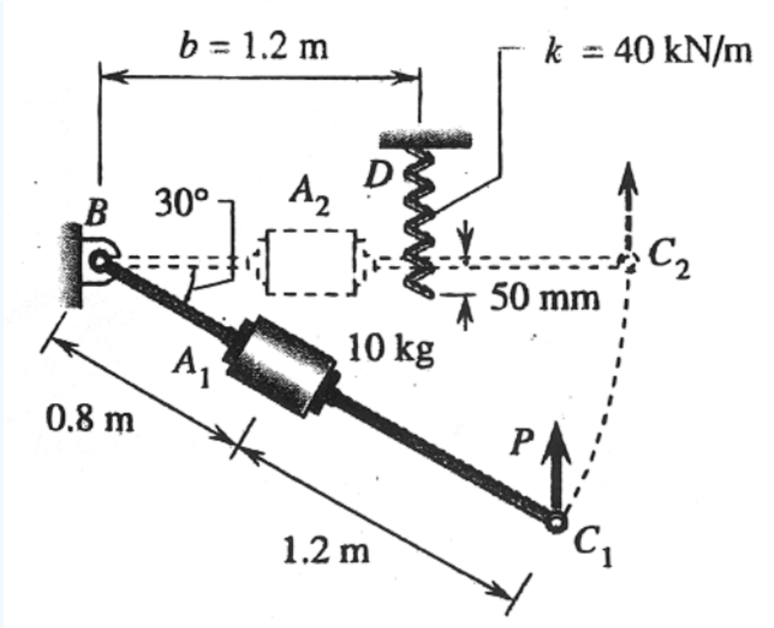


Figure 7.10.29 A rod with an attached mass is pivoted by application of a constant force.

(a) Determine the work done by the force $\mathbf{P}$ to rotate the rod from C_1 to C_2, in joules.

(b) Determine the magnitude of force $\mathbf{P}$, in newtons.

Problem 7.50

A centrifugal pump is driven by an electric motor as shown in the figure. Water flows steadily through the pump with the inlet and outlet conditions shown in the figure. The 440-ac-volt electric motor receives 42 kW of electrical power and delivers 40 kW of shaft power to the pump under steady-state conditions. The motor rotates at 1750 rpm and has a power factor of unity. Assume that water can be modeled as an incompressible substance with constant specific heats, and assume changes in gravitational potential energy are negligible.

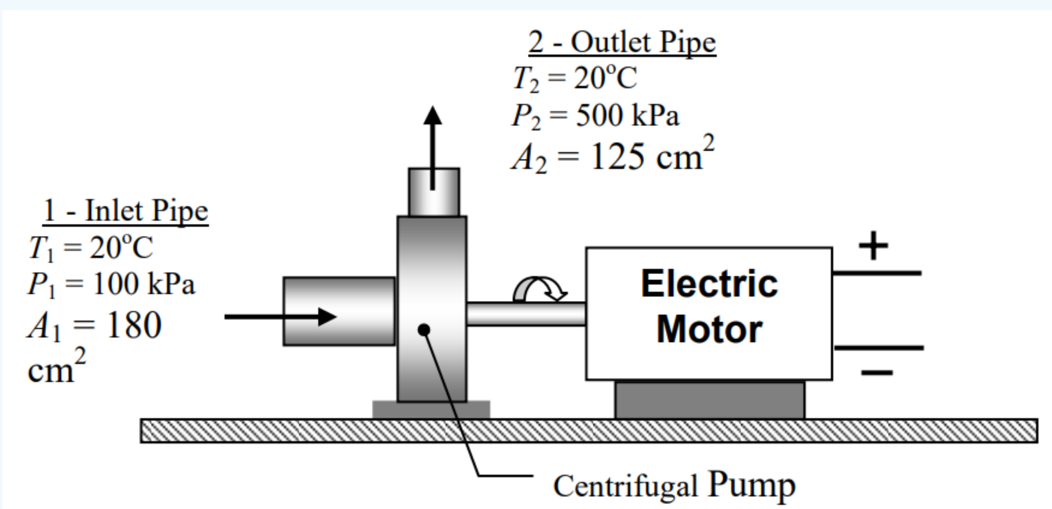


Figure 7.10.30 An electric motor powers a centrifugal pump for water.

(a) Determine the direction and magnitude of the heat transfer rate for the pump, in kilowatts.

(b) Determine the torque transmitted by the motor shaft to the pump, in $\mathrm{N-m}$.

(c) Determine the electric current supplied to the motor, in amps.

Problem 7.51

A piston-cylinder device contains helium gas. Initially, the gas has a pressure of $70\ \mathrm{psia}$, a temperature of $600^\circ\mathrm{R}$, and a volume of $7\ \mathrm{ft}^3$. During a process where $PV = C$, a constant, the helium is expanded to a final volume of $28\ \mathrm{ft}^3$. Assume that helium gas can be modeled as an ideal gas with constant specific heats and assume that changes in kinetic and potential energy are negligible.

Determine the direction and the magnitude of the work and the heat transfer for the helium gas, in $\mathrm{ft \cdot lbf}$.

Problem 7.52

Air is contained inside of a piston-cylinder device that also contains an electric resistance heating element (see the figure). The cylinder walls and piston are heavily insulated giving an adiabatic boundary. The air expands from State 1 to State 2 in a constant pressure (isobaric) process.

During the expansion process electrical energy is supplied to the resistance heating element. For purposes of this analysis, you may assume that the heating element has negligible mass. Assume that air can be modeled as an ideal gas with room temperature specific heats. Also assume changes in kinetic and gravitational potential energy are negligible.

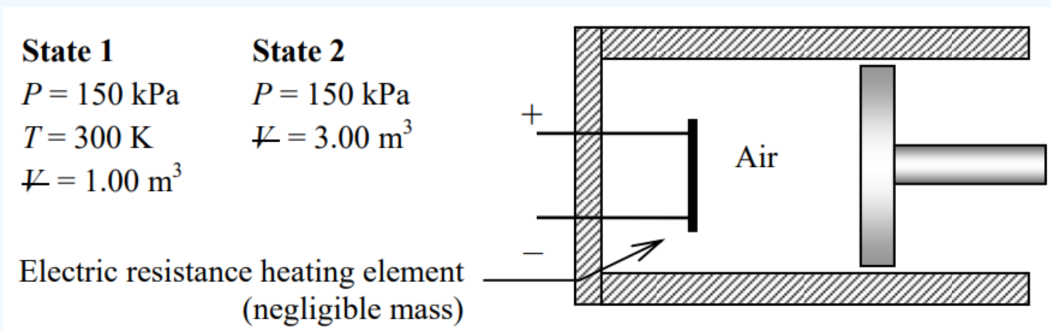


Figure $7.10.31$: State information of air in a piston-cylinder device containing a resistance heating element.

(a) Determine the temperature of the gas in State 2.

(b) Determine the direction and magnitude of the transfer of energy by electric work for the process, in kJ.

Problem 7.53

The Collar A is released from rest at the position shown in the figure and slides up the fixed rod under the action of a constant force P applied to the cable. The rod is inclined at 30° from the horizontal as shown in the figure, and the position of the small pulley B is fixed. When the collar has traveled 40 inches along the rod to position D, the spring is compressed 6 inches, the cable makes a 90° angle with the rod (see dashed line), and the collar is still moving with an unknown velocity.

The mass of the collar is $30\ \mathrm{lbm}$ and the constant force $P = 50\ \mathrm{lbf}$. The spring has a stiffness $\mathrm{k} = 200\ \mathrm{lbf/ft}$. Assume that friction between the collar and rod is negligible.

Determine the speed of the collar, in $\mathrm{ft/s}$, when the collar reaches Point D, i.e. the cable is coincident with the dashed line in the figure.

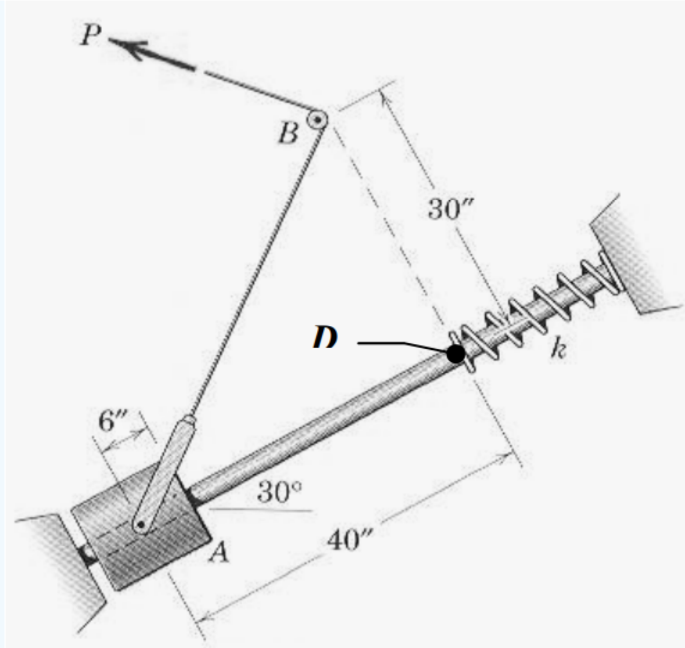


Figure 7.10.32 System consisting of a rod, spring, and collar with attached cable.

Problem 7.54

A hydroelectric turbine-generator produces an electric power output of 20 MW (megawatts). Water enters the turbine penstock at Point 1 and exits the turbine at Point 2 as shown in the figure. The known information at the inlet and exit are shown in the figure. The turbine-generator operates adiabatically at steady-state conditions. Do not neglect kinetic or gravitational potential energy unless you can substantiate your assumption.

Assume water can be modeled as an incompressible substance with room-temperature specific heats.

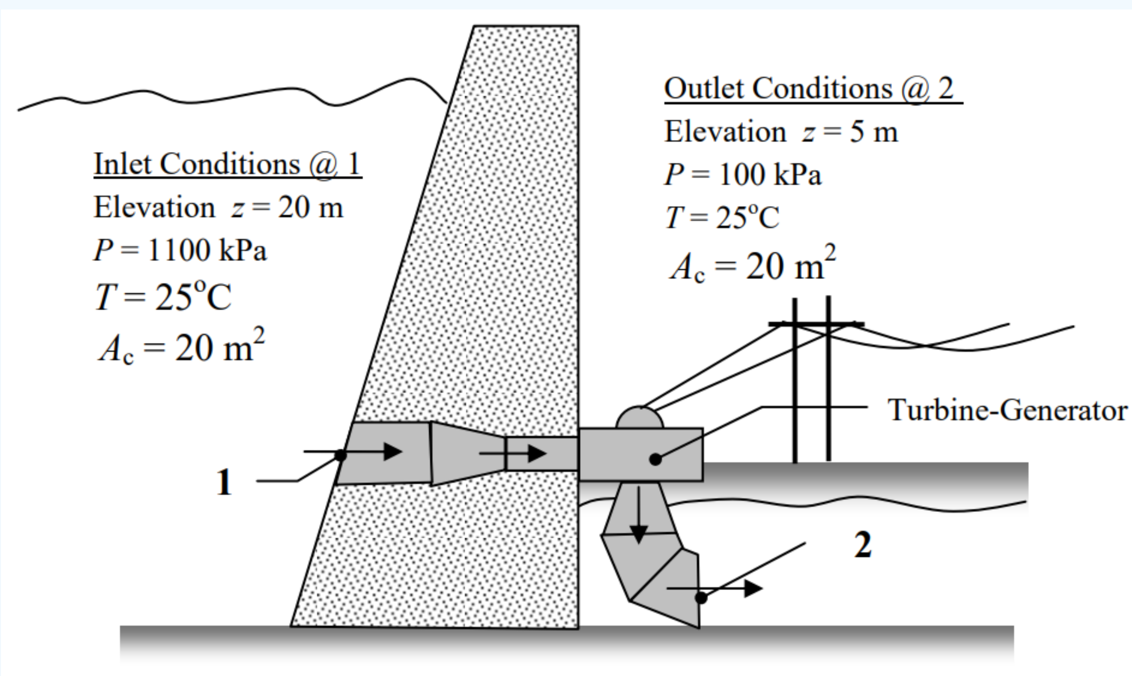


Figure 7.10.33 Water powers a turbine-generator attached to a dam.

(a) Determine the mass flow rate of the water through the turbine-generator, in kg/s.

(b) If a shaft *inside* the turbine-generator system transmits 22 MW of power at a rotational speed of 100 rpm, determine the torque in the shaft.

Problem 7.55

An ideal gas is contained in a simple piston-cylinder device and executes the three-step process shown in the table.

State 1	$P_1 = 100$ kPa; $V_1 = 1.00$ m^3; $T_1 = 300$ K
Process $1 \rightarrow 2$	Constant-pressure (isobaric) expansion
State 2	$V_2 = 2.00$ m^3
Process $2 \rightarrow 3$	Constant-temperature (isothermal) expansion where $PV = C$.
State 3	$V_3 = 3.00$ m^3
Process $3 \rightarrow 4$	Constant-pressure (isobaric) compression
State 4	$V_4 = V_1 = 1.00$ m^3

(a) Determine the temperature of the gas at state 2.

(b) Sketch the process on a $P - V$ diagram. Clearly label the four states, 1, 2, 3, and 4 and the connecting processes

(c) Using your sketch from part (b) above, identify the area on the diagram that represents the work done during process $2 \rightarrow 3$ by shading or cross-hatching the area.

Problem 7.56

A 455 cubic inch Pontiac engine (A) is connected to a TH400 automatic transmission (B) which sends power out to the rear wheels (C). A transmission cooler heat exchanger (D) is used to remove heat generated by frictional losses within the transmission. A liquid coolant circulates through a closed loop to transfer the energy from the transmission to the cooler heat exchanger. Steady-state performance data is measured in the lab and the results are shown in the table:

Power Measurements	
Engine output shaft @ 3800 rpm	390.0 hp
Transmission output shaft	350.4 hp
Coolant Temperatures	
Cooler inlet / transmission outlet	300°F
Cooler outlet / transmission inlet	237°F

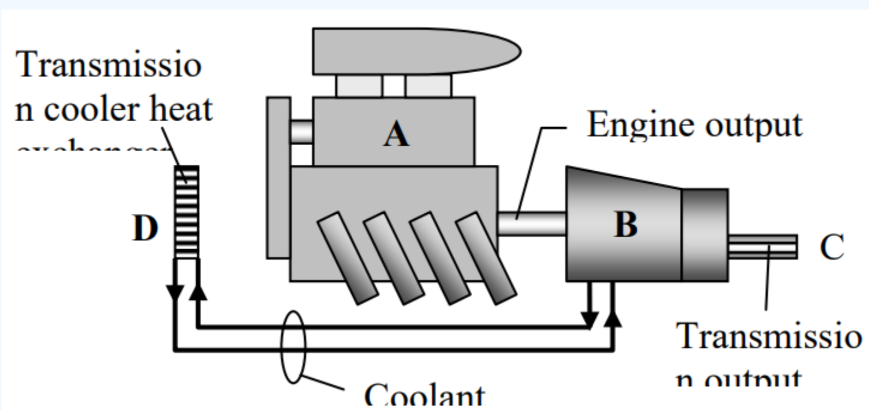


Figure $7.10.34$ Engine is connected to a transmission and trasnmission coolant system.

If needed, you may assume the transmission coolant fluid can be modeled as an incompressible substance with a density of $56.8\ \mathrm{lb_m/ft^3}$ and a specific heat $0.4286\ \mathrm{Btu/\left(lb_m \cdot {}^\circ F\right)}$.

a) Determine the *torque* produced by the motor at the engine output shaft, in $\mathrm{ft \cdot lb_f}$.

b) Determine the *heat transfer rate* out of the transmission cooler heat exchanger, in $\mathrm{Btu/s}$. Assume that there is negligible heat transfer from the surfaces of the transmission casing and the coolant lines.

c) Determine the *mass flow rate of coolant* through the coolant lines, in $\mathrm{lb_m/s}$. Assume that pressure drops inside the coolant loop circuit are negligible, i.e. pressure inside the coolant loop is uniform.

d) Experience has shown that the heat transfer from the transmission casing may *not* be negligible. To check this out, estimate the *heat transfer rate from the surface of the transmission* by convection. Assume that the surface area of the casing is $8.68\ \mathrm{ft^2}$, the surface temperature is $80^\circ\mathrm{F}$, the temperature of the surrounding air is $70^\circ\mathrm{F}$, and the convective heat transfer coefficient is $0.0144\ \mathrm{Btu/\left(ft^2 \cdot s \cdot {}^\circ F\right)}$.

Problem 7.57

In a belated move to surpass the U.S. space program, chipmunks have decided to place a chipmunk in space. The planned launch vehicle for the chipnaut is a potato slingshot as shown in the figure.

For the launch, the chipnaut first climbs out on a limb and takes a position immediately above the potato slingshot. Then his launch crew pulls back the slingshot into the firing position as shown. Once the slingshot is fired, the chipnaut awaits the potato "booster". When the potato booster arrives, it picks up (impacts) the chipnaut without touching the tree launch platform and carries the chipnaut upward to the heavens.

Your job is to predict how the flight will proceed. You may assume that all motion is in the vertical direction. Additional information is provided below:

Mass of the potato $= 1.0\ \mathrm{kg}$	Elastic Band: Spring Constant $k = 500\ \mathrm{N/m}$
Mass of the chipnaut $= 2.0\ \mathrm{kg}$	Total length (unstretched) = 1.0 m
	Total length installed (with initial stretch) = 1.3 m
	Total length stretched for firing = 2.0 m

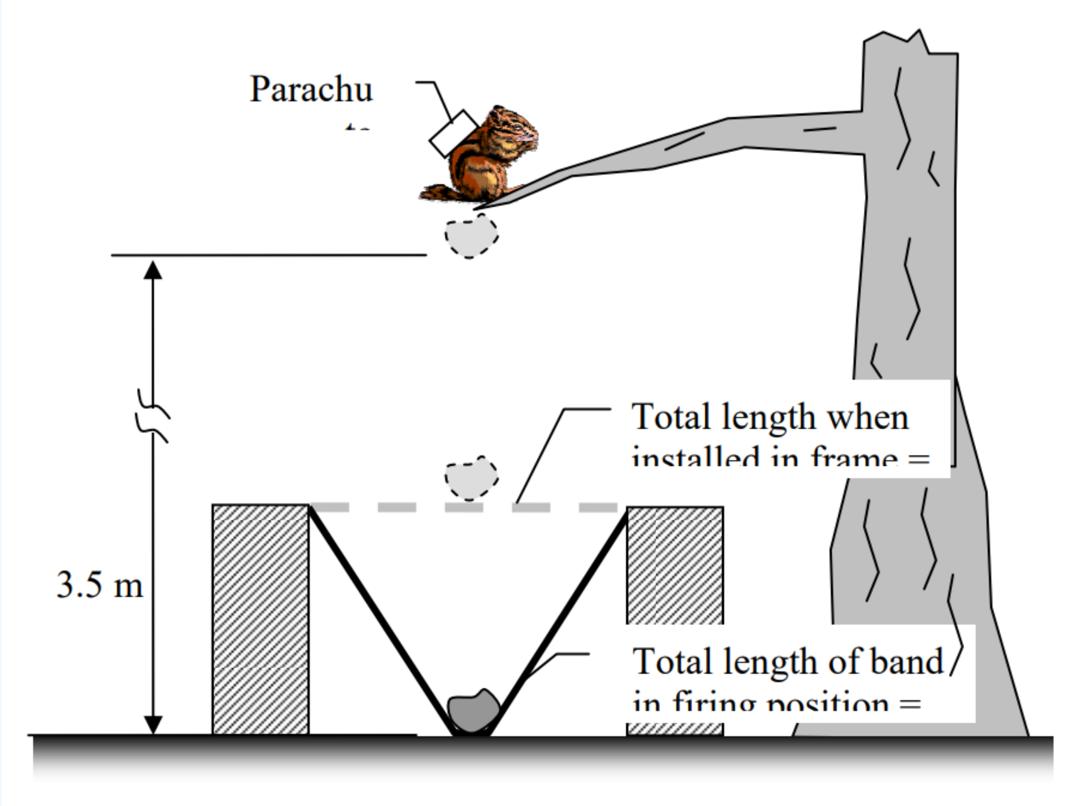


Figure 7.10.35 Chipnaut and booster prior to launch.

a) Determine the velocity, in $\mathrm{m/s}$, of the potato "booster" immediately before it picks up the chipnaut.

b) Determine the maximum elevation, in meters, that can be achieved by the "booster" carrying the chipnaut. (Please note that some chipnauts have been concerned about remaining conscious during the flight. Future test flights will investigate this potential problem.)

(Please note that some chipnauts have been concerned about remaining conscious during the flight. Future test flights will investigate this potential problem.)

Problem 7.58

The ion sputtering process creates a new surface layer of material on an object by bombarding the surface with ions of the desired material. The top half of the device in the figure is the vacuum chamber and the lower half is the piston-cylinder device used to raise or lower the target. The position of the piston is altered by adding energy to the gas using an electric resistance heating element or removing energy by heat transfer using a cold plate in the cylinder wall. The known information is shown below.

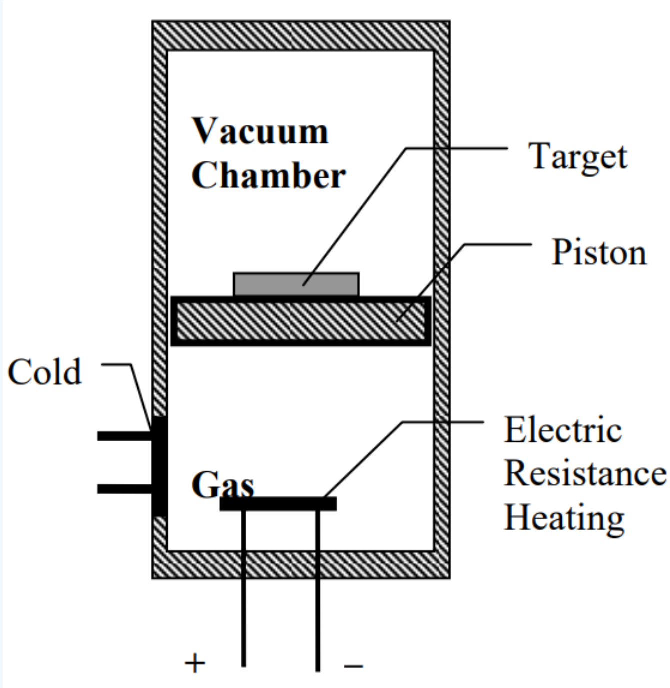


Figure $7.10.36$: A cylinder consisting of a vacuum chamber and a cylinder-piston device of gas, aligned vertically.

Given Information:

$$\begin{aligned} m_{\mathrm{P}} &= \text{Mass of the piston} \\ m_{\mathrm{T}} &= \text{Mass of the target} \\ m_{\mathrm{G}} &= \text{Mass of the gas} \\ A_{\mathrm{P}} &= \text{Area of the piston} \end{aligned}$$

$$\begin{aligned} T_1 &= \text{Initial temperature of the gas} \\ T_2 &= \text{Final temperature of the gas} \\ P_1 &= (m_T + m_P)\, g / A_P, \text{ the initial pressure in the gas.} \\ z_1 &= \text{Initial elevation of the piston} \end{aligned}$$

$$\begin{aligned} g &= \text{Acceleration of gravity} \\ c_{\mathrm{p}} \,\&\, c_v &= \text{Specific heats of the gas} \end{aligned}$$

$$\begin{aligned} W_{\text{elect, in}} &= \text{Electric work into the gas} \\ Q_{\text{cold, out}} &= \text{Heat transfer out of the gas and into the cold plate.} \end{aligned}$$

Select an appropriate system or systems and determine the change in elevation of the target, $\Delta z = z_2 - z_1$, in terms of some or all of the given information.

Assume that the piston is frictionless and initially stationary. The change in elevation occurs very slowly with negligible change in kinetic energy of the piston. In addition, the piston, the cylinder wall, and the vacuum chamber wall (all the cross-hatched regions) are made of material that provides an adiabatic boundary and does not change temperature. The gas can be modeled as an ideal gas with room-temperature specific heats.

? Problem 7.59

The piston cylinder device shown below contains nitrogen. The nitrogen undergoes a volume-change process where $P\mathcal{V} = C$. Other information about the process is shown below. Assume nitrogen can be modeled as ideal gas with room-temperature specific heats.

$$\text{Nitrogen}\,(\mathrm{N}_2):$$

$$c_v = 0.743 \text{ kJ}/(\text{kg} \cdot \text{K})$$
$$c_p = 1.04 \text{ kJ}/(\text{kg} \cdot \text{K})$$
$$R = 0.298 \text{ kJ}/(\text{kg} \cdot \text{K})$$

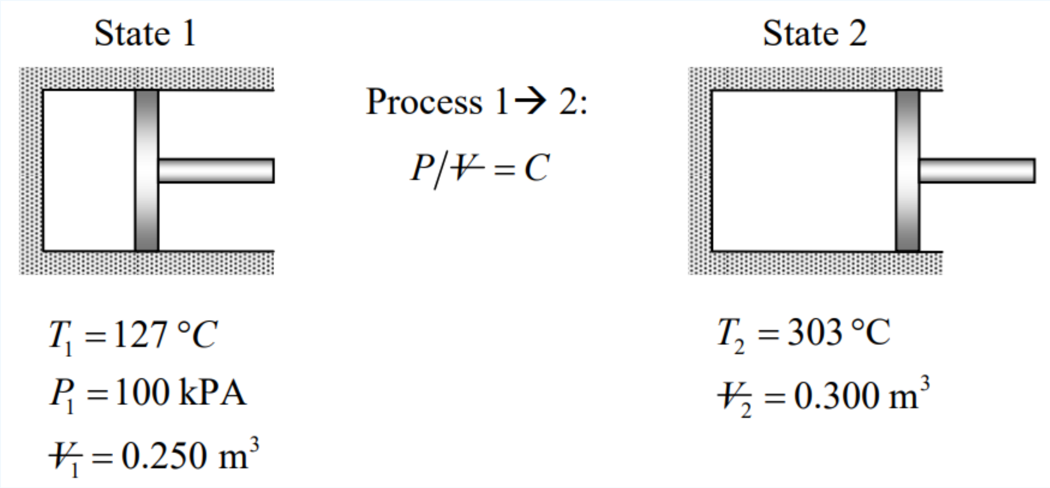


Figure 7.10.37: Initial and final states of gas in a piston-cylinder device.

(a) Determine the mass of nitrogen in the piston-cylinder device, in kg.

(b) Determine the work transfer of energy for the gas during this process, in kJ.

(c) Determine the heat transfer of energy for the gas during this process, in kJ.

(d) Sketch the process on a P-$V\!\!\!\!-$ (pressure-volume) diagram *and* show the work for the process.

? Problem 7.60

A hot-water heating system is shown in the figure below. The circulating pump is located in the basement of the building and the hot-water radiator is located on an upper floor. Under steadystate conditions, the radiator delivers 3.0 kW by heat transfer to the surroundings.

Pertinent operating conditions are shown in the table. Assume that water can be treated as an incompressible substance with room-temperature specific heats.

$$[\text{ Liquid Water Properties:} \quad c_p = 4.18 \text{ kJ}/(\text{kg} \cdot \text{K}) \text{ and } \rho = 997 \text{ kg}/\text{m}^3 \,]$$

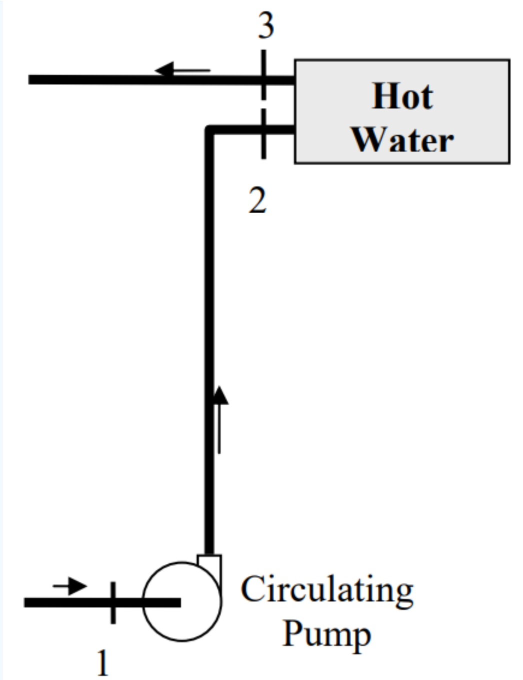


Figure $7.10.38$ Water is pumped upwards before being heated.

Operating Conditions

State	T (°C)	P (kPa)	z (m)	A (m^2)
1	60	100	10.0	0.0020
2	60	125	30.0	0.0020
3	40	125	30.0	0.0020
Heat transfer from the pipes and the pump is negligible.				
The only significant heat transfer occurs from the radiator.				

(a) Determine the mass flow rate through the radiator, in $\mathrm{kg/s}$.

(b) Determine the shaft power supplied to the pump, in kW, to move the water up to the radiator.

(c) Estimate the surface area of the radiator. Assume that convection heat transfer is the primary mechanism, the convection heat transfer coefficient $h = 50\ \mathrm{W}/\left(\mathrm{m}^2 \cdot {}^\circ\mathrm{C}\right)$, the room temperature is $22^\circ\mathrm{C}$ and the average radiator temperature is $50^\circ\mathrm{C}$.

Problem 7.61

A common safety device utilized in mountainous areas is a "runaway truck" ramp used to stop a truck without functional brakes. This device consists of a long, upward-sloped ramp covered in gravel and a bumper attached to a spring. (See the figure below.).

A runaway truck weighing $45,000\ \mathrm{lb_f}$ pulls onto the ramp at $100\ \mathrm{ft/s}$ (just over $68\ \mathrm{mph}$). The ramp is $250\ \mathrm{ft}$ long. The bumper weighs $500\ \mathrm{lb_f}$. The spring is initially undeflected. As a last resort, the spring is attached to an immense, immovable concrete barrier.

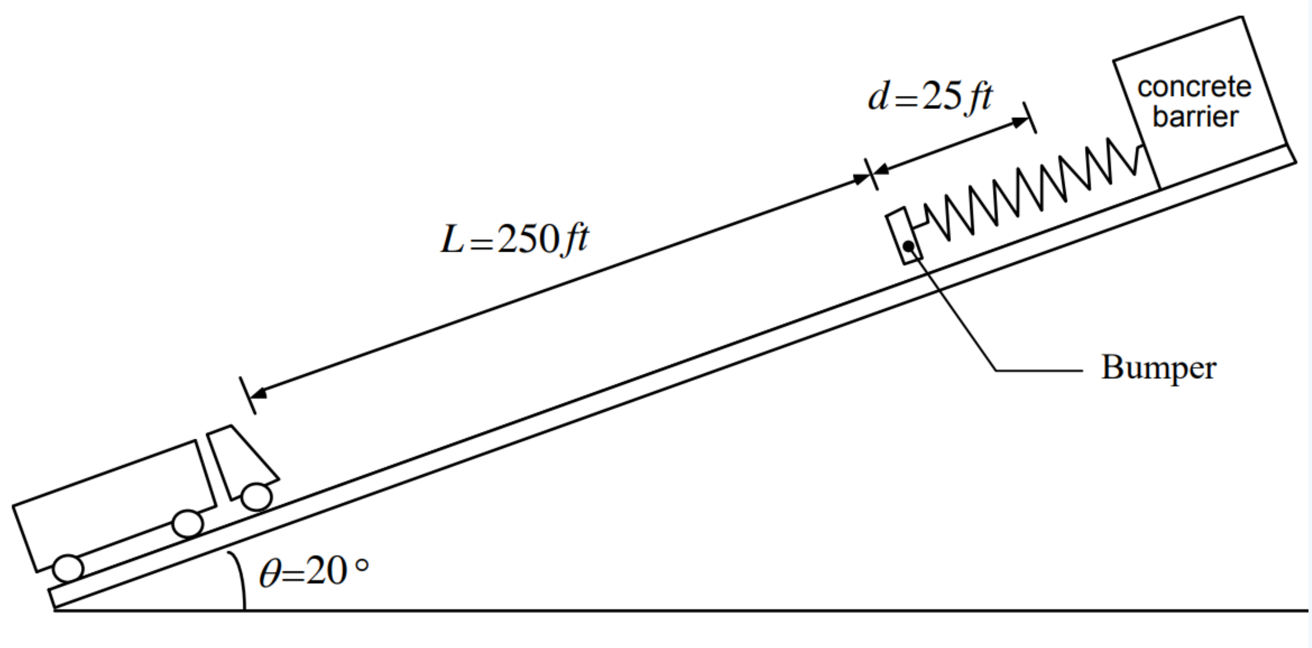


Figure $7.10.39$: A truck faces upslope on a long ramp with a spring-mounted bumper at the top.

(a) Determine the average friction force the ramp exerts on the truck, in newtons, as it climbs the ramp if the truck velocity is 30 ft/s just before it strikes the bumper.

(b) Determine the value of the spring constant k, in $\text{lb}_\text{f}/\text{ft}$, necessary to bring the truck to rest in 25 ft after the truck strikes and sticks to the bumper. Neglect frictional effects.

Problem 7.62

Block A with mass m_A is released from rest in the position shown. It slides a distance L down a smooth incline before hitting and sticking to Block B. Block B is initially at rest and has mass m_B.

Determine the equations necessary to find the maximum distance the spring deflects, d.

Assume m_A, m_B, k, L and θ are known.

Do not solve these equations. Your solution should consist of a list of equations and unknowns.

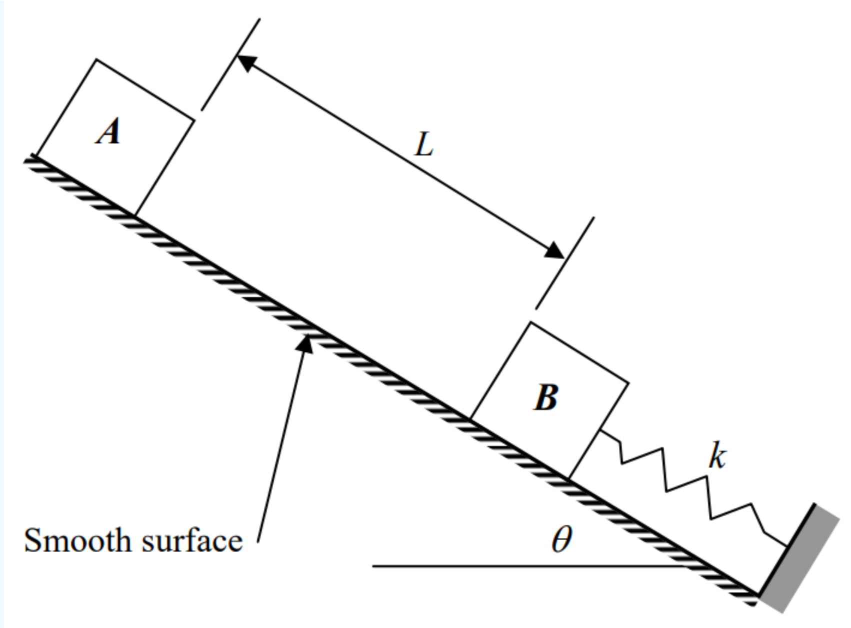


Figure 7.10.40: A ramp with a spring-mounted block at the bottom and a free-sliding block at the top.

Problem 7.63

The 10-kg collar is attached to two identical springs and slides on the smooth vertical rod as shown in the figure. The spring constant for each spring is $k = 800\ \mathrm{N/m}$, and the unstretched length of each spring is $0.3\ \mathrm{m}$. In position A, the collar has a velocity $V_1 = 2\ \mathrm{m/s}$ in the direction shown.

It is desired to modify this device by applying a constant force $\mathbf{F}$ to the collar as it moves from A to B so that the velocity of the collar at position B will be zero, i.e. $V_2 = 0$.

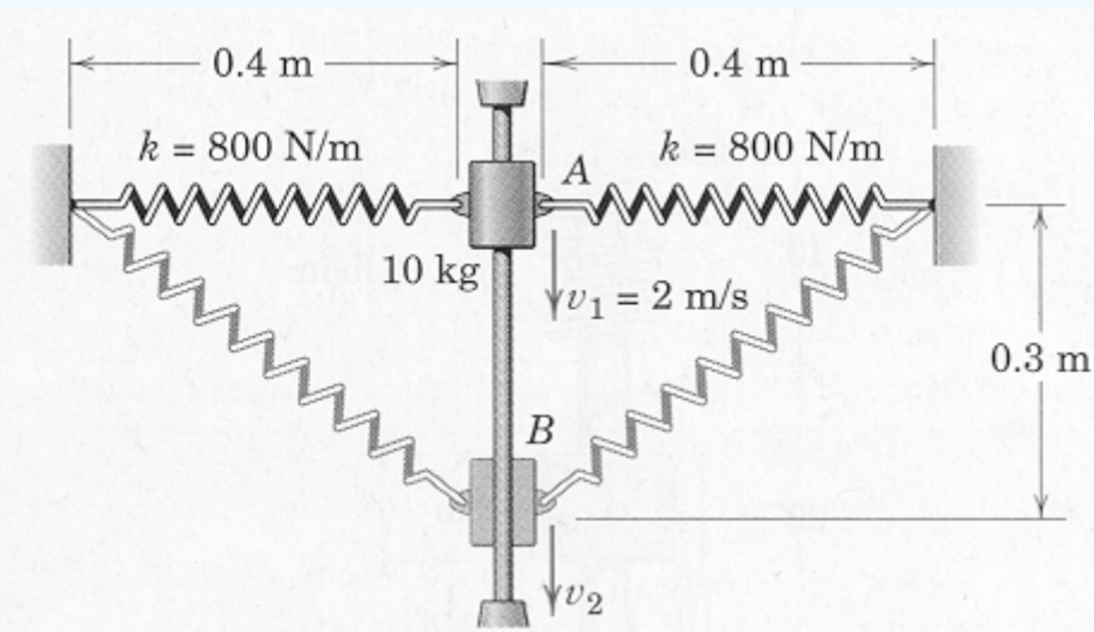


Figure 7.10.41: A sliding collar is attached to two springs.

(a) Determine the direction and magnitude, in newtons, of the constant force $\mathbf{F}$ that must be applied to the collar as it moves from A to B so that $V_2 = 0$.

(b) Will the collar *stop* moving once it reaches position B even though $V_2 = 0$? Explain the basis for your answer. (Even without a numerical answer, full credit will be given for part (b) if a clear explanation of *how* you would determine the answer is given.)

Problem 7.64

A small steam turbine is connected to an air compressor through a gear reducer as shown in the figure. A gear reducer is a device used to change the shaft rotation speed when two devices must be connected but operate at different speeds.

- Steam enters the turbine at $110^\circ\mathrm{C}$ with a specific enthalpy $h_1 = 2691.5\ \mathrm{kJ/kg}$ and exits the turbine at a pressure of $100\ \mathrm{kPa}$ and a specific enthalpy $h_2 = 2675.5\ \mathrm{kJ/kg}$. The turbine shaft rotates at 2000 rpm.
- Air enters the compressor at a mass flow rate of $70\ \mathrm{kg/min}$ at $P_3 = 100\ \mathrm{kPa}$ and $T_3 = 300\ \mathrm{K}$ and exits the compressor at $P_4 = 500\ \mathrm{kPa}$ and $T_4 = 460\ \mathrm{K}$. The compressor shaft rotates at 600 rpm. A ssume that air can be modeled as an ideal gas with constant specific heats.

Assume all devices shown in the figure—turbine, compressor and gear reducer—operate adiabatically at steady-state conditions with negligible changes in kinetic and gravitational potential energy.

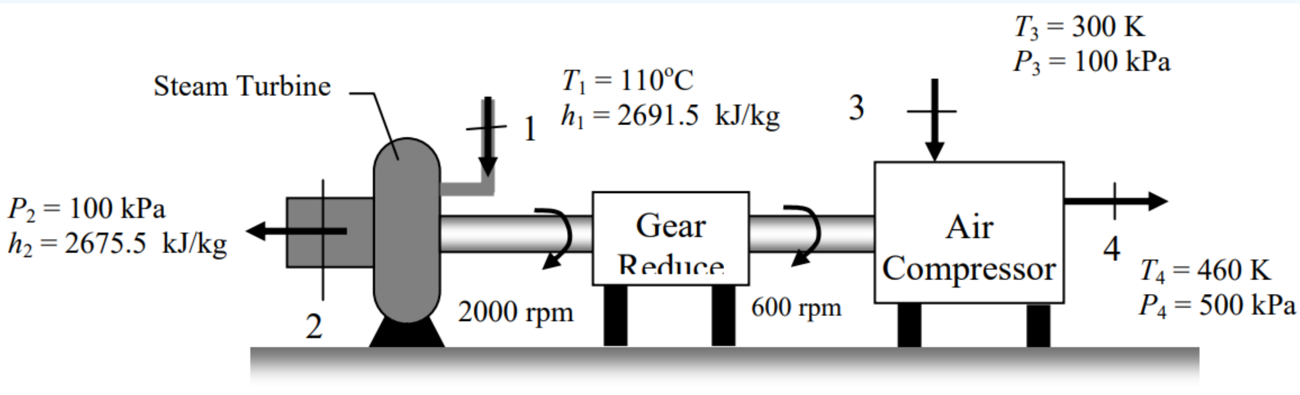


Figure 7.10.42 System consisting of a steam turbine, gear reducer, and air compressor, all sharing a common shaft.

(a) Determine the mass flow rate of steam into the turbine, in $\mathrm{kg/min}$.

(b) Determine the shaft power required by the air compressor, in kW.

(c) Determine the torque, in $\mathrm{N \cdot m}$, transmitted by the air-compressor shaft.

Problem 7.65

A piston-cylinder device contains carbon dioxide (CO_2) gas initially occupying the volume V_1 at the pressure P_1 and temperature T_1 indicated below. The gas undergoes the process described below:

State 1:	$P_1 = 150\ \mathrm{kPa}; \quad T_1 = 400\ \mathrm{K}; \quad V_1 = 0.5\ \mathrm{m}^3$
Process 1 $\rightarrow$ 2:	Quasistatic process where $P = \left(300\mathrm{kPa/m}^3\right) V$
State 2:	$V_2 = 1.0\ \mathrm{m}^3$

Assume that carbon dioxide can be modeled as an ideal gas with constant specific heats and that changes in kinetic and gravitational potential energy are negligible for the process.

Determine the work and heat transfer of energy for process $1 \rightarrow 2$. Indicate both the direction and the magnitude (in kilowatts) of each.

Problem 7.66

The collar C slides on the curved rod in the vertical plane under the action of a constant force F in the cord guided by the small pulleys at D. The collar has a mass of 0.70 kg and slides without friction.

If the collar is released from rest at A, determine the value of the constant force F that will result in the collar striking the stop at B with a velocity of $4\ \mathrm{m/s}$.

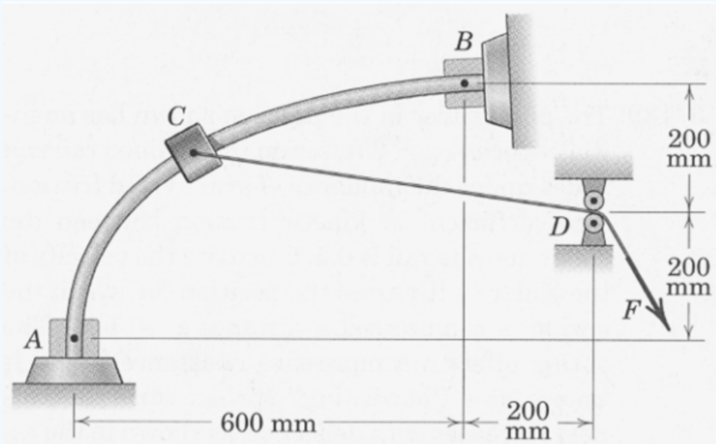


Figure $7.10.43$ A collar with an attached cable that passes over a pulley slides along a curved rod.

Problem 7.67

The system shown at right is the back end of a jet aircraft engine. Operating information about the system is shown in the table and figure. A ir flows steadily through the system. Assume changes in gravitational potential energy are negligible and air can be modeled as an ideal gas with room temperature specific heats.

State	T (°C)	P (kPa)	V (m/s)	A_c (m^2)
1	600	800	$V_1 \approx V_2$	$\cdots$
2	???	800	$V_2 \approx V_3$	$\cdots$
3	1300	600	\(V_{3} << V_{4}\)	$\cdots$
4	950	100	???	???

Turbine: steady-state and adiabatic

Nozzle: steady-state and adiabatic

Heat exchanger: steady-state

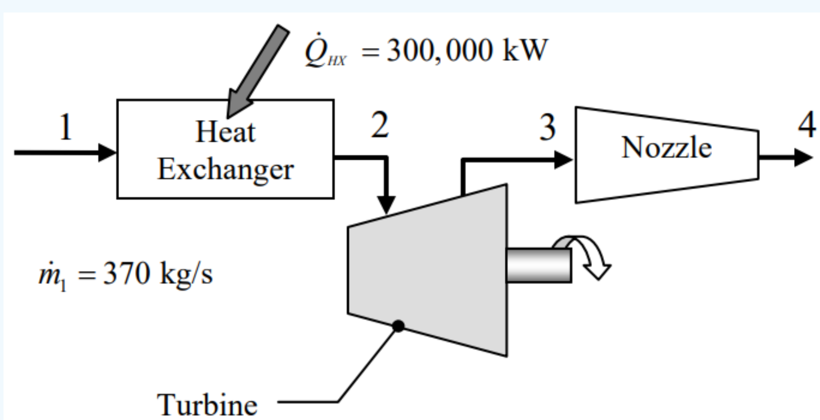


Figure $7.10.44$ System consisting of a heat exchanger, turbine, and nozzle.

(a) Determine the velocity of the air leaving the nozzle, in $\mathrm{m/s}$.

(b) Determine the cross-sectional area A_c at the nozzle outlet, in m^2

(c) Determine the shaft power out of the turbine, in kW.

Problem 7.68

A well-insulated copper tank of mass $13\ \mathrm{kg}$ contains $4\ \mathrm{kg}$ of liquid water. Initially, the temperature of the copper is $27^\circ\mathrm{C}$ and the temperature of the water is $50^\circ\mathrm{C}$. As the tank and its contents come to equilibrium, an electrical resistor of negligible mass transfers $100\ \mathrm{kJ}$ of energy to the contents of the tank. Assume copper and liquid water can be modeled as incompressible substances.

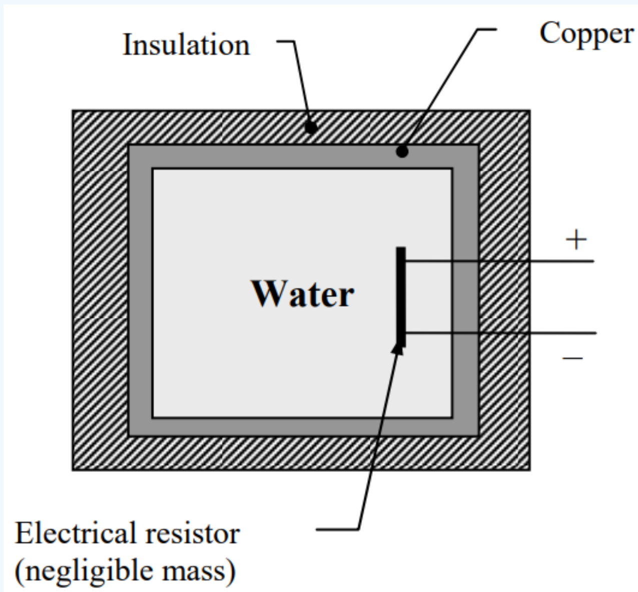


Figure $7.10.45$: Insulated tank contains water and an electrical resistor.

(a) Determine the final temperature of the tank and water.

(b) If current through the resistor is 0.5 amps and the applied voltage is 110 volts, determine (i) the electrical power supplied to the resistor and (ii) how long the resistor was "on" to deliver $100\ \mathrm{kJ}$ of electrical energy.

Problem 7.69

A spring-loaded boot-on-a-stick kicks a marble as shown in the figure. Initially both the boot and marble are stationary. To load the device, the boot is swung up to the position shown and the uncompressed spring on the ceiling is compressed a distance d. The stationary boot is then released, swinging down and to the left before kicking the marble. The mass of the boot and marble are m_b and m_m, respectively, and the spring has a stiffness k. The stick of length L has negligible mass and is hinged to a frictionless pin at A.

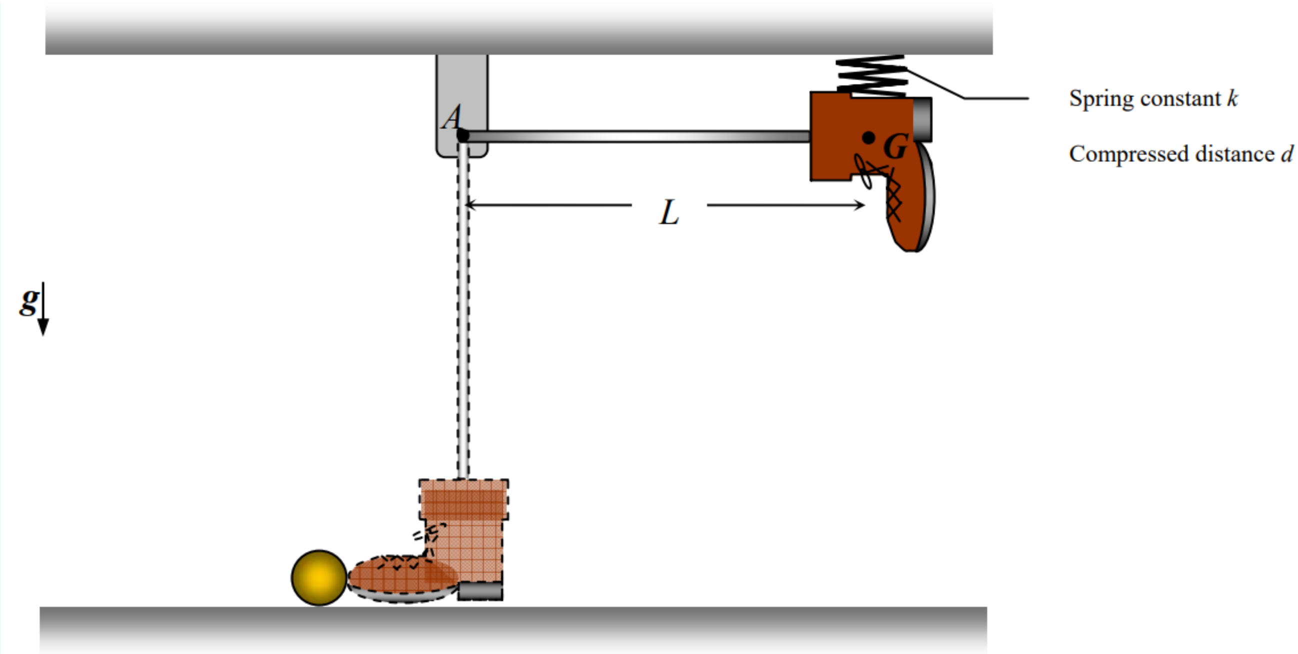


Figure 7.10.46: A boot on a pivoting rod is loaded via spring before it swings down to kick a marble.

(a) Find an expression for the velocity of the boot just before it kicks the marble.

(b) Assuming the boot and the marble stick together, find an expression for the velocity of the marble immediately after it has been kicked.

(c) If the spring was initially compressed a distance $d/3$ before the device was loaded, i.e. before it was compressed a distance d as described above, would the velocity found in part (a) increase, decrease or remain the same? Why? [A clear, concise, correct explanation without equations is acceptable.]

Problem 7.70

A typical cylinder for a Cummins Model H diesel engine is shown in the figure at right. Details of the compression process are shown below. The piston-cylinder volume contains air. For modeling purposes, you may assume that the air can be modeled as an ideal gas with room-temperature specific heats.

State 1:	$P_1 = 100 \text{ kPa}; T_1 = 320 \text{ K}; \forall_1 = 300 \text{ cm}^3$
Process 1 $\rightarrow$ 2:	Compression process with $P\forall^{1.3} = C$
State 2:	$\forall_2 = (1/16)\forall_1$

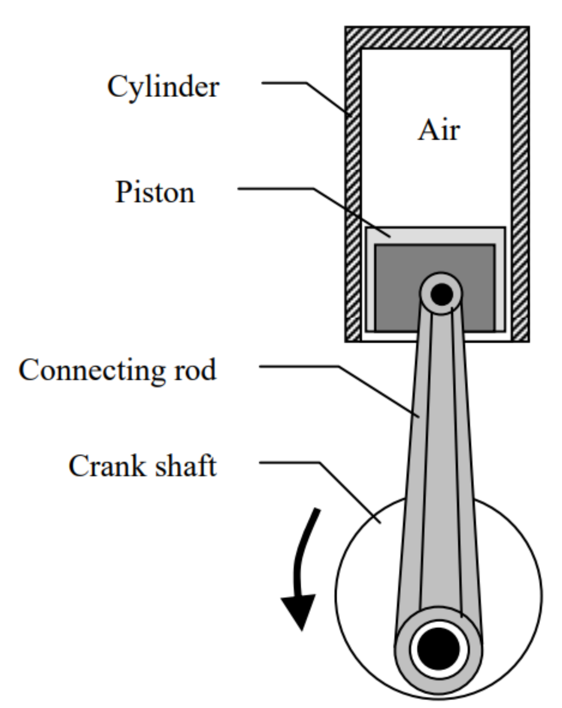


Figure $7.10.47$: A piston connected by a rod to a crankshaft moves up and down in a cylinder of air.

(a) Determine the final pressure and temperature.

(b) Determine the heat transfer and work transfer of energy for the air during the compression process.

(c) Sketch the process on a P-$\forall$ diagram. What, if anything, is the significance of the area under the process curve?

Problem 7.71

High-pressure hot water is mixed with $0.20\ \mathrm{ft}^3/\mathrm{min}$ of high-pressure cold water in a showerhead as shown in order to produce a comfortable shower temperature of $110^\circ\mathrm{F}$. The mixing process can be modeled as adiabatic with negligible kinetic and potential energies of the fluid streams. Assume liquid water can be modeled as an incompressible substance with room-temperature specific heats.

Find the required flow rate of hot water in $\mathrm{ft}^3/\mathrm{min}$.

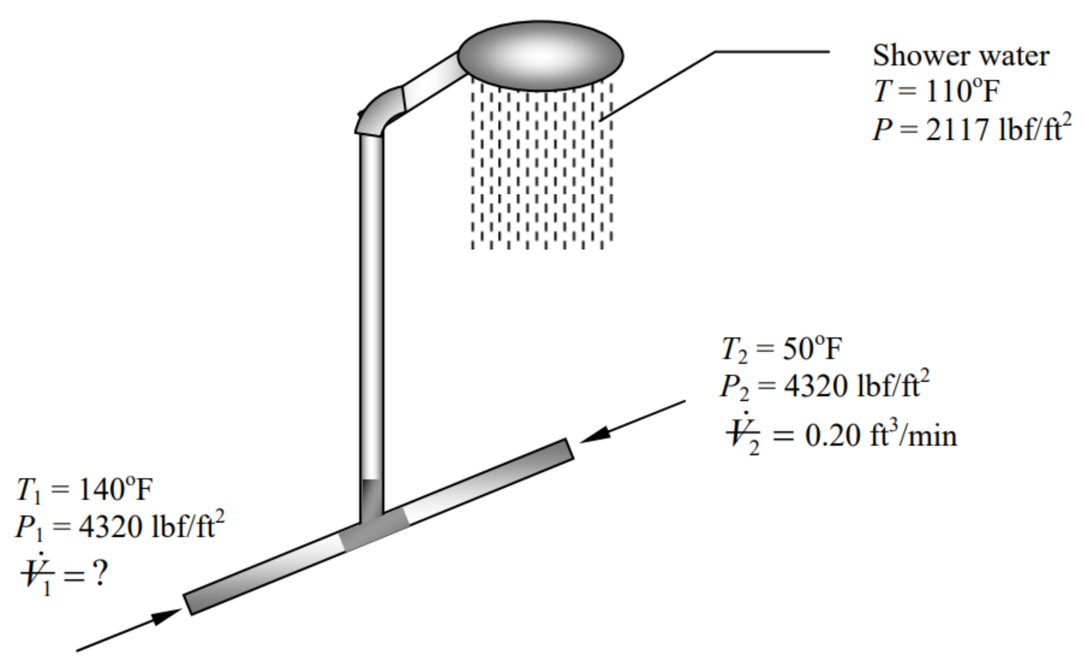


Figure 7.10.48: Mixing tee of hot and cold water for a showerhead.

Problem 7.72

A hydraulic power system operates at steady-state conditions and consists of an electrically driven hydraulic pump connected to a hydraulic motor by a two pipes carrying the hydraulic fluid. (See figure below.) The electric power input to the hydraulic pump is 9.0 kW.

For purposes of analysis, assume that changes in potential energy are negligible, the hydraulic fluid lines are well insulated, and the fluid can be modeled as an incompressible substance with the properties of liquid water.

$$[\text{ Liquid Water Properties: } c_{\text{p}} = 4.18 \text{ kJ}/(\text{kg}\cdot\text{K}) \text{ and } \rho = 997 \text{ kg}/\text{m}^3\ ]$$

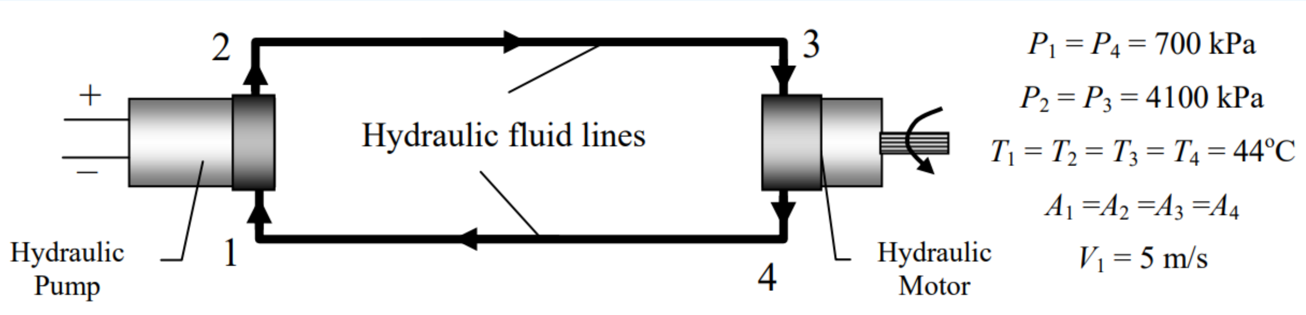


Figure 7.10.49: Two hydraulic fluid lines connect a hydraulic pump and a hydraulic motor.

(a) Assuming that the pump operates adiabatically, determine the mass flow rate of fluid through the pump, in kg/s.

(b) Assuming the hydraulic motor loses 1.0 kW by heat transfer, determine the shaft power out of the hydraulic motor, in kW.

(c) Estimate the convection heat transfer coefficient, h_{conv}, in $\text{W}/\left(\text{m}^2\cdot\text{K}\right)$ for the motor. The motor heat transfer is 1.0 kW, the room air temperature is 24°C, the motor surface area is 0.22 m^2, and the motor surface temperature is 44°C.

CHAPTER OVERVIEW

8: Entropy Production and Accounting

8.1: Four Questions

Entropy and the Second Law of Thermodynamics is one of the most misunderstood concepts in science. Outside the sphere of technology, it has been used to argue both for and against the existence of a god. It has found applications in thermodynamics, information theory, sociology, and economics. A detailed definition is often based on probabilities, energy distributions, and explanations of the microscopic behavior of the fundamental particles (atoms and molecules) that make up a substance. Engineering definitions of entropy often rely on extensive discussions of ideal thermodynamic cycles and investigations of what constitutes the "best" possible performance for a cycle. As we will discover shortly, there is no "conservation of entropy" principle. Experience has shown that entropy is continually produced in the world and the lack of a conservation principle or the ability to even partially consume (or destroy) entropy places strict limitations on what processes are possible for any given system. Because the entropy accounting principle corresponds to the Second Law of Thermodynamics, a fundamental law of physics, it cannot be proven or developed from other more fundamental principles. In the discussion that follows, we will attempt to minimize the explanations up front and move quickly to develop an entropy accounting principle. Then using your past experience with modeling systems in terms of mass, charge, momentum, and energy, we can take our time and explore the consequences of this powerful, new concept.

As before with every accounting concept for a new property, there are four questions that must be answered. When applied to entropy, the questions become:

1. What is entropy?
2. How can entropy be stored in a system?
3. How can entropy be transported?
4. How can entropy be created or destroyed?

Once we answer these questions, we will have the appropriate accounting equation for entropy.

8.1.1 What is entropy?

Entropy is a property that allows us to quantify the Second Law of Thermodynamics, one of the most significant laws of physics. So before we can talk about entropy, we must state the Second Law of Thermodynamics. But before we do that, we will consider some of our everyday experiences that are related to this new property called entropy.

Everyday experiences

As we go through our everyday tasks, our actions are based on our observations and expectations about how the physical world behaves. We may be very conscious of these assumptions or we may just go with our intuition, but in either case we expect our physical world to behave in a predictable fashion. Consider the following anecdotes and see how they match with your experience:

- You buy a cup of hot coffee, return to your office and set it on your desk. If you forget it and rediscover it one hour later, what would you expect to find — a piping hot cup of coffee that is warmer than when you first set it down, a room-temperature cup of coffee, or a cup of iced coffee? None of these scenarios violate any of the physical laws we have studied to this point.
- You find some pennies lying on an asphalt parking lot next to your car. Miser that you are, you bend over and pick up a penny and notice how warm it feels. As you stand up, another penny suddenly jumps up and lands in your hand. This second penny feels noticeably cooler than the first one did. Should you believe your eyes? From an energy standpoint, you hypothesize that the jumping process was adiabatic and the penny's internal energy (and temperature) decreased with a corresponding increase in its gravitational potential energy increase. Does this seem reasonable from an energy standpoint?
- You are a science fair judge and come upon an interesting project describing a method for charging a battery. A student has taken a $300\ \mathrm{k\Omega}$ resistor and connected it in series with a rechargeable, 9-volt-DC battery. Using a propane blowtorch, he holds a flame under the resistor and claims that the battery is charging. What do you think? Is this possible? Should you give him the prize for best science project or best hoax?
- On Friday evening you and some friends get some old 8-mm films and decide to watch the story of Simon Legree, Little Nell, and the Canadian Mountie. In typical fashion, at some point in the story Little Nell has been tied to the railroad tracks at the end of a railroad trestle. As the train just reaches the trestle, the Canadian Mountie blows up the trestle and as the trestle collapses the train arcs across the ravine and dissolves into a rock wall. Nothing about this seems too unusual except the passing thought of "what am I doing watching this?" Suddenly your buddy gets the bright idea to run the entire movie backwards. As you watch the improbable actions, you find the scene with the crashing train especially amusing when it is run backwards. Why? Why does running this backwards so catch our attention?

Without even knowing it, you began your study of thermodynamics and the Second Law of Thermodynamics when you were just an infant. Chances are your parents repeated a popular Mother Goose rhyme:

> "Humpty Dumpty sat on a wall.
> Humpty Dumpty had a great fall.
> All the king's horses and all the king's men
> Couldn't put Humpty Dumpty together again."

This is a great life lesson about our experience that certain processes, like breaking an egg, cannot be reversed. (Maybe that's why you find it intriguing to watch videos or movies run in reverse.)

All of these anecdotes speak to our expectation that there is a preferred direction for certain processes, and that certain processes are just not within our experience and appear to be at least highly improbable if not impossible.

The collective experience of scientists and engineers can be distilled into four formal statements about our expectations as to the behavior of the physical world:

- **Spontaneous Processes** — Spontaneous processes have a preferred direction of change.
- **Power Cycles (Heat Engines) —** The maximum thermal efficiency of a power cycle is always less than 100%. (This is called the *Kelvin-Planck Statement of the Second Law*.)
- **Heat Transfer —** It is impossible to operate any device in such a manner that the sole effect is the heat transfer of energy from a low-temperature body to another body at a higher temperature. (This is called the *Clausius Statement of the Second Law*.)
- **Final Equilibrium States —** A closed, adiabatic system with no work transfer of energy has a preferred final equilibrium state.

Experience has shown that if any of these statements in false, then the other three are also false.

Reversible and Irreversible Processes[1]

A key concept in discussing the behavior of systems is the idea of reversibility. An **internally reversible process** is defined as follows:

> A system executes an internally reversible process if at any time during the process the state of the system can be made to retrace its path exactly.

The concept of reversibility by its definition requires *restorability*. Any process that is not internally reversible is **internally irreversible**. When used without a qualifier, the term "reversible process" will be assumed here to refer to an internally reversible process.

In practice, what does "internally reversible" mean? Assume that a system undergoes an arbitrary process and we record the state of the system (i.e. *all* of its properties) and all interactions with the surroundings as a function of time. If the process is internally reversible, it should be possible to get the system to essentially run backwards in time by merely reversing the direction of the interactions at the boundary of the system.

Based on the definition of an internally reversible process, there are three additional consequences that will be stated here without proof:

- A work transfer of energy for an internally reversible process has the same magnitude but opposite direction if the process is reversed.
- A heat transfer of energy for an internally reversible process has the same magnitude but opposite direction if the process is reversed.
- An internally reversible process occurs in such a fashion that the system is always infinitesimally close to being in equilibrium, i.e. an internally reversible process is also a quasiequilibrium process.

An internally reversible process is a useful fiction in the study of real processes. A real process can only approach an internally reversible process in the limit as all sources of irreversibility (dissipative effects) within the system are eliminated.

Determining whether a given process is internally reversible or not is best done by identifying any source of irreversibility within the system. The presence of *any* irreversibility within the system makes the process internally irreversible. Irreversibilities arise from two sources:

1. Inherent dissipative effects within the system
2. A non-quasiequilibrium process.

Recall that a quasiequilibrium process was originally defined as a process that proceeds in such a manner that the process is infinitesimally close to a state of equilibrium at all times. Thus a quasiequilibrium process qualifies as an internally reversible process and can be recognized by its slow, carefully controlled execution. Any work interaction that can be carried out in a quasiequilibrium fashion is a candidate work transfer of energy for an internally reversible process. Examples might include any processes in which the mechanical energy of a system is conserved (i.e. processes for which the work-energy principle is valid). The behavior of a simple electrical circuit that contains only ideal capacitors and inductors would also qualify as an internally reversible process.

Most irreversibility is the result of dissipative effects that we commonly experience. Examples of these include:

1. electric resistance
2. inelastic deformation
3. viscous flow of a fluid (flow with fluid friction)
4. solid-solid friction (dry friction)
5. heat transfer across a finite temperature difference or as the result of a finite temperature gradient
6. hysteresis effects
7. shock waves
8. internal friction, e.g. internal damping of a vibrating system
9. unrestrained expansion of a fluid
10. fluid flow through valves and porous plugs (throttling)
11. spontaneous chemical reactions
12. mixing of dissimilar gases or liquids
13. osmosis
14. dissolving of one phase into another phase
15. mixing of identical fluids initially at different pressures and temperatures

Notice that all of these effects are within your everyday experience and that they cover a range of physical and chemical effects. By carefully examining a system for any of these dissipative effects it is possible to determine whether a system is internally irreversible or internally reversible.

[1] Adapted from K. Wark and D. Richards, *Thermodynamics*, 6th ed., McGraw-Hill, Inc., New York, 1999.

Second Law of Thermodynamics

Sadi Carnot laid the groundwork for the Second Law of Thermodynamics in the 1800s by studying the performance of steam engines. Sadi Carnot's thoughts have come down to us in the form of two statements about the performance of power cycles referred to as the Carnot Principles:

- Principle I — The thermal efficiency of an internally *irreversible* power cycle is always less than the thermal efficiency of an internally *reversible* power cycle that transfers energy by heat transfer at the same boundary temperatures.
- Principle II — All internally reversible power cycles that transfer energy by heat transfer at the same boundary temperatures have the same thermal efficiency.

Although he did not give a complete statement of the Second Law of Thermodynamics, his understanding of reversible and irreversible processes was crucial. Both of these statements can be shown to be straightforward consequences of applying the entropy accounting equation that we are formulating.

We will make an axiomatic statement of the Second Law of Thermodynamics. As with the other fundamental laws of physics, the Second Law cannot be proven from more fundamental principles and embodies the collective experience and wisdom of the scientists and engineers. Our statement of the **Second Law of Thermodynamics** is embodied in the following three statements.

1. There exists an extensive property called **entropy**, S.
2. Entropy is transported across the boundaries of a closed system by heat transfer. The **entropy transport rate with heat transfer**, $\dot{S}_Q$, at a boundary is defined by the equation:

$$\dot{S}_Q \equiv \frac{\dot{Q}_j}{T_{b,\,j}}$$

where $\dot{Q}_j$ is the heat transfer rate at boundary j and $T_{b,\,j}$ is the thermodynamic temperature of the boundary surface j.

3. Entropy can only be produced except in the limit of an *internally reversible process* where the entropy production rate is zero:

$$\dot{S}_{gen} \geq 0$$
$$\text{where} \begin{cases} \dot{S}_{gen} > 0 \text{ for an internally } \textit{irreversible} \text{ process} \\ \dot{S}_{gen} = 0 \text{ for an internally } \textit{reversible} \text{ process} \end{cases}$$

Entropy

Entropy is an extensive property of a system that can be produced and is transported in a manner consistent with the Second Law of Thermodynamics. The dimensions of entropy are $[\text{Energy}]/[\text{Temperature}]$. Typical units for entropy are kJ/K in SI and $\text{Btu}/{}^\circ\text{R}$ in USCS.

Our study of entropy will lead to a general accounting principle that is helpful to engineers in a number of ways:

- Provides a way to establish a "thermodynamic" temperature scale that is independent of the specific thermometer used to measure temperature.
- Provides a way to determine, given a specific system, which of the many possible processes that satisfy the conservation of energy are in fact possible.
- Provides criteria for the theoretical "best" performance against which real systems can be compared.
- Provides a way to assess the usefulness (quality) of energy.
- Provides additional information to relate and predict the thermophysical properties of a substance, e.g. u, h, v, T, P and s.

The study of the property entropy is intimately related to the study of what processes are possible and preferred and how systems can evolve with time.

8.1.2 How can entropy be stored in a system?

Entropy is stored with mass. The entropy of a system is calculated as follows:

$$S_{sys} = \int_{V_{sys}} s\rho \, dV$$

where s is the **specific entropy**, the entropy per unit mass. If the specific entropy is spatially uniform inside the system, this integral simplifies as follows:

$$S_{sys} = \int_{V_{sys}} s\rho \, dV = s \underbrace{\int_{V_{sys}} \rho \, dV}_{m_{sys}} = m_{sys} s \qquad \begin{array}{c}\text{Spatially-uniform} \\ \text{specific entropy } s\end{array}$$

The dimensions on specific entropy are $[\text{Entropy}]/[\text{Mass}]$. Typical units for specific entropy are $\text{kJ}/(\text{kg}\cdot\text{K})$ in SI and $\text{Btu}/\,(\text{lbm}\cdot{}^\circ\text{R})$ in USCS.

8.1.3 How can entropy be transported?

Entropy can be transported across the boundary of a system by two different mechanisms—heat transfer and mass transfer.

Entropy transport by heat transfer

As stated in the Second Law of Thermodynamics, entropy is carried with heat transfer and the rate of transfer is defined as follows:

$$\dot{S}_Q \equiv \frac{\dot{Q}_j}{T_{b,\,j}} \qquad \begin{array}{c}\text{Heat Transfer Rate} \\ \text{of Entropy at Surface } j\end{array}$$

where $\dot{Q}_j$ is the heat transfer rate at boundary j and $T_{b,\,j}$ is the thermodynamic temperature of the boundary surface j. The net transport of entropy into a system by heat transfer at N surfaces is the sum of the heat transfer rates of entropy at all the boundary surfaces:

$$\dot{S}_{Q,\text{ net in}} = \sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,\,j}}$$

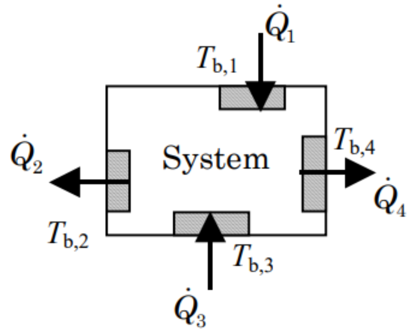


Figure 8.1.1: Net rate of entropy transport with heat transfer for a system with heat transfer at four different boundary temperatures

$$\dot{S}_{Q,\text{ net in}} = \frac{\dot{Q}_{1,\text{ in}}}{T_{b,\,1}} - \frac{\dot{Q}_{2,\text{ out}}}{T_{b,\,2}} + \frac{\dot{Q}_{3,\text{ in}}}{T_{b,\,3}} - \frac{\dot{Q}_{4,\text{ out}}}{T_{b,\,4}}$$

Figure 8.1.1 shows an example of a system with four different heat transfers of entropy, each occurring at a different boundary temperature. This sign convention for heat transfer of entropy is the same as for heat transfer of energy.

The dimensions on the heat transfer rate of entropy are \([\text{Energy}][\text{Time}]^{-1}[\text {Temperature}]^{-1}\). Typical units are $\mathrm{kJ}/(\mathrm{s} \cdot \mathrm{K})$ or kW/K in SI and $\mathrm{Btu}/(\mathrm{s} \cdot {}^\circ\mathrm{R})$ in USCS. Note that the temperature of the boundary where the heat transfer occurs must be measured in absolute units, K or ${}^\circ\mathrm{R}$. (The precise definition of what we mean by a "thermodynamic" or "absolute" temperature will be addressed shortly.

Entropy transport by mass flow

As our previous experience with other extensive properties has shown, any mass that crosses the boundary of a system carries with it extensive properties. Entropy is no exception. The rate at which entropy is transported across a boundary by mass flow is the product of the mass flow rate and the specific entropy, s, of the mass at the boundary:

$$\dot{S}_{\text{mass flow}} = \dot{m} s$$

The net rate at which entropy is carried into a system by mass flow is

$$\dot{S}_{\text{mass flow, net in}} = \sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e$$

The dimensions and units on the rate of entropy transport with mass flow is the same as the that for the rate of entropy transport with heat transfer.

8.1.4 How can entropy by generated or consumed?

Based on the Second Law of Thermodynamics, we say that entropy can only be produced within a system and in the limit of an internally reversible process entropy it is conserved. This is a very important result and gives the entropy accounting principle its power:

$$\dot{S}_{gen} \geq 0$$
$$\text{where} \begin{cases} \dot{S}_{gen} > 0 \text{ for an internally } \mathit{irreversible} \text{ process} \\ \dot{S}_{gen} = 0 \text{ for an internally } \mathit{reversible} \text{ process} \end{cases}$$

Experience has shown us that the entropy production term is always greater than or equal to zero. The presence of any irreversibility within the system results in entropy production during the process. Experience has also shown that entropy is produced in every real process. Thus an internally reversible process can be viewed as a limiting and ideal process that can only be approached.

Internally reversible processes (any process where $\dot{S}_{gen} \equiv 0$) play an important role in the design and analysis of real systems. First, it serves as an example of the theoretical "best" performance possible. Second, it often represents the only conditions under which we can actually do the calculations to predict the behavior of the system. Third, when combined with experimentally determined "correction factors", the internally reversible process plays a central role in predicting the actual behavior of a real systems.

8.1.5 Putting it all together — Entropy Accounting Equation

Applying the accounting framework to entropy, we know that

$$\frac{dS_{sys}}{dt} = \dot{S}_{Q,\text{ net in}} + \dot{S}_{\text{mass flow, net in}} + \dot{S}_{gen}$$

Now collecting all of the results developed above, we have the rate form of the entropy accounting equation:

$$\frac{dS_{sys}}{dt} = \sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,\,j}} + \sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e + \dot{S}_{gen}$$

where $\dot{S}_{gen} \geq 0$ and $\dot{S}_{gen} = 0$ for an internally reversible process.

In words, Eq. $8.1.10$ says the time rate of change of the entropy of the system equals the net rate of entropy transport into the system with heat transfer plus the net rate of entropy transport into the system with mass flow plus the rate of entropy generation (or production).

8.2: Empirical and Thermodynamic Temperature

The Second Law of Thermodynamics defines the transport of entropy in terms of the heat transfer and the thermodynamic temperature on the boundary where the heat transfer occurs. What precisely is a *thermodynamic temperature*? Because of its importance in the definition of entropy and the technological importance of temperature measurement, we will address the issue of temperature measurement and temperature scales before we proceed further.

8.2.1 Temperature and Thermal Equilibrium

The first question one might ask is "what is temperature?" Most of us have a very common-sense understanding of this. We understand temperature to be a characteristic of a system that is related to how "hot" or "cold" it feels. As these sensations are relative, it is highly possible that different individuals might have different perceptions of the temperature of any given object.

Temperature is intimately related with the concept of thermal equilibrium. When two objects are brought into contact for a sufficiently long period of time, their properties will eventually stop changing and we say that the two objects are in *thermal equilibrium*. Temperature is the property of the objects that indicates whether they are in thermal equilibrium or not. When we use a thermometer to compare the temperature of two objects, we are assuming that if both objects are independently in thermal equilibrium with the thermometer (have the same temperature as the thermometer), the two objects would also be in thermal equilibrium with each other (have the same temperature). This empirical result is known as the Zeroth Law of Thermodynamics. If this were not true, it would be impossible to use a thermometer to measure temperatures.

8.2.2 Empirical Temperature

To measure the temperature of a system, we need a *thermometer* with a property that changes with temperature (a *thermometric property*), we need a set of *fixed points* for reference, and we need a *temperature scale* to interpolate between the fixed points.

One of the most common thermometers is a liquid-in-glass thermometer where the liquid is mercury or alcohol. Imagine that you are given a mercury-in-glass thermometer without a scale engraved on the glass and asked to use it to measure the temperature of various objects. From experience, you know that the mercury in the thermometer (and the glass) will expand with increasing temperature. To quantify the temperature, you must establish a scale and some fixed reference points. To do this, you get a thermos bottle and fill it with a mixture of ice and water. Then you set a pan of water on the stove and get it boiling. You dip the bulb of the thermometer into the ice-water mixture, wait until the mercury stops moving, and scratch a line on the stem of the thermometer. You then repeat the process with the boiling water. You now have two fixed points - the boiling point of water and the ice-point of water. If you arbitrarily assign a temperature of 0° to the ice-point mark and a temperature of 180° to the boiling point of water you have two fixed points. If we divide the distance between the two marks on the thermometer stem into 90 equally spaced divisions, we now have a scale to interpolate the temperature anywhere between these two fixed points. This will work fine for comparing the temperature of various objects as long as their temperatures fall within the given range.

We are all familiar with numerous devices for measuring temperatures and each one depends on the thermometric behavior of a substance, e.g. the expansion of mercury and glass (mercury-in-glass thermometer), the change in electrical resistance (electric resistance thermometer), or the pressure-temperature behavior of a gas (Constant-volume ideal-gas thermometer). With each of these devices it is possible to establish an empirical temperature scale by assigning fixed temperatures to repeatable physical phenomena and then using a thermometer to interpolate between these points. Typical reference points include the boiling point of water at one atmosphere; the ice-point of water (an equilibrium mixture of ice, liquid water, and saturated air) at one atmosphere; and the melting points of various metals at one atmosphere.

8.2.3 Thermodynamic Temperature

One of the major achievements of thermodynamics has been the development of an absolute or thermodynamic temperature scale that is *independent* of any substance. Following a suggestion made by Lord Kelvin in 1848, we can establish such a temperature scale using the Second Law of Thermodynamics.

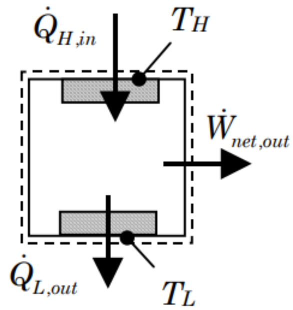


Figure 8.2.1: Power cycle with heat transfer of entropy at two different surfaces.

To see what this means, consider a power cycle as shown in Figure 8.2.1. The power cycle receives energy by heat transfer across a boundary at temperature T_H and rejects energy by heat transfer across a boundary at temperature T_L. If we write the rate form of the entropy accounting equation for the power cycle, we have the following result:

$$\underbrace{\cancelto{=0}{\frac{dS_{sys}}{dt}}}_{\text{Steady-state}} = \frac{\dot{Q}_{H,\text{ in}}}{T_H} - \frac{\dot{Q}_{L,\text{ out}}}{T_L} + \dot{S}_{gen} \quad \rightarrow \quad \frac{\dot{Q}_{L,\text{ out}}}{T_L} = \frac{\dot{Q}_{H,\text{ in}}}{T_H} + \dot{S}_{gen}$$

Recall that we are looking for a way to define a temperature scale that is independent of any specific physical thermometer or thermometric property.

If we rearrange Eq. 8.2.1 to find the ratio of temperatures, we have the following relationship between the ratio of heat transfer rates and the ratio of temperatures:

$$\frac{\dot{Q}_{L,\text{ out}}}{\dot{Q}_{H,\text{ in}}} = \frac{T_L}{T_H} + \dot{S}_{gen}\left(\frac{T_L}{\dot{Q}_{H,\text{ in}}}\right)$$

If we restrict ourselves to an internally reversible power cycle, $\dot{S}_{gen} = 0$ and this equation simplifies to:

$$\left(\frac{\dot{Q}_{L,\text{ out}}}{\dot{Q}_{H,\text{ in}}}\right)_{\substack{\text{internally}\\\text{reversible}}} = \frac{T_L}{T_H}$$

This result is independent of the physical properties of the working fluid of the power cycle and only requires that the power cycle operate in an internally reversible fashion. This satisfies our criterion for establishing a thermodynamic temperature scale. Equation 8.2.3 is the defining equation for thermodynamic temperature.

The ratio of any two temperatures on a **thermodynamic temperature scale** is equal to the ratio of the heat transfer rates for an internally reversible power cycle (heat engine) that operates between the same two temperatures. The minimum temperature on any absolute temperature scale is zero degrees, often called "absolute zero."

The *Kelvin temperature scale* is the thermodynamic temperature scale used with the SI system of units. (The *Rankine temperature scale* is the thermodynamic temperature scale used with the USCS system of units.) On the Kelvin scale, the triple point (tp) of water is assigned a temperature of *exactly* $T_{tp} = 273.16$ K. Thus, the defining equation for all other temperatures on this scale is

$$T = (273.16\text{ K})\left(\frac{\dot{Q}_T}{Q_{tp}}\right)_{\substack{\text{internally}\\\text{reversible}}}$$

where $\dot{Q}_T$ and $\dot{Q}_{tp}$ are the heat transfer rates on the boundary of an internally reversible power cycle that occur at temperature T and at T_{tp}, respectively.

There are four temperature scales that are commonly used in engineering work. The Celsius scale and the Fahrenheit scale are empirical scales and originally were established by interpolating between two fixed points. The Kelvin scale and the Rankine scale are both thermodynamic (absolute) temperature scales. As demonstrated above, an absolute scale is set when a numerical value is assigned to one fixed reference point, e.g. the triple point.

The relationship between temperatures and temperature differences on the four scales are described by the following equations:

$$\begin{array}{lll} \text{Temperatures} & \dfrac{T_{\mathrm{R}}}{{}^{\circ}\mathrm{R}}=\mathbf{1.8}\left(\dfrac{T_{\mathrm{K}}}{\mathrm{K}}\right) & \\ & \dfrac{T_{\mathrm{K}}}{\mathrm{K}}=\dfrac{t_{\mathrm{C}}}{{}^{\circ}\mathrm{C}}+\mathbf{273.15} & \dfrac{T_{\mathrm{R}}}{{}^{\circ}\mathrm{R}}=\dfrac{t_{\mathrm{F}}}{{}^{\circ}\mathrm{F}}+\mathbf{459.67} \\ & \dfrac{t_{\mathrm{C}}}{{}^{\circ}\mathrm{C}}=\dfrac{1}{\mathbf{1.8}}\left(\dfrac{t_{\mathrm{F}}}{{}^{\circ}\mathrm{F}}-\mathbf{32}\right) & \dfrac{t_{\mathrm{F}}}{{}^{\circ}\mathrm{F}}=\mathbf{1.8}\left(\dfrac{t_{\mathrm{C}}}{{}^{\circ}\mathrm{C}}\right)+\mathbf{32} \\ \\ \text{Temperature differences} & \dfrac{\Delta T_{\mathrm{R}}}{\Delta T_{\mathrm{K}}}=\dfrac{\Delta t_{\mathrm{F}}}{\Delta t_{\mathrm{C}}}=\mathbf{1.8} & \\ & \Delta T_{\mathrm{K}}=\Delta t_{\mathrm{C}} & \Delta T_{\mathrm{R}}=\Delta t_{\mathrm{F}} \end{array}$$

where T_{K}, T_{R}, t_{C}, and t_{F} are temperature values measured on the Kelvin, Rankine, Celsius, and Fahrenheit scales, respectively. The bold numbers in Eq. $8.2.5$ are exact.

In practice, the International Practical Temperature Scale (IPTS) is used to establish a practical scale for temperature measurement. The IPTS is based on of a number of easily reproducible and fixed points with assigned temperature values and prescribed instruments and formulas for interpolating between the points. The assigned temperature values are equal to the best experimental values of the thermodynamic temperatures of the fixed points.

Assigned temperatures of some fixed points used in defining the International Practical Temperature Scale of 1968 (IPTS-68)

Fixed Point	T_{68}/K
Triple point of hydrogen	13.80
Boiling point of neon	27.102
Triple point of oxygen	54.361
Triple point of water	273.16
Boiling point of water	373.15
Freezing point of zinc	692.73
Freezing point of silver	1235.08
Freezing point of gold	1337.58

8.3: Entropy Accounting Equation

The recommended starting point for any problem that requires the application of the Second Law of Thermodynamics, the determination of the entropy production, or the change in entropy of a system, is the rate form of the entropy accounting equation derived earlier:

$$\frac{dS_{sys}}{dt}=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\sum_{\text{in}}\dot{m}_i s_i-\sum_{\text{out}}\dot{m}_e s_e+\dot{S}_{gen}$$

In applying this equation to describe the behavior of a system, there are several modeling assumptions that are commonly used. These are described in detail in the following paragraphs. As always, you should focus on understanding the physics underlying the assumption and how they are used. *Do not just memorize the simplified equations.*

Typical modeling assumptions

Steady-state system: If a system is operating under steady-state conditions, all intensive properties and interactions are independent of time. Thus, the entropy of the system is constant, $S_{sys}=$ constant. When applied to the entropy accounting equation you have the following

$$\underbrace{\cancelto{=0}{\frac{dS_{sys}}{dt}}}_{\text{Steady-state}}=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\sum_{\text{in}}\dot{m}_i s_i-\sum_{\text{out}}\dot{m}_e s_e+\dot{S}_{gen}$$

$$0=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\sum_{\text{in}}\dot{m}_i s_i-\sum_{\text{out}}\dot{m}_e s_e+\dot{S}_{gen}$$

In words, the net heat transfer rate of entropy into the system plus the net mass transport rate of entropy into the system plus the net generation (production) rate of entropy must equal zero.

Closed system: A closed system has no mass flow across its boundary. With this constraint, the entropy accounting equation simplifies as follows:

$$\frac{dS_{sys}}{dt}=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\underbrace{\cancelto{=0}{\sum_{\text{in}}\dot{m}_i s_i-\sum_{\text{out}}\dot{m}_e s_e}}_{\text{Closed system}\,\rightarrow\,\text{no mass flow}}+\dot{S}_{gen}$$

$$\frac{dS_{sys}}{dt}=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\dot{S}_{gen}$$

Finite-time, closed system: For a closed system over a finite-time interval, you first apply the closed system assumption and then integrate the equation over the specified time interval:

$$\frac{dS_{sys}}{dt}=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\cancelto{=0}{\sum_{in}\dot{m}_i s_j-\sum_{out}\dot{m}_e s_e}+\dot{S}_{gen}$$

$$\frac{dS_{sys}}{dt}=\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}+\dot{S}_{gen}$$

$$\int_{t_1}^{t_2}\left(\frac{dS_{sys}}{dt}\right)dt=\int_{t_1}^{t_2}\left(\sum_{j=1}^{N}\frac{\dot{Q}_j}{T_{b,\,j}}\right)dt+\int_{t_1}^{t_2}\dot{S}_{gen}\,dt$$

$$S_{sys,\,2}-S_{sys,\,1}=\sum_{j=1}^{N}\left[\int_{t_1}^{t_2}\frac{\dot{Q}_j}{T_{b,\,j}}dt\right]+S_{gen}$$

In words, this says the *change* in the entropy of the system equals the net heat transfer of entropy into the system plus the amount of entropy produced inside the system.

Internally reversible process: When a process is internally reversible there is no entropy production and $\dot{S}_{gen} \equiv 0$. Under these conditions the entropy accounting equation reduces to the following:

$$\frac{dS_{sys}}{dt} = \sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,\,j}} + \sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e + \underbrace{\cancel{\dot{S}_{gen}}^{=0}}_{\text{Internally reversible process}}$$

$$\frac{dS_{sys}}{dt} = \sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,\,j}} + \sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e$$

This raises an interesting point. Unless you can assume that a process is internally reversible, all you know is that the entropy production rate is greater than or equal to zero, $\dot{S}_{gen} \geq 0$. This means that when you are looking for equations to help solve a problem, the entropy accounting equation brings with it a built-in unknown. For example, if you have three equations and four unknowns before you apply the entropy accounting equation, after you apply the entropy accounting equation you will have four equations and five unknowns unless you can assume an internally reversible process. Actually, the situation is not quite as bleak as this appears. Although you may not know its exact value, you do know that the entropy production rate cannot be negative. In addition, as we will show shortly, any real system must have a value that is positive and will approach zero as we approach the best or optimum behavior for the specified conditions.

Assumptions about heat transfer and work transfer of entropy:

Heat transfer of entropy — In this course, we will usually make one of three assumptions about the heat transfer of entropy for a system:

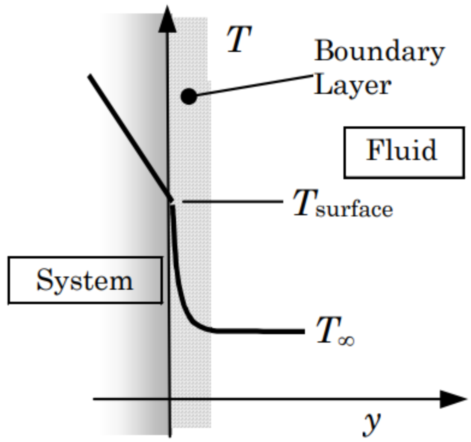
1. There is no heat transfer and thus there is no heat transfer of entropy.
2. The heat transfer rate $\dot{Q}_j$ and/or the surface temperature $T_{b,\,j}$ may be unknown and thus the rate of entropy transfer with heat transfer is also unknown. In this case, the entropy accounting equation may provide an additional equation that relates these two variables.
3. The heat transfer rate $\dot{Q}$ and the surface temperature $T_{b,\,j}$ are both specified and thus the rate of entropy transfer with heat transfer is known.

Please remember that entropy transfer with heat transfer can only be defined with respect to a boundary. If you move the boundary you may change the heat transfer rate or the surface temperature at which it occurs. Without clearly indicating your system boundary, it is impossible to apply any of these assumptions.

In applying the conservation of energy equation, you routinely calculated the *net heat transfer rate* for a system by summing the heat transfer rates over the system boundary. Because the entropy transfer rate with heat transfer depends on both the heat transfer rate *and* the boundary temperature where the heat transfer occurs, you must be careful to only sum heat transfer rates that occur at the same temperature when determining the heat transfer of entropy. For example,

$$\sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,\,j}} = \frac{1}{T_b} \sum_{j=1}^{N} \dot{Q}_j = \frac{\dot{Q}_{\text{net, in}}}{T_b} \qquad \text{only if } T_{b,\,j} = T_b \text{ for all surfaces.}$$

Another common problem is assigning a boundary temperature when a system exchanges energy by heat transfer with a fluid-convection heat transfer. To help us answer this question, we need to examine the temperature variation within the system and the fluid near the system boundary. Figure $8.3.1$ shows the temperature distribution normal to the boundary in the system and the surrounding fluid. If the convection heat transfer is occurring from the system to the surrounding fluid, the temperature distribution appears as shown in the figure. The temperature decreases inside the system as we approach the boundary. At the boundary (the interface between the system and the surrounding fluid), the temperature is T_{surface}. Immediately adjacent to the system is a layer of air commonly referred to as the *boundary layer*. The boundary layer is a layer of air across which the temperature changes from T_{surface} to T_∞, the fluid temperature away from the wall.



Convection heat transfer through the boundary layer: $\dot{Q}_{\text{convection}} = hA_{\text{surface}}\left(T_{\text{surface}} - T_\infty\right)$

Entropy production rate within the boundary layer:

$$\underbrace{\overset{=0}{\cancel{\frac{dS_{sys}}{dt}}}}_{\text{Steady-state}} = \frac{\dot{Q}_{\text{conv, in}}}{T_{\text{surface}}} - \frac{\dot{Q}_{\text{conv, out}}}{T_\infty} + \dot{S}_{\text{gen}} \quad \rightarrow \quad \dot{S}_{\text{gen}}\Big|_{\substack{\text{boundary}\\ \text{layer}}} = \dot{Q}_{\text{conv, in}}\left[\frac{1}{T_\infty} - \frac{1}{T_{\text{surface}}}\right]$$

Figure $8.3.1$: Entropy production at a system boundary with convection heat transfer

If our system does *not* include the boundary layer, then the correct temperature to use when calculating the heat transfer of entropy for the system is $T_b = T_{\text{surface}}$. If the system does include the boundary layer, then the correct temperature to use when calculating the heat transfer of entropy is $T_b = T_\infty$. However, you should note that because the latter system contains additional material (the boundary layer) it may have a different entropy production rate and different entropy transfer rates. As shown in Figure $8.3.1$, the process of steady-state heat transfer through the boundary results in an entropy production rate.

The reason this discussion is important is that frequently in applying the entropy accounting equation you will not know a surface temperature and you will only know (or can reasonably assume) an ambient fluid temperature, say room temperature. In these cases, it is perfectly acceptable to include the boundary layer inside your system so that you know a boundary temperature to calculate the heat transfer of entropy. However, please remember that the entropy production you calculate for a system that includes the boundary layer will be different and greater that what you would calculate for a smaller system that did not include the boundary layer.

Work transfer of entropy — I have included this here to reinforce a point that *there is no transport of entropy with work*. Let me say that one more time: *there is no transport of entropy with work.*

This leads us to another way to distinguish between a work transfer of energy and a heat transfer of energy. A heat transfer of energy *always* carries with it an amount of entropy; while a work transfer of energy never does.

Assumptions about the substance:

As we discovered with conservation of energy, we will need to evaluate thermophysical properties — u, h, s, T, P, ρ, and v. This requires empirical knowledge about the behavior of the material within the system. In the last chapter we introduced two different substance models and shortly we will extend these to include the calculation of entropy changes.

Problem Solving with the Entropy Accounting Equation

In practice, we will typically apply the entropy accounting equation to three different types of problems:

- problems where we are explicitly asked to determine the entropy production rate,
- problems where we are asked to determine if a given process is possible, i.e. the entropy production rate is within acceptable limits, $\dot{S}_{gen} \geq 0$, and
- problems where we are asked to determine the "best" performance that is theoretically possible.

For the first type of problem where we are asked explicitly to determine the entropy production rate, we will typically apply both the conservation of energy and the entropy accounting equations. In addition, we must be able to determine values for every term in the entropy accounting equation except for the entropy production rate.

For the second type of problem where we are asked to determine if a given process or device is possible, we again must be able to determine the entropy production rate. Questions about whether a given process or device is possible almost always require that we find the entropy generation rate for the process. Not every process that satisfies conservation of energy also satisfies the entropy accounting equation. Failure to satisfy either of these physical laws means that a given process is impossible.

For the third type of problem, we are asked to determine the "best" possible performance for the device or process. Again, the key to answering these questions involves the entropy accounting equation. To answer these questions you need to combine the conservation of energy and the entropy accounting equation so that the desired performance can be studied as a function of the entropy generation rate. (Recall that the entropy generation rate is the only quantity in either the conservation of energy or the entropy accounting equation that has any physical limitation placed on its value.) To find the "best" performance possible, you must study the desired performance of the device over the range of possible entropy-generation values and see what constitutes the "best" performance for the specified operating conditions, e.g. maximum power out or minimum power in for a system. Although you are encouraged to investigate specific cases to test this conclusion, experience has shown that *the "best" performance that is theoretically possible always occurs when the device or process is internally reversible*.

The following examples serve two functions. First, they demonstrate the mechanics of how to apply entropy accounting with the various modeling assumptions. Second, they are designed to help you learn more about the importance of the property entropy and its production. Please read the problems carefully and, where requested, answer the various questions to the best of your ability.

Example — The case of the *hot* resistor!

A 9-volt battery energizes a $200\text{-}\Omega$ resistor. The resistor operates at steady-state conditions. Convection heat transfer occurs between ambient air at $25^{\circ}\mathrm{C}$ and the resistor. The resistor surface area is $2.5\ \mathrm{cm}^2$, and the convection heat transfer coefficient is $10\ \mathrm{W}/\left(\mathrm{m}^2\cdot{}^{\circ}\mathrm{C}\right)$.

Determine the (a) magnitude and direction of the heat transfer rate, (b) the entropy generation rate for the resistor alone, and (c) the entropy generation rate for an enlarged system that includes the resistor and the boundary layer of air surrounding the resistor.

Solution

Known: A resistor is energized by a battery and operates at steady-state conditions.

Find: (a) Heat transfer rate, in W.

(b) Entropy generation rate for the resistor alone, in W/K.

(c) Entropy generation rate for an enlarged system that includes the convection boundary layer

Given:

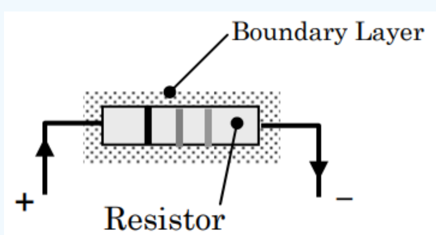


Figure $8.3.2$: Current passes through a resistor, which is surrounded by a boundary layer with the air.

Ambient air temperature: $T_{\infty}=25^{\circ}\mathrm{C}$

Convection heat transfer coefficient: $h=10\ \mathrm{W}/\left(\mathrm{m}^2\cdot{}^{\circ}\mathrm{C}\right)$

Surface area: $A=2.5\ \mathrm{cm}^2=2.5\times10^{-4}\ \mathrm{m}^2$

DC voltage across resistor: $\Delta V = 9$ volts

Resistor resistance: $R = 200\ \Omega$

Steady-state system

Analysis:

Strategy $\rightarrow$ Questions about heat transfer rates usually require application of conservation of energy.

Questions about entropy production always require application of the entropy accounting equation.

System $\rightarrow$ Begin with just the resistor.

Property to count $\rightarrow$ Energy and then entropy.

Time interval $\rightarrow$ Because it described the system as steady-state, we can assume an infinitesimal time interval.

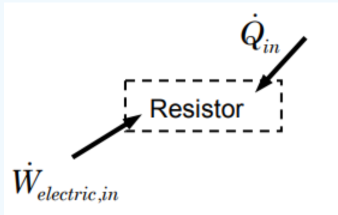


Figure 8.3.3: Closed system consisting only of the resistor.

(a) To solve for the heat transfer rate we will treat the resistor alone as a closed system and write the conservation of energy equation for this system:

$$\underbrace{\cancelto{=0}{\frac{dE_{sys}}{dt}}}_{\text{Steady-state}} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \underbrace{\cancelto{=0}{\sum_{\text{in}} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz\right) - \sum_{\text{out}} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz\right)}}_{\text{Closed system}}$$

$$0 = \dot{Q}_{\text{in}} + \dot{W}_{\text{electric, in}} \quad \rightarrow \quad \dot{Q}_{\text{in}} = -\dot{W}_{\text{electric, in}}$$

As might be expected, we discover that the heat transfer rate in equals the negative of the electric power in. Alternatively, we could say the electrical power in equals the heat transfer rate out.

Assuming that the resistor obeys Ohm's Law, we can calculate the electrical power and the heat transfer rate as follows:

$$\dot{Q}_{\text{in}} = -\underbrace{\dot{W}_{\text{electric, in}}}_{=i\cdot\Delta V} = -i \cdot \Delta V = -\left(\frac{\Delta V}{R}\right) \cdot \Delta V = -\frac{(\Delta V)^2}{R} = -\frac{(9\text{ V})^2}{(200\ \Omega)} = -0.405\text{ W}$$

Thus, the heat transfer rate *out* of the resistor is 0.405 W.

(b) To find the entropy generation rate we will use the same system as above and apply the entropy accounting equation:

$$\underbrace{\cancelto{=0}{\frac{dS_{sys}}{dt}}}_{\text{Steady-state}} = \sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,j}} + \underbrace{\cancelto{=0}{\sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e}}_{\text{Closed system}} + \dot{S}_{\text{gen}} \quad \rightarrow \quad 0 = \frac{\dot{Q}_{\text{in}}}{T_{\text{surface}}} + \dot{S}_{\text{gen}} \quad \rightarrow$$

$$\underbrace{-\frac{\dot{Q}_{\text{in}}}{T_{\text{surface}}}}_{\substack{\text{Heat transfer of entropy} \\ \textit{out}\text{ of the system}}} = \dot{S}_{\text{gen}}$$

This says that the entropy transfer rate out of the system with heat transfer equals the entropy generation rate inside the system. Notice that even though our original guess for the direction of the heat transfer rate was incorrect, we are still using the same assumptions for applying the entropy balance. (Changing signs and directions in the middle of a problem is a frequent source of errors in problem solving.)

To solve for the entropy generation rate we need to find the surface temperature of the resistor, T_b, because the system boundary coincides with the surface of the resistor. To do this we can make use of the convection heat transfer relationship as follows:

$$\dot{Q}_{\text{in}} = h \cdot A \cdot (T_\infty - T_{\text{surface}}) \quad \rightarrow \quad T_{\text{surface}} - T_\infty = \frac{-\dot{Q}_{\text{in}}}{h \cdot A} = \frac{-(-0.405 \text{ W})}{\left(10 \dfrac{\text{W}}{\text{m}^2 \cdot {}^\circ\text{C}}\right)(2.5 \times 10^{-4} \text{ m}^2)} = 162^\circ\text{C}$$

$$T_{\text{surface}} = (T_{\text{surface}} - T_\infty) + T_\infty \quad \rightarrow \quad T_{\text{surface}} = 162^\circ\text{C} + 25^\circ\text{C} = 187^\circ\text{C}$$

From a practical standpoint, this surface temperature is unacceptably high. However, it does demonstrate a significant problem in miniaturizing electronic components-maintaining acceptable operating temperatures.

Now that we know the surface temperature, we can calculate the entropy generation rate for the resistor:

$$\dot{S}_{\text{gen}}\Big|_{\text{Resistor}} = \frac{-\dot{Q}_{\text{in}}}{T_{\text{surface}}} = \frac{-(-0.405 \text{ W})}{(187 + 273)\text{K}} = \frac{0.405 \text{ W}}{460 \text{ K}} = 0.880 \times 10^{-3} \frac{\text{W}}{\text{K}}$$

This is the entropy production rate inside the resistor and is the direct result of the irreversible conversion of electrical energy into thermal energy. For resistors, this is sometimes referred to as *Joule heating*. Note that the surface temperature value was converted to the Kelvin temperature scale, a thermodynamic temperature scale, before it was used to calculate the entropy transfer rate with heat transfer. Had we used the temperature in Celsius degrees, we would have obtained a different and *incorrect* value for the entropy transfer rate!

(c) To solve for the entropy generation rate of the enlarged system that includes the boundary layer we need to redefine our system. (See the figure below.) If we apply conservation of energy to this system, we find the same result that we found for the resistor alone: $-\dot{Q}_{\text{in}} = \dot{W}_{\text{electric, in}}$.

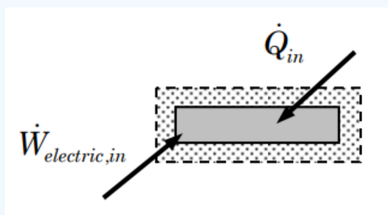


Figure 8.3.4: Closed system consisting of the resistor and its surrounding boundary layer.

The entropy accounting equation will also look the same with one significant exception—the boundary temperature at which the heat transfer occurs. For the enlarged system, the entropy generation rate for this enlarged system is as follows:

$$\dot{S}_{\text{gen}}\Big|_{\substack{\text{Resistor+} \\ \text{Boundary Layer}}} = \frac{-\dot{Q}_{\text{in}}}{T_\infty} = \frac{-(-0.405 \text{ W})}{(25 + 273)\text{K}} = \frac{0.405 \text{ W}}{298 \text{ K}} = 1.359 \times 10^{-3} \frac{\text{W}}{\text{K}}$$

where the boundary temperature was taken as the ambient air temperature. Notice that as one might expect the entropy production rate for the enlarged system is greater than the entropy production rate for just the resistor.

Comment:

Although we have answered all the questions, this is such a rich problem and we've already invested time getting started so let's see what else we can learn.

Can we explain precisely where the additional entropy production is coming from? YES!

Consider the convection boundary layer as a system by itself. Notice that the wires carrying the electrical power to the resistor pass through this system with no net transfer of electrical power; however, there are two heat transfers. At the inner surface of the boundary layer, the system exchanges energy by heat transfer $\dot{Q}_{\text{in, surface}}$ with the resistor at T_{surface}, and at the outer surface of the boundary layer, the system exchanges energy by heat transfer $\dot{Q}_{\text{out, }\infty}$ with the surroundings at T_∞.

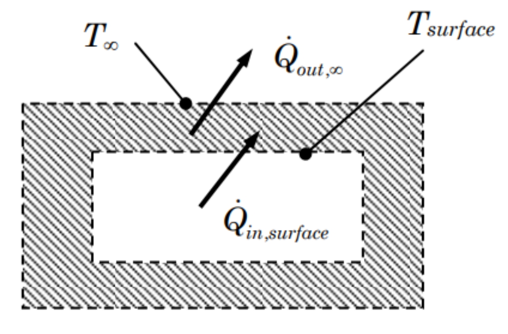


Figure 8.3.5: System consisting only of the convection boundary layer.

Now if we write the conservation of energy and the entropy accounting equation for this closed system we have

$$\underbrace{\overbrace{\frac{dE_{sys}}{\cancel{dt}}}^{=0}}_{\text{Steady-state}} = \dot{Q}_{\text{in, surface}} - \dot{Q}_{\text{out, }\infty} \qquad \rightarrow \qquad \dot{Q}_{\text{out, }\infty} = \dot{Q}_{\text{in, surface}}$$

$$\underbrace{\overbrace{\frac{dS_{sys}}{\cancel{dt}}}^{=0}}_{\text{Steady-state}} = \frac{\dot{Q}_{\text{in, surface}}}{T_{\text{surface}}} - \frac{\dot{Q}_{\text{out, }\infty}}{T_\infty} + \dot{S}_{\text{gen}} \qquad \rightarrow \qquad \frac{\dot{Q}_{\text{out, }\infty}}{T_\infty} = \frac{\dot{Q}_{\text{in, surface}}}{T_{\text{surface}}} + \dot{S}_{\text{gen}}$$

Note that from the entropy balance, the heat transfer of entropy out of the system cannot be less than the heat transfer of entropy into the system and it increases by the entropy production in the system.

Combining these results gives us the entropy production rate for the boundary layer:

$$\begin{aligned} \dot{S}_{\text{gen}}\Big|_{\substack{\text{Boundary}\\ \text{Layer}}} &= \frac{\dot{Q}_{\text{out, }\infty}}{T_\infty} - \frac{\dot{Q}_{\text{in, surface}}}{T_{\text{surface}}} \\ &= \dot{Q}_{\text{in, surface}}\left[\frac{1}{T_\infty} - \frac{1}{T_{\text{surface}}}\right] = (0.405\text{ W})\left[\frac{1}{298\text{ K}} - \frac{1}{460\text{ K}}\right] = 0.479\times10^{-3}\ \frac{\text{W}}{\text{K}} \end{aligned}$$

Notice that this entropy production is the result of a steady-state heat transfer across a layer of air with a finite-temperature difference. (Recall that heat transfer across a finite difference was on the list of dissipative effects.) Also notice that the *three* entropy production rates have a well-defined relationship:

$$\underbrace{\dot{S}_{\text{gen}}\Big|_{\substack{\text{Resistor+}\\ \text{Boundary Layer}}}}_{=1.359\times10^{-3}\ \frac{\text{W}}{\text{K}}} = \underbrace{\dot{S}_{\text{gen}}\Big|_{\text{Resistor}}}_{=0.880\times10^{-3}\ \frac{\text{W}}{\text{K}}} + \underbrace{\dot{S}_{\text{gen}}\Big|_{\text{Boundary Layer}}}_{=0.479\times10^{-3}\ \frac{\text{W}}{\text{K}}}$$

because the enlarged system is just the sum of the two subsystems. Note that *entropy cannot be generated on the boundary of a system*, as a boundary is an infinitesimally thin surface with no mass.

Would it be possible to reverse the direction of the heat transfer of energy for the resistor and get electrical power out of this hot resistor? NO!

So why not? Let's begin by revisiting our original system that consisted only of the resistor and reverse the direction of the heat transfer rate and the electrical power.

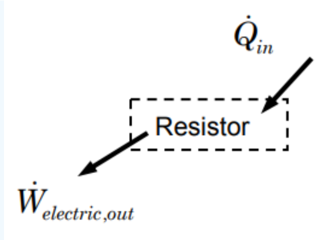


Figure 8.3.6: System from Figure 8.3.3 with the direction of electrical power reversed.

Again we can apply conservation of energy and entropy accounting to give the following results:

$$\left.\begin{array}{l}\underbrace{\overset{=0}{\cancel{\frac{dE_{sys}}{dt}}}}_{\text{Steady-state}} = \dot{Q}_{\text{in}} - \dot{W}_{\text{electric, out}} \quad \rightarrow \quad \dot{W}_{\text{electric, out}} = \dot{Q}_{\text{in}} \\ \underbrace{\overset{=0}{\cancel{\frac{dS_{sys}}{dt}}}}_{\text{Steady-state}} = \frac{\dot{Q}_{\text{in}}}{T_{\text{surface}}} + \dot{S}_{\text{gen}} \quad \rightarrow \quad \dot{Q}_{\text{in}} = -\left(T_{\text{surface}} \cdot \dot{S}_{\text{gen}}\right)\end{array}\right| \rightarrow \quad \dot{W}_{\text{electric, out}} = -\underbrace{\left(T_{\text{surface}} \cdot \dot{S}_{\text{gen}}\right)}_{\substack{T_{\text{surface}}>0 \\ \dot{S}_{\text{gen}} \geq 0}}$$

$$\leq 0$$

Thus the *maximum* electric power *out* of this resistor is *no* power. [So I guess you need to give that kid at the science fair from Section 8.1.1 the prize for the best hoax.]

Surely with modern technology we can build a steady-state device that receives energy by heat transfer at a single temperature $\mathbf{T_b}$ and completely converts that energy into a work transfer of energy out of the system. NO!

Wow, this seems like a pretty strong restriction. Are you sure? Review the development above. Although our system diagram is labeled as a resistor with electrical power out, we could easily rewrite it for *any steady-state system* and for *net power out*. The only other restriction is that we have a net heat transfer into the system and that all heat transfer occurs at a *single* boundary temperature. Under these conditions we get the same result.

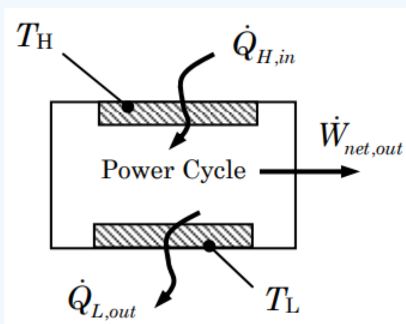


Figure 8.3.7: Basic structure of a power cycle.

Remember the Kelvin-Planck Statement in Section 8.1.1 that said it was impossible to have a power cycle that is 100% efficient. *If* we had a steady-state device, like a *closed-loop, steady-state power* cycle that received a heat transfer of energy $\dot{Q}_{H,\text{ in}}$ at a single temperature T_{H} and converted all of it to a net work out of the system $\dot{W}_{\text{net, out}}$, its thermal efficiency would be $\eta = \dot{W}_{\text{net, out}} / \dot{Q}_{H,\text{ in}} = 100\%$. But we just showed that this was impossible! So I guess Kelvin and Planck were right. If you think the key is in whether there is some heat transfer out of the system, you are correct. (We'll investigate this later, or you can pursue it on your own now by trying to determine the minimum heat transfer rate out of a system when the two boundary temperatures and the heat transfer rate into the system are fixed.)

So what have we learned here?

First, it seems that work and heat transfer, while both being energy transfer mechanisms, are not interchangeable. Clearly we can build a steady-state device that completely converts work to heat transfer at a single temperature, but doing the reverse is impossible.

Second, we have seen how some fairly straightforward applications of the entropy accounting equation have demonstrated that a whole class of devices is impossible. If we are given sufficient information to evaluate the entropy production, then we can determine whether a certain process is possible, internally reversible, or impossible. If we do not have sufficient information to calculate the entropy production, we can still use the entropy accounting equation and the restriction on entropy production to determine a range of physically possible solutions. As we will show shortly, the entropy accounting equation will also help us determine the theoretically "best" possible performance of a device.

Example — Entropy and the "perfect" motor

An electric motor operating at steady state draws a current of 10 amps with a voltage of 220 volts. The power factor is one. The output shaft rotates at $1800 \mathrm{RPM}=188.5 \mathrm{rad} / \mathrm{s}$with a torque of $10 \mathrm{~N} \cdot \mathrm{m}$ applied to the external load. The heat transfer from the motor occurs by convection to the surroundings. The convection heat transfer coefficient, $h_{\text {conv }}$, is $20 \mathrm{~W} /\left(\mathrm{m}^2 \cdot \mathrm{K}\right)$ and the surface area of the motor is $A=0.300 \mathrm{~m}^2$.

Determine

(a) the entropy production rate for the motor, in W / K, and

(b) the maximum theoretically possible shaft power output for this device, i.e. what's the best possible performance?

Solution

Known: A motor operates at steady-state conditions.

Find: (a) The heat transfer rate from the motor, in W.

(b) The entropy production rate for the motor, in W / K.

(c) The maximum theoretically possible shaft power output for this device, in W.

Given:

Shaft Information

$\tau=10 \mathrm{~N} \cdot \mathrm{m}$

$\omega=188.5 \mathrm{rad} / \mathrm{s}$

Electrical Information

$\Delta \mathrm{V}_{\text {effective }}=220$ volts

$\mathrm{i}_{\text {effective }}=10$ amps

Power factor = 1

Steady state operation

$\mathrm{h}_{\text {conv }}=20 \mathrm{~W} /\left(\mathrm{m}^2 \cdot{ }^{\circ} \mathrm{C}\right)$

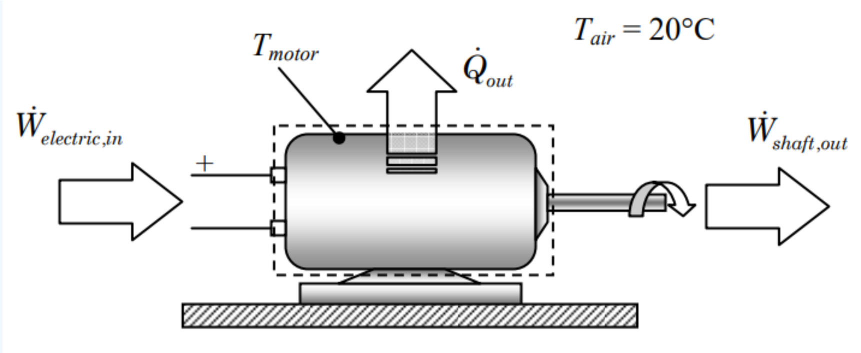


Figure 8.3.8: System consiting of the motor only.

Analysis:

Strategy $\rightarrow$ Calculating heat transfer may require using conservation of energy and/or the convection heat transfer equation. Calculating entropy production rate always requires the entropy balance

System $\rightarrow$ Take just the motor as a closed, non-deforming system.

Property to count $\rightarrow$ Entropy and energy

Time interval $\rightarrow$ Steady-state behavior as per problem statement.

(a) To begin the analysis we refer to the closed system containing the motor shown in the figure above by the dashed line. Writing the energy balance for this closed system to find the heat transfer rate, we have the following:

$$\underbrace{\cancelto{=0}{\frac{dE_{sys}}{dt}}}_{\text{Steady-state}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} - \dot{Q}_{\text{out}} \quad \rightarrow \quad \dot{Q}_{\text{out}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}}$$

To go further requires that we make use of the defining equations for electric and shaft power:

$$\dot{W}_{\text{electric, in}} = i_{\text{effective}} \cdot \Delta V_{\text{effective}} \cdot \begin{pmatrix} \text{Power} \\ \text{Factor} \end{pmatrix} = (10\text{ A}) \cdot (220\text{ V}) \cdot (1) = 2200\text{ W}$$

$$\dot{\text{W}}_{\text{shaft, out}} = \tau \cdot \omega = (10\text{ N} \cdot \text{m}) \cdot \left(188.5\ \frac{\text{rad}}{\text{s}}\right) = 1885\text{ W}$$

Substituting this back into the energy balance gives the heat transfer rate as follows:

$$\dot{Q}_{\text{out}} = \dot{W}_{\text{electric, in}} - \dot{W}_{\text{shaft, out}} = (2200 - 1885)\text{W} = 315\text{ W}$$

Thus the heat transfer rate *out* of the system is 315 W, or 14.32% of the electrical power input. (If we had started to solve for the heat transfer rate by first writing the convection heat transfer relation, we would have quickly realized that the motor temperature was unknown. Requiring another equation, we would then have turned to the conservation of energy.)

(b) Now to find the entropy production rate, we use the same closed system but write the entropy accounting equation:

$$\underbrace{\cancelto{=0}{\frac{dS_{sys}}{dt}}}_{\text{Steady-state}} = -\frac{\dot{Q}_{\text{out}}}{T_{\text{motor}}} + \dot{S}_{\text{gen}} \quad \rightarrow \quad \dot{S}_{\text{gen}} = \frac{\dot{Q}_{\text{out}}}{T_{\text{motor}}}$$

To proceed, we need to find the temperature of the motor surface. We will do this using the convection heat transfer relationship as follows:

$$\dot{Q}_{\text{out}} = h_{\text{conv}} \cdot A \cdot (T_{\text{motor}} - T_{\text{air}}) \quad \rightarrow \quad T_{\text{motor}} = \frac{Q_{\text{out}}}{(h_{\text{conv}} \cdot A)} + T_{\text{air}}$$

$$T_{\text{motor}} = \frac{(315 \text{ W})}{\left(20 \dfrac{\text{W}}{\text{m}^2 \cdot {}^\circ\text{C}}\right) \cdot (0.300 \text{ m}^2)} + 20^\circ\text{C} = 52.5^\circ\text{C} + 20^\circ\text{C} = 72.5^\circ\text{C}$$

Now to compute the entropy production rate, we have the following:

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}_{\text{out}}}{T_{\text{motor}}} = \frac{315 \text{ W}}{(72.5 + 273) \text{ K}} = \frac{315 \text{ W}}{345.5 \text{ K}} = 0.912 \frac{\text{W}}{\text{K}}$$

(c) The final question asks us to consider the maximum shaft power output that is theoretically possible under the specified operating conditions-steady-state, adiabatic operation. Based on your personal experience, you might conclude that the maximum shaft output would occur when the heat transfer rate goes to zero. But what makes this the maximum value possible? Why couldn't you transfer energy into the system by heat transfer, i.e. $\dot{Q}_{\text{out}} < 0$, and increase the shaft power out of the motor?

To find the answer to this question we will make use of both the conservation of energy and the entropy accounting equations developed previously:

$$\dot{W}_{\text{shaft, out}} = \dot{W}_{\text{electric, in}} - \dot{Q}_{\text{out}} \quad \text{and} \quad \frac{\dot{Q}_{\text{out}}}{T_{\text{motor}}} = \dot{S}_{\text{gen}}$$

We will assume that only the electrical power input is fixed. It then appears that shaft power output only depends on the heat transfer rate, and it in turn depends on the entropy production rate. The only one of these terms that we can say anything about is the entropy production rate. So our goal should be to relate the shaft power out to the entropy production rate.

To do this, we combine these two equations by eliminating the heat transfer rate from as follows:

$$\dot{W}_{\text{shaft, out}} = \dot{W}_{\text{electric, in}} - \underbrace{\dot{Q}_{\text{out}}}_{=T_{\text{motor}} \cdot \dot{S}_{\text{gen}}} = \dot{W}_{\text{electric, in}} - \underbrace{\left(T_{\text{motor}} \cdot \dot{S}_{\text{gen}}\right)}_{\substack{T_{\text{motor}} > 0 \\ \dot{S}_{\text{gen}} \geq 0}} \quad \rightarrow \quad \dot{W}_{\text{shaft, out}} \leq \dot{W}_{\text{electric, in}}$$

Thus, the shaft power output will always be less than or equal to the electrical power input. (This proves that those of you who wanted to build a fire and make $\dot{Q}_{\text{out}} < 0$ to increase the shaft power output are out of luck.)

Comment:

Why doesn't a perfect motor, one with $\dot{W}_{\text{shaft, out}} = \dot{W}_{\text{electric, in}}$, violate Kelvin-Planck's prohibition against a 100% efficient power cycle?

The Kelvin-Planck prohibition only applies to the steady-state conversion of a heat transfer of energy completely into a work transfer of energy. For the motor there is no heat transfer of energy *into* the system.

I'm still confused. What exactly is entropy production?

Equation 8.3.2 gives us some additional insight into the meaning of entropy production. For this motor operating under the specified conditions, the theoretically possible maximum shaft power out of the system is equal to the electric power into the system. Using this, we can rewrite Eq. 8.3.2 as follows:

$$\left.\begin{array}{l} \left.\dot{W}_{\text{shaft, out}}\right|_{\text{actual}} = \dot{W}_{\text{electric, in}} - \left(T_{\text{motor}} \cdot \dot{S}_{\text{gen}}\right) \\ \left.\dot{W}_{\text{shaft, out}}\right|_{\substack{\text{max} \\ \text{possible}}} \equiv \dot{W}_{\text{electric, in}} \end{array}\right| \quad \rightarrow \quad \left.\dot{W}_{\text{shaft, out}}\right|_{\text{actual}} = \left.\dot{W}_{\text{shaft, out}}\right|_{\substack{\text{max} \\ \text{possible}}} - \left(T_{\text{motor}} \cdot \dot{S}_{\text{gen}}\right)$$

Now solving for the entropy production rate within the motor, we have the following relation:

$$\dot{S}_{\text{gen}}\Big|_{\text{motor}} = \frac{\left[\dot{W}_{\text{shaft, out}}\Big|_{\substack{\text{max}\\ \text{possible}}} - \dot{W}_{\text{shaft, out}}\Big|_{\text{actual}}\right]}{T_{\text{motor}}} \geq 0$$

Now, what does this tell us about entropy production?

- First, the entropy production rate is a measure of how far a process deviates from *ideal* behavior.
- Second, ideal behavior corresponds to an entropy production rate of zero and this can only occur for an *internally reversible* process.
- Third, when viewed as the difference between the maximum possible power output and the actual power output, the entropy production is also a measure of the irreversible loss of the *potential to do work.*

To grasp this last point, think about the flow of energy through the motor. Initially, the energy enters the motor as electrical work and leaves the system as shaft work *and* heat transfer. Experience has shown that a work transfer of energy is clearly more valuable than an equal heat transfer of energy. Why? Because we can do anything with a work transfer of energy, including driving an ideal DC generator that converts the shaft power back into electrical power and supplying the electrical power to an electric resistor which converts electrical work back into a heat transfer of energy. However, as will be shown in the next example, it is impossible to convert all of the heat transfer from the motor (or any other system) completely into a work transfer of energy.

? Exercise — Work is more valuable than heat transfer? I don't believe it!

To investigate the relative "value" of work transfers of energy and heat transfers of energy, consider two steady-state closed systems shown in the diagrams:

- A "work converter" that receives a work transfer of energy and then "converts" that into a work transfer *and* a heat transfer of energy out of the system. Heat transfer occurs by convection heat transfer between the work converter at T_{surface} and the surroundings at temperature T_{o}.
- A "heat converter" that receives and rejects energy by heat transfer at boundary temperatures T_{surface} and T_{o}, respectively, and has a net work output.

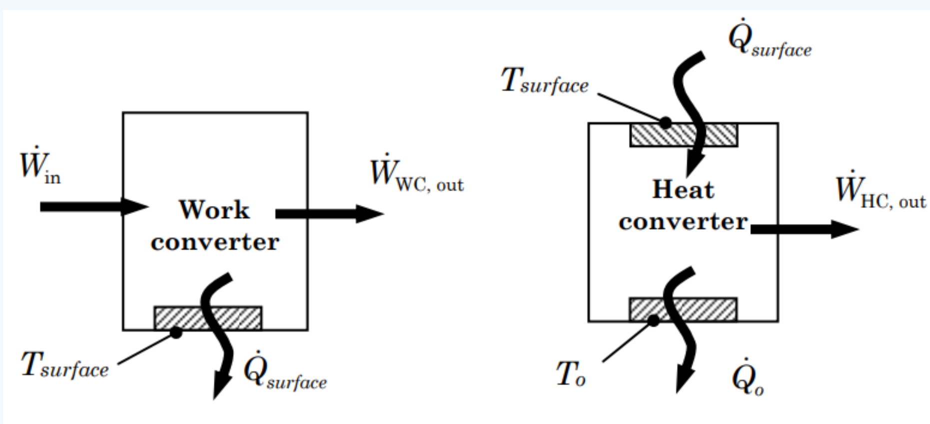


Figure 8.3.9: Structure of a work converter and a heat converter.

Answer the following questions:

(a) Starting with the conservation of energy and entropy accounting equations shown below, develop an expression for the power out of the work converter as a function of the surface temperature, the entropy generation rate, and the power input, i.e. $\dot{W}_{\text{WC, out}} = f\left(T_{\text{surface}},\ \dot{S}_{gen,\,\text{WC}},\ \dot{W}_{\text{in}}\right)$

$$\frac{dE_{sys}}{dt} = \dot{W}_{\text{in}} - \dot{W}_{\text{WC, out}} - \dot{Q}_{\text{surface}} \qquad \frac{dS_{sys}}{dt} = -\frac{\dot{Q}_{\text{surface}}}{T_{\text{surface}}} + \dot{S}_{gen,\,\text{WC}}$$

Answer

$$\dot{W}_{\text{WC, out}} = \dot{W}_{in} - \left(T_{\text{surface}} \cdot \dot{S}_{gen,\text{WC}}\right)$$

(b) What fraction of the power input to the work converter can in theory be returned to the surroundings as power out from the work converter, i.e. what's the maximum value of the ratio $\dot{W}_{\text{WC, out}}/\dot{W}_{\text{in}}$?

(c) Starting with the conservation of energy and entropy accounting equations, develop an expression for the power out of the engine as a function of the two boundary temperatures, the entropy generation rate, the heat transfer rate into the engine, i.e. $\dot{W}_{\text{HC, out}} = f\left(\dot{Q}_{\text{surface}}, T_{\text{o}}, T_{\text{surface}}, \dot{S}_{gen,\text{ HC}}\right)$

$$\frac{dE_{sys}}{dt} = \dot{Q}_{\text{surface}} - \dot{Q}_{\text{o}} - \dot{W}_{\text{HC, out}} \qquad \frac{dS_{sys}}{dt} = \frac{\dot{Q}_{\text{surface}}}{T_{\text{surface}}} - \frac{\dot{Q}_{\text{o}}}{T_{\text{o}}} + \dot{S}_{gen,\text{ HC}}$$

Answer

\(\dot{W}_{\text {HC, out}} = \dot{Q}_{\text {surface}} \left[1-\frac{T_{\text{o}}}{T_{\text {surface}}}\right] - \left(T_{\text{o}} \cdot \dot{S}_{gen, \text { HC}}\right) \nonumber \]

(d) What fraction of the heat transfer input to the heat converter can in theory be returned to the surroundings as power out from the heat converter, i.e. what's the maximum value of the ratio $\dot{W}_{\text{HC, out}}/Q_{\text{surface}}$ assuming the T_{surface} and T_{o} are fixed ?

From parts (b) and (d) above we learn two things:

- given a work transfer of energy and an ideal (internally reversible) work converter, we can completely convert all of the work transfer of energy into the system into an equal amount of work out of the system. Under the worst possible conditions, all of the energy into the work converter would leave the system as heat transfer, and
- given a heat transfer of energy and an ideal (internally reversible) heat converter, we can at best only turn a fraction of the heat transfer of energy into the system into a work transfer of energy out of the system.

Now let's consider what happens when energy flows through a non-ideal work converter. The figure below gives a graphical interpretation of the energy flow through a work converter combined with a heat converter.

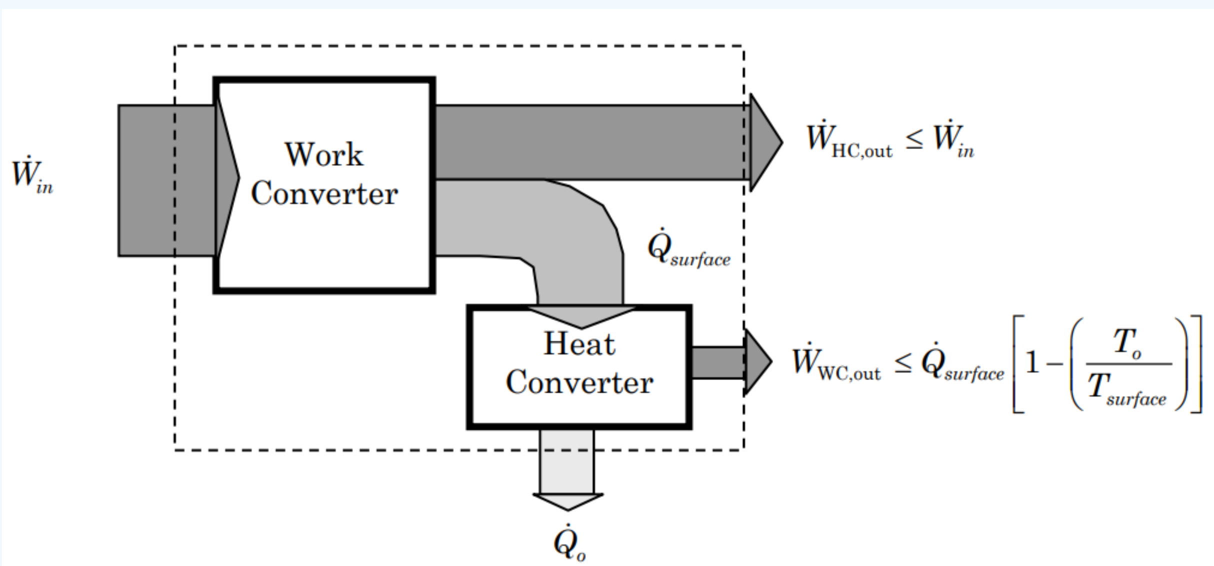


Figure 8.3.10: Energy flow through a system consisting of a work converter and a heat converter, whose input is the heat output of the work converter.

With an ideal work converter, $\dot{S}_{gen,\text{ WC}} = 0$ and all of the work transfer of energy into the system leaves the system as an equal work transfer of energy. With a non-ideal work converter, $\dot{S}_{gen,\text{ WC}} > 0$ and some of the energy leaves the system by heat transfer. To convert this heat transfer back into work we feed it into the heat converter. Even under the best of conditions only a fraction of the heat transfer of energy entering the heat converter can be converted into a work transfer of energy out of the system.

The work that could be recovered by combining the work output from both converters is

$$\begin{aligned}\dot{W}_{\text{combined}} &= \dot{W}_{\text{WC, out}} + \dot{W}_{\text{HC, out}} = \underbrace{\dot{W}_{\text{HC, out}}}_{\substack{\text{Actual power out}\\ \text{of the work converter}}} + \underbrace{\dot{Q}_{\text{surface}}\left[1-\left(\frac{T_{\text{o}}}{T_{\text{surface}}}\right)\right] - T_{\text{o}}\dot{S}_{gen,\text{ HC}}}_{\substack{\text{Actual power out}\\ \text{of the heat converter}}} \\ &= \underbrace{\left[\dot{W}_{\text{WC, out}} + \dot{Q}_{\text{surface}}\right]}_{=\dot{W}_{\text{in}}} - \underbrace{\dot{Q}_{\text{surface}}}_{=T_{\text{surface}}\dot{S}_{gen,\text{ WC}}} \cdot \left(\frac{T_{\text{o}}}{T_{\text{surface}}}\right) - T_{\text{o}}\dot{S}_{gen,\text{ HC}} = \dot{W}_{\text{in}} - \left(T_{\text{surface}}\dot{S}_{gen,\text{ HC}}\right)\left(\frac{T_{\text{o}}}{T_{\text{surface}}}\right) \\ &\qquad - T_{\text{o}}\dot{S}_{gen,\text{ HC}} \\ &= \dot{W}_{\text{in}} - T_{\text{o}}\left(\dot{S}_{gen,\text{ WC}} + \dot{S}_{gen,\text{ HC}}\right) \le \dot{W}_{\text{in}}\end{aligned}$$

Now, what's the impact of entropy production in the work converter? Any entropy production in the work converter results in a heat transfer out of the work converter. If we could convert all of the heat transfer back into work, there would be no problem. Unfortunately this is not the case. Even if we assume an ideal heat converter to recover the maximum amount of work from the heat transfer, we only recover part of the work transfer of energy supplied to the work converter:

$$\dot{W}_{\text{combined}}\Big|_{\substack{\text{Ideal}\\ \text{Heat Converter}}} = \dot{W}_{\text{in}} - T_{\text{o}}\left(\dot{S}_{gen,\text{ WC}} + \underbrace{\cancelto{=0}{\dot{S}_{gen,\text{ HC}}}}_{\text{Ideal Heat Converter}}\right) = \dot{W}_{\text{in}} - T_{\text{o}}\dot{S}_{gen,\text{ WC}} < \dot{W}_{\text{in}}$$

Thus, any entropy production within the work converter will reduce our *potential to do work*. This reinforces the point that it pays us at least thermodynamically to minimize entropy production because anytime entropy is produced we lose the capacity to do some work. Economically, this may not be the best approach; however, as the cost of energy increases there is greater economic incentive to reduce entropy production. *Thermoeconomics* is the discipline that attempts to assign the *true value* to various forms of energy. By combining conservation of energy and the entropy accounting principle, engineers have developed a new extensive property called *exergy* or *availability* that describes the work potential of any quantity or transfer of energy. Thus, it is possible to price energy and energy transfers based on its work potential.

? Exercise — Steady-state Heat Transfer Through a Wall

Energy flows steadily through a cylindrical "plug" with diameter $D = 0.5$ m and length $L = 0.25$ m. The heat transfer rate and surface temperature at Surface 1 are 500 kW and 300 K, respectively. The surface temperature at Surface 2 is 400 K. Assume steady-state behavior.

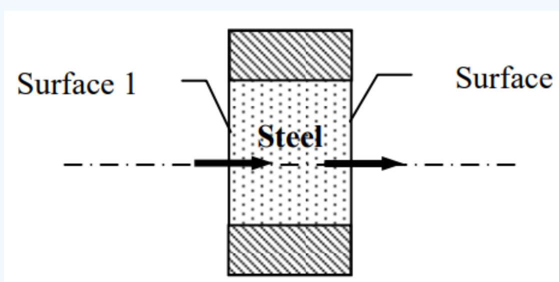


Figure 8.3.11: Energy passes through the axis of a cylindrical steel plug.

(a) Determine the heat transfer rate at Surface 2, in W.

Answer

500 kW

(b) Using the entropy balance, determine the entropy production rate within the steel plug, in W/K.

Answer

0.4167 kW/K

(c) What would happen to the rate of entropy production as the temperature difference across the steel plug gets very small? What is the entropy production rate as the temperature difference goes to zero? [Hint: Replace T_2 by the expression $T_2 = T_1 - \Delta T$. Then examine the entropy production as ΔT gets small.]

(d) If the boundary temperatures remained unchanged, would it be possible for the heat transfer to flow in the opposite direction? Yes or no? Why?

Example — Behavior of a closed system with no energy transfer

(a) Sketch a closed system that has no energy transfers with the surroundings.

(b) Simplify the rate form of the conservation of mass, conservation of energy, and the entropy balance for this system and write the resulting equation in the blank column of the table:

Mass	$\frac{\mathrm{dm_{sys}}}{\mathrm{dt}} = \sum_{\text{in}} \dot{m}_i - \sum_{\text{out}} \dot{m}_e$	
Energy	$\frac{\mathrm{dE_{sys}}}{dt} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \sum_{\text{in}} \dot{m}_i \left(h + \frac{V^2}{2} + gz\right)_i - \sum_{\text{out}} \dot{m}_e \left(h + \frac{V^2}{2} + gz\right)_e$	
Entropy	$\frac{dS_{\text{sys}}}{dt} = \sum_j \frac{\dot{Q}_j}{T_{\text{b, j}}} + \sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e + \dot{S}_{gen}$	

(c) Now using this information, plot the system energy E_{sys}, the system mass m_{sys}, and the system entropy S_{sys} as a function of time on the graphs below. (The small dot at $t = 0$ on each graphs represents the initial value of m_{sys}, E_{sys}, and S_{sys}.)

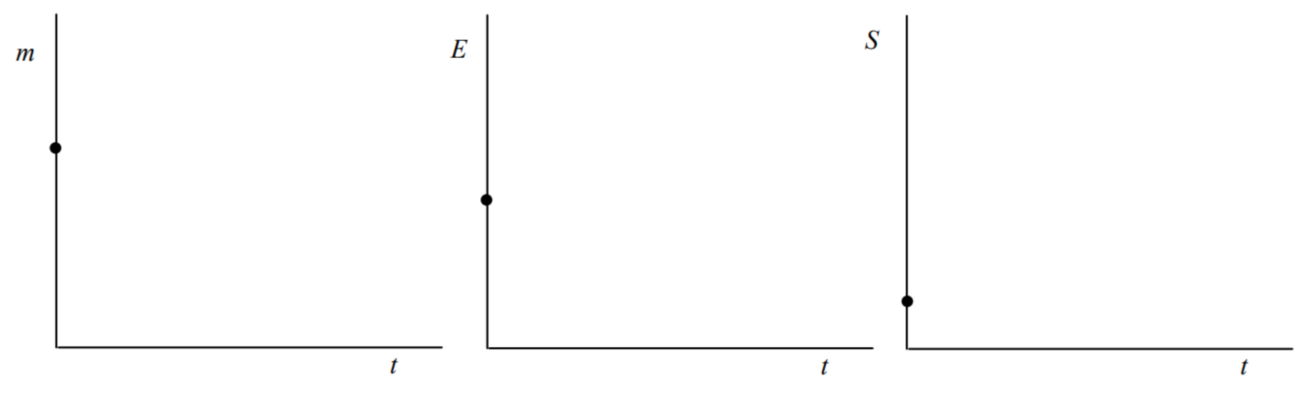


Figure 8.3.12: Axes to plot system mass, energy, and entropy.

What happens for very long times, i.e. as time goes to infinity? (For example, if you assumed that the system finally reached a steady-state, equilibrium value, what must happen to S_{sys}? What restriction does this place on the shape of your curve for S vs. t?)

8.4: Thermodynamic Cycles, Revisited

We concluded Chapter 7 with a discussion of the performance of thermodynamic cycles. We will now investigate what if anything the entropy accounting equation can tell us about the performance of thermodynamic cycles.

8.4.1 Power Cycles

Consider a power cycle that operates between a high temperature T_{H} and a lower temperature \(T_{\mathrm{L}\). (See Figure 8.4.1.) Recall that a power cycle is a thermodynamic cycle that has a net power output while exchanging energy by heat transfer at its boundaries. Our goal is to examine how changing the boundary temperatures and the entropy production rate influence the cycle performance — the thermal efficiency of the power cycle.

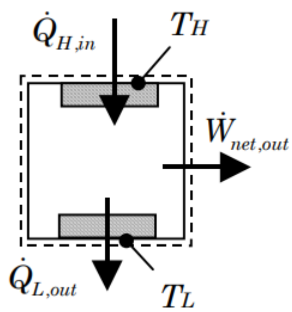


Figure 8.4.1: Power cycle operating between T_{H} and T_{L}.

The first thing we need to do is see what conservation of energy and entropy accounting tell us about this power cycle. First, we apply the conservation of energy equation:

$$\underbrace{\cancel{\frac{dE_{\mathrm{sys}}}{dt}}^{=0}}_{\text{Steady-state}}=\dot{Q}_{\mathrm{H,\,in}}-\dot{Q}_{\mathrm{L,\,out}}-\dot{W}_{\mathrm{net,\,out}} \quad\rightarrow\quad \dot{W}_{\mathrm{net,\,out}}=\dot{Q}_{\mathrm{H,\,in}}-\dot{Q}_{\mathrm{L,\,out}}$$

Not surprisingly, we find that the net power out is just the difference between the heat transfer rate into the cycle and the heat transfer rate out of the cycle. Next we apply the entropy accounting equation:

$$\underbrace{\cancel{\frac{dS_{\mathrm{sys}}}{dt}}^{=0}}_{\text{Steady-state}}=\frac{\dot{Q}_{\mathrm{H,\,in}}}{T_{\mathrm{H}}}-\frac{\dot{Q}_{\mathrm{L,\,out}}}{T_{L}}+\dot{S}_{gen} \quad\rightarrow\quad \frac{\dot{Q}_{\mathrm{L,\,out}}}{T_{\mathrm{L}}}=\frac{\dot{Q}_{\mathrm{H,\,in}}}{T_{\mathrm{H}}}+\dot{S}_{gen}$$

Here we discover that the entropy transfer rate out of the cycle by heat transfer at T_{L} equals the entropy transfer rate into the cycle at T_{H} plus the rate of entropy generation inside the power cycle.

If we rewrite Eq. 8.4.2 and solve for the entropy production rate we have

$$\dot{S}_{gen}=\frac{\dot{Q}_{\mathrm{L,\,out}}}{T_{\mathrm{L}}}-\frac{\dot{Q}_{\mathrm{H,\,in}}}{T_{\mathrm{H}}}\geq 0$$

To keep the entropy generation rate in the discussion, we can combine Eqs. 8.4.1 and 8.4.2 by eliminating the heat transfer out of the system, $\dot{Q}_{\mathrm{L,\,out}}$, from both equations:

$$\left.\begin{array}{r}\dot{Q}_{\mathrm{L,\,out}}=\dot{Q}_{\mathrm{H,\,in}}-\dot{W}_{\mathrm{net,\,out}}\\ \dot{Q}_{\mathrm{L,\,out}}=T_{\mathrm{L}}\left(\dfrac{\dot{Q}_{\mathrm{H,\,in}}}{T_{\mathrm{H}}}+\dot{S}_{gen}\right)\end{array}\right\} \quad\rightarrow\quad \begin{aligned}\dot{Q}_{\mathrm{H,\,in}}-\dot{W}_{\mathrm{net,\,out}}&=T_{\mathrm{L}}\left(\frac{\dot{Q}_{\mathrm{H,\,in}}}{T_{\mathrm{H}}}+\dot{S}_{gen}\right)\\ \dot{W}_{\mathrm{net,\,out}}&=\dot{Q}_{\mathrm{H,\,in}}-T_{\mathrm{L}}\left(\frac{\dot{Q}_{\mathrm{H,\,in}}}{T_{\mathrm{H}}}+\dot{S}_{gen}\right)\end{aligned}$$

Rearranging gives the following relation for the net power out of the power cycle:

$$\dot{W}_{\text{net, out}} = \dot{Q}_{\text{H, in}}\left(1 - \frac{T_{\text{L}}}{T_{\text{H}}}\right) - \left(T_{\text{L}}\dot{S}_{gen}\right)$$

Solving for the thermal efficiency of the power cycle gives the following:

$$\eta = \frac{\dot{W}_{\text{net, out}}}{\dot{Q}_{\text{H, in}}} = \left[1 - \frac{T_{\text{L}}}{T_{\text{H}}}\right] - \underbrace{\left(\frac{T_{\text{L}}\dot{S}_{gen}}{\dot{Q}_{\text{H, in}}}\right)}_{\text{Always} \geq 0} \leq 1$$

Now we are in a position to examine the performance of our power cycle.

? Investigating Power Cycle Performance

Using Eq. 8.4.5, answer the following questions about power cycle performance before continuing your reading:

(a) If the temperatures and input heat transfer rate are fixed, how can we increase the thermal efficiency?

(b) Does a reversible or an irreversible power cycle give the best performance?

(c) What is the theoretical maximum value of the thermal efficiency if the temperatures and input heat transfer rate are fixed? (This is known as the *Carnot efficiency*.) Is it surprising that your answer only depends on temperatures?

(d) How can we increase the numerical value of the maximum thermal efficiency?

(e) A typical steam power plant is a closed-loop, periodic cycle where water circulates in a closed loop formed by a boiler, a steam turbine, a condenser, and a boiler feedpump. The high-temperature heat transfer of energy into the cycle occurs in the boiler, and the low-temperature heat transfer of energy out of the cycle occurs in the condenser.

- What are the physical limits on the values of T_{H}, the high temperature at which the power cycle receives energy by heat transfer?
- What are the physical limits on the values of T_{L}, the low temperature at which the power cycle rejects energy by heat transfer?

If you did not try to answer the questions above, please go back and try them before reading further. This is a great opportunity to learn something new on your own. Don't miss out!

Now let's review what you should have discovered from your investigation of power cycle performance. If the input heat transfer rate and the boundary temperatures are fixed, the only way to improve the performance of this power cycle is to reduce the entropy production rate. In the limit of an internally reversible process, the thermal efficiency reaches a maximum value called the *Carnot efficiency*:

$$\eta_{\max} = 1 - \frac{T_L}{T_H} < 1 \qquad \begin{array}{c}\text{Carnot}\\ \text{Efficiency}\end{array}$$

where T_L and T_H are both thermodynamic temperatures. Thus, the efficiency of an ideal power cycle that exchanges energy by heat transfer with the surroundings at two temperatures only depends on the temperatures. [Carefully note that Eq. 8.4.6 is only valid for a power cycle with heat transfer at *two* temperatures. A different equation must be developed for power cycles that have heat transfer at more than two temperatures.] This is exactly the conclusion that Sadi Carnot summarized in the Carnot Principles given earlier in Section 8.1.1.

Stop for a moment and consider the significance of the Carnot efficiency. This is a fairly astounding result! It implies that no matter what kind of power cycle you construct, no matter what working fluid you select, and no matter how much money you spend, your power cycle efficiency can never exceed that predicted by the Carnot efficiency. And furthermore, I can tell you right now with a very simple calculation the maximum possible efficiency you can ever achieve if you will just give me the two temperatures your cycle operates between.

Example — Efficiencies of some ideal power cycles

Calculate the Carnot efficiency for the following power cycles that operate between two temperatures. Note that typically the actual efficiency of a power cycle is approximately $25\text{-}50\%$ of the ideal value.

(a) A steam power plant receives heat transfer in the boiler at $1000^{\circ}\mathrm{F}$ $(1460^{\circ}\mathrm{R})$ and rejects energy by heat transfer at $140^{\circ}\mathrm{F}$ $(600^{\circ}\mathrm{R})$.

(b) The first American nuclear-powered merchant ship was the *N. S. Savannah*. The nuclear--powered propulsion system used nuclear fission to provide the heat source to boil water in the reactor. The water in the boiler received energy by heat transfer at a mean temperature of $508^{\circ}\mathrm{F}$ $(968^{\circ}\mathrm{R})$ and the heat transfer was rejected in the condenser at a temperature of $347^{\circ}\mathrm{F}$ $(807^{\circ}\mathrm{R})$.

Why would anyone settle for such a low efficiency? What compensating advantage did a nuclear-fueled system have for a ship?

(c) Your neighbor has a solar-powered power cycle that receives energy by heat transfer from a solar collector on her roof. The maximum temperature of the solar collector is $95^{\circ}\mathrm{C}$ $(368\ \mathrm{K})$ and the power cycle rejects energy by heat transfer to a pond in her backyard at $15^{\circ}\mathrm{C}$ $(288\ \mathrm{K})$.

How could you justify spending money to build a system with such a low efficiency? (Remember the actual efficiency is typically $25\text{-}50\%$ of the ideal value.)

Now how can we improve the ideal efficiency of our power cycle? Since the ideal efficiency only depends on the temperatures, we should increase T_{H} and decrease T_{L}. Although this is a perfectly reasonable idea, it turns out in practice that physical considerations limit our ability to arbitrarily change these temperatures.

The high temperature T_{H} is limited by two factors. First, it is limited by the temperature of the heat source. For example, a combustion process found in a coal-fired boiler will typically produce a higher temperature than will a flat-plate solar collector. Second, it is limited by the material properties of the physical components of the cycle.

Material limitations are probably the most significant ones in limiting the efficiency of practical power cycles. At some point the steel used to manufacture a boiler tube for a fossil-fueled power plant will melt. However, we can never even get close to this temperature because the water is pressurized and the strength of the tubes which depends on temperature must be sufficient to resist the water pressure. (An interesting phenomenon called *creep* is of great interest to boiler designers. Just like a stretched rubber band will slowly deform with time, boiler tubes that are pressurized will also slowly stretch or creep with time. Extensive testing programs are used to ensure that boilers do not fail due to creep.) When designing gas-turbine power plants, operating temperatures in the combustor are limited by the ability of the first row of turbine blades to withstand the hot gases coming out of the combustor. This discussion about how materials limit performance underscores the importance of materials science in engineering. (Classroom design that only occurs on paper seems separated from real materials, but *any* engineer who has had to build what they designed understands the crucial role materials play in engineering.)

The low temperature T_{L} is not typically subject to material property limitations; however, it is still limited. When building a power cycle that must reject energy by heat transfer, the temperature T_{L} must be greater than the temperature of the ultimate energy sink. On the surface of the Earth the ultimate energy sink is the atmosphere, a lake, a river, or an ocean. These all have temperatures that are close to what is commonly called ambient temperature. To obtain a temperature significantly below ambient temperature requires the use of a refrigeration system and is not practical. Thus, T_{L} is limited to the lowest temperature that naturally occurs in the surroundings of the power cycle.

8.4.2 Refrigeration and Heat Pump Cycles

Now we want to look at the performance of refrigeration and heat pump cycles. As we stated earlier, these are sometimes called "reversed" cycles because all of the heat transfer and work transfer interactions are reversed from that of a power cycle. Again we will restrict our discussion to cycles that exchange energy by heat transfer with the surroundings at only two temperatures, T_H and T_L.

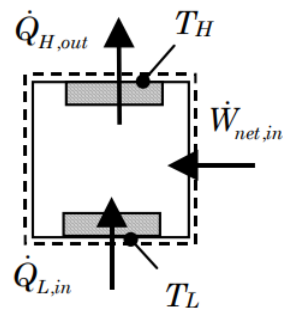


Figure 8.4.2: Refrigeration or Heat Pump Cycle operating between T_{H} and T_{L}.

Figure 8.4.2 shows a schematic that we will use to model the behavior of both a refrigeration cycle and a heat pump cycle. Remember that from a performance standpoint the only difference is which heat transfers we are interested in. For a refrigeration cycle, we are interested in the heat transfer into the cycle at a low temperature T_{L}. For a heat pump, we are interested in the heat transfer *out* of the cycle at the high temperature T_{H}.

Relations to predict the coefficient of performance (COP) for refrigeration cycles and heat pump cycles can be developed following an approach like that used in the previous section to develop the thermal efficiency for a power cycle. Only the final results for these cycles will be presented here:

Refrigeration cycle:

$$\mathrm{COP}_{\mathrm{Ref}} \equiv \frac{\dot{Q}_{\mathrm{L}}}{\dot{W}_{\text{net, in}}} = \frac{T_{\mathrm{L}}}{\left[(T_{\mathrm{H}} - T_{\mathrm{L}}) + T_{\mathrm{L}}\left(\dfrac{T_{\mathrm{H}}\dot{S}_{gen}}{\dot{Q}_{\mathrm{L}}}\right)\right]}$$

Heat pump cycle:

$$\mathrm{COP}_{\mathrm{HP}} \equiv \frac{\dot{Q}_{\mathrm{H}}}{\dot{W}_{\text{net, in}}} = \frac{T_{\mathrm{H}}}{\left[(T_{\mathrm{H}} - T_{\mathrm{L}}) + T_{\mathrm{H}}\left(\frac{T_{\mathrm{L}}\dot{S}_{gen}}{\dot{Q}_{\mathrm{H}}}\right)\right]}$$

As before, we will now investigate the ideal performance of these cycles.

? Performance of Refrigeration Cycles and Heat Pump Cycles —

Please answer the following questions before continuing. Because of the similarities between these two cycles it makes sense to consider their performance in parallel.

Refrigeration Cycles [Use Eq. 8.4.7]	**Heat Pump Cycles** [Use Eq. 8.4.8]
(a) Assuming that the two boundary temperatures and the heat transfer rate into the cycle are fixed, how can you increase the $\mathrm{COP}_{\mathrm{Ref}}$?	(a) Assuming that the two boundary temperatures and the heat transfer rate out of the cycle are fixed, how can you increase the $\mathrm{COP}_{\mathrm{HP}}$?
(b) Assuming that the two boundary temperatures and the heat transfer rate into the cycle is fixed, what is the maximum possible $\mathrm{COP}_{\mathrm{Ref}}$?	(b) Assuming that the two boundary temperatures and the heat transfer rate out of the cycle is fixed, what is the maximum possible $\mathrm{COP}_{\mathrm{HP}}$?
(c) Based on this result, why do you think that cryogenic refrigerators are very expensive to run?	(c) Based on this result, why do you think that electric heat pumps are an easier sell in Knoxville, TN than they are in Minneapolis, MN?

Again, please try your hand at answering these questions before you proceed.

Before we examine the ideal performance of a refrigeration cycle it will pay us to look at a typical refrigerator or freezer and see how this familiar device relates to our refrigeration cycle. Figure 8.4.3 shows the main components of a refrigerator or freezer.

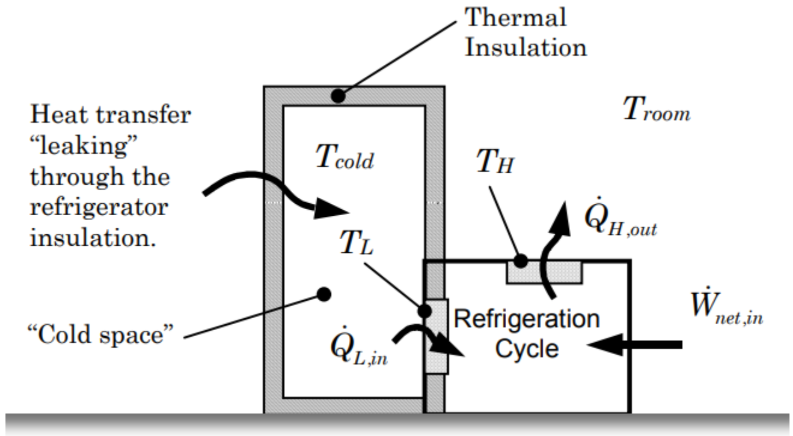


Figure 8.4.3: How a refrigeration cycle is used to keep the inside of a refrigerator or freezer cold.

A simple refrigerator consists of an insulated box and a refrigeration cycle. The low-temperature boundary of the refrigeration cycle is inside the insulated box, and typically forms a portion of the inner wall of the insulated box. The power input $\dot{W}_{\text{net, in}}$ is achieved by plugging the cycle into a wall outlet. The heat transfer of energy out of the refrigeration cycle occurs at a high temperature T_{H} boundary surface that is physically outside the insulated box. Before a refrigerator is turned on, $T_{\text{cold}} = T_{\text{L}} = T_{\text{H}} = T_{\text{room}}$. After the refrigerator is turned on, T_{H} starts to increase above T_{room} and T_{L} starts to drop below T_{room}. The system including the "cold space" reaches steady-state when the "heat" leaking through the thermal insulation equals the $\dot{Q}_{\text{L, in}}$ entering the refrigeration cycle. At steady-state, $T_{\text{cold}} > T_{\text{L}}$ and $T_{\text{H}} > T_{\text{room}}$. Obviously, if we had a perfect thermal insulator then all we would have to do is cool the "cold space" to the appropriate temperature and shut down the cycle. Unfortunately there is no perfect thermal insulator, and your refrigerator cycles on and off to maintain a specified interior temperature.

The ideal COP for a refrigeration cycle that exchanges energy by heat transfer at only two boundary temperatures occurs for an *internally reversible* cycle:

$$\text{COP}_{\text{Ref, max}} = \frac{T_{\text{L}}}{T_{\text{H}} - T_{\text{L}}} = \frac{T_{\text{L}}/T_{\text{H}}}{1 - T_{\text{L}}/T_{\text{H}}} \qquad \begin{array}{c}\text{Ideal}\\ \text{Refrigerator Cycle}\end{array}$$

The high temperature for a typical refrigeration system is the lowest naturally occurring temperature in the surroundings, typically room temperature. Thus, for a fixed T_{H}, the ideal COP of a refrigeration cycle decreases as the cold temperature drops.

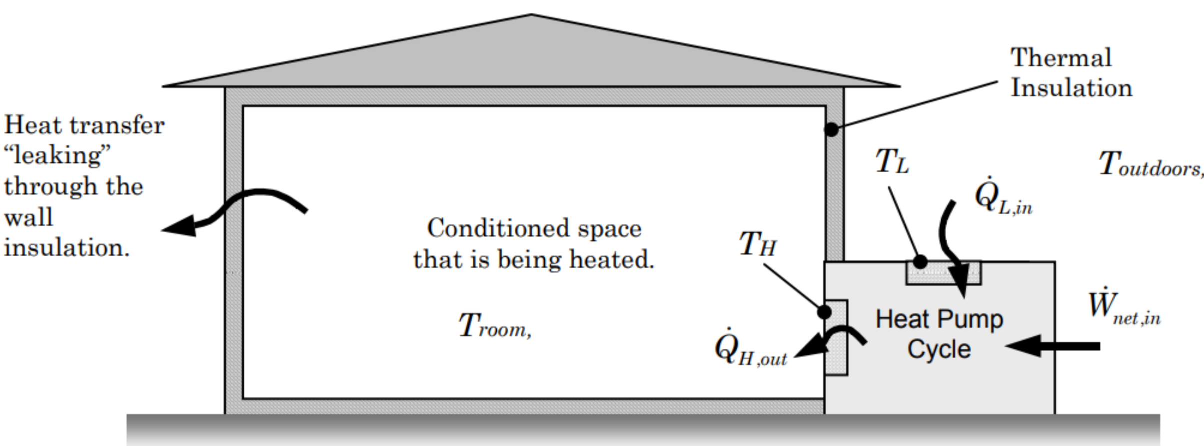


Figure 8.4.4: How a heat pump cycle is used to heat a house.

Figure 8.4.4 shows a schematic of a heat pump and a house that it is heating. As you can see, the heat-pump cycle rejects energy by heat transfer into the conditioned space at a boundary with high temperature T_{H} and receives energy from the outdoors at a boundary with a low temperature T_{L} outside the conditioned space. At steady state conditions, "heat" leaks out of the room through the walls and there is a heat transfer of energy into the room (and out of the cycle) at $T_{\mathrm{H}} > T_{\text{room}}$. The cycle also receives a heat transfer of energy into the cycle (and out of the surroundings) at $T_{\mathrm{L}} < T_{\text{outdoors}}$ as well as a net power input. (On very cold days, the outside heat transfer surface may be below freezing and frost up. In cold climates, heat pumps must have a defrost capability just like the freezer compartment in a refrigerator.)

The ideal COP for a heat pump cycle that exchanges energy by heat transfer at only two boundary temperatures occurs for an internally reversible cycle: \[\mathrm{COP}_{\text{HP, max} = \frac{T_{\text{H}}}{T_{\text{H}}-T_{\text{L}}} = \frac{1}{1-T_{\text{L}} / T_{\text{H}}} \quad\quad \begin{array} \text{Ideal} \\ \text {Heat Pump Cycle} \nonumber \] The high temperature for a residential heat pump system is something above the air temperature needed for human comfort. The low temperature, because it must be below the ambient air temperature outdoors, is at the mercy of the daily weather and average climatic conditions. Some heat pump installations use ground water the as source of energy at T_L to minimize these fluctuations.

For any installation, the ideal COP will always decrease as the outdoor temperature drops. This means that when you need it the most a heat pump has the worst performance. Because of this, most heat pump installations include a set of electric resistance heating coils to provide auxiliary heating of the air on the coldest days. In fact, as T_{L} drops the performance of an expensive electric heat pump approaches that of a less expensive electric resistance furnace. So why do people buy heat pumps? If your alternative is an electric resistance furnace, an electric heat pump is much cheaper to operate. Also the same equipment can be used to air condition the house. (What would you have to do physically to change a heat pump system into an air-conditioner?)

8.5: Entropy and the Substance Models

To apply the entropy accounting equation, we often need to determine the value of the specific entropy s of the substances found in or crossing the boundary of our system. The purpose of this section is to extend the two different substance models we studied previously—ideal gases and incompressible substances with room-temperature specific heats—to cover entropy.

8.5.1 Relating s to T, P, u, h, and v — The Tds relations

Before we give the working relationships for calculating changes in specific entropy as a function of other specific properties, we need to consider how, in fact, these properties are related.

Consider, if you will, a closed system that contains a *simple, compressible substance*. At this point you should be asking yourself, "What exactly is a simple, compressible substance?" A **simple, compressible substance** is a substance for which the only pertinent reversible (or quasiequilibrium) work mode is PdV work (compression and expansion work). There are substances (for example, a steel bar) that can undergo two different quasiequilibrium work processes—PdV work and elastic (or spring) work; however, in a given problem only one of the quasiequilibrium work modes may be important. (A steel bar where only elastic work is important would be called a *simple, elastic substance*). We will not belabor this point further except to say that both of our substance models apply to a simple, compressible substance.

Now consider the behavior of this system over a small time interval:

$$\begin{aligned} \frac{dE_{sys}}{dt} &= \dot{Q}_{\text{in}} + \dot{W}_{\text{in}} \quad \rightarrow \quad dE = \delta Q_{\text{in}} + \delta W_{\text{in}} \\ \frac{dS_{sys}}{dt} &= \frac{\dot{Q}}{T_b} + \dot{S}_{gen} \quad \rightarrow \quad dS = \frac{\delta Q_{\text{in}}}{T_b} + \delta S_{gen} \end{aligned}$$

In addition we will make the following assumptions:

- the process is internally reversible, therefore all intensive properties are spatially uniform during the process,
- the substance temperature T and the boundary temperature T_{b} are the same,
- the only possible work mode is PdV work, and
- the changes in kinetic and gravitational potential energy are negligible for the process.

Under these conditions, Eq. $8.5.1$ simplifies as below:

$$\left.\begin{aligned} d\cancel{E}^{=dU} &= \delta Q_{\text{in}} + \delta \cancel{W_{\text{in}}}^{=-Pd\forall} \\ dS &= \frac{\delta Q_{\text{in}}}{\cancel{T_b}^{=T}} + \underbrace{\delta \cancel{S_{gen}}^{=0}}_{\substack{\text{Internally} \\ \text{Reversible}}} \end{aligned}\right| \quad \rightarrow \quad dU = \delta \cancel{Q_{\text{in}}}^{=TdS} - Pd\forall$$

This finally gives us a relationship between the internal energy, temperature, entropy, pressure, and volume of the system:

$$dU = TdS - Pd\forall$$

For our purposes, we will rewrite this in terms of intensive properties as follows:

$$\left.\begin{aligned} &dU = TdS - Pd\forall \\ &d(mu) = Td(ms) - Pd(mv) \\ &mdu = mTds - mPdv \end{aligned}\right| \quad \rightarrow \quad du = Tds - Pdv$$

A related equation can be obtained by combining Eq. $8.5.3$ with the defining equation for enthalpy as follows:

$$\begin{gathered} \left.\begin{aligned} u &= Tds - Pdv \\ h &= u + Pv \end{aligned}\right| \quad \rightarrow \quad \underbrace{d(h - Pv)}_{=u} = Tds - Pdv \\ dh - [Pdv + vdP] = Tds - Pdv \\ dh = Tds + vdP \end{gathered} \tag{8.5.1}$$

When rearranged in the "standard" form, Eqs. $8.5.3$ and $8.5.4$ are known collectively as the Tds relations for a simple, compressible substance:

$$\begin{array}{ll} Tds = du + Pdv & \text{Tds relations for a} \\ Tds = dh - vdP & \text{simple, compressible substance} \end{array}$$

These two equations are valid for any simple, compressible substance and are the means for relating the change in specific entropy to other more easily measured properties.

8.5.2 Δs — Ideal Gas Model

Starting with the first Tds relation, we find the following for an ideal gas:

$$ds = \frac{du}{T} + \frac{P}{T}dv \quad \text{where} \quad du = c_v dT \quad \text{and} \quad \frac{P}{T} = \frac{R}{v}$$

After substitution, this gives up

$$ds = c_v \frac{dT}{T} + R\frac{dv}{v} \qquad \text{Ideal gas}$$

This expression is good for *any* ideal gas. A similar expression can be obtained starting with the second Tds relation as follows:

$$ds = \frac{dh}{T} - \frac{v}{T}dP \quad \text{where} \quad dh = c_P dT \quad \text{and} \quad \frac{v}{T} = \frac{R}{P}$$

After substitution, this finally gives us:

$$ds = c_P \frac{dT}{T} - R\frac{dP}{P} \qquad \text{Ideal gas}$$

which is also good for any ideal gas.

If we restrict ourselves to the ideal gas model with room-temperature specific heats, Eqs. $8.5.6$ and $8.5.7$ can be integrated to give the following two equations:

$$\Delta s = s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) \quad \left| \quad \begin{array}{c} \text{Ideal gas model} \\ \text{with room-temperature} \\ \text{specific heats} \end{array}\right.$$

and

$$\Delta s = s_2 - s_1 = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \quad \left| \quad \begin{array}{c} \text{Ideal gas model} \\ \text{with room-temperature} \\ \text{specific heats} \end{array}\right.$$

Either of these equations can be used to calculate the change in specific entropy between two states. Note that unlike specific internal energy u and specific enthalpy h, the specific entropy depends on *two* properties: T and v, or T and P. You will find both of these equations in the summary table for the ideal gas model (found in Section 7.4.3 of this text).

Under some operating conditions, we will find that the change in specific entropy is zero, i.e. $\Delta s = 0$. This will happen frequently enough that special equations are often developed for these conditions. Starting with Eq. $8.5.8$ we have the following:

$$\cancel{\Delta s}^{=0} = c_v \ln\left(\frac{\cancel{T_2}^{=T_{2s}}}{T_1}\right) + R\ln\left(\frac{\cancel{v_2}^{=v_{2s}}}{v_1}\right) \quad \rightarrow \quad \ln\left(\frac{T_{2s}}{T_1}\right) = -\frac{R}{c_v}\ln\left(\frac{v_{2s}}{v_1}\right)$$

$$\text{but } \frac{R}{c_v} = \frac{c_p - c_v}{c_c} = \frac{c_p}{c_v} - 1 = k - 1 \quad \rightarrow \quad \ln\left(\frac{T_{2s}}{T_1}\right) = -(k-1)\ln\left(\frac{v_{2s}}{v_1}\right)$$

This gives us finally

$$\frac{T_{2s}}{T_1} = \left(\frac{v_{2s}}{v_1}\right)^{1-k} \quad \text{when } s_2 = s\left(T_{2s},\ v_{2s}\right) = s\left(T_1,\ v_1\right) = s_1$$

where T_{2s} and v_{2s} are the temperature and specific volume, respectively, at state 2 when $s_2 = s_1$. A similar expression can be developed from Eq. $8.5.9$ and is presented here without development:

$$\frac{T_{2s}}{T_1} = \left(\frac{P_{2s}}{P_1}\right)^{\frac{k-1}{k}} \quad \text{where } s_2 = s\left(T_{2s}, P_{2s}\right) = s\left(T_1, P_1\right) = s_1$$

By combining Eqs $8.5.10$ and $8.5.11$, the following equation can be developed which is valid between any two states or along any process where s is a constant:

$$Pv^k = \text{ Constant} \qquad \begin{matrix} \text{Ideal gas model} \\ \text{with room-temperature specific heats} \\ \text{and } \Delta s = 0 \end{matrix}$$

If you were to remember one equation for conditions with $\Delta s = 0$, I would recommend Eq. $8.5.12$. Combined with the ideal gas law, $Pv = RT$, you can recover the other equations with a little algebra.

Any process in which the specific entropy remains constant is called an **isentropic process**. Eqs. $8.5.10$, $8.5.11$, and $8.5.12$ are all valid for isentropic processes where the substance can be modeled as an ideal gas with room-temperature specific heats.

8.5.3 Δs — Incompressible Substance Model

Starting with the first Tds relation, we find the following for an incompressible substance:

$$ds = \frac{du}{T} + \frac{P}{T}dv \quad \text{where} \quad du = cdT \quad \text{and} \quad v = \text{ constant}$$

After substitution, this finally gives us

$$ds = c\frac{dT}{T} \quad \text{Incompressible substance}$$

This expression is good for any incompressible substance.

If we restrict ourselves further to the incompressible substance model with room-temperature specific heats we can integrate Eq. $8.5.13$ giving us the following relationship:

$$\Delta s = s_2 - s_1 = c\ln\left(\frac{T_2}{T_1}\right) \qquad \begin{matrix} \text{Incompressible substance model} \\ \text{with} \\ \text{room-temperature specific heats} \end{matrix}$$

If you model something as an incompressible substance with room-temperature specific heats, use Eq. $8.5.14$ to calculate the change in specific entropy. This equation can also be found in the summary table for the incompressible substance model (from Section 7.4.3 of this text).

Under some operating conditions, we will find that the change in specific entropy is zero, i.e. $\Delta s = 0$. Under these conditions, Eq. $8.5.14$ reduces to the following:

$$\overbrace{\cancel{\Delta s}}^{=0} = c\ln\left(\frac{\overbrace{\cancel{T_2}}^{=T_{2s}}}{T_1}\right) \qquad \begin{matrix} \text{Incompressible substance model} \\ \text{with} \end{matrix}$$

$$T_{2s} = T_1 \qquad \begin{matrix} \text{room-temperature specific heats} \\ \text{and } \Delta s = 0 \end{matrix}$$

Thus, for an isentropic process of an incompressible substance the temperature of the substance is constant.

8.5.4 Examples

In the following examples, the only thing that is new is the use of the ideal gas model and incompressible substance model as required to relate changes in specific entropy to other intensive properties.

Example — Pumping Kerosene, Revisited

Revisit the Pumping Kerosene example in Section 7.4.3 of this text. If the pumping occurs in an adiabatic, reversible process, determine (a) the temperature of the kerosene leaving the pump and (b) the power required to operate the pump, in $\mathrm{ft \cdot lbf/lbm}$.

Solution

We will use exactly the same system as selected before and will also assume that kerosene can be modeled as an incompressible substance with room-temperature specific heats.

Applying the conservation of energy equation and the incompressible substance model to this system gives us the same result as before:

$$\begin{aligned}\frac{\dot{W}_{\text{pump, in}}}{\dot{m}} &= \underbrace{(h_2 - h_1)}_{=c(T_2-T_1)+v(P_2-P_1)} + \underbrace{\cancelto{=0}{\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right)}}_{\text{Same reasoning as before}} + g(z_2 - z_1) \\ &= c(T_2 - T_1) + v(P_2 - P_1) + g(z_2 - z_1)\end{aligned}$$

Now applying the entropy accounting equation to this system, we have

$$\underbrace{\cancelto{=0}{\frac{dS_{sys}}{dt}}}_{\text{Steady-state}} = \underbrace{\cancelto{=0}{\sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,j}}}}_{\text{Adiabatic}} + \underbrace{\dot{m}_1 s_1 - \dot{m}_2 s_2}_{\dot{m}_1=\dot{m}_2=\dot{m}} + \dot{S}_{gen} \quad \rightarrow \quad \underbrace{\cancelto{=0}{\dot{S}_{gen}}}_{\substack{\text{Reversible}\\ \text{process}}} = \dot{m}(s_2 - s_1) \quad \rightarrow \quad s_2 - s_1 = 0$$

To go further we must apply the incompressible substance model and evaluate the change in specific entropy:

$$\left.\begin{array}{r}\text{Substance model:} \quad \Delta s = c\ln\left(\frac{T_2}{T_1}\right) \\ + \\ \text{Process information:} \quad \Delta s = 0\end{array}\right| \Rightarrow \quad \cancelto{=0}{\Delta s} = c\ln\left(\frac{T_2}{T_1}\right) \quad \rightarrow \quad T_2 = T_1$$

Thus the outlet temperature is the same as the inlet temperature and the work for the pump can be found as follows from our earlier efforts:

$$\frac{\dot{W}_{\text{pump, in}}}{\dot{m}} = c\,\underbrace{\cancelto{=0}{(T_2 - T_1)}}_{\substack{T_2-T_1 \text{ for this}\\ \text{process}}} + \frac{1}{\rho}(P_2 - P_1) + g(z_2 - z_1) = (126.6 + 15.0)\,\frac{\mathrm{ft \cdot lbf}}{\mathrm{lbm}} = 142\,\frac{\mathrm{ft \cdot lbf}}{\mathrm{lbm}}$$

Comment:

Note that if the process is internally reversible, it only takes approximately 43% of the power input that was required when the kerosene temperature increased only $0.5°\mathrm{F}$.

If we revisit the entropy production equation, we can rearrange it so that the impact of entropy production on temperature change is easily seen:

$$\begin{array}{r}\dot{S}_{gen} = \dot{m}(s_2 - s_1) \\ s_2 - s_1 = c\ln\left(\frac{T_2}{T_1}\right) \\ T_2 = T_1 + \Delta T\end{array} \quad \rightarrow \quad \frac{\dot{S}_{gen}}{\dot{m}} = c\ln\left(1 + \frac{\Delta T}{T_1}\right) \quad \text{or} \quad T_2 - T_1 = T_1 \cdot \left[\exp\left(\frac{\dot{S}_{gen}}{\dot{m}c}\right) - 1\right]$$

Combining this with the general equation for the power into a pump gives the following result:

$$\frac{\dot{W}_{\text{pump, in}}}{\dot{m}} = c\left(T_2 - T_1\right) + \frac{1}{\rho}\left(P_2 - P_1\right) + \left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) + g\left(z_2 - z_1\right)$$

$$= cT_1 \cdot \left[\exp\left(\frac{\dot{S}_{gen}}{\dot{m}c}\right) - 1\right] + \frac{1}{\rho}\left(P_2 - P_1\right) + \left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) + g\left(z_2 - z_1\right)$$

Now using the equation above, determine the theoretical minimum amount of power that must be supplied to the pump. [Just an equation is acceptable.]

For a pump, the "best" performance corresponds to minimizing the power required to operate the pump. What type of process gives you this "best" performance?

Example — Stirring Things Up

A rigid tank contains 5 kg of nitrogen (N_2) initially at a pressure of 100 kPa and a temperature of 27°C. The tank also contains a paddle wheel that has negligible mass and volume. The walls of the tank are made of an insulating material. The paddle wheel operates for several minutes and does 500 kJ of work on the gas inside the rigid tank. Determine the (a) final pressure and temperature of the gas, and (b) the amount of entropy produced during this process. Assume N_2 can be modeled as an ideal gas with room-temperature specific heats.

Solution

Known: Nitrogen gas is contained in a rigid tank and is stirred by a paddle wheel in the tank.

Find: (a) The final temperature and pressure of the gas.

(b) The entropy produced during the stirring process.

Given:

Initial conditions of N_2 gas:
$T_1 = 27^\circ\text{C}$
$P1 = 100 \text{ kPa}$
$m_1 = 5 \text{ kg}$

Work done by paddle wheel on the gas: $W_{\text{Paddle-wheel, in}} = 500 \text{ kJ}$

Paddle wheel mass and volume are negligible.

Tank is rigid and made of insulating material.

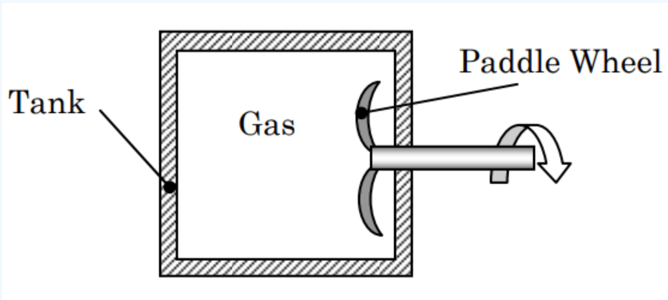


Figure 8.5.1: Tank contains a gas and a paddle wheel.

Analysis:

Strategy $\rightarrow$ Assume we will need to use energy because we want to know the change in a property of gas when work is done on it. And we will need to use entropy because we are asked about the entropy produced.

System $\rightarrow$ Treat the gas in the tank as a closed system.

Property to count $\rightarrow$ Mass and energy.

Time interval $\rightarrow$ Finite-time interval since beginning and end state are implied.

Starting with the conservation of energy equation for a closed-system, finite-time system we have the following:

$$\underbrace{\cancel{\Delta E}^{=\Delta U}}_{\substack{\text{Neglecting changes in}\\ \text{kinetic and gravitational}\\ \text{potential energy.}}} = \cancel{Q_{\text{in}}}^{=0} + \cancel{W_{\text{in}}}^{=W_{\text{PW, in}}} \quad \rightarrow \quad \Delta U = W_{\text{PW, in}}$$

(Don't despair if you cannot write down this closed-system, finite-time form immediately. Just start with the complete equation and apply the two assumptions — closed system and finite time. You should only start with a simplified form if you explicitly state the simplifications for the equation. Note that I clearly stated what I assumed before I wrote down the equation.)

Now if we apply the ideal gas model we have the following:

$$\Delta U = m\Delta u = mc_v\left(T_2 - T_1\right) \qquad \text{where } c_v = 0.743\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}$$

where the specific heat value is taken from the tables of thermophysical properties for N_2.

Substituting this back into the energy equation, we can solve for the final temperature as follows:

$$mc_v\left(T_2 - T_1\right) = W_{\text{PW, in}} \quad \rightarrow \quad T_2 = T_1 + \frac{W_{\text{PW, in}}}{mc_v}$$

$$T_2 = \left(27^{\circ}\text{C}\right) + \frac{500\ \text{kJ}}{(5\ \text{kg})\left(0.743\ \dfrac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)} = (27 + 134.6)^{\circ}\text{C} = 161.7^{\circ}\text{C}$$

Solving for the final pressure, we can make use of the ideal equation and ratios:

$$\begin{array}{l} P_2 V\!\!\!\!-_2 = m_2 R T_2 \\ P_1 V\!\!\!\!-_1 = m_1 R T_1 \end{array} \quad \rightarrow \quad \frac{P_2\, \cancel{V\!\!\!\!-_2}}{P_1\, \cancel{V\!\!\!\!-_1}} = \frac{\cancel{m_2}\, \cancel{R}\, T_2}{\cancel{m_1}\, \cancel{R}\, T_1} \quad \rightarrow \quad \frac{P_2}{P_1} = \frac{T_2}{T_1}$$

$$P_2 = P_1\left(\frac{T_2}{T_1}\right) = (100\ \text{kPa})\left(\frac{161.7 + 273}{27 + 273}\right) = 145\ \text{kPa}$$

Now to find the entropy produced, we must apply the entropy accounting equation to this same system. This time we will start with the general rate form and simplify it explicitly:

$$\frac{dS_{sys}}{dt} = \underbrace{\cancel{\sum_{j=1}^{N} \frac{\dot{Q}_j}{T_{b,\,j}}}}_{\text{Adiabatic}} + \underbrace{\cancel{\sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \dot{m}_e s_e}^{=0}}_{\text{Closed system}} + \dot{S}_{gen}$$

$$\frac{dS_{sys}}{dt} = \dot{S}_{gen} \quad \rightarrow \quad \int_{t_1}^{t_2} \left(\frac{dS_{sys}}{dt}\right) dt = \int_{t_1}^{t_2} \dot{S}_{gen}\ dt \quad \rightarrow \quad \underbrace{\Delta S_{sys}}_{\substack{\text{Change in entropy}\\ \text{of the system}}} = \underbrace{S_{gen}}_{\substack{\text{Entropy generated}\\ \text{inside the system}}}$$

Now to go further we must use the ideal gas model as follows:

$$S_{gen} = \Delta S = m\Delta s = m\underbrace{\left[c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\cancel{\left(\frac{v_2}{v_1}\right)}^{v_2 = v_1}\right]}_{\Delta s \text{ from ideal gas model}} = mc_v \ln\left(\frac{T_2}{T_1}\right)$$

$$= (5\ \text{kg})\left(0.743\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ \ln\left(\frac{161.7 + 273}{27 + 273}\right) = 1.378\ \frac{\text{kJ}}{\text{K}}$$

Comments:

(1) The fact that entropy is produced in this process indicates that doing paddle wheel work on an ideal gas is an *irreversible* process. The source of the irreversibility is the fluid friction between the surface of the paddle wheel and the gas. If the gas was frictionless, the paddle wheel would just slide without friction through the gas and thus do no work on the gas.

(2) Could you have used the other equation for Δs to calculate the entropy change? Absolutely and you should get exactly the same result:

$$S_{gen} = m\Delta s = m\left[c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)\right]$$

$$\begin{aligned} S_{gen} &= (5\text{ kg})\left[\left(1.04\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ \ln\left(\frac{161.7+273}{27+273}\right) - \left(0.2968\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\ \ln\left(\frac{145}{100}\right)\right] \\ &= (5\text{ kg})\left[\ 0.385709 \quad - \quad 0.110280\ \right]\frac{\text{kJ}}{\text{kg}\cdot\text{K}} = 1.377\,\frac{\text{kJ}}{\text{K}} \end{aligned}$$

Note that this answer differs from the earlier calculation due to round off errors in the problem. Because entropy calculations often require taking the difference of two terms of equal magnitude, it is suggested that you carry several more significant figures in the intermediate calculations than you present in the final answer.

(3) How would your solution have changed if the mass and volume of the paddle wheel had not been negligible?

System $\rightarrow$ must include paddle wheel that is modeled as an incompressible substance.

Conservation of energy $\rightarrow$ would have to include internal energy change of paddle wheel in ΔU_{sys}

Entropy accounting $\rightarrow$ would have to include entropy change of the paddle wheel in ΔS_{sys}.

(4) How would the original solution change if an electric resistance heater of negligible mass had been used instead of a paddle wheel? A review of this problem should show you that nothing about the solution would change. All of the answers would be the same.

(5) How would the solution change if the energy had been added to the rigid tank by heat transfer instead of the paddle wheel?

Conservation of energy $\rightarrow$ No change. Final P and T would be identical.

Entropy accounting $\rightarrow$ $S_2 - S_1 = \int_{t_1}^{t_2}\left(\frac{\dot{Q}_{\text{in}}}{T_b}\right)dt + S_{gen} \quad \rightarrow \quad S_{gen} = \underbrace{(S_2 - S_1)}_{\substack{\text{Same as as in problem} \\ \text{with paddle wheel}}} - \int_{t_1}^{t_2}\left(\frac{\dot{Q}_{\text{in}}}{T_b}\right)dt$

So the entropy production will be less than that found in the original problem since the change in entropy of the system only depends on the end states and the end states are the same in both processes. From an energy utilization standpoint, the fact that the paddle wheel and the electric resistor produce more entropy means that they have *destroyed more potential to do work* than would have been destroyed by the heat transfer process. If we can assume that the temperature of the gas is spatially uniform during the heating process and $T_b = T$, then

$$\frac{dU}{dt} = \dot{Q}_{\text{in}} \quad \rightarrow \quad dU = \dot{Q}_{\text{in}}\,dt \quad \rightarrow \quad \frac{1}{T}dU = \frac{\dot{Q}_{\text{in}}}{T}dt \quad \rightarrow \quad mc_v\frac{dT}{T} = \frac{\dot{Q}_{\text{in}}}{T}dt$$

and

$$S_{gen} = m\left(s_2 - s_1\right) - \underbrace{\int_{T_1}^{T_2}\left(\frac{mc_v}{T}\right)dt}_{= mc_v \ln\left(\frac{T_2}{T_1}\right)} = m\left[\left(s_2 - s_1\right) - c_v\ln\left(\frac{T_2}{T_1}\right)\right]$$

$$= m\left\{\underbrace{\left[c_v\ln\left(\frac{T_2}{T_1}\right) + R\ln\cancel{\left(\frac{v_2}{v_1}\right)}^{v_2=v_1}\right]}_{\Delta s\text{ for an ideal gas}} - c_v\ln\left(\frac{T_2}{T_1}\right)\right\} = 0$$

Wow, what happened to the entropy production? It would seem that we have placed enough constraints on the process that it is now internally reversible. In practice, it would be *impossible* to heat the gas in such a fashion that there are no internal temperature gradients. With this in mind the answer represents a minimum estimate of the entropy that would be produced during the heat transfer process.

Experience has shown that anytime we use a work transfer of energy to accomplish something that could have been done by an equal heat transfer of energy, the process with work transfer of energy will produce more entropy. In fact, this is another way to determine if a given process is reversible or irreversible.

Example — Mixing things up

Revisiting "Mixing things up" in Section 7.4.3 of the text. We will now calculate the entropy production for the mixing process.

We will begin with the entropy accounting equation, since we already know the final temperature and pressure of the mixture.

Applying the entropy accounting equation to the closed, adiabatic system consisting of the two tanks gives the following:

$$\frac{dS_{sys}}{dt} = \underbrace{\overbrace{\sum_{j=1}^{N} \cancel{\frac{\dot{Q}_j}{T_{b,j}}}}^{=0}}_{\text{Adiabatic tanks}} + \underbrace{\overbrace{\sum_{\text{in}} \dot{m}_i s_i - \sum_{\text{out}} \cancel{\dot{m}_e s_e}}^{=0}}_{\text{Closed system}} + \dot{S}_{gen} \quad \rightarrow \quad \dot{S}_{sys} dt = \dot{S}_{gen} \quad \underbrace{\rightarrow}_{\substack{\text{Integrating}\\ \text{for finite time}}} \quad \Delta S = S_{gen}$$

Now to evaluate the entropy generation, we need to evaluate the change in entropy for the mixture. As in the past, you are advised to arrange the equation so that you can calculate the *change in specific entropy*. Here is an example of this approach.

$$\begin{aligned} S_{gen} &= \Delta S = S_2 - S_1 \\ &= m_2 s_2 - (m_{A,1} s_{A,1} + m_{B,1} s_{B,1}) = \underbrace{(m_{A,1} + m_{B,1})}_{=m_2} s_2 - (m_{A,1} s_{A,1} + m_{B,1} s_{B,1}) = m_{A,1}(s_2 - s_{A,1}) \\ &\quad + m_{B,1}(s_2 - s_{B,2}) \end{aligned}$$

To go further requires that we use the ideal gas model.

$$\begin{aligned} s_2 - s_{A,1} = c_P \ln\left(\frac{T_2}{T_{A,1}}\right) - R \ln\left(\frac{P_2}{P_{A,1}}\right) &= \left(1.04 \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) \ln\left(\frac{367}{300}\right) - \left(0.2968 \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) \ln\left(\frac{236}{150}\right) \\ &= 0.0751338 \frac{\text{kJ}}{\text{kg}\cdot\text{K}} \\ s_2 - s_{B,1} = c_P \ln\left(\frac{T_2}{T_{B,1}}\right) - R \ln\left(\frac{P_2}{P_{B,1}}\right) &= \left(1.04 \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) \ln\left(\frac{367}{400}\right) - \left(0.2968 \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) \ln\left(\frac{236}{300}\right) = \\ &\quad -0.0183295 \frac{\text{kJ}}{\text{kg}\cdot\text{K}} \end{aligned}$$

Now substituting these values back into the entropy balance we have the following:

$$\begin{aligned} S_{gen} &= m_{A,1}(s_2 - s_{A,1}) + m_{B,1}(s_2 - s_{B,1}) \\ &= (1\ \text{kg})\left(0.0751338 \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) + (2\ \text{kg})\left(-0.0183295 \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right) = 0.03847 \frac{\text{kJ}}{\text{K}} \end{aligned}$$

So as you might expect, entropy is produced as a result of the mixing process.

If you really want a challenge, prove that the final equilibrium temperature and pressure must be the values that you calculated earlier. Why exactly do these have to be *the* final equilibrium values? As you might guess, it has something to do with the entropy production. To do this you must assume do the following:

(1) Assume that each tank can have its own final temperature and pressure, i.e. $P_{A,2}$, $P_{B,2}$, $T_{A,2}$, and $T_{B,2}$.

(2) Reformulate the problem so that after you pick a final pressure and temperature for, say, Tank A, you can then calculate the corresponding pressure and temperature in Tank B.

(3) Now vary the final pressure and temperature for Tank A over all possible values and solve for S_{gen} .

(4) Find the values for the final pressure and temperature in Tank A that maximizes the entropy generation.

8.6: Problems

In the following problems, all references to an ideal gas and an incompressible substance refer to the models with room-temperature specific heats.

Problem 8.1

A gearbox operating at steady-state conditions receives 2 hp along the input shaft and delivers 1.9 hp along the output shaft. The outer surface of the gearbox is at $105^{\circ}\mathrm{F}$. The temperature of the air in the room is $70^{\circ}\mathrm{F}$.

(a) Determine the rate of heat transfer, in $\mathrm{Btu/h}$. Indicate the direction

(b) Determine the rate of entropy production, in $\mathrm{Btu}/(\mathrm{h}\cdot{}^{\circ}\mathrm{R})$, within the system consisting of the gearbox and the shafts. (Sketch the system.)

(c) Consider the layer of air immediately adjacent to the gearbox. It receives energy by heat transfer at $105^{\circ}\mathrm{F}$ and loses energy by heat transfer at $70^{\circ}\mathrm{F}$. Determine the steady-state rate of entropy production, in $\mathrm{Btu}/(\mathrm{h}\cdot{}^{\circ}\mathrm{R})$, within the air layer. (Sketch the system.)

(d) Now consider an enlarged system that consists of both the gearbox and shafts and the air layer. This system loses energy by heat transfer to the surroundings at $70^{\circ}\mathrm{F}$. Determine the steady-state rate of entropy production, in $\mathrm{Btu}/(\mathrm{h}\cdot{}^{\circ}\mathrm{R})$, for this combined system. (Sketch the system.)

(e) Discuss how your result for Part (d) compares with your answers for Part (b) and Part (c).

Problem 8.2

An inventor claims to have developed a new device that operates at steady-state conditions and produces both shaft power and electrical power. A schematic of the device is shown in the figure with the known operating conditions and proposed energy transfers.

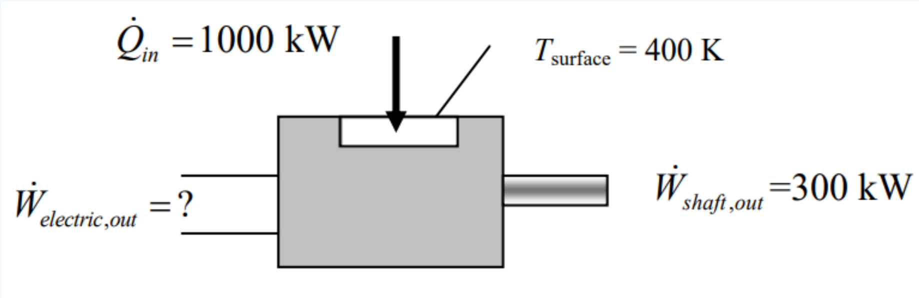


Figure $8.6.1$: Device with heat input that produces shaft power and electrical power.

(a) Determine the electrical power output from the device, in kW.

(b) Determine the entropy production rate for the device, in $\mathrm{kW/K}$.

(c) Based upon your answer to part (b), do believe that this device is possible? Explain the rationale for your answer.

Problem 8.3

An electric transformer is used to step down the voltage from 220 to 110 volts (AC). The current on the high-voltage side is 23 A and on the low voltage side it is 43 A. The power factor is one for both sides of the transformer. The transformer operates under steady-state conditions with a surface temperature of $40^{\circ}\mathrm{C}$. Determine (a) the heat-transfer rate for the device, in watts, and (b) the entropy production rate, in $\mathrm{W/K}$.

Problem 8.4

An electric motor operates under steady-state conditions and draws $3 \mathrm{~kW}$ of electric power. Ten percent of the electrical power supplied to the motor is lost to the surroundings by heat transfer. The surface temperature of the motor is $45^{\circ} \mathrm{C}$. Determine (a) the shaft power delivered by the motor in kW and (b) the entropy production rate for the motor, in kW / K.

Problem 8.5

A soldering iron draws \(0.10 \mathrm{~A}\) from a 110-V circuit at steady-state conditions. The operating temperature of the soldering iron is $105^{\circ} \mathrm{C}$. Determine the entropy production rate for the soldering iron in W / K.

Problem 8.6

A transmission consists of two gearboxes connected by an intermediate shaft. A torque of $220 \mathrm{~ft} \cdot \mathrm{lbf}$ is applied to the input shaft which rotates at $200 \mathrm{~rpm}$. The intermediate shaft and the output shaft rotate at $160 \mathrm{~rpm}$ and $128 \mathrm{~rpm}$, respectively. *Each* gearbox transmits only 95% of the shaft power supplied to it. The remainder of the energy is lost to the surroundings by heat transfer. The surface temperature of each gearbox is measured to be $120^{\circ} \mathrm{F}$, and the ambient air temperature is $70^{\circ} \mathrm{F}$.

Determine (a) the torque for the intermediate and output shafts, in $\mathrm{ft} \cdot \mathrm{lbf}$, (b) the entropy production rate for each gearbox individually and for the overall transmission, in \(\mathrm{Btu} / \left(h \cdot { }^{\circ} \mathrm{R}\right)\).

Problem 8.7

An inventor claims to have a invented a device that takes in $10 \mathrm{~kW}$ by heat transfer at $500 \mathrm{~K}$, rejectes energy by heat transfer at $300 \mathrm{~K}$ and produces $5 \mathrm{~kW}$ of power. As a U.S. Patent Examiner, you must determine if this device is possible or a hoax. Based on the description in the patent application, it appears that it is a closed, steady-state device. What do you think? Is it possible? Explain your reasoning.

Problem 8.8

A steady-state heat pump is designed to reject energy by heat transfer at a rate of $20,000 \mathrm{~Btu} / \mathrm{h}$ at a temperature of $90^{\circ} \mathrm{F}$ and requires an electrical power input equivalent to $5,000 \mathrm{~Btu} / \mathrm{h}$. Heat transfer into the system occurs at a temperature of $40^{\circ} \mathrm{F}$.

Determine (a) the COP for the heat pump and (b) the entropy generation rate for the heat pump, in $\mathrm{Btu} /\left(\mathrm{h} \cdot{ }^{\circ} \mathrm{R}\right)$. (c) Would you describe the heat pump as operating reversibly or irreversibly, or is it impossible to operate as specified?

Problem 8.9

A heat pump with a COP of 3 receives energy from the outdoors at $30^{\circ} \mathrm{F}$ and rejects energy to the air inside the house at $72^{\circ} \mathrm{F}$. The heat pump rejects energy to the air inside the house at the rate of $100,000 \mathrm{~Btu} / \mathrm{h}$.

(a) Determine the following:

- the power of the motor required to operate the heat pump, in horsepower,
- the rate of heat transfer from the outdoors, in $\mathrm{Btu} / \mathrm{h}$, and
- the rate of entropy production for the heat pump, in \(\mathrm{Btu} / \left(\mathrm{h} \cdot { }^{\circ} \mathrm{R}\right)\).

(b) Determine the maximum possible COP for a heat pump operating between these temperatures and the power of the motor, in hp, required to operate this *ideal* heat pump.

(c) Some people would consider the "extra" electrical energy required to run the real heat pump when compared with the power required to operate the *ideal* heat pump as being wasted, since it can't be used for anything else. If the ideal or best possible performance is associated with an internally reversible cycle and this cycle produces no entropy, entropy production may be a measure of energy waste. To check this out, investigate the validity of the following equation using your results from parts (a) and (b):

$$\frac{\left(\dot{W}_{\text{actual}}-\dot{W}_{\text{ideal}}\right)}{T_{\text{outdoors}}}=\dot{S}_{\text{production}}$$

where all power values are in $\mathrm{Btu/h}$, temperatures are in $^{\circ}\mathrm{R}$, and the entropy production rate is what you calculated in part (a). Is this result correct?

Problem 8.10

A system executes a power cycle. During each cycle, the system receives \(2000 \mathrm{~kJ}\) of energy by heat transfer at a temperature of $500\ \mathrm{K}$ and discharges energy by heat transfer at a temperature of $300\ \mathrm{K}$. There is no other heat transfer of energy.

(a) Assuming that the cycle has a thermal efficiency of 25%, determine the work out per cycle, in kJ, and the amount of entropy produced per cycle, in $\mathrm{kJ/K}$.

(b) Assuming that the cycle rejects $900\ \mathrm{kJ}$ of energy by heat transfer, determine the work out per cycle, in kJ, the amount of entropy produced per cycle, in $\mathrm{kJ/K}$, and the thermal efficiency.

(c) Assuming that the cycle is internally reversible, i.e. rate of entropy production is zero, calculate the work out per cycle, in kJ, and the thermal efficiency for this cycle.

(d) Compare your answers to Parts (a), (b), and (c). What does this tell you about the three cycles? Is it possible to build a power cycle that operates between the same two temperatures and is more efficient than the one you examined in Part (c)?

Problem 8.11

A geothermal power plant utilizes an underground source of hot water at $160^{\circ}\mathrm{C}$ as the heat source for a power cycle. The power plant boiler receives energy by heat transfer at a rate of $100\ \mathrm{MW}$ from the hot water source at $T_{H,\text{ Source}}=160^{\circ}\mathrm{C}$ The power plant condenser rejects energy by heat transfer at the rate of $78\ \mathrm{MW}$ to the ambient air at $T_{L,\text{ sink}}=15^{\circ}\mathrm{C}$.

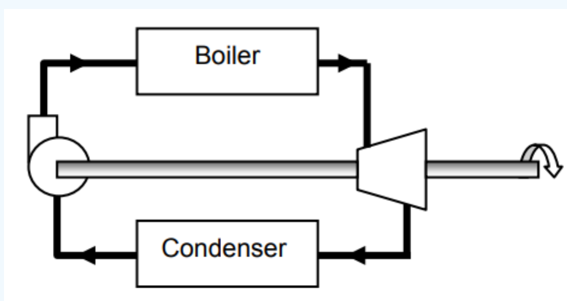


Figure $8.6.2$: Geothermal power plant consisting of a boiler and condenser, with an output of shaft work.

(a) Determine the net power output, in MW, and the thermal efficiency for this power cycle (heat engine) expressed as a percent.

(b) Determine the theoretical maximum thermal efficiency for a power cycle (heat engine) that operates between these two temperatures: $T_{\text{boiler}}=T_{H,\text{ source}}$ and $T_{\text{condenser}}=T_{L,\text{ sink}}$. Give your answer as a percent and compare it to your result from part (a).

(c) In reality, the rate of heat transfer is proportional to the temperature difference available to "drive" the heat transfer, i.e. $Q\propto\Delta T$. Practically this means to receive energy by heat transfer from a thermal source at temperature $T_{H,\text{ source}}$, the surface temperature of the boiler T_{boiler} must be less than the source temperature. Similarly to reject energy by heat transfer to a thermal sink at temperature $T_{L,\text{ sink}}$, the surface temperature of the condenser $T_{\text{condenser}}$ must be greater than the sink temperature.

As a first guess, assume that a temperature difference of $5^{\circ}\mathrm{C}$ is required, and determine the theoretical maximum thermal efficiency for a power cycle that operates between these new more realistic temperatures:

$$T_{\text{boiler}}=T_{H,\text{ Source}}-5^{\circ}\mathrm{C}\quad\text{and}\quad T_{\text{condenser}}=T_{L,\text{ sink}}+5^{\circ}\mathrm{C}.$$

How does the efficiency of this more realistic cycle compare with your answers to part (a) and part (b)?

Problem 8.12

A reversed power cycle operates between a high temperature of $50^{\circ}\mathrm{C}$ and a low temperature of $5^{\circ}\mathrm{C}$. Determine the best possible coefficient for this reversed power cycle (a) if it is operated as a heat pump cycle and (b) if it operated as a refrigeration cycle.

Problem 8.13

A heat pump receives energy by heat transfer from outside air at T_{outdoors} and rejects energy to a dwelling at T_{room}. Starting with the conservation of energy and entropy accounting equation for the steady-state heat pump, develop an expression for the COP of this heat pump similar to Eq. 8.4.7. Show your work.

Problem 8.14

A refrigeration cycle receives energy by heat transfer from a freezer compartment at T_{freezer} and rejects energy to the kitchen at T_{room}. Starting with the conservation of energy and entropy accounting equation for the steady-state refrigeration cycle, develop an expression for the COP of this cycle similar to Eq. 8.4.7. Show your work.

Problem 8.15

Liquid water flows steadily through a small centrifugal pump at a volumetric flow rate of $6.0\ \mathrm{m}^3/\mathrm{min}$. The water enters the pump at $100\ \mathrm{kPa}$ and $27^{\circ}\mathrm{C}$ and leaves the pump at a pressure of $400\ \mathrm{kPa}$. Inlet and outlet areas are identical and changes in potential energy are negligible. Assume that water can be modeled as an incompressible substance.

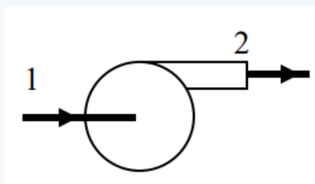


Figure 8.6.3: Water flows steadily through a centrifugal pump.

(a) If the pump is adiabatic and internally reversible:

- Determine the change in specific entropy, $s_2 - s_1$, for the water as it flows through the pump, in $\mathrm{kJ}/(\mathrm{kg}-\mathrm{K})$. [Hint: Apply the entropy accounting equation to the pump, make the appropriate modeling assumptions, and solve for $s_2 - s_1$.]
- Determine the shaft power input under these conditions in kW. [Hint: After applying the conservation of energy equation and appropriate modeling assumptions, don't forget to see what your result for Δs from above along with the incompressible substance model tells you about how the pressure and/or temperature of the water may change.]

(b) Now assume the pump operates adiabatically and that the temperature of the water increases as it flows through the pump, $T_2 - T_1 = 0.05^{\circ}\mathrm{C}$:

- Determine the entropy production rate for the pump, in kW/K. Is this process internally reversible or internally irreversible? How can you tell?
- Determine the shaft power input under these conditions, in kW.

(c) Compare your answers from Part (a) and (b).

- Which operating condition requires the larger power input? Why?
- Do you think it would be possible to reduce the shaft power further by operating this same pump under steady-state, adiabatic conditions so that the water temperature would decrease as it flows through the pump, e.g. $T_2 - T_1 = -0.05^{\circ}\mathrm{C}$?

Problem 8.16

The nozzle in a turbojet engine receives air at $180 \mathrm{kPa}$ and $707^{\circ} \mathrm{C}$ with a velocity of $70 \mathrm{~m} / \mathrm{s}$. The air expands adiabatically in a steady-state process to an outlet pressure of $70 \mathrm{kPa}$. The mass flow rate of air is $3.0 \mathrm{~kg} / \mathrm{s}$. Assume that air can be modeled as an ideal gas with room temperature specific heats.

(a) If the expansion process is internally *reversible*, determine the outlet air temperature in ${ }^{\circ} \mathrm{C}$ and the outlet velocity of the air, in m / s. [Hint: Apply the entropy accounting equation along with the ideal gas model.]

(b) If the expansion process is internally *irreversible* and T_2 is $527^{\circ} \mathrm{C}$, determine the entropy production rate for the nozzle, in kW / K, and determine the outlet velocity, in m / s.

(c) Compare and discuss your results, especially T_2 and V_2, in terms of the entropy production rate for each process.

Problem 8.17

An electric water heater having a 100-liter capacity employs an electric resistor to heat the water from \(18^{\circ} \mathrm{C}\) to $60^{\circ} \mathrm{C}$. The outer surface of the resistor remains at an average temperature of $97^{\circ} \mathrm{C}$ during the heating process. Heat transfer from the outside of the water heater is negligible, and the energy and entropy storage in the resistor and the tank holding the water are insignificant. Model the water as an incompressible substance.

(a) Determine the amount of electrical energy, in kJ, required to heat the water.

(b) Determine the amount of entropy produced, in kJ / K, within the water only, i.e. take the water as the system.

(c) Determine the amount of entropy produced, in kJ / K, within the overall water heater including the resistor, i.e. take the overall water heater including the resistor as the system.

(d) Why do the results of (b) and (c) differ? What is within the system for (c) that was excluded in (b)?

Problem 8.18

Air enters a shop air compressor with a steady flow rate of $0.7 \mathrm{~m}^3 / \mathrm{s}$ at $32^{\circ} \mathrm{C}$ and \(0.95 \mathrm{~bars}\). The air leaves the compressor at a pressure of 15 bars. Assume air can be modeled as an ideal gas and that changes in kinetic and gravitational potential energy are negligible. Determine the minimum power requirement to drive the adiabatic compressor, in kW. [Hint: How does varying the entropy generation rate affect the power input to the compressor? What are the limiting values on the entropy generation rate?]

Problem 8.19

A rigid air tank has a volume of $1.0 \mathrm{~m}^3$ and contains air at $27^{\circ} \mathrm{C}$ and $3400 \mathrm{kPa}$. Should the tank wall fail catastrophically, the tank would explode and cause considerable damage. To estimate the amount of energy that could be transferred from the air to the surroundings in an explosion, we will estimate the work done by the expanding gas. To model the expansion process, assume that the gas acts like a closed system and expands adiabatically and reversibly until the gas pressure matches the ambient air pressure of $100 \mathrm{kPa}$.

(a) Determine the work done by the gas on the surroundings during this expansion process, in kilojoules.

(b) Determine the temperature of the air after this hypothetical expansion process.

(c) How conservative are your results from Part (a)? Would you expect the actual blast to transfer more or less energy to the surroundings?

Problem 8.20

A short pipe and valve connect two heavily insulated tanks. Tank A has a volume of $1.0 \mathrm{~m}^3$ and Tank B has a volume of $2.0 \mathrm{~m}^3$. Tank A initially contains carbon dioxide at $400 \mathrm{~K}$ and $300 \mathrm{kPa}$. Tank B is initially evacuated. Once the value is opened, the carbon dioxide expands into Tank B.

Determine (a) the final equilibrium pressure and temperature of the carbon dioxide and (b) the entropy produced within the gas during this expansion process, in $\mathrm{kJ/K}$. [Assume carbon dioxide can be modeled as an ideal gas.]

Problem 8.21

The figure below shows a steady-state gas turbine power plant consisting of a compressor, a heat exchanger, and a turbine. Air enters the compressor with a mass flow rate of $3.9\ \mathrm{kg/s}$ at $0.95\mathrm{bar}, 22^{\circ}\mathrm{C}$ and exits the turbine at 0.95 bar, $421^{\circ}\mathrm{C}$. Heat transfer to the air as it flows through the heat exchanger occurs at an average temperature of $488^{\circ}\mathrm{C}$. The compressor and turbine operate adiabatically. Assume that air behaves like an ideal gas and assume changes in kinetic and gravitational potential energy are negligible.

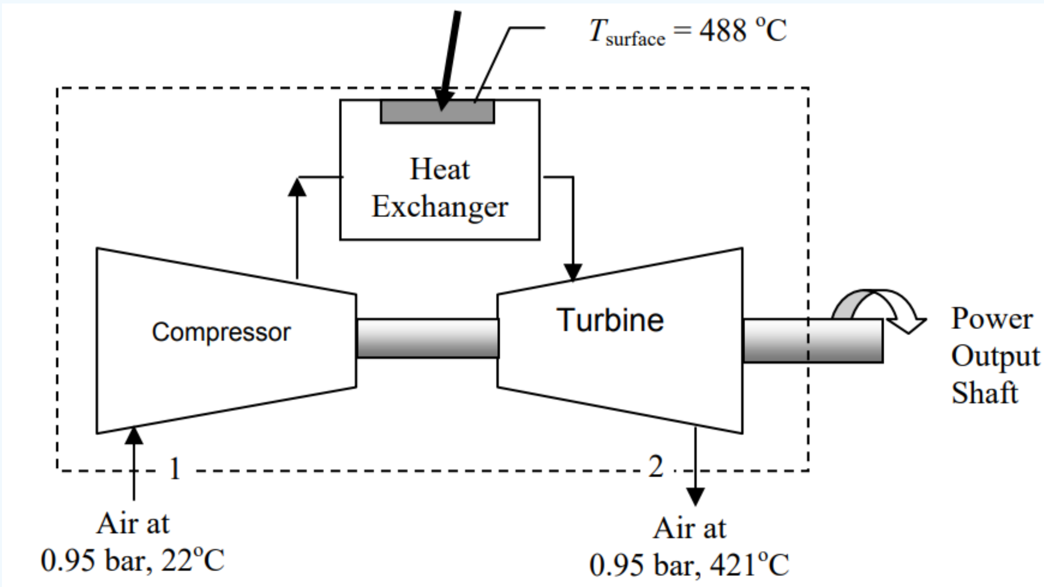


Figure $8.6.4$: Turbine power plant consisting of a compressor, heat exchanger, and turbine, with an output of shaft work.

Determine the maximum theoretical value for the net power that can be developed by the power plant, in MW.

[Hint: Consider the following steps.

(1) Apply the conservation of energy equation to develop an equation that relates the net power out to the heat transfer rate into the system.

(2) Now apply the entropy accounting equation to find a relationship between the heat transfer rate into the system and the entropy production rate for the system.

(3) Now vary the entropy production rate over its possible values and examine how it changes the net power out of the system.

(4) Determine the maximum value of the net power out of the system. Clearly indicate why your result is the maximum value.]

Problem 8.22

A short pipe and valve connect two heavily insulated tanks. Each tank has a volume of $0.5\ \mathrm{m}^3$. Tank A initially contains nitrogen at $150\ \mathrm{kPa}$ and $300\ \mathrm{K}$. Tank B initially contains nitrogen at $50\ \mathrm{kPa}$ and $300\ \mathrm{K}$. Suddenly, the valve is opened and the two gases are allowed to mix.

Determine (a) the final pressure and temperature of the mixture and (b) the entropy produced during this mixing process. [Assume nitrogen can be modeled as an ideal gas.]

Problem 8.23

The air trapped in the piston cylinder of an air compressor occupies an initial volume of $42 \mathrm{in}^3$ (cubic inches) when the piston is at the bottom of its stroke. The air has a temperature and pressure of $70^{\circ} \mathrm{F}$ and 15 psi $\left(\mathrm{lbf} / \mathrm{in}^2\right)$, respectively.

When the piston moves to the top of its stroke, the air is compressed to a volume of \(7.0 \mathrm{~in.}^{3}\). The compression process occurs so fast that heat transfer during the compression process is negligible. If necessary, assume that air can be modeled as an ideal gas and that changes in kinetic and gravitational potential energy for the gas are negligible.

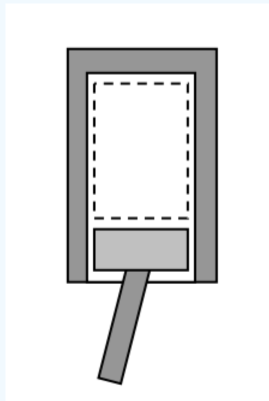


Figure $8.6.5$: Air in a piston cylinder.

(a) If the compression process is reversible, determine the temperature and the pressure of the gas after the compression. In addition, calculate the work done on the gas during the compression process, in Btu.

(b) Now assume that the compression process is irreversible, how will this change the final temperature and pressure of the gas after the compression and the work done on the gas. Clearly indicate whether the values increase or decrease as compared to the values for the reversible process?

(c) Why would an engineer care about the values for a reversible process?

Problem 8.24

An 18-kg lead casting at $200^{\circ} \mathrm{C}$ is quenched in a tank containing $0.03 \mathrm{~m}^3$ of liquid water initially at $25^{\circ} \mathrm{C}$. The water tank is insulated immediately after the casting is dropped into the water. Determine (a) the final equilibrium temperature of the lead, in K, and (b) the entropy generation for the lead-water system, in kJ / K. (c) Is this process reversible, irreversible, or impossible? [Assume lead and liquid water can both be modeled as incompressible substances.

Problem 8.25

A new device is proposed as a steady-state air heater (see figure). Air enters the heater (1) at $400 \mathrm{~K}$ and $200 \mathrm{kPa}$ with a volumetric flow rate of $1000 \mathrm{~m}^3 / \mathrm{min}$ It leaves the heater (2) at $500 \mathrm{~K}$ and $190 \mathrm{kPa}$. The heater is powered by electricity and has two different operating modes. Electricity costs $\$ 0.08$ per kilowatt-hour.

Mode I - Steady-state, adiabatic operation with no heat transfer on the surface of the device, $Q_{o, \text { in }}=0$.

Mode II - Steady-state, internally reversible operation with heat transfer on the surface at a boundary temperature of $T_o=300 \mathrm{~K}$.

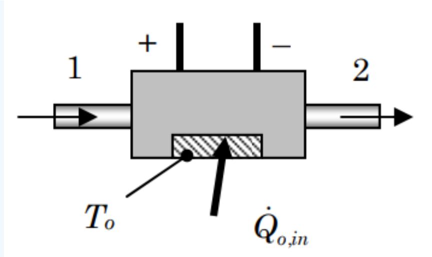


Figure 8.6.6: A steady-state electric air heater.

(a) For Mode I, determine the electric power required to operate the heater, in kW, and the entropy generation rate, in $\mathrm{kW/K}$.

(b) For Mode II, determine the electric power required to operate the heater, in kW, the entropy generation rate, in $\mathrm{kW/K}$, and the heat transfer rate, in kW.

(c) For an 8-hour day, how much would it cost to operate the air heater in each mode? Any advice to the plant engineer about which mode of operation should be used?

CHAPTER OVERVIEW

9: Appendices

9.1: Appendix A- Solving Engineering Problems - A Problem-Solving Heuristic

Engineering problem solving is based on the study of models that describe real systems. In every case, the real system must be modeled by making simplifying assumptions before any mathematical or empirical analysis can be performed. Realistic and useful answers can only be obtained if the modeling assumptions "catch" the important features of the problem. The behavior of any model is constrained by the physical laws it incorporates and the modeling assumptions used in its development. Two different models for the same system may behave in entirely different ways. The engineers' job is to develop the "best" model for the problem at hand.

Because most mistakes are made in the process of developing the model it is essential that you learn to solve problems in a methodical fashion that documents your solution process including your modeling assumptions. Engineering calculations are part of the archival record of any engineering project and are frequently referred to years after the original work is completed. Many a junior engineer begins a new job by reviewing engineering calculations performed by others.

To help you develop your engineering problem solving skills, a multi-step process is proposed to help you (1) organize your thoughts, (2) document your solution, and (3) improve your ability to solve *new* problems. A summary of the steps is presented in Figure A-1. A sample problem showing the format can be found at the end of this appendix. As with any heuristic, this one does not guarantee a solution; however, its usefulness has been proven so frequently that we want you to use it in this course.

Figure A-1

SUMMARY OF PROBLEM SOLVING STEPS

KNOWN: In your own words, state briefly what is known. (Step #1)
FIND: State concisely what you are trying to find. (Step #2)
GIVEN: Translate the problem word statement into sketches and symbolic notation. All pertinent information given explicitly in the problem statement should be listed here. (Step #3)
ANALYSIS: Develop a model and solve for desired information.
• Develop a strategy. (STRATEGY) (Step #4)
• Make modeling assumptions. (Clearly identified.) (Step #5)
• Develop and solve the model. (Step #6)
    ◦ Develop symbolic solutions.
    ◦ Calculate numerical values.
    ◦ Check the reasonableness of your answers.
COMMENT: Discuss your results. (Step #7)

A more detailed discussion of each step is presented in the following sections. (Based on material in *Fundamentals of Engineering Thermodynamics* by M. J. Moran and H. N. Shapiro, J. Wiley & Sons, Inc., New York, 1988.)

KNOWN: In your own words, state briefly what is known. Read the problem statement and think about what it says. Do not just blindly copy the problem statement over again or list every detail of the problem. Construct a short sentence that summarizes the situation.

FIND: State concisely what you are trying to find. (If you don't know what you are looking for, how do you know when you've found it?) Do not just copy (a)...., (b)...., etc. from the problem and do not assume that you must find things in the order implied in the problem statement.

GIVEN: Translate the word statement of the problem into engineering sketches and symbolic notation. When completed, you should be able to throw away the original problem statement because you have recorded all of the pertinent information.

Draw and label a sketch of the physical system or device. (If you cannot visualize the problem, you probably can't solve it!) If you anticipate using a conservation or accounting principle, identify the boundaries (control surfaces) of the system you select for your analysis <u>and</u> identify the interactions between this system and the surroundings, e.g. forces, work, mass flow, etc.

Define symbols for the important variables and parameters of the problem. Record the numerical values given for the important variables and parameters.

Label the diagram with all relevant information from the problem statement. This is where you record all of the information explicitly given in the problem statement.

Be especially wary of making implicit assumptions as you prepare this section. Recognize the difference between information that is given explicitly in the problem and your interpretation of the information.

ANALYSIS: It is in this section that an appropriate mathematical model is developed and used to find the desired information. As you prepare this section, carefully annotate your solution with words that describe what you are doing. This commentary is invaluable in exposing your thought processes and if need be in recreating it at a later time.

- **Develop a strategy.** Every solution should include some initial statements that reveal your plan for solving the problem. As a starting point, clearly state what you believe to be the physical laws or concepts that will be important in solving this problem. What's the property to be counted? What's the appropriate system? What's the appropriate time period? What constitutive relationships may be required?
  Your initial strategy may not be the best approach or the only approach. It may not even be correct approach, but as you proceed through the analysis process your plans may change. As they do just document them.
  To stress the importance of consciously thinking about the problem, every analysis section should start with a brief subsection labeled STRATEGY.
- **Make modeling assumptions.** Every problem solution requires that you make modeling assumptions. These assumptions are based on the information given in the problem statement, your interpretation of the given information, and your understanding of the underlying phenomena. Every model begins with universally accepted natural laws, and the assumptions provide the traceable link between the fundamental laws and problem-specific model you have developed. *All assumptions should be clearly identified as they are applied*. You should be able to give a logical reason for every modeling assumption you make. If you cannot, it probably is an incorrect assumption.
  Some problem solving formats call for a separate section listing all assumptions before you begin your analysis. There are two problems with this approach. First, experience shows that it is often difficult to know exactly what assumptions to make until you are building the model. Secondly, separating the assumptions from their application in the model tends to hide how they influence the modeling process. If a summary list is desired, it should be prepared after the analysis is completed.
- **Develop symbolic solutions.** Symbolic solutions are critical in engineering analysis and should always be developed and examined <u>before</u> you insert numerical values. The physics is in the symbolic solution, not the numerical answer. If the symbolic solution is incorrect, there's no hope for the numbers. If possible, solve an equation for the unknown quantity and isolate it on one side of the equal sign. *It is desirable to work with symbolic equations as long as possible before substituting in numbers for many reasons.* Symbolic solutions are especially useful when you are looking for errors, for solving parametric problems where certain parameters change, and are much easier to modify as your model develops. Look for groups of terms or ways to rearrange you symbolic answer that simplify the equation and allow you to check for dimensional consistency. Groups of terms with physical meaning or logical intermediate values should be assigned a unique symbol. Numerical values for these intermediate answers can then be calculated and checked separately.
- **Calculate numerical values.** Examine your symbolic solution and see if it makes sense. Once you are satisfied with the symbolic solution, substitute in the numbers and calculate the numerical answer. It is good practice to identify the source, e.g. table, chart, or book, of all numerical data used in the solution, especially if it is not common knowledge. It is also good practice to calculate intermediate or partial numerical answers when you are faced with a very long computation or complicated equation. This prevents calculator errors from creeping into a problem and gives you an opportunity to check the answers against your physical intuition.
- **Check the reasonableness of your answers.** Once you have a numerical answer, consider the magnitude and sign of all values and decide whether they are reasonable. One way to do this is to compare your answer against the results of a simpler model or models that would be expected to bracket your answer. Try different units for the answer, say gallons per minute instead of liters per second, to match your experience.

As you prepare the analysis, *do not waste time recopying the solution over again if you reach a dead end or make a mistake*. Just cross out the error, clearly identify the mistake, and keep going. Textbook examples and professors' notes give the mistaken impression that problem solving is a linear process that follows a single path with no mistakes and no side trips. Everyone makes mistakes, takes unexpected side trips, and forgets to make an important assumption.

Successful problem solvers acknowledge these diversions and learn from them. You should never start a problem more than once; however, your solution may take several turns before you are satisfied with the answer. The record of your journey is important. Don't "clean up" the solution. Clean up your standard problem solving method because a sloppy solution is usually the result of sloppy thinking. Get in the habit of attacking every problem in the same way. Scrap paper is meant for doodles, not engineering calculations.

COMMENTS: Discuss your results briefly. Comment on what you learned, identify key aspects of the solution, and indicate how your model might be improved by changing assumptions. Consciously check the validity of your answer by considering simpler models. Don't wait for someone else (like your boss or instructor) to find an error in your work by performing a five-minute "back-of-the-envelope" calculation you could have performed before submitting your answer.

DON RICHARDS
BOX 160

HOMEWORK SOLUTIONS

ES201

HW Set #1

DUE DATE: 8 SEPTEMBER 1996

1.14 ________

PROBLEM 1.14 | 9/4/96 | D. RICHARDS — 160 | 1/2

KNOWN: The mass and weight are given for an object.

FIND:
- Local acceleration of gravity in ft/s^2
- Mass in lbm and weight in lbf at a location where $g = 32.2\ ft/s^2$

GIVEN:

$m = 10\ lb_m$

$W = 9.8\ lbf$

ANALYSIS:

Strategy → Because both m & W are given we should be able to use definition of W.

By definition ⟶ $W = mg$

So $$g = \frac{W}{m} = \frac{9.8\ lb_f}{10\ lb_m} = 0.98\ \frac{lb_f}{lb_m}$$

local gravitational field strength

Need to convert units to ft/s^2

acceleration

Approach #1

$$g = 0.98\ \frac{lbf}{lbm}\left[\frac{32.2\ lbm}{slug}\right]$$

$$= 31.6\ \frac{lbf}{slug}\left[\frac{1\ slug \cdot ft/s^2}{1\ lbf}\right]$$

$$= \underline{\underline{31.6\ ft/s^2}}$$

Approach #2

$$g = \left(0.98\ \frac{lbf}{lbm}\right)\left[\frac{32.2\ \frac{lbm \cdot ft}{s^2}}{lbf}\right]$$

$$= \underline{\underline{31.6\ \frac{ft}{s^2}}}$$

local acceleration of gravity $g = 31.6\ \frac{ft}{s^2}$

Now it same object with $g = 32.2\ ft/s$

Mass is unchanged, because mass is independent of gravity.

Solving for weight

Approach #1	Approach #2
$W = mg$	$m = \frac{W_1}{g_1} = \frac{W_2}{g_2}$
$= (10\ lbm)(32.2\ ft/s^2)$	$\therefore W_2 = \frac{g_2}{g_1} W_1$
$= 322\ \frac{lbm \cdot ft}{s^2}\left[\frac{1\ lbf}{32.2\ \frac{lbm \cdot ft}{s^2}}\right]$	$= \left(\frac{32.2\ \frac{ft}{s^2}}{31.6\ \frac{ft}{s^2}}\right) 9.8\ lbf$
$= \underline{\underline{10.0\ lbf}}$	$= \underline{\underline{9.99\ lbf}}$

COMMENT:

- Unit conversions are very important
- Mismatch in last error due to round-off error.
- No assumptions required.

9.2: Appendix B- Dimensions and Units

B.1 Dimensions and Units

Consider the following mathematical statement:

$$L = 3.00 \text{ m}$$

In words you might say, "The length L is 3.00 meters." Implicit in the mathematical expression is an indication of both a physical quantity and its size.

The nature or description of a physical quantity is known as its **dimension**. The size of the physical quantity is expressed as a certain number of standard reference amounts. These reference amounts are known as **units**. In the example above, the dimension is length and the units are meters. (Note that the unit symbol "m" is a mathematical entity, not an abbreviation.) For any given dimension, there are an infinite number of possible units.

B.2 Systems of Units and Dimensions

Experience has shown that there is a set of independent dimensions that can be used to represent all physical quantities. The members of this set are known as the **fundamental (primary) dimensions**. Once the fundamental dimensions have been selected, all other physical quantities are described in terms of **derived (secondary) dimensions**.

Once the fundamental dimensions have been chosen, it is then possible to select their corresponding units - the **base (primary) units**. And as you might expect the derived dimensions have a corresponding set of units — the **derived (secondary) units**.

There are many different systems of units and dimensions based upon the choice of fundamental dimensions. Two of the most common systems of units are the SI system and the American Engineering System. Table B.1 shows the base units for these two systems.

Table B.1 - SI and AES Base Units

Quantity	SI Base Units		AES Base Units	
	Name	**Symbol**	**Name**	**Symbol**
length	meter	m	foot	ft
mass	kilogram	kg	pound-mass	lbm
time	second	s	second	s
electric current	ampere	A	ampere	A
thermodynamic temperature	kelvin	K	Rankine	$^\circ$ R
amount of substance	mole	mol	pound-mole	lbmol
luminous intensity	candela	cd	candela	cd

B.3 Calculations with Dimensions and Units

Units would not provide significant problems if we did not have to use them in calculations. Unfortunately, unit errors are one of an engineer's worst enemies. Although they always seem like trivial mistakes in school, in practice the consequences can be catastrophic.

B.3.1 Dimensional Homogeneity

All theoretically derived equations that describe physical phenomena must be dimensionally homogeneous. An equation is **dimensionally homogeneous** if the dimensions of both sides of the equation are the same and all additive terms have the same dimensions.

B.3.2 Converting Units

The most common errors in using units occur when converting a physical quantity from one set of units to another set. When you convert units you are not changing the size of the physical quantity, only the numerical value associated with the units in which it is measured.

The relationship between two units for the same dimension are typically found in a handbook as an **equivalence relation**, such as $1\ \text{ft} = 12\ \text{in}$. Note again that the unit symbols are mathematical entities and cannot be neglected.

The key to converting units is to recall that multiplying a mathematical expression by unity (1) does not change the magnitude of the mathematical expression. A **unit conversion factor** equals unity and can be constructed from an equivalence relation. Example B.1 shows how to convert equivalence statements into unit conversion factors.

Example B.1

Convert the given equivalence relations into unit conversion factors.

$$\begin{aligned} 1\ \text{ft} = 12\ \text{in} \quad &\Rightarrow \quad 1 = 12\ \frac{\text{in}}{\text{ft}} \\ 1\ \text{slug} = 32.174\ \text{lbm} \quad &\Rightarrow \quad 1 = 32.174\ \frac{\text{lbm}}{\text{slug}} \\ 1\ \text{mol} = 0.001\ \text{kmol} \quad &\Rightarrow \quad 1 = 0.001\ \frac{\text{kmol}}{\text{mol}} \\ 1\text{N} = 1\ \frac{\text{kg}\cdot\text{m}}{\text{s}^2} \quad &\Rightarrow \quad 1 = 1\ \frac{\text{kg}\cdot\text{m}}{\text{s}^2} \end{aligned}$$

The left-hand column shows the equivalence relations from a handbook and the right-hand column shows the resulting unit conversion factors. Notice how it would be mathematically incorrect to just drop the unit symbols.

Example B.2 illustrates how to perform a simple unit conversion for pressure, now that we have the unit conversion factors.

Example B.2

Given a pressure of $13.0\ \text{lbf}/\text{in}^2$, convert the pressure to lbf/ft^2.

$$p = 13.0\ \frac{\text{lbf}}{\text{in}^2} = 13.0\ \frac{\text{lbf}}{\text{in}^2} \times \left(12\ \frac{\text{in}}{\text{ft}}\right)^2 = 1872.0\ \frac{\text{lbf}}{\text{ft}^2}$$

Now convert the pressure value to N/m^2.

$$p = 13.0\ \frac{\text{lbf}}{\text{in}^2} \times \underbrace{\left(4.448\ \frac{\text{N}}{\text{lbf}}\right)}_{1} \times \underbrace{\left(\frac{1\ \text{in}}{0.0254\ \text{m}}\right)^2}_{1} = 89,627\ \frac{\text{N}}{\text{m}^2}$$

If done, correctly the intermediate units should cancel. Check this out by drawing lines through the units that cancel.

Example B.2 is relatively straightforward. Sometimes, however, you are faced with a unit conversion that appears to be both a unit and a dimension conversion. However, it is impossible to convert dimensions. Example B.3 demonstrates this type of unit conversion.

Example B.3

The kinetic energy per unit mass for a baseball can be described by the expression $ke = (1/2)V^2$. Calculate the kinetic energy per unit mass in kilojoules per kilogram if the speed of the baseball is $10\ \text{m}/\text{s}$.

$$ke=\frac{V^2}{2}=\frac{\left(10\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2}{2}=50\ \frac{\mathrm{m}^2}{\mathrm{s}^2}$$

$$=50\frac{\mathrm{m}^2}{\mathrm{s}^2}\times\underbrace{\left[\frac{\left(\frac{\mathrm{kJ}}{\mathrm{kg}}\right)}{\left(\frac{\mathrm{kJ}}{\mathrm{kg}}\right)}\right]}_{1}=\left[50\frac{\mathrm{kJ}}{\mathrm{kg}}\right]\times\left[\frac{\frac{\mathrm{m}^2}{\mathrm{s}^2}}{\frac{\mathrm{kJ}}{\mathrm{kg}}}\right]=50\ \frac{\mathrm{kJ}}{\mathrm{kg}}\times\frac{\left[\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}\right]}{\mathrm{kJ}}$$

$$=50\ \frac{\mathrm{kJ}}{\mathrm{kg}}\times\frac{\left[\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}\right]}{\mathrm{kJ}}\times\underbrace{\left[\frac{\mathrm{kJ}}{1000\ \mathrm{J}}\right]}_{1}\times\underbrace{\left[\frac{\mathrm{J}}{\mathrm{N}\cdot\mathrm{m}}\right]}_{1}\times\underbrace{\left[\frac{\mathrm{N}}{\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}}\right]}_{1}=0.050\ \frac{\mathrm{kJ}}{\mathrm{kg}}$$

The procedure here was to multiply the calculated quantity in the second line by a unit conversion factor formed from the desired final units. Then it was just a matter of massaging the units until we eliminated all but the desired combinations. If it had not been possible to achieve this, then the proposed conversion is really a conversion of dimensions and that is impossible.

B.4 Handling Units in Equations

Unit errors can also occur within equations as you substitute numerical values into symbolic equations. To prevent this type of error you should follow three steps:

1. always write down the units with a number when you substitute in a numerical value,
2. write down your unit conversion factors and show the algebra that cancels out the units, and
3. show the unit conversions as a separate step.

The last step is not always necessary, but it is always a good idea when the conversions are complicated. In addition, it is a useful step for novices. This approach is demonstrated in Example B.4.

Example B.4

A tank contains $15\ \mathrm{mol}$ of an ideal gas. The pressure in the tank is $1500\ \mathrm{kPa}$ and the volume of the tank is $10\ \mathrm{m}^3$. The ideal gas constant is $8.314\ \mathrm{kJ}/(\mathrm{kmol}\cdot\mathrm{K})$. Determine the temperature of the gas in the tank.

The ideal gas equation is $pV=n\bar{R}T$.

We solve for $T=\frac{pV}{n\bar{R}}$

$$\therefore T=\frac{pV}{n\bar{R}}=\frac{(1500\ \mathrm{kPa})\times(10\ \mathrm{m}^3)}{(15\ \mathrm{kmol})\times\left(8.314\ \frac{\mathrm{kJ}}{\mathrm{kmol}\cdot\mathrm{K}}\right)}=\frac{15000\ \mathrm{kPa}\cdot\mathrm{m}^3}{124.71\ \frac{\mathrm{kJ}}{\mathrm{K}}}\times\underbrace{\left[\frac{\mathrm{kJ}}{\mathrm{kPa}\cdot\mathrm{m}^3}\right]}_{1}=120.3\ \mathrm{K}$$

Again, the check is to see if the appropriate units cancel out.

B.5 Weight and Mass

People frequently confuse the terms weight and mass. The weight of an object is the force exerted by the earth's gravitational field on the object. Mathematically, $W=mg$, where m is the mass of the object and g is the local gravitational field strength. The local gravitational field strength is also referred to as the local acceleration of gravity.

Standard values for the local gravitational field strength are

$$g=9.80665\ \mathrm{N}/\mathrm{kg}=32.174\ \mathrm{lbf}/\mathrm{slug}=1.000\ \mathrm{lbf}/\mathrm{lbm}.$$

Standard values for the local acceleration of gravity are

$$g=9.80665\ \mathrm{m}/\mathrm{s}^2=32.174\mathrm{ft}/\mathrm{s}^2.$$

TEST YOURSELF: Why do these two interpretations for g come up with similar numbers but different units?

Much of the confusion about mass and weight can be directly attributed to the fact that the mass and force units in the American Engineering System are both called "pounds." To eliminate this problem, it is highly recommended that you only talk about pound-force (lbf) or a pound-mass (lbm). You would never confuse a newton with a kilogram, but then they have different names. Unfortunately, you will still find "pound" and "lb" used frequently to mean both mass and weight. Always approach "pounds" with caution when doing calculations. Remember that the weight of an object is always a function of the local gravitational field strength, but its mass is independent of the gravitational field.

B.6 Problems

? Problem $B.1$

Calculate the magnitude of the physical quantities in the indicated units. Show all of your work, i.e. explicitly show the use of unit conversion factors and how the units cancel out. If it is impossible to make the conversions indicated, please state the reasons why you believe this to be the case. [Hint: It is frequently necessary to break a secondary unit down into its primary units or other secondary units in the process of converting to the desired secondary units. See Example B.3 in Appendix B of the ES201 notes.]

(a) Pressure: $100\ \mathrm{lbf/in^2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{N/cm^2} = \mathrm{bar}$

b) Energy per unit mass: $2000\ \mathrm{Btu/slug} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{ft \cdot lbf/lbm} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{kJ/kg}$

c) Product of pressure and volume: ["psi" equals "pounds-force per square inch," $\mathrm{lbf/in^2}$]

$3000\ \mathrm{psi \cdot ft} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{ft \cdot lbf} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{N/kg}$

? Problem $B.2$

On one of the moon landings, astronauts collected moon rocks for study back on earth. The weight of the rocks measured on the moon was $50\ \mathrm{lbf}$. The strength of the gravity on the moon is $1/6$ the value on earth. (Many times this is stated as "The acceleration of gravity on the moon is $1/6$ the value on earth.")

(a) Determine the mass of the rocks collected on the moon, in lbm and in slugs.

(b) Determine the mass and the weight of the rocks back on earth, in lbm and lbf, respectively.

(c) A newspaper report from a later moon landing reports that the astronauts collected "200 pounds of rocks." What, if any, additional information would you need in order to actually specify the amount of rocks they brought back? (Be careful of making *implicit* assumptions.)

? Problem $B.3$

The units for a physical quantity can often seem to be at odds with the description used for them. A good example is the common units for "energy per unit mass." Showing all the steps, develop and verify the following conversion factor: $1\ \mathrm{ft \cdot lbf/lbm} = 32.174\ \mathrm{ft^2/s^2}$.

Note that both $\mathrm{ft \cdot lbf/lbm}$ and $\mathrm{ft^2/s^2}$ are correct units for $[\mathrm{energy}]/[\mathrm{mass}]$.

? Problem $B.4$

Calculate the magnitude of the physical quantities in the indicated units. Show all of your work. [Do not just look up one single unit conversion factor in a table.] If it is impossible to make the conversions indicated, please state the reasons why you believe this to be the case.

a) Dynamic Viscosity: $15\ \mathrm{kg/(m \cdot s)} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{Pa \cdot s} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{slug/ft \cdot s}$

b) Pressure: $100\ \mathrm{lbf/in^2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{lbf/ft^2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{bar}$

c) Energy per unit mass: $2000\ \mathrm{ft \cdot lbf/slug} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{ft^2/s^2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ \mathrm{kJ/kg}$

d) Product of pressure and specific volume:

$3000\ \mathrm{bar} \cdot \mathrm{m}^3/\mathrm{kg} =$ _______________ $\mathrm{kJ}/\mathrm{kg} =$ _________________ Btu/s

Problem $B.5$

The following unit equivalence factors for moment of inertia were copied from a standard controls textbook: \[1 \mathrm{~lb} \cdot \mathrm{in} \cdot \mathrm{s}^{2}=386 \mathrm{~lb} \cdot \mathrm{in}^{2} \quad \text{ and } \quad 1 \mathrm{~g} \cdot \mathrm{cm} \cdot \mathrm{s}^{2}=980

\mathrm{~g} \cdot \mathrm{cm}^{2} \nonumber \]

A quick examination seems to indicate some strange results that are dimensionally inconsistent, e.g. $1\ \mathrm{in} \cdot \mathrm{s}^2 = 386$\(in^2 and $1\ \mathrm{cm} \cdot \mathrm{s}^2 = 980\ \mathrm{cm}^2$. How can this be? What if the author had distinguished between lbf (pound-force) and lbm (pound-mass) and between $\mathrm{g_f}$ (gram-force) and $\mathrm{g_m}$ (gram-mass)? Would that make the expressions above dimensionally correct? Please explain.

Problem $B.6$

A moon landing craft has a mass of $5000\ \mathrm{lbm}$ on the surface of the earth.

(a) Determine the following information when the object is on the surface of the earth:

- weight of the object in lbf
- mass of the object mass in slugs

(b) Determine the following information for the object when it sits on the surface of the moon if the strength of gravity on the moon is $1/6$ of the value on earth:

- weight of the object in lbf
- mass of the object in lbm and slugs

(c) An infrequently used unit of force, the poundal, is defined by the following expression:

$$1\ \text{poundal} = 1\ \mathrm{lbm} \cdot \mathrm{ft}/\mathrm{s}^2$$

Determine the weight of the object in poundals when it is on the surface of the earth and the surface of the moon.

Problem $B.7$

A moon landing craft has a mass of $5000\ \mathrm{kg}$ on the surface of the earth.

(a) Determine the following information when the object is on the surface of the earth:

- weight of the object in newtons
- mass of the object mass in kilograms.

(b) Determine the following information for the object when it sits on the surface of the moon if the strength of gravity on the moon is $1/6$ of the value on earth:

- weight of the object in newtons
- mass of the object in kilograms.

(c) Although it is not a standard unit of measure, you will sometimes find forces expressed in terms of kilogram-force, e.g. $1\ \mathrm{kgf} = 9.81\ \mathrm{kg} \cdot \mathrm{m}/\mathrm{s}^2$. Determine the weight of the object in kgf when it is on the surface of the earth and the surface of the moon.

Problem $B.8$

Engineers often define units which make their life easier and simplify calculations. Calculate the magnitude of the physical quantities in the indicated units. Consult a good engineering handbook to find the units of terms with which you are unfamiliar.

Show all of your work. [Do not just look up one single unit conversion factor in a table.] If it is impossible to make the conversions indicated, please state the reasons why you believe this to be the case.

(a) Unit of mass: 1 blob $= 1\ \mathrm{lbf}/\left(\mathrm{in/s^2}\right)$

100 blobs $=$ ________________ lbm $=$ ________________ kg

(b) Units of volume: acre-foot

10 acre-foot $=$ ________________ $\mathrm{ft}^3 =$ ________________ gal

(c) Unit of area: circular mils

1000 circular mils $=$ ________________ $\mathrm{in}^2 =$ ________________ mm^2

(d) Unit of electrical resistivity: microhms-cm

1000 microhms-cm $=$ ________________ ohms-in $=$ ________________ volt-cm/amp

(e) Unit of electrical inductance: 1 henry $= 1$volt-s/amp

100 henrys $=$ __________________ $\mathrm{joule/amp^2} =$ __________________ $\mathrm{hp\text{-}s^3/coulomb^2}$

Problem $B.9$

Units of energy and power frequently occur in many different forms. Calculate the magnitude of the physical quantities in the indicated units. Show all of your work. [Do not just look up one single unit conversion factor in a table.] If it is impossible to make the conversions indicated, please state the reasons why you believe this to be the case.

(a) 100 hp $=$ ________________ kW $=$ ________________ J/s

(b) 100 kW-h $=$ ________________ hp-s $=$ ________________ J

(c) 1000 lbf-in^3 $=$ ________________ bar-cm^2 $=$ ________________ ft-lbf

(d) 1000 J $=$ ________________ N-m $=$ ________________ Btu-h

(e) 1000 Btu $=$ ________________ J-ft $=$ ________________ hp-h

9.3: Appendix C- Summary of Conservation and Accounting Equations, Unit Conversions, Property Models, Thermophysical Property Data

C.1: Basic Conservation & Accounting Equations

Basic Conservation & Accounting Equations

Accounting Equation for Generic, Extensive Property B:

Extensive Property ----- $B_{sys}(t) = \iiint\limits_{V_{sys}} b_{(x,y,z,t)}\,\rho_{(x,y,z,t)}\,dV$

Accounting Equation ----- $\dfrac{dB_{sys}(t)}{dt} = \dot{B}_{in} - \dot{B}_{out} + \dot{B}_{gen} - \dot{B}_{cons}$

$$\underbrace{\frac{dB_{sys}}{dt}}_{\text{Accumulation}} = \underbrace{\underbrace{\left\{\dot{B}_{in} - \dot{B}_{out}\right\}}_{\text{non-flow boundaries}} + \underbrace{\left\{\sum_{in}\dot{m}_i b_i - \sum_{out}\dot{m}_e b_e\right\}}_{\text{flow boundaries}}}_{\text{Transport}} + \underbrace{\left\{\dot{B}_{\text{gen}} - \dot{B}_{\text{cons}}\right\}}_{\text{Generation/Consumption}}$$

Conservation of Linear Momentum: $\mathbf{P}_{sys} = \int\limits_{V_{sys}} \mathbf{V}\rho\, dV$

$$\frac{d\mathbf{P}_{sys}}{dt} = \sum_j \mathbf{F}_{\text{ext},\,j} + \left\{\sum_{in}\dot{m}_i\mathbf{V}_i - \sum_{out}\dot{m}_e\mathbf{V}_e\right\}$$

Conservation of Mass: $m_{sys}(t) = \iiint\limits_{V_{sys}} \rho_{(x,y,z,t)}\; dV$

$$\frac{dm_{sys}}{dt} = \sum_{in}\dot{m}_i - \sum_{out}\dot{m}_e$$

$$\text{where } \dot{m} = \int\limits_{A_C} \rho V_n\, dA_C = \underbrace{\rho A_C V_{\text{avg}}}_{\text{1-D Flow Assumption}} \qquad \text{(the mass flow rate)}$$

Conservation of Angular Momentum: $\mathbf{L}_{o,\,sys} = \int\limits_{V_{sys}} (\mathbf{r}\times\mathbf{V})\rho\, dV$

$$\frac{d\mathbf{L}_{o,\,sys}}{dt} = \sum_j \mathbf{M}_{o,\,j} + \left\{\sum_{in}\dot{m}_i\left(\mathbf{r}_i\times\mathbf{V}_i\right) - \sum_{out}\dot{m}_e\left(\mathbf{r}_e\times\mathbf{V}_e\right)\right\}$$

$$\text{where } \mathbf{M}_{o,j} = \mathbf{r}_j\times\mathbf{F}_j \quad \text{or} \quad M_{\text{couple},\,j}$$

Chemical Species Accounting: $m_j = n_j M_j$

$$\text{mass} \quad \rightarrow \quad \frac{dm_{j,\,sys}}{dt} = \sum_{in}\dot{m}_{j,i} - \sum_{out}\dot{m}_{j,e} + \left(\dot{m}_{j,\,gen} - \dot{m}_{j,\,cons}\right)$$

$$\text{molar} \quad \rightarrow \quad \frac{dn_{j,\,sys}}{dt} = \sum_{in}\dot{n}_{j,\,i} - \sum_{out}\dot{n}_{j,\,e} + \left(\dot{n}_{j,\,gen} - \dot{n}_{j,\,cons}\right)$$

Conservation of Energy:

$$E_{sys} = \int\limits_{V_{sys}} e\rho\, dV \quad \text{where} \quad e = u + \frac{V^2}{2} + gz + e_{\text{spring}} + \ldots$$

$$\frac{dE_{sys}}{dt} = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \left\{\sum_{in}\dot{m}_i\left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{out}\dot{m}_e\left(h_e + \frac{V_e^2}{2} + \right.\right.$$

Conservation of Charge: $q_{sys} = \int\limits_{V_{sys}} \tilde{q}\rho\, dV$

$$\frac{dq_{sys}}{dt} = \sum_{in}\dot{q}_i - \sum_{out}\dot{q}_e$$

Entropy Accounting: $S_{sys} = \int\limits_{V_{sys}} s\rho\, dV$ and $S_{gen} \geq 0$

$$\frac{dS_{sys}}{dt} = \sum_j \frac{\dot{Q}_j}{T_{b,\,j}} + \left\{\sum_{in}\dot{m}_i s_i - \sum_{out}\dot{m}_e s_e\right\} + \dot{S}_{gen}$$

C.2 Unit Conversions

Unit Conversions

Length

$$\begin{aligned} &1 \text{ ft} = 12 \text{ in} = 0.3048 \text{ m} = 1/3 \text{ yd} \\ &1 \text{ m} = 100 \text{ cm} = 1000 \text{ mm} = 39.37 \text{ in} = 3.2808 \text{ ft} \\ &1 \text{ mile} = 5280 \text{ ft} = 1609.3 \text{ m} \end{aligned}$$

Force

$$\begin{aligned} &1 \text{ N} = 1 \text{ kg}\cdot\text{m/s}^2 = 0.22481 \text{ lbf} \\ &1 \text{ lbf} = 1 \text{ slug}\cdot\text{ft/s}^2 = 32.174 \text{ lbm}\cdot\text{ft/s}^2 = 4.4482 \text{ N} \end{aligned}$$

Mass

$$\begin{aligned} &1 \text{ kg} = 1000 \text{ g} = 2.2046 \text{ lbm} \\ &1 \text{ lbm} = 16 \text{ oz} = 0.45359 \text{ kg} \\ &1 \text{ slug} = 32.174 \text{ lbm} \end{aligned}$$

Pressure

$$\begin{aligned} &1 \text{ atm} = 101.325 \text{ kPa} = 1.01325 \text{ bar} = 14.696 \text{ lbf/in}^2 \\ &1 \text{ bar} = 100 \text{ kPa} = 10^5 \text{ Pa} \\ &1 \text{ Pa} = 1 \text{ N/m}^2 = 10^{-3} \text{ kPa} \\ &1 \text{ lbf/in}^2 = 6.8947 \text{ kPa} = 6894.7 \text{ N/m}^2 \\ &\quad \left[\text{lbf/in}^2 \text{ often abbreviated as “psi”}\right] \end{aligned}$$

Unit Conversions	
Temperature Values $(\mathrm{T/K}) = (\mathrm{T/^\circ R})/1.8$ $(\mathrm{T/K}) = (\mathrm{T/^\circ C}) + 273.15$ $(\mathrm{T/^\circ C}) = [(\mathrm{T/^\circ F}) - 32]/1.8$ $(\mathrm{T/R}) = 1.8(\mathrm{T/K})$ $(\mathrm{T/^\circ R}) = (\mathrm{T/^\circ F}) + 459.67$ $(\mathrm{T/^\circ F}) = 1.8(\mathrm{T/^\circ C}) + 32$	**Energy** $1\ \mathrm{J} = 1\ \mathrm{N \cdot m}$ $1\ \mathrm{kJ} = 1000\ \mathrm{J} = 737.56\ \mathrm{ft \cdot lbf} = 0.94782\ \mathrm{Btu}$ $1\ \mathrm{Btu} = 1.0551\ \mathrm{kJ} = 778.17\ \mathrm{ft \cdot lbf}$ $1\ \mathrm{ft \cdot lbf} = 1.3558\ \mathrm{J}$
Temperature Differences $(\Delta\mathrm{T/^\circ R}) = 1.8(\Delta\mathrm{T/K})$ $(\Delta\mathrm{T/^\circ R}) = (\Delta\mathrm{T/^\circ F})$ $(\Delta\mathrm{T/K}) = (\Delta\mathrm{T/^\circ C})$	**Energy Transfer Rate** $1\ \mathrm{kW} = 1\ \mathrm{kJ/s} = 737.56\ \mathrm{ft \cdot lbf/s} = 1.3410\ \mathrm{hp} = 0.94782\ \mathrm{Btu/s}$ $1\ \mathrm{Btu/s} = 1.0551\ \mathrm{kW} = 1.4149\ \mathrm{hp} = 778.17\ \mathrm{ft \cdot lbf/s}$ $1\ \mathrm{hp} = 550\ \mathrm{ft \cdot lbf/s} = 0.74571\ \mathrm{kW} = 0.70679\ \mathrm{Btu/s}$
Volume $1\ \mathrm{m^3} = 1000\ \mathrm{L} = 10^6\ \mathrm{cm^3} = 10^6\ \mathrm{mL} = 35.315\ \mathrm{ft^3} = 264.17\ \mathrm{gal}$ $1\ \mathrm{ft^3} = 1728\ \mathrm{in^3} = 7.4805\ \mathrm{gal} = 0.028317\ \mathrm{m^3}$ $1\ \mathrm{gal} = 0.13368\ \mathrm{ft^3} = 0.0037854\ \mathrm{m^3}$	**Specific Energy** $1\ \mathrm{kJ/kg} = 1000\ \mathrm{m^2/s^2}$ $1\ \mathrm{Btu/lbm} = 25037\ \mathrm{ft^2/s^2}$ $1\ \mathrm{ft \cdot lbf/lbm} = 32.174\ \mathrm{ft^2/s^2}$
Volumetric Flow Rate $1\ \mathrm{m^3/s} = 35.315\ \mathrm{ft^3/s} = 264.17\ \mathrm{gal/s}$ $1\ \mathrm{ft^3/s} = 1.6990\ \mathrm{m^3/min} = 7.4805\ \mathrm{gal/s} = 448.83\ \mathrm{gal/\ min}$	

C.3 Substance Models

Two Substance Models (Constitutive Relations)		
	Equation of State	
	Ideal Gas Model with room-temperature specific heats	**Incompressible Substance Model** with room temperature specific heats
Used to model behavior of	gases and vapors	liquids and solids
Basic Model Assumptions	1. Pressure, volume, and temperature obey the ideal gas relation: $PV = NR_uT$ 2. Specific internal energy depends only on temperature, $u = u(T)$. 3. Molar mass of an ideal gas equals the molar mass of the real substance: $M_{\text{ideal gas}} = M_{\text{real stuff}}$ 4. The specific heats are independent of temperature, i.e. they are constants.	1. The density of the substance is a constant. 2. Specific internal energy depends only on temperature, $u = u(T)$. 3. Molar mass of an incompressible substance equals the molar mass of the real substance: $M_{\text{incomp substance}} = M_{\text{real stuff}}$ 4. The specific heats are independent of temperature, i.e. they are constants.
$P - T - \rho$ and $P - T - v$ relations	$P = \rho RT$ and $Pv = RT$ where $R = R_u/M$	$v = 1/\rho = \text{constant}$ Evaluated at room temperature
Specific heat relations	$c_p - c_v = R; \quad k = c_p/c_v$	$c_p = c_v = c$, a constant
c_p and c_v values	Evaluated at room temperature	Evaluated at room temperature
Δu — specific internal energy	$\Delta u = u_2 - u_1 = c_v(T_2 - T_1)$	$\Delta u = u_2 - u_1 = c(T_2 - T_1)$
Δh — specific enthalpy	$\Delta h = h_2 - h_1 = c_p(T_2 - T_1)$	$\Delta h = h_2 - h_1$ $= (u_2 + P_2v) - (u_1 + P_1v)$ $= (u_2 - u_1) + v(P_2 - P_1)$ thus $\Delta h = \Delta u + v\Delta P = c\Delta T + v\Delta P$
Δs — specific entropy *Note:* All temperatures are absolute values, i.e. K or °R, in the entropy relations	$\Delta s = s_2 - s_1$ $= c_p \ln(T_2 - T_1) - R\ln(P_2 - P_1)$ $= c_v \ln(T_2/T_1) + R\ln(v_2 - v_1)$	$\Delta s = s_2 - s_1$ $= c\ln(T_2/T_1)$

Ideal Gas Equation	
Molar Basis	**Mass Basis**
$PV = nRT$ $P\bar{v} = R_u T \quad \text{and} \quad P = \bar{\rho} R_u T$ where P = absolute pressure of gas $[\text{kPa or lbf/ft}^2]$ V = volume of gas $[\text{m}^3 \text{ or ft}^3]$ n = number of moles of gas [kmol or lbmol] R_u = universal gas constant (the same for every gas) $[\text{kJ}/(\text{kmol}\cdot\text{K}) \text{ or } (\text{ft}\cdot\text{lbf})/(\text{lbmol}\cdot{}^\circ\text{R})]$ T = absolute temperature of gas [K or °R] $\bar{\rho}$ = molar density $= 1/\bar{v}$ $[\text{kmol/m}^3 \text{ or lbmol/ft}^3]$ $\bar{v}$ = molar specific volume $[\text{m}^3/\text{kmol or ft}^3/\text{lbmol}]$	$PV = mRT$ $Pv = RT \quad \text{and} \quad P = \rho RT$ where P = absolute pressure of gas $[\text{kPa or lbf/ft}^2]$ V = volume of gas $[\text{m}^3 \text{ or ft}^3]$ m = mass of gas [kg or lbm] R = specific gas constant (different for each gas) $[\text{kJ}/(\text{kg}\cdot\text{K}) \text{ or } (\text{ft}\cdot\text{lbf})/(\text{lbmol}\cdot{}^\circ\text{R})]$ T = absolute temperature of gas [K or °R] ρ = density $= 1/v$ $[\text{kg/m}^3 \text{ or lbm/ft}^3]$ v = specific volume $[\text{m}^3/\text{kg or ft}^3/\text{lbm}]$
and $R_u = 8.314 \dfrac{\text{kJ}}{\text{kmol}\cdot\text{K}} = 8.314 \dfrac{\text{J}}{\text{mol}\cdot\text{K}} = 1545 \dfrac{\text{ft}\cdot\text{lbf}}{\text{lbmol}\cdot{}^\circ\text{R}}$	and $R = \dfrac{R_u}{M}$ where M = molecular weight (molar mass) of a specific gas

Thermophysical Property Data for Some Common Substances (SI Units)								
Gases (at 25°C and 1 atm)								
Substance		Molar Mass	$\dfrac{R}{\left[\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right]}$	$\dfrac{c_v}{\left[\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right]}$	$\dfrac{c_p}{\left[\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right]}$	k	$\dfrac{T_c}{\text{K}}$	$\dfrac{P_c}{\text{bar}}$
Acetylene	C_2H_2	26.04	0.3193	1.37	1.69	1.23	309	62.4
Air	--	28.97	0.2870	0.718	1.005	1.40	133	37.7
Ammonia	NH_3	17.04	0.4879	1.66	2.15	1.30	406	112.8
Carbon dioxide	CO_2	44.01	0.1889	0.657	0.846	1.29	304.2	73.9
Carbon monoxide	CO	28.01	0.2968	0.744	1.04	1.40	133	35.0
Ethane	C_2H_6	30.07	0.2765	1.48	1.75	1.18	305.4	48.8
Ethylene	C_2H_4	28.05	0.2964	1.23	1.53	1.24	283	51.2
Helium	He	4.003	2.077	3.12	5.19	1.67	5.2	2.3
Hydrogen	H_2	2.016	4.124	10.2	14.3	1.40	33.2	13.0
Methane	CH_4	16.04	0.5183	1.70	2.22	1.31	190.7	46.4
Nitrogen	N_2	28.01	0.2968	0.743	1.04	1.40	126.2	33.9
Oxygen	O_2	32.00	0.2598	0.658	0.918	1.40	\(154.4\	50.5
Propane	C_3H_8	44.09	0.1886	1.48	1.67	1.13	370	42.5
Refrigerant 134a	$C_2F_2H_2$	102.03	0.08149	0.76	0.85	1.12	374.3	40.6
Water (Steam)	H_2O	18.02	0.4614	1.40	1.86	1.33	647.3	220.9

Liquids				**Solids***		
Substance	Temp. (°C)	$\dfrac{\rho}{\left[\frac{\text{kg}}{\text{m}^3}\right]}$	$\dfrac{c_p}{\left[\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right]}$	Substance	$\dfrac{\rho}{\left[\frac{\text{kg}}{\text{m}^3}\right]}$	$\dfrac{c_p}{\left[\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right]}$
Ammonia	25	602	4.80	Aluminum	2,700	0.902
Benzene	20	879	1.72	Brass, yellow	8,310	0.400
Brine (20%NaCl)	20	1,150	3.11	Brick (common)	1,922	0.79

Ethanol	25	783	2.46	Concrete	2,300	0.653
Ethyl Alcohol	20	789	2.84	Copper	8,900	0.386
Ethylene Glycol	20	1,109	2.84	Glass, window	2,700	0.800
Kerosene	20	820	2.00	Iron	7,840	0.45
Mercury	25	13,560	0.139	Lead	11,310	0.128
Oil (light)	25	910	1.80	Silver	10,470	0.235
Refrigerant 134a	25	1,206	1.42	Steel (mild)	7,830	0.500
Water	25	997	4.18	* Evaluated at room temperature.		
Values adapted from K. Wark, Jr. and D. E. Richards, *Thermodynamics*, 6th ed. (McGraw-Hill, New York, 1999) and Y. A. Cengul and M. A. Boles, *Thermodynamics*, 4th ed. (McGraw-Hill, New York, 2002).						

Thermophysical Property Data for Some Common Substances (USCS Units)								
Gases (at 77°F and 1 atm)								
Substance		Molar Mass	$\frac{R}{\left[\frac{\mathrm{ft\cdot lbf}}{\mathrm{lbm\cdot{}^\circ R}}\right]}$	$\frac{c_v}{\left[\frac{\mathrm{Btu}}{\mathrm{lbm\cdot{}^\circ R}}\right]}$	$\frac{c_p}{\left[\frac{\mathrm{Btu}}{\mathrm{lbm\cdot{}^\circ R}}\right]}$	k	$\frac{T_c}{{}^\circ\mathrm{R}}$	$\frac{P_c}{\mathrm{atm}}$
Acetylene	C_2H_2	26.04	59.33	0.328	0.404	1.23	556	61.6
Air	--	28.97	53.33	0.171	0.240	1.40	239	37.2
Ammonia	NH_3	17.04	90.67	0.397	0.514	1.29	730	111.3
Carbon dioxide	CO_2	44.01	35.11	0.156	0.202	1.29	548	72.9
Carbon monoxide	CO	28.01	55.16	0.178	0.249	1.40	239	34.5
Ethane	C_2H_6	30.07	51.38	0.353	0.419	1.19	549	48.2
Ethylene	C_2H_4	28.05	55.08	0.294	0.365	1.24	510	50.5
Helium	He	4.003	386.0	0.744	1.24	1.67	9.3	2.26
Hydrogen	H_2	2.016	766.4	2.43	3.42	1.40	59.8	12.8
Methane	CH_4	16.04	96.32	0.407	0.531	1.30	344	45.8
Nitrogen	N_2	28.01	55.16	0.178	0.248	1.39	227	33.5
Oxygen	O_2	32.00	48.28	0.157	0.219	1.40	278	49.8
Propane	C_3H_8	44.09	35.04	0.355	0.400	1.13	666	42.1
Refrigerant 134a	$C_2F_4H_2$	102.03	15.14	0.184	0.203	1.10	672.8	40.1
Water (Steam)	H_2O	18.02	87.74	0.335	0.445	1.33	1165	218.0

Liquids				**Solids***		
Substance	Temp (°F)	$\frac{\rho}{\left[\frac{\mathrm{lbm}}{\mathrm{ft^3}}\right]}$	$\frac{c_p}{\left[\frac{\mathrm{Btu}}{\mathrm{lbm\cdot{}^\circ R}}\right]}$	Substance	$\frac{\rho}{\left[\frac{\mathrm{lbm}}{\mathrm{ft^3}}\right]}$	$\frac{c_p}{\left[\frac{\mathrm{Btu}}{\mathrm{lbm\cdot{}^\circ R}}\right]}$
Ammonia	80	37.5	1.135	Aluminum	170	0.215
Benzene	68	54.9	0.411	Brass, yellow	519	0.0955
Brine (20%NaCl)	68	71.8	0.743	Brick (common)	120	0.189
Ethanol	77	48.9	0.588	Concrete	144	0.156
Ethyl Alcohol	68	49.3	0.678	Copper	555	0.0917
Ethylene Glycol	68	69.2	0.678	Glass, window	169	0.191
Kerosene	68	51.2	0.478	Iron	490	0.107
Mercury	77	847	0.033	Lead	705	0.030
Oil (light)	77	56.8	0.430	Silver	655	0.056

Refrigerant 134a	32	80.9	0.318	Steel (mild)	489	0.119
Water	68	62.2	1.00	* Evaluated at room temperature.		
Values adapted from K. Wark, Jr. and D. E. Richards, *Thermodynamics*, 6th ed. (McGraw-Hill, New York, 1999) and Y. A. Cengul and M. A. Boles, *Thermodynamics*, 4th ed. (McGraw-Hill, New York, 2002).						

9.4: Supplementary Materials- Course Learning Objectives

Introduction, Basic Concepts, and Appendices A and B

1. Define, illustrate, compare and contrast the following terms and concepts:
    - engineering analysis vs. engineering design
    - algorithm vs. heuristic
    - model (as discussed in the notes)
    - system
        - surroundings
        - boundary (control surface)
        - closed system (control mass) vs. open system (control volume)
        - interactions between a system and its surroundings
        - isolated system
    - property
        - extensive vs. intensive
        - necessary and sufficient test for a property
    - state
    - process
        - cycle
        - steady-state system
            - finite-time process
            - transient
    - units and dimensions
        - primary vs. secondary dimensions
        - base units and derived units
        - unit conversion factor
    - weight and mass
        - molar mass (molecular weight)
        - amount of substance — mole (mol, kmol, lbmol, slugmol, etc.)
        - local gravitational field strength (standard values)
            - $g = 9.80665\ \mathrm{N/kg} = 1.0000\ \mathrm{lb_f/lb_m} = 32.174\ \mathrm{lb_f/slug}$
            - relationship to local gravitational acceleration: $g = 9.80665\ \mathrm{m/s^2} = 32.174\ \mathrm{ft/s^2}$
        - slug vs. pound-mass ($\mathrm{lb_m}$ or lbm) vs pound-force ($\mathrm{lb_f}$ or lbf)
    - continuum hypothesis
        - macroscopic vs. microscopic viewpoint
    - accounting concept
        - basic components
            - accumulation within the system
            - transport across system boundary
            - generation/consumption (production/destruction) within the system
    - finite-time vs. rate form
        - rate of accumulation (“rate of change”) vs. accumulation (“change in”)
        - transport rate vs. amount transported
        - generation (consumption) rate vs. amount generated (consumed)
            - conserved property vs. non-conserved property
    - conservation laws vs. accounting statements (balances)

2. Given a sufficient set of unit conversion factors, convert the numerical value of a physical quantity from one set of units to a different, specified set of units.
3. Explain in words the difference between the mass of an object and its weight. Demonstrate this understanding by applying the defining equation for weight, $W = mg$, and Newton's second law, $F = ma$ to solve problems involving weight, mass, and acceleration. Answers for mass, weigh, and acceleration must be given with appropriate and standard units.
4. List the seven components of the problem solving format (methodology for engineering problem solving), explain the significance of each part, and use the format correctly in your problem solutions.
5. Given a problem that can be solved by accounting for physical quantities or, if you are requested to use the accounting principle, apply the accounting principle to solve for the desired information. Be sure to clearly indicate the system of interest, the property (or stuff) to be counted, and the time period of interest. Problems should be worked showing sufficient steps so that the method used is clear.
6. Give both a written and a symbolic statement of the *rate form* of the accounting principle and the *finite-time form* of the accounting principle. Clearly indicate the accumulation, transport, and generation terms.
7. Explain the mathematical and the physical difference between the "rate of accumulation (or change)" term and the "transport rate" and "generation/consumption rate" terms in the rate form of the accounting principle. For example, what happens when you integrate a rate of change term as compared to integrating a transport term, e.g. dm_{sys}/dt vs. $\dot{m}$.
8. Given a list of physical quantities, determine which of them are properties and indicate whether they are intensive or extensive properties. Explain how you made your decisions.

Chapter 3 — Conservation of Mass and Chemical Species Accounting

1. Define, illustrate, compare and contrast the following terms and concepts:
   - Mass
     - mass vs. weight
     - units of measurement
       - mass: m $(\mathrm{kg}, \mathrm{g}, \mathrm{lbm}, \mathrm{slug})$
       - amount of substance: n $(\mathrm{kmol}, \mathrm{gmol}, \mathrm{lbmol}, \mathrm{slugmol})$; (most useful when chemical reactions are involved)
     - molecular weight (molecular mass): M $(\mathrm{kg}/\mathrm{kmol},\ \mathrm{g}/\mathrm{gmol},\ \mathrm{lbm}/\mathrm{lbmol})$
       - relationship between m and n
       - density & specific volume (how are they related?)
       - specific weight vs. specific gravity (how are they different?)
   - Application of Accounting Principle for Mass
     - rate of accumulation of mass within the system
     - amount of mass within the system: $\boxed{m_{sys} = \iiint\limits_{V} \rho \, dV}$
       - density vs. specific volume (ρ vs v)
     - transport rate of mass across system boundaries
       - mass flow rate: $\boxed{\dot{m} = \iint\limits_{A_{c}} \rho V_{n} \ dA} \quad (\mathrm{kg} / \mathrm{s}, \mathrm{~lbm} / \mathrm{s})$
       - volumetric flow rate: $\dot{V} = \iint\limits_{A_c} V_n \, dA$ where $A_c =$ cross-sectional area $\left(\mathrm{m}^3\ \mathrm{s},\ \mathrm{ft}^3/\mathrm{s}\right)$
       - molar flow rate : $\dot{n} = \dot{m}/M \quad (\mathrm{kmol}/\mathrm{s},\ \mathrm{mol}/\mathrm{s},\ \mathrm{lbmol}/\mathrm{s})$
       - local normal velocity: V_n
       - one-dimensional flow assumption
       - average velocity at a cross-section: V_{AVG}
         - mass and volumetric flow rate based on average velocity:

$$\dot{m} = \rho A_c V_{AVG} \quad \& \quad \dot{V} = A_c \mathrm{V}_{AVG}$$

     - generation/consumption rate of mass within the system
       - Empirical result ..... Mass is conserved! It's really a conservation principle!

- Conservation of mass equation:

$$\begin{array}{ll}\textbf{rate form} & \frac{dm_{sys}}{dt} = \sum \dot{m}_{\text{in}} - \sum \dot{m}_{\text{out}} \\ \textbf{finite-time form} & m_{\text{sys, final}} - m_{\text{sys, initial}} = \sum m_{\text{in}} - \sum m_{\text{out}}\end{array}$$

- Chemical Species (Compounds)
  - units of measurement --- same as for mass
    - $m_i =$ mass of component i
    - $n_i =$ moles of component i; $\quad n_i = m_i / M_i$
  - mixture composition

$$\text{moles of mixture:} \quad \boxed{n = \sum_{i=1}^{N} n_i} \quad \text{where } N = \text{number of components in the mixture}$$

$$\text{mole fractions:} \quad nf_i = \frac{n_i}{n_{\text{mix}}} \text{ and } \boxed{\sum_{i=1}^{N} nf_i = 1}$$

$$\text{mass of mixture:} \quad \boxed{m = \sum_{i=1}^{N} m_i}$$

$$\text{mass (weight) fractions:} \quad mf_i = \frac{m_i}{m_{\text{mix}}} \text{ and } \boxed{\sum_{i=1}^{N} mf_i = 1}$$

- Application of Accounting Principle for Chemical Species
  - rate of accumulation of component i within the system
    - amount of component i within the system:

$$m_{i,\,sys} = \iiint_V \rho_i \, dV \quad \text{and} \quad n_{i,\,sys} = \frac{m_{i,\,sys}}{M_i}$$

    - transport rate of component i across system boundaries
      - mass flow rate of component i: \dot{m}_{i} \quad (\mathrm{kg} / \mathrm{s}, \mathrm{~lbm} / \mathrm{s}, \mathrm{~slug} / \mathrm{s})\)
      - molar flow rate of component i: \dot{n}_{i} \quad (\mathrm{kmol} / \mathrm{s}, \mathrm{~mol} / \mathrm{s}, \mathrm{~lbmol} / \mathrm{s})\)
    - generation/consumption rate of species i within the system
      - chemical reactions and generation/consumption
        - balanced reaction equations and consumption/generation terms
      - generation (production or creation) rate: $\dot{m}_{i,\text{ gen}}$ or $\dot{n}_{i,\text{ gen}}$
      - consumption (destruction) rate: $\dot{m}_{i,\text{ cons}}$ or $\dot{n}_{i,\text{ cons}}$
- **chemical species accounting equation (mass basis)**

$$\begin{array}{ll}\textbf{rate form} & \frac{dm_{i,\,sys}}{dt} = \sum \dot{m}_{i,\text{ in}} - \sum \dot{m}_{i,\text{ out}} + \dot{m}_{i,\text{ gen}} - \dot{m}_{i,\text{ cons}} \\ \textbf{finite-time form} & m_{i,\text{ sys, final}} - m_{i,\text{ sys, initial}} = \sum_{\text{in}} m_{i,i} - \sum_{\text{out}} m_{i,e} + m_{i,\text{ gen}} - m_{i,\text{ cons}}\end{array}$$

- **chemical species accounting equation (mole basis)**

$$\begin{array}{ll}\textbf{rate form} & \frac{dn_{i,\,sys}}{dt} = \sum n_{i,\text{ in}} - \sum \dot{n}_{i,\text{ out}} + \dot{n}_{i,\text{ gen}} - \dot{n}_{i,\text{ cons}} \\ \textbf{finite-time form} & n_{i,\text{sys, final}} - n_{i,\text{ sys, initial}} = \sum n_{i,\text{ in}} - \sum n_{i,\text{ out}} + \mathrm{n}_{i,\text{ gen}} - \mathrm{n}_{i,\text{ cons}}\end{array}$$

- Constitutive relation
  - Examples: Ohm's Law, Ideal Gas Model
- Ideal Gas Model
  - universal gas constant R_u vs. specific gas constant R

2. Given one of the species accounting or conservation of mass equations and a list of assumptions, carefully indicate the consequences of each assumption. Typical assumptions include: steady-state, one-inlet/one-outlet, closed system, open system, and no chemical reactions.
3. Given one of the species accounting or conservation of mass equations, explain what each term represents physically and how it relates to the overall accounting framework discussed in Chapters 1 and 2.
4. Given information about the local velocity distribution and density distribution at the boundary of a system, calculate the mass flow rate and the volumetric flow rate at the boundary.
5. Given a mixture composition in terms of either mass (weight) fractions or mole fractions, determine the composition in the other measure. If total mass of the mixture is specified, determine the moles or kilograms of each component in the mixture. (Best done using a simple table format.)
6. Given a problem that can be solved using conservation of mass and species accounting, you should be able to do the following tasks:
   1. Select an appropriate system. Identify the system and its boundaries on an appropriate drawing. Describe the system and its boundaries in sufficient detail so that there is no confusion about your choice. Indicate whether the system is open or closed.
   2. Indicate the time interval appropriate for the problem (e.g. should you use the rate form or the finite-time form?).
   3. Clearly identify and count the number of unknowns you are trying to find. Define and use a unique symbol for each unknown.
   4. Develop a set of INDEPENDENT equations that are equal in number to the number of unknowns and are sufficient to solve for the unknowns. These equations are developed using conservation of mass, species accounting, and information given in the problem statement, e.g. physical constraints and constitutive equations.
   5. Solve for the unknown quantities.
   6. Substitute in the numerical values to find a numerical answer.
7. Given a problem with non-uniform chemical composition, i.e. a separation, distillation or a mixing problem, apply conservation of mass and species accounting to solve for the unknown mass flow rates or masses and mixture compositions.
8. Given a numerical value for one of the following quantities, determine the numerical value of the remaining quantities: density, specific volume, specific weight, specific gravity
9. Given any two of the following properties—pressure, temperature, and density (mass or molar)—use the ideal gas equation to find the unknown property.

Chapter 5 — Conservation of Linear Momentum

1. Define, explain, compare and contrast the following terms and concepts:
   - Particle vs. Extended Body
     - Rigid body
   - Kinematic relationships: position, velocity, and acceleration
   - Linear Momentum
     - linear momentum of a particle: $\mathbf{P} = m\mathbf{V}$
       - specific linear momentum: $\vec{V}$
       - units of linear momentum $(\mathrm{N \cdot s};\ \mathrm{lbf \cdot s})$
     - vector nature of linear momentum
     - inertial reference frame
   - Application of Accounting Principle for Linear Momentum
     - amount of linear momentum within the system: $\mathbf{P}_{sys} = \int_{V_{sys}} \mathbf{V}\rho\, dV$
     - transport rate of linear momentum across the system boundaries
       - external forces: $\sum \mathbf{F}_{\text{external}}$

- body forces
- surface (contact) forces
  - normal stress vs. shear stress
- mass transport of linear momentum: $\sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e$

- generation/consumption rate of linear momentum within the system
  - Empirical result ----- Linear momentum is conserved!

- Conservation of Linear Momentum Equation
  - rate form $\quad \dfrac{d\mathbf{P}_{\text{sys}}}{dt} = \sum \mathbf{F}_{\text{external}} + \sum_{\text{in}} \dot{m}_i \mathbf{V}_i - \sum_{\text{out}} \dot{m}_e \mathbf{V}_e$
- Impulse: $\quad \mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt$
- Impulsive Force: $\quad \mathbf{F}_{\text{avg}} = \dfrac{\mathbf{I}}{\Delta t} = \dfrac{1}{\Delta t} \int_{t}^{t+\Delta t} \mathbf{F}\, dt$
- Conservation of Linear Momentum and Newton’s Laws
- Center of mass
- Dry Friction (A useful constitutive relation)
  - static friction coefficient
  - kinetic (sliding) friction
- Relative velocity

2. Given a problem that can be solved using conservation of linear momentum, you should be able to do the following:
   1. Select an appropriate system that can be used to find the requested unknowns using the information given in the problem.
      - Clearly identify the system and its boundaries on an appropriate drawing.
      - Carefully label all transports of linear momentum with the surroundings. (This is commonly called a free-body diagram.)
   2. Indicate the time interval appropriate for the problem.
   3. Clearly identify and count the number of unknowns you are trying to find. Define and use a unique symbol for each unknown.
   4. Develop a set of INDEPENDENT equations that are equal in number to the number of unknowns and are sufficient to solve for the unknowns. These equations are developed using the conservation and accounting equations and the information given in the problem. Carefully indicate how the given information plus your assumptions are used to develop the problem-specific equations from the general accounting and conservation principles. (Recognize that in a two-dimensional problem, application of conservation of linear and angular momentum to a system can contribute at most three independent equations.)
   5. Solve for the unknown values.
3. Starting with the conservation of linear momentum equation, show what assumptions are necessary to develop the traditional result for a rigid body: $\mathbf{F} = m\mathbf{a}$.
4. Given information about the acceleration of an object as a function of time, use elementary calculus to develop an equation for the velocity and as a function of time.
5. Given information about the velocity of an object as a function of time, use elementary calculus to develop an equation for the position as a function of time.
6. Use the concepts embodied in the conservation of momentum equation, including transport and storage of linear momentum, to explain the behavior of a device or system.
7. Given a problem that involves friction, use both sliding and static friction forces where appropriate to explain the motion and/or forces in the system.
8. Given a problem with impulsive forces or loads, evaluate the *impulse* applied to the system, and if the time interval is known, determine the average value of the impulsive force over the time interval.
9. Given a problem where relative velocities are given or required, correctly convert relative velocities to absolute velocities for use in the conservation of linear momentum equation.

Chapter 6 — Conservation of Angular Momentum

1. Define, explain, compare and contrast the following terms and concepts:
   - Motion of a rigid body
     - Rectilinear translation vs. curvilinear translation
     - Rotation about a fixed axis
     - General motion
   - Rotational motion
   - Angular position: $\theta\ [\text{radians}]$
     - Angular velocity: $\omega\ [\text{radians}/\text{second}]$
     - Angular acceleration: $\alpha\ [\text{radians}/\text{seconds}^2]$
   - Angular momentum about origin O
     - angular momentum about the origin O for a particle: $\mathbf{L}_O = \mathbf{r} \times m\mathbf{V}$
     - specific angular momentum about the origin O: $\mathbf{l}_O = \mathbf{r} \times \mathbf{V}$ where $\mathbf{r}$ is the position vector with respect to the origin O
       - right-hand rule sign convention
       - vector nature of angular momentum
       - units of angular momentum $(\mathrm{N} \cdot \mathrm{m} \cdot \mathrm{s};\ \mathrm{lbf} \cdot \mathrm{ft} \cdot \mathrm{s})$
   - Application of Accounting Principle for Angular Momentum
     - rate of accumulation of angular momentum with the system
       - amount of angular momentum about origin O: $\mathbf{L}_{O,\, sys} = \int\limits_{\mathcal{V}_{sys}} (\mathbf{r} \times \mathbf{V}) \rho \, d\mathcal{V}$
       - mass moment of inertia about a single axis: $I_G = \int\limits_{\mathcal{V}_{sys}} r^2 \rho \, d\mathcal{V}$
       - relation between mass moment of inertia, angular momentum, and angular velocity
     - transport rate of angular momentum across system boundaries
       - transport with forces
         - torques or moments of an external force about origin O : $\sum \mathbf{r} \times \mathbf{F}_{\text{ext}}$
         - torque or moment of a couple: $\mathbf{M}_O$
     - mass transport of angular momentum about the O : $\sum \dot{m}(\mathbf{r} \times \mathbf{V})_{\text{in}} - \sum \dot{m}(\mathbf{r} \times \mathbf{V})_{\text{out}}$
     - generation/consumption of angular momentum within the system
       - Empirical Result ----- Angular momentum is conserved!
   - Conservation of Angular Momentum (about the origin O)
     - rate form: $$\frac{d\mathbf{L}_{O,\, sys}}{dt} = \underbrace{\sum \mathbf{M}_{O,\, \text{external}}}_{\text{Due to couples}} + \underbrace{\sum (\mathbf{r} \times \mathbf{F}_{\text{external}})}_{\text{Due to forces}} + \underbrace{\sum_{\text{in}} \dot{m}_i (\mathbf{r} \times \mathbf{V})_i - \sum_{\text{out}} \dot{m}_e (\mathbf{r} \times \mathbf{V})_e}_{\text{Due to mass transport}}$$
   - Angular Impulse
   - **SPECIAL CASE:** Plane, Translational Motion of a Closed, Rigid System
     - angular momentum about origin O : $\mathrm{L}_{O,\, sys} = \mathbf{r}_G \times \mathrm{m}\mathbf{V}_G$

       $$\text{where} \quad \begin{array}{ll} \mathbf{r}_G & = \text{the position vector of the center of mass with respect to the origin.} \\ \mathbf{V}_G & = \text{the velocity of the center of mass.} \end{array}$$
     - Conservation of Angular Momentum:

$$\begin{aligned}
\frac{d\mathbf{L}_{O,\,sys}}{dt} &= \sum \mathbf{r}\times\mathbf{F}_{\text{ext}} + \sum \mathbf{M}_O \\
\frac{d}{dt}(\mathbf{r}_G \times m\mathbf{V}_G) &= \sum \mathbf{r}\times\mathbf{F}_{\text{ext}} + \sum \mathbf{M}_O \\
\underbrace{\cancel{\left[\frac{d\mathbf{r}_G}{dt}\times m\mathbf{V}_G\right]}}_{\text{since } \mathbf{V}_G\times\mathbf{V}_G=0} + \left[\mathbf{r}_G \times m\frac{d\mathbf{V}_G}{dt}\right] &= \sum \mathbf{r}\times\mathbf{F}_{\text{ext}} + \sum \mathbf{M}_O \\
\mathbf{r}_G \times m\frac{d\mathbf{V}_G}{dt} &= \sum \mathbf{r}\times\mathbf{F}_{\text{ext}} + \sum \mathbf{M}_O
\end{aligned}$$

$$\text{where} \quad \begin{aligned} \mathbf{r} &= \text{the position vector with respect to the origin.} \\ \mathbf{r}_G &= \text{the position vector of the center of mass with respect to the origin.} \end{aligned}$$

2. Apply conservation of angular momentum to solve problems involving
   1. steady-state open or closed systems,
   2. static (stationary) closed systems,
   3. closed, stationary, rigid-body systems,
   4. translating, closed, rigid body systems, i.e. systems with $\omega = 0$ and $\alpha = 0$. (See item number 2 from the objectives for linear momentum to see necessary steps.)

Chapter 7 — Conservation of Energy

1. Define, illustrate, and compare and contrast the following terms and concepts:
   - Work-Energy Principle
     - relation to conservation of linear momentum
   - Energy
     - internal energy
       - specific internal energy: u
     - mechanical energy
     - gravitational potential energy
       - specific gravitational potential energy: g_z
     - kinetic energy
       - specific kinetic energy: $V^2/2$
     - spring energy
   - Work
     - mechanism for transferring energy
     - mechanical work vs. thermodynamic work
     - work (W) vs. power $(\dot{W})$
     - path function
     - reversible (quasiequilibrium) work vs. irreversible work types
       - compression/expansion (pdV) work
       - shaft work
       - elastic (spring) work
       - electric work/power
         - dc power
         - ac power:
           - effective vs maximum values
           - power factor
   - Heat transfer
     - mechanism for transferring energy

- heat transfer (Q) vs. heat transfer rate $(\dot{Q})$
- adiabatic surface or boundary
- path function
- types of heat transfer
  - conduction
  - convection
    - Newton's law of cooling
    - convection heat transfer coefficient
  - thermal radiation
- Application of Accounting Principle to Energy
  - rate of accumulation of energy within the system
    - amount of energy \quad $E_{sys} = \int_V e\rho\, dV$ where the specific energy is defined as $e = u + \left(V^2\right)/2 + gz$
  - transport rate of energy by heat transfer: Q ----- Heat transfer rate
  - transport rate of energy by work at non-flow boundaries: $\dot{W}$ ----- Power
  - transport rate of energy by work at flow boundaries: $\boxed{\sum (pv)\,\dot{m}_{\text{in}} - \sum (pv)\,\dot{m}_{\text{out}}}$ ----- Flow Power
  - transport rate of energy mass flow: $\boxed{\sum \dot{m}\left(u + \frac{V^2}{2} + gz\right)_{\text{in}} - \sum \dot{m}\left(u + \frac{V^2}{2} + gz\right)_{\text{out}}}$
- Rate form of Conservation of Energy

$$\frac{dE_{sys}}{dt} = \dot{Q}_{\text{Net, in}} + \dot{W}_{\text{Net, in}} + \sum\left(h + \frac{V^2}{2} + gz\right)\dot{m}_{\text{in}} - \sum\left(h + \frac{V^2}{2} + gz\right)\dot{\mathrm{m}}_{\text{out}}$$

where $h = u + pv$ is a new property called enthalpy

- Substance models
  - Ideal gas with room-temperature specific heats
  - Incompressible substance with room-temperature specific heats
- Thermodynamic cycles
  - Definition (three parts)
  - Classifications
    - Working fluid: single vs two-phase
    - Structure: Closed, periodic vs Closed-loop, steady-state
    - Purpose: Power vs Refrigeration vs Heat Pump cycles
  - Measures of Performance
    - General definition
      - Power cycles $\rightarrow$ Thermal efficiency η
      - Refrigeration cycle $\rightarrow$ Coefficient of Performance $\mathrm{COP}_{\text{ref}}$
      - Heat pump cycles $\rightarrow$ Coefficient of Performance COP_{hp}

2. Given a mechanical system consisting of particles, apply the Work-Energy Principle where appropriate to solve problems where changes in mechanical energy (kinetic, potential, and spring) can be balanced with mechanical work done on the system.
3. Given a closed or open system and sufficient information about the properties of the system, apply conservation of energy to determine changes in energy (rates of change) within the system and heat transfers (heat transfer rates) and work transfers of energy (power) with the surroundings.
4. Given sufficient information, determine the change in specific internal energy Δu and the change in Δh for a substance that can be modeled using one of the following substance models:
   Ideal gas with room-temperature specific heats

Incompressible substance with room-temperature specific heats

and use this information in conjunction to meet Objective 3 above.

5. Given the indicated information, calculate the magnitude and the direction of the associated work transfer of energy or power for the system:
   - Given a relation between system pressure and system volume, calculate the compression/expansion work for the system.
   - Given a torque and a rotational speed for a shaft, calculate the shaft power transmitted by the shaft.
   - Given an electric current and the corresponding voltage difference across the terminals, calculate the electric power supplied to or by the system. (You should be able to perform this calculation for both DC and AC systems.)
6. Given a numerical value for a typical energy or power quantity, make the appropriate unit conversions to change the units to the requested values, e.g. convert $\mathrm{ft}^2/\mathrm{s}^2$ to $\mathrm{Btu}/\mathrm{lbm}$.
7. Given a device that operates in a closed-periodic cycle or a closed-loop, steady-state cycle,
   - determine whether the device operates as a power cycle (heat engine) or a refrigerator or heat pump, and
   - calculate the appropriate measure of performance for the specific device, i.e. a thermal efficiency for a power cycle and a coefficient of performance (COP) for a refrigerator or heat pump.
8. List the appropriate assumptions to recover the mechanical energy balance from the general conservation of energy equation.

Chapter 8 — Entropy Production and Accounting

1. Define, illustrate, and explain the following terms and concepts:
   - Second Law of Thermodynamics
   - Reversible processes
     - internally *reversible* vs internally *irreversible*
   - Entropy
     - units: $\mathrm{kJ/K};\ \mathrm{Btu}/{}^\circ\mathrm{R}$
     - specific entropy: s
       - units: $\mathrm{kJ}/(\mathrm{K}\cdot\mathrm{kg});\ \mathrm{Btu}/\left({}^\mathrm{R}\cdot\mathrm{lbm}\right)$
   - Thermodynamic temperature
   - Application of Accounting Principle for Entropy
     - rate of accumulation of entropy within the system
       - amount of entropy within the system: $\boxed{S_{sys} = \int\limits_V s\rho\, dV}$
     - transport rate of entropy across system boundaries
       - transport rate of entropy by heat transfer: $\sum \frac{\dot{Q}_j}{T_{b,\,j}}$
       - transport rate of entropy by mass flow: $\sum\limits_{\text{in}} \dot{m}_i s_i - \sum\limits_{\text{out}} \dot{m}_e s_e$
     - production/consumption of entropy
       - EMPIRICAL EVIDENCE ----- Entropy can only be produced and in the limit of an internally reversible process entropy is conserved.
       - Rate of entropy production:

$$\dot{S}_{\text{gen}} \begin{cases} > 0 \text{ Internally irreversible} \\ = 0 \text{ Internally reversible} \end{cases}$$

   - Accounting Equation for Entropy
     - rate form: $\boxed{\frac{dS_{sys}}{dt} = \sum \frac{\dot{Q}_j}{T_{b,\,j}} + \sum\limits_{\text{in}} \dot{m}_i s_i - \sum\limits_{\text{out}} \dot{m}_e s_e + \dot{S}_{\text{gen}}}$
   - Carnot Efficiency for a Power Cycle

- Isentropic Process

2. Apply the accounting equation for entropy in conjunction with the conservation of energy equation to calculate the entropy generation rate or entropy generation for a steady-state device or cycle.
3. Given sufficient information, determine the specific entropy change Δs for a substance when one of the following models apply:
    - Ideal gas with room-temperature specific heats
    - Incompressible substance with room-temperature specific heats
4. Apply the entropy accounting equation in conjunction with the conservation of energy equation to calculate the entropy generation or the entropy generation rate for a system when all other necessary information is known.
5. Apply the accounting equation for entropy in conjunction with the conservation of energy equation to determine the theoretical "best" performance, i.e. theoretical maximum thermal efficiency or coefficient of performance for a cycle.
6. Determine if a specific device or system is operating in a reversible fashion, an irreversible fashion, or is not physically possible.
7. Evaluate the performance of a device or system when it is operating in an internally reversible fashion.

9.5: Conservation of Energy, the Work-Energy Principle, and the Mechanical Energy Balance

When students leave physics mechanics and are introduced to the more general Conservation of Energy principle, they often struggle to understand how it relates to the more restricted Work-Energy Principle and Mechanical Energy Balance. Or as students often more simply put it, “Which energy equation should I use?”

The following notes (1) illustrate how the more restricted principles are related back to the fundamental Conservation of Energy and (2) give students a list of modeling assumptions that can be recognized and applied to a specific problem to recover the Mechanical Energy Balance.

Because this course emphasizes constructing problem-specific solutions from fundamental principles, i.e. Conservation of Energy in this case, this material is best introduced after students have read the Section 7.1, 7.2, and 7.3 of Chapter 7 and before they are assigned any work-energy-type homework problems typically found in a basic mechanics course.

1. Conservation of Energy, the Work-Energy Principle, and the Mechanical Energy Balance — These notes describe (1) how the Work-Energy Principle is developed from the Conservation of Linear Momentum, (2) how the Mechanical Energy Balance can be developed from the Conservation of Energy, (3) how the Work-Energy Principle and the Mechanical Energy Balance are related, and (4) when should students use each. In addition, these notes also introduce spring energy as a type of mechanical energy.
2. Summary — short summary of the section above.
3. When can I start my analysis with the Mechanical Energy Balance for a closed system? — These notes provide three different approaches with modeling assumptions for reducing the general Conservation of Energy equation to the more restricted Mechanical Energy Balance. [These notes do make explicit use of the incompressible substance model to relate temperature change to internal energy change.]

1. Conservation of Energy, the Work-Energy Principle, and the Mechanical Energy Balance

In your study of engineering and physics, you will run across a number of engineering concepts related to energy. Three of the most common are Conservation of Energy, the Work-Energy Principle, and the Mechanical Engineering Balance. The Conservation of Energy is treated in this course as one of the overarching and fundamental physical laws. The other two concepts are special cases and only apply under limited conditions. The purpose of this note is to review the pedigree of the Work-Energy Principle, to show how the more general Mechanical Energy Balance is developed from Conservation of Energy, and finally to describe the conditions under which the Mechanical Energy Balance is preferred over the Work-Energy Principle.

Work-Energy Principle for a Particle

Consider a particle of mass m and velocity $\vec{V}_G$ moving in a gravitational field of strength $\vec{g}$ subject to a surface force $\vec{R}_{\text{surface}}$. Under these conditions, writing Conservation of Linear Momentum for the particle gives the following:

$$\frac{d}{dt}\left(m\vec{V}_G\right)=\vec{R}_{\text{surface}}+m\vec{g} \tag{9.5.1}$$

Forming the dot product of Eq. $(\mathrm{SM}.1.1)$ with the velocity of the particle and rearranging terms gives the rate form of the Work-Energy Principle for a particle:

$$\underbrace{\frac{d}{dt}\left(m\frac{\mathrm{V}^2}{2}\right)}_{\substack{\text{Kinetic}\\ \text{energy}}}+\frac{d}{dt}\underbrace{(mgz)}_{\substack{\text{Gravitational}\\ \text{potential}\\ \text{energy}}}=\underbrace{\vec{R}_{\text{surface}}\cdot\vec{V}_G}_{\substack{\text{mechanical}\\ \text{potential}\\ \text{energy}}} \quad\Rightarrow\quad \frac{d}{dt}\left(E_K+E_{GP}\right)=\dot{W}_{\text{mech, in}} \tag{9.5.2}$$

Recall that mechanical power is defined as $\dot{W}_{\text{mech, in}}=\vec{R}_{\text{surface}}\cdot\vec{V}_G$, the dot product of the surface force with the velocity of its point of application. Because a particle has no extent and only one velocity, the point of application of the surface force and the particle always have the same velocity.

Integrating Eq. $(\mathrm{SM}.1.1)$ with respect to distance or Eq. $(\mathrm{SM}.1.2)$ with respect to time gives the more familiar relation for change in kinetic energy, change in gravitational potential energy, and mechanical work:

$$\underbrace{\Delta\left(m\frac{V_G{}^2}{2}\right)}_{=E_K} + \underbrace{\Delta(mgz_G)}_{=E_{GP}} = \int_{t_1}^{t_2} \vec{R}_{\text{surface}} \cdot \underbrace{\vec{V}_G\, dt}_{=d\vec{s}} = \underbrace{\int_1^2 \underbrace{\vec{R}_{\text{surface}} \cdot d\vec{s}}_{=\delta W_{\text{mech in}}}}_{=W_{\text{mech, in}}} \tag{9.5.3}$$

$$\Downarrow$$

$$\Delta E_K + \Delta E_{GP} = W_{\text{mech, in}}$$

This is known as the **finite-time form of the Work-Energy Principle for a particle**. Recall that mechanical work is the time integral of the mechanical power and can be calculated as the dot product of the surface force and the displacement of its point of application. Again, the displacement of the point of application of the surface force is unambiguous for a particle because it is the same as the displacement of the particle.

Although the Work-Energy Principle uses energy language—energy, work, power—it adds nothing new that could not have been discovered through a careful application of the Conservation of Linear Momentum.

Conservation of Energy and the Mechanical Energy Balance for a Closed System

Writing Conservation of Energy for a closed system, we obtain the rate form of Conservation of Energy for a closed system:

$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} \tag{9.5.4}$$

Restricting ourselves to only three types of energy—internal energy U, kinetic energy E_K, and gravitational potential energy E_{GP}—we have the following result:

$$\frac{d}{dt}(U_{sys} + E_{K,\,sys} + E_{GP,\,sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}}$$

Although the distinctions are somewhat artificial, we will segregate the energy into two groups: thermal energy and mechanical energy. Internal energy is usually classified as *thermal energy* because changing internal energy of a system is often associated with a change in temperature. The other two energies are classified as *mechanical energy* because changing the kinetic energy or gravitational potential energy of a system can be done solely by the application of a surface force and its associated mechanical work. In addition, we will separate the work transfer-rate of energy (power) into two terms: mechanical work where there is an identifiable surface force and non-mechanical work, e.g. electrical work.

Using these new distinctions between mechanical and thermal phenomena, we can rewrite Eq. (SM.1.5) and group the mechanical and thermal terms as shown below:

$$\begin{aligned}
\frac{d}{dt}(U_{sys} + E_{K,\,sys} + E_{GP,\,sys}) &= \dot{Q}_{\text{net, in}} + \underbrace{\dot{W}_{\text{net, mech, in}} + \dot{W}_{\text{net, nonmech, in}}}_{=\dot{W}_{\text{net, in}}} \\
\underbrace{\tfrac{d}{dt}(E_{K,\,sys} + E_{GP,\,sys})}_{\substack{\text{Rate of change} \\ \text{of the} \\ \text{mechanical energy} \\ \text{in the system}}} &= \underbrace{\dot{W}_{\text{net, mech, in}}}_{\substack{\text{Transport rate} \\ \text{of energy by} \\ \text{mechanical work}}} + \underbrace{\left[\dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, nonmech, in}} - \frac{d}{dt}(U_{sys})\right]}_{\substack{\text{Net production rate} \\ \text{of mechanical energy} \\ \text{inside the system}}} \\
&\Downarrow \\
\frac{d}{dt}(E_{K,\,sys} + E_{GP,\,sys}) &= \dot{W}_{\text{net, mech, in}} + \dot{E}_{\text{net, mech, prod}}
\end{aligned}$$

This is called the rate form of the Mechanical Energy Balance for a closed system. It accounts for the storage, transport, and production or destruction of mechanical energy in a closed system. In words,

> *the time-rate-of-change of the mechanical energy in the system equals the net transport rate of energy with mechanical work (net mechanical power) into the system plus the net production rate of mechanical energy inside the system.*

In general, the net production rate term can take on both positive and negative values.

The introduction and presence of a production term does not violate Conservation of Energy because we are only counting one type of energy, and one of the characteristics of energy is that it can be stored in different ways. [Consider a marble rolling up and down the sides of a wooden salad bowl. If there are no losses, there is a continuous interchange between kinetic and gravitational potential energy and if one was only counting kinetic energy it would alternately appear to be produced and then destroyed. The idea of counting only one type of energy is analogous to the idea of counting only a single chemical species (Species Accounting) used earlier in our study of Conservation of Mass.

Mechanical Energy Balance = Work-Energy Principle?

We've already shown that the Work-Energy Principle for particle is a direct descendant of Conservation of Linear Momentum and the Mechanical Energy Balance for a closed system grew out of the Conservation of Energy. Rewritten below together, we see that they appear to be similar even though one is written for a particle and the other for a more general closed system:

$$\begin{aligned}&\text{Work-Energy Principle for a particle (Supplementary Materials 1.2):}\\&\qquad \frac{d}{dt}\left(E_K+E_{GP}\right)=\dot{W}_{\text{mech, in}}\end{aligned}$$

$$\begin{aligned}&\text{Mechanical Energy Balance for a closed system (Supplementary Materials 1.6):}\\&\qquad \frac{d}{dt}\left(E_{K,\,sys}+E_{GP,\,sys}\right)=\dot{W}_{\text{net, mech, in}}+\dot{E}_{\text{net, mech, prod}}\end{aligned}$$

The only significant difference between the two equations is the net production rate term for mechanical energy.

If we can find a set of conditions under which mechanical energy is neither produced nor destroyed, the Work-Energy Principle and the Mechanical Energy Balance contain the same information. So the important question is when does this occur? Without creating an inclusive list of conditions, we will state only one set of conditions:

> *Mechanical energy will be neither produced nor destroyed within a closed system if (1) the materials in the system are incompressible, (2) there is no internal friction or friction between parts of the closed system, and (3) only mechanical work occurs on the system boundary. (Note that this does not prohibit friction on the boundary of the system.)*

This set of conditions is consistent with the conditions usually invoked when applying the Work-Energy Principle. When these conditions are satisfied, the Mechanical Energy Balance reproduces the results of the Work-Energy Principle with the added advantage that it applies to any closed system. If there is the restriction on internal friction is relaxed, we will show later that mechanical energy can only be destroyed. With this in mind, conditions that do not produce or destroy mechanical energy frequently represent the *best* or *ideal* behavior for the system. (This idea will be explored further when we encounter the Second Law of Thermodynamics and the Entropy Accounting Principle.)

Adding Springs (Elastic Energy) to the Mechanical Energy Balance for a closed system

Now that we've shown the relationship between the Work-Energy Principle and the Mechanical Energy Balance, we wish to include an additional type of energy that can be handled within our Mechanical Energy Balance — elastic or spring energy.

Typically we will only consider true mechanical springs. Assuming a linear spring with no hysteresis or internal friction, the energy stored in a spring can be calculated from the following equation:

$$\begin{array}{lll} & E_{\text{Spring}} & =\dfrac{1}{2}k\left(x-x_0\right)^2\\ \text{where} & k & =\text{spring constant [Force/Length]}\\ & x_0 & =\text{unstretched length of the spring}\\ & x & =\text{stretched length of the spring}\end{array}$$

Note that an unstretched spring stores no mechanical energy, and that a linear, ideal spring stores energy when it compressed or stretched.

Including this additional form of mechanical energy, we have an **expanded rate form of the Mechanical Energy Balance for a closed system:**

$$\frac{d}{dt}\left(E_{K,\,sys}+E_{GP,\,sys}+E_{Spring,\,sys}\right)=\dot{W}_{\text{net, mech, in}}+\dot{E}_{\text{net, mech, prod}}$$

This is the most general form we will present.

Bottom Line — When should and can I use the Mechanical Energy Balance?

Although the general form can be useful, Eq. (SM.1.8) is most useful when we can assume that there is no mechanical energy production or destruction. Under these conditions, we have the expanded rate form of the Mechanical Energy Balance for a closed, incompressible system with no internal friction and only mechanical work:

$$\frac{d}{dt}\left(E_{K,\ sys}+E_{GP,\ sys}+E_{Spring,\ sys}\right)=\dot{W}_{\text{net, mech, in}}$$

Integrated with respect to time we recover the finite-time form:

$$\Delta E_{K,\ sys}+\Delta E_{GP,\ sys}+\Delta E_{Spring,\ sys}=W_{\text{net, mech, in}}$$

If your system contains only incompressible objects, there is no internal friction, and only mechanical work occurs on the boundary of the system then you can and should use Eq. (SM.1.9) or (SM.1.10) instead of the complete Conservation of Energy. This will also replace the Work-Energy Principle for a particle, unless you find an particular advantage in starting with the Conservation of Linear Momentum and integrating. When applied appropriately and correctly, the Mechanical Energy Balance as presented in Eq. (SM.1.9) and Eq. (SM.1.10) can do everything the Work-Energy Principle can and *more*.

---

2. Summary

Work-Energy Principle for a Particle

... Start with the conservation of linear momentum for a particle (Equation 9.5.1):

$$\frac{d}{dt}\left(m\vec{V}_G\right)=\vec{R}_{\text{surface}}+m\vec{g}$$

... Form the dot product with the velocity of the center of mass $\vec{V}_G$ and define kinetic energy, gravitational potential energy and mechanical power to obtain the rate-form of the work-energy principle for a particle (Equation 9.5.2):

$$\underbrace{\frac{d}{dt}\left(m\frac{\mathrm{V}^2}{2}\right)}_{\substack{\text{Kinetic}\\ \text{energy}}}+\frac{d}{dt}\underbrace{(mgz)}_{\substack{\text{Gravitational}\\ \text{potential}\\ \text{energy}}}=\underbrace{\vec{R}_{\text{surface}}\cdot\vec{V}_G}_{\substack{\text{mechanical}\\ \text{potential}\\ \text{energy}}}\quad\Rightarrow\quad\boxed{\frac{d}{dt}\left(E_K+E_{GP}\right)=\dot{W}_{\text{mech, in}}}$$

... Integrate the rate-form of the work-energy principle over a time interval to obtain the finite-time form of the work-energy principle for a particle (Equation 9.5.3):

$$\underbrace{\Delta\left(m\frac{{V_G}^2}{2}\right)}_{=E_K}+\Delta\underbrace{(mgz_G)}_{=E_{GP}}=\int_{t_1}^{t_2}\vec{R}_{\text{surface}}\cdot\underbrace{\vec{V}_G\,dt}_{=d\vec{s}}=\underbrace{\int_1^2\underbrace{\vec{R}_{\text{surface}}\cdot\,d\vec{s}}_{=\delta W_{\text{mech in}}}}_{=W_{\text{mech, in}}}$$

$$\Downarrow$$

$$\boxed{\Delta E_K+\Delta E_{GP}=W_{\text{mech, in}}}$$

Conservation of Energy and Mechanical Energy Balance for a Closed System

... Start with the rate form of the conservation of energy for a closed system:

$$\frac{d}{dt}\left(E_{sys}\right)=\dot{Q}_{\text{net, in}}+\dot{W}_{\text{net, in}}$$

... Classify energy into two types — mechanical vs. thermal. Mechanical energy can be accomplished without changing the temperature of the system, while thermal energy typically requires a change in temperature or pressure of the system. Regrouping terms we have the rate-form of the Mechanical Energy Balance (in general mechanical energy, like any one type of energy, is not conserved). (See Equation 9.5.4)

$$\frac{d}{dt}\left(U_{sys}+E_{K,\,sys}+E_{GP,\,sys}\right) = \dot{Q}_{\text{net, in}} + \underbrace{\dot{W}_{\text{net, mech, in}} + \dot{W}_{\text{net, nonmech, in}}}_{=\dot{W}_{\text{net, in}}}$$

$$\frac{d}{dt}\left(E_{K,\,sys}+E_{GP,\,sys}\right) = \underbrace{\dot{W}_{\text{net, mech, in}}}_{\substack{\text{Transport rate}\\ \text{of energy by}\\ \text{mechanical work}}} + \underbrace{\left[\dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, nonmech, in}} - \frac{d}{dt}\left(U_{sys}\right)\right]}_{\substack{\text{Net production rate of mechanical energy}\\ \text{inside the system}\\ (\text{May be } >,\, <,\, \text{or } =0)}}$$

$$\frac{d}{dt}\left(E_{K,\,sys}+E_{GP,\,sys}\right) = \dot{W}_{\text{net, mech, in}} + \dot{E}_{\text{net, mech, prod}}$$

The value of the mechanical energy production term depends on the process. If the system contains only incompressible substances, the mechanical energy production term is always less than or equal to zero, i.e. mechanical energy is destroyed in real processes. When the mechanical energy production/destruction term is zero the mechanical energy balance and the work-energy principle are identical.

When does the Mechanical Energy Balance = Work-Energy Principle?

When mechanical energy is neither created nor destroyed, the Mechanical Energy Balance reproduces the results of the Work-Energy Principle with the added advantage that it applies to any closed system. Although not all inclusive, one useful set of conditions under which this occurs follows:

> **Mechanical energy will be neither produced nor destroyed within a system if**

1. the system is **closed**,
2. all substances in the system are **incompressible**,
3. there is **no friction (dissipation) within or between parts** of the closed system, and
4. the only energy transfer on the system boundary is **mechanical work**. (Note that this **does not prohibit friction on the boundary system**.)

> **If <u>all</u> of these conditions apply to your system, you *may* and *should* start your analysis with the Mechanical Energy Balance and set the production term to zero!**

These conditions are consistent with the assumptions that you make in developing and applying the Work-Energy Principle. Conditions that do not produce or destroy mechanical energy frequently represent the *best* or *ideal* behavior for the system. In addition, many of the types of problems you solved in physics that involved conservative forces can be solved using the Mechanical Energy Balance.

A new form of mechanical energy — Springs (Elastic Energy)

For an ideal linear spring, i.e. no hysteresis or internal friction, the magnitude of the force exerted by the spring, $|F|$, is proportional to the compression/extension, δ, of the spring from its unstretched (free) length, i.e. $|F|=k|\delta|$. An uncompressed or unstretched spring has zero spring (elastic) energy. When a linear, ideal spring is deflected (compressed or stretched) it stores spring energy. Springs may also have kinetic, gravitational potential, and internal energy; however, the amount of spring (elastic) energy only depends on the deflection of the spring. The elastic energy stored in a linear, ideal spring can be calculated as follows:

$$E_{\text{Spring}} = \frac{1}{2}k\delta^2 = \frac{1}{2}k|x-x_o|^2$$

$$\begin{aligned} \text{where} \qquad k &= \text{spring constant [Force/Length]} \\ \delta &= |x-x_o| = \text{spring deflection (compression or extension) from its free length} \\ x_o &= \text{length of the unstretched spring, sometimes called the "free length"} \\ x &= \text{length of the stretched or compressed spring} \end{aligned}$$

When using conservation of energy (or the mechanical energy balance) to solve a problem with springs, it is usually advantageous to place the springs inside the system. If they remain outside, the spring force (a vector) does work on the system. When placed inside the system, effect of the springs is handled through the changing spring energy in the system.

Bottom Line — When should I use the Mechanical Energy Balance?

If your system is **(1) closed**, contains only **(2) incompressible objects**, has **(3) no internal friction** (friction/dissipation within or between parts of the system), and has **(4) only mechanical work transfers of energy on the boundary**, then mechanical energy is "conserved" and you can, may, and should start your analysis with the Mechanical Energy Balance (rate-form or finite-time form) assuming mechanical production/destruction is identically zero.

$$\boxed{\frac{d}{dt}\left(E_{K,\,sys}+E_{GP,\,sys}+E_{Spring,\,sys}\right)=\dot{W}_{\text{net, mech, in}}} \quad \text{or} \quad \boxed{\Delta E_{K,\,sys}+\Delta E_{GP,\,sys}+\Delta E_{Spring,\,sys}=W_{\text{net, mech, in}}}$$

3. Under what conditions is the Mechanical Energy Balance (mechanical energy conserved) valid for a closed system? What assumptions must I make?

Closed System Mechanical Energy Balance (with mechanical energy conserved)

$$\text{Rate Form} \qquad \frac{dE_{\text{sys, mech}}}{dt}=\frac{d}{dt}\left(E_{\text{Kinetic}}+E_{\text{Gravitational}}+E_{\text{Spring}}\right)=\dot{W}_{\text{mech, net, in}}$$

$$\text{Finite-Time Form} \qquad \Delta E_{\text{sys, mech}}=\Delta E_{\text{Kinetic}}+\Delta E_{\text{Gravitational}}+\Delta E_{\text{Spring}}=W_{\text{mech, net, in}}$$

In all cases where energy is to be counted, we ALWAYS start by applying the full Conservation of Energy Equation, usually in the rate form. Then we travel to the Mechanical Energy Balance (MEB) by one of three approaches:

APPROACH #1 - (Preferred Approach)

Starts with the Conservation of Energy Equation and **emphasizes mechanics assumption**s (no mention of heat transfer, etc.) to move directly to the mechanical energy balance (MEB), where mechanical energy is conserved:

Assume:

1. Closed system
2. Incompressible substance
3. Only mechanical work/power at boundaries
4. No internal friction, i.e. within bodies or between surfaces inside the system

APPROACH #2

Starts with the Conservation of Energy Equation and **emphasizes thermodynamics assumptions without a substance model:**

$$\frac{dE_{sys}}{dt}=\dot{Q}_{\text{net, in}}+\dot{W}_{\text{net, i}}+\sum_{\text{in}}\dot{m}_i\left(h_i+\frac{V_i^2}{2}+gz_i\right)-\sum_{\text{out}}\dot{m}_e\left(h_e+\frac{V_e^2}{2}+gz_e\right)$$

Assume:

1. Closed system
2. No heat transfer, i.e. adiabatic system
3. No change in internal energy, i.e. $\Delta U=0$

APPROACH #3

Starts with the Conservation of Energy Equation and emphasizes thermodynamics assumptions including the incompressible substance model:

$$\frac{dE_{sys}}{dt}=\dot{Q}_{\text{net, in}}+\dot{W}_{\text{net, in}}+\sum_{\text{in}}\dot{m}_i\left(h_i+\frac{V_i^2}{2}+gz_i\right)-\sum_{\text{out}}\dot{m}_e\left(h_e+\frac{V_e^2}{2}+gz_e\right)$$

Assume:

1. Closed system
2. Incompressible substance
3. No heat transfer, i.e. adiabatic system
4. Isothermal process, i.e. no temperature change $(\Delta T=0)$

9.6: Modeling Devices as Steady-State, Open Systems

One of the most common and technologically important applications of the Conservation of Energy is to analyze the behavior of devices that can be modeled as steady-state, open systems. The following notes discuss and illustrate common devices that can be modeled as steady-state, open systems. They also introduce a systematic way to describe and classify these devices.

1. Steady-State, Open Systems with activities — set of notes that introduces steady-state, open systems, and an organized method for identifying the purpose, physical features (design features), and operating conditions (modeling assumptions). As students complete the notes, they are asked to develop the device specific equations by starting with the full equations.
2. Examples of Common Steady-State, Open Systems

Steady-state Open Systems - Some Important Devices

You are surrounded by devices that under typical operating conditions can be modeled as steady-state, open systems. Typical devices that can be modeled as steady-state, open systems include

- turbines, pumps, compressors, fans, and blowers;
- nozzles and diffusers;
- throttling devices; and
- heat exchangers.

Although the study of individual devices is important, they are typically used in various combinations to produce electric power, chill the milk in your refrigerator, or whisk you away on a trip to Europe.

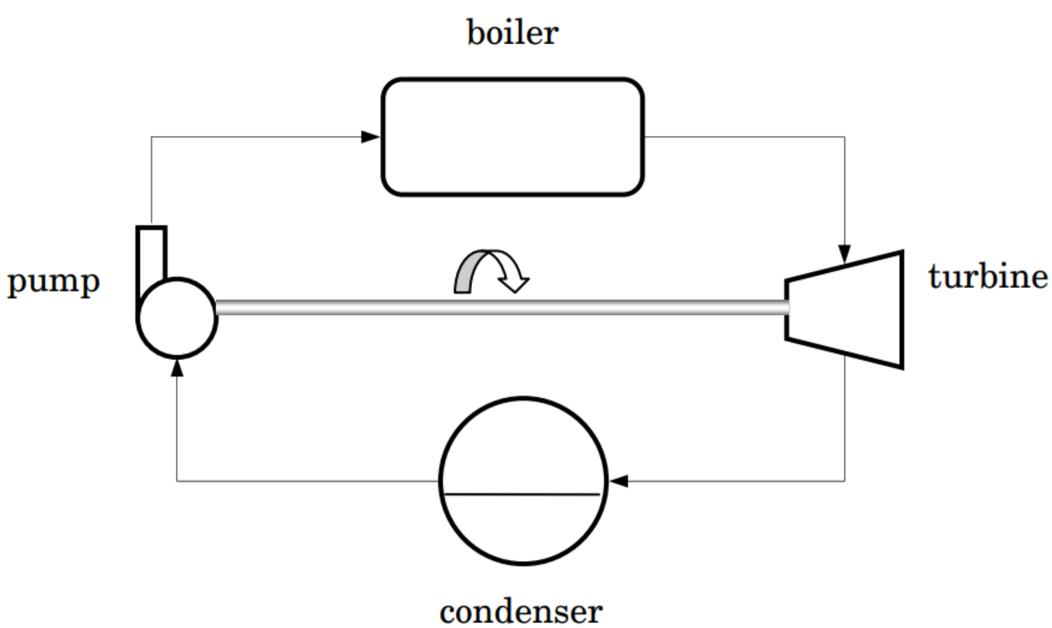


Figure SM.2.1: A simple steam power plant.

When you flip on the light switch in the morning, it is a good bet that the electricity you control was generated in a fossil-fueled steam power plant. The primary fuel for this power plant is typically natural gas or coal. The fuel is burned with air to heat high-pressure water in a boiler until it turns into steam. The high-pressure steam then expands through a turbine that drives an electrical generator. Low-pressure steam leaving the turbine is cooled and condensed back into a liquid and finally pumped back to the boiler to repeat the process. The complete power plant can be modeled as four steady-state, open systems - a boiler, a steam turbine, a condenser, and a water pump. (A schematic is shown in Figure SM.2.1.)

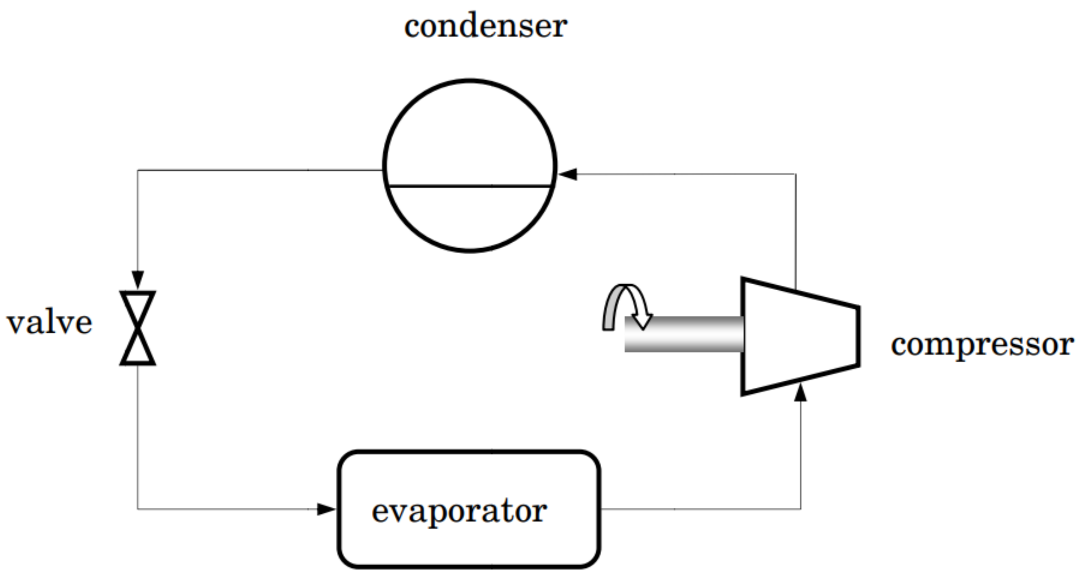


Figure SM.2.2: A mechanical vapor-compression refrigeration cycle.

When you grab milk out of the refrigerator, the cold milk is the result of a mechanical vapor-compression refrigeration cycle that keeps the contents of the refrigerator colder than the air in the kitchen. Again, this common device can be modeled as four steady-state, open systems - an evaporator, a compressor, a condenser, and a throttling valve. (A schematic is shown in Figure SM.2.2.) The evaporator receives energy by heat transfer from the contents of the refrigerator. Inside the evaporator a flowing, low-pressure refrigerant is boiled to produce a vapor. The refrigerant vapor leaving the evaporator is then compressed and fed to the condenser (a heat exchanger) where the high-pressure vapor is condensed back to a liquid. The liquid leaving the condenser then passes through a throttling valve where some of the liquid vaporizes cooling the refrigerant. This cold liquid-vapor refrigerant mixture then returns to the evaporator to repeat the process.

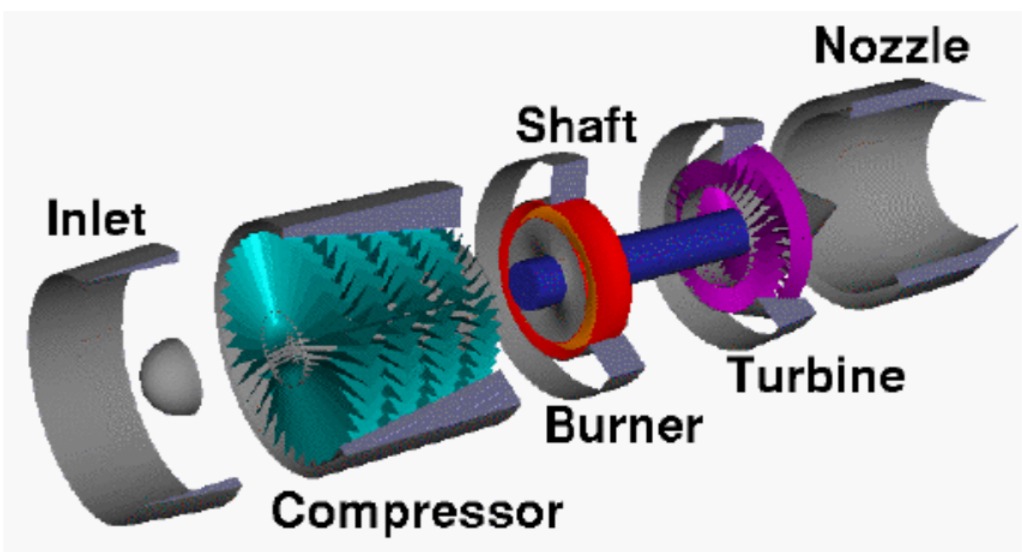


Figure SM.2.3: A gas-turbine jet engine.

When you travel in a modern jet liner, the plane is driven by a gas-turbine jet engine. At its most basic, the engine consists of five steady-state, open systems: an inlet diffuser, a compressor, a combustor or burner, a turbine, and a nozzle. (A schematic is shown in Figure SM.2.3.) In operation, the engine draws air in through a diffuser that decelerates the air and increases its pressure; next the air is compressed to a high pressure and fed into a combustion chamber. The high-pressure air is mixed with fuel in the combustion chamber and the combustion process produces high-pressure, high-temperature product gases. These gases expand through a turbine that drives the compressor. The hot gases leaving the turbine then expand through a nozzle to produce a high-speed exhaust stream. In a different incarnation as a stationary gas-turbine power plant, shaft power not thrust is the objective. To do this the nozzle is dispensed with and the turbine is enlarged to produce not thrust but shaft power to drive an electrical generator.

Even the humble forced-air furnace that keeps you warm in the winter makes use of steady-state, open systems. Air from a room is drawn back to the furnace through a return duct that is attached to a blower. The blower supplies air to a heat exchanger where the hot combustion gases warm the returned room air. The air leaving the furnace is then returned to the room through the supply air duct work.

Sometimes you will be asked to analyze a single component or device, say a pump or a heat exchanger, and sometimes you will be asked to consider several devices connected together. In either case, your ability to analyze the performance of the system will be enhanced if you understand the unique features or characteristics of each device.

To help you understand these devices and learn how to model them, we will study each device separately. As we do this we will identify its **purpose**, its essential **design** (or **physical**) **features**, and typical **operating conditions** (or **modeling assumptions**). We will also suggest a schematic diagram to represent each device.

Questions that you might ask to identify these things for a specific device are listed below:

Purpose	• What is this device supposed to do? • Why would you need one? • What happens to the fluid flowing through this device?
Physical Features (Design Features)	• What are the unique physical characteristics of this device, e.g. number of inlets and outlets? • What physical features does *every* one of these devices have? • What physical features do *most* of these devices have? • What can you say about the work term in the energy balance for this device, e.g. direction and magnitude?
Operating Conditions (Modeling Assumptions)	• What are the typical operating conditions for these devices, e.g. changes in kinetic energy negligible, constant pressure, adiabatic, one-dimensional flow, etc.? • What are the typical modeling assumptions that one would make in constructing a mathematical model to predict the performance of this device?

Physical features (design features) along with its **purpose** are *essential* features of the device. These characteristics should spring to your mind each time you think of this device. For example, a valve that also generates any shaft power is probably really a turbine. A electrical battery that must be plugged into a wall to get it to operate is probably not really a battery.

Operating conditions (modeling assumptions) indicate how the device usually operates. For example, the assumption of an adiabatic system is rarely a physical or design factor; however, it is frequently an operating condition or modeling assumption. To help you decide whether an attribute is a design factor or an operating condition, ask yourself the following question: "Would this device still be a (device name) if this condition was *not* satisfied?" If the condition is not essential, then you are probably considering an operating condition.

Example

Under some operating conditions, a simple electric motor can be modeled as a *closed*, *steady-state* system. Complete the following table by sketching a schematic diagram to represent an *electric motor* and then identifying its purpose, physical features and typical operating conditions:

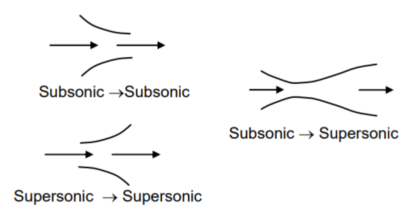
Device Name(s):	**Electric Motor**	
Purpose		Schematic:
Physical Features		

Device Name(s):	**Electric Motor**
Operating Conditions	*Best performance:* Reversible and adiabatic

Nozzles, Diffusers, and Throttling Valves

Nozzles, diffusers, and throttling valves all do the same thing — they alter the properties of a flowing fluid without any transfer of energy by work into or out of the system. The tables below provide additional information about each of these devices. Look for the similarities *and* the unique differences.

Device Name(s)	**Nozzles**
Purpose	Increase flow velocity (kinetic energy) while decreasing pressure in the direction of flow.
Physical Features	No work ($\dot{W}=0$) One-inlet / one-outlet
Operating Conditions	Steady-state system One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Inlet kinetic energy is negligible Negligible heat transfer for the system (adiabatic system, $\dot{Q}=0$) *Best performance:* Reversible and adiabatic



Example

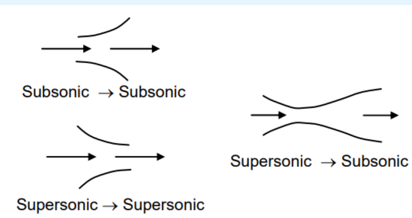
(1) Sketch a typical **nozzle** and label the inlet 1 and the outlet 2 .

(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

$$\frac{d}{dt}(m_{sys}) = \dot{m}_1 - \dot{m}_2$$

$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) - \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

Device Name(s)	**Diffusers**
Purpose	Increase pressure while decreasing flow velocity (kinetic energy) in the direction of flow.
Physical Features	No work ($\dot{W}=0$) One-inlet / one-outlet
Operating Conditions	Steady-state system



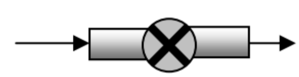
Operating Conditions	Steady-state system	
Device Name(s)	**Diffusers**	
	One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Outlet kinetic energy is negligible Negligible heat transfer for the system (adiabatic system, $\dot{Q}=0$) *Best performance:* Reversible and adiabatic	

Example

(1) Sketch a typical **diffuser** and label the inlet 1 and the outlet 2 .

(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

$$\frac{d}{dt}(m_{sys}) = \dot{m}_1 - \dot{m}_2$$
$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) - \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

Device Name(s)	Throttling Devices	
Purpose	Decrease pressure in the direction of flow.	
Physical Features	No work ($\dot{W}=0$) One-inlet / one-outlet	
Operating Conditions	Steady-state system One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Negligible changes in kinetic energy Negligible heat transfer for the system (adiabatic system, $\dot{Q}=0$)	

Example

(1) Sketch a typical **throttling valve** and label the inlet 1 and the outlet 2.

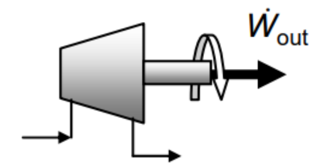
(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

$$\frac{d}{dt}(m_{sys}) = \dot{m}_1 - \dot{m}_2$$
$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) - \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

Turbines and Pumps, Compressors, Blowers, and Fans

Turbines and pumps, compressors, blowers and fans all do the same thing—they alter the properties of a flowing fluid by transferring energy by work into or out of the fluid. The tables below provide additional information about each of these devices. Look for the similarities *and* the unique differences.

Device Name(s)	**Turbines**	
Purpose	Produce mechanical power (shaft power) from a flowing stream of fluid.	

Physical Features	Mechanical power output ($\dot{W}_{\text{out}} > 0$) Often one-inlet/one-outlet	
Operating Conditions	Steady-state system One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Negligible changes in kinetic energy Negligible heat transfer for the system (adiabatic system, $\dot{Q} = 0$) *Best performance:* Reversible and adiabatic	

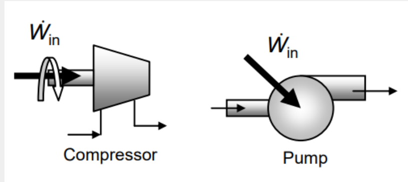
✔ Example

(1) Sketch a typical **turbine** and label the inlet 1 and the outlet 2.

(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

$$\frac{d}{dt}(m_{sys}) = \dot{m}_1 - \dot{m}_2$$

$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) - \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

Device Name(s)	**Pumps, Compressors, Blowers, and Fans**	
Purpose	Move, compress, and/or increase the pressure of a fluid.	
Physical Features	Mechanical power input \ ((\dot{W}_{\text{in}} >0)\) Usually one-inlet / one-outlet Pump $\rightarrow$ Liquids, large or small ΔP Compressors $\rightarrow$ gases, large ΔP Blowers $\rightarrow$ gases, small ΔP Fans $\rightarrow$ gases, very small ΔP	
Operating Conditions	Steady-state system One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Negligible changes in kinetic energy Negligible heat transfer for the system (adiabatic system, $\dot{Q} = 0$) *Best performance:* Reversible and adiabatic	

✔ Example

(1) Sketch a typical **compressor** and label the inlet 1 and the outlet 2.

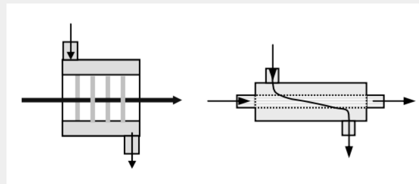
(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

$$\frac{d}{dt}(m_{sys}) = \dot{m}_1 - \dot{m}_2$$

$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) - \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

Heat Exchangers

Heat exchangers are devices designed to change the property of a flowing fluid by transferring energy by heat transfer into or out of a system. Some designs keep the fluids separate, such as in the radiator of your car. These are heat exchangers without mixing. Other times the fluids are typically mixed, such as in the plumbing to your shower—you adjust the temperature of the water by changing the flow rates of the hot and cold water. This is known as a heat exchanger with mixing. The tables below provide additional information about each of these devices. Look for the similarities *and* the unique differences.

Device Name(s)	**Heat Exchangers without Mixing**	
Purpose	Transfer thermal energy between fluid streams.	
Physical Features	No work ($\dot{W} = 0$) Separate flow paths for each stream (no mixing).	
Operating Conditions	Steady-state system One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Negligible changes in kinetic energy Negligible pressure drop for each stream (isobaric) Negligible heat transfer for the overall system (adiabatic system, $\dot{Q} = 0$) (This is not true if either fluid stream alone is the system.)	

Example

(1) Sketch a typical **heat exchanger without mixing** with two fluid streams — a hot stream and a cold stream. Label the inlet and the outlet of the cold stream C1 and C2. Label the inlet and the outlet of the hot stream H1 and H2.

(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

System is the Hot Stream

$$\frac{d}{dt}(m_{sys}) = \dot{m}_{H1} - \dot{m}_{H2}$$

$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_{H1}\left(h_{H1} + \frac{{V_{H1}}^2}{2} + gz_{H1}\right) - \dot{m}_{H2}\left(h_{H2} + \frac{{V_{H2}}^2}{2} + gz_{H2}\right)$$

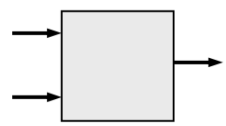
System is the Cold Stream

$$\frac{d}{dt}(m_{sys}) = \dot{m}_{C1} - \dot{m}_{C2}$$

$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_{C1}\left(h_{C1} + \frac{{V_{C1}}^2}{2} + gz_{C1}\right) - \dot{m}_{C2}\left(h_{C2} + \frac{{V_{C2}}^2}{2} + gz_{C2}\right)$$

System is the Complete Heat Exchanger

$$\frac{d}{dt}(m_{sys}) = \dot{m}_{C1} - \dot{m}_{C2} + \dot{m}_{H1} - \dot{m}_{H2}$$
$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_{C1}\left(h_{C1} + \frac{{V_{C1}}^2}{2} + gz_{C1}\right) - \dot{m}_{C2}\left(h_{C2} + \frac{{V_{C2}}^2}{2} + gz_{C2}\right) + \dot{m}_{H1}\left(h_{H1} + \frac{{V_{H1}}^2}{2} + gz_{H1}\right) - \dot{m}_{H2}\left(h_{H2} + \frac{{V_{H2}}^2}{2} + gz_{H2}\right)$$

Device Name(s)	**Heat Exchangers with Mixing**	
Purpose	Transfer thermal energy between fluid streams.	
Physical Features	Fluid streams mix No work ($\dot{W} = 0$)	
Operating Conditions	Steady-state system One-dimensional flow at inlets/outlets Negligible changes in gravitational potential energy Negligible changes in kinetic energy Negligible pressure drop (isobaric) Negligible heat transfer for the overall system (adiabatic system, $\dot{Q} = 0$)	

Example

(1) Sketch a typical heat exchanger with mixing with two fluid streams — a hot stream and a cold stream. Label the two inlet streams 1 and 2 and the outlet stream 3.

(2) Develop a model for this device using the information above to simplify the rate form of the conservation of mass and the conservation of energy equations.

$$\frac{d}{dt}(m_{sys}) = \dot{m}_1 + \dot{m}_2 - \dot{m}_3$$
$$\frac{d}{dt}(E_{sys}) = \dot{Q}_{\text{net, in}} + \dot{W}_{\text{net, in}} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) + \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right) - \dot{m}_3\left(h_3 + \frac{V_3^2}{2} + gz_3\right)$$

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