Stat 567
HW 4

53. If X is uniform over $(0,1)$, calculate $E[X^n]$ and $\mathrm{Var}(X^n)$.

55. Suppose that the joint probability mass function of X and Y is

$$P(X=i, Y=j) = \binom{j}{i} e^{-2\lambda}\lambda^j/j!, \quad 0 \le i \le j$$

(a) Find the probability mass function of Y.
(b) Find the probability mass function of X.
(c) Find the probability mass function of $Y - X$.

61. Let X and W be the working and subsequent repair times of a certain machine. Let $Y = X + W$ and suppose that the joint probability density of X and Y is

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda y}, \quad 0 < x < y < \infty$$

(a) Find the density of X.
(b) Find the density of Y.
(c) Find the joint density of X and W.
(d) Find the density of W.

76. Let X and Y be independent random variables with means μ_x and μ_y and variances σ_x^2 and σ_y^2. Show that

$$\mathrm{Var}(XY) = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2$$

79. With $K(t) = \log(E\left[e^{tX}\right])$, show that

$$K'(0) = E[X], \quad K''(0) = \mathrm{Var}(X)$$

53. $E[X^n] = \int_0^1 x^n \cdot 1 \cdot dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \underline{\underline{\frac{1}{n+1}}}$

$$E[X^{2n}] = \int_0^1 x^{2n}\, dx = \frac{x^{2n+1}}{2n+1} = \frac{1}{2n+1}$$

$$V[X^n] = E[X^{2n}] - (E[X^n])^2 = \frac{1}{2n+1} - \frac{1}{(n+1)^2}$$

$$= \frac{n^2 + 2n+1 - (2n+1)}{(n+1)^2(2n+1)}$$

$$= \underline{\underline{\frac{n^2}{(n+1)^2(2n+1)}}}$$

55. a) $P(Y=j) = \sum_{i=0}^{j} P(X=i, Y=j) = \sum_{i=0}^{j} \binom{j}{i} e^{-2\lambda} \frac{\lambda^j}{j!}$

$$= \frac{e^{-2\lambda}\lambda^j}{j!} \underbrace{\sum_{i=0}^{j} \binom{j}{i}}_{2^j} = \frac{e^{-2\lambda}(2\lambda)^j}{j!}, \; j \geq 0$$

Poisson (2λ)

b) $P(X=i) = \sum_{j=i}^{\infty} P(X=i, Y=j) = \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda} \frac{\lambda^j}{j!}$

Let $k = j-i$

$$= \sum_{k=0}^{\infty} \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!}$$

$$= e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\cancel{(k+i)!}}{i!\,k!} \frac{\lambda^{k+i}}{\cancel{(k+i)!}} = \frac{e^{-2\lambda}\lambda^i}{i!} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{e^{\lambda}}$$

$$= \frac{e^{-\lambda}\lambda^i}{i!}, \quad i \geq 0$$

Poisson(λ)

c) $$P(Y-X=k) = \sum_{i=0}^{\infty} P(X=i \cap Y=i+k)$$

$$= \sum_{i=0}^{\infty} \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!} = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{(as in (b))}$$

Poisson(λ)

61 a) $$f_X(x) = \int_x^{\infty} \lambda^2 e^{-\lambda y}\,dy = \lambda^2 \frac{e^{-\lambda y}}{(-\lambda)}\Bigg|_{y=x}^{\infty} = 0 - (-\lambda e^{-\lambda x})$$

$$= \lambda e^{-\lambda x}, \quad x > \infty$$

(Exponential(λ))

b) $$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\,dx = \lambda^2 e^{-\lambda y} x \Big|_{x=0}^{y}$$

$$= \lambda^2 y e^{-\lambda y}$$

Gamma($\alpha=2, \lambda$)

c) let $V = X$ $\quad$ $x = v$ $\quad$ $J^* = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$W = Y - X$ $\quad$ $Y = V + W$

$$g(v,w) = f(x,y)\cdot 1 = \lambda^2 e^{-\lambda(v+w)}, \quad 0 < v < v+w < \infty,$$

or $v > 0$, $w > 0$

d) Note that $g(v,w) = \lambda e^{-\lambda v} \cdot \lambda e^{-\lambda w}$

So $f_W(w) = \lambda e^{-\lambda w}$, $w > 0$

Exp(λ)

76. $V(XY) = E[X^2Y^2] - (E[XY])^2$

$= E[X^2]E[Y^2] - (E[X]E[Y])^2$ $\quad$ by independence

$= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2$

$= \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2$

79. $K'(t) = \dfrac{1}{E[e^{tX}]} \cdot E[Xe^{tX}] \Rightarrow K'(0) = E[X]$

$$K''(t) = \frac{E[e^{tX}]E[X^2e^{tX}] - E[Xe^{tX}]E[Xe^{tX}]}{(E[e^{tX}])^2} \Rightarrow K''(0) = \frac{E[X^2] - (E[X])^2}{1} = V[X]$$