Bruce Marron

Problem 1

Given

$$f(x,y) = \begin{cases} c(x+y) & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

from Midterm Exam,

$$c = 1,$$

$$E[X] = \int_0^1 x\, f_X(x)\, dx = 7/12$$

$$E[Y] = \int_0^1 y\, f_Y(y)\, dy = 7/12$$

$$E[X^2] = \int_0^1 x^2 f_X(x)\, dx = 5/12$$

$$E[Y^2] = \int_0^1 y^2 f_Y(y)\, dy = 5/12$$

$$Var[X] = E[X^2] - (E[X])^2 = 11/144$$

$$Var[Y] = E[Y^2] - (E[Y])^2 = 11/144$$

The correlation between X and Y,

$$\rho = \frac{COV[X,Y]}{Var[X]\, Var[Y]}$$

(OVER ⟹)

$$\mathrm{cov}[XY] = E[XY] - E[X]E[Y]$$

$$E[XY] = \iint xy\, f(x,y)\, dy\, dx$$

$$= \int_0^1\int_0^1 xy(x+y)\, dy\, dx$$

$$= \int_0^1\int_0^1 x^2y + xy^2\, dy\, dx$$

$$= \int_0^1 \frac{x^2}{2}y^2 + \frac{x}{3}y^3 \Big|_0^1\, dx$$

$$= \int_0^1 \frac{x^2}{2} + \frac{x}{3}\, dx$$

$$= \frac{x^3}{6} + \frac{x^2}{6}\Big|_0^1$$

$$= 2/6$$

$$\mathrm{cov}[XY] = E[XY] - E[X]E[Y]$$

$$= (2/6) - (7/12)(7/12)$$

$$= \frac{48-49}{144}$$

$$= -1/144$$

$$\therefore\ \rho = \frac{\mathrm{cov}[XY]}{\mathrm{Var}[X]\,\mathrm{Var}[Y]} = \frac{(-1/144)}{(11/144)(11/144)}$$

$$\boxed{= -1/121}$$

problem 2

Given

$p \equiv$ prob. A wins round

$(1-p) \equiv$ " B " "

Overall Winner $\equiv$ first player to win two more rounds than the other

$\equiv \left| \binom{\#wins}{A} - \binom{\#wins}{B} \right| = 2$

(a) Let

$1 \equiv$ A wins round

$0 \equiv$ A loses round

then A is overall winner as soon as either of following sequences occur,

$A_{OW} = \{0111,\ 1011,\ 111\}$

This implies that the $\binom{\#wins}{A} = \binom{\#wins}{B}$ prior to the start of the winning sequences. The length of the sequence of rounds played prior to start of a winning sequence is irrelevant.

$\therefore\ P(A_{OW}) = p(0111) + p(1011) + p(111)$

$= (2)(p^3)(1-p) + (p^3)$

(OVER ⇒)

(b) Let Ω_1 = state space for a triplet sequence
$= \{000, 001, 010, \ldots, 111\}$

$$X_1 = \begin{cases} 1 & \text{if } \omega = (111) \\ 0 & \text{otherwise} \end{cases}$$

$P(X_1 = 1) = 1/8$

$P(X_1 = 0) = 7/8$

$X_1 \sim \text{GEOM}(p = 1/8)$

$E[X_1] = 1/p = 1/\tfrac{1}{8} = 8$

For comparison, Eight rounds expected before see the sequence (111) and A wins if no other winning states possible.

Let Ω_2 = state space for a quadruple sequence
$= \{0000, 0001, 0010, \ldots, 1111\}$

Cardinality $\Omega_2 = 14 \Rightarrow$ collapse (0111) (1110) (1111) into one triplet state

$$X_2 = \begin{cases} 1 & \text{if } \omega = \{1011, 1101, 111\} \\ 0 & \text{otherwise} \end{cases}$$

$P(X_2 = 1) = 3/14$

$P(X_2 = 0) = 11/14$

$X_2 \sim \text{GEOM}(p = 3/14)$

$E[X_2] = 1/p = 1/\tfrac{3}{14} = 4.67$

~ five rounds expected before see either of the sequences (1011) (1101), (111) and A wins

$\therefore$ The expected number of games (rounds) played is 5.

Bruce Marron

Problem 3

Let

N = # customers entering store per day

$N \sim POIS(\lambda = 10)$

D = # dollars spent per customer

$D \sim UNIF(0, 100)$

$$T = \left(\sum_{i=1}^{N} D_i\right)$$ ← a cmpd rv

(a) $E[T] = E[E[D_i | N]] = E[N]\,E[D]$

$= (10)(50)$

$\boxed{= 500}$

(b) $Var[T] = E\left[Var[T|N]^{①}\right] + Var\left[E[T|N]^{②}\right]$

$$Var[T|N]^{①} = Var\left(\sum_{i=1}^{N} D_i \mid N\right)$$

$$= Var\left(\sum_{i=1}^{n} D_i\right)$$

$$= (n)(Var[D_i]) = (n)\frac{(100-0)^2}{12}$$

$= 833.33\, n$

(OVER ⇒)

②

$$E[T|N] = E\left(\sum_{i=1}^{N} D_i \,\middle|\, N\right)$$

$$= E\left(\sum_{i=1}^{n} D_i\right)$$

$$= (n)\, E[D_i]$$

$$= (n)(50)$$

$$= 50n$$

$$Var[T] = E[Var[T|N]] + Var[E[T|N]]$$

$$= E[833.33\, N] + Var[50\, N]$$

$$= 833.33\, E[N] + 50\, Var[N]$$

$$= (833.33)(10) + (50)(10)$$

$$\boxed{\doteq 8833}$$

Problem 4

Given

$$f_X(x) = 2kxe^{-2x^2}$$

$$Y = X^2$$

The density of Y can be found by the change-of-variable technique (Penn State, Online Course in Prob. Theory and Mathematical Stats https://onlinecourses.science.psu.edu/stat414/node/157)

$$Y = x^2$$
$$x = v(y) = y^{1/2}$$
$$v'(y) = \tfrac{1}{2} y^{-1/2}$$

$$f_Y(y) = f_X(v(y)) \cdot v'(y)$$
$$= 2k y^{1/2} e^{-2(y^{1/2})^2} \cdot \tfrac{1}{2} y^{-1/2}$$
$$\boxed{= ke^{-2y}}$$

Burke Maurer

Problem 5

Given for some rv X,

$E[X] = 3 = \mu$

$E[X^2] = 13$

Then

$Var[X] = E[X^2] - (E[X])^2$

$= 13 - 3^2$

$= 4 = \sigma^2$

For a lower bound on $P(-2 < X < 8)$ use Chebyshev's Inequality,

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

When $-2 < x < 8$,

$|X-3| \geq k$

$3-k < X < k+3$

$k = 5$

Then

$P(|X-3| \geq 5) \leq \frac{4}{25}$

$P(|X-3| < 5) \geq (1 - 4/25) = \boxed{21/25}$

∴ A lower bound on the value of X will be between -2 and 8 is at least $21/25 = 0.84$

Bruce Marron

Problem 6

Given

$X_1, X_2, \ldots X_{10}$ iid

$X_i \sim NORM(\mu = 70, \sigma = 15)$

$\sigma^2 = 225$

Let

$$\bar{X} = \left[\sum_{i=1}^{n} X_i / n \right] \equiv \text{sample mean}$$

$$S^2 = \left[\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{(n-1)} \right] \equiv \text{sample variance}$$

By Proposition 2.5 (Ross, p. 74)

1) $\bar{X}$ and S^2 are independent

2) $\bar{X} \sim NORM(\mu, \sigma^2/n)$

3) $V = \left[\frac{(n-1)}{\sigma^2} S^2 \right] \sim \chi^2_{n-1}$

then

$$P(68 < \bar{X} < 72, \sqrt{S^2} < 16) = P(68 < \bar{X} < 72) P(S^2 < 256)$$

by independence

(OVER ⇒)

$$P(68 < \bar{X} < 72) = \int_{68}^{72} f_X(x \mid \mu, \sigma^2/n)$$

$$= \int_{68}^{72} f_X(x \mid 70, 225/10)$$

$$\boxed{= 0.3267}$$

$$P(S^2 < 256) = P\left(\frac{\sigma^2}{n-1} V < 256\right)$$

$$= P\left(V < \left(\frac{9}{225}\right)(256)\right)$$

$$= P(V < 10.24)$$

$$= \int_{0}^{10.24} f_X(x \mid n-1)$$

$$= \int_{0}^{10.24} f_X(x \mid 9)$$

$$\boxed{= 0.6686}$$

$$\therefore P(68 < \bar{X} < 72, \sqrt{S^2} < 16) = P(68 < \bar{X} < 72)\, P(S^2 < 256)$$

$$= (0.3267)(0.6686)$$

$$\boxed{= 0.2184}$$

Bruce Manson

Problem 7

a) Given

$$P = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix}$$

states = 0, 1, 2

$$\begin{tikzcd} 0 \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & 1 \arrow[d, leftrightarrow] \\ & 2 \end{tikzcd}$$

i) There is one (1) communication class,

$\{0, 1, 2\}$ ← Recurrent

ii) All states are recurrent

iii) Because this Markov chain is regular,

" $\lim_{m \to \infty} P^m = \bar{P}$ where $\bar{P}$ is $n \times n$ matrix ea of whose rows is equal to p^T "

— Intro. to Dynamic Systems, Luenberger p. 231

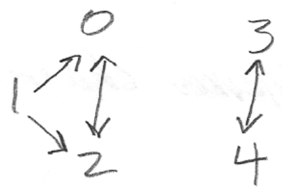
$$\boxed{p^T = [0.333, 0.333, 0.333]}$$

(OVER ⇒)

b) Given

$$P = \begin{bmatrix} .5 & 0 & .5 & 0 & 0 \\ .25 & .5 & .25 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & .5 & .5 \end{bmatrix}$$

states = 0, 1, 2, 3, 4



i. and ii) There are three (3) communication classes w/ the noted recurrent or transient states,

$\{0,2\} \leftarrow$ Recurrent

$\{1\} \leftarrow$ Transient

$\{3,4\} \leftarrow$ Recurrent

iii) The presence of a transient class means that the Markov chain is <u>not</u> <u>regular</u> and only the recurrent classes will reach equilibrium. Re-writing the matrix in canonical form gives,

$$P^* = \left[\begin{array}{c|c} P_1 & O \\ \hline R & Q \end{array}\right]$$

where P_1 is an $r \times r$ matrix of recurrent classes

R is a matrix representing the transition prob.s from transient to recurrent classes

Q is a substochastic matrix

$$\Downarrow$$

$$\begin{array}{c|c} & \begin{array}{ccccc} 0 & 2 & 3 & 4 & 1 \end{array} \\ \begin{array}{c} 0 \\ 2 \\ 3 \\ 4 \\ 1 \end{array} & \left[\begin{array}{cccc|c} .5 & .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 \\ 0 & 0 & .5 & .5 & 0 \\ \hline .25 & .25 & 0 & 0 & .5 \end{array}\right] \end{array}$$

Equilibrium for $\{0,2\}$
$p^T = [0.5, 0.5]$

Equilibrium for $\{3,4\}$
$p^T = [0.5, 0.5]$

Transition prob.s to recurrent classes

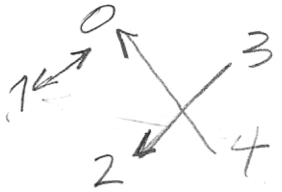
Final Exam
STAT 567

Bruce Mawron

c) Given

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1/4 & 3/4 & 0 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1/3 & 2/3 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \end{array}$$

states = 0, 1, 2, 3, 4



i and ii) There are three (3) communication classes w/ the noted recurrent or transient states,

$\{0,1\}$ ← Recurrent

$\{3,4\}$ ← Transient

$\{2\}$ ← Recurrent

iii) Again, the presence of a transient class means that the Markov chain is not regular and only the recurrent classes will reach equilibrium. In canonical form, which is the form as given,

$$\begin{array}{c|cc|c|cc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1/4 & 3/4 & 0 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ \hline 3 & 0 & 0 & 1/3 & 2/3 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Equilibrium for $\{0,1\}$
$p^T = [0.40, .60]$

Equilibrium for $\{2\}$
$p^T = [1]$

Transition prob.s to recurrent classes