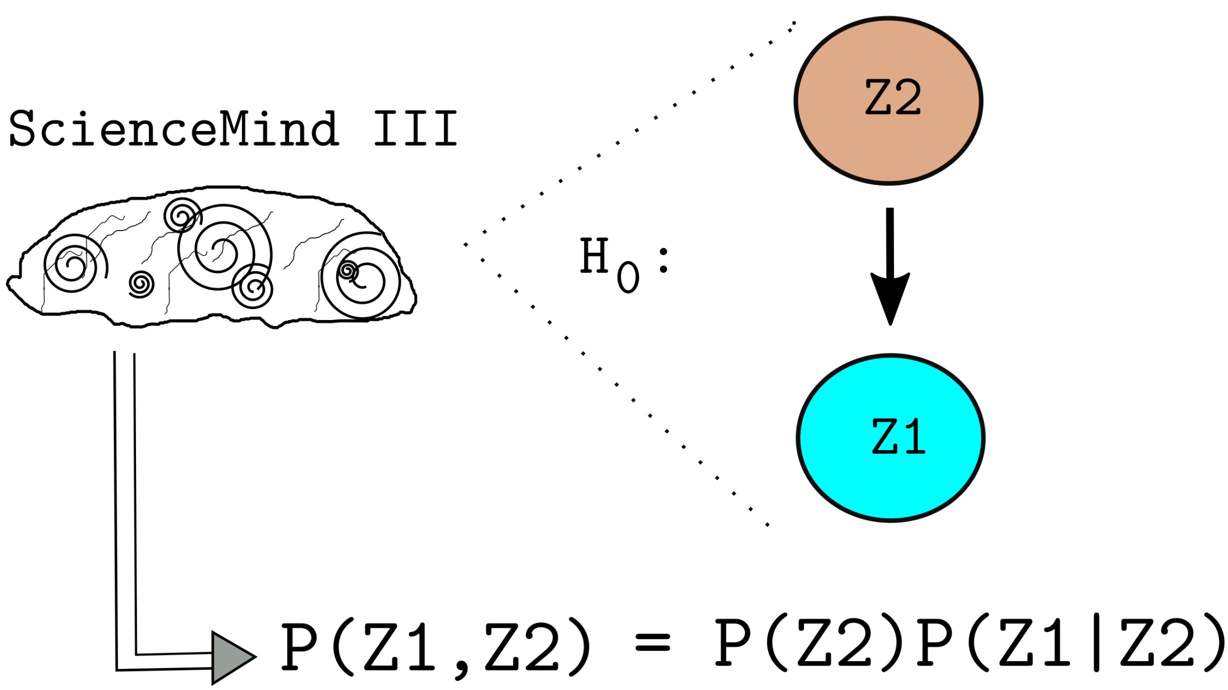
Probability

A Practical, Systems-Based Approach

Bruce D. Marron
UTECA, CDMX

Z1={observed system behavior}
Z2={possible causally-linked data}



This book was typeset using LaTeX software.

Preface

This book is dedeicated to hours of NOT understanding probability theory.

Table of Contents

Chapter 1

Introduction

Introduction to Probabi;lity Models, 10th edition by Sheldon Ross

1.1 Ross Examples

Suppose that n independent trials, each of which results in any of m possible outcomes with respective probabilities p1 , . . . , pm , $\sum pi = 1$, are continually performed. Let X denote the number of trials needed until each outcome has occurred at least once. Define the equation for X=n and explain the reasoning.

Let outcomes be $1, \ldots, m$ with probabilities $p_1, \ldots, p_m$ (independent across trials). For $n < m$, $\mathbb{P}(X = n) = 0$. For $n \geq m$

$$\boxed{\mathbb{P}(X = n) = \sum_{\varnothing \neq S \subseteq \{1,\ldots,m\}} (-1)^{|S|+1} \Big(\sum_{j \in S} p_j\Big) \Big(1 - \sum_{j \in S} p_j\Big)^{n-1}}$$

Why this is true (reasoning)

$X = n$ means: after $n-1$ trials at least one outcome is missing, and on trial n the **last** missing type appears for the **first** time. Equivalently, pick any nonempty set S of outcomes and consider the event that **all** outcomes in S have been missing up to time $n-1$ and one of them appears at time n. The probability that no outcome from S appears in a given trial is $1 - \sum_{j \in S} p_j$. So the chance they are all missing for the first $n-1$ trials and then one of them appears on trial n is

$$\Big(1 - \sum_{j \in S} p_j\Big)^{n-1} \Big(\sum_{j \in S} p_j\Big).$$

* But these events for different S overlap (inclusion–exclusion fixes the overcount), giving the alternating-sum formula above.

Coupon Collector Cheat Sheet (General Probabilities)

Problem: m outcomes with probabilities $p_1, \ldots, p_m > 0$, $\sum p_i = 1$.
X = number of trials until **all outcomes appear at least once**.
Define $q_S = \sum_{j \in S} p_j$ for subset $S \subseteq \{1, \ldots, m\}$.

Core Formulas

Quantity	Formula
PMF ($n \geq m$)	$\mathbb{P}(X = n) = \sum_{\varnothing \neq S} (-1)^{\lvert S \rvert + 1} \, q_S \, (1 - q_S)^{n-1}$
CDF	$\mathbb{P}(X \leq n) = \sum_{S} (-1)^{\lvert S \rvert} (1 - q_S)^n$
Expectation	$\mathbb{E}[X] = \sum_{\varnothing \neq S} (-1)^{\lvert S \rvert + 1} \frac{1}{q_S}$
Minimum trials	$X_{\min} = m$
Sanity Check	$\mathbb{P}(X = m) = m! \prod p_i$; $\sum_{n \geq m} \mathbb{P}(X = n) = 1$

Quick Insights

- Tail behavior dominated by largest $1 - q_{\{i\}} = 1 - \min p_i$; decays geometrically.
- Uniform probs: $p_i = 1/m \Rightarrow \mathbb{E}[X] = mH_m \approx m(\ln m + \gamma)$.
- Complexity: exact computation $O(2^m)$; feasible for $m \lesssim 20$.
- Simulation: use Monte Carlo for large m.
- First m trials all distinct probability: $m! \prod p_i$.

Algorithm (Bitmask / Combinations)

1. Enumerate all non-empty subsets S (bitmask or itertools.combinations).
2. Compute $q_S = \sum_{j \in S} p_j$.
3. Plug into PMF or expectation formulas.
4. Verify probabilities sum to 1 within tolerance.

Python Helper

```python
from itertools import combinations

def coupon_collector_pmf(p, n):
    m = len(p)
    prob = 0.0
    idx = list(range(m))
    for r in range(1, m+1):
        for subset in combinations(idx, r):
            q = sum(p[i] for i in subset)
            prob += ((-1)**(r+1)) * q * (1-q)**(n-1)
    return prob

def coupon_collector_expectation(p):
    m = len(p)
    exp_val = 0.0
    idx = list(range(m))
    for r in range(1, m+1):
        for subset in combinations(idx, r):
            q = sum(p[i] for i in subset)
            exp_val += ((-1)**(r+1)) * (1/q)
    return exp_val

# Example usage:
p = [0.2, 0.3, 0.5]
for n in range(3, 8):
    print(f"P(X={n}) = {coupon_collector_pmf(p, n):.5f}")
print("E[X] =", coupon_collector_expectation(p))
```

1.1.1 Example 2.5, p. 23

1.1.2 Excessive Elaborations

1.2 Ross Exercises Chapter 1

1.2.1 Examples

1.2.2 Excessive Elaborations

1.3 Third Principles

1.3.1 Examples

1.3.2 Excessive Elaborations

Chapter 2

Theory of Numbers

2.1 First Principles

2.1.1 Examples

2.1.2 Excessive Elaborations

2.2 Second Principles

2.2.1 Examples

2.2.2 Excessive Elaborations

2.3 Third Principles

2.3.1 Examples

2.3.2 Excessive Elaborations

Chapter 3

Irrational and Transcendent Numbers

3.1 First Principles

3.1.1 Examples

3.1.2 Excessive Elaborations

3.2 Second Principles

3.2.1 Examples

3.2.2 Excessive Elaborations

3.3 Third Principles

3.3.1 Examples

3.3.2 Excessive Elaborations

Index