

Statistics 100A

Homework 2 Solutions

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Chapter 2

9. *A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both. What percentage of its customers carry a credit card that the establishment will accept?*

The establishment will accept the credit card if it is American Express or VISA. Let A be the event a customer has American Express and let V be the event a customer has a VISA card. Then

$$P(A \cup V) = P(A) + P(V) - P(A \cap V) = 0.24 + 0.61 - 0.11 = \boxed{0.74}$$

12. *An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. These classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both French and German, and 6 that are in both Spanish and German. In addition, there are 2 students taking all 3 classes.*

Let S be the event that a student is in Spanish, let F be the event that a student is in French and let G be the event that a student is in German.

- (a) *If a student is chosen randomly, what is the probability that he or she is not in any of these classes?*

“Not in any of these classes” means “not in Spanish AND not in French AND not in German,” or, $P(S^c \cap F^c \cap G^c)$. By De Morgan’s laws, this is the same as finding $P((S \cup F \cup G)^c) = 1 - P(S \cup F \cup G)$.

To find $P(S \cup F \cup G)$, we must use Proposition 4.4, the principle of inclusion and exclusion.

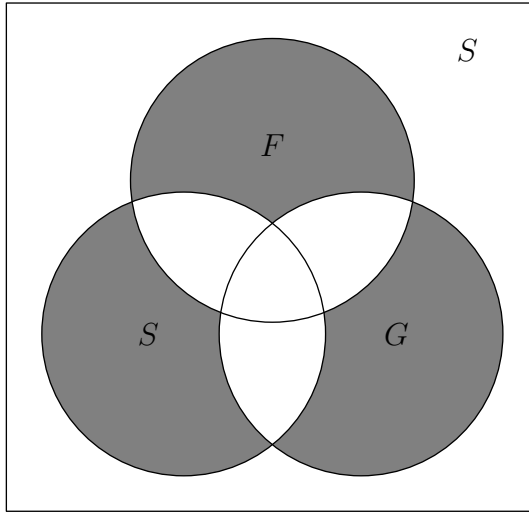
By Proposition 4.4 (or more specifically, Example 4a),

$$\begin{aligned} P(S \cup F \cup G) &= P(S) + P(F) + P(G) - P(S \cap F) - P(S \cap G) - P(F \cap G) \\ &\quad + P(S \cap F \cap G) \\ &= 0.28 + 0.26 + 0.16 - 0.12 - 0.06 - 0.04 + 0.02 \\ &= \boxed{0.50} \end{aligned}$$

Then, the probability that a student takes none of the classes is $1 - P(S \cup F \cup G) = 1 - 0.5 = \boxed{0.5}$.

- (b) *If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?*

This problem will be a mystery until you look at the Venn diagram! Do so!! We want to find the probability that a student only takes one of the courses.



We will start with the probability that a student takes any of the classes at all, $P(S \cup F \cup G)$ and from it, subtract off the areas that are not shaded in the diagram. First subtract off $P(S \cap F)$. Note that $P(S \cap F)$ includes part of the highest level intersection $P(S \cap F \cap G)$. Next, subtract off $P(F \cap G)$ but this *also* contains the highest level intersection which means we are subtracting out that intersection twice, so we add one $P(S \cap F \cap G)$ back. Now we subtract off $P(S \cap G)$, but again, this contains the highest order intersection, so we must add another one back again. This is the principle of inclusion and exclusion. Then,

$$\begin{aligned}
 P(\text{exactly one}) &= P(S \cup F \cup G) - P(S \cap F) - P(F \cap G) + P(S \cap F \cap G) \\
 &\quad - P(S \cap G) + P(S \cap F \cap G) \\
 &= P(S \cup F \cup G) - P(S \cap F) - P(F \cap G) - P(S \cap G) \\
 &\quad + 2P(S \cap F \cap G) \\
 &= 0.5 - 0.12 - 0.04 - 0.06 + 2 \cdot 0.02 \\
 &= \boxed{0.32}
 \end{aligned}$$

- (c) *If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?*

The phrase at least implies using the complement.

$$P(\text{at least 1 is taking a language class}) = 1 - P(\text{neither is taking a language class})$$

Since there are 100 students in the school and $P(\text{does not take a language class}) = 0.5$, there are 50 students that are not taking any language classes. We want to find the probability that we choose 2 of these students out of all possible choices of two students.

That is,

$$\frac{\binom{50}{2}}{\binom{100}{2}}$$

Then the probability that at least one of the two students is taking a language class is

$$1 - \frac{\binom{50}{2}}{\binom{100}{2}} \approx \boxed{0.753}$$

14. *The following data were given in a study of a group of 1000 subscribers to a certain magazine: in reference to job, marital status, and education there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect.*

Via the hint, let M denote the set of professionals, let W denote the set of married people, and let G be the set of college graduates. The hint then tells us to show that $P(M \cup W \cup G) > 1$.

$$\begin{aligned} P(M \cup W \cup G) &= P(M) + P(W) + P(G) - P(M \cap W) - P(M \cap G) - P(W \cap G) + \\ &\quad P(M \cap W \cap G) \\ &= 0.312 + 0.470 + 0.525 - 0.086 - 0.042 - 0.147 + 0.025 \\ &= 1.057 \\ &> 1 \end{aligned}$$

and probabilities cannot ever be greater than 1 or less than 0.

15. *If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt*

(a) *a flush?*

A flush consists of cards of all the same suit. First, of the 4 suits in a standard deck, we choose 1 of them for use in the flush. In each suit, there 13 cards. Of these 13 cards, we must choose 5 to make a full poker hand. Therefore, the probability of being dealt a flush is

$$P(\text{flush}) = \frac{\binom{4}{1} \binom{13}{5}}{\binom{52}{5}} \approx \boxed{0.002}$$

(b) *one pair?*

Surprisingly, this is more difficult to compute. First of all, from the 13 denominations in a standard deck of playing cards, we must select 1 denomination for the pair. Once this denomination is selected, we have 4 suits of this denomination to choose from. A pair consists of 2 cards, so from these 4 suits, we must choose two of them to be in the pair. So our answer will have the following in the answer

$$\binom{13}{1} \binom{4}{2}$$

We are not done yet.

Now consider the other 3 cards in the hand. From the 12 *remaining* denominations, we must choose 3 denominations for the other cards. Then, for each card separately, we have four possible suits to choose from for each of the three denominations and we must choose one for each card.

$$\binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1}$$

So the probability of being dealt a pair is

$$\frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}} \approx \boxed{0.4226}$$

(c) *Two pairs?*

First, from the 13 denominations, we must choose 2 denominations, one for each pair. Once of the denomination is chosen, there are 4 possible suits to choose from for this denomination, and we must choose 2 of them to form a pair. Again, for the second pair, there are 4 possible suits to choose from for its denomination, and we must choose 2 of them to form the second pair. So the following will be in our answer.

$$\binom{13}{2} \underbrace{\binom{4}{2}}_{\text{First pair}} \underbrace{\binom{4}{2}}_{\text{Second pair}}$$

Now we must draw the fifth card of a different denomination. Of the 11 remaining denominations, we must choose 1. Of the 4 suits available for that denomination, we choose 1 for the final card. Another way we can look at this is to consider that we have already drawn 2 denominations, both of which cannot be drawn again. That is, of the 52 cards originally in the deck, 8 of them are off limits since their denominations have already been dealt into a pair. Of the 44 remaining cards, we must draw 1.

$$\binom{11}{1} \binom{4}{1} = \binom{44}{1}$$

So the probability of being deal two pairs is

$$\frac{\binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{11}{1} \binom{4}{1}}{\binom{52}{5}} \approx \boxed{0.0475}$$

(d) *three of a kind?*

A three of a kind is when we have three of one denomination and two of some other two denominations. Of the 13 denominations, we choose 1 to use in the three of a kind. There are 4 suits of a denomination, and we must choose 3 of these suits to define a three of a kind.

$$\binom{13}{1} \binom{4}{3}$$

Of the 12 remaining denominations, we must choose 2 of them for the other 2 cards. Of the 4 suits of each denomination, we must choose one for each individual card. So the following will be in the answer

$$\binom{12}{2} \binom{4}{1} \binom{4}{1}$$

Then the probability of a three of a kind is

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}} = \boxed{0.0211}$$

(e) *a four of a kind?*

In a four of a kind, we have all four suits present of a particular denomination with one card of a different denomination. First, we draw one denomination of the 13 possible. Then, we choose all 4 of the possible suits for the denomination:

$$\binom{13}{1} \binom{4}{4}$$

For the final card, we choose a denomination from the remaining 12 denominations, and one suit out of the possible 4 for that denomination.

$$\binom{12}{1} \binom{4}{1}$$

Thus, the probability of drawing a four of a kind is given by

$$\frac{\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1}}{\binom{52}{5}} = \boxed{0.0002}$$

28. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 green balls is randomly selected, what is the probability that each of the balls will be

(a) *of the same color*

If we draw a ball from the urn, it is no longer available for subsequent draws. This is *sampling without replacement*. Of the 19 balls, we want to draw 3 of them.

The size of the sample space is $\binom{19}{3}$.

Since all three balls must be of the same color, we want to draw 3 of the red balls, or 3 of the green balls, or 3 of the blue balls:

$$\binom{5}{3} + \binom{6}{3} + \binom{8}{3}$$

So the probability of drawing three balls of the same color is,

$$\frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}} \approx \boxed{0.089}$$

Under sampling with replacement, we have the following. Each time the ball is drawn, it is replaced and can be drawn again. The probability of drawing 3 reds or 3 greens or 3 blues is,

$$\left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3 = \boxed{0.124}$$

(b) *of different colors.*

We are interested in drawing 1 red ball, 1 green ball and 1 blue ball:

$$\binom{5}{1} \binom{6}{1} \binom{8}{1}$$

Therefore, the probability of drawing three balls of different colors is,

$$\frac{\binom{5}{1} \binom{6}{1} \binom{8}{1}}{\binom{19}{3}} \approx \boxed{0.248}$$

Under sampling with replacement, the probability is

$$\frac{5}{19} \cdot \frac{6}{19} \cdot \frac{8}{19} = \boxed{0.035}$$

36. *Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they*

(a) *are both aces.*

There are 4 aces in a deck of cards. The probability that the first card is an ace is $\frac{4}{52}$. Once that ace is drawn, it cannot be chosen for the second draw (sampling without replacement). The probability that the second card is an ace (after the first card was an ace) is $\frac{3}{51}$. Then,

$$P(\text{two aces}) = \frac{4}{52} \cdot \frac{3}{51} = \boxed{0.0045}$$

The other way to consider this problem is as follows. There are 4 aces in a deck of cards and we must choose 2 of them. Therefore, out of $\binom{52}{2}$ possible hands of 2 cards, $\binom{4}{2}$ of them contain both aces.

$$\frac{\binom{4}{2}}{\binom{52}{2}} \approx \boxed{0.0045}$$

(b) *have the same value?*

There are 13 denominations (values) in a deck of cards. Both cards must be of the same denomination. Each denomination occurs 4 times (one per suit). This is really just a generalized version of the previous part.

Of the 13 denominations, we must choose 1, and then of the 4 suits of that particular denomination, choose 2 of them (a pair). That gives us

$$\frac{\binom{13}{1} \binom{4}{2}}{\binom{52}{2}} = \frac{13 \binom{4}{2}}{\binom{52}{2}} \approx \boxed{0.0588}$$

41. *If a die is rolled 4 times, what is the probability that 6 comes up at least once?*

Consider the complement: $P(6 \text{ comes up at least once}) = 1 - P(6 \text{ never appears on 4 rolls})$

Since rolls of a die are independent, the probability that a 6 will not show up on the $i = k$ th roll is the same as the probability that a 6 will show up on the $i \neq k$ th roll.

$$P(6 \text{ comes up at least once}) = 1 - \left(\frac{5}{6}\right) \left(\frac{5}{6}\right) \left(\frac{5}{6}\right) \left(\frac{5}{6}\right) = 1 - \left(\frac{5}{6}\right)^4 = \boxed{0.5177}$$

47. If there are 12 strangers in a room, what is the probability that no two of them celebrate their birthday in the same month?

First, let's find the size of the sample space. There are 12 months of the year and each of the people can initially have a birthday in any one of these 12 months. The size of the sample space is 12^{12} .

Now we impose the restriction that none of the 12 strangers has a birthday in the same month. Let's start with the first person. The first person can have a birthday in any of the 12 months. Once this month is "chosen", the next person cannot have a birthday in the same month. The second person can have a birthday in any one of 11 months and the third in any of 10 months and so on. Therefore, there are $12!$ ways for 12 strangers to have a birthday in a different month. So,

$$P(\text{each has a different birthmonth}) = \boxed{\frac{12!}{12^{12}} \approx 0.00005}$$

Chapter 2 Theoretical Exercises

11. If $P(E) = 0.9$ and $P(F) = 0.8$, show that $P(EF) \geq 0.7$. In general, prove Bonferroni's inequality, namely, $P(EF) \geq P(E) + P(F) - 1$.

Proof. Let E and F be events. Then by Proposition 4.3,

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

As with all probabilities, we know that $0 \leq P(E \cup F) \leq 1$ by Axiom 1. Therefore,

$$P(E) + P(F) - P(EF) \leq 1$$

and

$$P(E) + P(F) - 1 \leq P(EF)$$

which is equivalent to

$$P(EF) \geq P(E) + P(F) - 1$$

□

As a special case, let's consider $P(E) = 0.9$ and $P(F) = 0.8$. Then,

$$P(E \cup F) = 0.9 + 0.8 - \underbrace{P(EF)}_{\text{between 0 and 1}}$$

but $0 \leq P(E \cup F) \leq 1$ so $P(EF)$ is at least 0.7 and at most 1. Therefore, $0.7 \leq P(EF) \leq 1$ which under Axiom 1 becomes $\boxed{P(EF) \geq 0.7}$.

15. An urn contains M white and N black balls. If a random sample of size r is chosen, what is the probability that it contains exactly k white balls?

Let's start with the size of the sample space. Of the $M + N$ balls, we choose r balls. So the denominator is $\binom{M+N}{r}$.

Now, of the r balls that we draw, k must be white, which means that $r - k$ of the selected balls are black. So, of the M white balls, we select k of them and of the N black balls, we select $r - k$ of them. The numerator is then $\binom{M}{k} \binom{N}{r-k}$.

Then the probability is given by

$$\frac{\binom{M}{k} \binom{N}{r-k}}{\binom{M+N}{r}}$$

This is an example of the *hypergeometric* distribution, which we will study shortly.

Chapter 3 Problems

2. If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is $i, i = 2, 3, \dots, 12$?

In this problem you will construct a probability distribution. This problem is quite boring. Let X_1 be the face of the first die, and let X_2 be the face of the second die.

Finding Probability	Bayes Formula	Numerator	Denominator	Probability
$P(X_1 = 6 X_1 + X_2 = 1)$	$= \frac{P(X_1=6 \cap X_1+X_2=1)}{P(X_1+X_2=1)}$	not possible	0	0
$P(X_1 = 6 X_1 + X_2 = 2)$	$= \frac{P(X_1=6 \cap X_1+X_2=2)}{P(X_1+X_2=2)}$	not possible	$\frac{1}{36}$	0
$P(X_1 = 6 X_1 + X_2 = 3)$	$= \frac{P(X_1=6 \cap X_1+X_2=3)}{P(X_1+X_2=3)}$	not possible	$\frac{2}{36}$	0
$P(X_1 = 6 X_1 + X_2 = 4)$	$= \frac{P(X_1=6 \cap X_1+X_2=4)}{P(X_1+X_2=4)}$	not possible	$\frac{3}{36}$	0
$P(X_1 = 6 X_1 + X_2 = 5)$	$= \frac{P(X_1=6 \cap X_1+X_2=5)}{P(X_1+X_2=5)}$	not possible	$\frac{4}{36}$	0
$P(X_1 = 6 X_1 + X_2 = 6)$	$= \frac{P(X_1=6 \cap X_1+X_2=6)}{P(X_1+X_2=6)}$	not possible	$\frac{5}{36}$	0
$P(X_1 = 6 X_1 + X_2 = 7)$	$= \frac{P(X_1=6 \cap X_1+X_2=7)}{P(X_1+X_2=7)}$	$\frac{1}{36} : \begin{array}{ c c } \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{ c } \hline \bullet \\ \hline \end{array}$	$\frac{6}{36}$	$\frac{1}{6}$
$P(X_1 = 6 X_1 + X_2 = 8)$	$= \frac{P(X_1=6 \cap X_1+X_2=8)}{P(X_1+X_2=8)}$	$\frac{1}{36} : \begin{array}{ c c } \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{ c } \hline \bullet \\ \hline \end{array}$	$\frac{5}{36}$	$\frac{1}{5}$
$P(X_1 = 6 X_1 + X_2 = 9)$	$= \frac{P(X_1=6 \cap X_1+X_2=9)}{P(X_1+X_2=9)}$	$\frac{1}{36} : \begin{array}{ c c } \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{ c } \hline \bullet \\ \hline \end{array}$	$\frac{4}{36}$	$\frac{1}{4}$
$P(X_1 = 6 X_1 + X_2 = 10)$	$= \frac{P(X_1=6 \cap X_1+X_2=10)}{P(X_1+X_2=10)}$	$\frac{1}{36} : \begin{array}{ c c } \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{ c } \hline \bullet \\ \hline \end{array}$	$\frac{3}{36}$	$\frac{1}{3}$
$P(X_1 = 6 X_1 + X_2 = 11)$	$= \frac{P(X_1=6 \cap X_1+X_2=11)}{P(X_1+X_2=11)}$	$\frac{1}{36} : \begin{array}{ c c } \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{ c } \hline \bullet \\ \hline \end{array}$	$\frac{2}{36}$	$\frac{1}{2}$
$P(X_1 = 6 X_1 + X_2 = 12)$	$= \frac{P(X_1=6 \cap X_1+X_2=12)}{P(X_1+X_2=12)}$	$\frac{1}{36} : \begin{array}{ c c } \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{ c } \hline \bullet \\ \hline \end{array}$	$\frac{1}{36}$	1

9. Consider 3 urns. Urn A contains 2 white and 4 red balls; urn B contains 8 white and 4 red balls; and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white, given that exactly 2 white balls were selected?

By Bayes' Formula,

$$P(\text{Ball from A white} | 2 \text{ white balls selected}) = \frac{P(\text{Ball from A white} \cap 2 \text{ white balls selected})}{P(2 \text{ white balls selected})}$$

First, consider $P(\text{Ball from A white} \cap 2 \text{ white balls selected})$. This means that the ball chosen from A must be white and either the ball from B or C is white *and the other one is not*. The probability of drawing a white ball from A is $\frac{2}{6}$. Likewise, the probability of drawing a white ball from B, C is $\frac{8}{12}$ and $\frac{1}{4}$ respectively. The probability of not drawing the white ball from B, C is $\frac{4}{12}$ and $\frac{3}{4}$ respectively. So,

$$P(\text{Ball from A white} \cap 2 \text{ white balls selected}) = \frac{2}{6} \left(\frac{8}{12} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{4}{12} \right)$$

Now consider $P(2 \text{ white balls selected})$. There are 3 ways we can choose the two white balls: choose a white ball from A and B and a red ball from C, choose a white ball from B and C and a red ball from A and choose a white ball from A and C and a red ball from B. So the denominator is

$$\frac{2}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{4}{6} \cdot \frac{8}{12} \cdot \frac{1}{4} + \frac{2}{6} \cdot \frac{4}{12} \cdot \frac{1}{4}$$

The probability is

$$\frac{\frac{2}{6} \left(\frac{8}{12} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{4}{12} \right)}{\frac{2}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{4}{6} \cdot \frac{8}{12} \cdot \frac{1}{4} + \frac{2}{6} \cdot \frac{4}{12} \cdot \frac{1}{4}} = \boxed{\frac{7}{11}}$$

11. Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces; let A_s be the event that the ace of spades is chosen and let A be the event that at least one ace is chosen. Find

(a) $P(B|A_s)$

We must find the probability that both cards are aces *given* that the ace of spades is chosen. By Bayes' Formula,

$$P(B|A_s) = \frac{P(B \cap A_s)}{P(A_s)}$$

$B \cap A_s$ is the event that both of the cards are aces and one of them is the ace of spades. There are 3 ways for this to happen, an ace of spades and an ace of diamonds, an ace of spades and an ace of hearts or an ace of spades and an ace of clubs. There are $\binom{52}{2}$ ways of drawing 2 cards from a deck.

Therefore,

$$P(B \cap A_s) = \frac{3}{\binom{52}{2}}$$

The event A_s is the event that an ace of spades occurs. This can happen in one of 51 ways: an ace of spades and any other card in the deck. So,

$$P(A_s) = \frac{51}{\binom{52}{2}}$$

Therefore,

$$P(B|A_s) = \frac{\frac{3}{\binom{52}{2}}}{\frac{51}{\binom{52}{2}}} = \frac{3}{51} = \boxed{\frac{1}{17}}$$

(b) $P(B|A)$

By Bayes Rule:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$A \cap B$ is the event that at least one ace is chosen and both cards are aces which is redundant: $A \cap B = B$.

$P(B)$ is the probability that both cards are aces. Of the 4 aces in the deck, we must choose 2 of them. Therefore,

$$P(B) = \frac{\binom{4}{2}}{\binom{52}{2}}$$

$P(A)$ is the probability that at least one of the cards is an ace. We use the complement; $P(A) = 1 - P(\text{neither card is an ace})$. To find $P(\text{neither card is an ace})$, of the 48 non-aces, we want to choose 2 of them and of the 4 aces, we want to choose none of them. Then,

$$P(A) = 1 - P(\text{neither card is an ace}) = 1 - \frac{\binom{48}{2}}{\binom{52}{2}}$$

Then,

$$P(B|A) = \frac{P(B)}{P(A)} = \frac{\frac{\binom{4}{2}}{\binom{52}{2}}}{1 - \frac{\binom{48}{2}}{\binom{52}{2}}} = \frac{\binom{4}{2}}{\binom{52}{2} - \binom{48}{2}} = \boxed{0.0303}$$

20. *Fifty-two percent of the students at a certain college are females. Five percent of the students are majoring in computer science. Two percent of the students are women majoring in computer science. If a student is selected at random, find the conditional probability that,*

Let F be the event that the randomly selected student is female. Let C be the event that the randomly selected student is a computer science major.

We know that $P(F) = 0.52$ and $P(C) = 0.05$. Since the wording is “two percent of the students are women majoring in computer science”, we know that $P(F \cap C) = 0.02$. Had the wording said “two percent of the females are majoring in computer science” then we would know that $P(F|C) = 0.02$ but that is not the case here.

- (a) *this student is female, given that the student is majoring in computer science.*

Using Bayes Rule,

$$P(F|C) = \frac{P(F \cap C)}{P(C)} = \frac{0.02}{0.05} = \boxed{0.4}$$

- (b) *this student is majoring in computer science, given that the student is female.*

Using Bayes Rule,

$$P(C|F) = \frac{P(C \cap F)}{P(F)} = \frac{0.02}{0.52} = \boxed{0.038}$$

26. *Suppose that 5 percent of men and .25 percent of women are colorblind. A colorblind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females. What if the population consisted of twice as many males as females?*

Let M be the event that the randomly selected person is a man, then M^c is the event that the person is a woman. Let C be the event that the randomly selected person is color blind. Then $P(C|M) = 0.05$ and $P(C|M^c) = 0.0025$. Since the number of men and women is the same, $P(M) = P(M^c) = 0.5$. A color blind person is chosen.

$$P(M|C) = \frac{P(M \cap C)}{P(C)}$$

But what are the numerator and denominator? We do not know...yet.

For event C , there are two possibilities, a colorblind person is male M or female M^c , that is

$$P(C) = P(C \cap M) + P(C \cap M^c)$$

The key here is to play around with Bayes Rule to get the numerator. We know that

$$P(C|M) = \frac{P(C \cap M)}{P(M)} \Rightarrow P(C \cap M) = P(C|M)P(M)$$

So, $P(C \cap M) = P(C|M)P(M) = 0.05 \cdot 0.5 = 0.025$ and $P(C \cap M^c) = P(C|M^c)P(M^c) = 0.0025 \cdot 0.5 = 0.00125$

Therefore, the probability is

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} = \frac{0.5 \cdot 0.05}{0.5 \cdot 0.05 + 0.5 \cdot 0.0025} = \boxed{0.9524}$$

This problem is an example of the Law of Total Probability.

Now if the population consisted of twice as many women, then $P(M) = 0.67$, $P(M^c) = 0.33$

Then,

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} = \frac{0.05 \cdot 0.67}{0.05 \cdot 0.67 + 0.0025 \cdot 0.33} = \boxed{0.976}$$

36. Stores A , B , C have 50, 75, and 100 employees, and, respectively, 50, 60, and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C ?

There are a total of 225 employees among the three stores. The marginal probabilities for the stores are $P(A) = \frac{50}{225}$, $P(B) = \frac{75}{225}$ and $P(C) = \frac{100}{225}$. We also know that $P(W|A) = 0.5$, $P(W|B) = 0.6$, $P(W|C) = 0.7$.

Then,

$$\begin{aligned} P(W) &= P(W|A)P(A) + P(W|B)P(B) + P(W|C)P(C) \\ &= 0.5 \cdot \frac{50}{225} + 0.6 \cdot \frac{75}{225} + 0.7 \cdot \frac{100}{225} \\ &= 0.622 \end{aligned}$$

Then we can use Bayes' Rule

$$\begin{aligned} P(C|W) &= \frac{P(C \cap W)}{P(W)} \\ &= \frac{P(W|C)P(C)}{P(W)} \\ &= \frac{0.7 \cdot \frac{100}{225}}{0.622} \\ &= \boxed{0.500} \end{aligned}$$

55. In a class there are 4 freshmen boys, 6 freshmen girls, 6 sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

We do not know how many students in total there are in the class because we do not know how many sophomore girls are in the class. Let n be the number of sophomore girls in the class. Then,

$$P(F) = \frac{10}{16+n}, P(B) = \frac{10}{16+n}, P(F \cap B) = \frac{4}{16+n}$$

But we want sex and class to be independent so by independence, we need

$$P(F \cap B) = P(F)P(B) = \left(\frac{10}{16+n}\right)^2$$

So we set $\frac{4}{16+n} = \left(\frac{10}{16+n}\right)^2$ and solve for n .

$$\begin{aligned} \frac{4}{16+n} &= \left(\frac{10}{16+n}\right)^2 \\ 4 &= \frac{10^2}{16+n} \\ 4(16+n) &= 10^2 \\ n &= \frac{10^2}{4} - 16 \\ n &= 9 \end{aligned}$$

There would need to be $n = \boxed{9}$ sophomore girls in the class.

Chapter 3 Theoretical Exercises

1. If $P(A) > 0$, show that $P(AB|A) \geq P(AB|A \cup B)$

Proof. For this proof, use the “left hand side, then right hand side, meet in the middle” approach as I call it.

Left Hand Side: First note that by Bayes’ Rule, $P(A \cap B|A) = \frac{P((A \cap B) \cap A)}{P(A)}$. If we draw out the Venn diagram, we see that $\frac{P((A \cap B) \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$.

Right Hand Side: By Bayes’ Rule:

$$P(A \cap B|A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}$$

Meet in the Middle: We must do something with $\frac{P(A \cap B)}{P(A)}$ and $\frac{P(A \cap B)}{P(A \cup B)}$. Note that it is always true that $P(A \cup B) \geq P(A)$, therefore $\frac{1}{P(A)} \geq \frac{1}{P(A \cup B)}$. Thus,

$$\frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B)}{P(A \cup B)}$$

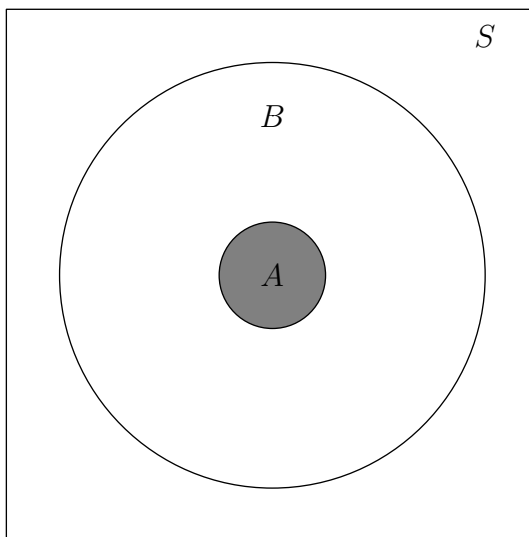
□

2. If $A \subset B$, express the following probabilities as simply as possible.

A Venn diagram can help with each case.

(a) $P(A|B)$

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$.



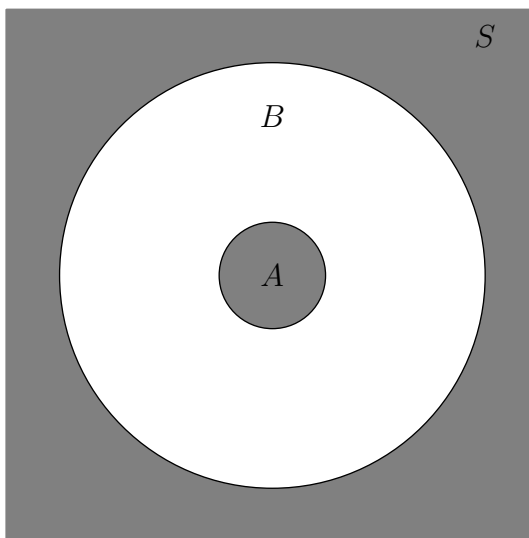
Since $A \subset B$ we know that $P(A \cap B) = P(A)$.

So we get

$$\boxed{\frac{P(A)}{P(B)}}$$

(b) $P(A|B^c)$

Again, we know that $P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}$.



But $P(A \cap B^c) = 0$ therefore we get $\boxed{0}$.

(c) $P(B|A)$

By now we know we must worry about mainly the intersection, $P(B \cap A)$. Since $A \subset B$ we know that $P(B \cap A) = P(A)$. So we get

$$\frac{P(A)}{P(A)} = \boxed{1}$$

(d) $P(B|A^c)$

By Bayes Rule.

$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)}$$

This creates a complicated region because we know that B contains A as well as *part of* A^c . We just stop at what we know,

$$\boxed{\frac{P(B \cap A^c)}{P(A^c)}}$$

9. Consider two independent tosses of a fair coin. Let A be the event that the first toss lands heads, let B be the event that the second toss lands heads and let C be the event that both land on the same side. Show that the events A , B and C are pairwise independent - but not independent.

The problem is asking to show that pairs of A , B and C are independent, but that A , B and C altogether are *not* independent.

We must show that $P(A \cap B) = P(A)P(B)$, $P(B \cap C) = P(B)P(C)$, $P(A \cap C) = P(A)P(C)$ but that $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

This problem is trivial because coin tosses are independent...

$$\begin{aligned} P(A \cap B) &= P(A)P(B) = 0.5 \cdot 0.5 = 0.25 \\ P(A \cap C) &= P(A)P(C) = 0.5 \cdot (0.5 \cdot 0.5) = 0.167 \\ P(B \cap C) &= P(B)P(C) = 0.5 \cdot 0.5 = 0.25 \end{aligned}$$

Let's dissect the intersection $A \cap B \cap C$ now. This means that the first toss shows a head, the second toss shows a head, and both tosses show the same side of the coin. C is automatically satisfied since B and A are satisfied,, so

$$P(A \cap B \cap C) = 0.5 \cdot 0.5 = 0.25$$

In general however, the probability that both coins will show the same side is

$$P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

So, $P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{16}$ but $P(A \cap B \cap C) = \frac{1}{4}$.

Therefore, $P(A \cap B \cap C) \neq P(A)P(B)P(C)$. Thus, A , B , and C are pairwise independent, but not independent.