Statistics 100A
Homework 1 Solutions

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Chapter 1

1. (a) *How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?*

   The first two places can contain any letter of alphabet, 1 of 26 possibilities. The last five places can contain any single digit number, 1 of 10 possibilities.

26	26	10	10	10	10	10

   If we consider each place an experiment with n_i outcomes, then the generalized basic principle of counting implies that the total number of possibilities

   $$\prod_{i=1}^{7} n_i = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^2 10^5 = \boxed{67{,}600{,}000}$$

   (b) *Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.*

   Again, the first two places can contain any letter of alphabet, 1 of 26 possibilities. Once we have drawn a letter for the first place, that letter is no longer available for the second place. The last five places can contain any single digit number, 1 of 10 possibilities. Once a number has been chosen for the first place, it is no longer available for the next place and so on.

26	25	10	9	8	7	6

   $$\prod_{i=1}^{7} n_i = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = \boxed{19{,}656{,}000}$$

2. *How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on a 3, the second on 4, the third on 3, and the fourth on a 1?*

   This is almost identical to the previous problem. Rather than each experiment being a place on a license plate, each experiment in this problem is the number of pips showing on a single die.

6	6	6	6

   which yields a result of $6^4 = \boxed{1{,}296}$.

5. *For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9; the second digit was either 0 or 1; the third digit was any integer between 1 and 9. How many area codes were possible? How many area codes starting with a 4 were possible?*

The first digit must be between 2 and 9 so there are 8 possibilities. The second digit must be 0 or 1 so there are 2 possibilities. Finally, the last digit can be any number between 1 and 0, so there are 9 possibilities.

8	2	9

which yields a result of $8 \cdot 2 \cdot 9 = \boxed{144}$.

To answer the second question, we just place a constraint on the first digit. Since the first digit must be 4, there is only possibility for the first digit, not 8 as in the previous part.

1	2	9

which yields a result of $1 \cdot 2 \cdot 9 = \boxed{18}$.

7. (a) *In how many ways can 3 boys and 3 girls sit in a row?*

The problem does not explicitly state how many chairs there are in the row, so we will assume that there are 6 chairs, one for each person. Since there is no restriction on gender, it is irrelevant to the problem. We might as well find the number of ways to seat 6 children in a row, which is $\boxed{6! = 720}$.

(b) *In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together.*

In other words, boys are seated in one section of the row, and girls are seated in the other section of the row. There are two possible ways to seat these groups

B	B	B	G	G	G

G	G	G	B	B	B

Within the boy section, there are 3! ways to arrange the boys. Within the girl section, there are 3! ways to arrange the girls. In total, there are $\boxed{2 \cdot 3! \cdot 3! = 72}$ ways to seat 3 boys and 3 girls in a row such that the boys sit together and the girls sit together.

(c) *In how many ways if only the boys must sit together?*

Now, only the boys must sit as a group and the girls fill in the empty chairs. There are several (4) ways this is possible

B	B	B	G	G	G

G	B	B	B	G	G

G	G	B	B	B	G

G	G	G	B	B	B

There are 4 ways to pick the group of seats the boys must occupy. Within this boy section, there are 3! ways to arrange the boys. Although the girls are spread throughout the row, there are still 3! ways to arrange the girls.

There are $\boxed{4 \cdot 3! \cdot 3! = 144}$ ways to arrange the boys and girls if the boys must sit together.

(d) *In how many ways if no two people of the same sex are allowed to sit together?*

This is just another way of saying "alternating seats boy/girl." There are two ways to alternate sexes:

G	B	G	B	G	B

B	G	B	G	B	G

Within each gender there are 3! ways to arrange the children. There are $\boxed{2 \cdot 3! \cdot 3! = 72}$ ways to arrange the boys and girls in an alternating way.

There is another way to solve this problem. First, we note that there is no gender restriction on the first person, so we can choose one of 6 children to sit in the first seat. The second seat must be for a student of the opposite sex, of which there are 3. The third seat again must be someone of the opposite sex of chair number 2, so there are 2 possibilities (1 was already chosen for chair number 1 and so on).

6	3	2	2	1	1

Using this method, there are $\boxed{6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 72}$ ways to arrange the boys and girls in an alternating way.

8. *How many different letter arrangements can be made from the letters*

(a) Fluke

The word FLUKE contains 5 distinct letters, so there are $\boxed{5! = 120}$ arrangements.

(b) Propose

Now we have the case where letters appear more than once. **The key here is to consider each character in entire word to be n distinct objects that are partitioned into non-overlapping subgroups, one for each *unique* letter in the word**. Let n be the number of letters in the word PROPOSE, so $n = 7$ and let r be the number of unique letters in the word PROPOSE, so $r = 5$. We now must find the size of each subgroup, n_i.

For P, $n_1 = 2$ and for O, $n_2 = 2$. For R, $n_3 = 1$; for S, $n_3 = 1$; for E $n_4 = 1$.

With this information, we use the multinomial coefficient:

$$\frac{n!}{n_1 \cdot n_2 \cdot \ldots \cdot n_r} = \frac{7!}{2!2!1!1!1!} = \frac{7!}{4} = \boxed{1{,}260}$$

(c) Mississippi

Again, we use the multinomial coefficient with $n = 11, r = 4$.

For M, $n_1 = 1$; for I, $n_2 = 4$; for S, $n_3 = 4$; for P, $n_4 = 2$.

$$\frac{n!}{n_1 \cdot n_2 \cdot \ldots \cdot n_r} = \frac{11!}{1!4!4!2!} = \boxed{34{,}650}$$

(d) Arrange

$$\frac{n!}{n_1 \cdot n_2 \cdot \ldots \cdot n_r} = \frac{7!}{2!2!} = \frac{7!}{4} = \boxed{1{,}260}$$

9. *A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?*

(why would a child care?)

We are given a set of 12 distinct blocks that are dividing into 4 non-overlapping groups. We use the multinomial coefficient.

$$\frac{n!}{n_1 \cdot n_2 \cdot \ldots \cdot n_r} = \frac{12!}{6!4!1!1!} = \boxed{27,\ 720}$$

15. *A dance class consists of 22 students, 10 women, and 12 men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?*

Of the 10 women in the class, we must *choose* 5 of them and of the 12 men in the class, we must *choose* 5 of them. We do not worry about the pairing off right now. We know the solution is going to contain the term

$$\binom{10}{5}\binom{12}{5}$$

Now what to do about the pairing? We know that we must pair 5 men to 5 women.

Consider the first man (or the first woman, whichever). He can be paired to any of 5 women, so there are 5 possible pairings for him. Once this man is paired to a woman, that man and woman are no longer available for pairing.

Consider the second man. Since one woman has already been paired, he has 4 possible pairings. The third man then has 3 possible pairings and so on. Thus, the number of pairings possible is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$. Then the total number of ways to choose 5 men and 5 women from 12 men and 10 women and then pair them off is

$$\binom{10}{5}\binom{12}{5} 5! = \boxed{23{,}950{,}080}$$

28. *If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?*

The problem is not clearly written, however, we just want to find the number of ways of placing 8 new teachers into 4 schools with no restriction on the number assigned to each school. We consider each of the 8 teachers, and the fact that they have 4 possible schools to be assigned to

4	4	4	4	4	4	4	4

So there are $\boxed{4^8 = 65,536}$ divisions possible.

In the second part of the problem, we still have 8 distinct teachers that must be divided into 4 non-overlapping groups (schools), but now we impose the restriction that each school receives exactly 2 teachers. We have jumped back to the multinomial coefficient. Here, $n = 8$ and $r = 4$ and $n_i = 2, 1 \le i \le r$.

$$\frac{n!}{n_1 \cdot n_2 \cdot \ldots \cdot n_r} = \frac{8!}{2!2!2!2!} = \frac{8!}{2^4} = \boxed{2{,}520}$$

Chapter 2 Problems

1. *A box contains 3 marbles, 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box, then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without first replacing the first marble.*

   Let R represent the red marble, G the green marble, and B the blue marble.

   The sample space consists of all possible results of this experiment.

   Consider first drawing the red marble. The second draw can be either the red, green or blue marble and so on. The sample space will have size $3 \cdot 3$ since there are three possibilities for the first draw and three possibilities for the second draw. The sample space is

   $$S = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),(B,R),(B,G),(B,B)\}$$

   If after the first draw we do not replace the first marble, then the sample space has size $3 \cdot 2$ since there are three possibilities for the first draw and only two for the second draw. Then the sample space is

   $$S = \{(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}$$

2. *A die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n? What is $(\bigcup_1^\infty E_n)^c$?*

   First let's consider the sample space. This is a fairly theoretical question, and it is not trivial. First, consider an integer n that represents the number of rolls necessary for a 6 to appear. n takes on an any positive integer value, $n \geq 1$. The sample space consists of all possible outcomes of this experiment which is difficult to write out, so we write it out abstractly instead. Let $(x_1, x_2, \ldots, x_{n-1}, x_n)$ be the vector of possible outcomes of the experiment. Each x_i is a number denoting the number of pips that show on the die on the ith experiment. We cannot stop here though. We must recall the constraints of the experiment. The experiment stops after a 6 is shown on the die and we consider this the nth run of the experiment. Thus, $x_n = 6$. All of the other x_is prior to the nth run must be between 1 and 5: $1 \leq x_i \leq 5, 1 \leq i \leq n-1$. A correct response should have all of these features: 1) definition of n, 2) vector of possible outcomes, 3) vector correctly noted with the constraints.

   For the second part of the problem, we note that E_n is the event that n rolls are necessary to complete the experiment. The sample space is simply the vector $\mathbf{x}$ for the particular value of n. In this particular context, this question is redundant.

   The final part of the question asks us to consider $(\bigcup_1^\infty E_n)^c$. The union inside represents all E_n, that is roughly speaking, the events that the experiment stops after some n. The complement then is the event that a 6 never appears.

3. *Two dice are thrown. Let E be the event that the sum of the dice is odd; let F be the event that at least one of the dice lands on 1; and let G be the event that the sum is 5. Describe the events $EF, E \cup F, FG, EF^c$, and EFG.*

   (Note that $EF = E \cap F$)

   EF is the event that the sum of the dice is odd *and* at least one of the dice lands on 1.

   $$EF = \{(1,2),(1,4),(1,6),(2,1),(4,1),(6,1)\}$$

   $E \cup F$ is the event that the sum of the dice is odd *or* at least one of the dice lands on 1.

   $$\begin{aligned} E \cup F &= \{(1,2),(1,4),(1,6),(2,1),(4,1),(6,1),(2,3),(2,5),(3,2),(3,4),(3,6),(4,3),(4,5), \\ &\quad (5,2),(5,4),(5,6),(6,3),(6,5)\} \end{aligned}$$

   FG is the event that the sum of the dice is odd *and* the sum is 5.

   $$FG = \{(1,4),(4,1)\}$$

   EFG is the event that the sum is odd *and* the sum is 5 *and* at least one of the dice shows a 1.

   $$EFG = \{(1,4),(4,1)\}$$

   EF^c is the event that the sum of the dice is odd *and* at least one of the dice is *not* a 1.

   $$EF^c = \{(2,3),(2,5),(3,2),(3,4),(3,6),(4,3),(4,5),(5,2),(5,4),(5,6),(6,3),(6,5)\}$$

6. *A hospital administrator codes incoming patients suffering from gunshot wounds according to whether or not they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of coding such a patient.*

   (a) *Give the sample space of this experiment.*

   The sample space consists of all possible outcomes of this experiment. That is, all possible combinations of insurance codings and condition codings.

   $$S = \{(0,g),(0,f),(0,s),(1,g),(1,f),(1,s)\}$$

   (b) *Let A be the event that the patient is in serious condition. Specify the outcomes in A.*

   Given that a person is in serious condition, there are only two possible outcomes in A: insured or not insured.

   $$S = \{(0,s),(1,s)\}$$

(c) *Let B be the event that the patient is uninsured. Specify the outcomes in B.*

Given that a person does not have insurance, there are only three possible outcomes: serious, good or fair.

$$S = \{(0, g), (0, f), (0, s)\}$$

(d) *Give all the outcomes in the event $B^c \cup A$.*

$B^c \cup A$ is the event that the patient has insurance (does not *not* have insurance) or is in serious condition.

$$S = \{(1, g), (1, f), (1, s), (0, s)\}$$

7. *Consider an experiment that consists of determining the type of job - either blue-collar or white-collar - and the political affiliation - Republican, Democratic, or Independent - of the 15 members of an adult soccer team. How many outcomes are*

   (a) *in the sample space*

   The sample space consists of all possible outcomes of this experiment. For each player there are $3 \cdot 2 = 6$ possible outcomes, so the total number of outcomes is $\boxed{6^{15}}$.

   (b) *in the event that at least one of the team members is a blue-collar worker.*

   The phrase *at least* immediately implies to use the complement. We could extract the result by starting with the entire sample space, and then excluding the case that all 15 players are *not* blue collar.

   How many outcomes are possible if all 15 players are not blue collar? $3 \cdot 1 = 3$ because we have reduced the levels of job type by 1 by considering only white collar.

   The total number of outcomes is then $\boxed{6^{15} - 3^{15}}$.

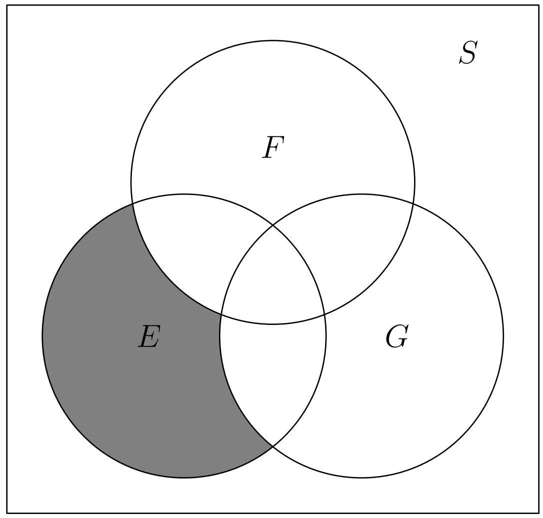
   (c) *in the event that none of the team members considers himself or herself an Independent?*

   If none of the team members considers themselves an Independent, then we have decreased the number of possible responses for political affiliation alone to 2. The total number of possible outcomes per player is then $2 \cdot 2 = 4$. For all 15 players, the total number of possible outcomes is $\boxed{4^{15}}$.

Chapter 2 Theoretical Exercises

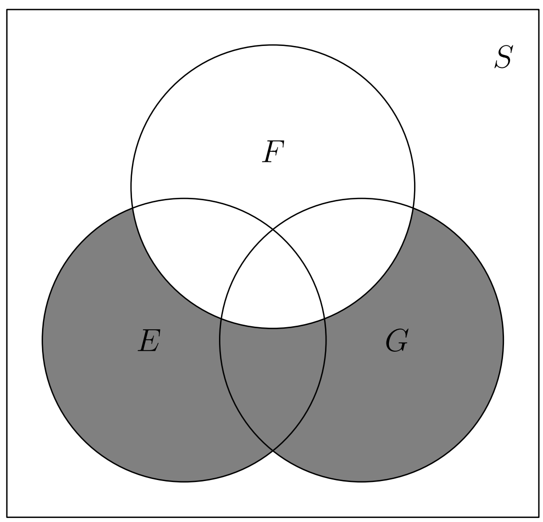
6. *Let E, F, and G be three events. Find expressions for the events so that of E, F, and G:*

   (a) *only E occurs*



If E occurs alone, that means that F and G do not occur $\Rightarrow$ $\boxed{EF^cG^c}$.

   (b) *both E and G but not F occur*



So, we get $\boxed{EF^cG}$.

   (c) *at least one of the events occurs.*

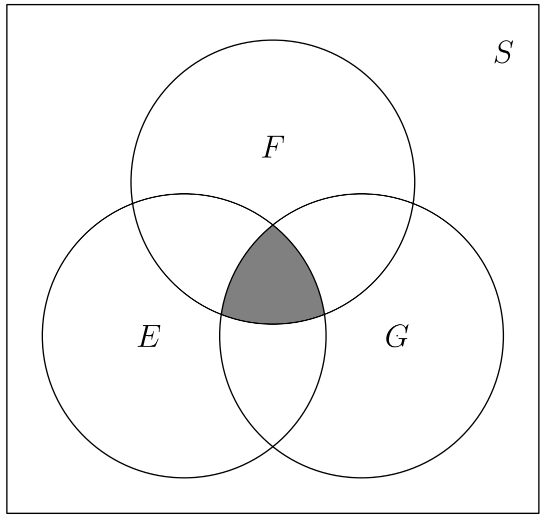
For these problems, it makes more sense to think about what to do first and perhaps then make the Venn diagram. The phrase "at least" implies we should consider the complement of the case "none of the events occur." We represent "none of the events occur" as not E, and not F and not G, $E^cF^cG^c$. Then, we just note the complement: $\boxed{(E^cF^cG^c)^c}$ which is the same as $\boxed{E \cup F \cup G}$ by De Morgan's laws.

   (d) *at least two of the events occur.*

"At least two of the events occur" means that two of the events occur, or all three events occur. There are several ways that two of the events can occur: EF, FG, EG. We connect these cases using the union. $\boxed{EF \cup FG \cup EG}$

(e) *all three occur*

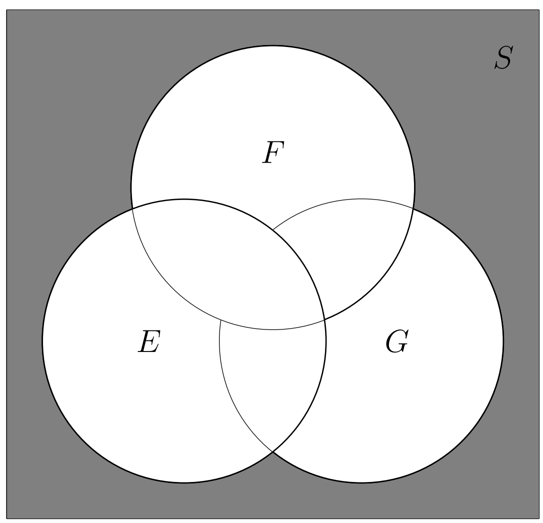
This can be visualized as



which can be represented as $\boxed{EFG}$.

(f) *None of the events occur.*

This can be represented as an unshaded Venn diagram



which means we are considering the complement of the union, $(E \cup F \cup G)^c$ which is the same as $\boxed{E^cF^cG^c}$ by deMorgan's Laws.

(g) *at most one of them occurs.*

This means that either one of the events occur (and the other two do not), or none of the events occurs which is $\boxed{EF^cG^c \cup E^cFG^c \cup E^cF^cG \cup E^cF^cG^c}$.

(h) *at most two of them occur.*

Either none of the events occur, one of the events occur (and the other two do not), or two of the events occur (and the other one does not). We just augment our response from the previous part with the case that two of the events occur.

$$\boxed{EF^cG^c \cup E^cFG^c \cup E^cF^cG \cup E^cF^cG^c \cup EFG^c \cup EF^cG \cup E^cFG}$$

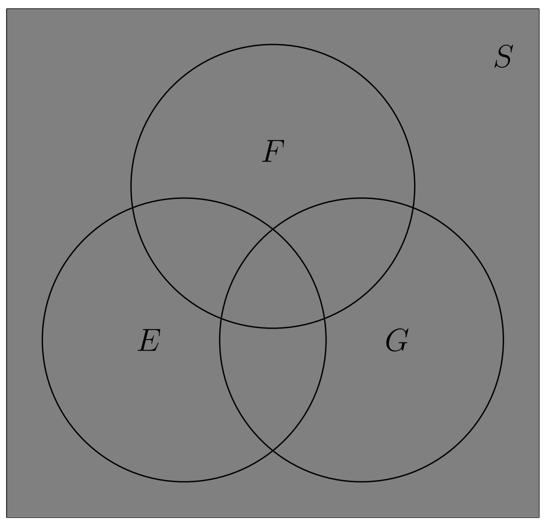
There is another, more compact way to do this by considering the complement. Since we are working with three events, the complement of "at most two events" is "not all three events" which is $\boxed{(EFG)^c}$.

(i) *exactly two of them occur.*

There are three ways two of the events can occur: EFG^c, EF^cG and E^cFG. We connect theses cases using the union: $\boxed{EFG^c \cup EF^cG \cup E^cFG}$.

(j) *at most three of them occur.*

This means that none of them occur, exactly one occurs (and the other two do not), exactly two occur (and the other does not), or all three occur. This exhausts all possibilities!



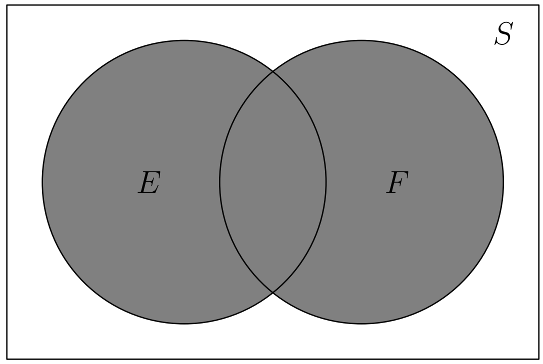
The region depicted is the entire sample space, $\boxed{S}$.

7. *Find the simplest expression for the following events:*

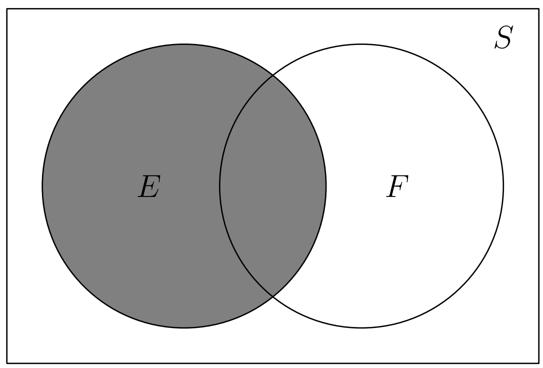
The easiest way to approach this problem is to draw a separate Venn diagram for each expression in the conjunction, and then write an expression for the intersection of all the diagrams.

(a) $(E \cup F)\,(E \cup F^c)$

The Venn diagram for $(E \cup F)$ is



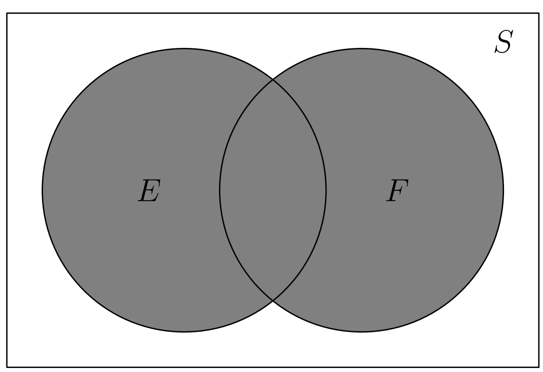
The Venn diagram for $(E \cup F^c)$ is



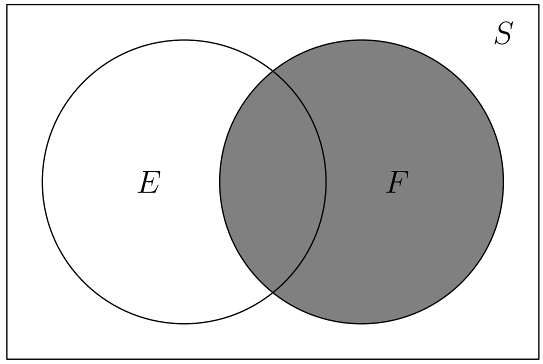
The region that appears in both diagrams (the intersection of both diagrams, if you will) if $\boxed{E}$.

(b) $(E \cup F)\,(E^c \cup F)\,(E \cup F^c)$

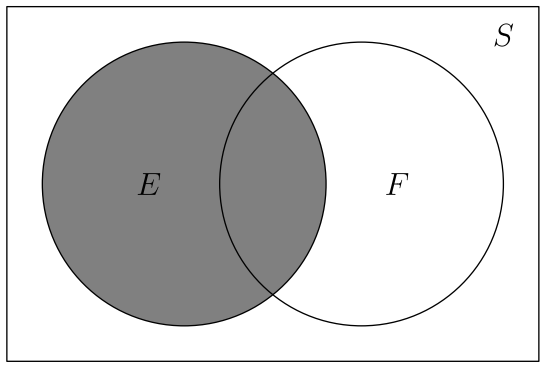
The Venn diagram for $(E \cup F)$ is



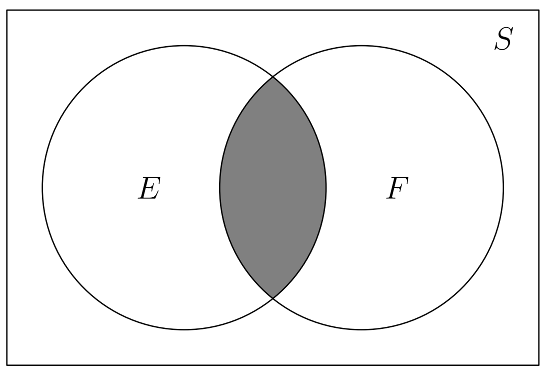
The Venn diagram for $E^c \cup F$ is



The Venn diagram for $E \cup F^c$ is

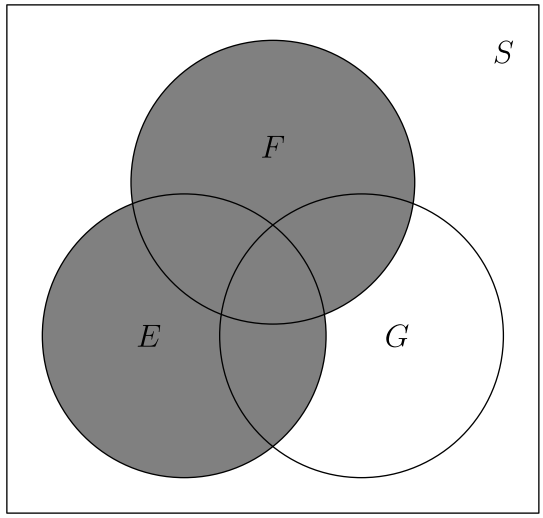


The intersection of all three diagrams is $\boxed{EF}$.

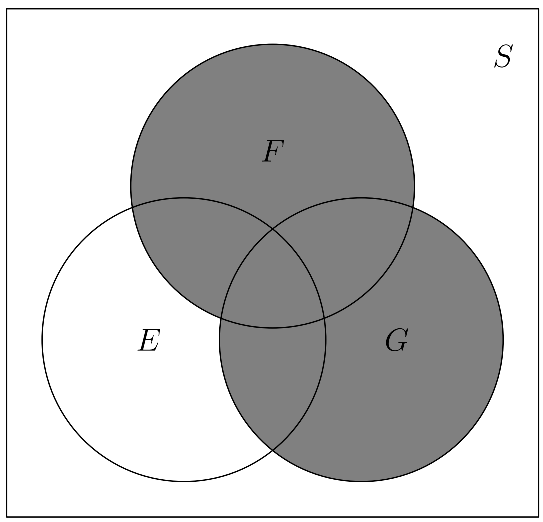


(c) $(E \cup F)(F \cup G)$

The Venn diagram for $(E \cup F)$ is



The Venn diagram for $F \cup G$ is



The intersection is $\boxed{F \cup EG}$.