10 pts

68. Let $X_1, X_2, \ldots, X_{10}$ be independent Poisson random variables with mean 1.

(a) Use the Markov inequality to get a bound on $P\{X_1 + \cdots + X_{10} \geq 15\}$.

(b) Use the central limit theorem to approximate $P\{X_1 + \cdots + X_{10} \geq 15\}$.

70. Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

Hint: Let X_n be Poisson with mean n. Use the central limit theorem to show that $P\{X_n \leq n\} \to \frac{1}{2}$.

8. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find (a) $E[X]$, (b) $E[X|Y = 1]$, (c) $E[X|Y = 5]$.

14. Let X be uniform over (0, 1). Find $E[X|X < \frac{1}{2}]$.

15. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty$$

Compute $E[X^2|Y = y]$.

68. a) $X = X_1 + \dots + X_{10} \sim \text{Poisson}(\lambda = 10)$

$$P(X \geq 15) \leq \frac{E[X]}{15} = \frac{10}{15} = 2/3$$

b) $X_i \sim \text{Poisson}(\lambda=1)$ so $\mu = 1,\ \sigma = 1$

$$\frac{X - n\mu}{\sigma\sqrt{n}} = \frac{X-10}{\sqrt{10}} \approx N(0,1)$$

$$P(X \geq 15) \approx P\left(Z \geq \frac{15-10}{\sqrt{10}}\right) = P(Z \geq 1.58)$$
$$= .057$$

70. $$\sum_{k=0}^{n} e^{-n}\frac{n^k}{k!} = P(X \leq n),\ \text{where } X \sim \text{Poisson}(\lambda = n)$$

think of X as $X_1 + \dots + X_n$, $X_i \sim \text{Poisson}(\lambda=1)$

So $$\frac{X - n\mu}{\sigma\sqrt{n}} = \frac{X-n}{\sqrt{n}\sqrt{n}} = \frac{X-n}{n} \approx N(0,1)$$

$$P(X \leq n) \approx P\left(Z \leq \frac{n-n}{n}\right) = P(Z \leq 0) = \frac{1}{2}$$

8. a) $X \sim \text{Geom}(p = \frac{1}{6})$ $\quad E[X] = \frac{1}{p} = 6$

b) $E[X|Y=1]$ $\quad P(X=1|Y=1) = 0,\ P(X=2|Y=1) = \frac{1}{6},\ P(X=3|Y=1) = \frac{5}{6}\frac{1}{6},$

$P(X=4|Y=1) = (\frac{5}{6})^2 \frac{1}{6}$, etc.

$$E[X|Y=1] = 2\cdot\frac{1}{6} + 3\cdot\frac{5}{6}\cdot\frac{1}{6} + 4\cdot\left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \dots = \sum_{x=2}^{\infty} x\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{x-2}$$

$$= \sum_{z=1}^{\infty} (z+1)\frac{1}{6}\left(\frac{5}{6}\right)^{z-1} = E[Z+1] \text{ where } Z \sim \text{Geom}(p=\tfrac{1}{6})$$
$$= 6+1 = 7$$

8c $E[X|Y=5]$

$P[X=1|Y=5] = \frac{1}{5}$

$P[X=2|Y=5] = \frac{4}{5}\cdot\frac{1}{5}$

$P[X=3|Y=5] = \left(\frac{4}{5}\right)^2\cdot\frac{1}{5}$

$P[X=4|Y=5] = \left(\frac{4}{5}\right)^3\cdot\frac{1}{5}$

$P[X=5|Y=5] = 0$

$P[X=6|Y=5] = \left(\frac{4}{5}\right)^4\cdot\frac{1}{6}$

$P[X=7|Y=5] = \left(\frac{4}{5}\right)^4\cdot\frac{5}{6}\cdot\frac{1}{6}$ etc.

So
$$E[X|Y=5] = \overbrace{\frac{1}{5} + 2\cdot\frac{4}{5}\cdot\frac{1}{5} + 3\cdot\left(\frac{4}{5}\right)^2\cdot\frac{1}{5} + 4\left(\frac{4}{5}\right)^3\cdot\frac{1}{5}}^{a}$$
$$+ \underbrace{6\left(\frac{4}{5}\right)^4\cdot\frac{1}{6} + 7\left(\frac{4}{5}\right)^4\cdot\frac{5}{6}\cdot\frac{1}{6} + \cdots}_{b} \qquad = \underline{5.8192}$$

$a = 1.3136 \qquad b = \sum_{k=6}^{\infty} k\left(\frac{4}{5}\right)^4\cdot\frac{1}{6}\left(\frac{5}{6}\right)^{k-6}$

$$= \left(\frac{4}{5}\right)^4 \sum_{z=1}^{\infty} (z+5)\,\frac{1}{6}\left(\frac{5}{6}\right)^{z-1}$$
$$= \left(\frac{4}{5}\right)^4 E[Z+5] = \left(\frac{4}{5}\right)^4[6+5] = 4.5056$$

14. $f(x \mid X<\frac{1}{2}) = \dfrac{f(x, X<\frac{1}{2})}{P(X<\frac{1}{2})} = \dfrac{1}{\frac{1}{2}} = 2, \quad 0<x<\frac{1}{2}$

$E[X \mid X<\frac{1}{2}] = \int_0^{\frac{1}{2}} x \cdot 2\, dx = x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{4}$

15. $f(x,y) = \dfrac{e^{-y}}{y} \quad 0<x<y$

$f_Y(y) = \int_0^y \dfrac{e^{-y}}{y}\, dx$

$= x\dfrac{e^{-y}}{y}\Big|_{x=0}^{y}$

$= e^{-y}$

$f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

$= \dfrac{e^{-y}/y}{e^{-y}} = \dfrac{1}{y},$

$0<x<y$

$E[X^2|y] = \int_0^y x^2 \cdot \frac{1}{y}\, dx = \frac{1}{y}\frac{x^3}{3}\Big|_{x=0}^{y}$

$= \frac{1}{y}\frac{y^3}{3} = \frac{y^2}{3}$