4

The Source Coding Theorem

▶ 4.1 How to measure the information content of a random variable?

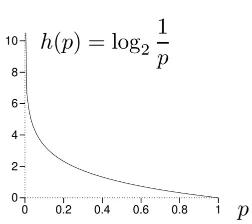
In the next few chapters, we'll be talking about probability distributions and random variables. Most of the time we can get by with sloppy notation, but occasionally, we will need precise notation. Here is the notation that we established in Chapter 2.

An ensemble X is a triple $(x, \mathcal{A}_X, \mathcal{P}_X)$, where the *outcome* x is the value of a random variable, which takes on one of a set of possible values, $\mathcal{A}_X = \{a_1, a_2, \ldots, a_i, \ldots, a_I\}$, having probabilities $\mathcal{P}_X = \{p_1, p_2, \ldots, p_I\}$, with $P(x = a_i) = p_i$, $p_i \geq 0$ and $\sum_{a_i \in \mathcal{A}_X} P(x = a_i) = 1$.

How can we measure the information content of an outcome $x = a_i$ from such an ensemble? In this chapter we examine the assertions

1. that the *Shannon information content*,
$$h(x = a_i) \equiv \log_2 \frac{1}{p_i}, \tag{4.1}$$
is a sensible measure of the information content of the outcome $x = a_i$, and

2. that the *entropy* of the ensemble,
$$H(X) = \sum_i p_i \log_2 \frac{1}{p_i}, \tag{4.2}$$
is a sensible measure of the ensemble's average information content.



p	$h(p)$	$H_2(p)$
0.001	10.0	0.011
0.01	6.6	0.081
0.1	3.3	0.47
0.2	2.3	0.72
0.5	1.0	1.0

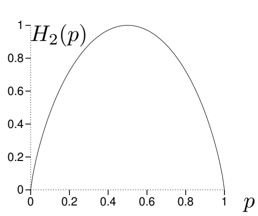


Figure 4.1. The Shannon information content $h(p) = \log_2 \frac{1}{p}$ and the binary entropy function $H_2(p) = H(p, 1-p) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)}$ as a function of p.

Figure 4.1 shows the Shannon information content of an outcome with probability p, as a function of p. The less probable an outcome is, the greater its Shannon information content. Figure 4.1 also shows the binary entropy function,
$$H_2(p) = H(p, 1-p) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)}, \tag{4.3}$$
which is the entropy of the ensemble X whose alphabet and probability distribution are $\mathcal{A}_X = \{a, b\}, \mathcal{P}_X = \{p, (1-p)\}$.