Network Informed Vector Autoregression

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Motivation

High dimensional time series, called *panels*, arise in many scientific disciplines such as neuroscience, finance, and macroeconomics. The daily closing price of each company in the S&P 500 over the last 10 years is an example of a panel of high dimensional time series. Often, **co-movements** within groups of the panel components occur. Extracting these groupings from the data provides a **course grained description** of the complex system in question and can aid with **prediction tasks**.

Introduction to NIRVAR

We model the co-movement between a panel of multiple time series via a **restricted vector autoregression** (VAR) model. The restrictions are determined by a **latent graphical structure** which induces sparsity in the VAR coefficient matrix. A novel **estimation method** is proposed that firstly **defines a latent graph** and secondly estimates the corresponding restricted VAR coefficient matrix via Ordinary Least Squares (OLS). We apply the model to the **prediction** of daily financial returns of a universe of equities and to forecasting US industrial production.

Main Contribution

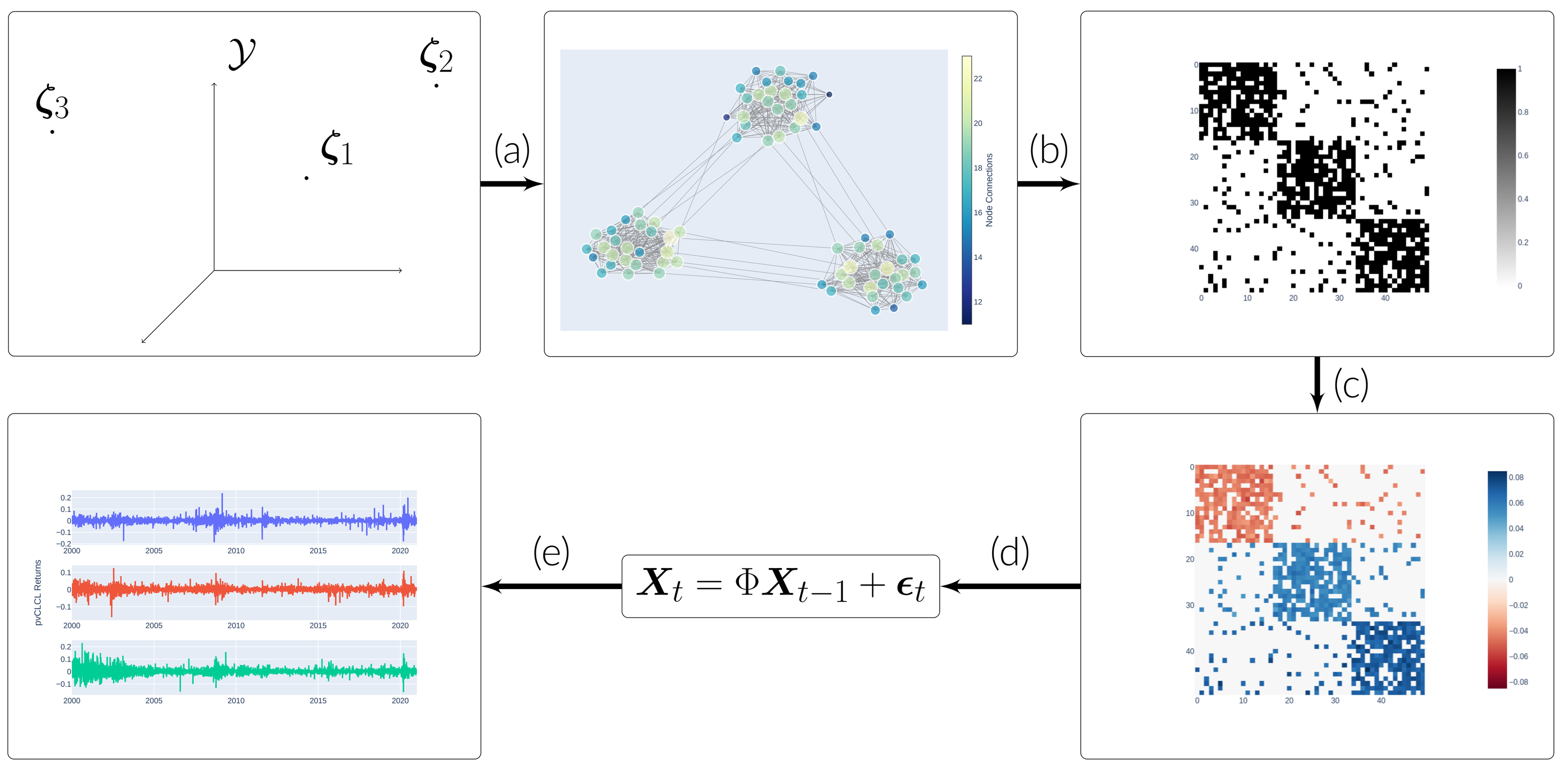
We introduce an **estimation framework for sparse VAR** models that firstly determines the restrictions to be placed on the VAR coefficients and secondly estimates the remaining unrestricted coefficients via OLS estimation. We call it **"Network Informed Restricted VAR (NIRVAR)"** estimation.

Model

Consider a zero-mean, second order stationary sequence $\{\boldsymbol{X}_t\}$ with $\boldsymbol{X}_t = (X_{1,t}, ..., X_{N,t})'$, where

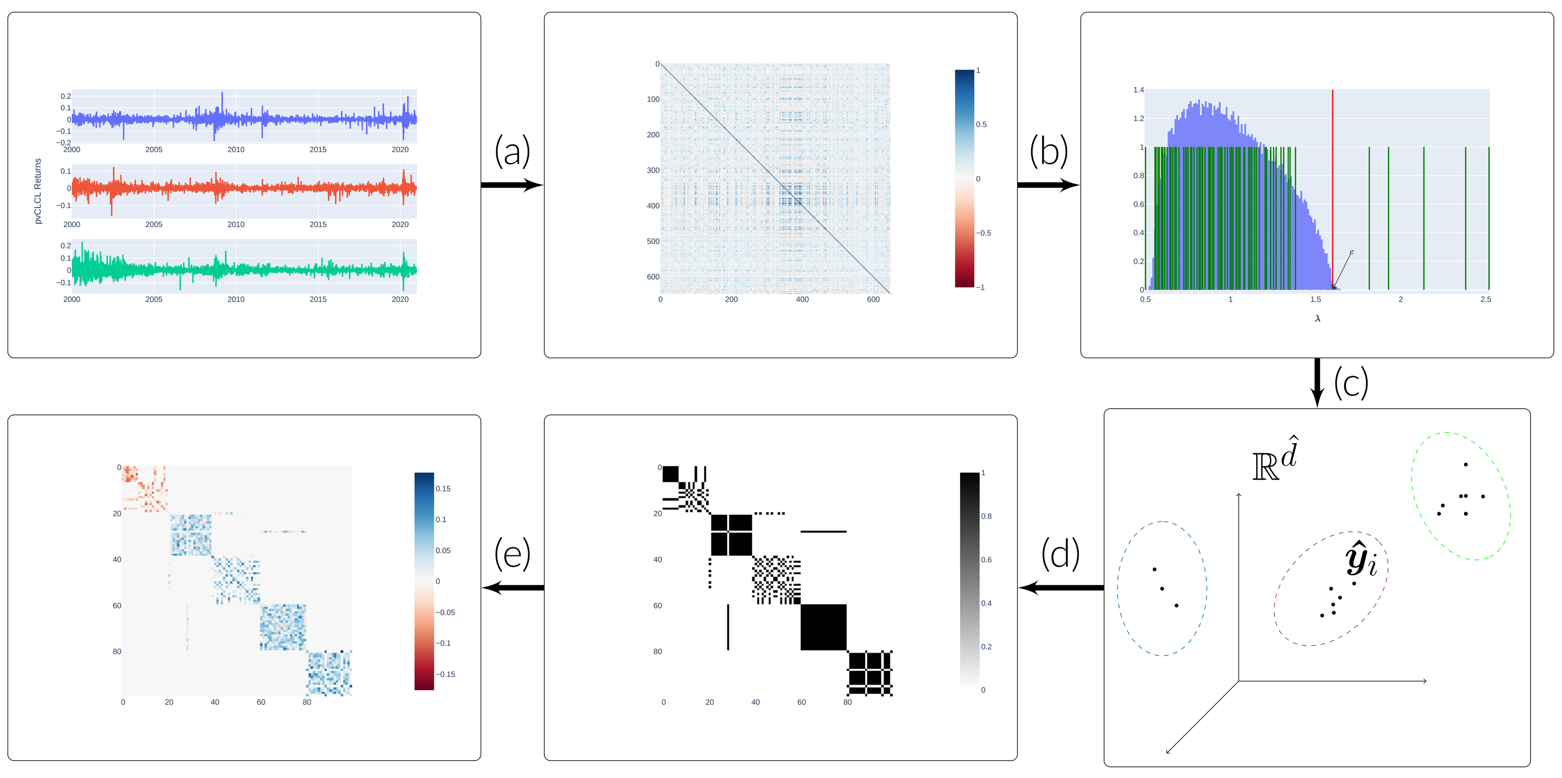
$$X_{i,t} = \sum_{j=1}^{N} A_{ij}\tilde{\Phi}_{ij}X_{j,t-1} + \epsilon_{i,t} \qquad \epsilon_{i,t} \sim \mathcal{N}(0, \sigma^2) \tag{1}$$

with $\tilde{\Phi}_{ij} \in \mathbb{R}$ and $A_{ij} \in \{0,1\}$. The matrix, $A \in \{0,1\}^{N\times N}$, **determines the non zero elements** of the restricted VAR coefficient matrix, $\Phi = A \odot \tilde{\Phi}$, where $\tilde{\Phi}$ is the $N\times N$ matrix with elements $\tilde{\Phi}_{ij}$ and $\odot$ denotes component-wise multiplication. One can consider A as being the **adjacency matrix** of a directed graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of N nodes and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is a set of directed edges. We define A_{ij} as being 1 if there exists an edge from j to i and 0 otherwise.



We model $\mathcal{G}$ as a **stochastic block model** (SBM) with $K \leq N$ blocks. The SBM is represented as a random dot product graph with each vertex having one of K **latent positions**, $\zeta_1, ..., \zeta_K$. The adjacency matrix corresponding to a random sample from the SBM is $A_{ij} \sim \text{Bernoulli}\left(\zeta'_{\tau(i)}\zeta_{\tau(j)}\right)$, where $\tau : \{1,\ldots,N\} \to \{1,\ldots,K\}$ is a block assignment function.

Estimation



The NIRVAR estimation steps (summarised in the above flow diagram) are as follows:

1. Compute the **correlation matrix**, S_T, between panel components.
2. **Estimate the dimension** of the latent space by counting the number, $\hat{d} \in \mathbb{N}$, of eigenvalues of S_T that are greater than the Marchencko-Pastur distribution cutoff.
3. Let $\Lambda, U \in \mathbb{R}^{N\times\hat{d}}$ be the matrices containing the $\hat{d}$ largest eigenvalues and corresponding eigenvectors of S_T. Compute the **embedding**, $\hat{Y} = U\Lambda^{1/2} \in \mathbb{R}^{N\times\hat{d}}$.
4. **Cluster** each $\hat{\boldsymbol{y}}_i \in \mathbb{R}^{\hat{d}}$ into $\hat{K} = \hat{d}$ groups using a Gaussian Mixture Model (GMM).
5. Define the binary matrix, $\hat{A}_{ij}$, as

$$\hat{A}_{ij} = \begin{cases} 1 & \text{if } \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{y}}_j \text{ are in the same GMM cluster} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

6. Set $\hat{\Phi}_{ij} = 0$ if $\hat{A}_{ij} = 0$ and estimate the remaining unrestricted parameters via **OLS**.

Extension to Multiple Edge Types

The NIRVAR model and estimation framework can easily be extended to accommodate multiple types of edges by modelling each edge type as a layer in a **multiplex graph**.

Properties of NIRVAR Estimator

Bias

Let Z_i and $\hat{Z}_i$ denote the subset of rows of the design matrix, $X \in \mathbb{R}^{N\times T}$, that correspond to the true and estimated predictors of $\boldsymbol{\phi}_i \equiv \Phi_{i\cdot} \in \mathbb{R}^N$. The OLS estimate of $\boldsymbol{\phi}_i$ is

$$\hat{\boldsymbol{\phi}}_i = \left(\hat{Z}_i\hat{Z}'_i\right)^{-1}\hat{Z}_i\left(Z'_i\boldsymbol{\phi}_i + \boldsymbol{\epsilon}_i\right) \tag{3}$$

where $\boldsymbol{\epsilon}_i := (\epsilon_{i,1}, \ldots, \epsilon_{i,T})'$. Since $\mathbb{E}(\boldsymbol{\epsilon}_i) = 0$, then

$$\mathbb{E}\left(\hat{\boldsymbol{\phi}}_i\right) = \left(\hat{Z}_i\hat{Z}'_i\right)^{-1}\hat{Z}_iZ'_i\boldsymbol{\phi}_i. \tag{4}$$

If $\hat{A}_{ij} = 0$ and $A_{ij} = 1$ for some j, then the **estimator is biased** since $\left(\hat{Z}_i\hat{Z}'_i\right)^{-1}\hat{Z}_iZ'_i \neq \mathbb{1}_N$, where $\mathbb{1}_N$ is the $N\times N$ identity matrix. This is **model misspecification**: removing one of the predictors of $\boldsymbol{\phi}_i$ from the OLS estimation procedure leads to a bias.

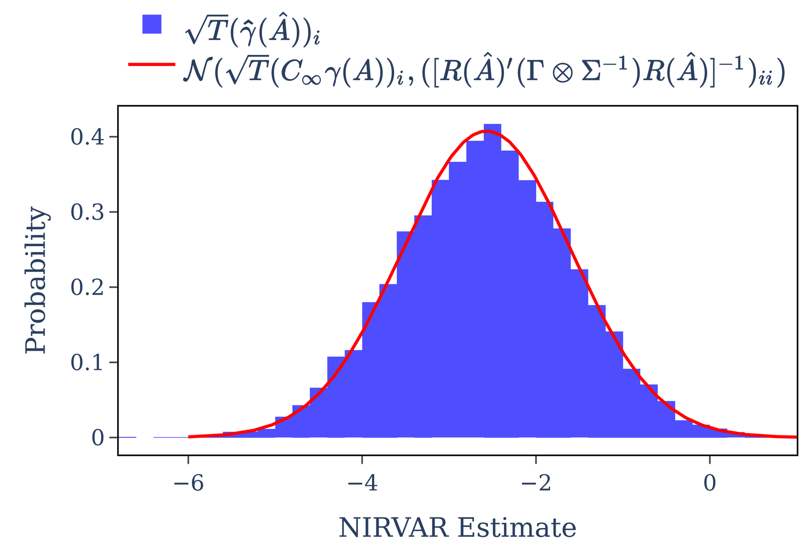


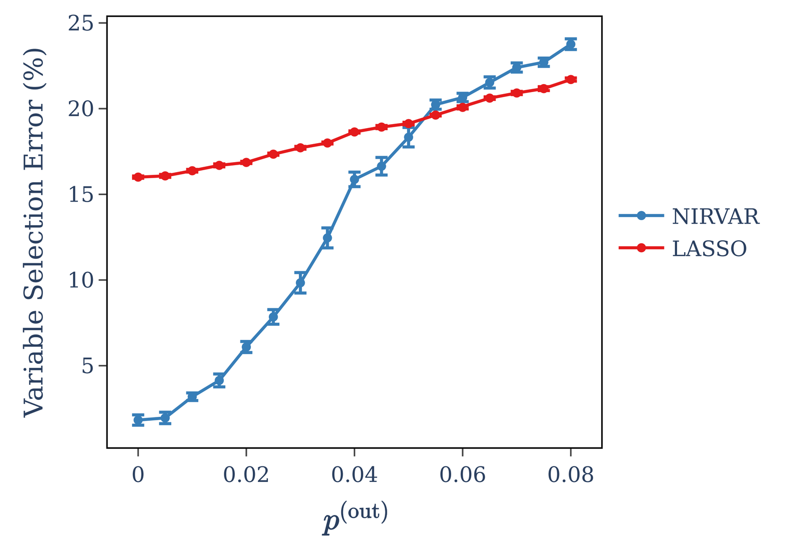
Figure 1. The blue histogram gives $\sqrt{T}\left(\hat{\Phi}_{ij} - \Phi_{ij}\right)$ for a particular choice of ij for 2000 simulated datasets. The red curve gives the corresponding asymptotic distribution.

Asymptotic Efficiency

If a proper subset of the true restrictions are known, the asymptotic variance of the NIRVAR estimator is greater than or equal to that of a restricted estimator that uses all restrictions.

Simulation Study

The NIRVAR estimator constructs a graph with $\hat{\boldsymbol{K}}$ **connected components**. It is not capable of reconstructing edges of the ground truth SBM that are between nodes in different blocks. We conducted a simulation study in which the probability, p_{out}, of an edge forming between nodes in different blocks was varied for the data generating process. We computed the percentage of incorrect entries of $\hat{A}$ as a function of $p^{(\text{out})}$. In particular, the percentage error was calculated as $100\times\sum_{i,j=1}^{N}\mathbb{1}\{\hat{A}_{ij} \neq A_{ij}\}/N^2$. The figure shows that the percentage of $\hat{\Phi}$ is lower than that of a LASSO estimator for $p_{\text{out}} < 0.05$.



Application to Financial Returns

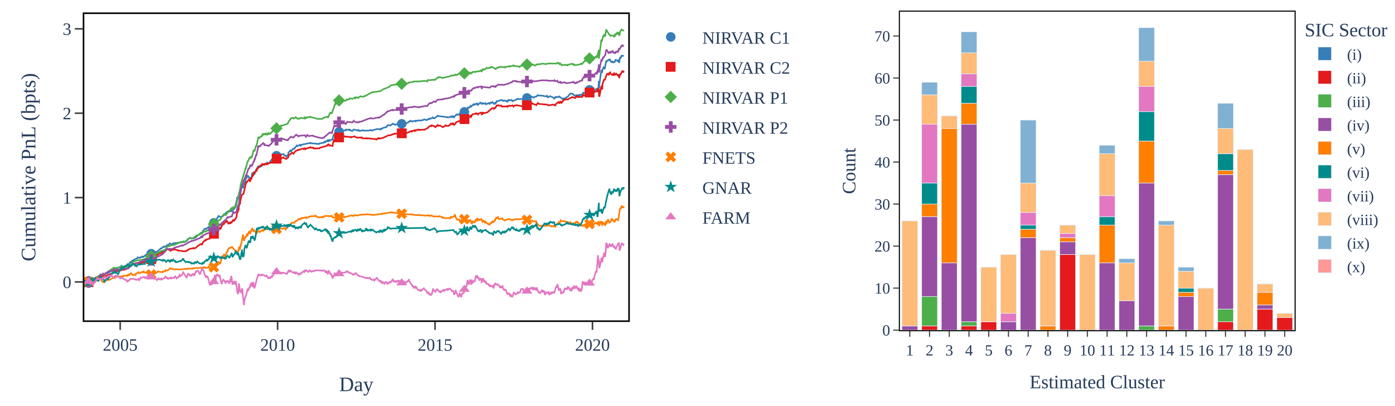
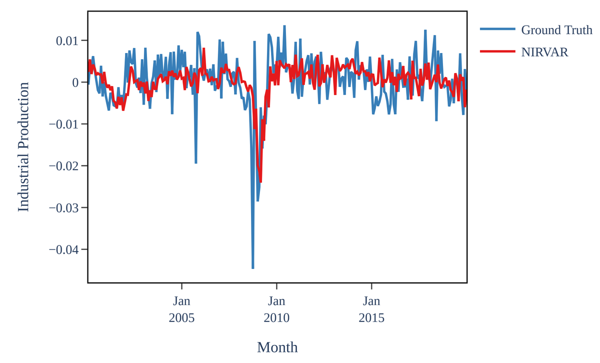


Figure 2. Left: normalised cumulative PnL in bpts for NIRVAR, FARM, FNETS, and GNAR. Right: comparison of the NIRVAR estimated clusters with the Standard Industrial Classification (SIC).

- We use NIRVAR estimation to **predict the daily financial returns** for a universe of 648 financial instruments.
- For each **backtesting** day we define $\text{PnL}_t = \sum_{i=1}^{N}\text{sign}\left(s_i^{(t)}\right)\times\text{fret}_i^{(t)}$, where $s_i^{(t)}$ is the predicted return of asset i on day t, and $\text{fret}_i^{(t)}$ is the realized return of asset i on day t.
- The **Sharpe ratio** (SR) [6] is a measure of risk-adjusted returns. Let $\text{PnL} \equiv \{\text{PnL}_t\}_{t=1,\ldots,T}$. The SR is defined as $\text{SR} = \text{mean}(\text{PnL})/\text{stdev}(\text{PnL})\times\sqrt{252}$. The NIRVAR P1 SR is **2.82**.

Forecasting US Industrial Production

FRED-MD is a publicly accessible database of monthly observations of macroeconomic variables [5]. The prediction task is one-step ahead forecasts of the first order difference of the logarithm of the monthly industrial production index. The figure shows the NIRVAR prediction against the realised value. The MSEs between the NIRVAR/FARM/FNETS/GNAR predictions and the realised values were 0.0087, 0.0089, 0.0096, and 0.0101, respectively.



Key Advantage of NIRVAR Estimation

Allows for a network based approach to determining restricted VAR coefficients when the underlying network is **unobserved**.

Future Directions

- Prove that NIRVAR **recovers the latent positions** of the SBM up to an orthogonal transformation.
- Modify the estimation method to allow for **vertices between blocks**.
- Extend the model to incorporate a **time varying adjacency matrix**.
- Incorporate a **factor model** into the NIRVAR framework.

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