

# A survey on the Square Peg Problem

Benjamin Matschke\*

March 14, 2014

## Abstract

This is a short survey article on the 102 years old Square Peg Problem of Toeplitz, which is also called the Inscribed Square Problem. It asks whether every *continuous* simple closed curve in the plane contains the four vertices of a square.

This article first appeared in a more compact form in Notices Amer. Math. Soc. 61(4), 2014.

## 1 Introduction

In 1911 Otto Toeplitz [66] formulated the following conjecture.

**Conjecture 1** (Square Peg Problem, Toeplitz 1911). *Every continuous simple closed curve in the plane  $\gamma : S^1 \rightarrow \mathbb{R}^2$  contains four points that are the vertices of a square.*

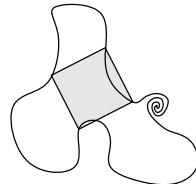


Figure 1: Example for Conjecture 1.

A continuous simple closed curve in the plane is also called a Jordan curve, and it is the same as an injective map from the unit circle into the plane, or equivalently, a topological embedding  $S^1 \hookrightarrow \mathbb{R}^2$ .

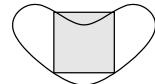


Figure 2: We do not require the square to lie fully inside  $\gamma$ , otherwise there are counter-examples.

---

\*Supported by Deutsche Telekom Stiftung, NSF Grant DMS-0635607, an EPDI-fellowship, and MPIM Bonn.

This article is licensed under a Creative Commons BY-NC-SA 3.0 License.

In its full generality Toeplitz' problem is still open. So far it has been solved affirmatively for curves that are “smooth enough”, by various authors for varying smoothness conditions, see the next section. All of these proofs are based on the fact that smooth curves inscribe generically an odd number of squares, which can be measured in several topological ways. None of these methods however could be made work so far for the general continuous case.

One may think that the general case of the Square Peg Problem can be reduced to the case of smooth curves by approximating a given continuous curve  $\gamma$  by a sequence of smooth curves  $\gamma_n$ : Any  $\gamma_n$  inscribes a square  $Q_n$ , and by compactness there is a converging subsequence  $(Q_{n_k})_k$ , whose limit is an inscribed square for the given curve  $\gamma$ . However this limit square is possibly degenerate to a point, and so far there is no argument known that can deal with this problem.

Suppose we could show that any smooth (or equivalently, any piecewise linear) curve  $\gamma$  that contains in its interior a ball of radius  $r$  inscribes a square of side length at least  $\sqrt{2}r$  (or at least  $\varepsilon r$  for some constant  $\varepsilon > 0$ ). Then the approximation argument would imply that any continuous curve had the same property. However it seems that we need more geometric than merely topological ideas to show the existence of *large* inscribed squares.

Other surveys are due to Klee and Wagon [33, Problem 11], Nielsen [57], Denne [9], Karasev [27, 2.6, 4.6], and Pak [59, I.3, I.4]. Jason Cantarella's homepage [4] offers some animations. A java applet and an extended version of this article is available on my homepage.

In order to raise awareness, let me put 100 euros on each of the Conjectures 1, 8, and 16. That is, you may earn 300 euros in total.

I want to thank the referees for many very useful comments.

## History of the Square Peg Problem

The Square Peg Problem first appears in the literature in the conference report [66] in 1911. Toeplitz gave a talk whose second part had the title “On some problems in topology”. The report on that second part is rather short:

- b) *Ueber einige Aufgaben der Analysis situs.* [...]
- b) Der Vortragende erzählt von zwei Aufgaben der Analysis Situs, zu denen er gelangt ist, und dann von der folgenden dritten, deren Lösung ihm nur für konvexe Kurven gelungen ist: Auf jeder einfach geschlossenen stetigen Kurve in der Ebene gibt es vier Punkte, welche ein Quadrat bilden. Diskussion: Die Herren Fueter, Speiser, Laemmel, Stäckel, Grossmann.

Here is an English translation:

- b) *On some problems in topology.* [...]
- b) The speaker talks about two problems in topology that he obtained, and then about the following third one, whose solution he managed to find only for convex curves: On every simple closed continuous curve in the plane there are four points that form a square. Discussions: Messrs. Fueter, Speiser, Laemmel, Stäckel, Grossmann.

It seems that Toeplitz never published a proof. In 1913, Arnold Emch [11] presented a proof for “smooth enough” convex curves. Two years later Emch [12] published a further proof

which requires a weaker smoothness condition. However he did not note that the special case of smooth convex curves already implies by a limit argument that all convex curves inscribe squares. In a third paper from 1916, Emch [13] proved the Square Peg Problem for curves that are piecewise analytic with only finitely many inflection points and other singularities where the left and right sided tangents at the finitely many non-smooth points exist.

Emch states in his second paper [12] that he had not been aware of Toeplitz' and his students' work and that the problem was suggested to him by Kempner. From 1906–1913 Toeplitz was a postdoc in Göttingen. Aubrey J. Kempner was an English mathematician who finished his Ph.D. with Edmund Landau in Göttingen in 1911. Afterwards he went to the University of Illinois in Urbana-Champaign and stayed there until 1925, according to <http://www.maa.org/history/presidents/kempner.html> (another biography of Kempner can be found on <http://www.findagrave.com/cgi-bin/fg.cgi?page=gr&GRid=13165695>, which claims different dates). Emch joined the faculty of the same university in 1911.

I will let the reader decide whether this is enough information on how this fits together and who considered the Square Peg Problem first. It is usually attributed to Toeplitz.

In 1929 Schnirelman proved the Square Peg Problem for a class of curves that is slightly larger than  $C^2$ . An extended version [64], which corrects also some minor errors, was published posthumously in 1944. Guggenheimer [18] states that the extended version still contains errors which he claims to correct. However in my point of view Schnirelman's proof is up to minor errors correct. His main idea is a bordism argument, below we give some details. Since the transversality machinery was not invented at this time, Schnirelman's proof contains many computations in explicit coordinates. Guggenheimer's main lemma on the other hand admits counter-examples, see [50, Section III.9]; he was not aware that squares can vanish pairwise when one deforms the curve.

Other proofs are due to Hebbert [21] when  $\gamma$  is a quadrilateral, Zindler [78] and Christensen [7] for convex curves, Jerrard [25] for analytic curves, Nielsen–Wright [58] for curves that are symmetric across a line or about a point, Stromquist [65] for locally monotone curves, Vrećica–Živaljević [70] for Stromquist's class of curves, Pak [59] for piecewise linear curves, Sagols–Marín [62], [63] for similar discretizations, Cantarella–Denne–McCleary [5] for curves with bounded total curvature without cusps and for  $C^1$ -curves, Makeev [44] for star-shaped  $C^2$ -curves that intersect every circle in at most 4 points (more generally he proved the Circular Quad Peg Problem 9 for such curves, see below), [51] for a technical open and dense class of curves and for continuous curves in certain bounded domains. In the next section we will review some of these special cases in more detail.

Pettersson, Tverberg and Östergård [60] have the latest result, which uses a computer: They showed that any Jordan curve in the  $12 \times 12$  square grid inscribes a square whose size is at least  $1/\sqrt{2}$  times the size of the largest axis parallel square that fits into the interior of the curve.

## Special cases

Let us discuss some of the above mentioned proofs in more detail.

**Emch's proof.** Let  $\gamma : S^1 \hookrightarrow \mathbb{R}^2$  be the given piecewise analytic curve. Fixing a line  $\tau$ , Emch considers all secants of  $\gamma$  that are parallel to  $\tau$  and he calls the set of all midpoints of

these secants the set of *medians*  $M_\tau$ . Under some genericity assumptions he proves that for two orthogonal lines  $\tau$  and  $\tau^\perp$ ,  $M_\tau$  intersects  $M_{\tau^\perp}$  in an *odd* number of points. Nowadays one could write this down homologically. These intersections correspond to inscribed rhombi, where the two intersecting secants are the two diagonals of the rhombus.

Now he rotates  $\tau$  continuously by  $90^\circ$  and argues that  $M_\tau \cap M_{\tau^\perp}$  moves continuously, where at finitely many times, two intersection points can merge and disappear, or two new intersection points can appear. When  $\tau$  is rotated by  $90^\circ$ , the one-dimensional family of intersections points closes up to a possibly degenerate union of circle components.

Since  $M_\tau \cap M_{\tau^\perp}$  is odd, Emch argues that an odd number of these components must be  $\mathbb{Z}/4\mathbb{Z}$ -invariant, meaning that if  $R_1R_2R_3R_4$  is a rhombus in such a component, then  $R_2R_3R_4R_1$  must also be in the same component. By the mean value theorem, when moving from  $R_1R_2R_3R_4$  to  $R_2R_3R_4R_1$  along a component of inscribed rhombi, at some point the diagonals must have equal length. That is, we obtain an inscribed square. This argument also implies that the number of inscribed squares is (generically) odd for Emch's class of curves.

**Schnirelman's proof.** Schnirelman proved the Square Peg Problem for a slightly larger class than  $C^2$  using an early bordism argument, which yields a very conceptual proof. His idea was that the set of inscribed squares can be described as a preimage, for example in the following way: Let  $\gamma : S^1 \hookrightarrow \mathbb{R}^2$  be the given curve. The space  $(S^1)^4$  parametrizes quadrilaterals that are inscribed in  $\gamma$ . We construct a so-called *test-map*

$$f_\gamma : (S^1)^4 \rightarrow \mathbb{R}^6 \tag{1}$$

that sends a 4-tuple  $(x_1, x_2, x_3, x_4)$  of points on the circle to the mutual distances between  $\gamma(x_1), \dots, \gamma(x_4) \in \mathbb{R}^2$ . Let  $V$  be the 2-dimensional linear subspace of  $\mathbb{R}^6$  that corresponds to the points where all four edges are of equal length and the two diagonal are of equal length. The preimage  $f_\gamma^{-1}(V)$  is parametrizing the set of inscribed squares, plus a few 'degenerate components'. The degenerated components consist of points where  $x_1 = x_2 = x_3 = x_4$ , these are the degenerate squares, and more generally of 4-tuples where  $x_1 = x_3$  and  $x_2 = x_4$ .

Now Schnirelman argues as follows: An ellipse inscribes exactly one square up to symmetry. Now deform the ellipse (via some smooth isotopy) into the given curve along other curves  $\gamma_t$ ,  $t \in [0, 1]$ . By smoothness these inscribed squares do not come close to the degenerate quadrilaterals during the deformation, that is, they do not shrink to a point. Thus the non-degenerate part of all preimages  $f_{\gamma_t}^{-1}(V)$  forms a 1-manifold that connects the solution sets for  $\gamma$  and the ellipse, and since 1-manifolds have always an even number of boundary points, the parities of the number of inscribed squares on  $\gamma$  and on the ellipse coincide.

Thus, any smooth curve inscribes generically an odd number of squares. Here we swept technical arguments concerning transversality under the rug, which is hopefully appreciated by the reader.

For general curves, it is difficult to separate the degenerate quadrilaterals in  $f^{-1}(V)$  from the squares we are interested in. This is the basic reason why the Square Peg Problem could not be solved completely with the current methods.

**Stromquist's criterion.** Stromquist's class of curves for which he proved the Square Peg Problem is very beautiful and it is the second strongest one: A curve  $\gamma : S^1 \hookrightarrow \mathbb{R}^2$  is called

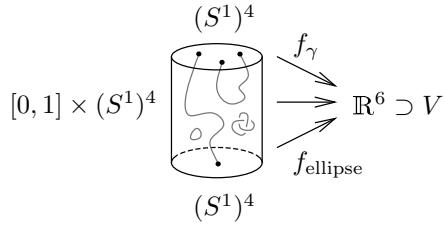


Figure 3: The bordism between the solution sets for  $\gamma$  and the ellipse; to simplify the figure we already modded out the symmetry group of the square and omitted the degenerate components.

*locally monotone* if every point of  $x \in S^1$  admits a neighborhood  $U$  and a linear functional  $\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $\ell \circ \gamma|_U$  is strictly monotone.



Figure 4: Example of a piece of a locally monotone curve. Note that Figure 1 is not locally monotone because of the spiral.

**Theorem 2** (Stromquist). *Any locally monotone embedding  $\gamma : S^1 \hookrightarrow \mathbb{R}^2$  inscribes a square.*

In his proof Stromquist also considers the set of inscribed rhombi first.

**Fenn's table theorem.** A beautiful proof for convex curves is due to Fenn [14]. It follows as an immediate corollary from his table theorem.

**Theorem 3** (Fenn). *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$  be a non-negative function that is zero outside of a compact convex disc  $D$  and let  $a > 0$  be an arbitrary real number. Then there exists a square in the plane with side length  $a$  and whose center point belongs to  $D$  such that  $f$  takes the same value on the vertices of the square.*

As the reader might guess Fenn's proof basically uses a mod-2 argument, showing that the number of such tables is generically odd.

The table theorem implies the Square Peg Problem for convex curves  $\gamma$  by constructing a height function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$  whose level sets  $f^{-1}(x)$  are similar to  $\gamma$  for all  $x > 0$ .

Zaks [77] found an analogous “chair theorem”, where instead of a square table he considers triangular chairs with a fixed direction. Kronheimer–Kronheimer [35] found conditions on  $\partial D$  such that the table/chair can be chosen such that all four/three vertices lie in  $D$ : Namely,  $\partial D$  should not inscribe a square/triangle of a smaller size. More table theorems are due to Meyerson [53], see also [54, 55] for interesting examples.

**An open and dense criterion.** In [51] the Square Peg Problem was proved for the so far weakest smoothness condition.

**Theorem 4.** Let  $\gamma : S^1 \hookrightarrow \mathbb{R}^2$  be a Jordan curve. Assume that there is an  $0 < \varepsilon < 2\pi$  such that  $\gamma$  contains no (or generically an even number of) special trapezoids of size  $\varepsilon$ . Then  $\gamma$  inscribes a square.

Here an inscribed *special trapezoid* is a 4-tuple of pairwise distinct points  $x_1, \dots, x_4 \in S^1$  lying clockwise on  $S^1$  such that the points  $P_i := \gamma(x_i)$  satisfy

$$\begin{aligned} \|P_1 - P_2\| = \|P_2 - P_3\| = \|P_3 - P_4\| &> \|P_4 - P_1\| \\ \text{and } \|P_1 - P_3\| &= \|P_2 - P_4\|. \end{aligned}$$

The *size* of this special trapezoid is defined as the length of the clockwise arc in  $S^1$  from  $x_1$  to  $x_4$ .

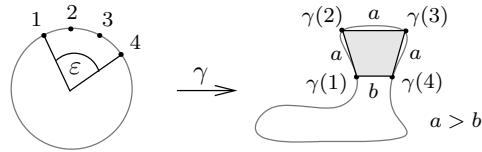


Figure 5: A special trapezoid of size  $\varepsilon$ .

The set of curves without inscribed special trapezoids of a fixed size  $\varepsilon$  is open and dense in the space of embeddings  $S^1 \hookrightarrow X$  with respect to the compact-open topology. This theorem is basically the exact criterion that one obtains by applying equivariant obstruction theory to the test-map (1). Vrećica and Živaljević [70] have been the first who applied obstruction theory to the Square Peg Problem and they proved it for Stromquist's class of locally monotone curves.

**An explicit open criterion.** All previous criteria on curves for which the Square Peg Problem was proved are defined by local smoothness conditions. The following criterion from [51] is a global one, which yields an open set of not necessarily injective curves in  $C^0(S^1, \mathbb{R}^2)$  with respect to the  $C^0$ -topology, or equivalently, the compact-open topology.

**Theorem 5.** Let  $A$  denote the annulus  $\{x \in \mathbb{R}^2 \mid 1 \leq \|x\| \leq 1 + \sqrt{2}\}$ . Suppose that  $\gamma : S^1 \rightarrow A$  is a continuous closed curve in  $A$  that is non-zero in  $\pi_1(A) = \mathbb{Z}$ . Then  $\gamma$  inscribes a square of side length at least  $\sqrt{2}$ .

It is open whether the outer radius  $1 + \sqrt{2}$  of  $A$  can be increased by some small  $\varepsilon > 0$ .

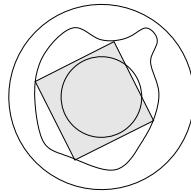


Figure 6: Example for Theorem 5.

The proof idea is very simple: If the annulus  $A$  is thin enough, then the set of squares with all vertices in  $A$  splits into two connected components: big squares and small squares. A generic curve that represents a generator of  $\pi_1(A)$  inscribes an odd number of big squares (and an even number of small squares).

## Related problems

**The number of inscribed squares.** Popvassilev [61] constructed for any  $n \geq 1$  a smooth convex curve that has exactly  $n$  inscribed squares, every square being counted exactly once and not with multiplicity. All but one of the  $n$  squares in his construction are non-generic. They will disappear immediately after deforming the curve by a suitable  $C^\infty$ -isotopy. An analog piecewise linear example was given by Sagols–Marín [63].

Van Lamoen [67] studied the geometric relationship between the inscribed squares in the union of three lines (an extended triangle).

In [51] the parity of the number of squares on generic smooth *immersed* curves in the plane was given, which depends not only on the isotopy type of the immersion but also on the intersection angles.

Van Heijst proves in his upcoming master thesis [22] that any real algebraic curve in  $\mathbb{R}^2$  of degree  $d$  inscribes either at most  $(d^4 - 5d^2 + 4d)/4$  or infinitely many squares. For this he makes use of Bernstein’s theorem, which states that the number of common zeros in  $(\mathbb{C}^*)^k$  of  $k$  generic Laurent polynomials in  $k$  variables with prescribed Newton polytopes equals the mixed volume of these polytopes.

**Inscribed triangles.** It is not hard to show that any smooth embedding  $\gamma : S^1 \rightarrow \mathbb{R}^2$  inscribes arbitrary triangles, even if we prescribe where one of the vertices has to sit. Moreover the set of all such inscribed triangles determines a homology class  $\alpha \in H_1(P_3, \mathbb{Z}) = \mathbb{Z}$ , where  $P_3$  is the set of 3-tuples of points on  $\gamma$  that lie counter-clockwise on the curve. The class  $\alpha$  turns out to be a generator, as one sees from inspecting the situation for the circle.

For continuous curves Nielsen [56] proved the following version of that.

**Theorem 6** (Nielsen). *Let  $T$  be an arbitrary triangle and  $\gamma : S^1 \rightarrow \mathbb{R}^2$  an embedded circle. Then there are infinitely many triangles inscribed in  $\gamma$  which are similar to  $T$ , and if one fixes a vertex of smallest angle in  $T$  then the set of the corresponding vertices on  $\gamma$  is dense in  $\gamma$ .*

**Inscribed rectangles.** Instead of squares one may ask whether any embedded circle in the plane inscribes a rectangle. If one does not prescribe the aspect ratio then the answer is affirmative.

**Theorem 7** (Vaughan). *Any continuous embedding  $\gamma : S^1 \hookrightarrow \mathbb{R}^2$  inscribes a rectangle.*

Vaughan’s proof, which appeared in Meyerson [53], is very beautiful:  $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$  acts on the torus  $(S^1)^2$  by permuting the coordinates, and the quotient space  $(S^1)^2/\mathbb{Z}_2$  is a Möbius strip. The proof of Theorem 7 uses that the map  $f : (S^1)^2/\mathbb{Z}_2 \rightarrow \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$  given by

$$f(x, y) = ((\gamma(x) + \gamma(y))/2, \|\gamma(x) - \gamma(y)\|),$$

must have a double point, otherwise it would extend to an embedding of  $\mathbb{R}P^2$  into  $\mathbb{R}^3$  by gluing to that Möbius strip the disc  $I \times \{0\}$ , where  $I \subset \mathbb{R}^2$  is the interior of  $\gamma$ . The double point corresponds to two secants of  $\gamma$  having the same length and the same midpoint. Hence this forms an inscribed rectangle.

If we furthermore prescribe the aspect ratio of the rectangle, then the problem is widely open, even for smooth or piecewise linear curves.

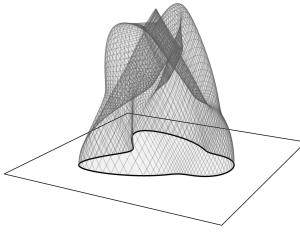


Figure 7: The image of  $f$ , a self-intersecting Möbius strip with boundary  $\gamma$ .

**Conjecture 8** (Rectangular Peg Problem). *Every  $C^\infty$  embedding  $\gamma : S^1 \rightarrow \mathbb{R}^2$  contains four points that are the vertices of a rectangle with a prescribed aspect ratio  $r > 0$ .*

This conjecture is highly interesting since the standard topological approach does not yield a proof: The equivariant homology class of the solution set, a  $\mathbb{Z}$ -valued smooth isotopy invariant of the curve, turns out to be zero. For example, an ellipse inscribes a positive and a negative rectangle. Stronger topological tools fail as well. It seems again that more geometric ideas are needed.

Equivalently we could state Conjecture 8 for all piecewise linear curves. Proofs exist only for the case  $r = 1$ , which is the *smooth* Square Peg Problem, for arbitrary  $r$  in case the curve is close to an ellipse, see Makeev [44] and Conjecture 9 below, and for  $r = \sqrt{3}$  in case the curve is close to convex, see [51].

A proof for the Rectangular Peg Problem was claimed by Griffiths [16], but it contains errors regarding the orientations. Essentially he calculated that the number of inscribed rectangles of the given aspect ratio counted with appropriate signs and modulo symmetry is 2, however zero is correct.

Using topology alone one can show the following: Assume that Conjecture 8 is false. Then for some ‘generic’ curve  $\gamma$  the 1-dimensional manifold of inscribed rectangles of arbitrary size contains a component homeomorphic to  $S^1$  that contains an odd number of squares up to symmetry. (In other words, this component is invariant under the symmetry group  $\mathbb{Z}/4\mathbb{Z}$  that permutes the vertices of a rectangle cyclicly.) This seems to put strong geometric conditions on  $\gamma$ , and possibly such curve do not exist at all.

**Other inscribed quadrilaterals.** It is natural to ask what other quadrilaterals can be inscribed into closed curves in the plane. Since the unit circle is a curve, those quadrilaterals must be circular, that is, they must have a circumcircle.

Depending on the class of curves that we look at, the following two conjectures seem reasonable.

**Conjecture 9** (Circular Quad Peg Problem). *Let  $Q$  be a circular quadrilateral. Then any  $C^\infty$  embedding  $\gamma : S^1 \rightarrow \mathbb{R}^2$  admits an orientation preserving similarity transformation that maps the vertices of  $Q$  into  $\gamma$ .*

Makeev [44] proved a first instance of this conjecture, namely for the case of star-shaped  $C^2$ -curves that intersect every circle in at most 4 points.

Furthermore, Karasev [31] proved that for any smooth curve and a given  $Q = ABCD$ , either this conjecture holds, or one can find two inscribed triangles similar to  $ABC$ , such that

the two corresponding fourth vertices  $D$  coincide (but  $D$  may not lie on  $\gamma$ ). The proof idea is a beautiful geometric volume argument. It should be stressed that most open problems discussed here are geometric problems rather than topological ones: We understand the basic algebraic topology here quite well, but not the restrictions on the topology that the geometry dictates. New geometric ideas such as Karasev's are needed.

**Conjecture 10** (Trapezoidal Peg Problem). *Let  $T$  be an isosceles trapezoid. Then any piecewise-linear embedding  $\gamma : S^1 \rightarrow \mathbb{R}^2$  inscribes a quadrilateral similar to  $T$ .*

The reason for restricting the latter conjecture to isosceles trapezoids, that is, trapezoids with circumcircle, is that all other circular quadrilaterals cannot be inscribed into very thin triangles. This was observed by Pak [59].

**Other inscribed polygons.** For any  $n$ -gon  $P$  with  $n \geq 5$  it is easy to find many curves that do not inscribe  $P$ . If we do not require all vertices to lie on  $\gamma$  then Makeev has some results for circular pentagons, see [47].

Alternatively, we can relax the angle conditions, that is, we require only that the edge ratios are the same as the ones in a given polygon  $P$ . Then as for the triangles above one can show that the set of such  $n$ -gons represents the generator of  $H_1(P_n; \mathbb{Z}) = \mathbb{Z}$ , where  $P_n$  is the set of  $n$ -tuples of points on  $\gamma$  that lie counter-clockwise on the curve; see Meyerson [52], Wu [71], Makeev [47], and Vrećica–Živaljević [70].

**Higher dimensions.** In higher dimensions one may ask whether any  $(n - 1)$ -sphere that is smoothly embedded in  $\mathbb{R}^n$  inscribes an  $n$ -cube in the sense that all vertices of the cube lie on the sphere. However, most smooth embeddings  $S^{n-1} \hookrightarrow \mathbb{R}^n$  do not inscribe an  $n$ -cube for  $n \geq 3$ , in the sense that these embeddings form an open and dense subset of all smooth embeddings in the compact-open topology, a heuristic reason being that the number of equations to fulfill is larger than the degrees of freedom. An explicit example are the boundaries of very thin simplices, as was noted by Kakutani [26] for  $n = 3$ . Hausel–Makai–Szűcs [20] proved that the boundary of any centrally symmetric convex body in  $\mathbb{R}^3$  inscribes a 3-cube.

If we do not want to require further symmetry on the embedding  $S^{n-1} \hookrightarrow \mathbb{R}^n$ , then crosspolytopes are more suitable higher analogs of squares: The regular  $n$ -dimensional crosspolytope is the convex hull of  $\{\pm e_i\}$  where  $e_i$  are the standard basis vectors in  $\mathbb{R}^n$ .

**Theorem 11** (Makeev, Karasev). *Let  $n$  be an odd prime power. Then every smooth embedding  $\Gamma : S^{n-1} \rightarrow \mathbb{R}^n$  contains the vertices of a regular  $n$ -dimensional crosspolytope.*

The  $n = 3$  case was posed as Problem 11.5 in Klee & Wagon [33]. This was answered affirmatively by Makeev [46]. Karasev [28] generalized the proof to arbitrary odd prime powers. Akopyan and Karasev [1] proved the same theorem for  $n = 3$  in case  $\Gamma$  is the boundary of a simple polytope by a careful and non-trivial limit argument from the smooth case.

Gromov [17] proved an similar theorem for inscribed simplices.

**Theorem 12** (Gromov). *Any compact set  $S \subset \mathbb{R}^d$  with  $C^1$ -boundary and non-zero Euler characteristic inscribes an arbitrary given simplex up to similarity on its boundary  $\partial S$ .*

**Circumscribing problems.** Instead of inscribing polytopes into surfaces we can ask which polytopes  $P$  can circumscribe a given surface  $S$  in the sense that  $P$  contains  $S$  and every facet of  $P$  touches  $S$ .

The most prominent result is the following theorem.

**Theorem 13** (Kakutani, Yamabe–Yujobô). *Any compact convex body in  $\mathbb{R}^n$  has an  $n$ -dimensional circumscribing cube.*

The problem was posed by Rademacher. Kakutani [26] proved the  $n = 3$  case. Yamabe and Yujobô [72] generalized this to arbitrary dimensions. The latter proof is in particular interesting since it uses a clever and simple induction without any oddness arguments.

More generally they proved that for any map  $f : S^{n-1} \rightarrow \mathbb{R}$  there exist  $n$  pairwise orthogonal points on the sphere that are mapped by  $F$  to the same value. This has been a motivation for many similar problems and theorems:

**Conjecture 14** (Knaster [34]). *For any map  $f : S^{n-1} \rightarrow \mathbb{R}^m$  and any  $n - m + 1$  points  $x_0, \dots, x_{n-m} \in S^{n-1}$  there exists a rotation  $\rho \in SO(n)$  such that  $f(\rho(x_0)) = \dots = f(\rho(x_{n-m}))$ .*

Hopf [24] proved the special case of two points, that is, for  $n - m + 1 = 2$  (he actually proved a more general version for functions on Riemannian manifolds, see also Akopyan–Karasev–Volovikov [2]). Floyd [15] proved it for the special case  $(n, m) = (3, 1)$ . For all other pairs  $(n, m)$  Knaster’s conjecture is either open or false; counter-examples have been found by Makeev [41], Babenko–Bogatyř [3], Chen [6], Kashin–Szarek [32], and Hinrichs–Richter [23]. See Hinrichs–Richter [23] and Liu [37] for more detailed overviews of known results. Thus for  $n - 1 > m$  the following problem is more sensible:

**Problem 15.** *Given a triple positive integers  $(n, k, m)$ , classify all point configurations  $x_1, \dots, x_k \in S^{n-1}$  with the following property: For any map  $f : S^{n-1} \rightarrow \mathbb{R}^m$  there exists a rotation  $\rho \in SO(n)$  such that  $f(\rho(x_1)) = \dots = f(\rho(x_k))$ .*

Many such point configurations  $x_1, \dots, x_k$  have been established: Dyson [10] showed that for  $(n, k, m) = (3, 4, 1)$  some level set of  $f$  has to contain the four vertices of a square whose midpoint is the origin. Again, this can be easily proved by a parity argument. Livesay [39] generalized this to rectangles with a prescribed aspect ratio. Here the number of solutions is *even*. Livesay circumvents this difficulty by reducing the problem to the intermediate value theorem. Yang [76] proved that for  $n = m + 2$  and  $k = 3$ , the vertices of an equilateral triangle of geodesic edge length at most  $2\pi/3$  satisfy the property of Problem 15. Further Knaster type theorems, some of them requiring additional symmetry of  $f$ , are due to Makeev [42, 43, 49], Volovikov [68, 69], Crabb–Jaworowski [8], Karasev–Volovikov [30], Karasev [29], and Liu [37, 38].

For  $m = 1$ , any of those point configurations in Problem 15 imply a corresponding circumscribing result: The particular point configuration  $(x_0, \dots, x_k)$  corresponds to the facet normals of the possibly unbounded circumscribing polyhedron, and the symmetry conditions on  $f$  also have to be put on the symmetry of the convex body that one wants to circumscribe.

Many other very interesting inscribing and circumscribing theorems and problems are due to Hausel–Makai–Szűcs [19, 20], Kuperberg [36], Makeev [40, 45, 46, 48], and Yang [73, 75, 74], and this list is not complete.

Let us finish with the following table problem on the sphere, which is in some sense the smallest open case of Problem 15.

**Conjecture 16** (Table problem on  $S^2$ ). Suppose  $x_1, \dots, x_4 \in S^2 \subset \mathbb{R}^3$  are the vertices of a square that is inscribed in the standard 2-sphere, and let  $h : S^2 \rightarrow \mathbb{R}$  be a smooth function. Then there exist a rotation  $\rho \in SO(3)$  such that  $h(\rho(x_1)) = \dots = h(\rho(x_4))$ .

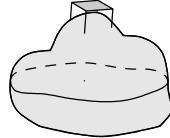


Figure 8: Intuition behind Conjecture 16: Think of a square table for which we want to find a spot on Earth such that all four table-legs are at the same height.

So far this has been proven only when  $x_1, \dots, x_4$  lie on a great circle, see Dyson [10], since this is the only case in which the generic number of solutions is odd. The critical points of  $h$  can be thought of as the spots on which you can put an infinitesimally small table. But it is not clear whether Morse theory could also describe the other level squares (i.e. arbitrarily sized squares on  $S^2$  whose vertices are at the same height) in a useful way.

Using topology alone one can show the following: Assume that the conjecture is false. Then for some ‘generic’ height function  $h$ , the 1-dimensional manifold of level squares contains a component homeomorphic to  $S^1$  that parametrizes an *odd* number of squares modulo symmetry. (In other words, this component is invariant under some reflection of the symmetry group  $D_8$ .) This seems to put a quite restrictive condition on  $h$ , but it’s not clear whether this is useful.

## References

- [1] A. Akopyan and R. N. Karasev. Inscribing a regular octahedron into polytopes. *Discrete Math.*, 313(1):122–128, 2013.
- [2] A. Akopyan, R. N. Karasev, and A. Volovikov. Borsuk–Ulam type theorems for metric spaces. arXiv:1209.1249, 2012.
- [3] I. K. Babenko and S. A. Bogatyř. Mapping a sphere into Euclidean space. *Math. Notes*, 46(3-4):683–686, 1990. In Russian: Matematicheskie Zametki 46 (1989), no. 3, 3–8.
- [4] J. Cantarella. Webpage on the square peg problem. [http://www.jasoncantarella.com/webpage/index.php?title=Square\\_Peg\\_problem](http://www.jasoncantarella.com/webpage/index.php?title=Square_Peg_problem), 2008.
- [5] J. Cantarella, E. Denne, and J. McCleary. Transversality in Configuration Spaces and the Square Peg Problem. In preparation.
- [6] W. Chen. Counterexamples to Knaster’s conjecture. *Topology*, 37:401–405, 1998.
- [7] C. M. Christensen. A square inscribed in a convex figure (in Danish). *Matematisk Tidsskrift B*, 1950:22–26, 1950.
- [8] M. C. Crabb and J. W. Jaworowski. Theorems of Kakutani and Dyson revisited. *J. Fixed Point Theory Appl.*, 5(2):227–236, 2009.
- [9] E. Denne. Inscribed squares: Denne speaks. <http://quomodocumque.wordpress.com/2007/08/31/inscribed-squares-denne-speaks/>, 2007. Guest post on Jordan S. Ellenberg’s blog *Quomodocumque*.
- [10] F. J. Dyson. Continuous functions defined on spheres. *Ann. Math.*, 54(2):534–536, 1951.
- [11] A. Emch. Some properties of closed convex curves in a plane. *Amer. J. Math.*, 35:407–412, 1913.
- [12] A. Emch. On the medians of a closed convex polygon. *Amer. J. Math.*, 37:19–28, 1915.
- [13] A. Emch. On some properties of the medians of closed continuous curves formed by analytic arcs. *Amer. J. Math.*, 38(1):6–18, 1916.
- [14] R. Fenn. The table theorem. *Bull. London Math. Soc.*, 2:7376, 1970.
- [15] E. E. Floyd. Real-valued mappings of spheres. *Proc. Amer. Math. Soc.*, 6:957–959, 1955.

- [16] H. B. Griffiths. The topology of square pegs in round holes. *Proc. London Math. Soc.*, 62(3):647–672, 1990.
- [17] M. L. Gromov. Simplexes inscribed on a hypersurface (Russian). *Matematicheskie Zametki*, 5:81–89, 1969.
- [18] H. W. Guggenheimer. Finite sets on curves and surfaces. *Israel J. Math.*, 3:104–112, 1965.
- [19] T. Hausel, E. Makai, Jr., and A. Szűcs. Polyhedra inscribed and circumscribed to convex bodies. In *Proceedings of the Third International Workshop on Differential Geometry and its Applications and the First German-Romanian Seminar on Geometry (Sibiu, 1997)*, volume 5 of *General Mathematics*, pages 183–190, 1997.
- [20] T. Hausel, E. Makai, Jr., and A. Szűcs. Inscribing cubes and covering by rhombic dodecahedra via equivariant topology. *Mathematika*, 47(1-2):371–397, 2002.
- [21] C. M. Hebbert. The inscribed and circumscribed squares of a quadrilateral and their significance in kinematic geometry. *Ann. of Math. (2)*, 16(1-4):38–42, 1914/15.
- [22] W. van Heijst, 2014. Master thesis, in preparation.
- [23] A. Hinrichs and C. Richter. The Knaster problem: more counterexamples. *Israel J. Math.*, 145:311–324, 2005.
- [24] H. Hopf. Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze (German). *Portugaliæ Mathematica*, 4:129–139, 1944.
- [25] R. P. Jerrard. Inscribed squares in plane curves. *Trans. Amer. Math. Soc.*, 98:234–241, 1961.
- [26] S. Kakutani. A proof that there exists a circumscribing cube around any bounded closed convex set in  $\mathbb{R}^3$ . *Ann. Math.*, 43(4):739–741, 1942.
- [27] R. N. Karasev. Topological methods in combinatorial geometry. *Russian Math. Surveys*, 63(6):1031–1078, 2008.
- [28] R. N. Karasev. Inscribing a regular crosspolytope. arXiv:0905.2671, 2009.
- [29] R. N. Karasev. Knaster’s problem for  $(Z_2)^k$ -symmetric subsets of the sphere  $S^{2^k-1}$ . *Discr. Comp. Geom.*, 44(2):429–438, 2010.
- [30] R. N. Karasev and A. Y. Volovikov. Knaster’s problem for almost  $(Z_p)^k$ -orbits. *Topology Appl.*, 157(5):941–945, 2010.
- [31] R. N. Karasev and A. Y. Volovikov. A note on Makeev’s conjectures. arxiv:1002.4070, 2010.
- [32] B. S. Kashin and S. J. Szarek. The Knaster problem and the geometry of highdimensional cubes. *Comptes Rendus de l’Académie des Sciences, Paris, Série I*, 336:931–936, 2003.
- [33] V. Klee and S. Wagon. *Old and new unsolved problems in plane geometry and number theory*. Dolciani Mathematical Expositions. The Math. Ass. America, 1996.
- [34] B. Knaster. Problème 4. *Colloquium Mathematicum*, 1:30–31, 1947. (French).
- [35] E. H. Kronheimer and P. B. Kronheimer. The tripos problem. *J. London Math. Soc. (2)*, 24(1):182–192, 1981.
- [36] G. Kuperberg. Circumscribing constant-width bodies with polytopes. *New York J. Math.*, 5:91–100 (electronic), 1999.
- [37] Y. Liu. On a property of functions on the sphere and its application. *Nonlinear Anal.*, 73:3376–3381, 2010.
- [38] Y. Liu. Some mapping theorems for continuous functions defined on the sphere. *Nonlinear Anal.*, 75(4):1881–1886, 2012.
- [39] G. R. Livesay. On a theorem of F. J. Dyson. *Ann. Math.*, 59(2):227–229, 1954.
- [40] V. V. Makeev. Dimension restrictions in problems of combinatorial geometry. *Sibirsk. Mat. Zh.*, 23(4):197–201, 222, 1982.
- [41] V. V. Makeev. Some properties of continuous mappings of spheres and problems in combinatorial geometry (Russian). *Geometric Questions in the Theory of Functions and Sets (Russian)*, Kalinin. Gos. Univ., Kalinin, 1986.
- [42] V. V. Makeev. Knaster’s problem on continuous maps of a sphere into Euclidean space. *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta imeni V. A. Steklova Akademii Nauk SSSR (LOMI)*, 167(Issled. Topol. 6):169–178, 193, 1988. Translation in J. Soviet Math. 52 (1990), no. 1, 2854–2860.
- [43] V. V. Makeev. Solution of the Knaster problem for polynomials of second degree on a two-dimensional sphere. *Matematicheskie Zametki*, 53(1):146–148, 1993. Translation in Math. Notes 53 (1993), no. 1-2, 106–107.
- [44] V. V. Makeev. On quadrangles inscribed in a closed curve. *Math. Notes*, 57(1-2):91–93, 1995.
- [45] V. V. Makeev. Three-dimensional polytopes inscribed in and circumscribed about convex compacta. II.

- Rossiiskaya Akademiya Nauk. Algebra i Analiz*, 13(5):110–133, 2001. Translation in St. Petersburg Math. J. 13 (2002), no. 5, 791–807.
- [46] V. V. Makeev. *Universally inscribed and outscribed polytopes*. PhD thesis, Saint-Petersburg State University, 2003.
- [47] V. V. Makeev. On quadrangles inscribed in a closed curve and the vertices of the curve. *J. Math. Sci.*, 131(1):5395–5400, 2005.
- [48] V. V. Makeev. Some geometric properties of closed space curves and convex bodies. *Rossiiskaya Akademiya Nauk. Algebra i Analiz*, 16(5):92–100, 2005. Translation in St. Petersburg Math. J. 16 (2005), 815–820.
- [49] V. V. Makeev. Inscribed and circumscribed polyhedra for a convex body and a problem on continuous functions on a sphere in Euclidean space. *Rossiiskaya Akademiya Nauk. Algebra i Analiz*, 18(6):187–204, 2006. Translation in St. Petersburg Math. J. 18 (2007), no. 6, 997–1009.
- [50] B. Matschke. Equivariant topology and applications. Diploma thesis, TU Berlin, 2008.
- [51] B. Matschke. *Equivariant topology methods in discrete geometry*. PhD thesis, Freie Universität Berlin, 2011.
- [52] M. D. Meyerson. Equilateral triangles and continuous curves. *Polska Akademia Nauk. Fundamenta Mathematicae*, 110(1):1–9, 1980.
- [53] M. D. Meyerson. Balancing acts. *Topology Proc.*, 6(1):59–75, 1981.
- [54] M. D. Meyerson. Convexity and the table theorem. *Pacific J. Math.*, 97(1):167–169, 1981.
- [55] M. D. Meyerson. Remarks on Fenn’s “the table theorem” and Zaks’ “the chair theorem”. *Pacific J. Math.*, 110(1):167–169, 1984.
- [56] M. J. Nielsen. Triangles inscribed in simple closed curves. *Geometriae Dedicata*, 43:291–297, 1992.
- [57] M. J. Nielsen. Web page on Figures Inscribed in Curves. <http://www.webpages.uidaho.edu/~markn/squares/>, 2000.
- [58] M. J. Nielsen and S. E. Wright. Rectangles inscribed in symmetric continua. *Geom. Dedicata*, 56(3):285–297, 1995.
- [59] I. Pak. Lectures on Discrete and Polyhedral Geometry. <http://www.math.ucla.edu/~pak/book.htm>, 2010.
- [60] V. H. Pettersson, H. Tverberg, and P. R. J. Östergård. A note on Toeplitz’ square problem. Submitted, 2013.
- [61] S. G. Popvassilev. On the number of inscribed squares of a simple closed curve in the plane. arxiv:0810.4806, 2008.
- [62] F. Sagols and R. Marín. The inscribed square conjecture in the digital plane. In *Combinatorial image analysis*, volume 5852 of *Lecture Notes in Comput. Sci.*, pages 411–424. Springer, 2009.
- [63] F. Sagols and R. Marín. Two discrete versions of the inscribed square conjecture and some related problems. *Theoret. Comput. Sci.*, 412(15):1301–1312, 2011.
- [64] L. G. Schnirelman. On some geometric properties of closed curves. (*in Russian*) *Usp. Mat. Nauk*, 10:34–44, 1944. Available at <http://ega.math.narod.ru/Nquant/Square.djv>. Posthumous reproduction and extension of the author’s original article in *Sbornik Rabot Matematicheskogo Razdela Sekcii Estestvennyh i Točnyh Nauk Komakademii*, Moscow, 1929.
- [65] W. R. Stromquist. Inscribed squares and square-like quadrilaterals in closed curves. *Mathematika*, 36:187–197, 1989.
- [66] O. Toeplitz. Ueber einige Aufgaben der Analysis situs. *Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Solothurn*, 4:197, 1911.
- [67] F. van Lamoen. Inscribed squares. *Forum Geom.*, 4:207–214 (electronic), 2004.
- [68] A. Y. Volovikov. A theorem of Bourgin-Yang type for  $\mathbb{Z}_p^n$ -action. *Matematicheskii Sbornik*, 183(7):115–144, 1992. Translation in Russian Acad. Sci. Sb. Math. 76 (1993), no. 2, 361–387.
- [69] A. Y. Volovikov. On a property of functions on a sphere. *Matematicheskie Zametki*, 70(5):679–690, 2001. Translation in Math. Notes 70 (2001), no. 5–6, 616–627.
- [70] S. Vrećica and R. T. Živaljević. Fulton–MacPherson compactification, cyclohedra, and the polygonal pegs problem. *Israel J. Math.*, 184(1):221–249, 2011.
- [71] Y.-Q. Wu. Inscribing smooth knots with regular polygons. *Bull. London Math. Soc.*, 36(2):176–180, 2004.
- [72] H. Yamabe and Z. Yujobō. On the continuous function defined on a sphere. *Osaka Math. J.*, 2(1):19–22, 1950.
- [73] C. T. Yang. On the theorem of Borsuk–Ulam, Kakutani–Yamabe–Yujobō and Dyson, I. *Ann. Math.*, 60(2):262–282, 1954.
- [74] C. T. Yang. Continuous functions from spheres to euclidean spaces. *Ann. Math.*, 62(2):284–292, 1955.
- [75] C. T. Yang. On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobō and Dyson, II. *Ann. Math.*, 62

- (2):271–283, 1955.
- [76] C. T. Yang. On maps from spheres to Euclidean spaces. *Amer. J. Math.*, 79:725–732, 1957.
  - [77] J. Zaks. The chair theorem. In *Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing*, pages 557–562, Baton Rouge, La., 1971. Louisiana State Univ.
  - [78] K. Zindler. Über konvexe Gebilde. *Monatshefte für Mathematik und Physik*, 31:25–56, 1921.

Benjamin Matschke  
Max Planck Institute for Mathematics, Bonn  
matschke@mpim-bonn.mpg.de