Topics in topological combinatorics
— *very* preliminary notes —

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Abstract

These are the lecture notes for a course on "Topics in topological combinatorics" held at MPIM Bonn in the summer term of 2015.

1 Introduction

What is topological combinatorics? In one short sentence, algebraic topology is the study of topological spaces via algebraic invariants such at homology and homotopy groups.

Combinatorics is more difficult to define. Some people would say combinatorics is the study of finite sets and structures thereof. In my opinion this is a bit vague, it can mean all and nothing, depending on the readers interpretation. Instead of classifying such structures in more detail, it seems more worthwhile to classify what one does with such structures:

1. One can prove their existence or non-existence.
2. One can construct structures.
3. One can count such structures.
4. One can be interested in extremal structures, some which maximise certain parameters.

There is a zoo of methods that can be used: Probabilistic methods, generating functions, analytic methods, algebraic methods, and so on. In these lectures we focus on topological methods. Naturally the methods one chooses depend very much on the structure that one tries to study. The topological method is very useful for geometric problems, obviously, but not exclusively. There are great examples of problems in which the topological method is applicable even though there is a priori no obvious indication for that. A perfect example for this is Lovász' lower bound for the chromatic number of a graph, which is in terms of the connectivity of a certain hom-complex associated to this graph.

To me, topological combinatorics is the study of combinatorial problems (in one of the four senses above, in particular 1. and 3.) using topological methods.

About the topics covered in this course. In these lectures we will cover some standard methods from topology and apply these to some beautiful problems in combinatorics.

The topological methods include: Bordism arguments, obstruction theory, characteristic classes, Fadell–Husseini index theory, Lyusternik–Schnirelmann category, Schwarz genus, Morse theory.

The applications are quite diverse: Some of these topics are more classical, some more recent; some more elementary, some more elaborate, and most of them are easily accessible and offer challenging open problems.

Prerequisites. We assume some background in algebraic topology. Having read a standard text book such as Spanier's "Algebraic Topology", Bredon's "Topology and Geometry", or Hatcher's "Algebraic Topology" will be more than enough. These lectures will review some standard concepts, but in a very concise manner, without too many proofs, but with links to the literature. In case my writing style is too difficult to follow right now, I recommend the wonderful and more elementary text books by Matoušek "Using the Borsuk–Ulam Theorem" and by de Longueville "A course in Topological Combinatorics".

2 Teaser

2.1 Topics.

The inscribed square problem. A Jordan curve is a continuous simple closed curve in the plane. Equivalently, it is an injective map $S^1 \to \mathbb{R}^2$.

Problem 2.1 (Toeplitz 1911). *Does every Jordan curve inscribe a square?*

More generally: Inscribing and circumscribing problems. The Knaster conjecture. (Borsuk–Ulam theorem. Hopf theorem. Kakutani–Yamabe–Yujobo theorem. Dyson's theorem. Livesay's theorem.)

The ham sandwich theorem. For the following theorem, let us define a *mass* to be a probability measure on the Borel σ-algebra of $\mathbb{R}^d$ such that hyperplanes are zero sets. A hyperplane H is said to *bisect* a mass μ if both halfspaces H^+ and H^- have the same measure, $\mu(H^+) = \mu(H^-)$.

Theorem 2.2 (Banach–Steinhaus 1938, Stone-Tukey 1942). *Any d masses in $\mathbb{R}^d$ can be bisected simultaneously using a single hyperplane.*

More generally: Measure partitions by several hyperplanes. Convex partitions. Partitions by fans, by spheres.

Gromov's waist of the sphere theorem.

Theorem 2.3 (Gromov 2003)**.** *Let $f : S^n \to S^k$ be a map with $n \geq k$. Then there exists a point $z \in S^k$ such that for any $\varepsilon > 0$,*

$$\mathrm{vol} U_\varepsilon(f^{-1}(z)) \geq \mathrm{vol} U_\varepsilon(S^{n-k}),$$

where U_ε denotes the ε-neighborhood in S^n, and vol *the n-dimensional standard rotation invariant measure on S^n.*

More general: Karasev–Volovikov (maps to arbitrary manifolds of even degree).

Tverberg's theorem.

Theorem 2.4 (Tverberg 1966)**.** *Let $p \geq 2$ and $d \geq 1$, and put $N := (p-1)(d+1)$. Then any set X of $N+1$ points in $\mathbb{R}^d$ can be partitioned into p parts, $X = X_1 \dot\cup \ldots \dot\cup X_p$, whose convex hulls intersect,*

$$\mathrm{conv}(X_1) \cap \ldots \cap \mathrm{conv}(X_p) \neq \emptyset.$$

More generally: Topological Tverberg, Colored Tverberg, Projective Tverberg. Other constrained Tverberg theorems.

Non-planarity of $K_{3,3}$ and K_5. Let K_n denote the *complete graph* on n vertices, and $K_{n,m}$ the *complete bipartite graph* on n and m vertices. A graph is called planar if it (regarded as a topological space) can be embedded into the plane. (For finite graphs, embeddings into the plane are the same as injective maps into the plane.)

Theorem 2.5 (Kuratowski 1930)**.** *K_5 and $K_{3,3}$ are non-planar.*

More generally: Van Kampen–Flores theorem. Sarkaria's coloring and embedding theorem.

Kneser conjecture.

Definition 2.6 (Kneser graph)**.** We define *Kneser graph* $KG_{n,k}$ to be the graph with vertex set $V = \binom{[n]}{k}$, two vertices $A, B \in V$ being connected by an edge if they are disjoint subsets of $[n]$.

Definition 2.7 (Chromatic number)**.** The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors one needs to color the vertices of G such that any two adjacent vertices obtain different colors.

In other words,

$$\chi(G) = \min\{k \mid \text{there exists a graph homomorphism } G \to K_k\}.$$

Kneser conjectured the following, which is now a theorem by Lovász [].

Theorem 2.8 (Lovász 78)**.** *For $n \geq k/2$, the chromatic number of the Kneser graph $KG_{n,k}$ is given by*

$$\chi(KG_{n,k}) = n - 2k + 1.$$

More generally: Hom complexes. Hypergraphs.

Nash equilibria, hex game. More generally: More players, higher dimensional hex game. Others?

Gromov–Milman conjecture. The following was a conjecture written up by Milman [] after discussions with Gromov. It was recently proved by Karasev and Dolnikov [13].

Theorem 2.9 (Dolnikov–Karasev 2011)**.** *For any $k \geq 1$ and any even $d \geq 2$ there exists $n = n(d,k)$ such that the following holds. Any homogeneous polynomial f of degree d on $\mathbb{R}^n$ can be restricted to a linear k-dimensional subspace V, such that $f|_V$ is proportional to the $d/2$'th power of $q|_V$, where q is the standard quadratic form $q(x) = x_1^2 + \ldots + x_n^2$.*

Maybe first in dimension 2, then for higher dimensions.

Upper bound theorem.

Definition 2.10 (Cyclic polytope)**.** For $n > d \geq 1$, the *cyclic polytope* $C(d,n)$ of dimension d on n vertices is defined as the convex hull of any n distinct points on the d-dimensional moment curve $t \mapsto (t, t^2, \ldots, t^d)$.

Definition 2.11 (f-vector)**.** Let P be a d-dimensional polytope. Its *f-vector* is the vector $f(P) = (f_0(P), \ldots, f_d(P))$, where $f_i(P)$ is the number of i-dimensional faces of P.

In general, the affine and projective type of $C(d,n)$ of course will depend on the n chosen points, but the combinatorial type does not.

Theorem 2.12 (Upper bound theorem, McMullen 1970)**.** *Let P be a d-dimensional polytope on n vertices. Then $f_i(P) \leq f_i(C_{d,n})$.*

More generally: Toric manifolds vs. polytopes. g-theorem.

Asynchronous computability. First: binary consensus.

First selection lemma.

Theorem 2.13 (First selection lemma, Gromov's version 2010)**.** *Let $f : \Delta_{[n]} \to \mathbb{R}^d$ be a map from the n-simplex to $\mathbb{R}^d$. Then some point $a \in \mathbb{R}^d$ is contained in the image of at least $c_d\binom{n}{d+1}$ many d-faces of $\Delta_{[n]}$, where $c_d > 0$ is an explicit constant that depends only on d.*

More generally: Gromov's proof, extension to manifolds.

Billiards.

Theorem 2.14 (Birkhoff 1927)**.** *Each smooth convex curve has at least $\varphi(n)$ different periodic billiard trajectories of length n.*

More generally: Higher dimensions: Farber–Tabachnikov, Karasev.

Equal area triangulations.

Theorem 2.15 (Monsky 1970)**.** *The square does not admit a triangulation with an odd number of triangles, all of which have equal area.*

More general: Higher dimensions.

Van der Waerden's theorem.

Theorem 2.16 (van der Waerden 1927)**.** *In any coloring of the integers with finitely many colors and any $k \geq 1$ there exists a monochromatic arithmetic progression of length k.*

Additionally: $\beta\mathbb{Z}$, Hales–Jewett.

Euclidean Ramsey theory.

Theorem 2.17 (Křiž 1992)**.** *Let X be the set of vertices of an trapezoid. Then for any r there exists $d = d(r)$ such that any r-coloring of $\mathbb{R}^d$ contains a monochromatic isometric copy of X.*

Also: Křiž 1991 (for X with solvable transitive isometry group).

Multiple coincidences.

Parametrized Borsuk–Ulam theorems. Applications: Transversal Helly and Tverberg theorems, projective versions.

2.2 Further topics.

1. Topological complexity
2. Discrete Morse theory
3. Polynomial method

3 Notation

$I = [0, 1]$

Homotopies $H : I \times X \to Y$, $H_t(x) := H(t, x)$.

4 The preimage method

Let us classify for a moment mathematical statements (theorems, problems, conjectures) by their sequence of top-level quantifiers, which is an alternating sequence of "$\forall$" and "$\exists$" symbols. For example the statement "there exist infinitely many primes" contains only one such quantifies, namely "$\exists \ldots$". The equivalent statement "for all primes there exist a larger one" is of the form "$\forall \ldots \exists \ldots$". If we break down primes into their definition, we can easily rewrite this statement into the form "$\forall \ldots \exists \ldots \forall \ldots$". Therefore this "classification" is not at all uniquely determined, but let's think about it nonetheless.

Suppose we want to prove a statement of the form "$\forall \ldots \exists \ldots$", or in words: "For each *instance* of the problem there exists a *solution*." The *preimage method* can deal with certain statements of this form. [1] The approach is to describe the solution set S for each instance i of the problem as a preimage $S = f_i^{-1}(Z)$, where $f_i : X \to Y \supseteq Z$ is a family of maps, one for each instance i. Then using some properties about the test-map f one can sometimes show that the preimage S can never be empty. In practise, f is often G-equivariant (see below for definitions) with respect to some natural symmetries of the underlying problem, and often f is quite well known (say up to G-homotopy) on some subset of X (for example its boundary).

The map f_i is called the *test map* for the problem instance i, as it

Example: Inscribed square problem, state test-map, its equivariante, give intuitive bordism arguement.

Example: Ham Sandwich theorem.

4.1 Basic notions from equivariant topology

Definition 4.1 (Topological group)**.** A *topological group* G is a group whose set of elements has additionally the structure of a topological space, such that the group

[1] Živaljević calls this the *Configuration space – test map scheme*. We call it preimage method just for the sake of brevity, and as it seems more intuitive.

multiplication $G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ is continuous (here, on $G \times G$ we use the product topology) and such that the inversion $G \to G : g \mapsto g^{-1}$ is continuous.

Each group can be given the discrete topology, which makes it into a topological group in a rather trivial way. In these notes (and almost always anywhere else) all finite groups will be endowed with this discrete topology.

Definition 4.2 (G-space)**.** A *group action* of some group G on a set X is a binary operation $\cdot : G \times X \to X$ such that $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for any $g_1, g_2 \in G$ and $x \in X$. If additionally G is a topological group, X a topological space, and $\cdot : G \times X \to X$ a continuous map, then X (or more precisely $(X, \cdot)$) is called a *G-space*.

For simplicity we often write gx for $g \cdot x$.

Definition 4.3 (G-equivariant maps)**.** A *G-equivariant map* (or G-map for short) $f : X \to Y$ between two G-spaces is a continuous map between the underlying topological spaces, such that for any $g \in G$ and $x \in X$, $g \cdot (f(x)) = f(g \cdot x)$.

So G-equivariant maps are those that preserve the symmetry. For a fixed topologcal group G, G-spaces and G-equivariant maps form a category Top_G. If G is clear from the context, we sometimes omit it, and call for example G-equivariant maps just *equivariant maps*.

TODO: Orbits $[x] = Gx$, free action, quotient X/G (or better $G\backslash X$), invariant subspace $Y \subset X$, fixed points X^G, X^H for $H \subseteq G$, diagonal action, $X \times_G Y$, restrictions res^G_H, inductions ind^H_G, G-homotopies, G-CW complexes X, relative G-CW complexes (X, A).

Equivariant maps as sections. Consider two G-spaces X and Y. G-maps $f : X \to Y$ can be interpreted as follows.

Let $p : X \times_G Y \to X/G$ be the map induced by projection to the first coordinate, i.e. $p([x, y]) = [x]$. Consider p as a bundle, with total space $X \times_G Y$ and base space X/G. The word bundle here means merely a surjective map. In general, p may not be locally trivial. A *section* of p is a map $s : X/G \to X \times_G Y$ such that $p \circ s = \mathrm{id}_{X/G}$. Notice that any G-map $f : X \to Y$ gives rise to a section $s_f : X/G \to X \times_G Y$ given by $s([x]) = [x, f(x)]$. (Check that this is indeed well-defined and continuous!) Under some technical conditions, this is indeed a bijective correspondence.

TODO: Define bundles beforehand, sections, and also discuss homotopies of sections. Also mention vector bundles, their zero sections, and singular sets of sections.

Proposition 4.4. *Suppose G is compact and X and Y Hausdorff G-spaces. Then the correspondence $f \mapsto s_f$ is a bijection between G-maps $f : X \to Y$ and sections of $p : X \times_G Y \to X/G$. In the same way, G-homotopies between maps $X \to Y$ correspond to homotopies of sections of p. In particular, $f_1, f_2 : X \to Y$ are G-homotopic if and only if the corresponding s_{f_1} and s_{f_2} are homotopic sections.*

Proof. See [10, Prop. 7.2, 7.3]. □

The following is an important special case.

Example 4.5 (G-maps from free G-spaces to representations.)**.** Consider a test-map $f : X \to Y \supseteq Z$ coming from the preimage method. In practise G is often a compact group, X is a finite G-CW complex (see below for the definition), and Y is some finite G-representation, i.e. a finite vector space Y together with a linear G-action given by a continuous group homomorphism $G \to \mathrm{GL}(Y)$. As an invariant subspace, Z is a linear subspace of Y. By possibly going over to the quotient Y/Z we may assume that $Z = 0$ is the origin of Y. By proposition 4.4, f corresponds to a section of $p : E \to B$ with $E := X \times_G Y$ and $B := X/G$.

Let us further assume that X is a *free* G-CW complex. Then E is a (local trivial) *vector bundle* whose fiber is Y. Let $S := f^{-1}(Z)$, which in the preimage method is the solutions set we are interested in. Clearly it is a G-invariant subspace of X (because Z is an invariant subspace of Y). For the corresponding section s_f, $S/G = s_f^{-1}(X \times_G Z) = s_f^{-1}(E_0)$, which is the singular set of s_f (where $E_0 \subset E$ is the zero-section, i.e. the collection of origins of of all fibers).

This opens the door for characteristic classes. (Let's postpone this.)

4.2 Formalizing the preimage method

The idea of the preimage method is to reduce a given problem to the problem of showing that a preimage under a certain test-map is non-zero. Often the latter problem can be written in the following form.

Definition 4.6 (Preimage problem)**.** Let G be a topological group, X a G-CW complex, Y a G-space (often a finite G-respresentation), Z a closed invariant subspace of Y. The *preimage problem* for (G, X, Y, Z, f) is the problem to show that for any G-map $f : X \longrightarrow_G Y$, the preimage $f^{-1}(Z)$ is non-empty. A preimage problem is written as $f : X \longrightarrow_G Y \supseteq Z$.

Equivariance is not the only property of the test-map that can be derived from the underlying problem. It is quite common that f is already known on some subset A of X, at least up to G-homotopy as a map $A \to Y \backslash Z$. Quite often, A is a neighborhood of some boundary compoments of X, or the non-free part of X.

Definition 4.7 (Relative preimage problem)**.** Let G be a topological group, (X, A) a relative G-CW complex, Y a G-space (often a finite G-respresentation), Z a closed invariant subspace of Y, and let $f_0 : A \longrightarrow_G Y \backslash Z$ be a G-equivariant map. The *relative preimage problem* for (G, X, A, Y, Z, f_0) is the problem to show that for any G-map $f : X \longrightarrow_G Y$, with $f(A) \subseteq Y \backslash Z$ and whose restriction to A is G-homotopic to f_0 as maps from A to $Y \backslash Z$, the preimage $f^{-1}(Z)$ is non-empty. A relative preimage problem is written as $f : (X, A) \longrightarrow_G Y \supseteq Z$, $f|_A \sim f_0$.

A relative preimage problem with $A = \emptyset$ is the same as an ordinary preimage problem in the sense of definition 4.6.

The careful reader might wonder: Well, maybe we can phrase or transform some geometric problems into a (relative) preimage problem, but how does this help? A few approaches will be treated in the next sections.

5 Method 1: Bordisms

Suppose we want to solve a relative preimage problem $f : (X, A) \longrightarrow_G Y \supset Z$, $f|_A \sim f_0$, where G is a finite group, X and Y are G-manifolds, possibly with boundary, A is a G-invariant union of boundary components of X, and $X \backslash A$ is a free G-space. (In these notes all manifolds are smooth, and smooth always means C^∞.)

TODO: Manifolds, smooth maps, tangent space $T_x M$, tangent bundle, differential $d_x f$, submanifold $Z \subseteq Y$ (with $\partial Z \subseteq \partial Y$), immersion, submersions, embeddings, regular points, regular values, ε-approximation, ε-homotopy, Whitney's embedding theorem, tubular neighborhoods, collars, G-manifolds. All manifolds in these notes are C^∞, Hausdorff and paracompact, possibly with boundary if not mentioned otherwise.

In case these notes are too dense to be understandable, let me refer to the wonderful differential topology books by Milnor [21] and Guillemin–Pollack [15].

5.1 Transversality

The test-map in the preimage method is often merely continuous, but not necessarily smooth. The following theorem will help.

Theorem 5.1 (Smooth approximation of continuous maps)**.** *Let $f : X \to Y$ be a continuous map between two manifolds, and let $A \subset X$ be a closed (possibly empty) subset of X such that $f|_A : A \to Y$ is smooth. For any $\varepsilon > 0$ there exists an ϵ-approximation g of f such that $f|_A = g|_A$ and g is smooth. Moreover g can be chosen to be homotopic to f relative to A (i.e. the homotopy is constant on A).*

Proof. See [**?**]. Proof idea: There exist a C^∞ "bump function", that is, smooth functions $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with arbitrarily small compact support $[-\varepsilon, \varepsilon]$ that integrate to 1. Replacing a continuous function $f : \mathbb{R} \to \mathbb{R}$ by its convolution with h, $f * h(x) := \int_{\mathbb{R}} f(x-t)h(t)\,dt$, makes it smooth around and the amount it changes f can be controlled. Using a partition of unity of X and a higher dimensional analogue of this convolution trick does it. □

Furthermore we will need to slightly modify the test-map such that $f^{-1}(Z)$ is a submanifold of X in a "generic way". For this, transversality is the key notion.

Definition 5.2 (Transversality, without boundaries)**.** Let $f : X \to Y$ be a continuous map between smooth manifolds without boundary and let $Z \subseteq Y$ be a submanifold without boundary. We say that f intersects Z *transversally* at the point $x \in X$, if $f(x) \in Z$, f is smooth in an open neighborhood U of x, and

$$d_x f(T_x X) + T_{f(x)} Z = T_{f(x)} Y,$$

that is, if the tangent space of Y at $f(x)$ is spanned by the tangent space of Z at $f(x)$ and the image of the tangent space of X at x. If f intersects Z transversally at each point $x \in f^{-1}(Z)$ we say f is *transversal to* Z, in symbols $f \pitchfork Z$.

More generally, X and Y may be manifolds with boundary:

Definition 5.3 (Transversality, with boundaries)**.** Let $f : X \to Y$ be a continuous map between smooth manifolds and let $Z \subseteq Y$ be a submanifold without boundary. We say that f is transversal to Z, in symbols $f \pitchfork Z$, if $f|_{X^\circ} \pitchfork Z$, $f|_{\partial X} \pitchfork Z$, and f is smooth in an open neighborhood of $f^{-1}(Z)$.

Note that $f|_{\partial X} \pitchfork Z$ is a condition about what f does with the tangent bundle of ∂X, which is a subbundle of $TX|_{\partial X}$ of rank one less, as it contains only vectors that are tangent to ∂X, and not those in $TX|_{\partial X}$ that are normal to ∂X.

Transversality is good for exactly the following purpose.

Theorem 5.4 (Smooth preimages)**.** *Let $f : X \to Y$ be transversal to $Z \subseteq Y$. Then $S := f^{-1}(Z)$ is a submanifold of X. Moreover, $\partial S = S \cap \partial X$, and if $S \neq \emptyset$ then* $\operatorname{codim}_X(S) = \operatorname{codim}_Y(Z)$*.*

Proof idea. Transversality and being a submanifold are local conditions, so we can restrict to charts, i.e. we may assume that X and Y are open subsets of $\mathbb{R}^{\dim X}$ and $\mathbb{R}^{\dim Y}$, respectively. One can use the implicit function theorem (from real analysis) twice: Once in order to write Z locally as the preimage of a regular value of a map from some open subset of Y to $\mathbb{R}^{\operatorname{codim}_Y Z}$; then also S can be locally written as a preimage of some open subset of X to the same $\mathbb{R}^{\operatorname{codim}_Y Z}$ by composing f with the latter map. And then once more to parametrize S locally by a euclidean space. TODO: Make this much more precise.

See [15, Chapters 1.4,1.5] for details. □

The last equality says that the codimension of S in X equals the codimension of Z in Y, or equivalently, $\dim X - \dim S = \dim Y - \dim Z$. In particular if $\dim X < \operatorname{codim}_Y(Z)$ then $S = \emptyset$. (This last statement follows also immediately from the definition of transversality.)

Theorem 5.5 (Transversal approximations)**.** *Any continuous map $f : X \to Y$ can be made transversal to any submanifold $Z \subseteq Y$ by an arbitrarily small ε-homotopy. If for a closed subset $A \subset X$, both $f : X \to Y$ and $f|_{\partial X} : \partial X \to Y$ are already transversal to Z at each $x \in A$ and $x \in A \cap \partial X$, respectively, then this homotopy can be chosen relative to A (i.e. constant on A).*

Proof. See [15, Chapter 2.3] for the case $G = 1$. This proof extends easily to the equivariant setting, as free continuous actions by finite groups on manifolds are properly discontinuous. □

This is actually a corollary of the following more general theorem, which will be useful when instead of making the test-map transversal by simply deforming it, we make it transversal by deforming the underlying geometric object it comes from.

In the following theorem, think of P as a parameter space, and $F : X \times P \to Y$ as a smoothly varying family of smooth maps $f_p := F(\cdot, p) : X \to Y$.

Theorem 5.6 (General transversality theorem)**.** *Let $F : X \times P \to Y$ be a smooth map and $Z \subseteq Y$, where P and Z have no boundary. If $F \pitchfork Z$, then there exists a $p \in P$ such that f_p is transversal to Z.*

Proof idea. Let $W := F^{-1}(Z)$, which is a submanifold by theorem 5.4. Let $\pi : W \to P$ be the composition $W \hookrightarrow X \times P \to P$ of the inclusion of W into $X \times P$ with the projection to the second coordinate. Sard's theorem states that the set of critical values of a smooth map has Lebesgue measure zero (with respect to some and hence any Riemann metric on the target manifold of the map). We apply Sard's theorem twice to find some $p \in P$ which is a regular value for both maps π and $\pi|_{\partial W}$. Any such p does the job. Proving transversality of $f_p \pitchfork P$ is now only a task of unravelling definitions and simple linear algebra. See [15, Chapter 2.3] for all details. $\square$

Now, when trying to deform a map $f : X \to Y$ to make it transversal to $Z \subset Y$, with this theorem you may choose deformations that are convenient for you. Intuitively, the more deformations you allow (i.e. the larger the dimension of P gets) the easier it should get to prove transversality of $F \pitchfork Z$.

Remark 5.7. Suppose we are given a preimage problem $f : X \longrightarrow_G Y \supseteq Z$, where X, Y, Z are manifolds, Z without boundary, where G is a finite group that acts properly discontinuously on $X \backslash A$ (in particular freely). Then all theorems above work as well in the G-equivariant setting. (Their proofs are local by nature and thus extend immediately.)

5.2 Bordism theory

Motivation from the preimage method. The preimage method applied to some geometric problem might yield a test-map $f : X \longrightarrow_G Y \supset Z$ which a priori might be difficult to study. A good approach to simplify this is the following. Try to "deform" the instance of the problem continuously to one for which the corresponding test-map is very easy to study. This deformation should yield a G-homotopy relative to $A \subseteq X$ from f to some much simpler map $g : X \to Y$, for which $g^{-1}(Z)$ can be easily described.

In the manifold situation (i.e. when X, Y, Z are all G-manifolds and G is a finite group that acts properly discontinuously on X), assume that both f and g are transversal to Z. Then, as we will see below, the preimages $f^{-1}(Z)$ and $g^{-1}(Z)$ differ only by a *bordism* in X.

The bordism relation.

Definition 5.8 (Singular manifolds)**.** Let X be a topological space. A *singular n-dimensional manifold* in X is a continuous map $s : S \to X$ from a compact n-dimensional manifold S to X. We denote it by (S, s). The *boundary* of (S, s) is the singular manifold $(\partial S, s|_{\partial S})$. The singular manifold (S, s) is called *closed* if $\partial S = \emptyset$.

It is convenient to also consider $(\emptyset, \emptyset)$ as a singular n-dimensional manifold, the empty singular n-manifold.

Definition 5.9 (Bordism relation)**.** A *bordism* between two closed n-dimensional singular manifolds (S_1, s_1), (S_2, s_2) in X is a compact $(n+1)$-dimensional singular manifold (M, m) such that ∂M can be identified with the disjoint union $S_1 \dot\cup S_2$ in such a way that $m|_{S_1} = s_1$ and $m|_{S_2} = s_2$. In that case, (S_1, s_1) and (S_2, s_2) are called *bordant*.

Theorem 5.10. *Bordism (i.e. being bordant) is an equivalence relation on the set of all closed n-dimensional singular manifolds of X.*

Definition 5.11 (Bordism groups)**.** Let X be a topological space. The equivalence classes of closed singular n-manifolds of X modulo the bordism relation are called n-dimensional *bordism classes*. The set of all such bordism classes is called the *unoriented bordism group* of X and it is denoted by $O_n(X)$.

Let us denote the bordism class of (S, s) by $[S, s]$, and let us write $0 := [\emptyset, \emptyset] \in O_n(X)$ for the bordism class of the empty singular n-manifold in X. If X is a manifold and $S \subseteq X$ a submanifold with inclusion map $i : S \to X$, we will sometimes abbreviate $[S, i]$ to $[S]$. (TODO: Maybe avoid this, as $[S]$ also denotes the fundamental class of S. Also shall we invent different notations for $[S, s]$, depending on whether we mean unoriented or oriented classes?)

Additive structure. For any two singular n-manifolds (S_1, s_1), (S_2, s_2) in X, one can define their *sum* to be the singular manifold

$$(S_1, s_1) + (S_2, s_2) := (S_1 \dot\cup S_2, s_1 \dot\cup s_2)$$

in X given by taking the disjoint union. This induces at group operation on $O_n(X)$.

Any (S, s) in X is bordant to itself: A trivial bordism is given by $(S \times I, (x, t) \mapsto s(x))$. Hence $2 \cdot [S, s] = 0$ for any $[S, s] \in O_n(X)$. Thus $[S, s]$ it it's own inverse. In particular, $O_n(X)$ is indeed a group, which is abelian and has exponent is 2.

Let $H_n(X; \mathbb{F}_2)$ denote singular homology of X with $\mathbb{F}_2$-coefficients, $\mathbb{F}_2$ being the field with two elements.

Theorem 5.12 (Thom homomorphism)**.** *The canonical map $O_n(X) \to H_n(X; \mathbb{F}_2)$ that sends $[S, s]$ to $s_*([S])$, where $[S] \in H_n(S; \mathbb{F}_2)$ is the mod-2 fundamental class of S, is a well-defined group homomorphism.*

In fact this homomorphism is indeed a natural transformation of generalized homology theories (see below), but this will not really be needed in the rest of these notes.

Product structure. Similarly for any two singular manifolds (S_1, s_1) in X_1 and (S_2, s_2) in X_2 one can define their *product* to be the singular manifold

$$(S_1, s_1) \times (S_2, s_2) := (S_1 \times S_2, s_1 \times s_2)$$

in $X_1 \times X_2$.

Remark 5.13 (Corners)**.** In case both singular manifolds are not closed, a technical problem arises: $S_1 \times S_2$ has corners. But we then either allow manifolds with corners, or we smoothen the differentiable structure at the corners, which can be done in a canonical way; see Milnor [**?**].

This gives rise to a product map

$$O_{n_1}(X_1) \otimes_{\mathbb{Z}} O_{n_2}(X_2) \to O_{n_1+n_2}(X_1 \times X_2).$$

Let us write $O_*(X) := \bigoplus_n O_n(X)$ and $O_* := O_*(\mathrm{pt})$. With this product, O_* becomes an $\mathbb{F}_2$-algebra, and $O_*(X)$ becomes an O_*-module.

How to be used in the preimage method. Let G be a finite group. A *G-manifold* X is a manifold with a smooth G-action on it. A *relative G-manifold* (X, A) is a G-space X together with a closed invariant subspace A such that $X \backslash A$ is a G-manifold without boundary. We call such (X, A) *free*, if furthermore G acts freely on $X \backslash A$.

Theorem 5.14 (Preimage method via unoriented bordisms)**.** *Let (G, X, A, Y, Z, f_0) be a relative preimage-problem, where (X, A) is a relative free G-manifold, the closure of $X \backslash A$ in X being compact, Y a G-manifold without boundary, $Z \subseteq Y$ an invariant submanifold without boundary. TODO: Don't need f_0 in the following.*

1. *If $f_1, f_2 : X \to Y$ are smooth G-maps transversal to Z, which are homotopic via a (continuous) G-homotopy $H : I \times X \to Y$ such that $H|_{I \times A}$ maps into $Y \backslash Z$, then $[f_1^{-1}(Z)/G]$ and $[f_2^{-1}(Z)/G]$ are equal as elements in $O_*((X \backslash A)/G)$.*

2. *Let f_1, f_2 be as before, except that f_2 is only required to be a continuous G-map and the condition $f_2 \pitchfork Z$ is dropped. Assume that $[f_1^{-1}(Z)/G]$ is non-zero in $O_*((X \backslash A)/G)$. Then $f_2^{-1}(Z)$ is non-empty.*

Proof. Let us write the proof first non-equivariantly (i.e. as if $G = 1$) to simplify notation.

To prove 1.), let $S_1 := f_1^{-1}(Z)$, $S_2 := f_2^{-1}(Z)$, and let $H : I \times X \to Y$ be the homotopy between $f_1, f_2 : X \to Y$ from the assumption. As f_1 and f_2 are transversal to Z, $H|_{(\partial I) \times X}$ is already transversal to Z. Moreover $H|_{I \times A}$ is already transversal to Z for the trivial reason that they do not intersect. Using Theorem 5.5 we can make H transversal to Z via a small ε-homotopy relative to $(\partial I) \times X \cup I \times A$. Let us call this new transversal homotopy $\widetilde{H}$. Then $B := \widetilde{H}^{-1}(Z)$ is a submanifold of $I \times (X \backslash A)$ whose boundary is $\partial B = \{0\} \times S_1 \,\dot\cup\, \{1\} \times S_2$. If $p : I \times X \to X$ denotes the projection to the second coordinate, then $(B, p|_B)$ is a bordism between $(S_1, incl)$ and $(S_2, incl)$. TODO: Improve notation. This proves the first part.

To prove 2.), note that $f_2(A) \cap Z = \emptyset$. We can deform f_2 to a map $f_\varepsilon : X \to A$ by an ε-homotopy relative to A, such that $f_\varepsilon \pitchfork Z$. Then we apply the first part of the theorem to f_1 and f_ε, which yields $[S_\varepsilon] = [S_1] \neq 0$, where $S_\varepsilon = f_\varepsilon^{-1}(Z)$. In particular, $S_\varepsilon \neq \emptyset$, so we may choose an element $x_\varepsilon \in S_\varepsilon$, which of course satisfies $f_\varepsilon(x_\varepsilon) \in Z$. Letting ε go to zero, we obtain a sequence of x_ε in $X \backslash A$. As the closure of $X \backslash A$ is compact, this sequence has a limit point $x \in X$. As $Z \subseteq Y$ is a closed subset, a continuity argument implies that $f_2(x) \in Z$.

If G is non-trivial, then essentially the same proof works: In particular, all homotopies have to be chosen to be G-equivariant using Remark 5.7. $\square$

Remark 5.15 (on (X, A) being free)**.** The assupmtion in Theorem 5.14 that (X, A) needs to be free may look somewhat restrictive. In practise however it turns out that most test-maps are uniquely given up to G-homotopy on the non-free part X_{nf} of X as maps to $Y \backslash Z$ (an exception is the inscribed square problem 2.1, where even $f(X_{nf}) \subseteq Z$). So often we can simply enlarge A to contain the non-free part of X, which makes (X, A) free. In case you still cannot make your (X, A) free, one approach is to look at all restrictions of f of the form $f^H := f|_{X^H} : X^H \to Y^H$, whose restriction to A^H is $f_0^H : A^H \to Y^H \backslash Z^H$, and then argue inductively over all closed subgroups H of G. See Bredon's equivariant obstruction theory [5] (which may be part of later lectures).

Exercises

Exercise 5.1. Show that Theorem 5.14 becomes false if we remove the assumption that $X \backslash A$ has compact closure.

Exercise 5.2. Show that Theorem 5.14 becomes false if we remove the assumption that Z is a closed subset of Y.

5.3 Oriented bordisms

TODO: Define oriented manifolds.

Definition 5.16. Let X be a topological space. An *oriented singular n-manifold* (S, s) in X is a singular n-manifold in X for which S is an orientable manifold and the orientation is fixed.

The boundary of (S, s) is an oriented singular $(n-1)$-manifold with a canonical orientation, the *boundary orientation*, which is given as follows. A basis $(v_1, \ldots, v_{n-1})$ of $T_x \partial S$ is defined to be positively oriented if and only if $(w, v_1, \ldots, v_{n-1})$ is a positively oriented basis of $T_x S$ for some (and hence any) outwards showing tangent vector $w \in T_x S$.

Let (S, s) be an oriented singular n-manifold in X. We denote by $-(S, s)$ the same singular n-manifold however with the opposite orientation. The sum of two oriented singular n-manifolds is defined as in the unoriented case. Two closed oriented singular n-manifolds (S_1, s_1) and (S_2, s_2) are called *bordant* if there exists an oriented singular $(n+1)$-manifold (M, m) in X with $\partial M = S_1 - S_2$ and $m|_{S_1} = s_1$ and $m|_{S_2} = s_2$.

Definition 5.17 (Oriented bordism groups)**.** The equivalence classes of closed oriented singular n-manifolds of X modulo the (oriented) bordism relation are called *oriented bordism classes*. The set of all such classes is called the *oriented bordism group* of X and is denoted by $SO_n(X)$.

Note that in the oriented setting, we allow less manifolds, but they have an extra structure, and the bordism relation is more restrictive. Thus there exists a natural map $SO(X) \to O(X)$ by forgetting the orientation, but it is neither injective nor surjective.

We use the analog notation as in the unoriented case, such as $SO_*(X) = \bigoplus_n SO_n(X)$ and $SO_* = SO_*(\mathrm{pt})$. There is again a Thom homomorphism

$$SO_*(X) \to H_*(X; \mathbb{Z}).$$

Free equivariant setting. Let G be a finite group and consider (X, A) a relative oriented G-manifold (i.e. a G-manifold such that $X \backslash A$ is oriented), where as always A is allowed to be empty. The *orientation character* $\omega_{X,A} : G \to \{\pm 1\}$ is the homomorphism from G to the group $\{\pm 1\}$ that sends $g \in G$ to $+1$ if and only if the map $g \cdot \_ : X \to X$ preserves the orientation (and $\omega_{X,A}(g) = -1$ if and only if $g \cdot \_ : X \to X$ reverses the orientation). Let us denote by $\mathbb{1}_G$ the *trivial character*, $\mathbb{1}_G(g) = +1$ for all $g \in G$. The quotient $(X, A)/G := (X/G, A/G)$ is a relative manifold, which is canonically oriented if $\omega_{X,A} = \mathbb{1}_G$.

Preimage orientation. If X, $Y \subset Z$ are all oriented manifolds and $f : X \to Y$ is transversal to Z, then $S := f^{-1}(Z)$ inherits a canonical *preimage orientation*: First for a basis $v_1, \ldots, v_{\dim X}$ of T_xX write $o_X(v_1, \ldots, v_{\dim X}) = +1$ if this basis is positively oriented and -1 it if is negatively oriented. Similarly define o_Y and o_Z. If $x \in S \subseteq X$ and $w_1, \ldots, w_{\dim S}$ is a basis of T_xS, then extend it to a basis $v_1, \ldots, v_{\operatorname{codim}_X S}, w_1, \ldots, w_{\dim S}$ of T_xX. Let $u_1, \ldots, u_{\dim Z}$ be a basis of $T_{f(x)}Z$. Define

$$\begin{aligned} o_S(w_1, \ldots, w_{\dim S}) \ :=\ & o_X(v_1, \ldots, v_{\operatorname{codim}_X S}, w_1, \ldots, w_{\dim S}) \cdot \\ & o_Y(d_xf(v_1), \ldots, d_xf(v_{\operatorname{codim}_X S}), u_1, \ldots, u_{\dim Z}) \cdot \\ & o_Z(u_1, \ldots, u_{\dim Z}). \end{aligned} \tag{1}$$

This o_S yields the promised canonical preimage orientation o_S. It coincides with the preimage orientation convention in [15, Chapter 3.2].

Example 5.18 (0-dimensional case)**.** To be precise we should mention explicitly how (1) reads in case $\dim S = 0$. Then S is a disjoint union of points in X, and giving it an orientation means *by definition* to associate to each $x \in S$ a sign, either $+1$ or -1. The sign of $x \in S$ in its preimage orientation is then $+1$ if and only if for a positively oriented basis $v_1, \ldots, v_{\dim X}$ of T_xX and a positively oriented basis $u_1, \ldots, u_{\dim Z}$ of $T_{f(x)}Z$, $d_xf(v_1), \ldots, d_xf(v_{\dim X}), u_1, \ldots, u_{\dim Z}$ is a positively oriented basis of $T_{f(x)}Y$.

If X, Y, Z are furthermore G-manifolds, X with a free G-action, then S is also a free G-manifold and its orientation character is given by the equation

$$\omega_S = \omega_X \cdot \omega_Y \cdot \omega_Z.$$

So S/G together with the inclusion $i : S/G \hookrightarrow X/G$ always represents an element $[S/G, i] \in O_*(X/G)$. If furthermore $\omega_S = \mathbb{1}_G$, then we can even regard $[S/G, i] \in SO_*(X/G)$ (which maps to the previous one in $O_*(X/G)$ by forgetting the orientation of S/G).

Theorem 5.19 (Preimage method via oriented bordisms)**.** *The analog of Theorem 5.14 also holds with O_* replaced by SO_* under the following extra conditions: (X, A), Y, Z need to be oriented and their orientation characters need to satisfy $\omega_{X,A} \cdot \omega_Y \cdot \omega_Z = \mathbb{1}_G$.*

Example 5.20 (0-dimensional case)**.** If X/G is path-connected then $O_0(X) = \mathbb{Z}/2$ and $SO_0(X) = \mathbb{Z}$. So in the situation when $\dim S = \dim X - \operatorname{codim}_Y Z$ is zero, the bordism class of $[S/G, i] \in O_0(X)$ is the parity of the number of points in S/G, and if $\omega_S = \mathbb{1}_G$ then $[S/G, i] \in SO_0(X)$ is the number of points in S/G counted with signs. The signs were explained in Example 5.18

5.4 Framed bordisms

Not sure whether we'll need this. (Except for showing that the preimage method fails for the mass partition problem with one mass in $\mathbb{R}^4$ and four hyperplanes (for this a test-map $X \longrightarrow_G Y \backslash Z$ *does* exist).)

6 Immediate applications

6.1 Fundamental theorem of algebra

Theorem 6.1 (Fundamental theorem of algebra)**.** *Each non-constant complex polynomial $p \in \mathbb{C}[x]$ in one variable has a root.*

Proof. Say $p(x) = \sum_{i=0}^{d} a_i x^i$ is of degree $d \geq 1$, and we may assume $a_d = 1$. First regard p as a function $p : \mathbb{C} \to \mathbb{C}$. Take the one-point compactification of $\mathbb{C}$, which we write as $\mathbb{C} \cup \{\infty\}$, which can be identified with the 2-sphere S^2. ($\mathbb{C}$ embeds smoothly into S^2 for example via the stereographic projection, whose image is all of S^2 minus one point, say the "north pole".) One checks that p extends to a continuous (and in fact even smooth) map $p : S^2 \to S^2$ by defining $p(\infty) := \infty$. The preimage $S_p := p^{-1}(0)$ is the set of roots, which we want to prove non-empty.

A very simple polynomial of degree d is $q(x) = (x-1)(x-2) \cdot \ldots \cdot (x-d)$. $S_q = q^{-1}(0)$ consists of the d points $1, \ldots, d$. S^2 is an oriented manifold, and $\{0\}$ is an oriented submanifold (as all 0-dimensional manifolds are). Thus S_q can be given a preimage orientation, which is $+$ for each of the d points of S_q. Hence S_q represents $[S_q] = d \in H_0(S_d; \mathbb{Z}) \cong \mathbb{Z}$, which is a non-zero class by the assumption $d \geq 1$.

Now deform p into q via a homotopy, for example the homotopy that deforms the coefficients of p linearly to the coefficients of q works. This homotopy is clearly continuous when restricted to $\mathbb{C}$. One checks that it is also continuous at ∞ (essentially because the d'th coefficient never vanishes).

$\square$

6.2 Borsuk–Ulam theorem

Theorem 6.2 (Borsuk–Ulam, 1933, version 1)**.** *For any map $\varphi : S^d \to \mathbb{R}^d$ there exists $x \in S^d$ such that $\varphi(x) = \varphi(-x)$.*

Proof. Given φ, the probably most obvious way to write the solution set S of all $x \in S^d$ with $\varphi(x) = \varphi(-x)$ as a preimage is the following. $S_\varphi = f_\varphi^{-1}(\Delta_{(\mathbb{R}^d)^2})$, where $f_\varphi : S^d \to (\mathbb{R}^d)^2$ is given by $f_\varphi(x) = (\varphi(x), \varphi(-x))$ and $\Delta_{(\mathbb{R}^d)^2} = \{(y, y) \mid y \in \mathbb{R}^d\}$ is the diagonal of $(\mathbb{R}^d)^2$.

There is a natural symmetry that we can make use of: $\mathbb{Z}/2 = \langle\varepsilon\rangle$ acts on S^d via the so-called *antipodal action* $\varepsilon \cdot x := -x$, and on $(\mathbb{R}^d)^2$ by permuting the coordinates, $\varepsilon \cdot (y_1, y_2) = (y_2, y_1)$. Clearly f_φ is $\mathbb{Z}/2$-equivariant and $\Delta_{(\mathbb{R}^d)^2} \subset (\mathbb{R}^d)^2$ is an invariant subspace. The antipodal $\mathbb{Z}/2$-action on the closed manifold S^d is free, so we are in the best possible setting for applying Theorem 5.14.

A very simple map is $\varphi' : S^d \to \mathbb{R}^d$ given by $(x_0, \dots, x_d) \mapsto (x_1, \dots, x_d)$, which forgets the first coordinate, and "makes the sphere flat". Clearly, the solution set for φ' is $S_{\varphi'} = \{\pm e_0\}$, $e_0 = (1, 0, \dots, 0)$. A quick calculation shows that f'_φ is transversal to $\Delta_{(\mathbb{R}^d)^2}$:

$$d_{e_0} f_{\varphi'}(T_{e_0} S^2) = \{(v, -v) \in T_0(\mathbb{R}^2)^2\}$$

and

$$T_0 \Delta_{(\mathbb{R}^d)^2} = \{(v, v) \in T_0(\mathbb{R}^d)^2\},$$

which together span all of $T_0(\mathbb{R}^d)^2$, and by symmetry the same holds at $-e_0$. Thus $S_{\varphi'}$ represents $1 \in H_0(S^2/_{\mathbb{Z}Z/2}; \mathbb{F}_2) = \mathbb{F}_2$.

As φ and φ' are homotopic maps (e.g. via a linear homotopy) and this lift to a $\mathbb{Z}/2$-homotopy $f_\varphi \sim_{\mathbb{Z}/2} f_{\varphi'}$, S_φ is non-empty. $\square$

Remark 6.3. It is a bit more efficient to compose the test-map f_φ from the proof with the projection $\pi : (\mathbb{R}^d)^2 \to (\mathbb{R}^d)^2/Z \cong \mathbb{R}^d$, the latter isomorphism being given by $(y_1, y_2) \mapsto y_1 - y_2$, as then we can write S as the preimage of zero under the composition $g_\varphi = \pi \circ f_\varphi$. Explicitly, $g_\varphi : S^d \to \mathbb{R}^d$ sends x to $\varphi(x) - \varphi(-x)$. If we let $\mathbb{Z}/2$ act antipodally on S^d and on $\mathbb{R}^d$, then g_φ is $\mathbb{Z}/2$-equivariant (as π and f_φ are).

There are many "equivalent" versions of Borsuk–Ulam's theorem.

Theorem 6.4 (Borsuk–Ulam theorem, equivalent versions)**.** *Let $\mathbb{Z}/2$ act on S^n and $\mathbb{R}^n$ as usual antipodally, i.e. $\varepsilon \cdot x = -x$. Then the following statements hold.*

1. *Any $\mathbb{Z}/2$-map $f : S^n \to \mathbb{R}^n$ has 0 in its image.*

2. *There is no* $\mathbb{Z}/2$*-map* $g : S^n \to S^{n-1}$.

3. *Let* $A_1, \dots, A_n$ *be closed subsets of* S^n *that cover* S^n*, i.e.* $\bigcup_i A_i = S^n$*. Then there exists an* $i \in [n]$ *and* $x \in S^n$ *such that* $x, -x \in A$.

Proof. Theorem 6.2 $\Rightarrow$ (1): Use $\varphi(x) := f(x)$.

(1) $\Rightarrow$ Theorem 6.2: Use $f(x) := \varphi(x) - \varphi(-x)$.

(1) $\Rightarrow$ (2): Use $f(x) := g(x)$, regarded as a map to $\mathbb{R}^n$.

(2) $\Rightarrow$ (1): Use $g(x) := f(x)/||f(x)||$.

Theorem 6.2 $\Rightarrow$ (3): Use $f(x) := (\mathrm{dist}(x, A_i))_{i\in[n]}$.

(3) $\Rightarrow$ (2): Use $A_i := g^{-1}(B_i)$, where $B_1, \dots, B_n$ is a covering of closed subsets of S^{n-1} such that no B_i contains a pair of antipodal points. (Such B_i do indeed exist, for example take a regular n-simplex centered at the origin of $\mathbb{R}^n$ and project its n facets radially to S^n.)

□

6.3 Brouwer's fixed point theorem

Theorem 6.5 (Brouwer's fixed point theorem, 1910)**.** *Any map* $\varphi : B^d \to B^d$ *from the d-dimensional ball to itself has a fixed point, i.e. a point* $x \in B^d$ *such that* $\varphi(x) = x$.

Remark 6.6. This theorem is often taught in a first course on algebraic topology, and the standard proof involves a trick that reduced the fixed point theorem to the problem of showing that there exists no extension of the identity map $S^{d-1} \to S^{d-1}$ to a map from the ball $B^d \to S^{d-1}$ (which can be seen by writing this extension problem into a commutative triangle, applying the $(d-1)$-dimensional homology functor, and observing a contradiction that the obtained triangle is not commutative anymore).

The proof given below is hoped to be "more conceptual" from the point of view of the preimage method, as it involves no trick. (Of course the reader is allowed to disagree.) It is a bit more technical and thus a bit longer, but these technicalities will appear again and again.

Proof. Given $\varphi : B^d \to B^d$, we can write its fixed points as a preimage $S_\varphi = f_\varphi^{-1}(Z)$, where $f_\varphi : B^d \to (B^d)^2$ is the map $x \mapsto (x, \varphi(x))$ and $Z = \{(y, y) \mid y \in B^d\}$ is the diagonal of $(B^d)^2$.

Now consider a very simple map $\varphi' : B^d \to B^d$ given by $x \mapsto -x$. We construct the analog testmap $f_{\varphi'}$, such that the fixed points of φ' are given by $S_{\varphi'} = f_{\varphi'}^{-1}(Z)$. Of course the only fixed point of φ' is 0, thus $S_{\varphi'} = \{0\}$. The important point here is that $|S_{\varphi'}| = 1$ is an *odd* number, which makes S_φ topologically detectable already without considering orientations. A quick calculation shows that the test-map $f_{\varphi'}$ is transversal to Z:

$$d_0 f_{\varphi'}(T_0 B^d) = \{(v, -v) \in T_0(B^d)^2 \mid v \in T_0 B^d\}$$

and

$$T_0 Z = \{(v, v) \in T_0(B^d)^2 \mid v \in T_0(B^d)\},$$

and their sum is all of $T_0(B^d)^2$. Thus $[S_{\varphi'}] \in H_0(B^d; \mathbb{F}_2) = \mathbb{F}_2$ represents the generator of $\mathbb{F}_2$.

In order to relate φ to φ', we deform them by a linear homotopy $\Phi : I \times B^d \to B^d$, $\Phi_t(x) = (1-t)\varphi(x) + t\varphi'(x)$. If φ has fixed points in $S^{d-1} = \partial B^d$, we are done. So assume that φ has no fixed points in S^{d-1}, then also each Φ_t has no fixed points in S^{d-1} for any t (this uses that B^d is a strictly convex), i.e. $\Phi_t(S^{d-1}) \cap Z = \emptyset$. Now we can apply Theorem 5.14 with $(X, A) = (B^d, S^{d-1})$, $Y = (B^d)^2$, $Z = Z$, $G = 1$, $f_1 = f_{\varphi'}$ and $f_2 = f_\varphi$.

□

Exercises

Exercise 6.1 (Intermediate value theorem)**.** Prove the intermediate value theorem using the preimage method. (The intermediate value theorem states that any map $f : I \to \mathbb{R}$ with $f(0) < 0$ and $f(1) > 0$ has zero in its image.)

Exercise 6.2. Come up with one or more(?) analogs of the intermediate value theorems in higher dimensions.

Exercise 6.3. Show that there exists a map $\varphi :^\circ B^d \to^\circ B^d$ from the *open* unit ball to itself without any fixed points. Nonetheless try to apply the preimage method as in the above proof of the Brouwer fixed point theorem 6.5 and say why it fails. In particular, describe the bordism.

6.4 Gromov's waist of the sphere theorem

Let us come to a much more involved application. We content ourselves with the topological part of its proof.

Gromov [14] proved the following version of the Borsuk–Ulam theorem.

Theorem 6.7 (Gromov's waist of the sphere theorem, 2003)**.** *Let $\varphi : S^n \to \mathbb{R}^k$ be a continuous map where $n \geq k \geq 0$. Then there exists a point $z \in \mathbb{R}^k$ such that for any $\varepsilon > 0$,*

$$\mathrm{vol}_n\Big(U_\varepsilon\big(\varphi^{-1}(z)\big)\Big) \geq \mathrm{vol}_n\Big(U_\varepsilon\big(S^{n-k}\big)\Big).$$

Here, vol_n denotes the standard measure on S^n, $U_\varepsilon(X)$ denotes the ε-neighborhood of a subset $X \subseteq S^n$ with respect to the standard metric on S^n, and S^{n-k} is the $(n-k)$-dimensional equator of S^n.

In the case $n = k$, the waist of the sphere theorem 6.7 follows easily from the Borsuk–Ulam theorem 6.2: Simply take a point $z \in \mathbb{R}^k$ whose preimage contains a pair of antipodal points; this pair *is* a 0-dimensional equator.

Thus we may assume $n > k$. Gromov's proof of theorem 6.7 splits into a topological and an analytic part. The topological part is the following mass partition theorem, for which we need some preparation.

Let $\mathrm{Conv}(S^n)$ denote the set of all closed convex subsets of $C \subset S^n$ with $C \neq S^n$. Let $\mathrm{Conv}^*(S^n)$ be its subset of sets with positive volume. The Hausdorff metic makes $\mathrm{Conv}(S^n)$ into a metric space. Any map $c : \mathrm{Conv}^*(S^n) \to S^n$ is called a *center map*. A *partition of S^n into q convex sets* is a family of subsets $C_1, \ldots, C_q \in \mathrm{Conv}(S^n)$ with pairwise disjoint interior such that $S^n = \bigcup_i C_i$.

Theorem 6.8 (A mass partition theorem)**.** *Let $\varphi : S^n \to \mathbb{R}^k$ be a map, $n > k$, and let $c : \mathrm{Conv}^*(S) \to S^n$ be a center map. Then for any $q = 2^\ell$ there exists a partition of S^n into q convex sets $C_1, \ldots, C_q$ with*

$$\varphi(c(C_1)) = \ldots = \varphi(c(C_q))$$

and

$$\mathrm{vol}(C_1) = \ldots = \mathrm{vol}(C_q).$$

Moreover the set C_i can be required to lie in the ε-neighborhood of some k-dimensional equator $E_i \subset S^n$ in case $q \geq q_0(\varepsilon)$.

Gromov used theorem 6.8 with $\varepsilon \to 0$ together with rather involved isoperimetric inequalitie to prove his waist of the sphere theorem 6.7. We will skip this analytic part as it is quite disjoint from the rest of these lectures. The interested reader is referred to the original paper by Gromov [14] or the survey paper by Memarian [20].

Proof of Theorem 6.8. Every point $x \in S^n$ determines an orthogonal hyperplane through the origin, which bisects S^n into two convex pieces. Two more points on the sphere, one for each of the two pieces, will yield a convex partition of S^n into four pieces. Iterating this, we obtain a map

$$p : X := (S^n)^{q-1} \to \mathrm{Conv}(S^n)^q.$$

This map is equivariant with respect to the following natural actions. Let T be the complete binary tree of height $\ell - 1$, which has q leaves. The interior nodes of T naturally correspond to the $q-1$ sphere factors of X, and the q leaves correspond to the convex sets in the associated partition of S^n. Let the interior nodes be labelled by $N_1, \ldots, N_{q-1}$, where N_1 shall denote the root. Let the leaves of T be labelled by $L_1, \ldots, L_q$. The symmetry group of T is the 2-Sylow subgroup $G := \mathbb{Z}_2 \wr \ldots \wr \mathbb{Z}_2$ of the symmetric group S_q (in fact this is not even needed). G acts on X by permuting the S^n factors in the same ways at it acts on the interior nodes of T, with an additional antipodal action on an S^n-factor whenever its children are exchanged, such that the partition $p(x)$ for $x \in X$ stays the same up to permutation of the indices. G also acts on $\mathrm{Conv}(S^n)^q$ by permuting the $\mathrm{Conv}(S^n)$ factors in the same way as it acts on the leaves of T. This makes p into a G-equivariant map.

We would like to define a test-map

$$f : (S^n)^{q-1} \longrightarrow_G (\mathbb{R}^k \times \mathbb{R})^q$$

whose i'th coordinate at $x = (x_1, \ldots, x_{q-1})$ is given by

$$(\varphi(c(p_i(x))), \mathrm{vol}(p_i(x))), \tag{2}$$

such that the preimage of $\Delta := \Delta_{(M\times\mathbb{R})^q}$ corresponds to the partitions of S^n into q convex sets of equal volume whose center points have all equal g-images. However c is not continuous at some of the convex sets with zero volume. Fortunately this is only a technical problem, which we can circumvent as follows. We replace c in (2) by a slightly deformed map c': First, for any convex set $C \in \mathrm{Conv}(S^n)$, let γ_C be the shortest geodesic on S^n between $\gamma_C(0) = \pm x_1$ and $\gamma_C(1/2q) = c(C)$, where the sign in front of the vector x_1 (in the sphere corresponding to the root of T) depends on whether the leaf of T corresponding to the convex set C is on the left or on the right side of the root. If $\mathrm{vol}(C) = 0$ then γ_C might not be defined except at its end point $\gamma_C(0)$, but we will use such γ_C only at 0. We then define

$$c'(C) := \begin{cases} c(C) & \text{if } \mathrm{vol}(C) \geq 1/2q, \\ \gamma_C(\mathrm{vol}(C)) & \text{if } \mathrm{vol}(C) \leq 1/2q. \end{cases}$$

The so defined $f : x \mapsto \big(\varphi(c'(p_k(x))), \mathrm{vol}(p_k(x))\big)_k$ is indeed continuous and $f^{-1}(\Delta)$ is the set of convex equipartitions of S^n such that g maps all centers of the convex parts to the same point in M.

Lemma 6.9. *There exists a G-map $f' : (S^{k+1})^{q-1} \longrightarrow_G (\mathbb{R}^{k+1})^q$ that is transversal to $\Delta := \Delta_{(\mathbb{R}^{k+1})^q}$ and such that $(f')^{-1}(\Delta)$ consists of a single G-orbit.*

Proof. Consider the map $f' : (S^{k+1})^{q-1} \to (\mathbb{R}^{k+1})^{\oplus q}$ given by

$$x \mapsto \Big(\sum_{N_i \in P_j} \pm \mathrm{pr}_{S^{k+1}\to\mathbb{R}^{k+1}}(x_i)\Big)_{j=1\ldots q},$$

where $\mathrm{pr}_{S^{k+1}\to\mathbb{R}^{k+1}} : S^{k+1} \to \mathbb{R}^{k+1}$ is the standard projection to the first $k+1$ coordinates; for every leaf L_j, P_j is the set of interior nodes in T that lie on the shortest path from the root N_1 to L_j, and the sign at $N_i \in P_j$ depends on whether the path P_j continues at the right or the left subtree at node N_i.

The sum of all q $\mathbb{R}^{k+1}$-coordinates of this test-map is zero, since the sum for P_j cancels with the sum for the reflected P_j. So it's image can intersect Δ at most in the origin. Furthermore, f' is G-equivariant, and $(f')^{-1}(0) = \{(0,\ldots,0,\pm 1)\}^{q-1}$ is the set of $(q-1)$-tuples x such that every x_i is the north or the south pole of S^{k+1}. Modulo G this is exactly one preimage, and f' is indeed transversal to Δ. $\square$

This lemma together with the preimage method (theorem 5.14) applied to $f|_{(S^{k+1})^{q-1}}$ and f' shows that $f^{-1}(\Delta) \neq \emptyset$. This proves the first part of Theorem 6.8.

More generally, we may restrict the configuration space $(S^n)^{q-1}$ to some G-invariant subspace $(S^{k+1})^{q-1}$. Here, G-invariance means that we can choose the $(k+1)$-dimensional equators $S^{k+1} \subseteq S^n$ independently as long as they agree on each height (i.e. the distance to the root N_1 within T). Choosing these equators well-distributed enough will assure the ε-neighborhood condition in theorem 6.8. $\square$

7 Topic: Embeddings and graph colorings

Embeddings.

Definition 7.1 (Embedding dimension)**.** Let K be a simplicial complex (or an arbitrary topological space). The *embedding dimension* $\dim_{\mathrm{emb}}(K)$ of K is smallest integer d such that K embeds continuously into $\mathbb{R}^d$.

Kneser graphs.

Definition 7.2 (Kneser graph)**.** Given integers $n \geq k \geq 1$, the *Kneser graph* $KG_{n,k}$ is defined as the graph with vertex set $\binom{[n]}{k}$ (i.e. all k-subsets of an n-set), and two vertices $A, B \subset [n]$ are connected by an edge if and only if $A \cap B = \emptyset$.

Often on requires $2k \leq n$, otherwise $KG_{n,k}$ has no edges.

Definition 7.3 (Chromatic number)**.** Let G be a finite graph. The *chromatic number* $\chi(G)$ of G is the minimal integer n such that the vertices of G can be colored with n colors such that any two adjacent vertices are colored differently.

Embeddings versus graph colorings.

Definition 7.4 (Generalised Kneser graphs)**.** Let N be a finite set. The *generalized Kneser graph* $KG(N)$ of N is defined as the graph whose vertex set is N, and two vertices $A, B \in N$ are connected by an edge if and only if $A \cap B = \emptyset$.

For $N = \binom{[n]}{k}$, $KG(N)$ is simply the Kneser graph $K_{n,k}$.

Definition 7.5 (Minimal non-faces)**.** Let K be a simplicial complex on vertex set $[n] = \{1, \ldots, n\}$. The set of *minimal non-faces* $N(K)$ of K is the set of all $\sigma \subseteq [n]$ such that $\sigma \notin K$ but all proper faces $\tau \subsetneq \sigma$ lie in K.

Theorem 7.6 (Sarkaria's coloring and embedding theorem)**.** *Let K be a simplicial complex on n vertices. Let $N = N(K)$ be the set of minimal non-faces of K, $KG(N)$ be the associated Kneser graph, and $\chi(KG(N))$ its chromatic number. Then*

$$\dim_{\mathrm{emb}}(K) + \chi(KG(N)) \geq n - 1.$$

So intuitively, the minimal embedding dimension $\dim_{\mathrm{emb}}(K)$ of K and $\chi(KG(N))$ cannot both be small. An upper bound on one of the quantities $\dim_{\mathrm{emb}}(K)$ and $\chi(KG(N))$ yields a lower bound for the other one! This is so amazing as usually for any of the two numbers $\dim_{\mathrm{emb}}(K)$ and $\chi(KG(N))$, upper bounds are much easier to obtain, as we only have to exhibit an explicit embedding $K \hookrightarrow \mathbb{R}^d$ and an explicit coloring $KG(N) \to K_m$, respectively. (TODO: This notation for a coloring is not defined yet.) Taking $K = (\Delta^{n-1})_{(k-2)}$, the $(k-2)$-skeleton of the $(n-1)$-dimensional simplex yields the following two immediate corollaries.

Theorem 7.7 (Van Kampen–Flores)**.** *For $n \geq 2k+1$, $k \geq 2$,* $\dim_{\mathrm{emb}}((\Delta^{n-1})_{(k-2)}) = 2k-3$*. Equivalently, for any $d \geq 1$, $(\Delta^{2d+2})_{(d)}$ does not embed into $\mathbb{R}^{2d}$.*

The following theorem was the famous Kneser conjecture, and first proved by Lovász.

Theorem 7.8 (Lovász, 1978)**.** *For $n \geq 2k+1$, $k \geq 1$, $\chi(KG_{n,k}) = n - 2k + 2$.*

Proofs of theorems 7.7 and 7.8, simultaneously. For $k = 1$, theorem 7.8 is trivially true, as $KG_{n,1}$ is the complete graph on n vertices. So we may assume $k \geq 2$ from now on. Consider the complex $K = (\Delta^{n-1})_{(k-2)}$, whose vertex set can be identified by $[n]$.

Any finite simplicial complex of dimension $d = k-2$ embeds into $\mathbb{R}^{2d+1}$. [The follows from transversality. We can even easily construct an explicit linear embedding: For example place its vertices along the moment curve $\gamma_{2d+1}(t) = (t, t^2, \ldots, t^{2d+1})$ and extend this map linearly to the other faces of the complex. This gives indeed an embedding, as any $2d+2$ distinct points on γ_{2d+1} are affinely independent (exercise! Use the explicit formula for the determinant of Vandermonde matrices). A more slick but less explicit way is to send the vertices of K to point in $\mathbb{R}^d$ all of whose coordinates are algebraically independent, and then extend this embedding linearly to all of K.] In particular,

$$\dim_{\mathrm{emb}}(K) \leq 2(k-2) + 1 = 2k - 3. \tag{3}$$

Next, $N(K) = \binom{[n]}{k}$, and the associated generalized Kneser graph is $KG(N(K)) = KG_{n,k}$. A coloring of $KG_{n,k}$ with $n-2k+2$ colors is given by the map $\binom{[n]}{k} \to [n-2k+2]$ that sends a k-set $A \subset [n]$ to $\min(\min(A), n-2k+2)$. [This is indeed a graph coloring of $KG_{n,k}$: If two k-sets A,B get the same color, then either $\min A = \min B$ or $A, B \subset \{n-2k+2 \ldots, n\}$ (which has $2k-1$ elements); in either case it follows $A \cap B \neq \emptyset$.] Hence

$$\chi(KG_{n,k}) \leq n - 2k + 2. \tag{4}$$

Plugging (3) and (4) into the bound of Theorem 7.6 yields

$$n - 1 \geq \dim_{\mathrm{emb}}(K) + \chi(KG_{n,k}) \geq n - 1.$$

Therefore (3) and (4) hold with equality. □

It remains to prove Sarkaria's coloring and embedding theorem. The following proof is a simplified version of the proof in Matoušek [18]. But first, some definitions.

Definition 7.9 (Face poset, order complex, barycentric subdivision)**.** Let K be a simplicial complex. We denote by $P(K)$ it's face poset, i.e. the set of faces of K *except* for the empty set, ordered by inclusion.

Let P be a poset. The *order complex* $\Delta(P)$ of P is defined as the simplicial complex, whose vertices are the elements of P and whose faces are the chains in P.

The *barycentric subdivision* $\mathrm{sd}K$ of K is defined to as $\Delta(P(K))$.

If $||.||$ denotes the geometric realization functor, i.e. $||K||$ is the geometric realization of the abstract simplicial complex K, then $||K|| = ||\mathrm{sd}K||$.

Definition 7.10 (Join)**.** The *join* $K * L$ of two abstract simplicial complexes K and L is the abstract simplicial complex whose vertex set $V(K * L) := V(K) \dot{\cup} V(L)$ and whose faces are $F_K \dot{\cup} F_L$ for $F_K \in K$ and $F_L \in L$.

The *join* $X*Y$ of two topological spaces is defined to be the quotient of $X \times I \times Y$ that identifies $(x, 0, y) \sim (x', 0, y)$ and $(x, 1, y) \sim (x, 1, y')$ for any $x, x' \in X$ and $y, y' \in Y$.

Clearly there is a canonical homeomorphism $||K * L|| = ||K|| * ||L||$.

Simplices of dimension n are denoted by Δ^n, and simplices on the vertex set V are denoted by Δ_V. Let us stress the index shift $\Delta^{n-1} = \Delta_{[n]}$.

Proof of theorem 7.6. Let $P = 2^{[n]}$ denote the face poset of $\Delta_{[n]}$. Consider K as a subposet of P, and let $L := \Delta(P \backslash K)$ be the order complex of the subposet $P \backslash K$.

There is a canonical injective simplicial map

$$\mathrm{sd}\Delta_{[n]} \hookrightarrow \mathrm{sd}K * L,$$

which sends a vertex $v \in \mathrm{sd}\Delta_{[n]}$, $\emptyset \neq v \subseteq [n]$, to itself in K if it is a face of K, and otherwise to its copy in L. Moreover there is a simplicial map

$$L \hookrightarrow \mathrm{sd}\Delta_{[m]}, \quad m := \chi(KG(N(K))),$$

which is given as follows. Each vertex $v \in L$ is an element of $P \backslash K$, i.e. a face F_v of $\Delta_{[n]}$ which is not a face of K. This face F_v contains at least one minimal nonface of K. Let $N_v = \{F \in N(K) \mid F \subseteq F_v\}$ be the set of all minimal nonfaces of K contained in F_v, and let $C_v \subseteq [m]$ be the set of all colors appearing in N_v. Then $v \mapsto C_v$ defines our simplicial map $L \to \mathrm{sd}\Delta_{[m]}$.

Putting both maps together we obtain a simplical map

$$\varphi' : \mathrm{sd}\Delta_{[n]} \to \mathrm{sd}K * \mathrm{sd}\Delta_{[m]}$$

which gives a continuous map

$$\varphi : \Delta_{[n]} \to K * \Delta_{[m]}.$$

We claim that for any two *disjoint* faces $F, F' \in \Delta_{[n]}$, $\varphi(F) \cap \varphi(F') = \emptyset$. As φ' is simplicial, the claim is equivalent to saying that any two vertices $v, v' \in \mathrm{sd}\Delta_{[n]}$ with $v \cap v' = \emptyset$ (as subsets of $[n]$) satisfy $\varphi'(v) \neq \varphi'(v')$. (Is it okay to leave this as an exercise?)

Composing with the given embedding $K \hookrightarrow \mathbb{R}^d$ we obtain a map $\Delta_{[n]} \to \mathbb{R}^d * \Delta_{[m]} \hookrightarrow \Delta^{d+m}$, such that the images of any ptwo disjoint faces do not intersect. This contradicts the topological Radon theorem **??** (which is the topological Tverberg theorem **??** for $p = 2$ parts). $\square$

TODO: Van Kampen obstruction for embeddings, hom complex obstruction for graph colorings, more precise analogy between embeddings and colorings.

8 Short topic: Equal area triangulations

Definition 8.1 (Geometric triangulation)**.** An n-simplex in $\mathbb{R}^d$ is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^d$. A *geometric triangulation* of a compact subset $X \subseteq \mathbb{R}^d$ is finite set of simplicies $T = (\sigma_i)_{i \in I}$ with disjoint interior, whose union is X, and such that the intersection of any two simplices $\sigma_i \cap \sigma_j$ in T is either empty or equal to some $\sigma_k \in T$.

Theorem 8.2 (Thomas 1968, Monsky 1970)**.** *Each triangulation of the unit square $[0,1]^2$ with all triangles of equal area must have an even number of triangles.*

This was proved by Thomas [28] in case all vertices of the triangulation have rational coordinates. His proof was extended by Monsky [23] to the general case.

The proof relies on the following lemma by Sperner.

Lemma 8.3 (Sperner's lemma)**.** *Denote the n facets of the $(n-1)$-simplex $\Delta_{[n]}$ by $F_1, \ldots, F_n$, F_i being the facet opposite to the vertex i.*

Let T be a triangulation of a simplex $\Delta_{[n]}$ (i.e. a subdivision of the simplicial complex $\Delta_{[n]}$) whose vertices are colored with n colors, such that no vertex of T in the facet F_i has color i $(1 \leq i \leq n)$. Then some facet of T has vertices of all colors.

Proof. Consider the coloring of T as a simplicial map $c : T \to \Delta_{[n]}$. The assumption on the coloring is equivalent to $c(F_i) \subseteq F_i$ for each facet F_i of $\Delta_{[n]}$. In particular, c restricts to a continuous map $\partial c : \partial T \to \partial \Delta_{[n]}$.

Let H be the linear homotopy between c and the identity $\mathrm{id}_{\Delta_{[n]}}$. It restricts to linear homotopies between $c|_{F_i}$ and id_{F_i} for each i. Thus at each time $t \in I$, H_t sends $\partial \Delta_{[n]}$ into $\partial \Delta_{[n]}$.

Therefore $\deg(\partial c) = \deg(\mathrm{id}_{\partial \Delta_{[n]}}) = 1$, which is non-zero. Hence, ∂c is not contractible, so it has no extension to a map $\Delta_{[n]} \to \partial \Delta_{[n]}$. In particular, the image of c must contain some (actually any) interior point x of $\Delta_{[n]}$. Now a face $F \in T$ has x in its image if and only if F contains vertices of all n colors. $\square$

We further need a p-adic valuation on $\mathbb{R}$: The p-adic valuation $|\,.\,|_p : \mathbb{Q} \to \mathbb{R}_{\geq} 0$ is the function given by $|x|_p := p^{-n}$, when x is of the form $x = p^n \frac{a}{b}$ with $p \nmid a, b$. It satisfies

1. $|x|_p = 0$ if and only if $x = 0$,
2. $|xy|_p = |x|_p \cdot |y|_p$,
3. $|x+y|_p \leq \max(|x|_p, |y|_p)$, with equality if $|x|_p \neq |y|_p$ (but not only if).

Theorem 8.4. *The p-adic valuation on $\mathbb{Q}$ extends to all of $\mathbb{R}$, satisfying the same three properties.*

Sketch of proof. Use Zorn's lemma: It yields a maximal field extension $k/\mathbb{Q}$ within $\mathbb{R}$ on which there exists a valuation $|\,.\,|_k$ that extends $|\,.\,|_p$ on $\mathbb{Q}$. We claim that $k = \mathbb{R}$: Assume the contrary and let $x \in \mathbb{R} \backslash k$.

- If $K = k(x)$ is a finite extension k, then $|\,.\,|_k$ can be extended to a valuation $|\,.\,|_K$ on K via $|\alpha|_K = (N_{K/k}(\alpha))^{1/n}$, where n is the degree of α over k, and $N_{K/k}$ is the norm.

- If $K = k(x)$ is transcendental over k, then $|\,.\,|_k$ can be extended to a valuation $|\,.\,|_K$ on $k[x]$ via the Gauss norm $|\sum_{i=0}^{d} c_i x^i|_K = \max |c_i|_k$ and further on $k(x)$ via $|a/b|_K := |a|_K/|b|_K$.

This proves $k = \mathbb{R}$. An alternative short proof uses that $\bar{Q}_p$ (the algebraic closure of the p-adic numbers) is as a field isomorphic to $\mathbb{C}$ (which however is also proved via the axiom of choice). □

Remark 8.5. If one counts the "rainbow facets" of T in lemma 8.3 with appropriate signs, the number of such equals 1. For proof see the easy exercise 8.1.

Proof of theorem 8.2. Let the unit square be triangulated using m triangles of equal area $A = 1/m$.

Let $|\,.\,|$ denote an extension of the 2-adic valuation $|\,.\,|_2$ on $\mathbb{Q}$ to all of $\mathbb{R}$. Let $x = (x_1, x_2)$ be a vertex of the triangulation of the square, and put $\hat{x} = (x_1, x_2, x_3)$ with $x_3 := 1$ (you may think of them as projective coordinates). Define a coloring $c : \mathbb{R}^2 \to [3]$ as follows. $c(x)$ is the minimal index i such that $|x_i| = \max(|x_1|, |x_2|, |x_3|)$.

If $c(x) = k \neq 3$, then $|x_k| > |x_3| = 1$.

Let's understand the coloring at the boundary of the unit square $\partial[0,1]^2$:

- Along $1 \times [0,1]$ and $[0,1] \times 1$, we see only colors 1 and 2.

- Along $[0,1] \times 0$, we see only colors 1 and 3.

- Along $0 \times [0,1]$, we see only colors 2 and 3.

We consider the triangulation of $[0,1]^2$ as a triangulation of a triangle with three edges $F_1 = 0 \times [0,1]$, $F_2 = [0,1] \times 0$, and $F_3 = 1 \times [0,1] \cup [0,1] \times 1$. According to Sperner's lemma 8.3 one of the triangles is rainbow colored, say with vertices x^1, x^2, x^3, with $c(x^i) = i$. Its area equals

$$\frac{1}{m} = \frac{1}{2} \det(\hat{x}^1, \hat{x}^2, \hat{x}^3)$$

and hence $|m| = |2|/|\det(\hat{x}^1, \hat{x}^2, \hat{x}^3)|$. We claim that

$$|\det(\hat{x}^1, \hat{x}^2, \hat{x}^3)| = |\sum_{\sigma \in S_3} \pm \hat{x}^1_{\sigma(1)} \hat{x}^1_{\sigma(3)} \hat{x}^3_{\sigma(3)}| = |\hat{x}^1_1| \cdot |\hat{x}^2_2| \cdot |\hat{x}^3_3| \geq 1^3 = 1.$$

The second equation needs an explaination: It will follow from additive property of the valuation once we show that the summand for $\sigma = \mathrm{id}_{[n]}$ has the unique largest valuation: By the definition of the coloring, $|\hat{x}^i_{\sigma(i)}| \leq |\hat{x}^i_i|$, and this inequality is strict if $\sigma_i < i$. Moreover for any permutation $\sigma \neq \mathrm{id}_{[n]}$ there exists an i such that $\sigma(i) < i$.

Thus, $|m| \leq |2|$, which means that m is even.

□

Exercises

Exercise 8.1. Prove Sperner's lemma 8.3 instead using the preimage method. (Hint: Use $(X, A) = (\Delta_{[n]}, \partial\Delta_{[n]})$, $Y = \Delta_{[n]}$ and Z the midpoint of $\Delta_{[n]}$.)

Exercise 8.2. Prove Sperner's lemma combinatorially: First prove that the number of rainbow n-faces of T equals the number of rainbow $(n-1)$-faces of $T|_{F_i}$ modulo 2. Then proceed by induction.

Exercise 8.3. Show that more generally any equal area triangulation of the d-cube $[0,1]^d$ has a number of d-simplices that is divisible by $d!$. (This was proved by Mead 1979 [19]. Hint: The proof above for $d = 2$ is written in such a way that one should have an obvious guess for the coloring also for arbitrary d. It indeed works. The non-obvious problem here is how to apply Sperner's lemma, i.e. to show the existence of a rainbow facet.)

9 Method 2: Cohomological methods

We have seen so far how apply to certain geometric problems the preimage method (see secion 4.2), obtaining a test-map

$$f : X \to_G Y \supseteq Z,$$

and how to show that $f^{-1}(Z) \neq \emptyset$ using Bordism theory in case the involved spaces are G-manifolds (see section 5).

Now it is time to turn to methods that work for more general G-spaces X, such as G-CW complexes.

9.1 The classifying space BG

Infinite joins. For any family of topological spaces $(X_i)_{i\in J}$, the *join* $X = \mathop{\ast}_{i\in J} X_j$ is defined as the quotient

$$\{(t_i, x_i)_{i\in J} \mid x_i \in X_i, t_i \in [0,1], \sum_i t_i = 1, \text{ and only finitely many } t_i \text{ are nonzero}\}/\sim,$$

by the equivalence relation

$$(t_i, x_i)_J \sim (t'_i, x'_i)_J \text{ if and only if for all } i \in J,\ t_i = t'_i \text{ and } (t_i \neq 0 \Rightarrow x_i = x'_i).$$

We write the element represented by $(t_i, x_i)_J$ for simplicity as $\sum_{i\in J} t_i x_i$, with the intuition that a coordinate x_i is irrelevant if $t_i = 0$. The join X becomes a topological space by giving it the coarsest topology such that the canonical maps $\tau_k : X \to [0,1]$, $\tau_k(\sum t_i x_i) := t_k$ and $p_k : \tau^{-1}(]0,1]) \to X_k$, $p_k(\sum t_i x_i) := x_k$ are continuous.

The spaces EG and BG. In this section, G may be an arbitrary compact Hausdorff topological group. We define EG as the countably infinite join

$$EG := \mathop{\ast}_{i \in \mathbb{Z}_{\geq 0}} G = G * G * G * \dots .$$

G acts on EG diagonally from the *right* (this is notationally convenient). The quotient space $BG := EG/G$ is called the *classifying space* of G.

In this formulation it is obvious that EG and BG are constructed functorially, that is, if H is another compact Hausdorff group and $f : G \to H$ a homomorphism (of topological groups, i.e. a continuous group homomorphism), then there are canonical maps $Ef : EG \to EH$ and $Bf : BG \to BH$.

In case G is a finite group, EG is in an obvious way a contractible free G-CW complex, and up to G-homotopy equivalence EG is uniquely characterized by these properties.

Borel construction. Let X be a G-CW complex (not necessarily free). G acts diagonally on the product $EG \times X$, and we denote by $X_G := EG \times_G X$ the quotient space. There is a canonical map $p : X_G \to BG$, which is given by sending a representative $(e, x) \in EG \times X$ to the point $\pi(e) \in BG$, where $\pi : EG \to BG = EG/G$ is the quotient map. This p is in fact a fibration with fiber X,

$$X \hookrightarrow X_G \xrightarrow{p} BG, \tag{5}$$

where the inclusion depends on a basepoint $e_0 \in EG$ and is given by $x \mapsto [e_0, x]$. This fibration 5 (and sometimes also X_G alone) is called the *Borel construction*.

Intuitively, the Borel construction replaces X by a free G-space without forgetting too much information about X.

Example 9.1 (Free G-CW complexes)**.** If X is a *free* G-CW complex, then there is another fibration, $EG \hookrightarrow X_G \xrightarrow{q} X/G$, the inclusion being given by sending $e \in EG$ to $[e, x_0]$, $x_0 \in X$ being a basepoint, and the projection $q : X_G \to X/G$ being the canonical one, $[e, x] \mapsto [x]$. As EG is contractible, the long exact sequence in homotopy groups shows that q is a weak homotopy equivalence (i.e. it induces isomorphisms between homotopy groups $q_* : \pi_n(X_G) \xrightarrow{\cong} \pi_n(X/G)$).

Equivariant cohomology. There are several equivariant cohomology theories. The following is a standard one, which is also called equivariant bundle cohomology or Borel cohomology. We simply call it equivariant cohomology.

The *equivariant cohomology* of a G-space X with coefficients in some G-module M is defined as $H^*_G(X; M) := H^*(X_G; M)$, where the latter is the ordinary cohomology of X with local coefficients in M. Explicitly, let $C_*(EG \times X)$ denote the ordinary singular chain complex of $EG \times X$ with $\mathbb{Z}$-coefficients, which is in a natural way a G-module. The set of G-module homomorphisms $\hom_G(C_n(EG \times X), M)$ is an abelian group, and with varying n this forms a cochain complex, whose cohomology is called $H^*(X_G; M) =: H^*_G(X; M)$.

When M is $\mathbb{Z}$ with the trivial G-action on it, we also write $H^*_G(X)$ for $H^*_G(X; \mathbb{Z})$.

Example 9.2 (Equivariant cohomology of a point)**.** Let pt denote a point, with the trivial G-action on it (well, there is no other). Note that $\mathrm{pt}_G = BG$. Then by definition, $H^*_G(\mathrm{pt}; M) = H^*(BG; M)$.

For finite groups G, there is a relation to the group cohomology $H^*(G; M)$, which is defined as follows: Let $\ldots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ be any projective resolution of the trivial G-module $\mathbb{Z}$ (via projective G-modules P_n), remove G from that sequence, apply the functor $\hom_G(\_, M)$ which yields a cochain complex, and whose cohomology is by definition the group cohomology $H^*(G; M)$. (This is of course only a definition up to natural isomorphism, as it depends on the choice of the resolution.)

Now observe that the augmented chain complex $\ldots \to C_1(EG) \to C_0(EG) \to \mathbb{Z} \to 0$ *is* a projective (and even free) resolution of the trivial G-module $\mathbb{Z}$. Using this resolution, the definitions of $H^*(BG; M)$ and $H^*(G; M)$ coincide. Therefore we have a natural isomorphism

$$H^*_G(\mathrm{pt}; M) := H^*(BG; M) \cong H^*(G; M).$$

For any G-space X projects to a point $p_X : X \to_G \mathrm{pt}$, we obtain a natural homomorphism $p_X^* : H^*_G(\mathrm{pt}; M) \to H^*_G(X; M)$. That is, we may regard $H^*_G(X; M)$ as a $H^*_G(\mathrm{pt}; M)$-module, or in other words, as a $H^*(G; M)$-module.

Example 9.3 (Equivariant cohomology of a free G-CW complex)**.** If X is a free CW-complex, then by Example 9.1 the canonical map $X_G \to X/G$ is a weak homotopy equivalence, which therefore induces a natural isomorphism $H^*_G(X; M) \cong H^*(X/G; M)$.

More generally, whenever h^* is a generalized cohomology theory (i.e. it satisfies all Eilenberg–Steenrod axioms except for possibly the dimension axiom), then we may define another equivariant cohomology theory h^*_G via

$$h^*_G(X) := h^*(X_G).7$$

In this course we will usually use

How to be used in the preimage method. Let G be a compact Hausdorff group, and (G, X, A, Y, Z, f_0) be a preimage problem, where now X, A, Y, Z may be arbitrary G-spaces. Suppose there exists an extension $f : X \to Y$ of the map $f_0 : A \to Y \backslash Z$ whose preimage $f^{-1}(Z)$ is empty. We want to show that such a map cannot exist. Using this map f and the unique maps $p_X : X \to \mathrm{pt}$, $p_{Y\backslash Z} : Y\backslash Z \to \mathrm{pt}$, we obtain a commutative diagram of G-maps,

$$\begin{tikzcd}
 & A \arrow[dl, hook', "i"'] \arrow[dr, "f_0"] & \\
X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y\backslash Z \arrow[dl, "p_{Y\backslash Z}"] \\
 & \mathrm{pt} &
\end{tikzcd}$$

which induces a commutative diagram in equivariant cohomology (actually with arbitrary coefficients),

$$\begin{tikzcd}
 & H^*_G(A) & \\
H^*_G(X) \arrow[ur, "i^*"] & & H^*_G(Y\backslash Z) \arrow[ll, "f^*"'] \arrow[ul, "f_0^*"'] \\
 & H^*(G) \arrow[ul, "p_X^*"] \arrow[ur, "p_{Y\backslash Z}^*"'] &
\end{tikzcd}$$

In particular when $A = \emptyset$ a common approach is to find some element $\alpha \in H^*(G)$ with $p^*_{Y\backslash Z}(\alpha) = 0$ but $p^*_X(\alpha) \neq 0$, which contradicts the commutativity of the diagram and thus the existence of f. This motivates the following definition.

Definition 9.4 (Fadell–Husseini index)**.** Let X be a G-space and M a G-module. The *Fadell–Husseini index* of X with coefficients in M is defined as

$$\mathrm{ind}_{G,M}(X) := \ker\left(p_X^* : H^*_G(\mathrm{pt}; M) \to H^*_G(X; M)\right),$$

which is an ideal in $H^*_G(\mathrm{pt}; M) = H^*(G; M)$. If h^* is a generalized cohomolgoy theory, then we analogously define

$$\mathrm{ind}_{G,h^*}(X) := \ker\left(p_X^* : h^*_G(\mathrm{pt}) \to h^*_G(X)\right),$$

In the following let ind_G denote the Fadell–Husseini index of G-spaces with respect to either $H^*(\_; M)$ or a multiplicative generalized cohomology h^*.

Theorem 9.5 (Basic properties)**.** *Let X, Y be G-spaces.*

1. *If $f : X \to_G Y$ is a G-map then*

$$\mathrm{ind}_G(X) \supseteq \mathrm{ind}_G(Y).$$

2. *If $X = X_1 \cup X_2$ and (X, X_1, X_2) is an excisive triple, then*

$$\mathrm{ind}_G(X_1) \cdot \mathrm{ind}_G(X_2) \subseteq \mathrm{ind}_G(X).$$

Corollary 9.6. *If $f : X \to_G Y$ is a G-map and $Z \subseteq Y$ a closed G-invariant subset of Y, and U an arbitrary open neighborhood of $f^{-1}(Z)$, then*

$$\mathrm{ind}_G(U) \cdot \mathrm{ind}_G(Y \backslash Z) \subseteq \mathrm{ind}_G(X).$$

In case $\mathrm{ind}_G = \mathrm{ind}_{G,h^*}$ and h^* is a *continuous* multiplicative generalized cohomology theory, in the sense that for any closed subset $A \subset X$, $h^*(A) = \lim_{U \supset A} h^*(U)$ (the direct limit is over all open neighborhoods U of A), then in corollary 9.6 we even obtain the inequality

$$\mathrm{ind}_{G,h^*}(f^{-1}(Z)) \cdot \mathrm{ind}_{G,h^*}(Y \backslash Z) \subseteq \mathrm{ind}_{G,h^*}(X).$$

Lemma 9.7 (Connectivity bound)**.** *If X is an n-connected G-space, then*

$$\mathrm{ind}_{G,M}(X) \subseteq H^{*\geq n+2}(G; M),$$

meaning that all elements in $\mathrm{ind}_{G,M}(X)$ *of degree at most $n+1$ are zero.*

Proof. Diagram chase the two Leray–Serre spectral sequences for the Borel constructions associated to X and pt. TODO: More details. $\square$

If Y is a G-representation of a compact Lie group G, then there exists a G-invariant inner product on Y. The unit sphere of Y with respect to such an inner product is denoted by $S(Y)$, which is called the *representation sphere* of Y, and we have $Y \backslash \{0\} \simeq_G S(Y)$.

Theorem 9.8 (Representations of elementary abelian groups)**.** *Let G be an elementary abelian group, i.e. $G = (\mathbb{Z}/p)^k$ for some prime p and some $k \geq 0$. Let Y be a finite G-representations with fixed point set $Y^G = \{0\}$. Give $\mathbb{F}_p = \mathbb{Z}/p$ the trivial G-module structure. Then* $\mathrm{ind}_{G,\mathbb{F}_p}(Y \backslash 0) \neq 0$.

As a corollary we obtain the following theorem.

Theorem 9.9. *Let G be an elementary abelian group, X be an n-connected free G-space and Y an n-dimensional real G-representation with only fixed-point $Y^G = 0$. Then for any map $f : X \to_G Y$, $f^{-1}(0) \neq \emptyset$.*

Remark 9.10. If X is a free G-CW complex, which is n-connected, then also its $n+1$-skeleton $X^{(n+1)}$ is n-connected. In particular in the setting of theorem 9.9, we even obtain $f^{-1}(0) \cap X^{(n+1)} \neq \emptyset$.

10 Topic: Tverberg type theorems

10.1 The topological Tverberg theorem

Let us come to a few applications of theorem 9.9.

Theorem 10.1 (Tverberg '66)**.** *Let $d \geq 1$, $q \geq 2$, and define $N := (q-1)(d+1)$. Any $N+1$ points in $\mathbb{R}^d$ can be partitioned into q parts whose convex hulls have a point in common.*

The case $q = 2$ was already due to Radon [25], and the general statement was first conjectured by Birch [4] and proved by Tverberg [29].

In case $q = p^k$ is a prime power, Tverberg's theorem has the following topological generalization.

Theorem 10.2 (Topological Tverberg theorem)**.** *Let $d \geq 1$, $q = p^k$ be a prime power, and as before $N := (q-1)(d+1)$. For any continuous map $f : \Delta^N \to \mathbb{R}^d$ from the N-simplex to $\mathbb{R}^d$ there exists q pairwise disjoint faces $F_1, \ldots, F_q \subset \Delta^N$ such that their images have a point in common,*

$$f(F_1) \cap \ldots \cap f(F_q) \neq \emptyset.$$

This was proved by Bárány–Shlosman–Szűcs [2] when q is a prime, and for arbitrary prime powers by Özaydin [24].

For prime powers q, Tverberg's theorem 10.1 is the special case of the topological Tverberg theorem 10.2 for affine maps f. Indeed, for any $N+1$ given points $x_0, \ldots, x_N$ in $\mathbb{R}^d$ there is an affine map $f : \Delta^N \to \mathbb{R}^d$ such that the vertices of Δ^N are send bijectively to $x_0, \ldots, x_N$. And the convex hull of any subset $(x_i)_{i \in J}$ is then the same as the f-image of the face of Δ^N whose vertices are send to $(x_i)_{i \in J}$.

Proof of theorem 10.2. TODO $\square$

10.2 The center point theorem

As a corollary of Tverberg's theorem or its topological counter part we obtain the following center point theorem, which is due to Rado [].

Theorem 10.3 (Center point theorem)**.** *Let μ be a probability measure on the Borel σ-algebra of $\mathbb{R}^d$. Then there exists a point $c \in \mathbb{R}^d$ such that any half space H^+ containing c has measure at least $\mu(H^+) \geq \frac{1}{d+1}$.*

Any such c will be called a *center point* of μ. The set of all center points of μ is a closed convex set.

Proof. We may approximate μ using $N+1$ points $x_0, \ldots, x_N$, where $N = (q-1)(d+1)$ and q is a large prime power. Do it in such a way that the x_i are in general position. Using the (topological) Tverberg theorem, we obtain a partition $\{x_i\} = X_1 \dot\cup \ldots \dot\cup X_q$ whose convex hulls contain some point c_q. All but at most d of the X_i will be the vertex sets of some d-simplex σ_i in $\mathbb{R}^d$. Any half space H^+ containing c_q will also contain also a vertex of each such σ_i. Thus H^+ will contain at least $q-d$ vertices of $x_0, \ldots, x_N$, which asymptotically for large q is the $d+1$'st fraction of $N+1$. For larger and larger q, we obtain a sequence $(c_q)_q$, which will have a limit point c (via some compactness argument). By some limit argument, any such limit point c does it. $\square$

10.3 The center transversal theorem

The center point theorem yields a center point for a single measure μ, whereas the ham sandwich theorem 2.2 was about d-measures that can be bisected using a $(d-1)$-dimensional hyperplane.

These two statements are not only formally similar, one can interpolate between them via the following center transversal theorem. It simultaneously generalizes both, the center point and the ham sandwich theorem. It is due to Dolnikov [11, 12] and Živaljević–Vrećica [32], who discovered it independently.

Theorem 10.4 (Center transversal theorem)**.** *Let $\mu_1, \ldots, \mu_m$ be m probability measures in $\mathbb{R}^d$. Then there exists a $(m-1)$-dimensional affine subspace $C \subset \mathbb{R}^d$ (the center transversal) such that any half space H^+ containing C satisfies $\mu_i(H^+) \geq \frac{1}{d-m+2}$ for each $i = 1, \ldots, m$.*

For the proof we need a lemma. For $n \geq k \geq 0$, let $G_{n,k}$ denote the *Grassmann manifold* (also called the Grassmannian) of all linear k-subspaces of $\mathbb{R}^n$. $G_{n,k}$ is a closed manifold of dimension $k(n-k)$. Let $\gamma_{n,k} : E\gamma_{n,k} \to G_{n,k}$ denote the tautological rank k vector bundle over $G_{n,k}$: The fiber over a point $H \in G_{n,k}$ is simply H regarded as a k-subspace $H \subseteq \mathbb{R}^n$.

Lemma 10.5. *The $(n-k)$-fold Withney sum $\gamma_{n,k}^{\oplus(n-k)}$ does not admit a nowhere vanishing section.*

Proof of lemma 10.5. We use the preimage method: Let $s : G_{n,k} \to E(\gamma_{n,k}^{\oplus(n-k)})$ be the following section. At a point $H \in G_{n,k}$, which is a k-subspace of $\mathbb{R}^n$, let $p_H : \mathbb{R}^n \to H$ denote the orthogonal projection with respect to the standard inner product of $\mathbb{R}^n$, and define $s(H) := (p_H(e_{k+1}), \ldots, p_H(e_n))$. Clearly the only zero of s is the subspace $H^* = \langle e_1, \ldots, e_k \rangle$. One checks that s is continuous and transversal to the zero-section. Using theorem 5.14 it follows that any generic section has an odd number of zeros, and any section has some zero. □

Proof of theorem 10.4. Let $n := d$ and $k := d - m + 1$. For any linear k-subspace $H \in G_{n,k}$, push forward the mass μ_i to H, $\mu_{i,H} := (p_H)_* \mu_i$, and consider the set $C_{i,H} \subset H$ of center points for $\mu_{i,H}$. By theorem 10.3, $C_{i,H} \neq \emptyset$. Now construct a section $c_i : G_{n,k} \to E\gamma_{n,k}$ such that for any $H \in G_{n,k}$, $c_i(H)$ lies in an ε-neighborhood of $C_{i,H}$ (it's a technicality that this is indeed possible). Consider the section $s = (c_2 - c_1, \ldots, c_m - c_1)$ of $\gamma_{n,k}^{n-k}$. By lemma 10.5, it has a zero $H^* \in G_{n,k}$. Let C be the affine subspace of $\mathbb{R}^n$, which is orthogonal to H and which goes through $c_1(H^*) = \ldots = c_m(H^*)$. Then C is ε-close to being a center transversal for $\mu_1, \ldots, \mu_m$. Letting $\varepsilon \to 0$ and a compactness argument finished the proof. □

11 Methods: Volume arguments

- Milnor's proof of Hairy ball theorem (and Brouwer's fixed point theorem as a corollary).

- Karasev's result on inscribed circular quadrilaterals
- Some Knaster type theorems

12 Method 2: Obstruction theory and characteristic classes

TODO

13 Method: Polynomial ham sandwich method

In recent years a new "divide and conquer" method has been introduced to combinatorial geometry.

It it based on the polynomial ham sandwich theorem. For this section, a *mass* μ is a probability measure on $\mathbb{R}^d$ which is absolutely continuous with respect to the Lebesgue measure (i.e. there is a Lebesgue measurable density function f such that $\mu(A) = \int_A f \, d\lambda$).

A *hypersurface* of degree D in $\mathbb{R}^d$ is the zero-set $Z(f) = \{x \mid f(x) = 0\}$ of a non-zero polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree D.

A hypersurface $Z(f)$ is said to *bisect* a mass μ if $\mu(\{f > 0\}) = \mu(\{f < 0\})$.

Theorem 13.1 (Polynomial ham sandwich theorem, Stone–Tuckey)**.** *Let $\mu_1, \ldots, \mu_m$ be m masses in $\mathbb{R}^d$ and let D be a positive integer satisfying $\binom{D+d}{d} > m$. Then there exists hypersurface of degree at most D that bisects all m masses μ_i simultaneously.*

In other words, on can bisect any m masses in $\mathbb{R}^d$ simultaneously using a hypersurface of degree $O_d(m^{1/d})$.

The proof of theorem 13.1 is analogous to the one of the ordinary ham sandwich theorem 2.2: For this simply note that the space of all real polynomials of degree D in d variables is a vector space of dimension $\binom{D+d}{d}$, and we can make again use of the antipodal action.

Alternatively, consider the Veronese map $v : \mathbb{R}^d \to \mathbb{R}^{\binom{D+d}{d}-1}$, whose coordinate functions vary over all non-constant monomials of degree at most D in d variables. Apply the ordinary ham sandwich theorem to the push forwards $v_*(\mu_1), \ldots, v_*(\mu_m)$, which yields a bisection hyperplane $H \subset \mathbb{R}^{\binom{D+d}{d}-1}$. Then $v^{-1}(H)$ is a hypersurface of degree at most D which bisects all given masses $\mu_1, \ldots, \mu_m$.

Corollary 13.2 (Cell decomposition)**.** *Let S be a finite set of points in $\mathbb{R}^d$ and let $m \geq 1$. Then there exists a non-zero polynomial f of degree $O_d(m^{1/d})$, such that $\mathbb{R}^d \backslash Z(f)$ is a union of m open sets C_i (possibly empty, not necessarily connected) such that for all i, $\partial C_i \subseteq Z(f)$ and $|S \cap C_i| \leq O_d(|S|/m)$.*

The polynomial ham sandwich theorem was first used as a method in combinatorial geometry by Guth and Katz, who (up to a logarithmic factor) solved the Erdös distance problem.

Problem 13.3 (Erdös distinct distances problem)**.** *Let $g(N)$ denote the minimal number of distinct distances $|\{|x_i - x_j|\}|$ among any N pairwise distinct points $x_1, \dots, x_N$ in the plane. How does $g(N)$ behave asymptotically?*

A square grid shows $g(N) = O(N/\sqrt{\log(N)})$. Guth and Katz proved $g(N) = \Omega(N/\log(N))$.

A more didactic application is the following classical Szemeredi–Trotter theorem.

Theorem 13.4 (Szemeredi–Trotter)**.** *Let P be a finite set of points and L be a finite set of lines in the plane. Denote by $I(P, L) := |\{(p, \ell) \mid p \in P, \ell \in L, p \in \ell\}|$ be the number of incidences. Then*

$$I(P, L) = O(|P|^{2/3}|L|^{2/3} + |P| + |L|).$$

First a very simple bound:

Lemma 13.5. $I(P, L) \leq |L| + |P|^2$.

Proof. Split $L = L' \cup L''$ according to whether a line $\ell \in L$ is incident to at most one or more than one points of P. Clearly $I(P, L') \leq |L|$. Now each $p \in P$ can be incident to at most $|P| - 1$ lines of L''. Thus $I(P, L'') \leq |P|^2$. □

Proof of Szemeredi–Trotter theorem 13.4. We may assume $|P| \leq |L|$ (otherwise use projective duality to arrive at this situation). Further we may assume $\sqrt{|L|} \leq |P|$, otherwise the result follows from the lemma.

Choose an $m \geq 1$ and the cell decomposition corollary 13.2. It yields a polynomial f of degree $D = O(m^{1/2})$ and a partition $\mathbb{R}^d \backslash Z(f) = \bigcup C_i$ into m open sets. Let $P_0 := P \cap Z(f)$, $P_i := P \cap C_i$, and $L_0 = \{\ell \in L \mid \ell \subseteq Z(f)\}$.

$$I(P, L) = \sum_i I(P_i, L) + I(P_0, L_0) + I(P_0, L \backslash L_0).$$

First,

$$I(P_0, L_0) \leq |P_0| \cdot |L_0| \leq |P| \cdot D.$$

Second,

$$I(P_0, L \backslash L_0) \leq |L \backslash L_0| \cdot D \leq |L| \cdot D.$$

Third,

$$\sum_i I(P_i, L) = \sum_i I(P_i, L_i) \leq \sum_i |L_i| + |P_i|^2,$$

where L_i is the set of lines in L that intersect points in P_i. A line ℓ can appear in at most D such L_i! Thus

$$\sum_i |L_i| \leq D \cdot |L|.$$

It remains

$$\sum_i |P_i|^2 \le O(|P|/m) \cdot \sum_i |P_i| = O(|P|^2/m).$$

Choosing $m = \lceil (|P|^2/|L|)^{2/3} \rceil$ will yield the theorem. $\square$

14 Short topic: Kakutani–Yamabe–Yujobo

Rademacher asked whether every closed convex set in $\mathbb{R}^n$ can be circumscribed with an n-cube. This was proven by Kakutani in the 3-dimensional case and by Yamabe–Yujobo in general. It follows from the following theorem.

Theorem 14.1. *For any continuous map $f : S^n \to \mathbb{R}$ there exist $n+1$ points $x_0, \ldots, x_n$ perpendicular to each other on S^n such that*

$$f(x_0) = \ldots = f(x_n).$$

The theorem has a rather strange proof: Let $L = \{(x,t) \in S^n \times I \mid f(x) = t\}$ be the graph of f. Then the theorem follows from the following Lemma.

Lemma 14.2. *Let L be a closed subset of $S^n \times I$ that intersects any continuous curve from $S^n \times \{0\}$ to $S^n \times \{1\}$. Then L contains $n+1$ points $(x_0, t^*), \ldots, (x_n, t^*)$ for some $t^* \in I$ such that $x_0, \ldots, x_n$ are pairwise perpendicular.*

Proof. We prove the lemma by induction on n. For $n = 0$, the lemma is true. Now assume that the lemma is true for $n-1$.

Let us replace L by a small closed ε-neighborhood. If L is disconnected then we remove all but one components such that it still satisfies the condition of the lemma.

Let p_0 and p_1 be the points of L with minimal and maximal t-coordinates, respectively. We can connect p_0 and p_1 by a path $p : I \to L$, $p(0) = p_0$, $p(1) = p_1$. We construct a map

$$i : S^{n-1} \times I \to S^n \times I$$

that sends the slices $S^{n-1} \times \{t\}$ in a continuous fashion homeomorphically to $S^{n-1}_{p(t)}$, which is the equator in the slice of $S^n \times I$ that contains $p(t)$ such that all points in this equator are orthogonal to $p(t)$.

Applying the lemma to $S^{n-1} \times I \supset L' := i^{-1}(L)$ yields points $(x_0, t^*), \ldots, (x_{n-1}, t^*) \in S^{n-1} \times I$ such that $x_0, \ldots, x_{n-1}$ are pairwise perpendicular. Shifting this solution with the above map i to $S^n \times I$ and appending the point $p(t^*)$ yields a solution for n. $\square$

15 Topic: The Gromov–Milman conjecture

15.1 Dvoretzky's theorem and a polynomial version.

Theorem 15.1 (Dvoretzky)**.** *For any integer $k \ge 1$ and $\varepsilon > 0$ there exists an $N = N(k, \varepsilon) = \exp^{O(k/\varepsilon^2)}$ such that the following holds. For any centrally symmetric convex* *body K in $\mathbb{R}^N$, there exists a k-dimensional linear subspace $V \subset \mathbb{R}^N$ and a ball $B \subset V$ of some radius centered at the origin such that*

$$B \subseteq K \cap V \subseteq (1+\varepsilon)B.$$

Gromov and Milman asked whether for convex bodies K defined via polynomials one may find a cut $K \cap V$ which is *exactly* a ball of some radius.

They conjectured the following theorem, which was proved by Dolnikov and Karasev [13].

Theorem 15.2 (Dolnikov–Karasev, polynomial Dvoretzky theorem)**.** *For any $k \geq 1$ and even $d \geq 2$ there exists an $n = n(d,k)$ such that the following holds. For any homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d there exists a k-dimensional linear subspace $V \subseteq \mathbb{R}^n$ such that with respect to orthonormal coordinates of V,*

$$f|_V(y_1, \ldots, y_k) = r \cdot (y_1^2 + \ldots + y_k^2)^{d/2}, \text{ for some } r \in \mathbb{R}.$$

In other words, if we consider $f : \mathbb{R}^n \to \mathbb{R}$ as a function on $\mathbb{R}^n$, there exists an orthogonal k-frame $(v_1, \ldots, v_k) \in (\mathbb{R}^n)^k$ (i.e. $||v_i|| = 1$ and $\langle v_i, v_j \rangle = 0$ for $i \neq j$) and a real number $r \in \mathbb{R}$ such that

$$f(y_1 v_1 + \ldots + y_k v_k) = r \cdot (y_1^2 + \ldots + y_k^2)^{d/2}.$$

Example 15.3 ($d = 2$)**.** In theorem 15.2 we may take $n(d = 2, k) = 2k - 1$. In particular, any ellipsoid of dimension $2k-1$ can be cut with a k-plane through its center such that the intersection is a euclidean ball.

Proof of example 15.3. For $d = 2$, $f(x) = x^t Q x$ is a quadratic form (not necessarily positive definite), where Q is a symmetric real $n \times n$ matrix. Via an orthonormal change of basis, with new basis $w_1, \ldots, w_n$, we may write $f(\sum_{i=1}^n x_i w_i) = \sum_{i=1}^n a_i x_i^2$, with $a_1 \leq \ldots \leq a_n$. Now let $n = 2k-1$. Choose $\lambda_1, \ldots, \lambda_{2k-1} \geq 0$ such that $\lambda_i^2 + \lambda_{2k-i}^2 = 1$ and $a_i \lambda_i^2 + a_{2k-i}\lambda_{2k-i}^2 = a_k$ for all i. (This can be done, as a_k lies in the convex hull of a_i and a_{2k-i}.) Define a subspace $V = \langle v_1, \ldots, v_k \rangle$ via $v_i = \lambda_i w_i + \lambda_{2k-i} w_{2k-i}$. Then $f(\sum_{i=1}^k y_i v_i) = a_k(y_1^2 + \ldots + y_k^2)$. $\square$

Remark 15.4 (Versions of the polynomial Dvoretzky theorem for 1) odd d, and 2) over $\mathbb{C}$)**.** One may ask whether theorem 15.2 also holds for odd degrees d. And indeed it does, and one can prove much more: The statement of the theorem holds then with $r = 0$, i.e. f vanishes on some linear subspace V of dimension k completely. Moreover one can take $n(d,k) = k + \binom{d+k-1}{d}$.

Another version of theorem 15.2 comes from considering complex polynomials $f \in \mathbb{C}[x_1, \ldots, x_n]$ that are homogeneous of degree d. Here we allow even and odd d. In this setting one can find a complex linear subspace $V \subseteq \mathbb{C}^n$ of complex dimension k such that $f|_V = 0$. Similar as in the real case for odd d, here in the complex case $n(d,k) = k + \binom{d+k-1}{d}$ is sufficient.

15.2 A Borsuk–Ulam theorem for p-toral groups.

Definition 15.5 (p-toral group)**.** Let p be a prime. A p-toral group G is a group extension of a finite p-group F by a torus $T = (S^1)^n$. That is, G fits into a short exact sequence

$$0 \to T \to G \to F \to 0.$$

Theorem 15.6. *Let Y be a finite dimensional representation of a p-toral group G. Then there exists an integer $n = n(G, Y)$ such that for any n-connected free G-space X, any G-equivariant map $f : X \longrightarrow_G Y$ intersects the G-fixed point set Y^G, i.e. $f^{-1}(Y^G) \neq \emptyset$.*

So far, the bound $n = n(G, Y)$ is only known to exist, however no effective n is known, i.e. one which is computable for any given G and Y.

The proof of theorem 15.6 is based on the Segal conjecture, which was proved by Carlsson [7]. By this theorem, for any p-group G, the natural map from the Burnside ring $A(G)$ to the G-equivariant stable cohomotopy of EG^+,

$$A(G) = \pi_G^0(S^0) \hookrightarrow \hat{\pi}_G^0(S^0) \xrightarrow{\cong} \pi_G^0(EG^+) = \pi_s^0(BG^+) = \varprojlim_n \pi_s^0(B_nG^+),$$

is injective (the plus denotes taking the disjoint union with a point, which is the new base-point and obtains the trivial G-action). In the proof of theorem 15.6 one constructs a certain element $\alpha = \alpha(G, Y) \in A(G)$ that maps to zero in $\pi_G^0(S(Y/Y^G)^+)$, which depends only on G and Y, and one searches for an integer n for which α does not vanish in $\pi_s^0(B_{n+1}G^+)$. By Carlsson's theorem, such an integer $n = n(G, Y)$ exists, however so far no effective n doing this job is known.

If X is n-connected, there exists a G-map $E_{n+1}G \longrightarrow_G X$, where $E_{n+1}G$ is the $(n+1)$-skeleton of some EG (EG is a free G-CW complex that is contractible). Thus if there is a map $X \longrightarrow_G S(Y/Y^G)$, then we obtain a composition $\pi_G^0(S(Y/Y^G)^+) \to \pi_G^0(X^+) \to \pi_G^0(E_{n+1}G^+)$, which yields a contradiction for $n \geq n(G, Y)$ as α maps to zero in $\pi_G^0(S(Y/Y^G)^+)$ but to something non-zero in $\pi_G^0(E_{n+1}G^+)$.

15.3 The polynomial Dvoretzky theorem in degree $d = 4$.

We will prove theorem 15.2 only for $d = 4$. For larger d, a similar proof technique works, although it is technically more difficult.

First consider the Stiefel manifold $V_{n,k}$, which is the set of all orthogonal k-frames in $\mathbb{R}^d$,

$$V_{n,k} = \{(v_1, \ldots, v_k) \mid v_i \in \mathbb{R}^n, ||v_i|| = 1, \langle v_i, v_j \rangle = 0\},$$

together with a natural smooth structure coming from regarding $V_{n,k}$ as a submanifold of $(\mathbb{R}^n)^k$.

The orthogonal group $O(k)$ acts on $V_{n,k}$ freely by rotating the k-frames, and the quotient space $V_{n,k}/O(k)$ is the Grassmannian $G_{n,k}$, which parametrizes linear subspaces of $\mathbb{R}^n$ of dimension k. Let $\gamma_{n,k} : E\gamma_{n,k} \to G_{n,k}$ denote the *tautological rank k vector* *bundle* over $G_{n,k}$, whose fiber over a point $V \in G_{n,k}$ is V regarded as a k-subspace of $\mathbb{R}^n$.

For a k-dimensional vector space V, let $\mathrm{Sym}^d(V)$ denote its d-fold symmetric tensor product. It is just a coordinate free notation of the space of all homogeneous polynomials on $\dim V$ variables of degree d (regarded as functions on V).

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. Let $Y = \mathrm{Sym}^d(\mathbb{R}^k) \subseteq \mathbb{R}[x_1, \ldots, x_k]$ be the vector space of homogeneous polynomial in k variables of degree d. Associate to f a map

$$\varphi : V_{n,k} \longrightarrow_{O(k)} Y$$

given by sending $(v_1, \ldots, v_k) \in V_{n,k}$ to the polynomial $p(x_1, \ldots, x_k) = f(\sum x_i v_i)$ (here f is considered as a function $f : \mathbb{R}^n \to \mathbb{R}$). Clearly φ is $O(k)$-equivariant with respect to the natural $O(k)$-actions on $V_{n,k}$ and on Y. Note that the fixed points $p \in Y^{O(k)}$ are of the form $p = r \cdot (x_1^2 + \ldots + x_k^2)^{d/2}$ with $r \in \mathbb{R}$. Moreover, as $V_{n,k}$ is the total space of a tower of fibrations with fibers and base spaces of the form S^i with $n - k \leq i \leq n - 1$, $V_{n,k}$ is $(n - k - 1)$-connected (for this use Hurewicz's theorem and Leray–Serre spectral sequences). (And indeed, $V_{\infty,k} = \bigcup_n V_{n,k}$ is an $EO(k)$.) So we would like to use a Borsuk–Ulam type theorem to show that for large n, any such $O(k)$-map φ intersects Y^G, which then would imply theorem 15.2. Unfortunately, the analog of theorem 15.6 does not hold in general for $G = O(k)$. In fact, there does exist a map $V_{n,k} \longrightarrow_{O(k)} Y$ for odd k, even d, and arbitrarily large n.

In order to circumvent this problem, Dolnikov and Karasev [13] used not only the topological Borsuk–Ulam type theorem 15.6, but also an additional "geometric" idea, an averaging trick (already Milman thought about this idea, however without carrying it to the end). From my point of view, that's what makes this proof so beautiful and interesting. Here is how it works.

First we assume that $k = 2^\ell$ and $n = 2^h$ are powers of two (if not, make them larger). We restrict $O(k)$ to its subgroup $G := (\mathbb{Z}/2)^n \rtimes \mathrm{Syl}^2(n)$. Here, $(\mathbb{Z}/2)^n$ acts on $V_{n,k}$ by flipping the signs of the legs of k-frame, and $\mathrm{Syl}^2(S_k)$ is the 2-Sylow subgroup of the symmetric group S_k, which acts on a k-frame by permuting the legs. Clearly, G is a finite 2-group, and thus we may apply theorem 15.6 to φ. For large n, say $n \geq n(d, k)$, this implies that for any given f, φ sends some k-frame $(v_1, \ldots, v_k)$ to some polynomial $p \in Y^G$.

G-invariance of p means that $p = \sum_{i,j} a_{ij} x_i^2 x_j^2$ with $a_{ij} = a_{ji}$ and moreover $a_{ij} = a_{i'j'}$ if the distance of L_i and L_j is the same as the distance between $L_{i'}$ and $L_{j'}$, where L_i are the leaves of a complete binary tree of height h with n leaves.

TODO....

16 Literature for the course

Differential topology:

1. Guillemin and Pollack [15] (transversality)

2. Milnor [21] (transversality)

Equivariant topology:

1. tom Dieck [10] (equivariant topology bible)
2. Bredon [5] (equivariant cohomology, equivariant obstruction theory for non-free domains)
3. Conner and Floyd [8] (equivariant bordism theory)

Topological combinatorics:

1. Matoušek [18] (elementary text book)
2. de Longueville [9] (elementary text book)
3. Karasev [16] (survey article)
4. Živaljević [30, 31] (two survey articles)
5. Bárány [1] (survey article on Borsuk–Ulam and applications)
6. Bartsch [3] (critical point theory in equivariant setting)
7. Kozlov [17] (combinatorial algebraic topology)
8. Skopenkov [26] (embeddings and knots)

Algebraic topology: many standard books! (Bredon [6], Spanier [27], ..., Milnor [22])

17 Contents of lectures

1. Teaser: What is topological combinatorics, preimage method intuitively, inscribed squares + proof idea, ham sandwich + proof idea, many applications that may appear in the course.
2. G-spaces, formal preimage method.
3. Transversality, unoriented bordism.
4. Oriented bordism, first applications with proof: Borsuk–Ulam, fundamental theorem of algebra, Brouwer's fixed point theorem.
5. Embeddings and graph colorings: Van Kampen – Flores, Kneser conjecture, Sarkaria's coloring and embedding theorem.
6. Applications: Sperner's lemma, equal-area triangulations, Gromov waist of sphere theorem.

7. Cohomological methods, in particular Fadell–Husseini index, actions of elementary abelian groups.

8. Tverberg's theorem, topological Tverberg, center point theorem, center transversal theorem.

9. Polynomial ham sandwich method (Szemeredi–Trotter, mention Guth–Katz). Kakutani–Yamabe–Yujobo.

10. Plan: Gromov–Milman conjecture a.k.a. polynomial Dvoretzki theorem a.k.a. Dolnikov–Karasev theorem.

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