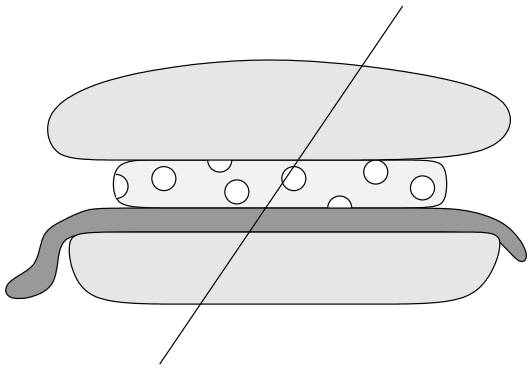
Equivariant Topology And Applications



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Equivariant Topology And Applications

Diploma Thesis
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Summary

This thesis deals with three topics in discrete geometry:

- Mass partitions by hyperplanes
- Polygons and tetrahedra inscribed in curves and surfaces
- The Topological Tverberg Problem

The methods to attack these subjects are as interesting as the problems themselves. The large appendices contain methods that I want to deal with separately for the sake of clarity.

Chapter I describes the configuration space-test map method, which can be an immensely useful proving scheme that builds a bridge from problems in discrete geometry and combinatorics to powerful methods of algebraic topology.

In *Chapter II* we deal with mass partitions by hyperplanes. We formalise some "elementary" inequalities for the smallest dimension, such that the partition problem is solvable. Especially Lemma 2.7 is new and interesting, since Ramos' results on mass partitions imply with the help of this lemma immediately all but one of the bounds that have been found so far. For more than five hyperplanes we obtain even new bounds. This however has to be checked, since Ramos did not state exactly the algorithm he used to make his calculations (I didn't find an algorithm that was fast enough). Then we write down known bounds coming from the Fadell–Husseini index and give two alternative proofs, which yield the same bound: We use at first another test map and then characteristic classes. Another very interesting approach will also be presented, which uses covering arguments and the ring structure of $H^*(\mathbb{R}P^d; \mathbb{F}_2)$. This is the most geometric approach. Finally, we deal with a very promising ansatz, which seems to be very strong and would work also in a more general setting, however the required calculations are out of reach at this stage.

In *Chapter III* we deal with inscribed polytopes. After the introduction, we prove that each circle with a symmetric distance function inscribes a triangle. In the smooth case we can do even more: There any closed curve contains a one-parameter family of (maybe skew) polygons

with arbitrarily many edges and arbitrary edge ratios. In the special case that all edges have the same length, we can prove a strong property of these one-parameter families, which in turn not only yields easily a new proof for the Square Peg Problem, but also lets us prove another nice fact: Every circle contains an equilateral triangle with respect to one metric that is also an isosceles triangle with respect to another metric. Then we show that in a paper of H. B. Griffiths, the proofs of three out of four theorems unfortunately contain errors, such that it still remains open, whether every smooth plane closed curve inscribes a rectangle with prescribed edge ratios. Finally, we prove a very positive result: Every compact surface with a symmetric distance function that in some small open neighborhood looks like a smoothly embedded disc has an inscribed tetrahedron, whose edge ratios can be prescribed subject to some restrictions (e. g. a regular tetrahedron does it).

Chapter IV is about the Topological Tverberg Problem. Here we explicitly calculate the obstruction cocycle whose cohomology class tells whether the test map of the Topological Tverberg Problem exists or not. This could yield a new and topological proof for the (affine) Tverberg Theorem.

The appendices are about more theoretical topics that I want to treat separately. In *Appendix A* we will prove a small Lemma that tells us what G-simplicial complexes deformation-retract to when we delete a subcomplex. Then we show that the deleted-product construction yields in some typical cases a better test map than the deleted-join construction. Furthermore we list some known topological methods to treat existence issues of maps.

In *Appendix B* we deal with the known Fadell–Husseini index and give short proofs for some of its known properties. Then we show that in practice, the Fadell–Husseini index often gives the same criterion for the existence of equivariant maps as characteristic classes.

In *Appendix C* we summarise the idea of equivariant obstruction theory and rediscover Bredon cohomology (in a practical way, such that we can state a slightly more general obstruction theory), which one needs to generalise the usual obstruction theory to non-free domains. Finally non-simple ranges will be dealt with.

Zusammenfassung (German summary)

Die Diplomarbeit widmet sich drei verschiedenen Themenkomplexen in der diskreten Geometrie:

- Massepartitionen (durch Hyperebenen)
- In Kurven und Flächen einbeschriebene Polygone und Tetraeder
- Das Topologische Tverbergproblem

Das Interesse liegt jedoch gleichermaßen auf der Seite der Methoden die nötig sind, um die wichtigen Fragestellungen in diesen Komplexen lösen zu können. Methoden, die ich isoliert darstellen will, befinden sich im Anhang, was deren Wichtigkeit jedoch nicht schmälern soll.

Kapitel I beschreibt kurz die Konfigurationsraum-Testabbildungsmethode, welche sich teilweise hervorragend dazu eignet, Fragestellungen aus der diskreten Geometrie und Kombinatorik in topologische umzuwandeln um sie mit Hilfe der Methoden aus der algebraischen Topologie zu lösen.

Im *Kapitel II* beschäftigen wir uns mit den Massepartitionen. Wir formalisieren "elementare" Abschätzungen für die kleinste Dimension, in der das Massepartitionsproblem lösbar ist. Insbesondere ist Lemma 2.7 neu und interessant, da es aus Ramos' Resultaten leicht viele erst später gefundene Resultate folgern läßt. Anschließend geben wir die bekannte Schranke an, die der Fadell–Husseini-Index liefert, und zeigen dass sowohl eine andere Testabbildung, als auch charakteristische Klassen die gleichen Ergebnisse liefern. Ein interessanter andersartiger Zugang, welcher die Ringstruktur von $H^*(\mathbb{R}P^d; \mathbb{F}_2)$ benutzt, wird dargestellt. Dann wird eine Stelle in einem Beweis von Ramos angegeben, die ich nicht verifizieren konnte, weswegen ich mir nicht sicher bin ob die neuen Ergebnisse stimmen, die die Lemmas 2.4 und 2.7 darauf aufbauend liefern. Abschließend wird ein vielversprechender Ansatz beschrieben, der sich allgemein gut eignen könnte um die Existenzfrage von Testabbildungen zu klären, jedoch sind die dazu benötigten Berechnungen noch außer Reichweite.

Im *Kapitel III* beweisen wir nach der Einleitung, dass jeder Kreis mit symmetrischer Distanzfunktion ein gleichseitiges Dreieck enthält. Im glatten Fall können wir sogar viel mehr: Da enthält jeder Kreis

sogar eine Einparameterfamilie von geschlossenen Streckenzügen mit beliebiger Kantenanzahl, sodass die Kanten vorgegebene Längenverhältnisse erfüllen. Im Spezialfall, dass alle Kanten gleichlang sind, können wir eine starke Schlußfolgerung ziehen, die mit einem Schlag einen neuen Beweis für das Square Peg Problem liefert, und auch zeigt, dass jeder glatt eingebettete Kreis ein Dreieck enthält, welches bezüglich einer Metrik gleichseitig und bezüglich einer weiteren gleichschenklig ist. Dann zeigen wir, dass in einem Paper von H. B. Griffiths leider drei von vier Sätzen fehlerhalt beweisen wurden und es deswegen immer noch ungewiss ist, ob jede glatte ebene geschlossene Kurve ein Rechteck mit vorgeschriebenen Seitenverhältnissen umschreibt. Abschließend wird bewiesen, dass jede kompakte Fäche mit symmetrischer Distanzfunktion, die wenigstens an einer kleinen offenen Menge von einer glatten Einbettung stammt, die Ecken eines Tetraeders mit vorgeschriebenen Seitenverhältnissen enthält, wobei an die Seitenverhältnisse noch Bedingungen geknüpft sind (z. B. ein regulärer Tetraeder tut's).

Im *Kapitel IV* berechnen wir explizit den Hinderniskozykel, dessen Kohomologieklasse angibt, ob die dem Problem entsprechende Testabbildung existiert oder nicht. Dies könnte einen neuen, topologischen Beweis für den (affinen) Tverbergsatz liefern, wie im Anschluß bemerkt wird.

Die Anhänge behandeln theoretischere Themen, die ich getrennt darstellen will. Im *Anhang A* beweisen wir ein kleines Lemma, welches beschreibt, auf was ein G-Simplizialkomplex deformationsretrahiert, wenn man einen Teilkomplex löscht. Weiterhin zeigen wir, dass Deleted-Product-Konstruktion in einigen typischen Fällen eine stärkere Testabbildung liefert als die Deleted-Join-Konstruktion. Anschließend listen wir bekannte topologische Methoden auf.

Im *Anhang B* behandeln wir den bekannten Fadell–Husseini-Index, geben kurz Beweise für bekannte Sachen, deren Beweise in der Literatur ausgelassen wurden und zeigen dass der Fadell–Husseini-Index in der Praxis oft das gleiche Kriterium für die Existenz von äquivarianten Abbildungen liefert, wie charakteristische Klassen.

Im *Anhang C* fassen wir die Grundidee der äquivarianten Hindernistheorie zusammen und erfinden die Bredonkohomologie neu (in einer problemorientierteren und dort leicht allgemeineren Version), die man benötigt, falls man die Hindernistheorie auf nichtfreie Wertebereiche erweitern will. Abschließend werden nichteinfache Wertebereiche behandelt.

Preliminaries

Prerequisites. The reader of this thesis is supposed to be familiar with very basic definitions of and facts about transformation groups (i. e. what are equivariant maps, diagonal actions..., see [**Die86**, Chap. I.1]), with fundamental algebraic topology tools (see e. g. [**Bre93**]) and their equivariant analoga (see [**Die86**, Ch. II.1] or [**AlPu93**, Ch. 1.1]). Knowing basic facts about representation theory of finite groups will not be necessary, but it helps to understand the underlying ideas, how we were able to decompose some of our representations as in Section 3.2 (see [**FuHa91**, first chapters] for an introduction). In Chapter III we will use basic methods from differential topology (see e. g [**GuPo74**]).

Notations. We will shortly write **iff** and " $\Longleftrightarrow$ " for "if and only if". **Maps** will always be assumed to be continuous functions. **Groups** will always be finite. A G**-CW-complex** is a CW-complex with a G action on it whose translations are mapping cells homeomorphically onto cells, and if $g \in G$ leaves a cell invariant then it *fixes* it. Further notations:

- $\mathbb{Z}_2$ — subgroup $\{+1, -1\}$ of the multiplicative group $(\mathbb{R}\backslash\{0\}, \cdot)$ of the reals.
- $\mathbb{F}_2$ — field with two elements, 0 and 1.
- S_n — symmetric group on n elements.
- σ^d — abstract d-dimensional simplex (the powerset of $\{0, \ldots, d\}$). We will usually denote its vertices just by the numbers $0, \ldots, d$ instead of $\{0\}, \ldots, \{d\}$.
- $\Delta_{\leq k}$ — (or "Δ_k" if no confusion) denotes the k-skeleton of an abstract or geometric simplicial complex or CW-complex Δ.
- $||\Delta||$ — the realisation of (= the topological space corresponding to) an abstract simplicial complex Δ.
- $\mathrm{sd}(K)$ — barycentric subdivision of a simplicial complex K.
- S^d — standard d-dimensional sphere $\{x \in \mathbb{R}^{d+1} | \ ||x|| = 1\}$.
- $S(Y)$ — unit sphere in an Euclidean vector space Y. The specific choice of the scalar product will be irrelevant, since we will only be interested in the topology of $S(Y)$. However, if Y is a G-space, we want this scalar product to be G-equivariant (such a scalar product

exists, by averaging an arbitrary scalar product over G.), such that $S(Y)$ is a G-invariant subspace of Y.

- Δ_{X^n} — diagonal in X^n: $\{(x, \ldots, x) \in X^n\}$.
- $U_\epsilon(X)$ — the ϵ-neighborhood of a subset X of a metric space.
- $\overline{U}_\epsilon(X)$ — denotes the closed ϵ-neighborhood of X.
- $\{*\}$ — toplogical space consting of one point.
- X^G — fixed points of X under G: $\{x \in X \mid Gx = x\}$.
- G_x — isotropy group of x: $\{g \in G \mid gx = x\}$.
- $f : X \longrightarrow_G Y$ — a G-equivariant map: $f(g \cdot x) = g \cdot f(x)$.
- $[X, Y]$ — homotopy classes of maps $X \longrightarrow Y$.
- $[X, Y]_0$ — homotopy classes of maps $X \longrightarrow Y$ in the pointed category.
- $[X, Y]_G$ — G-homotopy classes of G-maps $X \longrightarrow_G Y$.
- $H^*_G(X; M)$ — equivariant cohomology of X with coefficients in M (which cohomology depends on the chapter).
- $\mathrm{dom}(f)$ — domain of the map f.
- $\mathrm{im}(f)$ — image of the map f.
- $\ker(f)$ — kernel of the map f.
- pr_i — projection to the i'th factor: $X_1 \times \ldots \times X_n \longrightarrow X_i$.
- $\diamondsuit$ — end of proof

CHAPTER I

The Configuration Space – Test Map Method

In this chapter we describe the so called CS-TM method ([**Živ96**], [**Živ98**]), which is a general proving scheme for problems from discrete geometry. We will use it a lot in this thesis in many variations. First of all we formulate it in a general fashion to give then an easy illustrative example.

(1) Suppose we are given a problem for all of whose instances we are to show the existence of a solution. Every instance of the problem is supposed to have a natural set of candidates for a solution which we call the **configuration space**, and further a continuous (as always) **test map**, $t : X \longrightarrow Y$ measuring which candidate is a solution. That is, $x \in X$ shall be a solution iff $t(x) \in Z$, for the so called **test space** $Z \subset Y$.
(2) Assume there were a counter-example, that is, an instance of the problem, for which there is no solution. Then our test map t becomes a map $t : X \longrightarrow Y \backslash Z$.
(3) The test map usually has strong properties which are naturally inherited from the problem, such as symmetry (that is, $t : X \longrightarrow_G Y \backslash Z$ is then an equivariant map), monotonicity, differentiability, values on the boundary and so on.
(4) Deduce from these properties that no such map $t : X \longrightarrow Y \backslash Z$ exists. Hence there is no counter-example and this is what had to show.

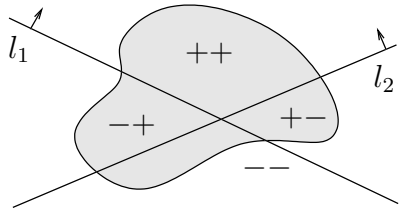
Here is an exemplary problem. We only sketch how the CS-TM method can be applied, an exact treatment will follow in Chapter II.

Problem. *Suppose we are given a mass*[1] *in the Euclidean plane* $\mathbb{R}^2$. *Show, that one can cut this mass into quarters using only two lines!*

The arrows in the following figure are showing the line orientations.

An *instance* of this problem is just a mass that we have to divide. Fix one. The space of *candidates* of a solution is just the space off all tuples of lines in $\mathbb{R}^2$, which are one-dimensional affine subspaces

[1] A concrete definition will be given in Section 1



of $\mathbb{R}^2$ together with orientations (one can also think of their defining half-planes). This is our *configuration space*, which we denote by X.

Next we want to test whether a pair of lines $(l_1, l_2) \in X$ equiparts the mass. For that to do, define a test map $t : X \to \mathbb{R}^4$ as follows. Any pair of oriented lines (l_1, l_2) defines four quadrants of $\mathbb{R}^2$. Denote the quadrants by $++$, $+-$, $-+$ and $--$ depending on whether it lies on the positive or negative side of each line. Now let $t_{++}(l_1, l_2)$ be the weight of the mass in the '$++$'-quadrant, and so on. These shall build the four components of t. Now we observe, that (l_1, l_2) forms an equipartition of the mass, iff $t(l_1, l_2)$ lies in the diagonal $\Delta := \{(x, x, x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^4$. Therefore Δ becomes our test space.

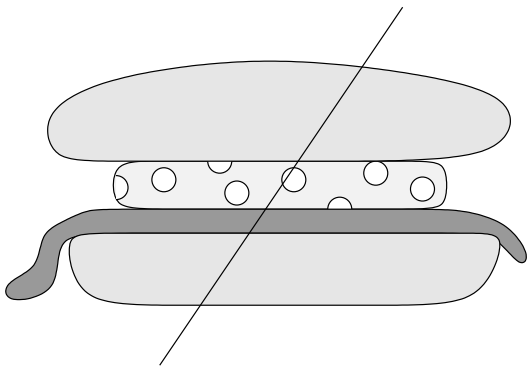
We know one very important property of our test map t, namely its symmetry: Let the group $(\mathbb{Z}_2)^2$ act on X by reversing the orientation of the lines respectively, and on $\mathbb{R}^4$ by acting on the single coordinates such that t becomes $(\mathbb{Z}_2)^2$-equivariant (that is $\varepsilon \cdot t(x) = t(\varepsilon \cdot x)$ for all $x \in X$, $\varepsilon \in (\mathbb{Z}_2)^2$).

Later we will show that such a map $t : X \longrightarrow_{(\mathbb{Z}_2)^2} \mathbb{R}^4 \backslash \Delta$ (avoiding the test space Δ!) does not exist. Therefore any test map coming from a mass has to intersect the test space, hence any mass admits an equipartition into four equal parts! And this is what the CS-TM method is all about.

CHAPTER II

The Mass Partition Problem

1. Introduction



The mass partition problem asks: For which positive natural numbers d, h and m it is possible to cut any m given masses in Euclidean d-space with h hyperplanes simultaneously into pieces, such that each of the m masses becomes bisected into 2^h equal parts. We call the triple (d, h, m) **admissible** if this equipartition works always. To make this precise, we need the following

Definition 1.1. A **mass** in $\mathbb{R}^d$ is a finite measure on the Borel σ-algebra, such that any hyperplane is a zero set.

Remarks 1.2.

- For instance we may take a measure μ defined by $\mu(A) := \lambda(A \cap M)$, where λ is the Lebesgue-measure in $\mathbb{R}^d$, and M is a measurable set of finite measure. Intuitively we are then looking for an equipartition of the *set* M.
- A more general mass may come from a density function f, which is simply a λ-integrable function $f : \mathbb{R}^d \to \mathbb{R}$. The corresponding measure μ is then $\mu(A) := \int_A f \mathrm{d}\lambda$.
- The measure should be finite, since otherwise we could not really cut them into *halves* (and quarters, and so on...).
- Hyperplanes have to be zero sets for μ, since otherwise later some problems would occur with the continuity of the test map. Imagine

this like the cake becomes bisected properly, it does not stick at the knife.

- The measure can be signed, however some proofs in later chapters will work only for unsigned measures. We will state later exactly, which arguments only work for unsigned measures.

A hyperplane is an affine subspace in $\mathbb{R}^d$ of codimension one. It can be written in the form

$$\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d = a_{d+1}\},$$

such that not all of the coefficients $a_1, \ldots, a_{d+1}$ are zero. We can normalise this coefficient vector $(a_1, \ldots, a_{d+1})$ without affecting the hyperplane. Hence we can define:

Definition 1.3. An **oriented hyperplane** H in $\mathbb{R}^d$ is an element $(a_1, \ldots, a_{d+1}) \in S^{d+1} \subset \mathbb{R}^{d+1}$. We think of H as the zero set of the function

$$\mathbb{R}^d \longrightarrow \mathbb{R} : \ (x_1, \ldots, x_d) \longmapsto a_1x_1 + \ldots + a_dx_d - a_{d+1}$$

together with the preimage orientation.[1] This can be a usual affine subspace of $\mathbb{R}^d$ of codimension one, or the empty set. H divides $\mathbb{R}^d$ into two **open half spaces**

$$H^+ := \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d > a_{d+1}\right\}$$

and

$$H^- := \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d < a_{d+1}\right\}.$$

In the extremal cases $H = (0, \ldots, 0, \pm 1)$, H^+ and H^- are $\mathbb{R}^d$ and $\emptyset$ respectively.

We say that H **bisects** the mass μ iff $\mu(H^+) = \mu(H^-)$. Similarly, hyperplanes $H_1, \ldots, H_h$ are said to be an **equipartition** of μ if all the 2^h orthants formed by them have the same measure under μ.

2. Elementary considerations

To find out, which triples (d, h, m) are admissible, it suffices to find the smallest d for a given (h, m), such that (d, h, m) is admissible, which we call

$$\Delta(h, m) = \min\{d \in \mathbb{Z}_{\geq 1} \mid (d, h, m) \text{ is admissible}\}. \tag{2.1}$$

[1] Another good way to think of it is the following. Let $\mathbb{R}^d$ sit in $\mathbb{R}^{d+1}$ as the set of vectors whose last coordinate is -1. Any $H \in S^{d+1}$ determines its orthogonal complement $H^\perp \subset \mathbb{R}^{d+1}$ together with an orientation. The affine subspace of $\mathbb{R}^d$ that we associate to H is then $\{x = (x_1, \ldots, x_d, -1) \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d - a_{d+1} = 0\} = H^\perp \cap \mathbb{R}^d$.

For if (d, h, m) is admissible then clearly $(d+1, h, m)$ is too: Just project all m given masses in $\mathbb{R}^{d+1}$ orthogonally down to $\mathbb{R}^d$ (as image measures), equipart them there, and pull these h hyperplanes in $\mathbb{R}^d$ back to hyperplanes in $\mathbb{R}^{d+1}$, which then are the desired equipartition of the original masses. For the existence of $\Delta(h, m)$ one can easily use the special case $h = 1$ inductively:

THEOREM 2.2 (Ham Sandwich). *$\Delta(1, m) = m$ for all $m \geq 1$, in particular one can bisect any m masses in $\mathbb{R}^m$ using one hyperplane.*

PROOF. $\Delta(1, m) < m$ is not possible, for if we put $d+1 \leq m$ small masses around the vertices of a d-simplex in $\mathbb{R}^d$, then no hyperplane can bisect all of them.
We will prove the other direction later several times. For a nice proof using the Borsuk-Ulam theorem see [**Mat03**, Ch. 3.1]. ∅

COROLLARY 2.3. *$\Delta(h, m) \leq 2^{h-1}m$ for all $h \geq 1$, $m \geq 1$.*

PROOF. Assume that $d = 2^{h-1}m$. Via Theorem 2.2 we can bisect all m masses simultaneously with one hyperplane. Using it again, we can bisect the $2m$ resulting masses again by another hyperplane. And so on... We are done after h steps. ∅

A bit more general, we have the following lemma ([**Had66**] and [**Ram96**] used the underlying idea to obtain new admissible triples).

LEMMA 2.4. *$\Delta(h, m) \leq \Delta(h-1, 2m)$ for all $h \geq 2$, $m \geq 1$ for all $h \geq 1$, $m \geq 1$.*

PROOF. If we are given m masses in Euclidean $\Delta(h-1, 2m)$-space, we can first bisect them using one hyperplane, since $\Delta(h-1, 2m) \geq \Delta(1, 2m) = 2m \geq m$. The resulting $2m$ masses can then be cut into equal parts using $h-1$ further hyperplanes, by the definition of $\Delta(h-1, 2m)$. ∅

Another easy inequality is the following [**Ram96**]:

LEMMA 2.5. *$\Delta(h, m) \geq m\frac{2^h-1}{h}$.*

PROOF. We have to find masses that do not admit an equipartition if the dimension is too small. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be the moment curve $t \mapsto (t, t^2, \ldots, t^d)$. Any hyperplane can intersect this curve γ in at most d points, since plugging in this curve into the hyperplane equation gives a non-zero polynomial of degree d, which has at most d real solutions. Thus h hyperplanes can intersect γ only in at most dh points.

If we put m pairwise non-intersecting intervals on the curve, and call them masses, then we want every mass to be cut in 2^h pieces, so we

need $2^h - 1$ division points on that curve for each mass. Summing them up, all h hyperplanes together have to intersect γ in at least $m(2^h - 1)$ points. Therefore, if we can find an equipartition of these m masses by h hyperplanes, then $dh \geq m(2^h - 1)$. ∅

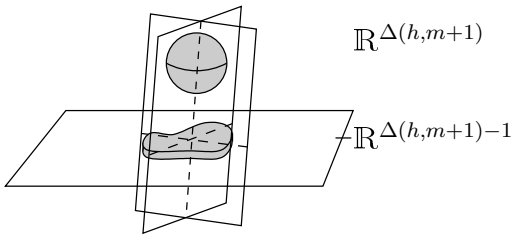
Remarks 2.6.

- A better lower bound for Δ is not known. Is there any?
- The smallest open cases are the exact values of
  - $\Delta(h = 2, m = 6) \in \{9, 10\}$,
  - $\Delta(h = 3, m = 3) \in \{7, 9\}$ and
  - $\Delta(h = 4, m = 1) \in \{4, 5\}$.

The next estimate will be again natural and easy, however it seems to be new (at least [**Had66**], [**Ram96**], [**MVZ06**] did not mention it). Indeed, all of the results of [**MVZ06**, Sect. 4] obtained by using Fadell–Husseini index theory (see Section 4), follow already from [**Ram96**] (see Section 7) with the help of the next lemma.

Lemma 2.7. $\Delta(h, m) \leq \Delta(h, m+1) - 1$. *That is, if* $(d+1, h, m+1)$ *is admissible, then so is* (d, h, m).

Proof. Assume we are given m masses in $\mathbb{R}^{\Delta(h,m+1)-1}$. Think of $\mathbb{R}^{\Delta(h,m+1)-1}$ as being embedded in $\mathbb{R}^{\Delta(h,m+1)}$ with last coordinate equal to zero. Add a ball with radius $\frac{1}{2}$ at the point $(0, \ldots, 0, 1) \in \mathbb{R}^{\Delta(h,m+1)}$ and view it as another mass. Thicken the first m masses by an $\varepsilon > 0$ into the direction of the last coordinate, such that they are now actually masses in $\mathbb{R}^{\Delta(h,m+1)}$ (recall that hyperplanes have to be zero sets for the masses, which is fulfilled by the new masses as the reader might check easily).



An example for $m = 1$ and $h = 2$.

We then find an equipartition of the $m + 1$ masses by h hyperplanes. All of the hyperplanes hit the point $(0, \ldots, 0, 1)$, therefore they intersect $\mathbb{R}^{\Delta(h,m+1)-1}$ in hyperplanes of $\mathbb{R}^{\Delta(h,m+1)-1}$. These yield an equipartition of the given m masses up to a small error which depends on the chosen ε. A limit argument finishes the proof (take a convergent *sub*sequence). ∅

Remarks 2.8.

- One might hope to strengthen this estimate to an inequality like $\Delta(h,m) \leq \Delta(h,m+x) - y$ with $y > x \geq 1$, where x and y may depend on h. For example $\Delta(h=2,m) \leq \Delta(h=2,m+2) - 3$ were a highly desirable result, since one could then show $\Delta(h=2,m)$ to be equal to $\lceil \frac{3}{2}m \rceil$.
- If the masses admit density functions (that is, they are absolutely continuous with respect to the Lebesgue measure), one can avoid the limit process by thickening the masses into the direction of the mid point of the ball [G. Ziegler, private communication]. For general masses however the thickened measures might not fulfill the requirement that hyperplanes are zero sets.

3. Test map for mass partitions

We now apply the CS-TM method. Assume we want to show a fixed triple (d,h,m) to be admissible. Assuming the contrary, we can find m masses $\mu_1,\ldots,\mu_m$ in $\mathbb{R}^d$ that do not allow for an equipartition by h hyperplanes. We will construct a function (the test map)

$$f : X \longrightarrow_{W_k} Y \backslash Z$$

for this setting.

X is the configuration space $(S^d)^h$ of h (oriented) hyperplanes in $\mathbb{R}^d$. Let $\mathbb{R}_{2^h}$ be the orthogonal complement of the all-one-vector $(1,\ldots,1)$ in $\mathbb{R}^{2^h}$. Define Y to be $(\mathbb{R}_{2^h})^m$. If we index the standard basis of $\mathbb{R}^{2^h}$ by $\{+,-\}^h$, we can define f as

$$f(H_1,\ldots,H_h) := \left(\mu_j(H_1^{\beta_1} \cap \ldots \cap H_h^{\beta_h}) - \frac{1}{2^h}\mu_j(\mathbb{R}^d)\right)_{\beta\in\{+,-\}^h}^{j\in\{1,\ldots,m\}}. \tag{3.1}$$

f is continuous by Lebesgue's dominated convergence theorem (here we need again our measures to be finite). Thanks to the correction term "$-\frac{1}{2^h}\mu_j(\mathbb{R}^d)$", f maps in fact into Y. Let $Z := \{0\} \subset (\mathbb{R}^{2^h})^m$ be the test space. This Z makes f mapping into $Y\backslash Z$ iff the masses $\mu_1,\ldots,\mu_m$ do not admit an equipartition. Thus if we can show that an equivariant map $X \longrightarrow_{W_k} Y\backslash Z$ does not exist, then we are done proving the admissibility of (d,h,m).

3.1. Equivariance. Now let's see how the group action looks like. Let $\mathbb{Z}_2$ be described as the subgroup $\{+1,-1\}$ of the multiplicative group $(\mathbb{R}\backslash\{0\},\cdot)$, and let S_h denote the symmetric group on h elements.

$\mathbb{Z}_2^h := (\mathbb{Z}_2)^h$ is acting on X as the antipodal action in each coordinate

$$(\varepsilon_1, \dots, \varepsilon_h) \cdot (x_1, \dots, x_h) := (\varepsilon_1 x_1, \dots, \varepsilon_h x_h),$$

and S_h is acting on X by interchanging coordinates

$$\pi \cdot (x_1, \dots, x_h) := (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(h)}).$$

The Weyl group $W_h = \mathbb{Z}_2^h \rtimes S_h$ merges these two group actions. It acts on X by

$$((\varepsilon_1, \dots, \varepsilon_h), \pi) \cdot (x_1, \dots, x_h) := (\varepsilon_1 x_{\pi^{-1}(1)}, \dots, \varepsilon_h x_{\pi^{-1}(h)}).$$

For this to be a left action, we have to define W_h's group operation by

$$((\varepsilon_1, \dots, \varepsilon_h), \pi) \cdot ((\delta_1, \dots, \delta_h), \tau) := ((\varepsilon_1 \delta_{\pi^{-1}(1)}, \dots, \varepsilon_h \delta_{\pi^{-1}(h)}), \pi \circ \tau).$$

W_h acts as well on $\mathbb{R}^{2^h}$ by acting on the indices

$$\begin{aligned}((\varepsilon_1, \dots, \varepsilon_h), \pi) \cdot (x_\beta)_{\beta=(\beta_1,\dots,\beta_h)\in\mathbb{Z}_2^h} &:= (x_\beta)_{(\varepsilon_1\beta_{\pi^{-1}(1)},\dots,\varepsilon_h\beta_{\pi^{-1}(h)})} \\ &= \left(x_{(\varepsilon_{\pi(1)}\beta_{\pi(1)},\dots,\varepsilon_{\pi(h)}\beta_{\pi(h)})}\right)_\beta .\end{aligned}$$

This action leaves $\mathbb{R}_{2^h}$ invariant, and taking the diagonal action we obtain an action of W_h on $Y = (\mathbb{R}_{2^h})^m$.

Exercise 3.2. *Convince yourself that under these W_h-actions on X and Y respectively, our induced test map f (see (3.1)) is indeed W_h-equivariant.*

3.2. Representation Y. We want to describe the W_k-representation Y as conveniently as possible. One could use elementary representation theory[2], but we can avoid it, since there is a nice basis of $\mathbb{R}_{2^h}$.

We define for any two elements $\alpha, \beta \in \mathbb{Z}_2$,

$$\alpha^\beta := \begin{cases} +1 & \text{if } \beta = +1, \\ \alpha \in \{-1, +1\} & \text{if } \beta = -1. \end{cases}$$

Then $\mathbb{R}^{2^h}$ has the following orthogonal basis (v_α), indexed by $\alpha \in \mathbb{Z}_2^h$,

$$v_\alpha := \left(\prod_{i=0}^{h} \alpha_i^{\beta_i}\right)_{\beta\in\mathbb{Z}_2^h},$$

where the right side states the components of this vector corresponding to the standard basis of $\mathbb{R}^{2^h}$, which we index by $\beta \in \mathbb{Z}_2^h$. One shows straight-forwardly that the v_α's are in fact pairwise orthogonal. Since

[2] $\mathbb{R}_{2^h}$ is the standard representation of the subgroup $\mathbb{Z}_2^h < W_k$, thus it splits into all $2^h - 1$ non-trivial irreducible $\mathbb{Z}_2^h$-representations since $\mathbb{Z}_2^h$ is Abelian. It then remains to check how they behave concerning to the subgroup $S_k < W_k$.

$v_{(+1,\dots,+1)}$ is the all-one-vector, the other v_α's span $\mathbb{R}_{2^h}$. Now $\mathbb{Z}_2^h$ acts on this basis by

$$\begin{aligned}(\varepsilon_1,\dots,\varepsilon_h)\cdot v_\alpha &= \left(\prod_{i=1}^h \alpha_i^{\beta_i}\right)_{(\varepsilon_1\beta_1,\dots,\varepsilon_h\beta_h)} = \left(\prod_{i=1}^h \alpha_i^{\varepsilon_i\beta_i}\right)_{(\beta_1,\dots,\beta_h)} \\ &= \prod_{i=1}^h \alpha_i^{\varepsilon_i}\cdot\left(\prod_{i=1}^h \alpha_i^{\beta_i}\right)_{(\beta_1,\dots,\beta_h)} = \prod_{i=1}^h \alpha_i^{\varepsilon_i}\cdot v_\alpha,\end{aligned} \tag{3.3}$$

while S_h acts on them by

$$\begin{aligned}\pi\cdot v_\alpha &= \left(\prod_{i=1}^h \alpha_i^{\beta_i}\right)_{(\beta_{\pi^{-1}(1)},\dots,\beta_{\pi^{-1}(h)})} = \left(\prod_{i=1}^h \alpha_i^{\beta_{\pi(i)}}\right)_{(\beta_1,\dots,\beta_h)} \\ &= \left(\prod_{i=1}^h \alpha_{\pi^{-1}(i)}^{\beta_i}\right)_{(\beta_1,\dots,\beta_h)} = v_{(\alpha_{\pi^{-1}(1)},\dots,\alpha_{\pi^{-1}(h)})} =: v_{\pi\cdot\alpha}.\end{aligned} \tag{3.4}$$

Finally, Y gets such a basis for each of its $\mathbb{R}_{2^h}$-factors. We will call these basis vectors v_α^j, for $\alpha\in\mathbb{Z}_2^h\backslash\{(+1,\dots,+1)\}$ and $j\in\{1,\dots,m\}$. The existence issues of our test map will be dealt with in the later sections.

3.3. Another test map. There is an in some respect simpler test map

$$f' : X' \longrightarrow_{W_h} Y'\backslash Z',$$

as long as we assume our masses to be *unsigned* measures. (Do not confuse the dash with a derivative of real functions. For derivatives we will always use $\mathrm{d}f$ in this thesis).

To define this test map properly, we need to add to our m measures $\mu_1,\dots,\mu_m$ a noise measure ν. By this we mean a measure ν with a density function, which is positive at each point in $\mathbb{R}^d$, and such that $\nu(\mathbb{R}^d)$ is very small. Instead of bisecting $\mu_1,\dots,\mu_m$, we will try to prove, that there is an equipartition of $\mu_1',\dots,\mu_m'$, where $\mu_i'(A) := \mu_i(A)+\nu(A)$. If we can do this for all noise measures ν, then by a compactness argument[3] we also find an equipartition of the given masses $\mu_1,\dots,\mu_m$.

We added noise, because now we have for each direction vector $v\in S^{d-1}\subset\mathbb{R}^d$ exactly one oriented hyperplane $H_v=(a_1,\dots,a_{d+1})\in S^d$

[3]The space of h hyperplanes in $\mathbb{R}^d$ is $(S^d)^h$, which is compact. Therefore, if we let ν become smaller and smaller, we get a (sub-)sequence of equipartitions of masses, which "converge" to our given masses $\mu_1,\dots,\mu_m$. Then use Lebesgue's dominated convergence theorem.

with that vector as its unique[4] oriented normal vector[5], such that H_v bisects μ_1'. This gives us a continuous[6] map $g : X' := (S^{d-1})^h \longrightarrow_{W_h} X$, sending $v \mapsto H_v$. Composing this map with f yields f'

$$f' : X' \xrightarrow{g}_{W_h} X \xrightarrow{f}_{W_h} Y.$$

As with f, we see immediately that f' has $Z' := Z = \{0\}$ in its image, iff $\mu_1', \ldots, \mu_m'$ admit an equipartition.

If we take a closer look, we see that all points in $\mathrm{im}(f')$ have the property that its coordinates concerning to the basis $\{v_\alpha^1\}$ are zero for all α which have only one "-1"-entry. That is why f' actually is a function

$$X' \longrightarrow_{W_h} Y' \backslash Z' \tag{3.5}$$

with

$$Y' := \left\{ \sum \lambda_\alpha^j v_\alpha^j \mid \lambda_\alpha^1 = 0 \text{ for all } \alpha \text{ with only one or no "-1"-entry} \right\}$$

and

$$Z' := \{0\},$$

when we assume, that $\mu_1', \ldots, \mu_m'$ do not admit an equipartition. Again, if we can show that this map does not exist, then we are done proving (d, h, m) to be admissible for unsigned measures.

This test map f' works just as well or better than f (but only for unsigned measures), since

Lemma 3.6. *If there is a map $X' \longrightarrow_{W_h} Y' \backslash Z'$, so there is as well a map $X \longrightarrow_{W_h} Y \backslash Z$.*

Whether the converse is also true is not clear. Characteristic classes and the cohomological index theory yield the same existence obstructions for both test maps.

Proof of Lemma 3.6. We identify S^d with the unreduced suspension of S^{d-1}, that is $(S^{d-1} \times I)/\sim$, where $I = [0, 1]$ is the unit interval, and $\sim$ identifies $S^{d-1} \times \{0\}$ and $S^{d-1} \times \{1\}$ to a point respectively. The antipodal action of $\mathbb{Z}_2$ on S^d becomes $(-1) \cdot [x, t] = [-x, 1 - t]$, where $(x, t) \in S^{d-1} \times I$ is any representative. Hence $X \cong_{W_h} ((S^{d-1} \times I)/\sim)^h$.

[4]This is the point where we need our original masses $\mu_1, \ldots, \mu_m$ to be unsigned

[5]The normal vector of H is simply the vector, that we obtain by normalising $(a_1, \ldots, a_d)$, which is defined for all but both degenerate hyperplanes $(0, \ldots, 0, \pm 1) \in S^d$.

[6]Again by Lebesgue's dominated convergence theorem...

By definition,

$$Y \cong Y' \oplus \operatorname{span}(\{v^1_{\alpha_i} \mid \alpha_i = (1, \ldots, \underbrace{-1}_{i\text{'th pos.}}, \ldots, 1)\}).$$

Assume, we are given a function $h' : X' \longrightarrow_{W_h} Y'$, then we can simply construct the following h out of it:

$$\begin{array}{llll} h: & ((S^{d-1} \times I)/\sim)^h & \longrightarrow_{W_h} & Y' \oplus \operatorname{span}(\{v^1_{\alpha_i} | \ldots\}) \\ & ([x_1, t_1], \ldots, [x_h, t_h]) & \longmapsto & \left(\prod_{i=1}^h t_i(1-t_i)\right) \cdot h'(x_1, \ldots, x_h) + \\ & & & \sum_{i=1}^h (t_i - \frac{1}{2}) v^1_{\alpha_i} \end{array}$$

h is

- well-defined because of the product term,
- W_h-equivariant and
- is avoiding $Z = \{0\}$ in its range as long as h' avoids $Z' = \{0\}$.

4. Applying the Fadell–Husseini index

Mani-Levitska, Vrećica and Živaljević applied the cohomological index theory to the test map (3.1) in [**MVZ06**] to obtain a very good upper bound for $\Delta(h, m)$, which is actually the best known *general* upper bound (well... nearly, see Section 7 and especially Subsection 7.1). There are only a few cases, in which better bounds are known (see as well [**MVZ06**]).

For that to do, they reduced the group action to the torus subgroup $\mathbb{Z}_2^h \subset W_h$ and used $\mathbb{F}_2$-coefficients, because then both indices are easy to calculate using the available theory. By Corollary B3.6 the index of $X = (S^d)^h$ is

$$\operatorname{Index}_{\mathbb{Z}_2^h} X = \left\langle t_1^{d+1}, \ldots, t_h^{d+1} \right\rangle \subset \mathbb{F}_2[t_1, \ldots, t_h],$$

and by Theorem B3.7 and Equation (3.4) the index of $Y \backslash Z \simeq S(Y)$ is

$$\operatorname{Index}_{\mathbb{Z}_2^h} S(Y) = \left\langle \prod_{\substack{\alpha \in \{0,1\}^h: \\ \alpha \neq (0,\ldots,0)}} (\alpha_1 t_1 + \ldots + \alpha_h t_h)^m \right\rangle \subset \mathbb{F}_2[t_1, \ldots, t_h].$$

Lemma 4.1. $\operatorname{Index}_{\mathbb{Z}_2^h} X \supset \operatorname{Index}_{\mathbb{Z}_2^h} S(Y)$ *holds, iff each of the monomials of the expanded generating polynomial of* $\operatorname{Index}_{\mathbb{Z}_2^h} S(Y)$ *contains a variable with an exponent* $\geq d + 1$.

The algebraic calculations are sketched in [**MVZ06**, Sect. 4]. They show that for a given number of masses $m = 2^q + r$ $(0 \leq r < 2^q)$ the smallest dimension d, such that the above ideal inclusion does not hold,

is $d = 2^{h+q-1}+r$. In this case, the test map cannot exist (Lemma B2.1). Therefore we get

THEOREM 4.2 ([**MVZ06**]). $\Delta(h, m = 2^q + r) \leq 2^{h+q-1} + r$.

$\not\Phi$

Now let us see what changes if we take our second test map (3.5) instead of (3.1). As above we get similar indices:

$$\mathrm{Index}_{\mathbb{Z}_2^h} X' = \left\langle t_1^d, \ldots, t_h^d \right\rangle \subset \mathbb{F}_2[t_1, \ldots, t_h],$$

and

$$\mathrm{Index}_{\mathbb{Z}_2^h} S(Y') = \left\langle \prod_{\substack{\alpha \in \{0,1\}^h: \\ \alpha \neq (0,\ldots,0)}} (\alpha_1 t_1 + \ldots + \alpha_h t_h)^{m_\alpha} \right\rangle \subset \mathbb{F}_2[t_1, \ldots, t_h],$$

where

$$m_\alpha := \begin{cases} m & \text{if } \alpha \text{ has more than one “1”-entry,} \\ m-1 & \text{if } \alpha \text{ has exactly one “1”-entry.} \end{cases}$$

As above, we have that $\mathrm{Index}_{\mathbb{Z}_2^h} X' \supset \mathrm{Index}_{\mathbb{Z}_2^h} S(Y')$ holds iff each of the monomials of the generating polynomial of $\mathrm{Index}_{\mathbb{Z}_2^h} S(Y')$ contains a variable with an exponent $\geq d$. Since the generating polynomials of $\mathrm{Index}_{\mathbb{Z}_2^h} S(Y')$ and $\mathrm{Index}_{\mathbb{Z}_2^h} S(Y)$ differ by the factor $t_1 \ldots t_h$, this characterisation shows that

$$\mathrm{Index}_{\mathbb{Z}_2^h} X' \supset \mathrm{Index}_{\mathbb{Z}_2^h} S(Y') \iff \mathrm{Index}_{\mathbb{Z}_2^h} X \supset \mathrm{Index}_{\mathbb{Z}_2^h} S(Y).$$

Hence,

COROLLARY 4.3. *Using the Fadell–Husseini index, both test maps* (3.1) *and* (3.5) *yield the same upper bound for* $\Delta(h, m)$.

5. Applying the ring structure of $H^*(\mathbb{R}P^d)$

5.1. An alternative proof of the Ham Sandwich Theorem. There is a standard proof of the Ham Sandwich Theorem (Theorem 2.2) which uses the Borsuk-Ulam Theorem [**Mat03**, Ch. 3.1]. Now we want to give a nice alternative proof for the case that all masses are *unsigned* measures using the high cup length of projective spaces:

PROOF OF THEOREM 2.2 (HAM SANDWICH). We have to show that any d masses $\mu_1, \ldots, \mu_d$ in $\mathbb{R}^d$ can be bisected by a hyperplane. For all $i \in \{1, \ldots, d\}$ and $\varepsilon \in \{+, -\}$, let

$$A_i^\varepsilon := \left\{ H \in S^d \mid \mu_i(H^\varepsilon) > \frac{1}{2}\mu_i(\mathbb{R}^d) \right\}.$$

A hyperplane $H \in S^d$ bisects the masses iff it lies in none of the A_i^ε's. Therefore we have to show that these sets do not cover S^d. Each A_i^+ is contractible, since it deformation-retracts to the degenerate hyperplane $H_{+\infty} := (0, \ldots, 0, -1) \in S^d$, which satisfies $(H_{+\infty})^+ = \mathbb{R}^d$, by moving every hyperplane $H \in A_i$ parallely to infinity, such that H^+ increases monotonically to $\mathbb{R}^d$. In the same way, each A_i^- deformation-retracts to $H_{-\infty} := (0, \ldots, 0, +1) \in S^d$, which satisfies $(H_{-\infty})^+ = \emptyset$.

Let $\widetilde{A_i}$ be the projection of A_i^+ under the natural quotient/covering map $q : S^d \to \mathbb{R}P^d$. By definition, $A_i^+ = -A_i^-$ (as sets in S^d), therefore $q^{-1}(C_i) = A_i^+ \cup A_i^-$. That is, the A_i^ε's cover S^d iff the $\widetilde{A_i}$'s cover $\mathbb{R}P^d$. Since the A_i^ε's are open, contractible and do not contain antipodal points, the $\widetilde{A_i}$'s are contractible as well[7].

Let $\alpha \in H^1(\mathbb{R}P^d; \mathbb{F}_2) \cong \mathbb{F}_2$ be the one-element. Recall that [**Hat06**, Prop. 3.38, Ex. 3.40],

$$H^*(\mathbb{R}P^d; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/_{(\alpha^{d+1})}.$$

Since $\widetilde{A_i}$ is contractible, we conclude by the long exact cohomology sequence an isomorphism $H^1(\mathbb{R}P^d; \mathbb{F}_2) \cong H^1(\mathbb{R}P^d, \widetilde{A_i}; \mathbb{F}_2)$ induced by inclusion. Consider the following diagram, which is commutative by naturality of $\cup$:

$$\begin{tikzcd}
H^1(\mathbb{R}P^d, \widetilde{A_1}; \mathbb{F}_2) \otimes \ldots \otimes H^1(\mathbb{R}P^d, \widetilde{A_d}; \mathbb{F}_2) \arrow[r, "\cup \ldots \cup"] \arrow[d, "\cong"'] & H^d(\mathbb{R}P^d, \widetilde{A_1} \cup \ldots \cup \widetilde{A_d}; \mathbb{F}_2) \arrow[d] \\
H^1(\mathbb{R}P^d; \mathbb{F}_2) \otimes \ldots \otimes H^1(\mathbb{R}P^d; \mathbb{F}_2) \arrow[r, "\cup \ldots \cup"] & H^d(\mathbb{R}P^d; \mathbb{F}_2)
\end{tikzcd}$$

The vertical maps are induced by inclusions. The bottom map sends $\alpha \otimes \ldots \otimes \alpha \mapsto \alpha^d \neq 0$. But this map factors through the three other maps. Therefore $H^d(\mathbb{R}P^d, \widetilde{A_1} \cup \ldots \cup \widetilde{A_d}; \mathbb{F}_2)$ cannot be 0, hence the $\widetilde{A_i}$'s cannot cover $\mathbb{R}P^d$. $\not\Box$

Remarks 5.1.

- Even though this proof does not use the CS-TM method, one can see a connection to the Borsuk-Ulam theorem, when one looks closer. The Borsuk-Ulam theorem is equivalent to the statement that one cannot cover S^d by $2d$ sets of the form $A_1, \ldots, A_d, (-A_1), \ldots, (-A_d)$,

[7]More precisely: If $H : I \times A_i^+ \cup A_i^- \to A_i^+ \cup A_i^-$ is the $\mathbb{Z}_2$-deformation retraction of $A_i^+ \cup A_i^-$ to $\{H_{+\infty}, H_{-\infty}\}$ as described above, then $q \circ H \circ (id_I \times (q^{-1}))$ is a deformation retraction of $\widetilde{A_i}$ to $q(H_{+\infty})$, which is continuous (for q is a local homeomorphism) and well defined (since H is a $\mathbb{Z}_2$-homotopy).

where the A_i's are closed sets and satisfy $A_i \cap (-A_i) = \emptyset$ [**Mat03**, Ex. 11*, p. 29].

- Actually here we used the more general connection between the *cup length* of a space X, which is the maximal number of elements of $H^*(X)$ in positive degrees whose product is non-zero, and the *Lyusternik-Shnirel'man category* of X, which is the minimal number of open, in X contractible sets, which cover X: The cup length is always a lower bound for the LS-category. See [**DFN90**, §19] for more details, but a slightly different definition of the LS-category.

5.2. Generalisation to the case of two hyperplanes. The idea of the previous proof can immediately be used to prove the same lower bound for $\Delta(h = 2, m)$ (2.1) (but only for unsigned masses) that we already obtained in the Theorem 4.2 which in turn was proved using the cohomological index theory of Fadell and Husseini.

THEOREM 5.2. *The smallest dimension d, such that $(d, h = 2, m)$ is admissible, is $\Delta(h = 2, m = 2^q + r) \leq 2^{q+1} + r$ (where $q \geq 0$ and $0 \leq r < 2^q$). That is, any $2^q + r$ masses in $\mathbb{R}^{2^{q+1}+r}$ can be cut simultaneously into equal quarters using two hyperplanes.*

PROOF. Suppose we are given m masses $\mu_1, \ldots, \mu_m$ in $\mathbb{R}^d$. As before, under the assumption that there were no equipartitioning pair of hyperplanes, we want to construct a contradictory covering of $(S^d)^2$. For all $i \in \{1, \ldots, d\}$, $j \in \{1, 2\}$ and $\varepsilon \in \{+, -\}$, let

$$ {}^jA_i^\varepsilon := \left\{ (H_1, H_2) \in (S^d)^2 \mid \mu_i(H_j^\varepsilon) > \frac{1}{2}\mu_i(\mathbb{R}^d) \right\}. $$

Furthermore, for all $i \in \{1, \ldots, d\}$ let

$$ \begin{aligned} B_i^+ := \{(H_1, H_2) \in (S^d)^2 \mid \; & \mu_i(H_1^+ \cap H_2^+) + \mu_i(H_1^- \cap H_2^-) > \\ & \mu_i(H_1^+ \cap H_2^-) + \mu_i(H_1^- \cap H_2^+)\} \end{aligned} $$

and

$$ \begin{aligned} B_i^- := \{(H_1, H_2) \in (S^d)^2 \mid \; & \mu_i(H_1^+ \cap H_2^+) + \mu_i(H_1^- \cap H_2^-) < \\ & \mu_i(H_1^+ \cap H_2^-) + \mu_i(H_1^- \cap H_2^+)\}. \end{aligned} $$

If a pair of hyperplanes $(H_1, H_2) \in (S^d)^2$ equiparts all masses μ_i, then is does not lie in any of these A's and B's. Conversely, if (H_1, H_2) does not lie in any of the A's and B's, then both H_1 and H_2 bisect all masses (because of the A's), and together with $\mu_i(H_1^+ \cap H_2^+) + \mu_i(H_1^- \cap H_2^-) = \mu_i(H_1^+ \cap H_2^-) + \mu_i(H_1^- \cap H_2^+)$ (because of the B's) it follows that every mass μ_i becomes equiparted. Therefore we have to show that the A's and B's do not cover $(S^d)^2$.

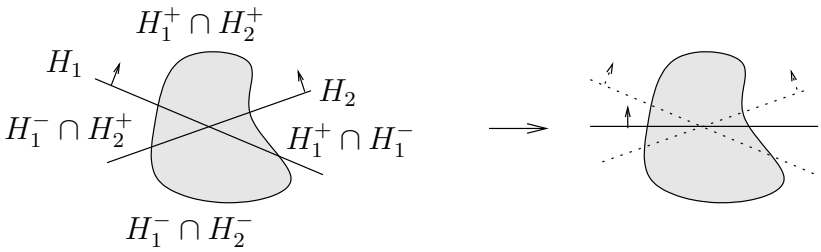
Using the same A_i^ε and $H_{\pm\infty}$ as in the previous proof, we get that

$$^1A_i^\varepsilon = A_i^\varepsilon \times S^d \simeq \{H_{\varepsilon\cdot\infty}\} \times S^d$$

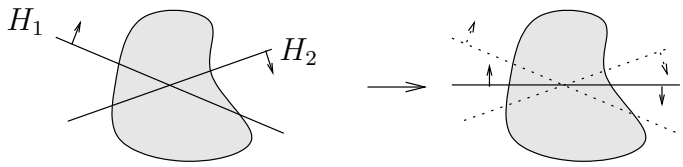
and

$$^2A_i^\varepsilon = S^d \times A_i^\varepsilon \simeq S^d \times \{H_{\varepsilon\cdot\infty}\}.$$

B_i^+ instead deformation-retracts to the diagonal $\Delta_{(S^d)^2} = \{(x,x) \in (S^d)^2\}$ by rotating the two hyperplanes of the pair $(H_1, H_2) \in (S^d)^2$ around their $(d-2)$-dimensional intersection *away from each other* until they become equal (see the following picture).



This can be done naturally enough, such that the resulting homotopy is in fact continuous (for this to work we have to note that no antipodal pair $(H, -H)$ is in B_i^+, and during the deformation retraction the pairs (H_1, H_2) stay in B_i^+. Both follows from the definition). Similarly B_i^- deformation-retracts to the *anti-diagonal* $\Delta'_{(S^d)^2} = \{(x,-x) \in (S^d)^2\}$ by rotating H_1 and H_2 as above *against each other*, such that they finally become their negatives.



Now let $q^2 : (S^d)^2 \longrightarrow (\mathbb{R}P^d)^2$ be the quotient/covering projection. For all $i \in \{1, \ldots, m\}$ and $j \in \{1, 2\}$, let $\widetilde{{}^jA_i} := q^2({}^jA_i^+)$ and $\widetilde{B_i} := q^2(B_i^+)$. We have that ${}^jA_i^+ \cap {}^jA_i^- = \emptyset$ and $B_i^+ \cap B_i^- = \emptyset$, all these set are open and the deformation retraction is symmetric. Therefore (as in the previous proof) $\widetilde{{}^1A_i}$ deformation-retracts to $\{*\} \times \mathbb{R}P^d$, $\widetilde{{}^1A_i}$ to $\mathbb{R}P^d \times \{*\}$, and $\widetilde{B_i}$ to $\Delta_{(\mathbb{R}P^d)^2} := \{(x,x) \in (\mathbb{R}P^d)^2\}$.

Now we will calculate all necessary cohomology groups. Everything will be done with $\mathbb{F}_2$-coefficients so we will omit that in our notations. By Künneth,

$$H^*((\mathbb{R}P^d)^2) \xleftarrow{\cong} H^*(\mathbb{R}P^d) \otimes H^*(\mathbb{R}P^d) \cong \mathbb{F}_2[\alpha, \beta]/_{(\alpha^{d+1}, \beta^{d+1})}$$

where the first isomorphism is induced by the two projections, and α corresponds to the first and β to the second factor. Consider the composition

$$H^*(\mathbb{R}P^d) \xrightarrow{pr_{1/2}^*} H^*((\mathbb{R}P^d)^2) \xrightarrow{i_2^*} H^*(\{*\} \times \mathbb{R}P^d),$$

where second map is induced by inclusion $i_2 : \{*\} \times \mathbb{R}P^d \longrightarrow (\mathbb{R}P)^d)$. If the first map is the one induced by projecting to the second factor, the whole composition is induced by the identity, therefore $i_2^*(\beta) = \beta$. If the first map is induced by projecting to the first factor, then the whole composition is induced by the constant map, therefore $i_2^*(\alpha) = 0$. The long exact sequence

$$\begin{aligned} H^*((\mathbb{R}P^d)^2) &\xrightarrow{\text{surj.}} H^*(\{*\} \times \mathbb{R}P^d) \xrightarrow{0} H^*((\mathbb{R}P^d)^2, \{*\} \times \mathbb{R}P^d) \\ &\xrightarrow{\text{inj.}} H^*((\mathbb{R}P^d)^2) \longrightarrow H^*(\{*\} \times \mathbb{R}P^d) \end{aligned}$$

has therefore a surjective first map, hence the second is 0, hence the next one is injective. The last one maps $\alpha \mapsto 0$ and $\beta \mapsto \beta$, therefore its kernel (= image of the injective map) is $\langle \alpha \rangle$, the ideal generated by α. Especially α is in the image. Analogously, β is in the image of the map

$$H^*((\mathbb{R}P^d)^2, \mathbb{R}P^d \times \{*\}) \longrightarrow H^*((\mathbb{R}P^d)^2).$$

Let us come to $\Delta := \Delta_{(\mathbb{R}P^d)^2} \xleftarrow{\cong} \mathbb{R}P^d$, where the right homeomorphism is given by $x \mapsto (x, x)$. Denote Δ's cohomology by $H^*(\Delta) = \mathbb{F}_2[\gamma]/_{(\gamma^{d+1})}$. Consider the composition

$$H^*(\mathbb{R}P^d) \xrightarrow{pr_{1/2}^*} H^*((\mathbb{R}P^d)^2) \xrightarrow{i_\Delta^*} H^*(\Delta) \xrightarrow{(x \mapsto (x,x))^*} H^*(\mathbb{R}P^d),$$

where the second map is again induced by inclusion. If the first map is pr_1^*, then the whole map is the one induced by the identity, therefore i_Δ^* maps $\alpha \mapsto \gamma$. Same is true if the first map is pr_2^*, hence i_Δ^* maps $\beta \mapsto \gamma$. Therefore we get an analogous long exact sequence

$$\begin{aligned} H^*((\mathbb{R}P^d)^2) &\xrightarrow{\text{surj.}} H^*(\Delta) \xrightarrow{0} H^*((\mathbb{R}P^d)^2, \Delta) \\ &\xrightarrow{\text{inj.}} H^*((\mathbb{R}P^d)^2) \longrightarrow H^*(\Delta), \end{aligned}$$

where now the kernel of the last map (= image of the injective map) is the ideal of all polynomials that have in each degree an even number of monomials (In fact, it is simply $\langle \alpha + \beta \rangle$). Especially $\alpha + \beta$ is in the

image of the injective map. Now consider the commutative diagram

$$\begin{array}{ccc}
\bigotimes_i H^*((\mathbb{R}P^d)^2, \widetilde{{}^1A_i}) \otimes H^*((\mathbb{R}P^d)^2, \widetilde{{}^2A_i}) \otimes H^*((\mathbb{R}P^d)^2, \widetilde{B_i}) & \xrightarrow{\cup} & H^*((\mathbb{R}P^d)^2, X) \\
\downarrow & & \downarrow \\
\bigotimes_i H^*((\mathbb{R}P^d)^2) \otimes H^*((\mathbb{R}P^d)^2) \otimes H^*((\mathbb{R}P^d)^2) & \xrightarrow{\cup} & H^*((\mathbb{R}P^d)^2),
\end{array}$$

where the vertical maps are induced by inclusion and X is the union of all $\widetilde{A}$'s and $\widetilde{B}$'s. If we insert at the bottom left $\bigotimes_i \alpha \otimes \beta \otimes (\alpha + \beta)$, then this has a preimage under the left vertical map. Therefore, if X were all of $(\mathbb{R}P^d)^2$, then $H^*((\mathbb{R}P^d)^2, X) = 0$ and $(\alpha\beta(\alpha + \beta))^m$ had to be zero in $H^*((\mathbb{R}P^d)^2)$, that is, $\alpha^m\beta^m(\alpha + \beta)^m$ must not contain a monomial $\alpha^i\beta^j$ such that $i, j \leq d$. This is apparently the same criterion that the Fadell–Husseini index gave us in Section 4 ($\langle t_1^m t_2^m (t_1 + t_2)^m \rangle \subset \langle t_1^{d+1}, t_2^{d+1} \rangle$ in $\mathbb{F}_2[t_1, t_2]$). Thus we get the same bound for $\Delta(h = 2, m)$ as in Theorem 4.2. ∅

6. Applying characteristic classes

In this section we show how characteristic classes can be applied to prove some triples (d, h, m) to be admissible. Assume for a contradiction (d, h, m) to be not admissible, hence we get a test map $f : X \longrightarrow_G Y \backslash \{0\}$ (3.1). To simplify calculations, that is to make them possible, we restrict the group of symmetry to $\mathbb{Z}_2^h$. Even though this approach can be seen in advance to work equally well as the Fadell–Husseini index method (see Section B4), the proof is probably more elementary, which hopefully justifies this section (in fact I first found this proof before seeing both approaches to be equivalent).

Our test map $f : X \longrightarrow_G Y \backslash \{0\}$ corresponds bijectively to the nowhere vanishing cross section

$$s : X/G \longrightarrow X \times_G Y : \ [x] \mapsto [x, f(x)]$$

of the vector bundle

$$p : X \times_G Y \longrightarrow X/G : \ [x, y] \mapsto [x].$$

See Section A4 for more details about this correspondence. Recall from Subsection 3.2, that Y has a basis

$$\left\{ v_\alpha^j \mid \alpha \in \mathbb{Z}_2^h \backslash \{(+1, \ldots, +1)\} \text{ and } j \in \{1, \ldots, m\} \right\}.$$

We write $Y = \bigoplus_{\alpha,j} V_\alpha^j$, where $V_\alpha^j := \mathbb{R} \cdot v_\alpha^j$ is the one-dimensional subspace of Y spanned by v_α^j. Let

$$p_\alpha^j := p|_{X \times_G V_\alpha^j} : \ X \times_G V_\alpha^j \longrightarrow X/G$$

be one-dimensional sub bundles of p. Their Whitney sum obviously yields $\bigoplus_{\alpha,j} p^j_\alpha = p$. So once we calculate the Stiefel–Whitney classes $\omega_1(p^j_\alpha)$, we get $\omega_n(p)$ by the Whitney sum formula [**MiSt74**, §4]:

$$\omega_n(p) = \prod_{\alpha,j} \omega_1(p^j_\alpha). \tag{6.1}$$

Now, $X/G = (\mathbb{R}P^d)^h$, hence

$$H^*(X/G;\mathbb{F}_2) = \mathbb{F}_2[x_1,\ldots,x_h]/_{\left(x_1^{d+1},\ldots,x_h^{d+1}\right)}.$$

Let

$$i_k:\ \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^d \hookrightarrow (\mathbb{R}P^d)^h:\ x \mapsto (\{*\},\ldots,\underbrace{x}_{k^{\text{th}}},\ldots,\{*\}),$$

be the k'th inclusion, where $\{*\} \in \mathbb{R}P^d$ is a base point. As in the previous section, it induces in cohomology,

$$i_k^*:\ H^1(X/G;\mathbb{F}_2) \longrightarrow H^1(\mathbb{R}P^1;\mathbb{F}_2) = \mathbb{F}_2:\ \lambda_1 x_1 + \ldots + \lambda_h x_h \longmapsto \lambda_k. \tag{6.2}$$

The following diagram shows a vector bundle morphism for all $k \in \{1,\ldots,h\}$,

$$\begin{tikzcd}
S^1 \times_{\mathbb{Z}_2} V_{\alpha_k} \arrow[r] \arrow[d, "pr_1/_G"'] \arrow[dr, phantom, "=: q^k_\alpha"] & X \times_G V^j_\alpha \arrow[d, "p^j_\alpha"] \\
\mathbb{R}P^1 \arrow[r, "i_j"] & X/G,
\end{tikzcd}$$

where V_{α_k} denotes the one-dimensional $\mathbb{Z}_2$-representation given by $(-1)\cdot x := \alpha_k \cdot x$. The bundle on the left is therefore the trivial bundle or the Möbius bundle over $\mathbb{R}P^1 = S^1$, depending on whether α_k is $+1$ or -1. Hence its first Stiefel–Whitney class is

$$\omega_1(q^k_\alpha) = \begin{cases} 0, & \text{if } \alpha_k = +1, \\ 1, & \text{if } \alpha_k = -1. \end{cases}$$

By $i_k^*(\omega_1(p^j_\alpha)) = \omega_1(q^k_\alpha)$ and (6.2),

$$\omega_1(p^j_\alpha) = \omega_1(q^1_\alpha)x_1 + \ldots + \omega_1(q^h_\alpha)x_h.$$

Hence (6.1) gives,

$$\omega_n(p) = \prod_{\substack{a\in\{0,1\}^h:\\ a\neq(0,\ldots,0)}} (a_1t_1 + \ldots + a_ht_h)^m \in \mathbb{F}_2[x_1,\ldots,x_h]/_{\left(x_1^{d+1},\ldots,x_h^{d+1}\right)}.$$

If $\omega_n(p)$ does not vanish, we get a contradiction to the existence of the section s of the bundle p (Proposition A5.1), and in this case we have

shown (d, h, m) to be admissible. Comparing this with Lemma 4.1, $\omega_n(p) \neq 0$ happens iff $\mathrm{Index}_{\mathbb{Z}_2^h} X \supset \mathrm{Index}_{\mathbb{Z}_2^h} S(Y)$. Therefore we got the same bound for $\Delta(h, m)$ as the Fadell–Husseini index.

7. Notes on Ramos' results

It is interesting to study E. Ramos' results [**Ram96**] on the mass partition problem, since together with Lemmas 2.7 and 2.4 they yield all explicitly known results for $\Delta(h, m)$, except for one, namely $\Delta(h = 2, m = 5) = 8$ [**MVZ06**]. Here, Ramos gives "only" $\Delta(h = 2, m = 5) \in \{8, 9\}$.

We can even deduce *new* bounds (Corollary 7.3). However, I was not able to check all of Ramos' results, since one needs a fast algorithm that computes a modified permanent mod 2, but I could not find one. More about this in Subsection 7.1.

The basic theorem that he used is a Borsuk–Ulam type theorem, which he proved elementarily, which is very interesting in its own. If one translates it into other terms, one can prove it quickly using characteristic classes, so we do this here:

Let $A = (a_{ij}) \in \mathbb{F}_2^{\ell \times k}$ and suppose

$$f : S^{n_1} \times \ldots \times S^{n_k} \longrightarrow_{\mathbb{Z}_2^k} \mathbb{R}^\ell$$

is an equivariant map, where $\ell = \sum n_i$, $\mathbb{Z}_2^k$ acts on $S^{n_1} \times \ldots \times S^{n_k}$ as usual, and on $\mathbb{R}^\ell = \mathrm{span}(v_1, \ldots, v_\ell)$ by

$$\varepsilon_j \cdot v_i = (-1)^{a_{ij}} \cdot v_i,$$

where ε_j is the generator of the j'th $\mathbb{Z}_2$-factor of $\mathbb{Z}_2^k$ and the product on the left is the group multiplication and the product on the right scalar multiplication. Let $\mathrm{perm}'(A) \in \mathbb{F}_2$ be the coefficient of $t_1^{n_1} \ldots t_k^{n_k}$ in the polynomial

$$\prod_{i=1}^{\ell} (a_{i1} t_1 + \ldots + a_{ik} t_k) .$$

THEOREM 7.1 (Theorem 3.1 in [**Ram96**]). *If* $\mathrm{perm}'(A) = 1$ *then* f *has a zero.*

Proof. Let $V_i := \operatorname{span}(v_i) \subset \mathbb{R}^l$ denote the one-dimensional irreducible subrepresentations of $\mathbb{R}^\ell$. Let

$$\begin{array}{c} (S^{n_1} \times \ldots \times S^{n_k}) \times_{\mathbb{Z}_2^k} V_i \\ \Big\downarrow p_i \\ \mathbb{R}P^{n_1} \times \ldots \times \mathbb{R}P^{n_k} \end{array}$$

be the line bundles induced by projecting to the first coordinate. As in Section 6 we see that the first Stiefel–Whitney class of p_i is

$$\omega_1(p_i) = \sum_{j=1}^{k} a_{ij} t_j \in \mathbb{F}_2[t_1, \ldots, t_k]/_{(t_1^{n_1+1}, \ldots, t_k^{n_k+1})}.$$

The Whitney sum $p := \sum_{i=1}^{\ell} p_i$ satisfies $\omega_\ell(p) = \prod \omega_1(p_i)$, which is non-zero since the coefficient of $t_1^{n_1} \ldots t_k^{n_k}$ is 1 by the assumption (actually this is the only possible non-zero coefficient in $\omega_1(p)$). Therefore, p does not admit a nowhere vanishing section (see Proposition A5.1) and hence f has a zero (see Appendix A4) ∅

Applying this to the mass partition problem one gets the following results [**Ram96**, p. 156] by restricting the test map (3.1) to smaller domains (but still products of spheres):

- $\Delta(h = 2, m = 3) = 5$,
- $\Delta(h = 3, m = 3) \in \{7, 8, 9\}$,
- $\Delta(h = 2, m = 5) \in \{8, 9\}$,

However this follows already from his further results and application of Lemma 2.7. The next theorem collects his further results, which rely on clever formula manipulations, and translates them into an easier language (in my point of view):

Theorem 7.2 (Around Theorem 6.5 in [**Ram96**]). *Let $m \geq 2$ and $h \geq 1$. Suppose we can find numbers $s \leq 1$, $m_i \geq 1$ and $t_i \geq 1$ for all $i \in \{1, \ldots, s\}$, such that*

- $\sum_{i=1}^{s} t_i = h$,
- $\sum_{i=1}^{s} m_i t_i = m(2^h - 1) - h$, *and*
- *the coefficient of* $(x_1 \ldots x_{t_1})^{m_1} \cdot \ldots \cdot (x_{h-(t_s-1)} \ldots x_{t_h})^{m_s}$ *is equal to 1 in the polynomial*

$$\prod_{\substack{\alpha \in \{0,1\}^h: \\ \alpha \neq (0,\ldots,0)}} (\alpha_1 x_1 + \ldots + \alpha_h x_h) \cdot \prod_{\substack{\alpha \in \{0,1\}^h: \\ \alpha \text{ is not special}}} (\alpha_1 x_1 + \ldots + \alpha_h x_h) \in \mathbb{F}_2[x_1, \ldots, x_h],$$

where α is called special if it has only one or no 1-entry, or if it is of the form $0^{t_1}\dots 0^{t_{p-1}}(1^q 0^{t_p-q})0^{t_{p+1}}\dots 0^{t_s}$ for some $p \in \{1,\dots,s\}, q \in \{0,\dots,t_p\}$.

Then $\Delta(h, 2^x m) \le 2^x(1+\max(m_i))$ for all $x \ge 0$.

Ramos found matching numbers s, m_i's and t_i's for $m = 2$ and $h \in \{1,\dots,5\}$. We want to list them ($x \ge 0$):

h=1,	(m_1,t_1)=(1,1)	$\Rightarrow$	$\Delta(1,2^{x+1})\le 2^x\cdot 2$
h=2,	(m_1,t_1)=(2,2)	$\Rightarrow$	$\Delta(2,2^{x+1})\le 2^x\cdot 3$
h=3,	(m_1,t_1)=(4,2),(m_2,t_2)=(3,1)	$\Rightarrow$	$\Delta(3,2^{x+1})\le 2^x\cdot 5$
h=4,	(m_1,t_1)=(8,2),(m_2,t_2)=(5,2)	$\Rightarrow$	$\Delta(4,2^{x+1})\le 2^x\cdot 9$
h=5,	(m_1,t_1)=(14,3),(m_2,t_2)=(11,1),(m_3,t_3)=(4,1)	$\Rightarrow$	$\Delta(5,2^{x+1})\le 2^x\cdot 15$

7.1. How to verify this. Ramos unfortunately did not state how he got these numbers, only that he could not manage to find explicit formulas. The best computer algorithm that I know is in the last case $h = 5$ too slow. The first product term is a Dickson polynomial and simplifies to

$$\prod_{\substack{\alpha\in\{0,1\}^h:\\ \alpha\neq(0,\dots,0)}} (\alpha_1 x_1 + \dots + \alpha_h x_h) = \sum_{\sigma\in S_h} x_{\sigma_1} x_{\sigma_2}^2 \dots x_{\sigma_h}^{2^{h-1}},$$

which makes $h!$ summands. But the second product term seems hard to handle with (it is the quotient of the Dickson polynomial by all non-zero terms coming from special α's, but the division is still not manageable fast enough). Also simplifying tricks that work to compute determinants efficiently do not apply, as it seems, even though we are working over $\mathbb{F}_2$.

That is, I do not know how to verify this. Assuming that it is true, let's look at the result for $h = 5$, which seems to break ranks, because the factor 15 is not only unequal to the expected 17, but also it is smaller than 2^{h-1}, such that it gives together with Lemma 2.7 and 2.4 for *all* $h \ge 5$ and $m \ge 2$ better results than the Fadell–Husseini index bound (Theorem 4.2):

Corollary 7.3 (of Ramos upper bound on $\Delta(5, 2^{x+1})$). *Let $h \ge 5$ and $m \ge 2$ and write $m = 2^q + r$ ($0 < r \le 2^q$) (Note that the bounds on r differ from the formula in Theorem 4.2). Then*

$$\Delta(h, m = 2^q + r) \le 2^{h-5}(7 \cdot 2^q + r).$$

$\not\Phi$

For $h \leq 4$ it gives only new results for $m = 2^{x+1}$, and together with Lemma 2.7 it gives the same bounds on the other m's as the Fadell-Husseini index, which sounds reasonable.

It might be interesting to run his algorithm again to check the case $h = 6$, since computers are faster today.

8. A promising ansatz using bordism theory

In this section we want to see how the maybe strongest topological approach using bordism theory works, and an idea in which way it might be computable. However, the idea is still under way.

We will need our masses to be of non-zero total weight, that is $\mu_i(\mathbb{R}^d) \neq 0$ for all i. Without loss of generality we can assume $\mu_i(\mathbb{R}^d) > 0$. We want to show the admissibility of (d, h, m), so suppose we are given m masses in $\mathbb{R}^d$. This gives us a test map (3.1) $f : X \longrightarrow_{S_h} Y$ (we could take the other test map (3.5) as well) and maybe it hits the test space $Z = \{0\}$. Let the non-free part of X be denoted by $X_{nf} := \{x \in X \mid G_x \neq 0\}$, and let $L := f^{-1}(Z)$ be the space of all solutions of the partitioning problem. Note that $X_{nf} \cap L = \emptyset$ (this requires our masses to have a non-zero total weight, since we do not want to allow equipartitions to contain two up to orientation equal hyperplanes), therefore we can make f by a small G-homotopy transversal to Z, such that L becomes a G-submanifold of X which stays away from X_{nf} since L is compact. A G-homotopy of f, which never lets X_{nf} hit Z, produces a G-bordism between the two solution sets L. So L is only given up to bordism class in a special bordism group that we will describe below. Two different configurations of m masses (all masses are assumed to be of positive total weight) give rise to exactly such a G-homotopy: Just take the linear homotopy between the two corresponding test maps f_1 and f_2! So every triple (d, h, m) gives us a unique bordism class $[L]$ in $X \backslash X_{nf}$, where our allowed bordisms have a special structure:

All of our manifolds X, Y and Z are oriented, therefore L is oriented as well using the preimage orientation. Let $\omega_X : G \longrightarrow \mathbb{Z}_2$ be the orientation character of X, that is

$$\omega_X(g) := \begin{cases} +1, & \text{if } g \text{ acts orientation preserving on } X, \\ -1, & \text{if } g \text{ acts orientation reversing on } X. \end{cases}$$

Similarly define ω_Y, ω_Z and ω_L. By definition of the preimage orientation,

$$\omega_L = \omega_X \cdot \omega_Y \cdot \omega_Z.$$

Definition 8.1. Let X be a G-space and $\omega : G \longrightarrow \mathbb{Z}_2$ some homomorphism. A **singular manifold** (M, f) in our bordism theory shall be an orientable free G-manifold M with orientation character ω together with a G-map $m : M \longrightarrow X$. Two of them, (M_1, m_1) and (M_2, m_2), are called **bordant** iff there is a third orientable free G-manifold with boundary and with orientation character ω mapping into X, such that its boundary is the union of M_1 and M_2 and the boundary map restricts to m_1 and m_2 respectively. The singular manifolds modulo the relation of being bordant defines our bordism group $\Omega_*^{G,\omega}(X)$, which is graded by dimension.

Therefore our $[L]$ lives in $\Omega_*^{G,\omega_X\omega_Y\omega_Z}(X\backslash X_{nf})$.

Remark 8.2. The reason why we want to use such a bordism theory is that it might give some new information. A simpler approach using *unoriented bordism does not yield any new upper bound for* $\Delta(h = 2, m)$, as I calculated. (It is not an attractive calculation, therefore we omit it. One can use the Thom-homomorphism and the key calculations can be found in [**Fed67**].) I also calculated some interesting examples where ω_L is the trivial character and L zero-dimensional. There, $\Omega_0^{G,\omega}(X\backslash X_{nf}) \cong \mathbb{Z}$, but unfortunately $[L]$ has always been zero.

Now, how to calculate $[L]$? Our idea now will be to use a product structure of $\bigoplus_\omega \Omega_*^{G,\omega}$! There is one problem (but which can be dealt with), which is that our X has a non-empty non-free part, but let's concern about this later. The background is that (Y, Z) has a nice and natural decomposition as a product of pairs of spaces, where we define the product in an unusual way: $(Y_1, Z_1) \times (Y_2, Z_2) := (Y_1 \times Y_2, Z_1 \times Z_2)$. Now, if $(Y, Z) = (Y_1, Z_1) \times (Y_2, Z_2)$ and f decomposes as (f_1, f_2) where $f_i : X \longrightarrow Y_i$, let $L_i := f_i^{-1}(Z_i)$, then apparently $L = L_1 \cap L_2$. In fact, the "$\cap$" gives us a product structure on $\bigoplus_\omega \Omega_*^{G,\omega}$ as long as we do everything transversally:

Take two bordism classes $[M_1^{n_1}, m_1] \in \Omega_{n_1}^{G,\omega_1}(X)$ and $[M_2^{n_2}, m_2] \in \Omega_{n_2}^{G,\omega_2}(X)$ and suppose our X^n is an n-dimensional compact orientable G-manifold with orientation character ω_X. Then define the **intersection product** by making m_1 and m_2 first of all equivariantly transversal to each other, take the preimage $m_1^{-1}(m_2(M_2))$ of the image of m_2 under m_1 and view it as a singular manifold of X by restricting m_1 to this submanifold of M_1. One can easily show that it is now a well-defined bordism element

$$[M_1, m_1] \bullet [M_2, m_2] \in \Omega_{n_1+n_2-n}^{G,\omega_1\omega_2\omega_X}(X).$$

But still, how does this help us? Well, we can restrict to equivariant homology by an equivariant Thom-homomorphism:

$$\tau : \Omega_*^{G,\omega} \longrightarrow H_*^G(X;\mathbb{Z}_\omega),$$

where $\mathbb{Z}_\omega$ is just the G-module $\mathbb{Z}$ together with the group action $g \cdot z := \omega(g) \cdot z$, where here we view $\omega(g)$ as $\pm 1 \in \mathbb{Z}$. In $\bigoplus_\omega H_*^G(X;\mathbb{Z}_\omega)$ we also have an intersection product, which we can simplest define using the cup product in cohomology. We therefore need equivariant Poincaré duality:

$$D^{-1} : H_*^G(X;\mathbb{Z}_\omega) \xrightarrow{\cong} H_G^*(X;\mathbb{Z}_\omega \otimes \mathbb{Z}_{\omega_X}).$$

Note that $\mathbb{Z}_\omega \otimes \mathbb{Z}_{\omega_X} \cong \mathbb{Z}_{\omega\cdot\omega_X}$. In equivariant cohomology we have a cup product. If X is free, then the equivariant cohomology is equivalent to the cohomology of X/G with the to $\mathbb{Z}_{\omega\cdot\omega_X}$ corresponding local coefficients. However, this is in general *not* a local coefficient system of *rings*, that is, we can still multiply, but we arrive in a different local coefficient system! Instead, in our situation we have the following cup product:

$$\cup : H_G^{k_1}(X, A_1;\mathbb{Z}_{\omega_1}) \otimes H_G^{k_2}(X, A_2;\mathbb{Z}_{\omega_2}) \longrightarrow H_G^{k_1+k_2}(X, A_1 \cup A_2;\mathbb{Z}_{\omega_1\cdot\omega_2}).$$

Pulling this back via Poincaré duality to H_*^G, we get there our desired intersection product, also denoted by "$\bullet$". Now, as in [**Bre93**, Sect. VI.11] one can show that the following diagram commutes (recall that $\dim X = n$):

$$\begin{tikzcd}
\Omega_{n-k_1}^{G,\omega_1}(X) \otimes \Omega_{n-k_2}^{G,\omega_2}(X) \arrow[r, "\bullet"] \arrow[d, "\tau\otimes\tau"'] & \Omega_{n-k_1-k_2}^{G,\omega_1\omega_2\omega_X}(X) \arrow[d, "\tau"'] \\
H_{n-k_1}^G(X;\mathbb{Z}_{\omega_1}) \otimes H_{n-k_2}^G(X;\mathbb{Z}_{\omega_2}) \arrow[r, "\bullet"] & H_{n-k_1-k_2}^G(X;\mathbb{Z}_{\omega_1\omega_2\omega_X}) \\
H_G^{k_1}(X;\mathbb{Z}_{\omega_1\omega_X}) \otimes H_G^{k_2}(X;\mathbb{Z}_{\omega_2\omega_X}) \arrow[r, "\cup"] \arrow[u, "D\otimes D"] & H_G^{k_1+k_2}(X;\mathbb{Z}_{\omega_1\omega_2}). \arrow[u, "D"]
\end{tikzcd}$$

8.1. Limits of topological methods. Grünbaum originally asked in [**Grü60**]: Given any dimension $d \geq 1$, is $(d, h = d, m = 1)$ admissible? The answer is "yes" for $d \leq 3$ and "no" for $d \geq 5$, but it is still an open problem for $d = 4$. Rade Živaljević has shown in [**Živ04**] that the test map

$$f : (S^4)^4 \longrightarrow_{\mathbb{Z}_2^4 \rtimes S_4} Y$$

exists, with the additional property, that f restricted to the non-free part of $(S^4)^4$ comes from an arbitrary non-zero *unsigned* mass in $\mathbb{R}^4$ (he states a weaker version, see his Theorem 5.9, his proof however generalises easily to the statement here by deforming the free part $(S^4)^4_\delta$ to a compact manifold with boundary and using relative Koschorke theory [**Kos81**]).

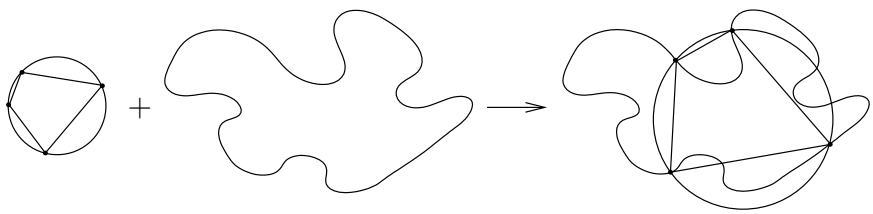
That is he showed that a pure topological approach does not work, at least not using this test map without more geometric ideas and provided that $(d = 4, h = 4, m = 1)$ is in fact admissible. Or does there exist a counter-example?

CHAPTER III

Inscribed Polygons and Tetrahedra

1. Introduction

To introduce this chapter we state an exemplary and in most instances open problem.



Let $V = \square ABCD$ be an arbitrary non-degenerate chordal quadrilateral in $\mathbb{R}^2$ (chordal means that the quadrilateral has to have a circumcircle) and consider an arbitrary plane injective closed curve $\gamma : S^1 \to \mathbb{R}^2$.

Question 1.1. *For each such pair (V, γ), is it possible to find an orientation preserving Euclidean transformation that maps all four points A, B, C and D into the image of γ?*

In this case we say that there is a quadrilateral similar (by which we mean orientation preserving similarities) to V **inscribed in** that curve, or the curve **contains** or **grips** a quadrilateral similar to V. In the case that V equals a square, this problem is known as the *Square Peg Problem*.

This problem is known to hold true for some special cases:

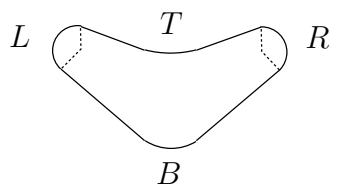
(1) When V is a square, and γ is smooth enough.
Shnirel'man [**Shn44**] proved this for piecewise analytic Jordan curves, and Stromquist [**Str89**] for locally monotone curves (that is, locally the scalar product of the curve with any non-zero vector is monotonically increasing; this is the case e. g. for C^1-curves without cusps). As far as I know, the latter is the strongest result for squares V that has been established so far.

(2) When V is any circular non-degenerate quadrilateral and γ is a C^4-generic smooth oval with four vertices[1] (see [**Mak05**]). Makeev in fact dealt with circular pentagons $ABCDE$. There are four similarities mapping $ABCD$ on the curve γ, while two of them map E outside of γ, and the other two map E inside of it.

There are also some negative results:

(1) If V is not a trapezoid, then one can find a very thin isosceles triangle (basis $\gg$ height), which does *not* grip V. If a trapezoid is not isosceles, then it cannot be inscribed into a circle. That is why one needs to put extra conditions on the curve like smoothness to be able to answer the above Question 1.1 with yes. Or one can simply restrict it to isosceles trapezoids, since counter-examples for them are not known, yet.

(2) There are curves, that do not grip a square that lies **inside**, by which we mean the square to be a subset of the closure of the interior of the curve. A nice example is the following heart-shaped curve:



For a proof, suppose V were an inscribed square. Then the arcs L and R cannot contain a vertex of V, and B and T at most two of them (the angles between the straight pieces of T and B respectively are chosen to be obtuse). Thus, V has two vertices on T, hence the interior of the edge connecting these two vertices lies outside the curve! The popular formulation "square pegs in round holes" is therefore a bit misleading.

(3) Griffiths answered in his often cited paper [**Gri91**] the question positively for rectangles V and regular C^1-curves γ. Unfortunately it appeared in my reading that there are some critical errors his calculations. Indeed his method cannot work without adding new major ideas, as we will show in Section 7. Thus at present it is not clear how one could rescue his ansatz.

[1]Points of the curve are called **vertices**, if the curvature radius has there a local extremum. A theorem of Blaschke states that any C^3-curve intersects a circle in at most as many points as its number of vertices.

It is interesting that even the special case, where V is a square and γ continuous, is still unsolved (see [**KlWa91**], also for more background information). We will show in Section 2 that the CS-TM method fails in this special case. A proof for γ a C^1-curve can be established with methods from differential topology, as we will see in Section 5. Finally, Section 3 shows that finding equilateral triangles is an "easy" problem.

2. Test maps for the Square Peg Problem

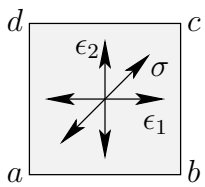
Suppose that $\gamma : S^1 \to \mathbb{R}^2$ is a continuous curve that does not grip a square. There is an obvious test map. $(S^1)^4$ parametrises via γ the space of all quadrilaterals on γ. If such a quadrilateral is a square, then its edge lengths are all equal, as well as both diagonal lengths, therefore we just measure this by the test map. However that does not characterise squares uniquely, since there are also degenerate quadrilaterals $ABCD$ with $A = C$ and $B = D$, which we therefore have to remove from our domain. This yields a test map

$$f : (S^1)^4 \backslash \{(x, y, x, y) \mid x, y \in S^1\} \longrightarrow_{D_8} \mathbb{R}^6 \backslash (\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}) \tag{2.1}$$

sending

$$(a, b, c, d) \longmapsto \big(d(a, b), d(b, c), d(c, d), d(d, a)\ ;\ d(a, c), d(b, d)\big)\,.$$

$D_8 = \mathbb{Z}_2^2 \rtimes S_2$ denotes the dihedral group, the symmetry group of a square, which is equal to the Weyl group W_2 from Section II.3.1. The generators ε_1 and ε_2 of the two $\mathbb{Z}_2$-copies and the generator of S_2 act as in the following picture:



Unfortunately such a test map (2.1) exists! One can show this as follows: There is up to symmetry and D_8-homotopy just one such map on the non-free part of the domain of f. To extend this map, one deformation-retracts $\mathrm{dom}(f)$ equivariantly to a D_8-CW-complex of dimension 3 by Lemma A1.1 of Appendix A. The map on the non-free part can now easily be extended to the rest by induction on the skeleta, since $\mathbb{R}^6 \backslash (\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2})$ deformation-retracts to a 3-sphere.

Instead of deleting anything from $(S^1)^4$ to obtain a larger ("higher-dimensional", by means of Lemma A1.1) domain for the test map, one can put extra conditions on the test map like requiring that certain

subset of the domain have to become mapped to special subsets of the range. I tried a lot, but all my enhanced test maps unfortunately exist.

3. Equilateral triangles on curves

In this section we present some pretty nice results on tracing *triangles* on curves. First we state a result by M. J. Nielsen.

THEOREM 3.1 (Nielsen [**Nie92**]). *Let T be an arbitrary triangle and $\gamma : S^1 \longrightarrow \mathbb{R}^2$ an injective plane closed curve. Then there are infinitely many triangles inscribed in (the image of) γ which are similar to T, and if one fixes a vertex of smallest angle in T then the set of the corresponding vertices on γ is dense in γ.* ∅

For our following result, we will restrict ourselves to equilateral triangles, to obtain a result for arbitrary curves.

THEOREM 3.2. *Let $d : S^1 \times S^1 \longrightarrow \mathbb{R}$ be a continuous function satisfying $d(x, y) = d(y, x)$. We regard d as a generalised metric. Then there are three points $x, y, z \in S^1$, not all of them equal, forming an equilateral triangle, that is $d(x, y) = d(y, z) = d(z, x)$.*

To illustrate that theorem, consider a continuous closed curve $\gamma : S^1 \longrightarrow M$ into any metric space M. Pulling back M's metric d_M to S^1 via

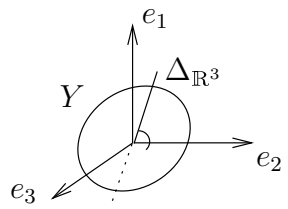
$$d_{S^1}(x, y) := d_M(\gamma(x), \gamma(y)),$$

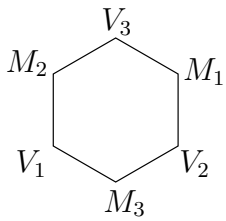
the theorem states that we can find an equilateral triangle on (the image of) γ with respect to the metric d_M. In general d neither needs to be positive definite, nor has to satisfy the triangle inequality. However, if d is positive definite, then the solution triangle always consists of three *pairwise* distinct points.

PROOF OF THEOREM 3.2. We use the CS-TM method. Assuming that there was a counter-example d, we let $X := \{(x, y, z) \in (S^1)^3 \mid x, y, z \text{ are not all equal}\} = (S^1)^3 \backslash \Delta_{(S_1)^3}$ be our configuration space, and construct our test map f to be the following composition

$$f : X \longrightarrow_{S_3} \mathbb{R}^3 \backslash \Delta_{\mathbb{R}^3} \simeq_{S_3} \mathbb{R}^2 \backslash \{0\} \simeq_{S_3} S^1, \tag{3.3}$$

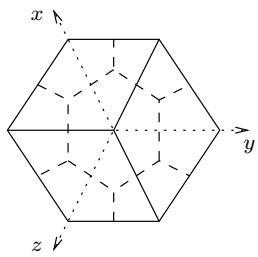
where the first map is given by $(x, y, z) \mapsto (d(y, z), d(z, x), d(x, y))$, the second one projects $\mathbb{R}^3 \backslash \Delta_{\mathbb{R}^3}$ to $\Delta_{\mathbb{R}^3}$'s orthogonal complement, and the last one normalises. The S_3-actions on $(S^1)^3$ and $\mathbb{R}^3$ are given in the same way by permuting coordinates, and on S^1 such that f is S_3-equivariant. (It is the induced S_3-action from $\mathbb{R}^3$ to the unit sphere of the orthogonal complement of $\Delta_{\mathbb{R}^3}$, which is an invariant subspace.) We call S^1 with this S_3-action Y.



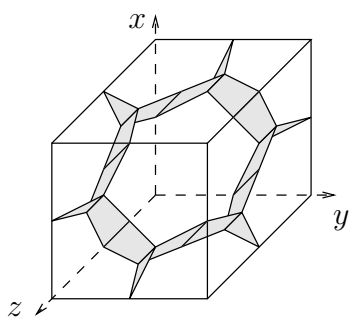


The right figure then is a S_3-cell structure of Y, where $\pi \in S_3$ acts on the vertices via $\pi \cdot M_i = M_{\pi(i)}$ and $\pi \cdot V_i = V_{\pi(i)}$. We want to show that such a test map (3) with this symmetry condition cannot exist.

In the following figure we see X, where $(S^1)^3$ is shown as a cube, where we have to imagine the opposite faces to be identified.



The cube is shown, such that one sees $\Delta_{(S^1)^3}$ as a point. We see from this figure that X deformation-retracts S_3-equivariantly to the two-dimensional CW-complex shown "from the top" as the dashed lines, which we call X'. The next figure shows more exactly, how the actual S_3-cell structure of X' looks like:



We want to apply relative obstruction theory. X and therefore X' have cells of isotropy group $\langle(12)\rangle, \langle(23)\rangle, \langle(13)\rangle$ and $\langle 0\rangle$, as one verifies quickly. We calculate $X^{\langle(12)\rangle} = \{(x, y, z) \in X \mid x = y\}$ and $Y^{\langle(12)\rangle} = \{V_3, M_3\}$. Since Y is symmetric in the M's and V's, we can assume that the test map (3) satisfies $f(x, x, z) = V_3$. Then by equivariance of f, we get $f(x, y, y) = V_1$ and $f(x, y, x) = V_2$. Therefore,

f is up to symmetry defined uniquely on the non-free part of X (or X', whatever one likes to use as f's domain). The non-free part of X' is a subcomplex, say A. It is an easy exercise (but longish to write everything down) to extend this $f|_A$ to the 1-skeleton of X', and to calculate the obstruction element $[o] \in H^2_{S_3}(X, A; \pi_1(Y))$ (relative equivariant obstruction theory (see Appendix C1) is applicable because Y is 1-simple, which means that $\pi_1(Y) = \mathbb{Z}$ is Abelian). If one extends f in the most obvious way and takes the same orientations as I did, then o is a coboundary iff the following integral linear equation system has a solution:

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

But is has none, since the sum of all equations modulo 3 gives $0 \equiv 1 \pmod 3$. Therefore our test map does not exist. $\not\Delta$

While this was a theorem which is applicable to any symmetric distance function, we can even find a *one-parameter family* of triangles which are similar to an arbitrary non-degenerate one, as long as our distance function is of a more special kind. Since this result can be generalised even more, such that it gives a proof for the Square Peg Problem for C^∞-curves in the plane, we deal with it in a new section.

4. Polygons on curves

From now on, we assume that we are given a curve $\gamma : S^1 \longrightarrow M$ on a *complete Riemannian manifold* M, which is an injective C^∞-immersion (an embedding). The completeness property of M is there (and only for this purpose), such that any two points on M with distance d can be connected by a geodesic of length d by the Hopf-Rinow Theorem [**HoRi31**]. I think this makes the result more geometric, even though in general these geodesics do not vary continuously (as a homotopy of paths) when the two end points are moved.

Let d_M be the metric on M induced by its Riemannian metric. We want to show how to find in the generic case a one-parameter family of n-gons (i. e. polygons on n vertices) with non-degenerate prescribed edge length ratios, which are allowed to be twisted and whose vertices lie counter-clockwise around γ. Note that in the case $n > 3$, the inner angles are not any more prescribed. For instance, if we choose $n = 4$ and require all edges to be of same length, then we will find a one-parameter family of rhombuses. What "generic" means will become

clear later. But we can drop this requirement on γ by saying that we will only find a one-parameter family of n-gons which satisfy *nearly* the prescribed edge length ratios.

The vertices of an n-gon P on γ can be parametrised like γ via the parameter space $(S^1)^n$. Identifying S^1 with $\mathbb{R}/\mathbb{Z}$ (as quotient of topological groups), an n-gon P parametrised by $(x_1, \ldots, x_n) \in (S^1)^n$ lies **counter-clockwise** on γ (with pairwise distinct vertices) iff there are representatives $\overline{x}_1, \ldots, \overline{x}_n \in \mathbb{R}$ of $x_1, \ldots, x_n$ satisfying $\overline{x}_1 < \ldots < \overline{x}_n < \overline{x}_1 + 1$. This n-gon is then described by $x_1 \in S^1$ together with $(\overline{x}_2 - \overline{x}_1, \ldots, \overline{x}_n - \overline{x}_{n-1}, \overline{x}_1 + 1 - \overline{x}_n) \in (\Delta^{n-1})^\circ$ (the interior of the $(n-1)$-dimensional standard simplex $\Delta^{n-1} = \operatorname{conv}\{e_1, \ldots, e_n\}$). Therefore the parameter space of the n-gons that lie counter-clockwise on γ is

$$P_O := S^1 \times (\Delta^{n-1})^\circ.$$

P_O stands for "oriented polygons". We will often view it as a subspace of $(S^1)^n$, the space of all "polygons" on γ. Note that $P_O \overset{pr_1}{\simeq} S^1$. We are interested in one-parameter families $\phi : S^1 \to P_O$ of n-gons with prescribed edge length ratios, which wind an odd no. of times around P_O.

Definition 4.1. An n-gon $P = (x_1, \ldots, x_n) \in (S^1)^n$ on $\gamma : S^1 \longrightarrow M$ is said to have **edge ratios** $\rho_1, \ldots, \rho_{n-1}$, iff for all $i \in \{1, \ldots, n-1\}$,

$$\rho_i \cdot d_M(\gamma(x_1), \gamma(x_n)) = d_M(\gamma(x_i), \gamma(x_{i+1})).$$

Conversely, $n-1$ positive reals $\rho_1, \ldots, \rho_{n-1}$ are called **edge ratios of a an n-gon**, iff each of the numbers $1, \rho_1, \ldots, \rho_{n-1}$ is less than the sum of the others, that is, there exists an n-gon with the ρ_i's as its edge ratios.

A one-parameter family of n-gons $\phi : S^1 \to P_O$ is said to **wind an odd no. of times around** P_O, iff the homotopy class $[\phi] \in [S^1, P_O] \cong [S^1, S^1] \cong \pi_1(S_1) \cong \mathbb{Z}$ is odd in $\mathbb{Z}$.

Theorem 4.2. *Let $n \in \mathbb{Z}_{\geq 3}$ be a given number of vertices and let $\rho_1, \ldots, \rho_{n-1} \in \mathbb{R}_{>0}$ be the edge ratios of an n-gon. If $\gamma : S^1 \longrightarrow M$ is a "generic" (see proof) C^∞-embedding of the circle into a (complete) Riemannian manifold, then the set of all n-gons that lie counter-clockwise on γ and which have the prescribed edge ratios $\rho_1, \ldots, \rho_{n-1}$ is a disjoint union of one-parameter families of the form $L_i : S^1 \longrightarrow P_O$. Furthermore, the number of such families that wind an odd no. of times around P_O, is odd.*

Before proving this theorem, we want to state two variants of it both of which get rid of the genericity condition on γ. The first version does this by changing the given ρ_i's by at most ε.

THEOREM 4.3 (Version B). *Let $n \in \mathbb{Z}_{\geq 3}$ and $\rho_1, \ldots, \rho_{n-1} \in \mathbb{R}_{>0}$ as above, let $\gamma : S^1 \longrightarrow M$ be an arbitrary C^∞-embedding and let $\varepsilon > 0$. Then we can change our ρ_i's each by at most ε, such that the number of one-parameter families $L_i : S^1 \longrightarrow S^1 \times (\Delta^{n-1})^\circ$ of n-gons with edge length ratios $\rho'_1, \ldots, \rho'_{n-1}$, that wind an odd no. of times around P_O, is odd.*

The next version turns out to be the useful one that we need to prove the Square Peg Problem for C^∞-curves. It finds a $\mathbb{Z}_4$-invariant subspace of one-parameter families of polygons, such that the edges of any polygon are all of equal length up to ε (that is, all $\rho_i \approx 1$).

THEOREM 4.4 (Version C). *Let $n \in \mathbb{Z}_{\geq 3}$, let $\gamma : S^1 \longrightarrow M$ be an arbitrary C^∞-embedding and let $\varepsilon > 0$. $\mathbb{Z}_n$ acts on $P_O \subset (S^1)^n$ by rotating the vertices $(x_1, \ldots, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1})$. Then there is a $\mathbb{Z}_n$-invariant subspace of P_O consisting of an odd number of one-parameter families of the form $L_i : S^1 \longrightarrow P_O$, each of which winds an odd no. of times around P_O, such that all these parametrised n-gons have edge length ratios $\rho_1, \ldots, \rho_{n-1} \in [1-\varepsilon, 1+\varepsilon]$.*

COROLLARY 4.5 (Corollary of Theorem 4.4). *If $n = 2^k$, one of the one-parameter families $L_i : S^1 \longrightarrow P_O$ in the previous theorem is by itself $\mathbb{Z}_n$-invariant.*

PROOF. ...because the subspace, that Theorem 4.4 gives, is a union of an *odd* number of disjoint one-parameter families. $\mathbb{Z}_{2^k}$ permutes the L_i's, and the orbits have an even length, except for those which stay fixed under the action. ∅

PROOF OF THEOREM 4.2. First of all, we define a function

$$\begin{array}{rlcl} r: & P_O \subset (S^1)^n & \longrightarrow & \mathbb{R}^{n-1} \\ & (x_1, \ldots, x_n) \in (S^1)^n & \longmapsto & \left(\frac{d_M(\gamma(x_i),\gamma(x_{i+1}))}{d_M(\gamma(x_1),\gamma(x_n))}\right)_{i \in \{1,\ldots,n-1\}}, \end{array}$$

which measures the edge ratios. The **genericity** of γ in the theorem means, that $\rho := (\rho_1, \ldots, \rho_{n-1})$ is a regular value of r. By Sard's Theorem [**Sar42**], this is really a generic (that is the usual) case. Let $L := r^{-1}(\rho) \subset P_O$ be the set of all polygons on γ, which lie counter-clockwise on this curve and whose edge ratios are as prescribed. It is a one-dimensional submanifold of P_O, which stays away from the topological boundary of P_O in $(S^1)^n$, that is there is a $\delta > 0$, such that $L \cap U_\delta((S^1)^n \backslash P_O)$ is empty:

For otherwise there were a sequence of n-gons in P_O with edge ratios ρ, such that one (and therefore all) edge lengths converge to zero. This is not possible, since all vertices lie counter-clockwise on

γ, ρ are edge ratios of an n-gon, and the curve is "locally straight" (since γ is embedded in a Riemannian manifold), which means that if the polygons become smaller, one edge becomes nearly as long as the sum of the others. How small they can become, such that they still have the prescribed edge ratios, has a continuous lower bound which depends only on the maximal local curvature of γ. This gives us δ.

Since L stays away from ∂P_O, and $\overline{P_O} = P_O \cup \partial P_O$ is compact, L is compact as well. Therefore L is a finite disjoint union of embedded circles in P_O. This finishes the technical part and we come to the heart of the proof. We want to show that the number of components of L, which wind an odd no. of times around P_O, is odd. We give two ways to show that; the first one will use more algebraic topology, the second one differential topology.

First way. We use Nash's embedding theorem [**Nas56**] to view M as being isometrically embedded in a Euclidean N-space $\mathbb{R}^N$, whose standard metric we denote by $d_{\mathbb{R}^N}$. We can deform the metric (distance function) on M to $d_(\mathbb{R}^N)$ by a linear homotopy, which changes nearly nothing in small neighborhoods. Without loss of generality, $N \geq 4$. Then γ is homotopic to a plane unit circle C via a smooth homotopy $H : I \times S^1 \longrightarrow \mathbb{R}^N$ which is at each time a smooth embedding of S^1 into $\mathbb{R}^N$. Then $\gamma(_) = H_0(_)$. We can extend our r to all of $I \times P_O$:

$$\begin{array}{rlcl} R: & I \times P_O \subset I \times (S^1)^n & \longrightarrow & \mathbb{R}^{n-1} \\ & (t, x_1, \dots, x_n) \in I \times (S^1)^n & \longmapsto & \left(\frac{d_{\mathbb{R}^N}(H_t(x_i), H_t(x_{i+1}))}{d_{\mathbb{R}^N}(H_t(x_1), H_t(x_n))} \right)_{i \in \{1,\dots,n-1\}} . \end{array}$$

We identify $r = R|_{\{0\}\times P_O}$ and $P_O \supset L = (R|_{\{0\}\times P_O})^{-1}(\rho) \subset \{0\} \times P_O$. One easily checks that $R|_{\{1\}\times P_O}$ is transversal to $\{\rho\} \in \mathbb{R}^{n-1}$ (that is, ρ is a regular value for $R|_{\{1\}\times P_O}$) and that $L' := (R|_{\{1\}\times P_O})^{-1}(\rho)$ is an embedded circle in $\{1\} \times P_O \cong P_O$, which winds exactly once around P_O (that is, given an arbitrary orientation, it represents a generator of $[S^1, P_O] \cong \pi_1(S^1) \cong \mathbb{Z}$). Making R transversal to ρ by changing it on $I^\circ \times P_O$ (see [**GuPo74**, Extension Thm., p. 72]), the preimage $R^{-1}(\rho)$ is an unoriented bordism between L and L' (we could do everything oriented, but it does not help), since it also stays away from ∂P as one can show similarly as above, using additionally the compactness of I. Now consider the Thom homomorphism [**BrDi70**, p. 20] $\mu : N_1(P_O) \longrightarrow H_1(P_O; \mathbb{Z}_2)$ from the first unoriented bordism group $N_1(P_O)$ to the first homology group of P_O with $\mathbb{Z}_2$ coefficients. It maps classes which can be represented by $S^1 \longrightarrow P_O$ to the mod-2 degree of the composed map $S^1 \longrightarrow P_O \xrightarrow{\simeq} S^1$, which lies in $\mathbb{Z}_2 \cong H_1(S^1; \mathbb{Z}_2) \cong H_1(P_O; \mathbb{Z}_2)$. Therefore it maps the bordism class

of L' to $(-1) \in \mathbb{Z}_2$, and so it does with L. The components $L_1, \ldots, L_k$ of L get mapped to

$$[L_i] \longmapsto \begin{cases} +1, & \text{if } L_i \text{ winds an even no. of times around } P_O, \\ -1, & \text{if } L_i \text{ winds an odd no. of times around } P_O. \end{cases}$$

Therefore, only the "odd" L_i's give a portion to $\mu([L]) = -1$, hence the number of them has to be odd. ⊘

Second way. The idea is to cut a generic slice out of this full torus $P_O = S^1 \times (\Delta^{n-1})^\circ$, which intersects L transversally in an odd number of points. Then we are done, since all the components of L which wind an odd no. of times around P_O give an odd number of intersections with that generic slice, and the even times winding components give an even number of intersections.

To begin with, let $i : (\Delta^n)^\circ \longrightarrow P_O$ be the inclusion $p \longmapsto (N, p)$, where $N \in S^1$ is some fixed point. We assume i to be transversal to L, otherwise we deform the map i inside a small ε neighborhood of $i^{-1}(L)$. Let $f : \Delta^N \longrightarrow (\Delta^N)^\circ$ be a map which is ε-close to the identity $\Delta^n \longrightarrow \Delta^n$ and such that it is the identity on an open neighborhood of $i^{-1}(L) \subset \Delta^n$. We want to show that $\mathrm{im}(i) \cap L$ is odd, or equivalently, $\mathrm{im}(i \circ f) \cap L$ is odd, or equivalently, $(r \circ i \circ f)^{-1}(\rho)$ has odd cardinality. We denote the latter composition as

$$h : \Delta^N \xrightarrow{f} \left(\Delta^N\right)^\circ \xrightarrow{i} P_O \xrightarrow{r} \mathbb{R}^{n-1}.$$

Since ρ is a regular value of h, the local degree of h at each point of the preimage $h^{-1}(\rho)$ is $+1$ or -1. They sum up to the winding number of $h|_{\partial\Delta^N}$ around $\rho \in \mathbb{R}^{n-1}$, denoted by $W(h|_{\partial\Delta^N}, \rho)$. Therefore, the number of preimages of ρ under h is odd, iff $W(h|_{\partial\Delta^N}, \rho)$ is odd. In fact, $W(h|_{\partial\Delta^N}, \rho) = \pm 1$ (depending on the chosen orientations), as long as ε is chosen small enough. We will leave this here as an exercise (which probably does not yield much insight, the idea behind this proof is more important). ⊘

This finishes the proof of Theorem 4.2 ⊘

Proof of Theorem 4.3. If $\rho := (\rho_1, \ldots, \rho_{n-1})$ is already a regular value of the ratio measuring function r from the previous proof, then the previous theorem applies. Otherwise we can change ρ by at most ε by Sard's Theorem [**Sar42**], such that it becomes a regular value. ⊘

Proof of Theorem 4.4. To prove this version, we deform the ratio measuring function r *equivariantly* by an ε-homotopy relative to

the set $\overline{U}_\delta(\partial P_O) \cap P_O$ for the given $\varepsilon > 0$, such that ρ becomes a regular value. If we take the first way in the proof of Theorem 4.2, we also need to extend this homotopy to all of R with the additional property that the homotopy is relative to $I \times (\overline{U}_\delta(\partial P_O) \cap P_O) \cup \{1\} \times P_O$ and such that ρ becomes a regular value of R. Both is possible, because the transversality is already given at the sets, where we want the homotopy to be constant, and for the remaining set we can do the deformation iterated locally and equivariantly, because the $\mathbb{Z}_n$-action on P_O is properly discontinuous and $\overline{P_O}$ is compact. ∅

Questions 4.6.

- *Can it actually happen that more than one ϕ wind an odd no. of times times around P_O?*
- *Can it actually happen that one ϕ winds more than once around P_O?*

A generalisation. We want to note an important generalisation of Corollary 4.5, which gives us to any S^1 that is embedded into a Riemannian manifold a one-parameter family (parametrised over S^1) of "up-to-ε"-regular n-gons, where n is an arbitrary *prime power*.

Lemma 4.7 (Corollary of the proofs in this section). *If n is a prime power, then one of the one-parameter families $\phi : S^1 \longrightarrow P_O$ in Theorem 4.4 is by itself $\mathbb{Z}_n$-invariant.*

Proof. This can be achieved by making the proofs of Theorem 4.2 and 4.4 oriented. For this to do, note that $P_O \subset (S^1)^n$ has the same orientation character as $\mathbb{R}^n$ (that is each left translation coming from the $\mathbb{Z}_n$-action is on both manifolds orientation preserving or on both manifolds orientation reversing), and $\mathbb{Z}_n$ acts trivially $\Delta_{\mathbb{R}^n}$. Therefore L has the trivial orientation character (that is all left translations are orientation preserving), since it is the preimage of $\Delta_{\mathbb{R}^n}$ of the function

$$f : P_O \longrightarrow_{\mathbb{Z}_n} \mathbb{R}^n : (x_1, \ldots, x_n) \mapsto (d_M(x_1, x_2), \ldots, d_M(x_N, x_1)).$$

Therefore we can say, how often the components L_i of L wind around P_O with orientation (as elements in $\mathbb{Z}$), once we chose a generator of $\pi_1(P_O) \cong \mathbb{Z}$. The sum of all these winding numbers must be 1 (or -1, if we chose the other generator), since so it is when γ is the unit circle in $\mathbb{R}^2$ and it does not change for different γ's by the argumentation of the proof of Theorem 4.2, first way.

Since $\pi_1(P_O) \cong \mathbb{Z}$ is Abelian, there is a natural way do identify $\pi_1(P_O, x)$ for different base points $x \in P_O$. Therefore it is clear, how a left translation of an element of $\mathbb{Z}_n$ induces an automorphism of $\pi_1(P_O)$. All these automorphisms are the identity (to see this, just

take the most obvious representative of a generator of $\pi_1(P_O)$ that is $\mathbb{Z}_n$-invariant. All of its left translated loops are apparently homotopic to each other).

Hence, since also the orientation character of L was trivial, the winding number of a component L_i is equal to the winding number of the left translated components. Now, since n is a power of a prime p, all $\mathbb{Z}_n$ orbits have a size that is divisible by p, except for the trivial orbit. Finally the sum of the winding numbers is ± 1, thus there must be a $\mathbb{Z}_n$-invariant component. ∎

5. A proof for the smooth Square Peg Problem

As a direct corollary of Corollary 4.5 or 4.7, we obtain a proof for the Square Peg Problem for C^∞-curves.

THEOREM 5.1. *Every C^∞-embedded circle in the plane inscribes a square whose vertices lie counter-clockwise on the curve.*

PROOF. Choose an $\varepsilon > 0$. By Corollary 4.5 we find a $\mathbb{Z}_4$-invariant one-parameter family $L_j : S^1 \to P_O$ $(n = 4)$ of "up-to-ε-rhombuses". Let $R \in S^1$ be an arbitrary up-to-ε-rhombus on L_j. Then $[1] \cdot R$ is again an up-to-ε-rhombus on L_j, but with interchanged diagonal lengths. We can now go along L_j from R to $[1] \cdot R$ by a path in P_O. At each time, we parametrise an up-to-ε-rhombus, and by the mean value theorem, one of these has to have equal diagonal lengths.

Now, letting $\varepsilon > 0$ go to 0, we find a sequence of up-to-ε-rhombuses in P_O with equal diagonal lengths. This sequence has a convergent subsequence in $(S^1)^4$, since this space in compact. The limit point is a quadrilateral with equal edge lengths and equal diagonal lengths. The edge lengths of this limit quadrilateral cannot be zero, since then the size of the largest inner angle of the approximated rhombuses of this subsequence has to converge apparently to π. (See [**Gri91**, Lem. 3.1] for a detailed proof. Here we need our curve again to be C^2 or something similar, pure continuity is not enough.) Since the vertices of each quadrilateral in the subsequence lie counter-clockwise on the curve, the limit quadrilateral has to do this as well, therefore this is our desired square. ∎

REMARK 5.2. It is not known but conjectured (at least by me, but probably by other mathematicians as well) that Theorem 5.1 should remain true for arbitrary circular quadrilaterals. In fact, there is no proof known for *any* other class of quadrilaterals than squares. Our

proof apparently does not generalise because of the lack of symmetry. Therefore Theorem 5.1 is a beautiful instance of a problem where symmetry is a basic property that is needed to prove something.

Finally, we want to state a small generalisation of the previous theorem (small in the sense that the same proof works).

THEOREM 5.3. *Every C^∞-embedded circle in a complete Riemannian manifold inscribes a square-like quadrilateral in the sense that all edges and all diagonals have the same length respectively.*

REMARKS 5.4.

- Here again we use the assumption on the manifold to be complete to not only find four points on it having the desired distance relations, but also to find geodesics connecting these points which realise the distances.
- Does Theorem 5.3 still hold true, if we change the definition of a square-like quadrilateral to being four points connected by distance realising geodesics of the same length, such that *all inner angles are equal*? The inner angles here are the smaller ones of both possibilities. The previous proof apparently does not work, since the distance realising geodesics do not move continuously when moving their end points (even if one can do unique choices) and therefore the inner angles behave in general non-continuously.
- The C^∞-condition can be replaced by C^2 using a limit argument, since the limiting square cannot be a point by the local straightness of γ. We can even allow piecewise C^2-curves, as long as no corner is a cusp, that is, the angles have to be positive.

6. Equilateral and isosceles triangles on curves

The generalisation of Corollary 4.5, Lemma 4.7, gives us the power to prove easily the following nice theorem:

THEOREM 6.1. *Suppose we are given a circle with two distance functions d_e and d_i where*

- *d_e is the restricted distance function coming from a smooth embedding of S^1 into a Riemannian manifold, and*
- *d_i is symmetric and continuous.*

Then there are three points $x, y, z \in S^1$ forming an equilateral triangle with respect to d_e and an isosceles triangle with respect to d_i.

PROOF. By Lemma 4.7, we find a $\mathbb{Z}_3$-invariant one-parameter family $L_i : S^1 \to P_O$ $(n = 3)$ of "up-to-ε-equilateral" triangles with respect

to d_e. Suppose no triangle on L_i would have two equal edges with respect to d_i, then we can order the edges by length. This order does not change, when we go along L_i by continuity of d_i. But $\mathbb{Z}_3$ just relabels the vertices of the triangles, so it does not keep this order invariant. This yields a contradiction to the $\mathbb{Z}_3$-invariance of L_i.

Letting ε go to zero, we get a converging subsequence of triangles (since the sequence is staying away from the boundary of P_O, since near the boundary the inner angles are becoming too obtuse, since d_e comes from a *smooth* embedding into a Riemannian manifold), whose limit triangle is what we wanted to find. ∎

Remark 6.2. It is interesting whether this theorem remains true if we also ask d_e to be an arbitrary symmetric continuous distance function, since the corresponding test map *does* exist, as P. Blagojević [private communication] has shown (there is a map on the non-free part of the domain, and the domain to a two-dimensional subcomplex, but the range is one-connected), so the CS-TM method gives no information!

7. Problems in Griffiths' paper

The reader of this section is assumed to be familiar with H. B. Griffiths paper [**Gri91**]. It has often been cited (see e. g. [**Mak95**], [**Pak08**], [**Mak05b**]), but it unfortunately contains some errors, which seem to invalidate the proofs of his Theorems A, B and C. While his underlying theory, a generalisation of the standard intersection number theory in differential topology [**GuPo74**], is valid and applicable in principle, Griffiths made some mistakes during the calculation that had to be done for applying the theory to the special problems of Theorems A and C. (Theorem B is probably the most interesting statement, but it is a special case of Theorem A, which Griffiths proved.)

First of all, we will give a short proof that his intersection number for Theorem B is zero instead of 16, therefore it does not imply a desired intersection. The argument is general enough to show the vanishing of the intersection number in a bunch of instances, for example Theorem C. Secondly, we will give a list of errors in [**Gri91**] that occured during his calculations for Theorems A and B.

7.1. Why Griffiths' ansatz cannot work on this problem. By saying ansatz we mean the following scheme:

(1) Reformulation of the given problem into the question whether two submanifolds Γ^4 and T^*_ρ intersect non-empty.

(2) Definition of an applicable intersection number (see [**Gri91**] section 4), that stays constant under a certain homotopy of Γ^4, that deforms it to a very nice manifold E^4.
(3) Showing that the intersection number between E^4 and T^*_ρ is non-zero.

In the situation of Theorem B, where we want to prove that every injective C^1-immersion Γ of S^1 into $\mathbb{R}^2$ grips a rectangle of fixed ratio ρ, the ambient space is $(\mathbb{R}^2)^4 =$ "set of all plane quadrilaterals", Γ^4 is the image of Γ in $\mathbb{R}^2$ to the forth power, and T^*_ρ is the set of all non-degenerate but possibly skew rectangles $(p, q, r, s) \in (\mathbb{R}^2)^4$ of aspect ρ (that is, $||p - q|| = ||r - s||$, and the segment joining the midpoints of the edges $\overline{pq}$ and $\overline{rs}$ is orthogonal to both and of length $\rho \cdot ||p - q||$).

Now, suppose $\Gamma : S^1 \longrightarrow \mathbb{R}^2$ is the inclusion of the unit circle in $\mathbb{R}^2$. Define a homotopy

$$\Gamma_t : S^1 \longrightarrow \mathbb{R}^2 : x \mapsto \left(1 - \frac{t}{2}\right) \cdot x.$$

It satisfies $\Gamma_0 = \Gamma$.

Let $\Delta := \{(x, x, x, x) \in (S^1)^4\}$ be the diagonal in $(S^1)^4$ and let $\varepsilon > 0$ be small. Then define $\phi : (S^1)^4 \longrightarrow [0, 1]$ to be a function, that maps all points (x, y, z, w) satisfying $d((x, y, z, w), \Delta) < \varepsilon$ to zero (d is some metric on $(S^1)^4$), and all points (x, y, z, w) satisfying $d((x, y, z, w), \Delta) > 2\varepsilon$ to one, and all other points in between. We construct a homotopy of Γ^4:

$$\begin{array}{rccl} H_t : & (S^1)^4 & \longrightarrow & (\mathbb{R}^2)^4 \\ & (x, y, z, w) & \longmapsto & \big(\Gamma(x), \Gamma(y), \Gamma(z), \Gamma_{t\cdot\phi(x,y,z,w)}(w)\big) \end{array}$$

This homotopy of $H_0 = \Gamma^4$ stays fixed at the ε-neighborhood of Δ, and if ε is small enough (depending on ρ), then the intersection of H_t and T^*_ρ is *empty* for each $t > 0$, since no skew rectangle can have three vertices x, y and z on the unit circle, and the forth one w in the interior. (Note that for each ρ there is an $\varepsilon > 0$, such that any non-degenerate rectangle of aspect ρ which has its first three vertices on the unit circle, has as an element of $(\mathbb{R}^2)^4$ a distance from Δ of at least 2ε. We choose such an ε). This means by definition [**Gri91**, Sect. 4], $I_{U_\varepsilon(\Delta)}(\Gamma^4, T^*_\rho) = 0$, for any small enough $\varepsilon > 0$.

A very similar construction can be done for his Theorem C showing his proof to be false. I do not know how to salvage the ideas. One probably has to add a new major argument (in the case that the theorems are indeed true).

7.2. Errors. Here are the mistakes in [**Gri91**] that were done during the calculations of the intersection number for Theorem A (and B).

(1) Griffiths' Lemma 1.3 must read:

LEMMA. *α, β and κ reverse the orientation of Γ^4. If n is odd, then α, β and κ reverse the orientation of $(\mathbb{R}^n)^4$. If n is even, then α, β and κ preserve the orientation of $(\mathbb{R}^n)^4$.*

PROOF. We will prove this only for κ: For all $p \in \Gamma$, let $v_p \in T_p\Gamma$ be a positively oriented tangent vector of Γ at p. Griffiths orientation of Γ^4 is then at $(p, q, r, s) \in \Gamma^4$ the columns of the matrix

$$\begin{pmatrix} v_p & 0 & 0 & 0 \\ 0 & v_q & 0 & 0 \\ 0 & 0 & v_r & 0 \\ 0 & 0 & 0 & v_s \end{pmatrix}.$$

Now κ maps (p, q, r, s) to (q, r, s, p), thus $\mathrm{d}k$ maps this basis above to the columns of

$$\begin{pmatrix} 0 & v_q & 0 & 0 \\ 0 & 0 & v_r & 0 \\ 0 & 0 & 0 & v_s \\ v_p & 0 & 0 & 0 \end{pmatrix}.$$

But a positive oriented basis of $T_{(q,r,s,p)}\Gamma^4$ is

$$\begin{pmatrix} v_q & 0 & 0 & 0 \\ 0 & v_r & 0 & 0 \\ 0 & 0 & v_s & 0 \\ 0 & 0 & 0 & v_p \end{pmatrix}.$$

Both bases are obtained from each other by three swaps of adjacent basis vectors, that is κ reverses the orientation of Γ^4.

The tangent bundle of $(\mathbb{R}^n)^4$ has a standard orientation, which is at each point the columns of the matrix

$$\begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}.$$

$\mathrm{d}\kappa$ maps the columns to the columns of the matrix

$$\begin{pmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ I_n & 0 & 0 & 0 \end{pmatrix}.$$

Both matrices are obtained from each other by swapping $n \cdot 3n \equiv n \mod 2$ adjacent columns. That is, κ preserves orientation of $(\mathbb{R}^n)^4$ iff n is even. $\not\Phi$

(2) Lemma 2.1., which does the main calculation for the proof of Theorem A and B, states that all sixteen local intersection numbers are constant, but this cannot be true, for at Section 7.1 of this article we describe, why the sum of them has to be zero.

His use of Lemma 1.3 in the proof of Lemma 2.1 would still work for odd n, since for proving Theorem A we can w.l.o.g. assume n to be odd. But there is another error: At page 653, line 10 from the bottom, and page 654, line 15 from the bottom ("$N_{1/\rho} = I(\kappa Q_{1/rho}, \kappa E^4, \kappa T^*_{1/\rho})$") he used the following argument:

If one knows the local intersection number $I(p, X, Y)$, where $p \in X \cap Y$ and $X, Y \subset Z$, and $X \pitchfork Y$, and one has a transformation ξ which maps $p \mapsto p'$, $X \xrightarrow{\cong} X'$, $Y \xrightarrow{\cong} Y'$ and $Z \xrightarrow{\cong} Z'$, then $I(p, X, Y) = I(p', X', Y')$ iff ξ affects the orientation of X and Z in the same way. But the latter condition must read: "... iff ξ preserves the orientation of exactly non or two of the three manifolds X, Y and Z."

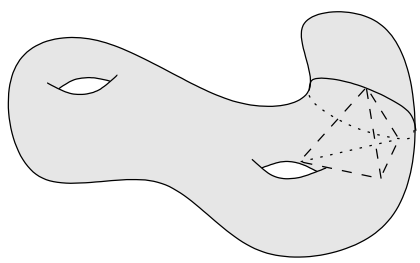
In the case $\rho = 1$, i. e. when we are looking for a square that inscribes a curve, it is a straightforward calculation that κ reverses the orientation of T^*_1, that is why

$$N_{1/\rho} = I(\kappa Q_{1/rho}, \kappa E^4, \kappa T^*_{1/\rho})$$

is wrong in this case.

8. Tetrahedra on surfaces

In this section we want to prove that any smoothly embedded compact surface S in a Riemannian manifold contains a regular tetrahedron, that is, four points of equal pairwise distance.

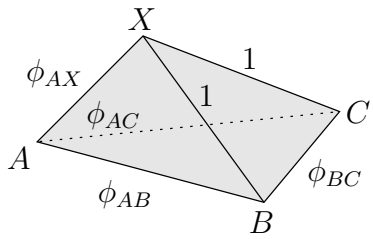


This can be seen as a higher dimensional variant of finding triangles on closed plane curves (Theorem 3.2). As an immediate corollary we get a lower bound for the topological Borsuk number $B(\mathbb{R}^3, d)$ for all metrics d that are coming from a Riemannian metric on some open ball of $\mathbb{R}^3$ (See [**Soi08**] for more about topological Borsuk numbers. Another result about tetrahedra on spheres was recently obtained by P. Blagojević and G. Ziegler [**BlZi08b**].)

In fact we can prove a much more general theorem, which finds a tetrahedron with prescribed edge ratios (under conditions) and a prescribed tip $X \in S$, where the metric on S is allowed to be more general:

Definition 8.1. Let T be a tetrahedron given by its edge lengths. It is called **realisable** iff there are four points in $\mathbb{R}^3$ having these edge lengths under the Euclidean metric. In this case it is called **non-flat** iff the affine hull of these four points is three dimensional. It is called **non-degenerate** iff all four points are pairwise distinct. We call two non-degenerate tetrahedra T_1 and T_2 **similar** iff all its edge lengths are proportional to each other.

Let $T = XABC$ be a *non-flat* tetrahedron with the following edge lengths (it is scaled in such a way, that $|XB| = 1$):



satisfying

- $\phi_{AX} \leq 1$ and
- $\phi_{AB} = \phi_{AC}$.

Note that this description is symmetric in B and C.

Let a smooth compact surface S be endowed with a continuous distance function $d : S \times S \longrightarrow \mathbb{R}$ (a generalised metric) which is

- symmetric,
- positive ($d(x, y) \geq 0$; and $d(x, y) = 0$ iff $x = y$), and
- there is an $X \in S$ and an open neighborhood U of X such that $d|_{U \times U}$ is the metric (distance function) coming from a Riemannian metric (This condition can be weakened: We only need to assume that there is an $X \in S$, an open neighborhood U of X and a smooth

embedding of U into a Riemannian manifold M, such that the $d|_{U\times U}$ is the distance function of M pulled back to U).

Note that the triangle inequality is not assumed.

THEOREM 8.2 (Tetrahedra on surfaces). *Let T, (S,d) and $X \in U \subset S$ as above. Then we can find three more points A, B and C on S such that the tetrahedron $XABC$ is similar to T according to the metric d.*

Before proving it we want to state a technical lemma:

LEMMA 8.3 (Approximation lemma). *If we can find a sequence of non-degenerate tetrahedra $(XA_nB_nC_n)_{n\in\mathbb{N}}$ on S, such that the edge ratios of $XA_nB_nC_n$ converge (as points in $\mathbb{R}^5$ since we have six edges) to the edge ratios of T, then there is a subsequence converging to a non-degenerate tetrahedron satisfying the edge ratios of T.*

PROOF OF LEMMA. The space of all tetrahedra on S is $S\times S\times S\times S$, which is compact. Therefore we get a converging subsequence in this space. Since T is non-flat, there is an $N \in \mathbb{N}$ and a small ε such that no tetrahedron of the subsequence from the index N on can lie fully in the ε-neighborhood of X, since tetrahedra in these neighborhoods have to get "arbitrarily flat" for $\varepsilon \to 0$ (Exercise; this uses that d is induced by a Riemannian metric around X). From this N on, all the tetrahedra have an edge whose length is at least ε. The convergence of the edge ratios then tells us that the subsequence has to converge to a non-degenerate tetrahedron. ∎

PROOF OF THEOREM 8.1. Using the lemma we can assume without loss of generality that

- $\phi_{AX} < 1$: If $\phi_{AX} = 1$ then take a solution tetrahedron for all $\phi_{AX} < 1$ for which T stays non-flat.
- d^2 is a C^∞-function. This is true on $U\times U$ (recall: U is a neighborhood of X, such that $d|_{U\times U}$ is induced by a Riemannian metric). So take a smaller ε-neighborhood of X, and for all $\delta > 0$ approximate d by a distance function d_δ with smooth d_δ^2, such that d_δ is equal to d on $U_\varepsilon(X)\times U_\varepsilon(X)$ and d_δ differs from d pointwise by at most δ. Let $\delta \to 0$, find tetrahedra for these approximated d_δ's and apply the approximation lemma.

Let $Y \in S$ be a point of maximal distance to X. Let $I \subset S$ be a smooth interval between X and Y, which is a geodesic on U. We will restrict to tetrahedra $XABC$ such that A lies on I. So let $K := I\times S\times S$

be the configuration space of triples (A, B, C), and define a function

$$\begin{array}{llll} f\colon & K & \longrightarrow & \mathbb{R}^6 \\ & (A,B,C) & \longmapsto & \begin{pmatrix} \phi_{AX}^{-2} d^2(A,X), & d^2(B,X), & d^2(C,X), \\ \phi_{AB}^{-2} d^2(A,B), & \phi_{AC}^{-2} d^2(A,C), & \phi_{BC}^{-2} d^2(B,C) \end{pmatrix}. \end{array}$$

As the reader might guess, $\mathbb{Z}_2$ acts on K by interchanging the two copies of S, $\tau \cdot (A, B, C) := (A, C, B)$ (τ shall be the generator of $\mathbb{Z}_2$), and on $\mathbb{R}^6$ by $\tau \cdot (a, b, c, d, e, f) := (a, c, b, e, d, f)$. Hence f becomes $\mathbb{Z}_2$-equivariant (here we need that the edge ratios are symmetric in B and C). We want to show that the preimage of the diagonal $\Delta_{\mathbb{R}^6}$ under f is non-empty. To do this we will first examine the preimage $L := f^{-1}(\Delta)$ of $\Delta := \Delta_{\mathbb{R}^3} \times \Delta_{\mathbb{R}^3} = \{(a, a, a, b, b, b) \in \mathbb{R}^6 \mid a, b \in \mathbb{R}\}$, which will be nearly a one-dimensional submanifold of K, after we made f nearly everywhere transversal to Δ. Imagine L to be the set of all tetrahedra that satisfy already the right ratios between the edges of the triangle ABC and separately between the three edges at X.

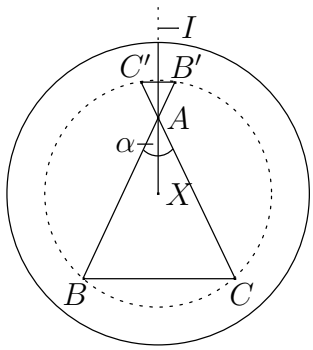
The non-free part of K is $K_{nf} := \{(A, B, C) \in K \mid B = C\}$. We have that

$$f(K_{nf} \backslash \{(X, X, X)\}) \cap \Delta = \emptyset, \tag{8.4}$$

since the only triples (A, B, B) getting mapped to Δ are those with $d(A, B) = 0$ (since $d(B, B) = 0$), therefore they satisfy $A = B$. Since $\phi_{AX} < 1$ and $\phi_{AX}^{-2} d^2(A, X) = d^2(B, X)$, the only (A, B, B) that becomes mapped under f to Δ is (X, X, X).

Unfortunately f is in general not transversal to Δ on the complement of $K_{nf} \backslash \{(X, X, X)\}$, so we have to make it transversal. Since we can use our approximation lemma, the only problem arises around (X, X, X).

First of all, suppose d were flat on a small ε-neighborhood of X, that is U_ε is isometric to an ε-ball in $\mathbb{R}^2$. Then f is *not* transversal to Δ on $U_\varepsilon(X)$ but let us still look at its preimage of Δ:



For each $A \in I \cap U_{\phi_{AX}\cdot\varepsilon}(X)$ we can construct four distinct solutions (A, B, C), which decompose into two $\mathbb{Z}_2$-orbits, except at $A = X$ we have just one solution. Let $V = K \cap U_{\phi_{AX}\cdot\varepsilon}(X) \times U_\varepsilon(X) \times U_\varepsilon(X)$. Then modulo $\mathbb{Z}_2$, $L \cap V$ consists of two paths both starting at (X, X, X).

Now we want to define a technical approximation $f'' : K \cap V \longrightarrow \mathbb{R}^2$ of f, which will be transversal to Δ, still $\mathbb{Z}_2$-equivariant and the preimage of Δ will look similar.

Let $f' : V \longrightarrow \mathbb{R}^6$ be $f|_V$ except that we change the fourth coordinate from $\phi_{AB}^{-2} d^2(A, B)$ to $(1 + \varepsilon_2)\phi_{AB}^{-2} d^2(A, B)$ for small $\varepsilon_2 > 0$. This is now transversal to Δ on $K \backslash \{(X, X, X)\}$ (which is an elementary but very longish calculation, we want to omit that), but not anymore $\mathbb{Z}_2$-equivariant. Therefore we define a further function $f'' : V \longrightarrow \mathbb{R}^6$ in the following way: For points (A,B,C) satisfying $A = X$, $B = X$, $C = X$, $\measuredangle(AXB) = 0$, $\measuredangle(AXC) = 0$ or $\measuredangle(BXC) = 0$, we define $f''(A, B, C) = f(A, B, C)$. All other (A, B, C) are decomposed into two connected components: These A, B, C lie clockwise or counterclockwise around X (and $\mathbb{Z}_2$ interchanges both components). In the first case and if $\measuredangle(AXB)$, $\measuredangle(AXC)$ and $\measuredangle(BXC) \geq \varepsilon_3$ for a small ε_3 we define $f''(A, B, C) = f'(A, B, C)$, and if some angle was smaller than ε_3 we approximate between f and f'.[2] In the other case we extend f'' $\mathbb{Z}_2$-equivariantly, that is $f''(A, B, C) := \tau \cdot f''(A, C, B)$.

Since δ was chosen small enough, f'' equals f' around the "clockwise part" of $f'^{-1}(\Delta)$. Therefore modulo $\mathbb{Z}_2$, $f''^{-1}(\Delta)$ looks around (X, X, X) also like two paths starting from (X, X, X).

If ε was chosen small enough, then these paths only move a little when one is taking the given metric d instead of the flat one, that we assumed (the omitted calculations can easiest be done with the flat metric, to avoid difficult estimations).

Now we can extend f'' to all of K such that it differs at no point outside of V more from f than it maximally did on V. The preimage $L'' := f''^{-1}(\Delta)$ now is a one-dimensional $\mathbb{Z}_2$-manifold (except at (X, X, X)), with boundary in $\partial K = \{X\} \times S \times S \cup \{Y\} \times S \times S$. Now if all approximations have been chosen close enough, $L'' \cap \{Y\} \times S \times S = \emptyset$, since $d(A, X)$ is (nearly) maximal, so there is no possible solution for B and C since $\phi_{AX} < 1$. Therefore, L'' is a disjoint union of circles

[2]In detail: Let $\phi : [0, \varepsilon_3]^3 \to \mathbb{R}$ be the function which takes $(\varepsilon_3, \varepsilon_3, \varepsilon_3)$ to 1, all points with one zero coordinate to 0, and approximate all other values smoothly such that all outwards directed partial derivatives on the boundary of $[0, \varepsilon_3]^3$ are zero. Extend ϕ to $\phi' : [0, \infty[^3 \to \mathbb{R}$ by $\phi'(a, b, c) := \phi(\min(a, \varepsilon_3), \min(b, \varepsilon_3), \min(c, \varepsilon_3))$. Then if A, B, C lie clockwise around X, define $f''(A, B, C) := (1 - \phi'(\measuredangle(AXB), \measuredangle(AXC), \measuredangle(BXC))) \cdot f(A, B, C) + \phi'(\measuredangle(AXB), \measuredangle(AXC), \measuredangle(BXC)) \cdot f'(A, B, C)$.

and the two paths starting from (X, X, X). Since L'' is compact, both paths have to end, therefore they connect again! Both paths start with both extremal kinds of tetrahedra having the right edge ratios separately in the triangle ABC and at the point X. Therefore along the path, the right ratio between an edge of the triangle and an edge at X has to occur. This is our desired tetrahedron. ∅

8.1. Version of Theorem 8.1 for *arbitrary* edge ratios. Suppose that our given surface S sits in fact in $\mathbb{R}^3$, by which we mean an injective continuous map $i : S \to \mathbb{R}^3$, which is furthermore smooth on a small non-empty open subset $U \subset S$, and suppose that the distance function d is then given by pulling back the Euclidean metric on $\mathbb{R}^3$ via i. In this case, we can transform *any* non-flat tetrahedron in $\mathbb{R}^3$ by an orientation preserving similarity such that all vertices map to $i(S)$. In particular we do not need to assume anymore the given tetrahedron T to fulfill symmetric edge ratios.

THEOREM 8.5. *Let $i : S \to \mathbb{R}$ and U as above. Let $T = XABC \subset \mathbb{R}^3$ be a non-flat tetrahedron satisfying $||XA|| \leq ||XB||$ and $||XA|| \leq ||XC||$. Then there is an orientation preserving similarity of $\mathbb{R}^3$ mapping the vertices X, A, B and C of T to the $i(S)$ and in fact the image of X can be arbitrarily prescribed on U.*

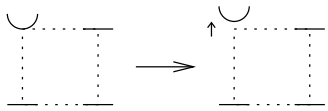
PROOF. The proof works very similar to the proof of Theorem but instead of L we are only looking at the tetrahedra in L that have the same orientation as the given tetrahedron T, this set shall be called $L' \subset L$. Then, L' is again (under some transversality assumptions) a compact manifold, except at (X, X, X), but only two paths instead of four are leaving from (X, X, X), which therefore have to close. Here we did not need the $\mathbb{Z}_2$-equivariance, since we have the orientation, which selects two of four paths. ∅

9. Cross polytopes on spheres

After the previous section it seems natual to ask in a similar fashion for octahedra.

QUESTION 9.1. *Does every smoothly embedded S^{n-1} in $\mathbb{R}^n$ circumscribe an n-dimensional cross polytope?*

Guggenheimer [**Gug65**] proved the answer to be yes, but unfortunately his proof contains an error: He says only that there were a Main Lemma similar to his first one, which he used to proof the Square Peg Problem, but unfortunately for this Main Lemma there is already a counter-example, see the next figure.



In this example, the rank of Δ is 3.

One could strengthen the assumption of the Main Lemma ("If Δ is invertible..."), but then it would not imply $G_2(n)$ to be arcwise connected anymore (p. 107, line 6-7 from the bottom). An argumentation about the parity of the number of squares on curve seems to be crucial to make Guggenheimer's proof of the Square Peg Problem work, and this was already done by Shnirel'man [**Shn44**]. Now for the existence of crosspolytopes one would need an argument that is similar to the parity argument. To be precise the argument were of the following kind:

Take a "generic" smooth embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$ for $n \geq 3$, and take as the configuration space of cross polytopes on this S^2 simply $X := \{(x_1, \ldots, x_{2n(n-1)}) \in (S^2)^{2n(n-1)} \mid x_1, \ldots x_{2n(n-1)} \text{ pw. distinct}\}/_G$, where the symmetry group of the regular cross polytope $G := \mathbb{Z}_2^n \rtimes S_n$ acts on $(S^2)^{2n(n-1)}$ as it acts on the vertices of the regular cross polytope (instead we could also take the subgroup $H := \ker\{G \to \mathbb{Z}_2 : \varepsilon_i \mapsto -1, \pi \mapsto \operatorname{sign}(\pi)\}$ of elements of G whose left translations on the regular cross polytope preserve the orientation). Measure by a test map $f : X \to Y$ ($Y = \mathbb{R}^{2n(n-1}$ some appropriate G-representation) the edges of the cross polytopes on S^2 (for $n \geq 3$, cross polytopes in $\mathbb{R}^n$ are determined to be regular, iff all vertices are pairwise distinct and all the edges are equally long), and let $L := f^{-1}(\Delta)$ be the set of regular cross polytopes inscribed in S. Let $l : L \hookrightarrow S^2$ be the inclusion map, then the unoriented bordism class $[L, l] \in N_1(X)$ is well defined. If we could show $[L, l] \neq 0$, we were done showing the answer to the above question to be "yes". But unfortunately it is zero for $n = 3$, as I checked.

CHAPTER IV

The Topological Tverberg Problem

1. Introduction

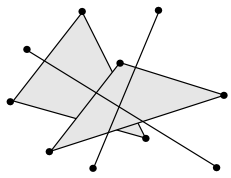
A well-known theorem in discrete geometry is Radon's Theorem ($d \geq 1$):

THEOREM 1.1 (Radon). *Every set of $d+2$ points in $\mathbb{R}^d$ can be divided into two disjoint subsets whose convex hulls have a non-empty intersection.*

This bound is tight: Consider $d+1$ vertices of a standard simplex in $\mathbb{R}^d$. Here are the two essentially distinct examples in dimension $d = 2$:

In [**Tve66**] and [**Tve81**] Tverberg generalised Radon's Theorem asking for not only two disjoint subsets but for $p \geq 2$ of them, whose convex hulls intersect in at least one point.

THEOREM 1.2 (Tverberg). *Every set of $(d+1)(p-1)+1$ points in $\mathbb{R}^d$ can be divided into p pairwise disjoint subsets, whose convex hulls have a non-empty intersection.*



An example for $p = 4$ and $d = 2$.

And again this bound is tight: Take $(d+1)(p-1)$ points, and place $(p-1)$ of them close to each of the $(d+1)$ vertices of a simplex in $\mathbb{R}^d$. This works, because for any vertex of the simplex we find a part of any partition into p disjoint subsets, which does not contain a point

close to this vertex. Therefore the convex hulls of the subsets have no common point.

Our aim (which we won't reach during this thesis) is to prove this topologically by examination of the following topological generalisation:

CONJECTURE 1.3 (Topological Tverberg). *Let $N := (d+1)(p-1)$ and $f : ||\Delta^N|| \to \mathbb{R}^d$ be a continuous map from the standard simplex $||\Delta^N||$ into $\mathbb{R}^d$. Then we find p disjoint faces $F_1, \ldots, F_p$ of Δ^N, such that $f(||F_1||) \cap \ldots \cap f(||F_p||)$ is non-empty.*

We call such a partition a **Tverberg partition**. If we further suppose f to be an affine map in this conjecture, this is an equivalent version of the (affine) Tverberg theorem 1.2. Conjecture 1.3 was proved for *prime powers* p, see among others [**Vol96**] for an elegant and short proof. Özaydin, who proved the same at first, did unfortunately not publish his preprint [**Öza87**].

Schöneborn and Ziegler have shown that the Topological Tverberg is equivalent to the so called *Winding Number Conjecture* [**ScZi05**]. Furthermore they show that this (Conj. 1.3) remains equivalent, when we add the condition that the faces $F_1, \ldots, F_p$ have to be at most d-dimensional. We can reformulate this by saying, the following conjecture is equivalent to the Topological Tverberg:

CONJECTURE 1.4 (d-Skeleton-Conjecture). *Let $N := (d+1)(p-1)$ and $f : ||\Delta^N_{\leq d}|| \to \mathbb{R}^d$ be a map from the d-skeleton $||\Delta^N_{\leq d}||$ of the N-dimensional standard simplex into $\mathbb{R}^d$. Then we can find a Tverberg partition, that is p disjoint faces $F_1, \ldots, F_p$ of $\Delta^N_{\leq d}$, such that $f(||F_1||) \cap \ldots \cap f(||F_p||)$ is non-empty.*

This may seem pretty obvious, but it isn't, since it is not apparent how to construct a Tverberg partition whose faces are all at most d-dimensional out of a Tverberg partition that contains higher dimensional faces. See [**ScZi05**, Prop. 2.2] for a proof.

Another useful observation [**Lon02**, Prop. 2.5] is the following

PROPOSITION 1.5. *If the Topological Tverberg (Conj. 1.3) holds true for $p \geq 2$ and $d \geq 2$, so it does for $p' = p$ and $d' = d - 1$.*

PROOF. Given a map $f' : ||\Delta^{N'}|| \to \mathbb{R}^{d'}$ with $N' = (d'+1)(p'-1) = d(p-1)$, embed $\mathbb{R}^{d'}$ into $\mathbb{R}^d$ by adding a 0-coordinate at the end, and construct a map $f : ||\Delta^{N'+p-1}|| \to \mathbb{R}^d$ by sending the N'-dimensional front face (given by the first $N'+1$ vertices) via f' to $\mathbb{R}^{d'}$, and the last $p-1$ vertices to arbitrary points in $\mathbb{R}^d$ with positive last coordinate, and on the rest of the simplex $||\Delta^{N'+p-1}||$ by linear extension.

Then any Tverberg partition $F_1, \ldots, F_p$ for f gives a Tverberg partition $F'_1, \ldots, F'_p$ for f' by intersecting the F_i's with the N'-dimensional front face of $\Delta^{N'+p-1}$, since at least one of the F_i's lies completely in this front face. See [**Lon02**, Prop. 2.5]) for more details. ⊘

According to J. Matoušek [**Mat03**, p. 154], the validity of the Topological Tverberg for arbitrary p is one of the most challenging problems in this field (of topological combinatorics). Hopefully one can understand it better sometime in the near future, since if it would turn out to be wrong for non prime powers, then it might give a fancy relationship between number theory and geometry.

2. Test maps for the Topological Tverberg

Fix numbers $d \geq 1$ and $p \geq 2$, let $N = (d+1)(p-1)$ as above, and assume that there is a map $f : ||\Delta^N|| \to \mathbb{R}^d$ that does not admit a Tverberg partition. We then have to construct a contradiction.

For this to do, we construct the following test map out of f, which is the restricted p-fold cartesian product of f:

(2.1)
$$f^p_\Delta : \bigcup_{\substack{p \text{ disj. faces} \\ F_1,\ldots,F_p \subset \sigma^N}} ||F_1|| \times \ldots \times ||F_p|| \longrightarrow_{S_p} (\mathbb{R}^d)^p \backslash \{(x, \ldots, x) \mid x \in \mathbb{R}^d\}.$$

Let X be the domain of f^p_Δ, Y be $(\mathbb{R}^d)^p$, and Z the diagonal
$$Z = \{(x, \ldots, x) \mid x \in \mathbb{R}^d\}.$$

All of them are S_p-spaces via permutating the coordinates. Clearly f^p_Δ is S_p-equivariant, and f^p_Δ is avoiding Z in its image, since we assumed f not to admit a Tverberg partition. $Y \backslash Z$ S_p-deformation-retracts first to $Z^\perp \backslash \{0\}$ and then to the unit sphere in $Z^\perp$, which we will identify with S^{dp-d-1}. The resulting S_p-homotopy equivalence is given by

$$\begin{array}{lcl} Y \backslash Z & \xrightarrow{\simeq}_{S_p} & S^{dp-d-1} \\ (x_1, \ldots, x_p) & \longmapsto & N(x_1 - \overline{x}, \ldots, x_p - \overline{x}), \end{array} \tag{2.2}$$

where $N : x \mapsto \frac{x}{||x||}$ is the normalising function, and $\overline{x} := \frac{1}{p} \sum_{i=1}^p x_i$ is the average function.

Remarks 2.3.

- Without adding extra conditions to our test map $X \longrightarrow_{S_p} S^{dp-d-1}$ it seems not to be possible to state an analogous relation about these test maps for different dimensions as we have in Proposition 1.5. That is, if we have a test map for p' and d', it is not clear how to construct a test map for $p = p'$ and $d = d' + 1$.

- In the literature (see e.g. [**Mat03**] for an overview) one finds as well another test map $f_\Delta^{*p} : ||(\sigma^N)^{*p}_{\Delta(2)}|| \longrightarrow_{S_p} (\mathbb{R}^d)^{*p}_\Delta$, which uses *deleted joins*, where certain indices are easier to calculate in the prime case, which then give a short contradiction for the existence of such a test map. Deleted joins are usually easier to deal with than deleted products. However, the deleted join construction looses more information about the problem, since a "product test map" induces a "join test map" (see Appendix A2), so our product test map (2.1) is stronger (or equipollent).

3. Applying obstruction theory

Now X is a *free* S_n-space of dimension $dp - d$. Therefore we can apply usual equivariant obstruction theory (see Appendix C), since there is only one obstruction to deal with, the primary obstruction $[o] \in H^{dp-d}_{S_n}(X; \pi_{dp-d-1}(Y \backslash Z))$.

3.1. Orientations. Before calculating this obstruction class, we have to deal with all orientation issues. Let us begin with X. It is a CW-complex whose cells $||F_1|| \times \ldots \times ||F_p||$ we will simply write as ordered p-tuples $(F_1, \ldots, F_p)$, whose elements F_i are pairwise disjoint subsets of $\{0, \ldots, N\}$. The dimension of each cell $(F_1, \ldots, F_p)$ is equal to $(\sum_i \#F_i) - p$. Each $F_i \subset \{0, \ldots, N\}$ get its standard orientation (by the order of the vertices which are ordered by $<$). If we let S_p act on the cells of X, then each orbit contains a unique cell $(F_1, \ldots, F_p)$ which satisfies $\min(F_1) < \ldots < \min(F_p)$. We give these cells the *direct sum orientation*, and to all other cells the unique orientation that makes X into a S_p-CW-complex (i. e. these orientations that make all left translations on X via elements in S_p orientation preserving. We cannot simply take for each cell the direct sum orientation, since then there would be orientation reversing left translations).

Now let's discuss Y and Z. $Y = (\mathbb{R}^d)^p$, so we simply give it the direct sum orientation (where each $\mathbb{R}^d$ gets its standard orientation). Z has as a vector space at each point a tangent space, which is naturally isomorphic to Z itself. So let's define the basis

$$((e_1, \ldots, e_1), \ldots, (e_d, \ldots, e_d))$$

of Z to be a positively oriented basis of the tangent space $T_P Z$ for all $P \in Z$, where $(e_1, \ldots, e_d)$ is the standard basis of $\mathbb{R}^d$. Now orient $Z^\perp \subset Y$ in such a way, that $Z \oplus Z^\perp = Y$ as oriented vector spaces, i. e. the direct sum orientation of $Z \oplus Z^\perp$ shall be the same as the orientation of Y. The unit sphere in $Z^\perp$, which we denoted by S^{dp-d-1}, is the preimage of 1 under the map $Z^\perp \to \mathbb{R} : x \mapsto ||x||^2$, so we

might give it the preimage orientation (by assuming $\mathbb{R}$ to be standard oriented). That is, at each point P of the sphere, an outer normal vector at P together with a positive oriented basis of $T_P S^{dp-d-1}$ has to yield a positive oriented basis of $T_P Z^\perp$.

3.2. The equivariant cellular cochain complex. How does the S_p-action on S^{dp-d-1} affect the orientation? Suppose $\tau \in S_p$ is a transposition. This τ preserves the orientation of Y, iff d is even, as one checks easily[1]. It preserves always the orientation of Z, because Z stays fixed under S_p. Together this implies that τ preserves the orientation of $Z^\perp$, iff again d is even. Since outer normal vectors of S^{dp-d-1} are mapped by $\mathrm{d}\tau$ to outer normal vectors, τ also preserves the orientation of S^{dp-d-1} iff d is even. Since transpositions generate S_p, we get, that any $\rho \in S_p$ preserves the orientation of S^{dp-d-1}, iff d is even or $\operatorname{sign}(\rho) = 1$. The action on S^{dp-d-1} induces an action on $\pi_{dp-d-1}(S^{dp-d-1}) \cong \mathbb{Z}$:

LEMMA 3.1. *Let $\rho \in S_p$ and $z \in \pi_{dp-d-1}(S^{dp-d-1}) \cong \mathbb{Z}$. Then:*

$$\rho \cdot z = \begin{cases} z & \text{if } d \text{ is even,} \\ \operatorname{sign}(\rho) \cdot z & \text{if } d \text{ is odd.} \end{cases}$$

PROOF. Follows from the action on S^{dp-d-1}. ∎

Hence, the equivariance of a cochain $c \in C^*_{S_n}(X; \pi_{dp-d-1}(S^{dp-d-1}))$ means

$$c(\rho(e)) = \begin{cases} c(e) & \text{if } d \text{ is even,} \\ \operatorname{sign}(\rho) \cdot c(e) & \text{if } d \text{ is odd,} \end{cases} \tag{3.2}$$

for each cell e of X of suitable dimension and $\rho \in S_p$. The next lemma states, how the coboundary operator δ looks like in the cellular equivariant cochain complex $C^*_{S_n}(X; \pi_{dp-d-1}(S^{dp-d-1}))$.

LEMMA 3.3. *Let $c \in C^*_{S_n}(X; \pi_{dp-d-1}(S^{dp-d-1}))$ and*

$$e = (F_1, \ldots, F_p) \in C^{S_n}_*(X; \pi_{dp-d-1}(S^{dp-d-1})),$$

such that $\min F_1 < \ldots < \min F_p$. *Then*

$$(\delta c)(e) = c(\partial e) = \sum_{i=1}^{p} \sum_{k=0}^{\dim F_i} \varepsilon_{i,k} \cdot c(F_1, \ldots, [v_i^0, \ldots, \widehat{v_i^k}, \ldots, v_i^{\dim F_i}], \ldots, F_p),$$

where

$$F_i = [v_i^0, \ldots, v_i^{\dim F_i}],\ v_i^0 < \ldots < v_i^{\dim F_i}$$

[1]A standard basis of $T_P Y = Y$ gets mapped by $\mathrm{d}\tau$ to a basis of $T_{\tau(P)} Y$, which differs from the standard basis by $d^2 \equiv d \mod 2$ transpositions.

and

$$\varepsilon_{i,k} = \begin{cases} (-1)^{\sum_{j=1}^{i-1} \dim F_j + k} & \text{if } k > 0 \text{ or } \dim F_i \text{ is even,} \\ (-1)^{\sum_{j=1}^{i-1} \dim F_j + \sharp\{j \mid \dim F_j \text{ is odd, and } v_i^0 < v_j^0 < v_i^1\}} & \text{if } k = 0 \text{ and } \dim F_i \text{ is odd.} \end{cases}$$

Proof. If all cells of X had the direct sum orientation of the F_i's, where the F_i's had their standard orientation, then the formula for δc would be correct, if we just set $\varepsilon_{i,k} = (-1)^{\sum_{j=1}^{i-1} \dim F_j + k}$; compare [**Hat06**, Prop. 3B1]. Therefore we just need to figure out how the signs $\varepsilon_{i,k}$ change when we use our given orientation. Since $\min F_1 < \ldots < \min F_p$, the orientation of e is equal to its direct sum orientation. The same is true for all its boundary cells that resulted from deleting one of the vertices v_k^i of an F_i with $k > 0$. But a boundary cell that resulted from deleting a v_k^i with $k = 0$ may have a different orientation than its direct sum orientation, since its orientation was given by reordering a positive standard basis of the direct sum orientation, such that two blocks of standard basis vectors corresponding to F_i and F_j with $i < j$ become swapped, iff $v_i^0 > v_j^0$. Hence, when we take e and delete from its i'th factor F_i the first vertex, we have to swap the $\dim F_i$ standard basis vectors (concerning to the direct sum orientation) corresponding to the factor F_i with all such vectors corresponding to factors F_j with $v_i^0 < v_j^0 < v_i^1$, to finally get a positive standard basis for our chosen orientation. This makes

$$\dim F_i \cdot \sum_{j:\ v_i^0 < v_j^0 < v_i^1} \dim F_j$$

transpositions of vectors, which yields the claimed signs. ∅

Remark 3.4. The formula

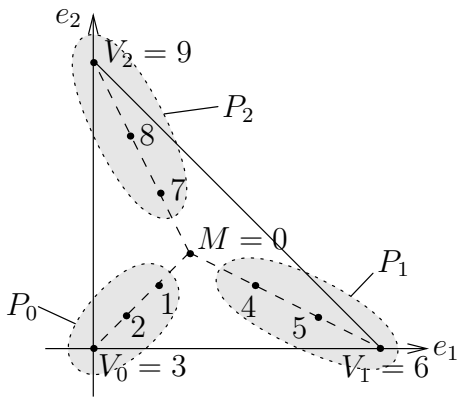
$$(-1)^{\sum_{j=1}^{i-1} \dim F_j + \sharp\{j \mid \dim F_j \text{ is odd, and } v_i^0 < v_j^0 < v_i^1\}}$$

can be rewritten as

$$(-1)^{\sum_{j:\ j<i \text{ or } (j>i \text{ and } v_j^0 < v_i^1)} \dim F_j}$$

3.3. The map to be extended. Now we plan to construct a map $G : X_{\leq dp-d-1} \longrightarrow_{S_p} Y \backslash Z$, where $X_{\leq dp-d-1}$ is the $(dp-d-1)$-skeleton of X. The obstruction cocycle$o$ of extending this map to all of X will then tell us our primary obstruction $[o]$ for finding a test map (2.1). Let $||\sigma^d||$ be the standard simplex in $\mathbb{R}^d$ with vertices $V_0, \ldots, V_d$, which has its 0'th vertex V_0 at $0 \in \mathbb{R}^d$, and its i'th vertex V_i at e_i, the ith standard basis vector, for $i \in \{1, \ldots, d\}$. We denote its vertices by $\{V_0, \ldots, V_d\}$ to avoid notational confusions. Let $M = (\frac{1}{d+1}, \ldots, \frac{1}{d+1}) \in \mathbb{R}^d$ be $||\sigma^d||$'s mid-point. Connect M with each of the vertices $V_0, \ldots, V_d$

by an imaginary line segment, and put into the interior of each of the $d+1$ segments $p-2$ points, and together with the vertices $V_0, \ldots, V_d$ these form $d+1$ sets $P_0, \ldots, P_d$ of $p-1$ points each. Now put the labels $0, \ldots, N = (d+1)(p-1)$ on M and the $(d+1)(p-1)$ other points on the segments, like the following picture for $d = 2$ shows:



That is, M gets the label 0, the points of P_0 get the labels $1, \ldots, p-1$ (beginning with the point of P_0 closest to M), then the points of P_1 get the labels $p, \ldots, 2p-2$, and so on. Define $\overline{P_i} := P_i \cup \{M\}$.

Now define an *affine* map $g : ||\sigma^N|| \to \mathbb{R}^d$, which sends each of the vertices called $0, \ldots, N$ to the points $\bigcup_{i=0}^d P_i \cup \{M\}$, such that i gets mapped to the point with label i. Then g is uniquely defined by linear extension of the values of g on the vertices. Define $G := g^p|_{\ldots} : X_{\leq dp-d-1} \to Y$ to be the restriction of the p'th cartesian product to $X_{\leq dp-d-1}$. This is actually a map

$$G : X_{\leq dp-d-1} \longrightarrow_{S_p} Y \backslash Z,$$

since G does not hit Z: This statement is equivalent to saying, that there is no partition of the points $\bigcup_{i=0}^d P_i \cup \{M\}$ into p disjoint subsets $F_1, \ldots, F_p$, such that at least one point is not used (since we only look at the $(dp-d-1)$-skeleton of X), and the convex hulls of $F_1, \ldots, F_p$ intersect non-empty. Assume that such a partition would exist. Let w.l.o.g. $\overline{P_0}$ contain the unused point. Then for each $\overline{P_0}, P_1, \ldots, P_d$ there is an i, such that this set of $\overline{P_0}, P_1, \ldots, P_d$ does not intersect F_i. Therefore the convex hulls of the F_i have no common point.[2]

[2]Remark: This reasoning would not work for $G = g^p$ without restricting it, since there is very apparently a partition of the points $\bigcup_{i=0}^d P_i \cup \{M\}$ into p sets, whose convex hulls intersect non-empty. That is, some top-dimensional cell of X would hit Z. But otherwise the Tverberg-problem were simply solved by a counter-example.

Now, let o be the obstruction cocycle of extending the map $G : X_{\leq dp-d-1} \longrightarrow_{S_p} Y\backslash Z$ to all of X. o is an equivariant cellular cochain in $C^{dp-d}_{S_n}(X; \pi_{dp-d-1}(Y\backslash Z))$.

3.4. Deducing the primary obstruction for a test map (2.1). The following theorem will do the main calculation:

THEOREM 3.5. *Let $e = (F_1, \ldots, F_p)$ be a $(dp-d)$-cell of X. Then $o(e) \neq 0$, iff one of the F_i's identifies under g with $\{M\}$ and for all other F_i's and $j \in \{0, \ldots, d\}$, $\sharp(g(F_i) \cap P_j) = 1$.*
If e is of this form, and by equivariance of o (3.2) assume $g(F_1)$ to be $\{M\}$, then $o(e) = 1$.

PROOF. First, we want to show with a degree argumentation that if $e = (F_1, \ldots, F_p)$ is not of this special form, then $o(e) = 0$. If e is not of this form, then there is an $i \in \{1, \ldots, p\}$ and a $j \in \{0, \ldots, d\}$, such that $g(F_i) \cap \overline{P_j} = \emptyset$. By equivariance of o let's assume $i = 1$, and by the symmetry of the following argumentation we assume $j = 0$ (if $j \neq 0$, then in the following argumentation we have to exchange the all-one vector $\mathbb{1}$ by a vector $M - V_j$, or a positive scalar of it). Hence $g(F_1) \cap \overline{P_0} = \emptyset$, or in other words, $g(F_i) \subset P_1 \cup \ldots \cup P_d$.

Let $\mathbb{1} = (1, \ldots, 1) \in \mathbb{R}^d$ and $w = (0, \ldots, 0, \mathbb{1}) \in Y = (\mathbb{R}^d)^p$.

CLAIM 3.6. *There are no two points $A, B \in \mathbb{R}^d$, such that $B = r \cdot \mathbb{1} + A$, $r > 0$ and $A \in \operatorname{conv}(g(F_1)) \cap \operatorname{conv}(g(F_{p-1}))$ and $B \in \operatorname{conv}(g(F_p))$.*

Here we need the assumption $d \geq 2$.

PROOF OF CLAIM. For all $k \in \{0, \ldots, d\}$ there is an F_i whose image under g does not contain any point of P_k, since each P_k contains only $p-1$ points. Therefore

$$A, B \in \mathbb{R}_+(1, \ldots, 1) + \operatorname{conv}\{\overline{P_0}, \ldots, \widehat{\overline{P_k}}, \ldots, \overline{P_d}\}.$$

And hence, A and B lie on the line h through all the points of $\overline{P_0}$, since this line has direction $\mathbb{1}$. Since $A \in \operatorname{conv}(g(F_1)) \cap \operatorname{conv}(g(F_{p-1}))$ and $g(F_i) \subset P_1 \cup \ldots \cup P_d$, we have $A \subset \operatorname{conv}(P_1 \cup \ldots \cup P_d)$. Since $B \in \mathbb{R}_+\mathbb{1} + A$, the same is true for B: $B \subset \operatorname{conv}(P_1 \cup \ldots \cup P_d)$. Since every $g(F_1), \ldots, g(F_p)$ has to contain A or B, and A and B lie in $h \cap \operatorname{conv}(P_1 \cup \ldots \cup P_d)$, every $g(F_1), \ldots, g(F_p)$ has to intersect each $P_1, \ldots, P_d$ at least once. But each P_i has only $p-1$ points, contradiction. This proves the claim 3.6. $\not\Leftarrow$

The claim implies, that $G(\partial e)$ does not intersect $\mathbb{R}_+ w + Z \subset Y$ (even the "linear extension" of G to all of X does not map any point

of the whole cell e to $\mathbb{R}_+ w + Z$; this is what the claim states). This is equivalent to saying, that the composition

$$X_{\leq dp-d-1} \xrightarrow{G}_{S_p} Y\backslash Z \xrightarrow{\simeq}_{S_p} Z^{\perp}\backslash\{0\} \xrightarrow{\simeq}_{S_p} S^{dp-d-1}$$

maps no point of $\partial(e)$ to $N(w-\overline{w})$. Here again, N is the normalising function and $\overline{w} = \frac{1}{p}\sum_{i=1}^{p} w_i$ the average function. Therefore, $G \circ \partial e$ is nullhomotopic, for $S^{dp-d-1}\backslash\{*\}$ is contractible. Hence $o(e) = 0$, which finishes the first part.

Now let $e = (F_1, \ldots, F_p)$, such that $g(F_1) = \{M\}$ and each of $g(F_2), \ldots, g(F_p)$ contains from each $P_0, \ldots, P_d$ exactly one point. We will calculate $o(e) = [G \circ \partial e] \in \pi_{dp-d-1}(Y\backslash Z) \cong \pi_{dp-d-1}(S^{dp-d-1}) \cong \mathbb{Z}$ as the degree of the function $\partial e \xrightarrow{G} Y\backslash Z \xrightarrow{\phi} S^{dp-d-1}$. We just need to figure out, how often $\phi \circ G$ intersects $\widetilde{w} := N(w-\overline{w})$ counted with the local degree. Let $(F_1', \ldots, F_p')$ be a cell in the boundary of e (that is, $F_i' = F_i$ for all except one i. The other $F_i' = F_i\backslash$ one vertex), whose image under $\phi \circ G$ intersects $\widetilde{w}$. The preimage of $\phi^{-1}(\widetilde{w})$ are points of the form $(A, \ldots, A, B) \in (\mathbb{R}^d)^p$. Since $g(F_1') = \{M\}$, $A = M$. Hence $B \in \mathbb{R}_+\mathbb{1} + M$, therefore $g(F_p')$ has to contain at least one point of each $P_1, \ldots, P_d$. As well, since $A = M = g(F_1')$, any $g(F_2'), \ldots, g(F_{p-1}')$ has to contain at least one point of each $P_0, \ldots, P_d$. It remains at most one point for the deleted point (remember that $(F_1', \ldots, F_p')$ came from $(F_1, \ldots, F_p)$ by deleting one vertex. Summing up, we have the following situation:

- $g(F_1') = g(F_1) = \{M\}$.
- $g(F_i') = g(F_i)$ contains of each $P_0, \ldots, P_d$ exactly one point, for all $i \in \{2, \ldots, p-1\}$.
- $F_p' = [\widehat{v_p^0}, v_p^1, \ldots, v_p^d] = F_p\backslash\{v_p^0\}$, whose image under g contains of each $P_1, \ldots, P_d$ exactly one point.

$G(F_1', \ldots, F_p')$ contains exactly one point of the form $(A, \ldots, A, B) \in (\mathbb{R}^d)^p$, which is equivalent to saying, that $(F_1', \ldots, F_p')$ intersects under the map $\phi \circ G$ our chosen point $\widetilde{w} \in S^{dp-d-1}$ exactly once. Let $\widetilde{v} := (\phi \circ G)^{-1}(\widetilde{w})$ be this point. It remains to show, that the local degree of $\partial e \xrightarrow{\phi \circ G} S^{dp-d-1}$ at this intersection is equal to $(-1)^{dp}$. The boundary cell $(F_1', \ldots, F_p')$ fulfills $\min F_1' < \ldots < \min F_p'$, for $F_p < F_p'$, therefore its given orientation is the same as the direct sum orientation of $F_1' \times \ldots \times F_p'$. The same was true for e. Therefore the coefficient $[(F_1', \ldots, F_p') : e]$ of $(F_1', \ldots, F_p')$ in ∂e is concerning to Lemma 3.3 or [**Hat06**, Prop. 3B1],

$$[(F_1', \ldots, F_p') : e] = (-1)^{\sum_{j=1}^{p-1} \dim F_j} = (-1)^{d(p-2)} = (-1)^{dp} \tag{3.7}$$

It remains to show, that the composition $\psi : (F_1', \dots, F_p') \overset{j}{\hookrightarrow} \partial e \overset{\phi \circ G}{\longrightarrow} S^{dp-d-1}$ is transversal at $\widetilde{v}$ and preserves orientation at this point iff $(-1)^{dp} = 1$.

The tangent space $T_{\widetilde{v}}(F_1', \dots, F_p') \cong \mathbb{R}^{dp-d}$ has the column vectors of the following matrix as a positive oriented basis

$$\begin{pmatrix} I_{\dim F_1'} & & & 0 \\ & I_{\dim F_2'} & & \\ & & \ddots & \\ 0 & & & I_{\dim F_p'} \end{pmatrix},$$

where I_n is the n'th identity matrix, which correspond here to positive oriented bases of the standard simplices $\operatorname{conv}(0, e_1, \dots, e_{\dim(F_i')})$. This gets mapped by

$$\mathrm{d}_{\widetilde{v}}(G \circ j) : T_{\widetilde{v}}(F_1', \dots, F_p') \longrightarrow T_{G(\widetilde{v})}(Y \backslash Z)$$

to the columns of

$$B := \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_p \end{pmatrix}, \tag{3.8}$$

where $B_i = \Big(g(v_i^1) - g(v_i^0) \ \dots \ g(v_i^{\dim F_i'}) - g(v_i^0) \Big)$ is a $p \times (\dim F_i)$-matrix. Note that B_0 is a $p \times 0$-matrix.

Claim 3.9. *The columns of B get mapped by*

$$\mathrm{d}_{G(\widetilde{v})}\phi : T_{G(\widetilde{v})}(Y \backslash Z) \longrightarrow T_{\widetilde{w}} S^{dp-d-1}$$

to a positive basis, iff

$$\operatorname{sign} \circ \det \underbrace{\left(\begin{array}{c|c|cccc} I_d & & B_1 & & & 0 \\ I_d & & & B_2 & & \\ \vdots & w & & & \ddots & \\ I_d & & 0 & & & B_p \end{array}\right)}_{M} = 1.$$

Proof. First of all, we can replace w in this matrix by $\widetilde{w} = N(w_1 - \overline{w}, \dots, w_d - \overline{w})$, since subtracting $(\overline{w}, \dots, \overline{w})$ doesn't affect the determinant (note that $(\overline{w}, \dots, \overline{w})$ is a linear combination of the first p columns of the matrix), and normalising columns doesn't change the sign of the determinant.

Now, the columns of M with $\widetilde{w}$ instead of w are a positive basis of $T_{G(\widetilde{v})}(Y \backslash Z)$, iff we get a positive basis of $Z^\perp$ when we project all

but the first d columns to $Z^\perp$ (by definition of the orientation of $Z^\perp$ and noting that the first d columns of the matrix are a positive basis for Z). And this is a positive basis for $Z^\perp$, iff all but the first vector (that is w) projected to $T_{\widetilde{w}}S^{dp-d-1}$ are a positive basis of $T_{\widetilde{w}}S^{dp-d-1}$, by definition of the orientation of S^{dp-d-1}. That is, we have shown, that $\operatorname{sign}\circ\det M = 1$, iff the columns of B form a positive basis of $T_{\widetilde{w}}S^{dp-d-1}$, after projecting them first down to $Z^\perp$ and then to $T_{\widetilde{w}}S^{dp-d-1}$. But this is exactly what $\mathrm{d}_{G(\widetilde{v})}\phi$ does, so this proves Claim 3.9. ⊘

By Claim 3.9 it remains to show that $\operatorname{sign}\circ\det M = (-1)^{dp}$. First of all note that g maps all $||F_2||, \dots, ||F_p||$ orientation preserving from $||\sigma^d||$ to $\mathbb{R}^d$. Therefore, $\operatorname{sign}\circ\det B_i = 1$ for all $i \in \{2, \dots, p-1\}$, and we can replace all of the $B_2, \dots, B_p$ by I_d's (we just have to perform column transformations in M). Thus we have to show,

$$\operatorname{sign}\circ\det\left(\begin{array}{c|c|ccc|c} I_d & 0 & 0 & & & 0 \\ \hline I_d & 0 & I_d & & & \\ \vdots & \vdots & & \ddots & & \\ I_d & 0 & & & I_d & \\ \hline I_d & \mathbb{1} & 0 & & & B_p \end{array}\right) = (-1)^{dp}.$$

Shifting the w-column $d(p-2)$ positions to the right, and noting that $\operatorname{sign}\circ\det(\mathbb{1}\ B_p) = 1$, we get the equivalent statement

$$(-1)^{d(p-2)}\cdot\operatorname{sign}\circ\det\left(\begin{array}{c|ccc|c} I_d & 0 & & & 0 \\ \hline I_d & I_d & & & \\ \vdots & & \ddots & & \\ I_d & & & I_d & \\ \hline I_d & 0 & & & I_d \end{array}\right) = (-1)^{dp},$$

which is apparently true. This proves Theorem 3.5. ⊘

3.5. The final integral linear equation system. We sum up this section in the following theorem.

THEOREM 3.10. *Let parameters $d \geq 1$ and $p \geq 2$ for the Topological Tverberg Problem (Conj. 1.3) be given. Then there is a test map $X \longrightarrow_{S_p} Y\backslash Z$ (see 2.1), iff there is an integral solution $c \in C^{dp-d-1}_{S_n}(X; \pi_{dp-d-1}(Y\backslash Z))$ of the integral linear equation system $\delta c = o$, where*

- *c is equivariant by means of Equation* (3.2),
- *δ was calculated in Lemma 3.3, and*
- *o was calculated in Theorem 3.5.* ⊘

Remark 3.11. Özaydin has already shown in his unpublished preprint [**Öza87**], that the obstruction class is zero, iff p is a prime power with different methods (he used the transfer homomorphism to see that the obstruction class is zero, iff all restricted obstruction classes are zero when restricting to all Sylow subgroups of S_p. Now for non-prime powers there even exists a constant test map, when restricting to any Sylow subgroup.)

However this direct approach may give a topological proof of the original affine Tverberg even for non-prime powers p, using the following key observation [B. Hanke, private communication].

Lemma 3.12. *If the (affine) Tverberg Theorem were wrong for some parameters $d \geq 1$ and $p \geq 2$, then the above equation system $\delta c = o$ would have a solution c which takes only values in $\{0, +1, -1\}$.*

Proof. If there were a counter-example to the Tverberg Theorem, then this would give another affine test map $f' : X \longrightarrow_{S_p} Y \backslash Z$, different from our constructed affine test map f. The obstruction cocycle for f' is apparently zero. Therefore our obstruction cocycle o for the map f is the coboundary of the difference cochain d of the affine homotopy between $f_{X_{dp-d-1}}$ and $f'_{X_{dp-d-1}}$. But an affine map from a prism over a product of simplices can wind around a point with orientation only 0, $+1$ or -1 times. Hence d takes only values in $\{0, +1, -1\}$.

In this argumentation one surely has to take care that the used linear homotopy hits Z only with the top-dimensional cells. For this to do just deform f a bit affinely and move the affine homotopy between f and f' according to it. ∅

The remaining task is now to see whether the now restricted equation system has a solution (hopefully it has none...).

APPENDIX A

Elementary Approaches

A1. A useful lemma

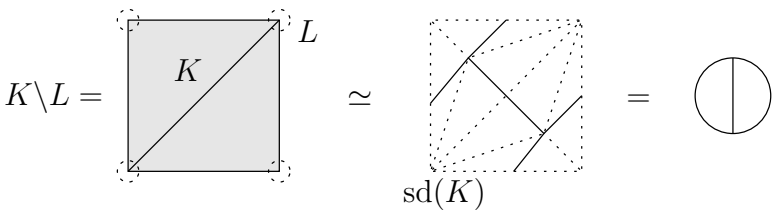
Let G be a finite group. A simple but very useful lemma is the following:

LEMMA A1.1. *Let K be a k-dimensional simplicial G-complex and $L \subset K$ a G-invariant subcomplex that contains the ℓ-skeleton of K. Then $K \backslash L$ deformation-retracts G-equivariantly to a subset that is again a simplicial G-complex of dimension $\leq k - \ell - 1$.*

PROOF. Subdivide K and L barycentrically. Define K' to be the subcomplex of $\mathrm{sd}(K)$, that contains all the cells whose intersection with L is empty. Then $K \backslash L$ deformation-retracts to K' in a natural way: All cells of $\mathrm{sd}(K) \backslash \mathrm{sd}(L)$ are already in K', or have some boundary cells in $\mathrm{sd}(L)$ and some in K'. The cells of the second kind can easily be pushed into K' by a linear homotopy. This deformation retraction is of course a G-homotopy.

The dimension statement is clear by definition of the barycentric subdivision: The cells of K' are chains in the order complex[1] of K, which do not contain cells of $L \supset K_{\leq d}$. So all elements of this chain have to be at least $(\ell + 1)$-dimensional, therefore all chains have a length[2] less than $k - \ell - 1$. ∎

Exemplarily let $K = S^1 \times S^1$ be the torus with $\mathbb{Z}_2$ action which changes both S^1-factors, and let L be the base point:

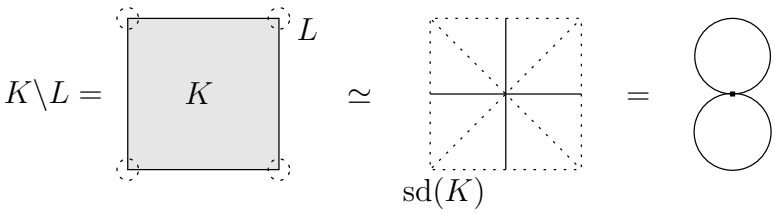


[1]The order complex of a simplicial complex is the poset of all cells, except the empty cell, ordered by inclusion.

[2]The length of a chain is equal to the number of its elements minus one. Here it is as well equal to the dimension of the cell corresponding to the chain.

Remarks A1.2.

- When the maximal faces of K are not all of the same dimension n (i. e. K is not *pure*), one can clearly relax the condition on L a bit: If L contains all cells e of K, which have a surrounding cell $f \supset e$ in K with $\dim(f) \geq \dim(e) + x$, then $K \backslash L$ deformation-retracts G-equivariantly to a simplicial complex of dimension $\leq x - 1$. The previous proof works as well for this statement.
- A similar argument works for CW-complexes with G-action translating cells to cells[3], as long as it admits a "useful" barycentric subdivision, e. g. if the complex is regular and the cells are G-homeomorphic to unit discs of G-representations. This works because the subdivision becomes a G-CW-complex! The above example simplifies a bit:



A2. Deleted products vs. deleted joins

This section is to show that sometimes one can see already in advance test maps coming from deleted product constructions to be better than these coming from deleted joins, that is, they give a stronger necessary criterion for the non-existence of counter-examples of the given problem.

Suppose we are given a space X, an S_p-invariant subspace X_0 of X^p and integers $d \geq 1$, $p \geq 2$ and $k \geq 2$. Suppose we want to show, that there is no map

$$f : X \longrightarrow \mathbb{R}^d$$

satisfying the condition:

If $(x_1, \ldots, x_p) \in X_0$, then $f(x_1), \ldots, f(x_p)$ are k-wise distinct,

that is, no k of them are equal.

Example A2.1. An instance of this situation is the Topological Tverberg Problem (Conjecture IV.1.3) with

- d and p are the usual parameters,

[3]Note that this is much more general than G-CW-complexes, since the latter also requires elements of G, which let a cell invariant, to *fix* this cell.

- $X = ||\sigma^N||$,
- $X_0 = \{(x_1, \ldots, x_p) \mid \text{the } x_i \text{ lie in pairwise distinct faces of } ||\sigma^N||\}$, and
- $k = p$.

The k**-fold deleted product** of $\mathbb{R}^d$ is defined by

$$(\mathbb{R}^d)^p_{\Delta(k)} := \{(x_1, \ldots, x_p) \in (\mathbb{R}^d)^p \mid x_1, \ldots, x_p \text{ are } k\text{-wise distinct}\},$$

and the k**-fold deleted join** of $\mathbb{R}^d$ by

$$(\mathbb{R}^d)^p_{\Delta(k)} := \{\textstyle\sum \lambda_i x_i \in (\mathbb{R}^d)^{*p} \mid \text{if } \lambda_1 = \ldots = \lambda_p \text{ then } x_1, \ldots, x_p \text{ are } k\text{-wise distinct}\}.$$

Lemma A2.2. *Suppose there is a "product test map"*

$$f^\times : X_0 \longrightarrow_{S_p} (\mathbb{R}^d)^p_{\Delta(k)},$$

then there is also a "join test map"

$$f^* : X_0^* \longrightarrow_{S_p} (\mathbb{R}^d)^{*p}_{\Delta(k)}$$

where X_0^* is the "joinified" version of X_0:

$$X_0^* := \left\{ \sum_{i=1}^p \lambda_i x_i \mid (x_1, \ldots, x_p) \in X_0, 0 \le \lambda_i \le 1, \sum \lambda_i = 1 \right\} \subset X^{*p}.$$

Proof. Construct f^* as

$$f^*(\lambda_1 x_1 + \ldots + \lambda_p x_p) := \sum_{i=1}^p \lambda_i \cdot \left(\left(p^p \prod_{j=1}^p \lambda_j \right) \cdot f_i^\times(x_1, \ldots, x_p) \right).$$

The operations in the big parentheses are the usual multiplication and power in $\mathbb{R}$ and the scalar multiplication between $\mathbb{R}$ and $\mathbb{R}^d$. f^* is well-defined because of the product term. Also, f^* is mapping into $(\mathbb{R}^d)^{*p}_{\Delta(k)}$ as long as $f^\times$ is mapping into $(\mathbb{R}^d)^p_{\Delta(k)}$. We could also have omitted the multiplication with p^p in the definition of f^*, but it is there to make the following diagram commute:

$$\begin{tikzcd} X_0 \arrow[r, "f^\times"] \arrow[d] & (\mathbb{R}^d)^p_{\Delta(k)} \arrow[d] \\ X_0^* \arrow[r, "f^*"] & (\mathbb{R}^d)^{*p}_{\Delta(k)} \end{tikzcd}$$

The vertical maps are the embeddings $(x_1, \ldots, x_p) \longmapsto \frac{1}{p}x_1 + \ldots + \frac{1}{p}x_p$. ∅

A3. Inductive construction of maps

The inductive method of constructing equivariant maps $f : X \longrightarrow_G Y$ is very useful, especially in connection with relative equivariant obstruction theory (Appendix C), since at each step we can deal with a *free* action on the part of the domain, where the map is to be extended.

Suppose we are given G-spaces X and Y. Let

$$I_X := \{G_x \mid x \in X\}$$

be the set of all **isotropy groups** of X, $G_x := \{g \in G \mid gx = x\}$. The inclusion relation of subgroups of G makes I_X into a poset. Since $G_{gx} = gG_xg^{-1}$, I_X is closed under conjugation of subgroups, that is: $H \in I_X, g \in G \Rightarrow gHg^{-1} \in I_X$. Let U be an up-set of I_X ($H \in U$ and $H \subset H'$ then $H' \in U$), which is closed under conjugation. Let $H \in I_X \backslash U$ be maximal (under inclusion), and denote by $(H) := \{gHg^{-1} \mid g \in G\}$ the conjugation class of H. Then

$$U' := U \cup (H)$$

is another up-set of I_X which is closed under conjugation. Denote by

$$X(U) := \{x \in X \mid G_x \in U\}$$

the set of all points in X whose isotropy group lies in U. Now, suppose we want to extend a map

$$f : X(U) \longrightarrow_G Y$$

to $X(U')$. Denote by $NH := \{g \in G \mid gHg^{-1} = H\}$ the normaliser of H in G and by $X^H := \{x \in X \mid Hx = \{x\}\}$ the fixed points of X under H.

Proposition A3.1 ([**Die86**, Prop. I(7.4)]). *The G-equivariant extensions of a map $f : X(U) \longrightarrow_G Y$ to $X(U')$ correspond bijectively to the (NH/H)-equivariant extensions of $f|_{X(U)^H} : X(U)^H \longrightarrow_{NH/H} Y$ to $X(U')^H$.*

Note that

- NH/H acts freely on $X(U')^H \backslash X(U)^H$.
- Maps $f : X(U) \longrightarrow_G Y$ are actually mapping into $Y(U)$, since U is an up-set.
- Maps $g : X^H \to Y$ are actually mapping into Y^H.

Proof. For the forward direction we just restrict the map $f : X(U') \longrightarrow_G Y$ to $X(U')^H$. The other direction works by extending the map $X(U')^H \longrightarrow_{NH/H} Y$ to all of $X(U') = G \cdot (X(U')^H)$ G-equivariantly (this is a unique description). The reader might fill in the details or look at [**Die86**, Prop. I(7.4)]. ∎

A4. Equivariant maps and cross sections

There is an elementary way of viewing equivariant maps as sections of certain fibre bundles (with restrictions). This opens bundle techniques such as characteristic classes to treat the existence issue of equivariant maps. Suppose we are given a G-map $f : X \longrightarrow_G Y$. This induces a map $s_f := (id, f)/_G$:

$$s_f : X/G \longrightarrow (X \times Y)/G : [x] \longmapsto [x, f(x)],$$

where $X \times Y$ gets the diagonal action $g \cdot (x, y) := (g \cdot x, g \cdot y)$. s is a section in the bundle $p : (X \times Y)/G \longrightarrow X/G$ with fibre Y defined by $p := pr_1/G : [x, y] \mapsto [x]$. Actually, s maps into

$$M(X, Y) := \{[x, y] \in (X \times Y)/G \mid G_x \subset G_y\},$$

where $G_x := \{g \in G \mid g \cdot x = x\}$ is the isotropy group of X at x, and analogously G_y. $M(X, Y)$ usually is not as *nice* as the fibre bundle $(X \times Y)/G$. However, if X is a free G-space, then both spaces coincide.

Now we state a classifying result. For a proof and a more general version see [**Die86**, Ch. I.7, (7.2) and (7.3)].

Proposition A4.1. *Let G be a compact group and X and Y Hausdorff G-spaces. Then the correspondence $f \longmapsto s_f$ is a bijection between G-maps $f : X \longrightarrow Y$ and sections of $p|_{M(X,Y)} : M(X, Y) \longrightarrow X/G$.*

$\not\Phi$

Note that this bijection is obvious except for continuity issues.

Remark A4.2. If G does not act freely on X then $p|_{M(X,Y)}$ is usually a rather complicated bundle. Instead one can make a "kind of Borel"-construction. Indeed a G-map $X \longrightarrow_G Y$ induces a map $EG \times_G X \longrightarrow EG \times_G (X \times Y) : [e, x] \mapsto [e, (x, f(x))]$, which is a section in the fibre bundle

$$p' : EG \times_G (X \times Y) \longrightarrow EG \times_G X : [e, (x, y)] \mapsto [e, x]$$

with typical fibre Y. This in fact really generalises the construction of p if we deal with G-CW-complexes: If X is a free G-CW-complex, then we have a bundle homotopy equivalence:

$$\begin{tikzcd}
EG \times_G (X \times Y) \arrow[r, "pr_2/_G"] \arrow[d, "p'"'] & X \times_G Y \arrow[d, "p"'] \\
EG \times_G X \arrow[r, "pr_2/_G"] & X/G,
\end{tikzcd}$$

where the top arrow is induced by projecting to the second factor, which is in fact a homotopy equivalence by the long exact homotopy sequence ($EG \simeq \{*\}$) and Whitehead's Theorem. Hence p has a section iff p' does.

A5. Cross sections and characteristic classes

Suppose, we want to disprove the existence of a G-map $f : X \longrightarrow_G Y\backslash\{0\}$, where X is a free G-space and Y is a real linear G-representation. From the last section it is enough to show the non-existence of a nowhere vanishing section in the *vector bundle* $p : (X \times Y)/G \longrightarrow X/G$. Here we want to list necessary conditions for the existence of such a cross section, all of which can be found in [**MiSt74**].

Proposition A5.1. *Suppose $p : E \longrightarrow B$ is a vector bundle of rank n which admits a nowhere vanishing cross section. Then:*

- *The n'th Stiefel–Whitney class $\omega_n(p) \in H^n(B; \mathbb{Z}_2)$ is zero.*
- *If p is orientable, then its Euler class $e(p) \in H^n(B; \mathbb{Z})$ is zero.*

Proof. See [**MiSt74**, §4.4, §9.7]. ∅

Remark A5.2. If X is not free, but Y still a G-representation, we can use the remark of the last section: If we can show that the vector bundle p' of Remark A4.2 does not admit a nowhere vanishing section using characteristic classes, then there cannot be a map $X \longrightarrow_G Y$.

APPENDIX B

Cohomological Index Theory

B1. Introduction

Let G be a compact Lie group[1]. The *ideal-valued cohomological index* Index_G, also called **Fadell–Husseini index** [**FaHu88**], [**Živ98**], is a contravariant functor from the category of G-spaces and G-maps to the category of ideals in a fixed ring $H^*_G(\{*\})$ (which depends on G and the chosen equivariant cohomology) and inclusions.

First of all, we want to define the *equivariant bundle cohomology* H^*_G using the Borel construction [**Die86**, Ch. III], [**AlPu93**, Ch. 1.1, 1.2], which will be the equivariant cohomology we deal with in this index theory. Let $\Bbbk$ be any principal ideal domain (a ring with unit would be enough to define H^*_G, but in Section B3 this stronger assumption on $\Bbbk$ is needed) and let $p : EG \to BG$ be the universal G-bundle, where EG is any contractible free G-CW-complex and G acts on EG from the right. Suppose we are given a (left) G-space X, then consider the associated fibre bundle with fibre X

$$p_X : X_G := EG \times_G X \longrightarrow BG,$$

where $EG \times_G X$ is defined to be $EG \times X$ modulo the diagonal action $(e, x) \sim (eg^{-1}, gx)$. The quotient map

$$q_X : EG \times X \longrightarrow X_G$$

is a G-bundle which induces the following commutative diagram

$$\begin{array}{ccc} EG \times X & \xrightarrow{pr_1} & EG \\ \Big\downarrow{\scriptstyle q_x} & & \Big\downarrow{\scriptstyle p} \\ X_G & \xrightarrow[pr_1/G]{\phi_X :=} & BG, \end{array}$$

and we get *the* classifying map ϕ_X of X_G (it is unique up to homotopy).

[1] In this thesis we are only interested in discrete groups G.

Definition B1.1. The **equivariant bundle cohomology** $H^*_G(X;\Bbbk)$ of X is defined to be $H^*(X_G;\Bbbk)$, the usual cohomology[2] of X_G with coefficients in a principal ideal domain $\Bbbk$. It is a module over

$$H^*_G(\{*\};\Bbbk) = H^*(\{*\}_G;\Bbbk) = H^*(BG;\Bbbk)$$

via the map $H^*_G(\{*\};\Bbbk) \longrightarrow H^*_G(X;\Bbbk)$, which is induced by the projection $X \longrightarrow \{*\}$ and the cup product in $H^*_G(X;\Bbbk) = H^*(X_G;\Bbbk)$.

For a nice and short overview about properties of this multiplicative equivariant cohomology, see [**Die86**, Ch. III].

The classifying map $\phi_X : X_G \longrightarrow BG$ gives rise to a unique map $H^*_G(\{*\}) = H^*(BG) \xrightarrow{\phi^*_X} H^*(X_G) = H^*_G(X)$ and hence to the following

Definition B1.2. The **Fadell–Husseini index** (ideal-valued cohomological index) $\mathrm{Index}^{\Bbbk}_G(X)$ is defined as the ideal

$$\mathrm{Index}^{\Bbbk}_G(X) := \ker(\phi^*_X) \subset H^*_G(\{*\}).$$

If there is no ambiguity with $\Bbbk$, we will simply write $\mathrm{Index}_G(X)$.

Now we want to list important properties of this index as stated in [**FaHu88**] and [**Živ98**, Sect. 2]. I give proofs only for things that I could not find in the literature.

B2. Basic properties of the index

Compare this section with [**FaHu88**, Sect. 2]. The following lemma will be *the* necessary criterion for the existence of G-maps, that the index gives us.

Lemma B2.1 (Functoriality/Monotonicity). *Let $f : X \longrightarrow_G Y$, then*

$$\mathrm{Index}_G(X) \supset \mathrm{Index}_G(Y),$$

that is, Index_G *is in fact a contravariant functor from the category of G-spaces and G-homotopy classes of G-maps to the category of ideals in $H^*_G(\{*\})$ and inclusions.*

Proof. This follows from the commutative diagram

$$\begin{array}{ccccc} EG \times X & \xrightarrow{id_{EG}\times f} & EG \times Y & \xrightarrow{pr_1} & EG \\ \downarrow{\scriptstyle q_x} & & \downarrow{\scriptstyle q_y} & & \downarrow{\scriptstyle p} \\ X_G & \xrightarrow{(id_{EG}\times f)/_G} & Y_G & \xrightarrow[pr_1/_G]{\phi_Y =} & BG, \end{array}$$

[2]It turns out that in general Alexander-Spanier cohomology H^* works the best [**Bor60**, Ch. IV]. However in this thesis we are only interested in deformation retracts of CW-complexes, hence singular cohomology is the same.

and the uniqueness (up to homotopy) of the classifying map ϕ_X (just apply $H^*(_)$ to the bottom row). ∅

LEMMA B2.2 (Additivity). *If $X_1 \cup X_2 = X$ is excisive (that is, the interiors of X_1 and of X_2 cover X), then*

$$(\mathrm{Index}_G X)^2 \subset \mathrm{Index}_G X_1 \cdot \mathrm{Index}_G X_2 \subset \mathrm{Index}_G X.$$

PROOF. The first "$\subset$" follows from Lemma B2.1. To prove the second "$\subset$", let $x_1 \in \mathrm{Index}_G X_1$, define $\overline{x}_1 := \phi_X^*(x_i) \in H_G^*(X)$ and let $i_1 : X_1 \hookrightarrow X$ be the inclusion. By uniqueness of ϕ_{X_1} up to homotopy, the following diagram is commutative:

$$\begin{tikzcd} H_G^*(\{*\}) \arrow[r, "\phi_X^*"] \arrow[dr, "\phi_{X_1}^*"'] & H_G^*(X) \arrow[d, "i_1^*"] \\ & H_G^*(X_1) \end{tikzcd}$$

Since $\phi_{X_1}^*(x_1) = 0$, $i_1(\overline{x}_1) = 0$. By the long exact sequence of the pair (X, X_1)

$$H_G^*(X_1) \xleftarrow{i_1^*} H_G^*(X) \xleftarrow{j_1^*} H_G^*(X, X_1),$$

where $j_1 : (X, \emptyset) \to (X, X_1)$ is the inclusion of pairs, we see that $\overline{x}_1 = j_1^*(y_1)$ for some $y_1 \in H_G^*(X, X_1)$. Using analogous definitions, we have as well $\overline{x}_2 = j_2^*(y_2)$ for some $y_2 \in H_G^*(X, X_2)$. The cup product

$$\cup : H_G^*(X, X_1) \otimes H_G^*(X, X_2) \longrightarrow H_G^*(X, X_1 \cup X_2) = H_G^*(X, X) = 0$$

makes $y_1 \cup y_2 = 0$, and therefore $\overline{x}_1 \cup \overline{x}_2 = 0$, hence $x_1 \cup x_2 \in \mathrm{Index}_G(X)$, by naturality of $\cup$. ∅

LEMMA B2.3 (Continuity). *Let (X, A) be a G-pair. Under some mild conditions*[3]*, there is an open set U, such that $A \subset U \subset \overline{U} \subset X$ and* $\mathrm{Index}_G \overline{U} = \mathrm{Index}_G A$.

∅

B3. Calculating the index

In this section, we restrict the coefficients $\Bbbk$ to an arbitrary field.

PROPOSITION B3.1 ([**FaHu88**, Prop. 3.1]).

$$\mathrm{Index}_{G_1 \times G_2}(X_1 \times X_2) \cong \mathrm{Index}_{G_1}(X_1) \otimes H_{G_2}^*(\{*\}) + H_{G_1}^*(\{*\}) \otimes \mathrm{Index}_{G_2}(X_2)$$

∅

[3]See [**FaHu88**, p. 74-75, 2. (c)] with $h = H_G^*$ and $B = \{*\}$.

EXPLANATION. First note that $B(G_1 \times G_2)$ can be taken to be $B(G_1) \times B(G_2)$. Since $\Bbbk$ is a field, Künneth gives $H^*(B(G_1 \times G_2)) \cong H^*(B(G_1)) \otimes H^*(B(G_1))$ as graded $\Bbbk$-algebras. The isomorphism in the proposition comes from restricting the Künneth isomorphism.

REMARK B3.2. This is the basic proposition for all the following calculations. While "$\supset$" is apparently true, "$\subset$" needs again $\Bbbk$ to be a field, to make vector space arguments applicable. We want to remark that the proposition remains true, if we only assume $\Bbbk$ to be a principal ideal domain and the $H^*(BG_i)$'s to be *free* $\Bbbk$-modules. The proof of "$\subset$" then uses Smith normal form.

COROLLARY B3.3 ([**FaHu88**, 3.2]). *If $H^*(BG_1) = \Bbbk[x_1, \ldots, x_k]$ and $H^*(BG_2) = \Bbbk[y_1, \ldots, y_l]$ are polynomial rings over $\Bbbk$, and let $\mathrm{Index}_{G_1} X_1 = \langle f_1, \ldots, f_m \rangle$ and $\mathrm{Index}_{G_2} X_2 = \langle g_1, \ldots, g_n \rangle$, then*

$$\begin{array}{ccccc} \mathrm{Index}_{G_1 \times G_2} X_1 \times X_2 & \cong & \langle f_1,\ldots,f_m,g_1,\ldots,g_n \rangle & & \\ \cap & & \cap & & \\ H^*(B(G_1 \times G_2)) & \cong & \Bbbk[x_1,\ldots,x_k,y_1,\ldots,y_l] & = & H^*(BG_1) \otimes H^*(BG_2). \end{array}$$

$\not\Phi$

EXAMPLE B3.4. Let $\mathbb{Z}_2$ act on S^d via the antipodal action. The universal $\mathbb{Z}_2$-bundle is $S^\infty \longrightarrow \mathbb{R}P^\infty$, therefore $H^*(B(\mathbb{Z}_2); \mathbb{F}_2) = \mathbb{F}_2[t]$. Now consider the following commutative diagram

$$\begin{tikzcd} & S^\infty \times S^d \arrow[r] \arrow[d] & S^\infty \arrow[d] \\ \mathbb{R}P^d \arrow[r, hook, "i"] & S^\infty \times_{\mathbb{Z}_2} S^d \arrow[r, "\phi_{S^d}"] & \mathbb{R}P^\infty. \end{tikzcd}$$

The inclusion i is a homotopy equivalence (by the long exact homotopy sequence of the Serre bundle $\mathbb{R}P^d \hookrightarrow S^\infty \times_{\mathbb{Z}_2} S^d \to S^\infty$) and $\phi_{S^d} \circ i : \mathbb{R}P^d \longrightarrow \mathbb{R}P^\infty$ is also the inclusion, whose kernel in cohomology is

$$\mathrm{Index}_{\mathbb{Z}_2}^{\mathbb{F}_2} S^d = \langle t^{d+1} \rangle \subset \mathbb{F}_2[t]. \tag{B3.5}$$

This important example can be extended via Corollary B3.3 to the product action of $(\mathbb{Z}_2)^k$ on $S^{d_1} \times \ldots \times S^{d_k}$:

COROLLARY B3.6.

$$\mathrm{Index}_{(\mathbb{Z}_2)^k}^{\mathbb{F}_2} S^{d_1} \times \ldots \times S^{d_k} = \langle t_1^{d_1+1}, \ldots, t_k^{d_k+1} \rangle \subset \mathbb{F}_2[t_1, \ldots, t_k].$$

$\not\Phi$

We will need the following index formula for $(\mathbb{Z}_2)^k$-representations.

THEOREM B3.7 ([**Živ98**, 2.12]). *Let $W = V_1 \oplus \ldots \oplus V_n$ be a $(\mathbb{Z}_2)^k$-representation, where V_i is the one-dimensional representation (note that any representation of Abelian finite groups has a decomposition into one-dimensional representations* [**FuHa91**]*) defined by*

$$(+, \ldots, +, \underbrace{-}_{j}, +, \ldots, +) \cdot v_i = (-1)^{a^i_j} \cdot v_i,$$

where the multiplication on the left side denotes the action of the generator $(+, \ldots, -, \ldots, +) \in (\mathbb{Z}_2)^k$ on $v_i \in V_i$, and the multiplication on the right is the scalar product in V_i, where $a^i_j \in \{0, 1\}$. Let $S(W)$ be the G-representation sphere $\{w \in W \mid ||w|| = 1\}$ of W. Then

$$\mathrm{Index}^{\mathbb{F}_2}_{(\mathbb{Z}_2)^k} S(W) = \left\langle \prod_{i=1}^{n} (a^i_1 t_1 + \ldots + a^i_k t_k) \right\rangle \subset \mathbb{F}_2[t_1, \ldots, t_k].$$

PROOF. There is a nice proof different from the one given in [**Živ98**] (but also look at this reference). We apply Lemma B4.2, prove the theorem for $n = 1$ as we did similarly in Chapter II.6 and use the Whitney sum formula for Stiefel–Whitney classes! $\not\Phi$

For index properties relating to joins see [**FaHu88**, 3.4–3.13][4] and for more examples see [**Živ98**, Sec. 2.2].

B4. Fadell–Husseini index vs. characteristic classes

We are interested in methods to disprove the existence of maps $X \longrightarrow_G Y$. Characteristic classes are for this purpose only applicable when Y is a representation sphere $S(V)$ of a G-representation V of dimension n. In this case, we will show that they yield the same criterion for the existence of a map (Proposition A5.1) as the Fadell–Husseini index (Lemma B2.1): Stiefel–Whitney classes are just as well as $\mathrm{Index}^{\mathbb{F}_2}_G$, and the Euler class as $\mathrm{Index}^{\mathbb{Z}}_G$ (as long as G acts orientation preserving on V).

Let $p' : EG \times_G (X \times V) \longrightarrow EG \times_G X$ be the corresponding vector bundle from Section A4, Remark A4.2.

THEOREM B4.1. *The following holds:*

- $\omega_n(p') = 0$, *iff* $\mathrm{Index}^{\mathbb{F}_2}_G X \supset \mathrm{Index}^{\mathbb{F}_2}_G Y$, *and*
- *Suppose G acts orientation preserving on V. Then $e_n(p') = 0$, iff* $\mathrm{Index}^{\mathbb{Z}}_G X \supset \mathrm{Index}^{\mathbb{Z}}_G Y$.

[4] Their remark 3.14 contains a wrong example.

To prove it, we will need the following lemma:

LEMMA B4.2. *Let $S(V)$ be a G-representation sphere whose underlying vector space V is n-dimensional. Then*

$$\mathrm{Index}_G^{\mathbb{F}_2} S(V) = \langle \omega(\phi_V) \rangle,$$

where $\phi_V : EG \times_G V \longrightarrow BG$ is the classifying map of V induced by projecting onto the first coordinate. The analogous result holds for $\mathrm{Index}_G^{\mathbb{Z}} S(V)$ and the Euler class $e(\phi_V)$, if G acts orientation preserving on V.

PROOF OF LEMMA B4.2. The sphere bundle

$$\phi_{S(V)} : EG \times_G S(V) \longrightarrow BG$$

associated to ϕ_V induces a Leray–Serre spectral sequence (see [**McC01**, Ch. 5, 6]) with E_2-page $E_2^{*,*} \cong H^*(BG) \otimes H^*(S(V))$ (as $H^*(BG)$-modules, $\mathbb{F}_2$-coefficients are understood). The edge homomorphism

$$H^*(BG) \xrightarrow{\cong} E_2^{*,0} \longrightarrow E_\infty^{*,0} \hookrightarrow H^*(EG \times_G S(V))$$

coincides with $\phi_{S(V)}^*$, therefore $\mathrm{Index}_G^{\mathbb{F}_2} S(V) = \ker(\phi_{S(V)})$ consists of all elements of $E_2^{*,0}$ which eventually become zero in this spectral sequence. This can happen only at the E_n-page. Let x be the generator of $H^n(S(V))$. We need the following two properties of this spectral sequence [P. Blagojević, private communication]:

- $d_{n+1}(x) = \omega_n(\phi_V)$, and
- d_{n+1} is a $H^*(BG)$-module homomorphism (in fact, $H^*(BG)$ acts horizontally on the spectral sequence).

Therefore, the image of $d_{n+1}^{*,n}$ is $H^*(BG) \cdot \omega_n(\phi_{S(V)}) = \langle \omega_n(\phi_{S(V)}) \rangle \subset H^*(BG)$. Analogous computations can be done with $\mathbb{Z}$-coefficients and the Euler class, as long as G acts orientation preserving on V. ∎

PROOF OF THEOREM B4.1. Let $\phi_X : EG \times_G X \longrightarrow BG$ be induced by projecting to the second coordinate, which coincides with the classifying map of the G-bundle $EG \times X \longrightarrow EG \times_G X$. Similar define ϕ_V and ϕ_Y. Now, consider the vector bundle map

$$\begin{tikzcd}
EG \times_G (X \times V) \arrow[r, "(pr_1, pr_3)/_G"] \arrow[d, "q"'] & EG \times_G V \arrow[d, "\phi_V"] \\
EG \times_G X \arrow[r, "\phi_X"] & BG,
\end{tikzcd}$$

where $q : [e, x, v] \mapsto [e, x]$. It induces a sphere bundle map

$$\begin{tikzcd} EG \times_G (X \times Y) \arrow[r, "(pr_1, pr_3)/_G"] \arrow[d, "p'"'] & EG \times_G Y \arrow[d, "\phi_Y"] \\ EG \times_G X \arrow[r, "\phi_X"] & BG. \end{tikzcd}$$

We have that

$$\mathrm{Index}_G^{\mathbb{F}_2} X = \ker(\phi_X^*),$$

and

$$\mathrm{Index}_G^{\mathbb{F}_2} Y = \ker(\phi_Y^*) = \langle \omega_n(\phi_V) \rangle,$$

by the previous lemma. By naturality of the characteristic classes,

$$\phi_X^*(\omega_n(\phi_V)) = \omega_n(q).$$

Therefore,

$$\begin{aligned} \omega_n(q) = 0 \quad &\iff \omega_n(\phi_V) \in \ker(\phi_X) = \mathrm{Index}_G^{\mathbb{F}_2} X \\ &\iff \mathrm{Index}_G^{\mathbb{F}_2} X \supset \langle \omega_n(\phi_V) \rangle \\ &\iff \mathrm{Index}_G^{\mathbb{F}_2} X \supset \mathrm{Index}_G^{\mathbb{F}_2} Y. \end{aligned}$$

$\not\Phi$

APPENDIX C

Equivariant Obstruction Theory

C1. ... for free domains

In this chapter we want to summarise shortly the equivariant obstruction theory. Let G be a finite group, $n \geq 1$ a natural number, (X, A) be a *free* relative G-CW-complex[1] and Y a path-connected G-CW-complex. Furthermore $\pi_1(Y)$ shall act trivially on $\pi_n(Y)$, hence we get a well-defined action of G on $\pi_n(Y)$, since then $\pi_n(Y)$ does not depend on a base point.

Let $C_*(X, A)$ denote the (usual non-equivariant) cellular complex of (X, A), which becomes a $\mathbb{Z}[G]$ module by the induced action of G on (X, A). If M is a $\mathbb{Z}[G]$-module, we can define the **equivariant cellular cochain complex**,

$$C_G^*(X, A; M) := \mathrm{Hom}_{\mathbb{Z}[G]}(C_*(X, A), M),$$

which can be seen as the subcomplex of all G-equivariant cochains of the usual cellular cochain complex $C^*(X, A; M)$. This also gives us the coboundary operator δ which finally yields the **equivariant cellular cohomology** $H_G^*(X, A; M)$.

Now, let $f : X_n \longrightarrow_G Y$ be given. We define the **obstruction cocycle** of f to be the cochain $o_f \in C_G^{n+1}(X, A; \pi_n(Y))$ given by

$$o_f(e) := [f|_{\partial(e)}] \in \pi_n(Y),$$

for all cells e of X, where we view $f|_{\partial(e)}$ as a map $S^n \longrightarrow Y$ (S^n is just being identified with $\partial(e)$ by the characteristic map of the cell e in X). It is in fact a cocycle.

Observation C1.1. *Observe that o_f is zero (in $C_G^{n+1}(X, A; \pi_n(Y))$) iff f is extendable to X_{n+1}.*

Theorem C1.2 (Equivariant obstruction theory for free domains). *Suppose we are given a map $f : X_n \longrightarrow_G Y$. Then the restricted map $f|_{X_{n-1}}$ can be extended G-equivariantly to X_{n+1} iff $[o_f] = 0$ as an element of $H_G^{n+1}(X, A; \pi_n(Y))$.*

[1] That is any (also non-free) G-space A to which one attaches free 0-cells, then free 1-cells, and so on. A is of course allowed to be empty. The k-skeleton of (X, A) is the union of A together with all $\leq k$-cells and denoted by $X_k = (X, A)_k$.

PROOF. Since the domain space X was assumed to be free, the equivariance of the map to be constructed is no real obstruction, that is, the equivariance can be plugged in without a problem into the standard proof of the usual non-free obstruction theory. For details see e. g. [**Die86**, Ch. II.3]. $\not\Phi$

We should not omit one important notion: Suppose $f, g : X_n \longrightarrow_G Y$ are G-homotopic on X_{n-1} by a homotopy $H : I \times X_{n-1} \longrightarrow_G Y$. Suppose we want to extend this homotopy to $I \times X_n$ (it is given already on the n-skeleton of $I \times X_n$ by H, f and g together), we therefore get an obstruction cocycle $o \in C_G^{n+1}(X, A; \pi_n(Y))$. Since the top-dimensional cells of $I \times X_n$ correspond to top-dimensional cells of X_n, we can view this o actually as an element $d(f, H, g) \in C_G^n(X, A; \pi_n(Y))$ and this is called the **difference cochain** of f and g given H. It has three important properties (see [**Die86**, II(3.13) and (3.14)]):

- $d(f, H, g) + d(g, H', h) = d(f, H + H', h)$ (where $H + H'$ is the concatenated homotopy),
- $\delta(d(f, H, g)) = o_f - o_g$, and
- For given $f : X_n \longrightarrow_G Y$ and $H : I \times X_{n-1} \longrightarrow_G Y$ with $f|_{X_{n-1}} = H_0$, and $d \in C_G^n(X, A; \pi_n(Y))$ there is a $g : X_n \longrightarrow_G Y$ satisfying $g|_{X_{n-1}} = H_1$ and $d(f, H, g) = d$.

The last two properties actually prove Theorem C1.2. The second one says, that $[o_f]$ only depends on $f|_{X_{n-1}}$.

If $\pi_n(Y)$ is the first non-trivial homotopy group of Y and if we are given a map $f_0 : A \longrightarrow_G Y$, then by an inductive construction there is a unique extension $f : X_{n-1} \longrightarrow_G Y$ up to G-homotopy rel A, and f is extendable to X_n. Hence, $o_f(e) \in C_G^{n+1}(X, A; \pi_n(Y))$ is the first non-trivial obstruction of extending f_0 to X_{n+1}, therefore it is called the **primary obstruction**.

For more about this equivariant obstruction theory, see [**Die86**, Ch. II.3].

C2. ... for non-free domains

Things become more complicated, if the given (X, A) is not free anymore. The main problem is that Observation C1.1 does not hold anymore: If $o_f(e) = 0$, then it does not imply in general that $f : X_n \longrightarrow_G Y$ is extendable over e, since it may happen that this so obtained $f|_{G \cdot e}$ is not equivariant! An element $g \in G$ which fixes such a cell e, also has to fix the image $f(e)$ to make the extended f equivariant.

We want to find an analog to Observation C1.1 for non-free X. For this to do, we have to define a new equivariant cellular cochain

complex. Let

$$I_{(X,A)} := \{G_x \mid x \in X \backslash A\}$$

be the set of all isotropy groups $G_x := \{g \in G \mid gx = x\}$ of $X \backslash A$. We have to add the assumption, that for all $H \in I_{(X,A)}$, Y^H is path-connected or empty (recall: $Y^H := \{y \in Y \mid hy = y \ \forall h \in H\}$), and $\pi_1(Y^H)$ is acting trivially on $\pi_n(Y^H)$.[2] If Y^H is non-empty, we define

$$\widetilde{\pi}_n(Y^H) := [S^n, Y^H] \cong \pi_n(Y^H).$$

For given (X, A), Y and n, let

$$\pi := \bigoplus_{\substack{H \in I_{(X,A)} \\ Y^H \neq \emptyset}} \widetilde{\pi}_n(Y^H).$$

Now we can define our appropriate "equivariant cellular cochain complex"

$$\widetilde{C}^k_G(X, A; Y, n) := \{c \in C^k_G(X, A; \pi) \mid c(e) \in \widetilde{\pi}_n(Y^{G_e}) \text{ for all } k\text{-cells } e \text{ in } (X, A)\}$$

The coboundary operator for $\widetilde{C}^*_G(X, A; Y, n)$ is given as follows: Suppose e is a $(k+1)$-cell whose boundary (as an element of $C_*(X, A; \mathbb{Z})$) is $\partial(e) = e_1 + \ldots + e_l \in C_*(X, A; \mathbb{Z})$. Then define the coboundary of $c \in \widetilde{C}^k_G(X, A; Y, n)$ by

$$\delta(c)(e) := \sum_{i=1}^{l} (i^{Y^{G_{e_i}}}_{Y^{G_e}})^* c(e_i),$$

where $i^{Y^{G_{e_i}}}_{Y^{G_e}} : Y^{G_{e_i}} \longrightarrow Y^{G_e}$ is the inclusion and $(i^{Y^{G_{e_i}}}_{Y^{G_e}})^*$ its induced map in $\widetilde{\pi}_n(_)$. Since $\delta \circ \delta = 0$, this defines a cohomology $\widetilde{H}^*_G(X, A; Y, n)$.

The **obstruction cocycle** o_f of a map $f : X_n \longrightarrow_G Y$ is now defined as the cochain $\widetilde{o}_f \in \widetilde{C}^k_G(X, A; Y, n)$ given by

$$\widetilde{o}_f(e) := [f|_{\partial(e)}] \in \widetilde{\pi}_n(Y^{G_e}) \subset \pi.$$

This is well-defined: Since every element $g \in G$ that fixes e also fixes $\partial(e)$ by continuity, $f|_{\partial(e)}$ is indeed mapping into Y^{G_e}.

Observation C2.1. *Observe that $\widetilde{o}_f$ is zero (in $C^{n+1}_G(X, A; \pi_n(Y))$) iff f is extendable to X_{n+1}.*

[2]If fact we will just need this condition for all $H \in I_{(X,A)}$ that are isotropy groups of cells in X of dimension n and $n+1$. This condition is only there to avoid the need of dealing with base points of Y^H, since we want make $[S^n, Y]$ into a group.

That we created the right setting can be seen from the following theorem, which is the analog of Theorem C1.2.

THEOREM C2.2 (Equivariant obstruction theory for non-free domains). *Suppose we are given a map $f : X_n \longrightarrow_G Y$. Then the restricted map $f|_{X_{n-1}}$ can be extended G-equivariantly to X_{n+1} iff $[\widetilde{o}_f] = 0$ as an element of $\widetilde{H}_G^{n+1}(X, A; Y, n)$.*

PROOF. The proof in [**Die86**, Ch. II.3, p. 115ff] can be appropriately modified. ∅

REMARKS C2.3.

- The uniqueness issue of the extension can be treated in a similar fashion by extending homotopies.
- The key to generalise the usual equivariant obstruction theory was just to find the right cohomology $\widetilde{H}_G^*(X, A; Y, n)$.
- Bredon invented the same (very similar) cohomology already in [**Bre67**], generalised it into an abstract setting ("generic coefficient systems") and has shown which properties of a cohomology theory his cohomology satisfies [**Bre67**, Ch. I].
- For more classifying theorems in obstruction theory, that can be deduced from Theorem C2.1, see [**Bre67**, Ch. II].
- His conditions that the Y^H shall be non-free for all H can be dropped. In particular, Y does not need to contain fixed-points (This is only of importance if one wants to assure the existence of at least one map $X \longrightarrow_G Y$, but for the extension process it is unnecessary).

C3. ... for non-simple ranges

First of all we will again assume (X, A) to be free, but the non-free case can be dealt with analogously. A necessary assumption on our range Y was, that it is n-simple, meaning that $\pi_1(Y)$ acts trivially on Y. If this assumption fails, we can still rescue one direction of Theorem C1.2 by taking instead of $\pi_n(Y)$ another coefficient group $\overline{\pi}_n(Y)$ for the cohomology:

Let $\overline{\pi}_n(Y) := \pi_n(Y)/N_Y$ be the quotient of $\pi_n(Y)$ by

$$N_Y := \left\{ \sum_i (\gamma_i \cdot \alpha_i - \alpha_i) \mid \gamma_i \in \pi_1(Y), \alpha_i \in \pi_n(Y) \right\} \subset \pi_n(Y),$$

where "$\cdot$" denotes the action of π_1 on π_n and "$+$" the group operation in π_n (which might be non-commutative if $n = 1$). In the case $n = 1$, N_Y is the commutator subgroup of $\pi_n(Y)$. The natural projection $\pi_n(Y) \longrightarrow \overline{\pi}_n(Y)$ gives then a new obstruction cocycle $\overline{o}_f \in C_G^{n+1}(X, A; \overline{\pi}_n(Y))$

out of the old one o_f (note that since Y is not n-simple, o_f is not well-defined, but $\overline{o}_f$ is). We then get a theorem similar to the above ones, but one direction is missing.

THEOREM C3.1 (Equivariant obstruction theory for *free* domains and *non-simple* ranges). *Suppose we are given a map $f : X_n \longrightarrow_G Y$, whose restriction $f|_{X_{n-1}}$ can be extended G-equivariantly to X_{n+1}. Then $[\overline{o}_f] = 0$ as an element of $H_G^{n+1}(X, A; \overline{\pi}_n(Y))$.*

$\not\Phi$

If (X, A) is non-free, but Y^H is path-connected for all H that are isotropy groups of n- or $(n+1)$-cells of X. We define

$$\overline{\pi} := \bigoplus_{\substack{H \in I_{(X,A)} \\ Y^H \neq \emptyset}} \overline{\pi}_n(Y^H).$$

and let $\overline{\widetilde{C}}_G^*(X, A; Y, n)$ be the analog of $\widetilde{C}_G^*(X, A; Y, n)$ from the previous section with coefficients in $\overline{\pi}$ and $\overline{\widetilde{H}}_G^*(X, A; Y, n)$ the corresponding cohomology. The obvious homomorphism $\pi \longrightarrow \overline{\pi}$ makes out of the in this situation non-well-defined obstruction cocyle $\widetilde{o}_f$ a well-defined cocyle $\overline{\widetilde{o}}_f$.

THEOREM C3.2 (Equivariant obstruction theory for *non-free* domains and *non-simple* ranges). *Suppose we are given a map $f : X_n \longrightarrow_G Y$, whose restriction $f|_{X_{n-1}}$ can be extended G-equivariantly to X_{n+1}. Then $[\overline{\widetilde{o}}_f] = 0$ as an element of $\overline{\widetilde{H}}_G^{n+1}(X, A; Y, n)$.*

$\not\Phi$

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