Two-dimensional "haul-out" model

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Implementation

The file haulout.2d.sim.R simulates data according to the model statement presented below, and haulout.2d.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let $\mathcal{S}$ be the support of the movement process and $\tilde{S}$ bethe support of haul-out sites. Here, both are defined in 2-dimensional space. Note that $\mathcal{S}$ and $\tilde{S}$ overlap, and that $\mathcal{S} \cap \tilde{S} = \tilde{\mathcal{S}}$. The domain defined by $\mathcal{S}$ therefore represents at-sea locations or locations of the individual while milling adjacent to the haul-out site. Also note that $\bar{\mathcal{S}}$, the complement of $\mathcal{S}$, represents inaccessible locations (i.e., land). Let $\mathbf{s}_t = (s_{1,t}, s_{2,t})'$ and $\boldsymbol{\mu}_t = (\mu_{1,t}, \mu_{2,t})'$, for $t = 1, \ldots, T$, be observed and true locations respectively.

$$
\begin{aligned}
\mathbf{s}_t &\sim \mathrm{N}(\boldsymbol{\mu}_t, \sigma^2 \mathbf{I}) \\
\boldsymbol{\mu}_t &\sim \begin{cases} \mathrm{Unif}(\tilde{\mathcal{S}}), & z_t = 1 \\ \mathrm{Unif}(\mathcal{S}), & z_t = 0 \end{cases} \\
z_t &\sim \mathrm{Bern}(p) \\
p &\sim \mathrm{Beta}(\alpha, \beta) \\
\sigma^2 &\sim \mathrm{IG}(r, q)
\end{aligned}
$$

Full conditional distributions

True locations ($\boldsymbol{\mu}_t$):

$$
\begin{aligned}
[\boldsymbol{\mu}_t | \cdot] &\propto [\mathbf{s}_t | \boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t | z_t, \mathcal{S}, \tilde{\mathcal{S}}] \\
&\propto [\mathbf{s}_t | \boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t | \tilde{\mathcal{S}}]^{z_t} [\boldsymbol{\mu}_t | \mathcal{S}]^{1-z_t}.
\end{aligned}
$$

For $z_t = 1$,

$$
\begin{aligned}
[\boldsymbol{\mu}_t | \cdot] &\propto [\mathbf{s_t} | \boldsymbol{\mu_t}, \boldsymbol{\sigma^2}][\boldsymbol{\mu}_t | \tilde{\mathcal{S}}]^{z_t} \\
&\propto [\mathbf{s}_t | \boldsymbol{\mu}_t, \sigma^2] \\
&\propto \exp\left\{ -\frac{1}{2} \left((\mathbf{s}_t - \boldsymbol{\mu}_t)' \left(\sigma^2 \mathbf{I}\right)^{-1} (\mathbf{s}_t - \boldsymbol{\mu}_t) \right) \right\} \\
&\propto \exp\left\{ -\frac{1}{2} \left(\mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \mathbf{s}_t - 2\mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t + \boldsymbol{\mu}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t \right) \right\} \\
&\propto \exp\left\{ -\frac{1}{2} \left(-2\mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t + \boldsymbol{\mu}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t \right) \right\} \\
&\propto \mathrm{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1})
\end{aligned}
$$

where $\mathbf{A} = \left(\sigma^2 \mathbf{I}\right)^{-1}$ and $\mathbf{b} = \mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1}$; therefore, $[\boldsymbol{\mu}_t | \cdot] = \mathrm{N}(\mathbf{s}_t, \sigma^2 \mathrm{I})$ for $z_t = 1$. Note that proposed values for μ_t not in $\tilde{\mathcal{S}}$ are rejected. For $z_t = 0$,

$$[\boldsymbol{\mu}_t | \cdot] = \mathrm{N}(\mathbf{s}_t, \sigma^2 \mathbf{I}).$$

In this case, note that proposed values for $\boldsymbol{\mu}_t$ not in $\mathcal{S}$ are rejected.

Haul-out indicator variable (z_t):

$$[z_t|\cdot] \quad \propto \quad [\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t}[\boldsymbol{\mu}_t|\mathcal{S}[z_t|p].$$

For all $\boldsymbol{\mu}_t \notin \tilde{\mathcal{S}}$, let $z_t = 0$. For all $\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}$, sample z_t from

$$\begin{aligned} [z_t|\cdot] &\propto [\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t}[\boldsymbol{\mu}_t|\mathcal{S}]^{1-z_t}[z_t|p] \\ &= \text{Bern}(\tilde{p}), \end{aligned}$$

where

$$\begin{aligned} \tilde{p} &= \frac{p[\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]}{p[\boldsymbol{\mu}_t|\tilde{\mathcal{S}}] + (1-p)[\boldsymbol{\mu}_t|\mathcal{S}]} \\ &= \frac{p\left(|\tilde{\mathcal{S}}|^{-1}\right)}{p\left(|\tilde{\mathcal{S}}|^{-1}\right) + (1-p)\left(|\mathcal{S}|^{-1}\right)}. \end{aligned}$$

The notation $|\cdot|$ denotes the area of the respective domain.

Probability of being hauled-out (p):

$$\begin{aligned} [p|\cdot] &\propto \prod_{t=1}^{T}[z_t|p][p] \\ &\propto \prod_{t=1}^{T} p^{z_t}(1-p)^{1-z_t}p^{\alpha-1}(1-p)^{\beta-1} \\ &\propto p^{\sum_{t=1}^{T} z_t}(1-p)^{\sum_{t=1}^{T}(1-z_t)}p^{\alpha-1}(1-p)^{\beta-1} \\ &= \text{Beta}\left(\sum_{t=1}^{T} z_t + \alpha, \sum_{t=1}^{T}(1-z_t) + \beta\right) \end{aligned}$$

Error in the observation process (σ^2):

$$\begin{aligned} [\sigma^2|\cdot] &\propto \prod_{t=1}^{T}[\mathbf{s}_t|\boldsymbol{\mu}_t, \sigma^2][\sigma^2] \\ &\propto \prod_{t=1}^{T} |\sigma^2\mathbf{I}|^{-1/2} \exp\left\{-\frac{1}{2}\left((\mathbf{s}_t - \boldsymbol{\mu}_t)'\left(\sigma^2\mathbf{I}\right)^{-1}(\mathbf{s}_t - \boldsymbol{\mu}_t)\right)\right\}\left(\sigma^2\right)^{-(q+1)} \exp\left\{-\frac{1}{\sigma^2 r}\right\} \\ &\propto \prod_{t=1}^{T} \left(\sigma^2\right)^{-1} \exp\left\{-\frac{1}{2\sigma^2}\left((\mathbf{s}_t - \boldsymbol{\mu}_t)'(\mathbf{s}_t - \boldsymbol{\mu}_t)\right)\right\}\left(\sigma^2\right)^{-(q+1)} \exp\left\{-\frac{1}{\sigma^2 r}\right\} \\ &\propto \left(\sigma^2\right)^{-(T+q+1)} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{\sum_{t=1}^{T}(\mathbf{s}_t - \boldsymbol{\mu}_t)'(\mathbf{s}_t - \boldsymbol{\mu}_t)}{2} + \frac{1}{r}\right)\right\} \\ &= \text{IG}\left(\left(\frac{\sum_{t=1}^{T}(\mathbf{s}_t - \boldsymbol{\mu}_t)'(\mathbf{s}_t - \boldsymbol{\mu}_t)}{2} + \frac{1}{r}\right)^{-1}, T+q\right). \end{aligned}$$

Note that the current version of haulout.2d.mcmc.R contains code for the conjugate update of σ^2 presented above, but this code is currently 'commented' out. Instead, error is modeled as $[\sigma|\cdot] \sim \text{Unif}(a, b)$, and the update for σ proceeds using Metropolis-Hastings.