Two-dimensional "haul-out" model with home base

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Implementation

The file haulout.homerange.sim.R simulates data according to the model statement presented below, and haulout.homerange.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let $\mathcal{S}$ be the support of the movement process and $\tilde{S}$ be the support of haul-out sites. Here, both are defined in 2-dimensional space. Note that $\mathcal{S}$ and $\tilde{S}$ overlap, i.e., $\tilde{\mathcal{S}} \subset \mathcal{S}$. The domain defined by $\mathcal{S}$ therefore represents at-sea locations or locations of the individual while milling adjacent to the haul-out site. Also note that $\bar{\mathcal{S}}$, the complement of $\mathcal{S}$, represents inaccessible locations (i.e., terrestrial sites that are not haul-outs). Let $\mathbf{s}_t = (s_{1,t}, s_{2,t})'$ and $\boldsymbol{\mu}_t = (\mu_{1,t}, \mu_{2,t})'$, for $t = 1, \ldots, T$, be observed and true locations respectively.

$$
\begin{aligned}
\mathbf{s}_t &\sim \mathcal{N}(\boldsymbol{\mu}_t, \sigma^2\mathbf{I}) \\
\boldsymbol{\mu}_t &\sim \begin{cases} \text{Unif}(\tilde{\mathcal{S}}) 1_{\{\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}\}}, & z_t = 1 \\ \mathcal{TN}(\boldsymbol{\mu}_0, \sigma_\mu^2 \mathbf{I})_{\mathcal{S}} 1_{\{\boldsymbol{\mu}_t \in S\}}, & z_t = 0 \end{cases} \\
z_t &\sim \text{Bern}(p) \\
p &\sim \text{Beta}(\alpha, \beta) \\
\sigma^2 &\sim \text{IG}(r, q) \\
\boldsymbol{\mu}_0 &\sim \text{Unif}(\tilde{\mathcal{S}}) \\
\sigma_\mu &\sim \text{Unif}(a, b)
\end{aligned}
$$

The latent variable z_t indicates when locations are at a haul-out ($z_t = 1$) or not at a haul-out ($z_t = 0$).

Full conditional distributions

True locations ($\boldsymbol{\mu}_t$):

$$
\begin{aligned}
[\boldsymbol{\mu}_t | \cdot] &\propto [\mathbf{s}_t | \boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t | z_t, \boldsymbol{\mu}_0, \sigma_\mu^2, \mathcal{S}, \tilde{\mathcal{S}}] \\
&\propto [\mathbf{s}_t | \boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t | \tilde{\mathcal{S}}]^{z_t} [\boldsymbol{\mu}_t | \boldsymbol{\mu}_0, \sigma_\mu^2, \mathcal{S}]^{1-z_t}.
\end{aligned}
$$

For $z_t = 1$,

$$
\begin{aligned}
[\boldsymbol{\mu}_t | \cdot] &\propto [\mathbf{s_t} | \boldsymbol{\mu_t}, \boldsymbol{\sigma^2}][\boldsymbol{\mu}_t | \tilde{\mathcal{S}}]^{z_t} \\
&\propto [\mathbf{s}_t | \boldsymbol{\mu}_t, \sigma^2] \\
&\propto \exp\left\{ -\frac{1}{2} \left((\mathbf{s}_t - \boldsymbol{\mu}_t)' \left(\sigma^2 \mathbf{I}\right)^{-1} (\mathbf{s}_t - \boldsymbol{\mu}_t) \right) \right\} 1_{\{\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}\}} \\
&\propto \exp\left\{ -\frac{1}{2} \left(\mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \mathbf{s}_t - 2\mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t + \boldsymbol{\mu}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t \right) \right\} 1_{\{\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}\}} \\
&\propto \exp\left\{ -\frac{1}{2} \left(-2\mathbf{s}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t + \boldsymbol{\mu}_t' \left(\sigma^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_t \right) \right\} 1_{\{\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}\}} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}) 1_{\{\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}\}}
\end{aligned}
$$

where $\mathbf{A} = \left(\sigma^2\mathbf{I}\right)^{-1}$ and $\mathbf{b} = \mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}$; therefore, $[\boldsymbol{\mu}_t|\cdot] = \mathcal{N}(\mathbf{s}_t, \sigma^2\mathrm{I})$ for $z_t = 1$. Note that 'proposed' values for $\boldsymbol{\mu}_t$ not in $\tilde{\mathcal{S}}$ are rejected, i.e., $[\boldsymbol{\mu}_t|\cdot] = \mathcal{TN}(\mathbf{s}_t, \sigma^2\mathrm{I})_{\tilde{\mathcal{S}}}$. For $z_t = 0$,

$$\begin{aligned}
[\boldsymbol{\mu}_t|\cdot] &\propto [\mathbf{s}_t|\boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t|\boldsymbol{\mu}_0, \sigma_\mu^2, \mathcal{S}]^{1-z_t} \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{s}_t - \boldsymbol{\mu}_t\right)'\left(\sigma^2\mathbf{I}\right)^{-1}\left(\mathbf{s}_t - \boldsymbol{\mu}_t\right)\right\}\exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_t - \boldsymbol{\mu}_0\right)'\left(\sigma_\mu^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_t - \boldsymbol{\mu}_0\right)\right\}1_{\{\boldsymbol{\mu}_t \in \mathcal{S}\}} \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\mathbf{s}_t - 2\mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t + \boldsymbol{\mu}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t\right)\right\} \times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_t'\left(\sigma_\mu^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t - 2\boldsymbol{\mu}_t'\left(\sigma_\mu^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_0 + \boldsymbol{\mu}_0'\left(\sigma_\mu^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_0\right)\right\}1_{\{\boldsymbol{\mu}_t \in \mathcal{S}\}} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1} + \boldsymbol{\mu}_0'\left(\sigma_\mu^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_t + \boldsymbol{\mu}_t'\left(\left(\sigma^2\mathbf{I}\right)^{-1} + \left(\sigma_\mu^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_t\right)\right\}1_{\{\boldsymbol{\mu}_t \in \mathcal{S}\}} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1})1_{\{\boldsymbol{\mu}_t \in \mathcal{S}\}}
\end{aligned}$$

where $\mathbf{A} = \left(\sigma^2\mathbf{I}\right)^{-1} + \left(\sigma_\mu^2\mathbf{I}\right)^{-1}$ and $\mathbf{b} = \mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1} + \boldsymbol{\mu}_0'\left(\sigma_\mu^2\mathbf{I}\right)^{-1}$. In this case, note that proposed values for $\boldsymbol{\mu}_t$ not in $\mathcal{S}$ are rejected, i.e.,, $[\boldsymbol{\mu}_t|\cdot] = \mathcal{TN}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1})_{\mathcal{S}}$.

Haul-out indicator variable (z_t):

$$[z_t|\cdot] \propto [\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t}[\boldsymbol{\mu}_t|\boldsymbol{\mu}_0, \sigma_\mu^2, \mathcal{S}]^{1-z_t}[z_t|p].$$

For all $\boldsymbol{\mu}_t \notin \tilde{\mathcal{S}}$, let $z_t = 0$. For all $\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}$,

$$[z_t|\cdot] = \mathrm{Bern}(\tilde{p}),$$

where

$$\begin{aligned}
\tilde{p} &= \frac{p[\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]}{p[\boldsymbol{\mu}_t|\tilde{\mathcal{S}}] + (1-p)[\boldsymbol{\mu}_t|\boldsymbol{\mu}_0, \sigma_\mu^2, \mathcal{S}]} \\
&= \frac{p\left(|\tilde{\mathcal{S}}|^{-1}\right)}{p\left(|\tilde{\mathcal{S}}|^{-1}\right) + (1-p)\mathcal{TN}(\boldsymbol{\mu}_t|\boldsymbol{\mu}_0, \sigma_\mu^2\mathbf{I})_{\mathcal{S}}}.
\end{aligned}$$

The notation $|\cdot|$ denotes the area of the respective domain.

Probability of being hauled-out (p):

$$\begin{aligned}
[p|\cdot] &\propto \prod_{t=1}^{T}[z_t|p][p] \\
&\propto \prod_{t=1}^{T}p^{z_t}(1-p)^{1-z_t}p^{\alpha-1}(1-p)^{\beta-1} \\
&\propto p^{\sum_{t=1}^{T}z_t}(1-p)^{\sum_{t=1}^{T}(1-z_t)}p^{\alpha-1}(1-p)^{\beta-1} \\
&= \mathrm{Beta}\left(\sum_{t=1}^{T}z_t + \alpha, \sum_{t=1}^{T}(1-z_t) + \beta\right)
\end{aligned}$$

Error in the observation process (σ^2):

$$
\begin{aligned}
[\sigma^2|\cdot] &\propto \prod_{t=1}^{T}[\mathbf{s}_t|\boldsymbol{\mu}_t,\sigma^2][\sigma^2] \\
&\propto \prod_{t=1}^{T}|\sigma^2\mathbf{I}|^{-1/2}\exp\left\{-\frac{1}{2}\left((\mathbf{s}_t-\boldsymbol{\mu}_t)'\left(\sigma^2\mathbf{I}\right)^{-1}(\mathbf{s}_t-\boldsymbol{\mu}_t)\right)\right\}\left(\sigma^2\right)^{-(q+1)}\exp\left\{-\frac{1}{\sigma^2 r}\right\} \\
&\propto \prod_{t=1}^{T}\left(\sigma^2\right)^{-1}\exp\left\{-\frac{1}{2\sigma^2}\left((\mathbf{s}_t-\boldsymbol{\mu}_t)'(\mathbf{s}_t-\boldsymbol{\mu}_t)\right)\right\}\left(\sigma^2\right)^{-(q+1)}\exp\left\{-\frac{1}{\sigma^2 r}\right\} \\
&\propto \left(\sigma^2\right)^{-(T+q+1)}\exp\left\{-\frac{1}{\sigma^2}\left(\frac{\sum_{t=1}^{T}(\mathbf{s}_t-\boldsymbol{\mu}_t)'(\mathbf{s}_t-\boldsymbol{\mu}_t)}{2}+\frac{1}{r}\right)\right\} \\
&= \text{IG}\left(\left(\frac{\sum_{t=1}^{T}(\mathbf{s}_t-\boldsymbol{\mu}_t)'(\mathbf{s}_t-\boldsymbol{\mu}_t)}{2}+\frac{1}{r}\right)^{-1}, T+q\right).
\end{aligned}
$$

Note that the current version of haulout.homerange.mcmc.R contains code for the conjugate update of σ^2 presented above, but this code is currently 'commented' out. Instead, error is modeled as $[\sigma|\cdot] \sim \text{Unif}(a,b)$, and the update for σ proceeds using Metropolis-Hastings.

Homerange center ($\boldsymbol{\mu}_0$):

$$
\begin{aligned}
[\boldsymbol{\mu}_0|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{\mu}_t|\boldsymbol{\mu}_0,\sigma_\mu^2,\mathcal{S}]^{1-z_t}[\boldsymbol{\mu}_0] \\
&\propto \prod_{\{z_t=0\}}\mathcal{TN}(\boldsymbol{\mu}_t|\boldsymbol{\mu}_0,\sigma_\mu^2\mathbf{I})_{\tilde{\mathcal{S}}}
\end{aligned}
$$

The update for $\boldsymbol{\mu}_0$ proceeds using Metropolis-Hastings. Note that the product is over all $\boldsymbol{\mu}_t$ for which $z_t = 0$.

Movement about homerange center (σ_μ^2):

$$
\begin{aligned}
[\sigma_\mu^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{\mu}_t|\boldsymbol{\mu}_0,\sigma_\mu^2,\mathcal{S}]^{1-z_t}[\sigma_\mu^2] \\
&\propto \prod_{\{z_t=0\}}\mathcal{TN}(\boldsymbol{\mu}_t|\boldsymbol{\mu}_0,\sigma_\mu^2\mathbf{I})_{\tilde{\mathcal{S}}}
\end{aligned}
$$

The update for σ_μ^2 proceeds using Metropolis-Hastings. Note that the product is over all $\boldsymbol{\mu}_t$ for which $z_t = 0$.