Two-dimensional "haul-out" model

Brian M. Brost

23 APR 2015

Implementation

The file haulout.2d.sim.R simulates data according to the model statement presented below, and haulout.2d.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let $\mathcal{S}$ be the support of the movement process and $\tilde{S}$ be the support of haul-out sites. Here, both are defined in 2-dimensional space. Note that $\mathcal{S}$ and $\tilde{S}$ overlap, and that $\mathcal{S} \cap \tilde{S} = \tilde{\mathcal{S}}$. The domain defined by $\mathcal{S}$ therefore represents at-sea locations or locations of the individual while milling adjacent to the haul-out site. Also note that $\bar{\mathcal{S}}$, the complement of $\mathcal{S}$, represents inaccessible locations (i.e., terrestrial sites that are not haul-outs). Let $\mathbf{s}_t = (s_{1,t}, s_{2,t})'$ and $\boldsymbol{\mu}_t = (\mu_{1,t}, \mu_{2,t})'$, for $t = 1, \ldots, T$, be observed and true locations respectively.

$$
\begin{aligned}
\mathbf{s}_t &\sim \mathrm{N}(\boldsymbol{\mu}_t, \sigma^2 \mathbf{I}) \\
\boldsymbol{\mu}_t &\sim \begin{cases} \mathrm{Unif}(\tilde{\mathcal{S}}), & z_t = 1 \\ \mathrm{Unif}(\mathcal{S}), & z_t = 0 \end{cases} \\
z_t &\sim \mathrm{Bern}(p) \\
p &\sim \mathrm{Beta}(\alpha, \beta) \\
\sigma^2 &\sim \mathrm{IG}(r, q)
\end{aligned}
$$

Full conditional distributions

True locations ($\boldsymbol{\mu}_t$):

$$
\begin{aligned}
[\boldsymbol{\mu}_t|\cdot] &\propto [\mathbf{s}_t|\boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t|z_t, \mathcal{S}, \tilde{\mathcal{S}}] \\
&\propto [\mathbf{s}_t|\boldsymbol{\mu}_t, \sigma^2][\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t}[\boldsymbol{\mu}_t|\mathcal{S}]^{1-z_t}.
\end{aligned}
$$

For $z_t = 1$,

$$
\begin{aligned}
[\boldsymbol{\mu}_t|\cdot] &\propto [\mathbf{s_t}|\boldsymbol{\mu_t}, \boldsymbol{\sigma^2}][\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t} \\
&\propto [\mathbf{s}_t|\boldsymbol{\mu}_t, \sigma^2] \\
&\propto \exp\left\{-\frac{1}{2}\left((\mathbf{s}_t - \boldsymbol{\mu}_t)'\left(\sigma^2\mathbf{I}\right)^{-1}(\mathbf{s}_t - \boldsymbol{\mu}_t)\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\mathbf{s}_t - 2\mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t + \boldsymbol{\mu}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t + \boldsymbol{\mu}_t'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_t\right)\right\} \\
&\propto \mathrm{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1})
\end{aligned}
$$

where $\mathbf{A} = \left(\sigma^2\mathbf{I}\right)^{-1}$ and $\mathbf{b} = \mathbf{s}_t'\left(\sigma^2\mathbf{I}\right)^{-1}$; therefore, $[\boldsymbol{\mu}_t|\cdot] = \mathrm{N}(\mathbf{s}_t, \sigma^2\mathrm{I})$ for $z_t = 1$. Note that proposed values for $\boldsymbol{\mu}_t$ not in $\tilde{\mathcal{S}}$ are rejected. For $z_t = 0$,

$$[\boldsymbol{\mu}_t|\cdot] = \mathrm{N}(\mathbf{s}_t, \sigma^2\mathbf{I}).$$

In this case, note that proposed values for $\boldsymbol{\mu}_t$ not in $\mathcal{S}$ are rejected.

Haul-out indicator variable (z_t):

$$[z_t|\cdot] \quad \propto \quad [\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t}[\boldsymbol{\mu}_t|\mathcal{S}]^{1-z_t}[z_t|p].$$

For all $\boldsymbol{\mu}_t \notin \tilde{\mathcal{S}}$, let $z_t = 0$. For all $\boldsymbol{\mu}_t \in \tilde{\mathcal{S}}$, sample z_t from

$$\begin{aligned}[z_t|\cdot] &\propto [\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]^{z_t}[\boldsymbol{\mu}_t|\mathcal{S}]^{1-z_t}[z_t|p] \\ &= \text{Bern}(\tilde{p}),\end{aligned}$$

where

$$\begin{aligned}\tilde{p} &= \frac{p[\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]}{p[\boldsymbol{\mu}_t|\tilde{\mathcal{S}}]+(1-p)[\boldsymbol{\mu}_t|\mathcal{S}]} \\ &= \frac{p\left(|\tilde{\mathcal{S}}|^{-1}\right)}{p\left(|\tilde{\mathcal{S}}|^{-1}\right)+(1-p)\left(|\mathcal{S}|^{-1}\right)}.\end{aligned}$$

The notation $|\cdot|$ denotes the area of the respective domain.

Probability of being hauled-out (p):

$$\begin{aligned}[p|\cdot] &\propto \prod_{t=1}^{T}[z_t|p][p] \\ &\propto \prod_{t=1}^{T} p^{z_t}(1-p)^{1-z_t}p^{\alpha-1}(1-p)^{\beta-1} \\ &\propto p^{\sum_{t=1}^{T} z_t}(1-p)^{\sum_{t=1}^{T}(1-z_t)}p^{\alpha-1}(1-p)^{\beta-1} \\ &= \text{Beta}\left(\sum_{t=1}^{T} z_t+\alpha, \sum_{t=1}^{T}(1-z_t)+\beta\right)\end{aligned}$$

Error in the observation process (σ^2):

$$\begin{aligned}[\sigma^2|\cdot] &\propto \prod_{t=1}^{T}[\mathbf{s}_t|\boldsymbol{\mu}_t,\sigma^2][\sigma^2] \\ &\propto \prod_{t=1}^{T}|\sigma^2\mathbf{I}|^{-1/2}\exp\left\{-\frac{1}{2}\left((\mathbf{s}_t-\boldsymbol{\mu}_t)'\left(\sigma^2\mathbf{I}\right)^{-1}(\mathbf{s}_t-\boldsymbol{\mu}_t)\right)\right\}\left(\sigma^2\right)^{-(q+1)}\exp\left\{-\frac{1}{\sigma^2 r}\right\} \\ &\propto \prod_{t=1}^{T}\left(\sigma^2\right)^{-1}\exp\left\{-\frac{1}{2\sigma^2}\left((\mathbf{s}_t-\boldsymbol{\mu}_t)'(\mathbf{s}_t-\boldsymbol{\mu}_t)\right)\right\}\left(\sigma^2\right)^{-(q+1)}\exp\left\{-\frac{1}{\sigma^2 r}\right\} \\ &\propto \left(\sigma^2\right)^{-(T+q+1)}\exp\left\{-\frac{1}{\sigma^2}\left(\frac{\sum_{t=1}^{T}(\mathbf{s}_t-\boldsymbol{\mu}_t)'(\mathbf{s}_t-\boldsymbol{\mu}_t)}{2}+\frac{1}{r}\right)\right\} \\ &= \text{IG}\left(\left(\frac{\sum_{t=1}^{T}(\mathbf{s}_t-\boldsymbol{\mu}_t)'(\mathbf{s}_t-\boldsymbol{\mu}_t)}{2}+\frac{1}{r}\right)^{-1}, T+q\right).\end{aligned}$$

Note that the current version of haulout.2d.mcmc.R contains code for the conjugate update of σ^2 presented above, but this code is currently 'commented' out. Instead, error is modeled as $[\sigma|\cdot] \sim \text{Unif}(a,b)$, and the update for σ proceeds using Metropolis-Hastings.