

# HAUL-OUT SITE ESTIMATION WITH A CONTINUOUS TIME HAUL-OUT PROCESS

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## Implementation

The file `haulout.dp.mixture.2.sim.R` simulates data according to the model statement presented below, and `haulout.dp.mixture.2.mcmc.R` contains the MCMC algorithm for model fitting. Dirichlet process mixture model implementation follows the blocked Gibbs sampler truncation approximation of Ishwaran and James (2001) and Gelman et al. (2014).

## Description

A Dirichlet process mixture model for haul-out site estimation. Unlike the model implemented in `haulout.dp.mixture.1.mcmc.R`, this version uses a probit semiparametric regression to estimate whether locations in the telemetry data set are hauled-out based on wet/dry information in a second data set. As such, this model accommodates the temporal misalignment between telemetry data and wet/dry data (i.e., \*.SEA records).

## Model statement

Let  $\mathbf{s}(t) = (s_1(t), s_2(t))'$  and  $\boldsymbol{\mu}(t) = (\mu_1(t), \mu_2(t))'$ , be observed and true locations of a single individual at some time  $t$ , respectively. Also let  $\boldsymbol{\mu}_{0,h} = (\mu_{0,1,h}, \mu_{0,2,h})'$ , for  $h = 1, \dots, H$ , be the locations of haul-out sites (i.e., cluster centroids), where  $H$  is the maximum number of haul-outs allowed per the truncation approximation of the Dirichlet process mixture model. The latent indicator variable  $z(t)$  denotes when location  $\mathbf{s}(t)$  is on a haul-out site ( $z(t) = 1$ ) or not ( $z(t) = 0$ ). When an individual is hauled-out, note that  $\boldsymbol{\mu}(t) = \boldsymbol{\mu}_{0,h_t}$ , where  $h_t$  acts as a classification variable that identifies the  $\boldsymbol{\mu}_{0,h}$  associated with each  $\boldsymbol{\mu}(t)$ . Furthermore, let  $\mathcal{S}$  be the support of the movement process and  $\tilde{\mathcal{S}}$  be the support of the haul-out sites (i.e., the Dirichlet process and all possible  $\boldsymbol{\mu}_{0,h}$ ). Note that  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  overlap, i.e.,  $\tilde{\mathcal{S}} \subset \mathcal{S}$ . The domain defined by  $\mathcal{S}$  therefore represents at-sea locations or locations of the individual while milling adjacent to the haul-out site. Also note that  $\bar{\mathcal{S}}$ , the complement of  $\mathcal{S}$ , represents inaccessible locations (i.e., terrestrial sites that are not haul-outs).

Information pertaining to the wet/dry status of the individual is available from a second data source,  $y(\tilde{t})$  (e.g., records from a \*.SEA file). Note that records in  $y$  are observed at times  $\tilde{t}$ , which may not be the same as the times  $t$  at which locations are collected. The binary data in  $y(\tilde{t})$  are modeled using an auxiliary variable  $\mathbf{v}$ , which itself is modeled semiparametrically as a function of  $\boldsymbol{\beta}$ , the 'fixed' effects that provide inference on covariates of biological interest in the matrix  $\mathbf{X}$ , and the 'random' effects  $\boldsymbol{\alpha}$  that describe non-linear trend or dependence in wet/dry status through the basis expansion  $\mathbf{W}$ . The auxiliary variable  $\mathbf{v}$  describes a continuous (in time) haul-out process, and links the wet/dry status of the individual with its

telemetry location.

$$\begin{aligned}
\mathbf{s}(t) &\sim \begin{cases} \mathcal{N}(\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I}), & z(t) = 1 \\ \mathcal{N}(\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I}), & z(t) = 0 \end{cases} \\
z(t) &\sim \begin{cases} 0, & v(t) \leq 0 \\ 1, & v(t) > 0 \end{cases} \\
y(\tilde{t}) &\sim \begin{cases} 0, & v(\tilde{t}) \leq 0 \\ 1, & v(\tilde{t}) > 0 \end{cases} \\
\mathbf{v} &\sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\alpha}, \mathbf{I}) \\
h_t &\sim \text{Cat}\left(\frac{\pi_h}{\sum_{\tilde{h}=1}^H \pi_{\tilde{h}}}\right) \\
\pi_h &\sim \text{Stick}(\theta) \\
\boldsymbol{\mu}_{0,h} &\sim \text{Unif}(\tilde{\mathcal{S}}) \\
\theta &\sim \text{Gamma}(r_\theta, q_\theta) \\
\boldsymbol{\beta} &\sim \mathcal{N}(\boldsymbol{\mu}_\beta, \sigma_\beta^2 \mathbf{I}) \\
\boldsymbol{\alpha} &\sim \mathcal{N}(\mathbf{0}, \sigma_\alpha^2 \mathbf{I}) \\
\sigma &\sim \text{Unif}(l_\sigma, u_\sigma) \\
\sigma_\mu &\sim \text{Unif}(l_{\sigma_\mu}, u_{\sigma_\mu}) \\
\sigma_\alpha^2 &\sim \text{IG}(r_\sigma, q_\sigma)
\end{aligned}$$

The concentration parameter  $\theta$  affects the clustering in the Dirichlet process mixture: smaller values yield fewer clusters with more observations per cluster, whereas larger values yield more clusters with fewer observations per cluster. Note that the lines in this model statement pertaining to  $h_t$ ,  $\pi_h$ , and  $\boldsymbol{\mu}_{0,h}$  comprise the stick-breaking representation of the Dirichlet process mixture model, i.e.,

$$\begin{aligned}
\boldsymbol{\mu}_{0,h} &\sim \mathbf{G} \\
\mathbf{G} &\sim \text{DP}(\theta, \mathbf{G}_0) \\
\mathbf{G}_0 &\sim \text{Unif}(\tilde{\mathcal{S}})
\end{aligned}$$

## Full conditional distributions

Haul-out site locations ( $\boldsymbol{\mu}_{0,h}$ ):

$$\begin{aligned}
[\boldsymbol{\mu}_{0,h}|\cdot] &\propto \prod_{\{t:h_t=h\}} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h}, \sigma^2]^{z(t)} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h}, \sigma^2, \sigma_\mu^2]^{1-z(t)} [\boldsymbol{\mu}_{0,h} | \tilde{\mathcal{S}}] \\
&\propto \prod_{\{t:h_t=h\}} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h}, \sigma^2 \mathbf{I})^{z(t)} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{1-z(t)} 1_{\{\boldsymbol{\mu}_{0,h} \in \tilde{\mathcal{S}}\}} \\
&\propto \prod_{\{t:h_t=h\}} \exp \left\{ -\frac{1}{2} (\mathbf{s}(t) - \boldsymbol{\mu}_{0,h})' (\sigma^2 \mathbf{I})^{-1} (\mathbf{s}(t) - \boldsymbol{\mu}_{0,h}) \right\}^{z(t)} \\
&\quad \exp \left\{ -\frac{1}{2} (\mathbf{s}(t) - \boldsymbol{\mu}_{0,h})' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} (\mathbf{s}(t) - \boldsymbol{\mu}_{0,h}) \right\}^{1-z(t)} 1_{\{\boldsymbol{\mu}_{0,h} \in \tilde{\mathcal{S}}\}} \\
&\propto \prod_{\{t:h_t=h\}} \exp \left\{ -\frac{1}{2} (\mathbf{s}(t)' (\sigma^2 \mathbf{I})^{-1} \mathbf{s}(t) - 2\mathbf{s}(t)' (\sigma^2 \mathbf{I})^{-1} \boldsymbol{\mu}_{0,h} + \boldsymbol{\mu}_{0,h}' (\sigma^2 \mathbf{I})^{-1} \boldsymbol{\mu}_{0,h}) \right\}^{z(t)} \times \\
&\quad \exp \left\{ -\frac{1}{2} (\mathbf{s}(t)' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} \mathbf{s}(t) - 2\mathbf{s}(t)' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} \boldsymbol{\mu}_{0,h} + \boldsymbol{\mu}_{0,h}' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} \boldsymbol{\mu}_{0,h}) \right\}^{1-z(t)} \\
&\quad \times 1_{\{\boldsymbol{\mu}_{0,h} \in \tilde{\mathcal{S}}\}} \\
&\propto \prod_{\{t:h_t=h\}} \exp \left\{ -\frac{1}{2} \left( -2 (\mathbf{s}(t)' (\sigma^2 \mathbf{I})^{-1} 1_{\{z(t)=1\}} + \mathbf{s}(t)' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} 1_{\{z(t)=0\}}) \boldsymbol{\mu}_{0,h} + \right. \right. \\
&\quad \left. \left. \boldsymbol{\mu}_{0,h}' \left( (\sigma^2 \mathbf{I})^{-1} 1_{\{z(t)=1\}} + (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} 1_{\{z(t)=0\}} \right) \boldsymbol{\mu}_{0,h} \right) \right\} 1_{\{\boldsymbol{\mu}_{0,h} \in \tilde{\mathcal{S}}\}} \\
&\propto \exp \left\{ -\frac{1}{2} \left( -2 \left( \sum_{\{t:h_t=h, z(t)=1\}} \mathbf{s}(t)' (\sigma^2 \mathbf{I})^{-1} + \sum_{\{t:h_t=h, z(t)=0\}} \mathbf{s}(t)' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} \right) \boldsymbol{\mu}_{0,h} + \right. \right. \\
&\quad \left. \left. \boldsymbol{\mu}_{0,h}' \left( n_{h,1} (\sigma^2 \mathbf{I})^{-1} + n_{h,0} (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1} \right) \boldsymbol{\mu}_{0,h} \right) \right\} 1_{\{\boldsymbol{\mu}_{0,h} \in \tilde{\mathcal{S}}\}} \\
&= \mathcal{N}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1}) 1_{\{\boldsymbol{\mu}_t \in \mathcal{S}\}}
\end{aligned}$$

where  $\mathbf{A} = n_{h,1} (\sigma^2 \mathbf{I})^{-1} + n_{h,0} (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1}$  and  $\mathbf{b}' = \sum_{\{t:h_t=h, z(t)=1\}} \mathbf{s}(t)' (\sigma^2 \mathbf{I})^{-1} + \sum_{\{t:h_t=h, z(t)=0\}} \mathbf{s}(t)' (\sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{-1}$ . Note that the product is over all  $\mathbf{s}(t)$  that are members of  $\boldsymbol{\mu}_{0,h}$ ,  $n_{h,1}$  is the number of members in cluster  $h$  where  $z(t) = 1$ , and  $n_{h,0}$  is the number of members in cluster  $h$  where  $z(t) = 0$ . Proposed values for  $\boldsymbol{\mu}_{0,h}$  not in  $\tilde{\mathcal{S}}$  are rejected, i.e.,  $[\boldsymbol{\mu}_{0,h}|\cdot] = \mathcal{TN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})_{\tilde{\mathcal{S}}}$ . For each of the unoccupied haul-out locations (i.e., clusters with no members),  $\boldsymbol{\mu}_{0,h}$  is sampled directly from the prior. When the support of  $\boldsymbol{\mu}_{0,h}$  exists as a linear feature (e.g., a shoreline), this update is easiest implemented using Metropolis-Hastings. See `haulouts.1.sim.R` and `haulouts.1.mcmc.R` for an example.

Probability mass for haul-out location  $\boldsymbol{\mu}_{0,h}$  ( $\pi_h$ ):

The stick-breaking representation of a Dirichlet process mixture consists of two components, namely a cluster weight and a cluster probability. Let  $\eta_h$  denote the weight assigned to cluster  $h$ , where  $\eta_h \sim \text{Beta}(1, \theta)$ . The associated full-conditional is

$$[\eta_h|\cdot] \sim \text{Beta} \left( 1 + n_h, \theta + \sum_{\tilde{h}=h+1}^H n_{\tilde{h}} \right), \text{ for } h = 1, \dots, H-1,$$

and  $\eta_H = 1$ . The parameter  $n_h$  denotes the number of observations allocated to cluster  $h$ . Note that  $\eta_h$  is sampled in order of decreasing  $n_h$ , i.e.,  $n_h$  is sorted largest to smallest and  $\eta_h$  is sampled in sequence. The

cluster probabilities ( $\pi_h$ ) are deterministic and calculated as

$$\pi_h = \eta_h \prod_{\tilde{h} < h} (1 - \eta_{\tilde{h}}).$$

The probabilities  $\pi_h$  are also calculated in order of decreasing  $n_h$ . See page 553 in Gelman et al. (2014) and Section 5.2 in Ishwaran and James (2001).

*Dirichlet process concentration parameter ( $\theta$ ):*

$$[\theta|\cdot] \propto \text{Gamma}(r_\theta + H - 1, q_\theta - \sum_{h=1}^{H-1} \log(1 - v_h)).$$

See page 553 in Gelman et al. (2014). Also see Escobar and West (1995) and West (1997?, white paper) for alternative full-conditionals for  $\theta$ .

*Haul-out classification variable ( $h_t$ ):*

$$\begin{aligned} [h_t|\cdot] &\sim [\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, z(t), \sigma^2, \sigma_\mu^2][h_t | \pi_h] \\ &\sim \text{Cat}\left(\frac{\pi_h [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, z(t), \sigma^2, \sigma_\mu^2]}{\sum_{\tilde{h}=1}^H \pi_{\tilde{h}} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, z(t), \sigma^2, \sigma_\mu^2]}\right) \\ &\sim \text{Cat}\left(\frac{\pi_h [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2]^{z(t)} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2, \sigma_\mu^2]^{1-z(t)}}{\sum_{\tilde{h}=1}^H \pi_{\tilde{h}} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2]^{z(t)} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2, \sigma_\mu^2]^{1-z(t)}}\right) \\ &\sim \text{Cat}\left(\frac{\pi_h \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2)^{z(t)} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{1-z(t)}}{\sum_{\tilde{h}=1}^H \pi_{\tilde{h}} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2)^{z(t)} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{1-z(t)}}\right). \end{aligned}$$

This update proceeds just as in multinomial sampling; see page 552 in Gelman et al. (2014).

*Latent haul-out indicator variable for telemetry locations ( $z(t)$ ):*

$$\begin{aligned} [z(t)|\cdot] &\propto [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h}, z(t), \sigma^2, \sigma_\mu^2][z(t) | v(t)] \\ &\propto [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2]^{z(t)} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2, \sigma_\mu^2]^{1-z(t)} [z(t) | v(t)] \\ &\propto \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I})^{z(t)} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{1-z(t)} \text{Bern}(z(t) | v(t)) \\ &\propto \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I})^{z(t)} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{1-z(t)} p(t)^{z(t)} (1 - p(t))^{1-z(t)} \\ &\propto (p(t) \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I}))^{z(t)} ((1 - p(t)) [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2, \sigma_\mu^2])^{1-z(t)} \\ &= \text{Bern}(\widetilde{p(t)}), \end{aligned}$$

where  $v(t) = \mathbf{x}'(t)\boldsymbol{\beta} + \mathbf{w}'(t)\boldsymbol{\alpha}$ ,  $p(t) = \Phi(v(t))$ , and

$$\widetilde{p(t)} = \frac{p(t) \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I})}{p(t) \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I}) + (1 - p(t)) \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})}.$$

*Auxiliary variable for continuous haul-out process ( $\mathbf{v}$ ):*

$$\begin{aligned}
[\mathbf{v}|\cdot] &\propto \prod_{\{t,\tilde{t}\}} [z(t), y(\tilde{t}) | \mathbf{v}] [\mathbf{v} | \boldsymbol{\alpha}, \boldsymbol{\beta}] \\
&\propto \prod_t \{[z(t) | v(t)] [v(t) | \boldsymbol{\alpha}, \boldsymbol{\beta}]\} \prod_{\tilde{t}} \{[y(\tilde{t}) | v(\tilde{t})] [v(\tilde{t}) | \boldsymbol{\alpha}, \boldsymbol{\beta}]\} \\
&\propto \prod_t \left\{ \left( 1_{\{z(t)=0\}} 1_{\{v(t) \leq 0\}} + 1_{\{z(t)=1\}} 1_{\{v(t) > 0\}} \right) \times \mathcal{N}(v(t) | \mathbf{x}(t)' \boldsymbol{\beta} + \mathbf{w}(t)' \boldsymbol{\alpha}, \mathbf{1}) \right\} \times \\
&\quad \prod_{\tilde{t}} \left\{ \left( 1_{\{y(\tilde{t})=0\}} 1_{\{v(\tilde{t}) \leq 0\}} + 1_{\{y(\tilde{t})=1\}} 1_{\{v(\tilde{t}) > 0\}} \right) \times \mathcal{N}(v(\tilde{t}) | \mathbf{x}(\tilde{t})' \boldsymbol{\beta} + \mathbf{w}(\tilde{t})' \boldsymbol{\alpha}, \mathbf{1}) \right\}
\end{aligned}$$

For wet/dry data ( $y$ ) observed at times  $\tilde{t}$ ,

$$\begin{aligned}
[v(\tilde{t})|\cdot] &\propto [y(\tilde{t}) | v(\tilde{t})] [v(\tilde{t}) | \boldsymbol{\alpha}, \boldsymbol{\beta}] \\
&\propto \left( 1_{\{y(\tilde{t})=0\}} 1_{\{v(\tilde{t}) \leq 0\}} + 1_{\{y(\tilde{t})=1\}} 1_{\{v(\tilde{t}) > 0\}} \right) \times \mathcal{N}(v(\tilde{t}) | \mathbf{x}(\tilde{t})' \boldsymbol{\beta} + \mathbf{w}(\tilde{t})' \boldsymbol{\alpha}, \mathbf{1}) \\
&= \begin{cases} \mathcal{TN}(\mathbf{x}(\tilde{t})' \boldsymbol{\beta} + \mathbf{w}(\tilde{t})' \boldsymbol{\alpha}, \mathbf{1})_{-\infty}^0, & y(\tilde{t}) = 0 \\ \mathcal{TN}(\mathbf{x}(\tilde{t})' \boldsymbol{\beta} + \mathbf{w}(\tilde{t})' \boldsymbol{\alpha}, \mathbf{1})_0^{\infty}, & y(\tilde{t}) = 1 \end{cases}
\end{aligned}$$

For the latent wet/dry status ( $z$ ) of telemetry locations at times  $t$ ,

$$\begin{aligned}
[v(t)|\cdot] &\propto [z(t) | v(t)] [v(t) | \boldsymbol{\alpha}, \boldsymbol{\beta}] \\
&\propto \left( 1_{\{z(t)=0\}} 1_{\{v(t) \leq 0\}} + 1_{\{z(t)=1\}} 1_{\{v(t) > 0\}} \right) \times \mathcal{N}(v(t) | \mathbf{x}(t)' \boldsymbol{\beta} + \mathbf{w}(t)' \boldsymbol{\alpha}, \mathbf{1}) \\
&= \begin{cases} \mathcal{TN}(\mathbf{x}(t)' \boldsymbol{\beta} + \mathbf{w}(t)' \boldsymbol{\alpha}, \mathbf{1})_{-\infty}^0, & z(t) = 0 \\ \mathcal{TN}(\mathbf{x}(t)' \boldsymbol{\beta} + \mathbf{w}(t)' \boldsymbol{\alpha}, \mathbf{1})_0^{\infty}, & z(t) = 1 \end{cases}
\end{aligned}$$

*Haul-out probability coefficients ( $\boldsymbol{\beta}$ ):*

$$\begin{aligned}
[\boldsymbol{\beta}|\cdot] &\propto [\mathbf{v}|\boldsymbol{\beta}, \boldsymbol{\alpha}] [\boldsymbol{\beta} | \boldsymbol{\mu}_{\beta}, \sigma_{\beta}^2] \\
&\propto \mathcal{N}(\mathbf{v} | \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\alpha}, \mathbf{1}) \mathcal{N}(\boldsymbol{\mu}_{\beta} | \mathbf{0}, \sigma_{\beta}^2 \mathbf{I}) \\
&\propto \exp \left\{ -\frac{1}{2} (\mathbf{v} - (\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\alpha}))' (\mathbf{v} - (\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\boldsymbol{\alpha})) \right\} \times \\
&\quad \exp \left\{ -\frac{1}{2} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})' (\sigma_{\beta}^2 \mathbf{I})^{-1} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta}) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} ((\mathbf{v} - \mathbf{W}\boldsymbol{\alpha}) - \mathbf{X}\boldsymbol{\beta})' ((\mathbf{v} - \mathbf{W}\boldsymbol{\alpha}) - \mathbf{X}\boldsymbol{\beta}) \right\} \times \\
&\quad \exp \left\{ -\frac{1}{2} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})' (\sigma_{\beta}^2 \mathbf{I})^{-1} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta}) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \left( -2 \left( (\mathbf{v} - \mathbf{W}\boldsymbol{\alpha}) \mathbf{X} + \boldsymbol{\mu}_{\beta}' (\sigma_{\beta}^2 \mathbf{I})^{-1} \right) \boldsymbol{\beta} + \boldsymbol{\beta}' (\mathbf{X}' \mathbf{X} + (\sigma_{\beta}^2 \mathbf{I})^{-1}) \boldsymbol{\beta} \right) \right\} \\
&= \mathcal{N}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where  $\mathbf{A} = \mathbf{X}' \mathbf{X} + (\sigma_{\beta}^2 \mathbf{I})^{-1}$  and  $\mathbf{b}' = (\mathbf{v} - \mathbf{W}\boldsymbol{\alpha})' \mathbf{X} + \boldsymbol{\mu}_{\beta}' (\sigma_{\beta}^2 \mathbf{I})^{-1}$ . Note that the matrices  $\mathbf{X}$  and  $\mathbf{W}$  contain covariates pertaining to times associated with both  $\mathbf{s}(t)$  and  $y(\tilde{t})$ .

*Random 'effects' for non-linear trend/dependence ( $\alpha$ ):*

$$\begin{aligned}
[\alpha|\cdot] &\propto [\mathbf{v}|\boldsymbol{\beta}, \alpha][\alpha | \mathbf{0}, \sigma_\alpha^2] \\
&\propto \mathcal{N}(\mathbf{v} | \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\alpha, \mathbf{1}) \mathcal{N}(\alpha | \mathbf{0}, \sigma_\alpha^{-1}\mathbf{I}) \\
&\propto \exp \left\{ -\frac{1}{2} (\mathbf{v} - (\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\alpha))' (\mathbf{v} - (\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\alpha)) \right\} \times \\
&\quad \exp \left\{ -\frac{1}{2} (\alpha - \mathbf{0})' (\sigma_\alpha^2 \mathbf{I})^{-1} (\alpha - \mathbf{0}) \right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where  $\mathbf{A} = \mathbf{W}'\mathbf{W} + (\sigma_\alpha^2 \mathbf{I})^{-1}$  and  $\mathbf{b}' = (\mathbf{v} - \mathbf{X}\boldsymbol{\beta})' \mathbf{W}$ . Note that the matrices  $\mathbf{X}$  and  $\mathbf{W}$  contain covariates pertaining to times associated with both  $\mathbf{s}(t)$  and  $y(\tilde{t})$ .

*Error in the observation process ( $\sigma$ ):*

$$\begin{aligned}
[\sigma|\cdot] &\propto \prod_t [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2]^{z(t)} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2, \sigma_\mu^2]^{1-z(t)} [\sigma] \\
&\propto \prod_t \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I})^{z(t)} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})^{1-z(t)}
\end{aligned}$$

The update for  $\sigma$  proceeds using Metropolis-Hastings.

*Homerange dispersion parameter ( $\sigma_\mu$ ):*

$$\begin{aligned}
[\sigma_\mu|\cdot] &\propto \prod_{\{t:z(t)=0\}} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2, \sigma_\mu^2][\sigma_\mu] \\
&\propto \prod_{\{t:z(t)=0\}} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I})
\end{aligned}$$

The update for  $\sigma_\mu$  proceeds using Metropolis-Hastings. Note that the product is over all  $t$  for which  $z(t) = 0$ .

*Variance of random effects ( $\sigma_\alpha^2$ ):*

$$\begin{aligned}
[\sigma_\alpha^2|\cdot] &\propto [\alpha|\mathbf{0}, \sigma_\alpha^2][\sigma_\alpha^2 | r_\sigma, q_\sigma] \\
&\propto \mathcal{N}(\alpha|\mathbf{0}, \sigma_\alpha^2 \mathbf{I}) \text{IG}(\sigma_\alpha^2 | r_\sigma, q_\sigma) \\
&\propto |\sigma_\alpha^2 \mathbf{I}|^{-1/2} \exp \left\{ -\frac{1}{2} \left( (\alpha - \mathbf{0})' (\sigma_\alpha^2 \mathbf{I})^{-1} (\alpha - \mathbf{0}) \right) \right\} \times \\
&\quad (\sigma_\alpha^2)^{-(q_\sigma+1)} \exp \left\{ -\frac{1}{r_\sigma \sigma_\alpha^2} \right\} \\
&\propto (\sigma_\alpha^2)^{-M/2} \exp \left\{ -\frac{1}{2\sigma_\alpha^2} \alpha' \alpha \right\} \\
&\quad (\sigma_\alpha^2)^{-(q_\sigma+1)} \exp \left\{ -\frac{1}{r_\sigma \sigma_\alpha^2} \right\} \\
&\propto (\sigma_\alpha^2)^{-(M/2+q_\sigma+1)} \exp \left\{ -\frac{1}{\sigma_\alpha^2} \left( \frac{\alpha' \alpha}{2} + \frac{1}{r_\sigma} \right) \right\} \\
&= \text{IG} \left( \left( \frac{\alpha' \alpha}{2} + \frac{1}{r_\sigma} \right)^{-1}, \frac{M}{2} + q_\sigma \right)
\end{aligned}$$

where  $M$  is the length of  $\alpha$  (or column dimension of  $\mathbf{W}$ ).

## References

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