Haul-out Site Estimation using a Truncation Approximation of a Dirichlet Process Mixture Model

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Description

A Dirichlet process mixture model for haul-out site estimation. Unlike the model implemented in haulout.dp.mixture.1.mcmc.R, this version uses a probit semiparametric regression to estimate whether locations in the telemetry data set are hauled-out based on wet/dry information in a second data. As such, this model accommodates the temporal misalignment between telemetry data and wet/dry data (i.e., *.SEA records).

Implementation

The file haulout.dp.mixture.2.sim.R simulates data according to the model statement presented below, and haulout.dp.mixture.2.mcmc.R contains the MCMC algorithm for model fitting. Dirichlet process mixture model implementation follows the blocked Gibbs sampler truncation approximation of Ishwaran and James (2001) and Gelman et al. (2014).

Model statement

Let $\mathbf{s}(t) = (s_1(t), s_2(t))'$ and $\boldsymbol{\mu}(t) = (\mu_1(t), \mu_2(t))'$, be observed and true locations of a single individual at some time t, respectively. Also let $\boldsymbol{\mu}_{0,h} = (\mu_{0,1,h}, \mu_{0,2,h})'$, for $h = 1, \ldots, H$, be the locations of haul-out sites (i.e., cluster centroids), where H is the maximum number of haul-outs allowed per the truncation approximation of the Dirichlet process mixture model. The latent indicator variable $z(t)$ denotes when location $\mathbf{s}(t)$ is on a haul-out site ($z(t) = 1$) or not ($z(t) = 0$). When an individual is hauled-out, note that $\boldsymbol{\mu}(t) = \boldsymbol{\mu}_{0,h_t}$, where h_t acts as a classification variable that identifies the $\boldsymbol{\mu}_{0,h}$ associated with each $\boldsymbol{\mu}(t)$. Furthermore, let $\mathcal{S}$ be the support of the movement process and $\tilde{S}$ be the support of the haul-out sites (i.e., the Dirichlet process and all possible $\boldsymbol{\mu}_{0,h}$). Note that $\mathcal{S}$ and $\tilde{S}$ overlap, i.e., $\tilde{\mathcal{S}} \subset \mathcal{S}$. The domain defined by $\mathcal{S}$ therefore represents at-sea locations or locations of the individual while milling adjacent to the haul-out site. Also note that $\bar{\mathcal{S}}$, the complement of $\mathcal{S}$, represents inaccessible locations (i.e., terrestrial sites that are not haul-outs).

Information pertaining to the wet/dry status of the individual is available from a second data source, $y(\tilde{t})$ (e.g., records from a *.SEA file). Note that records in y are observed at times $\tilde{t}$, which may not be the same as the times t at which locations are collected. The binary data in $y\left(\tilde{t}\right)$ are modeled using an auxiliary variable $u\left(\tilde{t}\right)$, which itself is modeled semiparametrically as a function of $\boldsymbol{\beta}$, the 'fixed' effects that provide inference on covariates of biological interest in the matrix $\mathbf{X}$, and the 'random' effects $\boldsymbol{\alpha}$ that describe non-linear trend or dependence in $y\left(\tilde{t}\right)$ through the basis expansion $\mathbf{W}$.

$$
\begin{aligned}
\mathbf{s}(t) &\sim \begin{cases} \mathcal{N}(\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I}), & z(t)=1 \\ \mathcal{N}(\boldsymbol{\mu}_{0,h_t}, \sigma^2 \mathbf{I} + \sigma_\mu^2 \mathbf{I}), & z(t)=0 \end{cases} \\
z(t) &\sim \begin{cases} 0, & u(t) \leq 0 \\ 1, & u(t) > 0 \end{cases} \\
y\left(\tilde{t}\right) &\sim \begin{cases} 0, & u\left(\tilde{t}\right) \leq 0 \\ 1, & u\left(\tilde{t}\right) > 0 \end{cases} \\
\left(\begin{array}{c} u(t) \\ u\left(\tilde{t}\right) \end{array} \right) &\sim \mathcal{N}\left(\left(\begin{array}{c} \mathbf{x}(t)'\boldsymbol{\beta} + \mathbf{w}'(t)\boldsymbol{\alpha} \\ \mathbf{x}\left(\tilde{t}\right)'\boldsymbol{\beta} + \mathbf{w}'\left(\tilde{t}\right)\boldsymbol{\alpha} \end{array} \right), \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{array} \right] \right) \\
h_t &\sim \mathrm{Cat}\left(\frac{\pi_h [\mathbf{s}(t) | \boldsymbol{\mu}_{0,h_t}, z(t), \sigma^2, \sigma_\mu^2]}{\sum_{\tilde{h}=1}^{H} \pi_{\tilde{h}} [\mathbf{s}(t) | \boldsymbol{\mu}_{0,\tilde{h}}, z(t), \sigma^2, \sigma_\mu^2]} \right) \\
\pi_h &= v_h \prod_{\tilde{h}<h} (1 - v_{\tilde{h}}) \\
v_h &\sim \mathrm{Beta}(1, \theta) \\
\boldsymbol{\mu}_{0,h} &\sim \mathrm{Unif}(\tilde{\mathcal{S}}) \\
\theta &\sim \mathrm{Gamma}(r, q) \\
\boldsymbol{\beta} &\sim \mathcal{N}(\boldsymbol{\mu}_\beta, \sigma_\beta^2 \mathbf{I}) \\
\boldsymbol{\alpha} &\sim \mathcal{N}(\mathbf{0}, \sigma_\alpha^2 \mathbf{I}) \\
\sigma &\sim \mathrm{Unif}(l_\sigma, u_\sigma) \\
\sigma_\mu &\sim \mathrm{Unif}(l_{\sigma_\mu}, u_{\sigma_\mu})
\end{aligned}
$$

The concentration parameter θ affects the clustering in the Dirichlet process mixture: smaller values yield fewer clusters with more observations per cluster, whereas larger values yield more clusters with fewer observations per cluster. Note that the lines in this model statement pertaining to h_t, π_h, v_h, and $\boldsymbol{\mu}_{0,h}$ comprise the stick-breaking representation of the Dirichlet process mixture model, i.e.,

$$
\begin{aligned}
\boldsymbol{\mu}_{0,h} &\sim \mathbf{G} \\
\mathbf{G} &\sim \mathrm{DP}(\theta, \mathbf{G}_0) \\
\mathbf{G}_0 &\sim \mathrm{Unif}(\tilde{\mathcal{S}})
\end{aligned}
$$

Full conditional distributions

Haul-out site locations ($\boldsymbol{\mu}_{0,h}$):

$$\begin{aligned}
[\boldsymbol{\mu}_{0,h}|\cdot] &\propto \prod_{\{t:h_t=h\}} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h},\sigma^2]^{z(t)}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h},\sigma^2,\sigma_\mu^2]^{1-z(t)}[\boldsymbol{\mu}_{0,h}] \\
&\propto \prod_{\{t:h_t=h\}} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h},\sigma^2\mathbf{I})^{z(t)}\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h},\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I})^{1-z(t)}1_{\{\boldsymbol{\mu}_{0,h}\in\tilde{\mathcal{S}}\}} \\
&\propto \prod_{\{t:h_t=h\}} \exp\left\{-\frac{1}{2}\left(\mathbf{s}(t)-\boldsymbol{\mu}_{0,h}\right)'\left(\sigma^2\mathbf{I}\right)^{-1}\left(\mathbf{s}(t)-\boldsymbol{\mu}_{0,h}\right)\right\}^{z(t)} \\
&\qquad \exp\left\{-\frac{1}{2}\left(\mathbf{s}(t)-\boldsymbol{\mu}_{0,h}\right)'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}\left(\mathbf{s}(t)-\boldsymbol{\mu}_{0,h}\right)\right\}^{1-z(t)} 1_{\{\boldsymbol{\mu}_{0,h}\in\tilde{\mathcal{S}}\}} \\
&\propto \prod_{\{t:h_t=h\}} \exp\left\{-\frac{1}{2}\left(\mathbf{s}(t)'\left(\sigma^2\mathbf{I}\right)^{-1}\mathbf{s}(t)-2\mathbf{s}(t)'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_{0,h}+\boldsymbol{\mu}_{0,h}'\left(\sigma^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_{0,h}\right)\right\}^{z(t)} \times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\mathbf{s}(t)'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}\mathbf{s}(t)-2\mathbf{s}(t)'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_{0,h}+\boldsymbol{\mu}_{0,h}'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_{0,h}\right)\right\}^{1-z(t)} \times \\
&\qquad 1_{\{\boldsymbol{\mu}_{0,h}\in\tilde{\mathcal{S}}\}} \\
&\propto \prod_{\{t:h_t=h\}} \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{s}(t)'\left(\sigma^2\mathbf{I}\right)^{-1}1_{\{z(t)=1\}}+\mathbf{s}(t)'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}1_{\{z(t)=0\}}\right)\boldsymbol{\mu}_{0,h}+\right.\right. \\
&\qquad \left.\left.\boldsymbol{\mu}_{0,h}'\left(\left(\sigma^2\mathbf{I}\right)^{-1}1_{\{z(t)=1\}}+\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}1_{\{z(t)=0\}}\right)\boldsymbol{\mu}_{0,h}\right)\right\}1_{\{\boldsymbol{\mu}_{0,h}\in\tilde{\mathcal{S}}\}} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{\{t:h_t=h,z(t)=1\}}\mathbf{s}(t)'\left(\sigma^2\mathbf{I}\right)^{-1}+\sum_{\{t:h_t=h,z(t)=0\}}\mathbf{s}(t)'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_{0,h}+\right.\right. \\
&\qquad \left.\left.\boldsymbol{\mu}_{0,h}'\left(n_{h,1}\left(\sigma^2\mathbf{I}\right)^{-1}+n_{h,0}\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_{0,h}\right)\right\}1_{\{\boldsymbol{\mu}_{0,h}\in\tilde{\mathcal{S}}\}} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1})1_{\{\boldsymbol{\mu}_t\in\mathcal{S}\}}
\end{aligned}$$

where $\mathbf{A}=n_{h,1}\left(\sigma^2\mathbf{I}\right)^{-1}+n_{h,0}\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=\sum_{\{t:h_t=h,z(t)=1\}}\mathbf{s}(t)'\left(\sigma^2\mathbf{I}\right)^{-1}+\sum_{\{t:h_t=h,z(t)=0\}}\mathbf{s}(t)'\left(\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I}\right)^{-1}$. Note that the product is over all $\mathbf{s}(t)$ that are members of $\boldsymbol{\mu}_{0,h}$, $n_{h,1}$ is the number of members in cluster h where $z(t)=1$, and $n_{h,0}$ is the number of members in cluster h where $z(t)=0$. Proposed values for $\boldsymbol{\mu}_{0,h}$ not in $\tilde{\mathcal{S}}$ are rejected, i.e., $[\boldsymbol{\mu}_{0,h}|\cdot]=\mathcal{TN}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1})_{\tilde{\mathcal{S}}}$. For each of the unoccupied haul-out locations (i.e., clusters with no members), $\boldsymbol{\mu}_{0,h}$ is sampled directly from the prior.

Latent haul-out indicator variable for locations ($z(t)$):

$$\begin{aligned}
[z(t)|\cdot] &\propto [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2]^{z(t)}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2,\sigma_\mu^2]^{1-z(t)}[z(t)|u(t)] \\
&\propto [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2]^{z(t)}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2,\sigma_\mu^2]^{1-z(t)}p(t)^{z(t)}(1-p(t))^{1-z(t)} \\
&\propto \left(p(t)[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2]\right)^{z(t)}\left((1-p(t))[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2,\sigma_\mu^2]\right)^{1-z(t)} \\
&= \text{Bern}(\widetilde{p(t)}),
\end{aligned}$$

where $u(t)=\mathbf{x}(t)'\boldsymbol{\beta}+\mathbf{w}(t)'\boldsymbol{\alpha}$, $\Phi^{-1}(p(t))=u(t)$, and

$$\widetilde{p(t)}=\frac{p(t)\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2\mathbf{I})}{p(t)\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2\mathbf{I})+(1-p(t))\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I})}.$$

Auxiliary variable for binary wet/dry data ($u(\tilde{t})$):

$$
\begin{aligned}
\left[u(\tilde{t})\,|\cdot\right] &\propto \left[y(\tilde{t})\,|u(\tilde{t})\right]\left[u(\tilde{t})\right] \\
&\propto \left(1_{\{y(\tilde{t})=0\}}1_{\{u(\tilde{t})\leq 0\}} + 1_{\{y(\tilde{t})=1\}}1_{\{u(\tilde{t})>0\}}\right) \times \mathcal{N}\left(u(\tilde{t}) \mid \mathbf{x}(\tilde{t})'\boldsymbol{\beta} + \mathbf{w}(\tilde{t})'\boldsymbol{\alpha}, \mathbf{1}\right) \\
&= \begin{cases} \mathcal{TN}\left(\mathbf{x}(\tilde{t})'\boldsymbol{\beta} + \mathbf{w}(\tilde{t})'\boldsymbol{\alpha}, \mathbf{1}\right)_{-\infty}^{0}, & y(\tilde{t}) = 0 \\ \mathcal{TN}\left(\mathbf{x}(\tilde{t})'\boldsymbol{\beta} + \mathbf{w}(\tilde{t})'\boldsymbol{\alpha}, \mathbf{1}\right)_{0}^{\infty}, & y(\tilde{t}) = 1 \end{cases}
\end{aligned}
$$

Haul-out classification variable (h_t):

$$
\begin{aligned}
[h_t|\cdot] &\sim \text{Cat}\left(\frac{\pi_h[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, z(t), \sigma^2, \sigma_\mu^2]}{\sum_{\tilde{h}=1}^{H}\pi_{\tilde{h}}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, z(t), \sigma^2, \sigma_\mu^2]}\right) \\
&\sim \text{Cat}\left(\frac{\pi_h[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2]^{z(t)}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2, \sigma_\mu^2]^{1-z(t)}}{\sum_{\tilde{h}=1}^{H}\pi_{\tilde{h}}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2]^{z(t)}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2, \sigma_\mu^2]^{1-z(t)}}\right) \\
&\sim \text{Cat}\left(\frac{\pi_h\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2)^{z(t)}\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t}, \sigma^2\mathbf{I} + \sigma_\mu^2\mathbf{I})^{1-z(t)}}{\sum_{\tilde{h}=1}^{H}\pi_{\tilde{h}}\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2)^{z(t)}\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,\tilde{h}}, \sigma^2\mathbf{I} + \sigma_\mu^2\mathbf{I})^{1-z(t)}}\right).
\end{aligned}
$$

This update proceeds just as in multinomial sampling; see page 552 in Gelman et al. (2014).

Probability mass for haul-out location $\boldsymbol{\mu}_{0,h}$ (π_h):

$$
\pi_h = v_h \prod_{\tilde{h}<h}(1 - v_{\tilde{h}}),
$$

where

$$
[v_h|\cdot] \sim \text{Beta}\left(1 + n_h, \theta + \sum_{\tilde{h}=h+1}^{H} n_{\tilde{h}}\right), \text{ for } h = 1, \ldots, H-1,
$$

and $v_H = 1$. This represents the stick-breaking construction of the Dirichlet process. The parameter n_h denotes the number of observations allocated to cluster h. Note that v_h is sampled in order of decreasing n_h, i.e., n_h is sorted largest to smallest and v_h is sampled in sequence. The probabilities π_h are calculated in order of decreasing n_h as well. See page 553 in Gelman et al. (2014) and Section 5.2 in Ishwaran and James (2001).

Dirichlet process concentration parameter (θ):

$$
[\theta|\cdot] \propto \text{Gamma}(r + H - 1, q - \sum_{h=1}^{H-1}\log(1 - v_h)).
$$

See page 553 in Gelman et al. (2014).

Haul-out probability coefficients ($\boldsymbol{\beta}$):

$$
\begin{aligned}
[\boldsymbol{\beta}|\cdot] &\propto [\mathbf{u}(\tilde{t})|\boldsymbol{\beta},\boldsymbol{\alpha}][\boldsymbol{\beta}] \\
&\propto \mathcal{N}\left(\mathbf{X}(\tilde{t})\boldsymbol{\beta}+\mathbf{W}(\tilde{t})\boldsymbol{\alpha},\mathbf{1}\right)\mathcal{N}(\mathbf{0},\sigma_\beta^2\mathbf{I}) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{u}(\tilde{t})-\left(\mathbf{X}(\tilde{t})\boldsymbol{\beta}+\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)\right)'\left(\mathbf{u}(\tilde{t})-\left(\mathbf{X}(\tilde{t})\boldsymbol{\beta}+\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)\right)\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\left(\mathbf{u}(\tilde{t})-\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)-\mathbf{X}(\tilde{t})\boldsymbol{\beta}\right)'\left(\left(\mathbf{u}(\tilde{t})-\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)-\mathbf{X}(\tilde{t})\boldsymbol{\beta}\right)\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\left(\mathbf{u}(\tilde{t})-\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)\mathbf{X}(\tilde{t})+\boldsymbol{\mu}_\beta\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}+\boldsymbol{\beta}'\left(\mathbf{X}(\tilde{t})'\mathbf{X}(\tilde{t})+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A}=\mathbf{X}(\tilde{t})'\mathbf{X}(\tilde{t})+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=\left(\mathbf{u}(\tilde{t})-\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)'\mathbf{X}(\tilde{t})+\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$.

Random 'effects' for non-linear trend/dependence ($\boldsymbol{\alpha}$):

$$
\begin{aligned}
[\boldsymbol{\alpha}|\cdot] &\propto \left[\mathbf{u}(\tilde{t})|\boldsymbol{\beta},\boldsymbol{\alpha}\right][\boldsymbol{\alpha}] \\
&\propto \mathcal{N}\left(\mathbf{X}(\tilde{t})\boldsymbol{\beta}+\mathbf{W}(\tilde{t})\boldsymbol{\alpha},\mathbf{1}\right)\mathcal{N}(\mathbf{0},\sigma_\alpha^{-1}\mathbf{I}) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{u}(\tilde{t})-\left(\mathbf{X}(\tilde{t})\boldsymbol{\beta}+\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)\right)'\left(\mathbf{u}(\tilde{t})-\left(\mathbf{X}(\tilde{t})\boldsymbol{\beta}+\mathbf{W}(\tilde{t})\boldsymbol{\alpha}\right)\right)\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\alpha}-\mathbf{0}\right)'\left(\sigma_\alpha^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\alpha}-\mathbf{0}\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A}=\mathbf{W}(\tilde{t})'\mathbf{W}(\tilde{t})+\left(\sigma_\alpha^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=\left(\mathbf{u}(\tilde{t})'-\mathbf{X}(\tilde{t})\boldsymbol{\beta}\right)'\mathbf{W}(\tilde{t})$.

Error in the observation process (σ):

$$
\begin{aligned}
[\sigma|\cdot] &\propto \prod_t [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2]^{z(t)}[\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2,\sigma_\mu^2]^{1-z(t)}[\sigma] \\
&\propto \prod_t \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2\mathbf{I})^{z(t)}\mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2\mathbf{I}+\sigma_\mu^2)^{1-z(t)}
\end{aligned}
$$

The update for σ proceeds using Metropolis-Hastings.

Homerange dispersion parameter (σ_μ):

$$
\begin{aligned}
[\sigma_\mu|\cdot] &\propto \prod_{\{t:z(t)=0\}} [\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2,\sigma_\mu^2][\sigma_\mu] \\
&\propto \prod_{\{t:z(t)=0\}} \mathcal{N}(\mathbf{s}(t)|\boldsymbol{\mu}_{0,h_t},\sigma^2\mathbf{I}+\sigma_\mu^2\mathbf{I})
\end{aligned}
$$

The update for σ_μ proceeds using Metropolis-Hastings. Note that the product is over all t for which $z(t)=0$.

References

Gelman, A., J.B. Carlin, H.S. Stern, D.B. Dunson, A. Vehtari, and D.B. Rubin. 2014. Bayesian data analysis. CRC Press.

Ishwaran, H., and L.F. James. 2001. Gibbs sampling methods for stick-breaking priors. Journal of the American Statistical Association 96: 161–173.