Poisson Generalized Linear Mixed Model for Count Data

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Description

A generarlized linear model for count data.

Implementation

The file poisson.glmm.sim.R simulates data according to the model statement presented below, and poisson.glmm.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let z_{ij}, for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, denote observed count data (i.e., z_i are integers greater than or equal to 0), where the index i denotes replicate observations within group j, and n_j is the number of observations in group j. Furthermore, let $\mathbf{x}_{ij}$ be a vector of p covariates (including the intercept) associated with z_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$
\begin{aligned}
z_{ij} &\sim \text{Pois}\left(\lambda_{ij}\right) \\
\log\left(\lambda_{ij}\right) &= \mathbf{x}'_{ij}\boldsymbol{\beta}_j \\
\boldsymbol{\beta}_j &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}_j$):

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \prod_{i=1}^{n_j} \left[z_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j\right] \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{i=1}^{n_j} \text{Pois}\left(z_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j\right) \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}
$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_\beta\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J} \left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
\propto \quad & \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
& \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
\propto \quad & \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
= \quad & \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$
\begin{aligned}
[\boldsymbol{\Sigma} \mid \cdot] \quad \propto \quad & \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right][\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu] \\
\propto \quad & \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right)\mathrm{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J}{2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{J}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
& \times |\mathbf{S}_0|^{-\frac{\nu}{2}}|\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\mathrm{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right) + \mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\mathrm{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\right) + \mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right)\boldsymbol{\Sigma}^{-1} + \mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)\boldsymbol{\Sigma}^{-1}\right]\right\} \\
= \quad & \mathrm{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, J+\nu\right).
\end{aligned}
$$