Binomial Generalized Linear Mixed Model with Varying Coefficients at Two Hierarchical Levels

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Description

A generalized linear mixed model for binomially distributed data with varying coefficients at two hiearchical levels.

Implementation

The file binomial.varying.coef.3.sim.R simulates data according to the model statement presented below, and binomial.varying.coef.3.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let z_{ijk} be the number of "successes" (i.e., z_{ijk} are integers greater than or equal to 0) out of N_{ijk} "trials", for groups $i = 1, \ldots, n$ and subgroups $j = 1, \ldots, J_i$ (the subgroups are nested withing groups). The index k, for $k = 1, \ldots, K_{ij}$, denotes replicate events within group i and subgroup j. Furthermore, let $\mathbf{x}_{ijk}$ be a vector of covariates associated with z_{ijk} and $\boldsymbol{\alpha}_{ij}$ be the corresponding vector of coefficients for subgroup j in group i. The vector $\boldsymbol{\beta}_i$ corresponds to group-level coefficients and $\boldsymbol{\mu}_\beta$ is a vector of population-level coefficients.

$$
\begin{aligned}
z_{ijk} &\sim \text{Binom}\left(N_{ijk}, p_{ijk}\right) \\
\text{logit}\left(p_{ijk}\right) &= \mathbf{x}'_{ijk}\boldsymbol{\alpha}_{ij} \\
\boldsymbol{\alpha}_{ij} &\sim \mathcal{N}\left(\boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \\
\boldsymbol{\beta}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1}_{\alpha_i} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right) \\
\boldsymbol{\Sigma}^{-1}_{\beta} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Subgroup-level regression coefficients ($\boldsymbol{\alpha}_{ij}$):

$$
\begin{aligned}
[\boldsymbol{\alpha}_{ij} \mid \cdot] &\propto \prod_{k=1}^{K_{ij}} \left[z_{ijk} \mid N_{ijk}, \mathbf{x}'_{ijk}\boldsymbol{\alpha}_{ij}\right] \left[\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right] \\
&\propto \prod_{k=1}^{K_{ij}} \text{Binom}\left(z_{ijk} \mid N_{ijk}, \mathbf{x}'_{ijk}\boldsymbol{\alpha}_{ij}\right) \mathcal{N}\left(\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right).
\end{aligned}
$$

The update for $\boldsymbol{\alpha}_{ij}$ proceeds using Metropolis-Hastings.

Group-level regression coefficients ($\boldsymbol{\beta}_i$):

$$
\begin{aligned}
[\boldsymbol{\beta}_i \mid \cdot] &\propto \prod_{j=1}^{J_i} [\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}] \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right] \\
&\propto \prod_{j=1}^{J_i} \mathcal{N}\left(\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \\
&\propto \exp\left\{\sum_{j=1}^{J_i}\left(-\frac{1}{2}\left(\boldsymbol{\alpha}_{ij}-\boldsymbol{\beta}_i\right)' \boldsymbol{\Sigma}_{\alpha_i}^{-1}\left(\boldsymbol{\alpha}_{ij}-\boldsymbol{\beta}_i\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J_i} \boldsymbol{\alpha}_{ij}' \boldsymbol{\Sigma}_{\alpha_i}^{-1}\right) \boldsymbol{\beta}_i + \boldsymbol{\beta}_i'\left(J_i \boldsymbol{\Sigma}_{\alpha_i}^{-1}\right) \boldsymbol{\beta}_i\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(-2\left(\boldsymbol{\mu}_\beta' \boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\beta}_i + \boldsymbol{\beta}_i'\left(\boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\beta}_i\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J_i} \boldsymbol{\alpha}_{ij}' \boldsymbol{\Sigma}_{\alpha_i}^{-1} - \boldsymbol{\mu}_\beta' \boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\beta}_i + \boldsymbol{\beta}_i'\left(J_i \boldsymbol{\Sigma}_{\alpha_i}^{-1} + \boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\beta}_i\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J_i \boldsymbol{\Sigma}_{\alpha_i}^{-1} + \boldsymbol{\Sigma}_\beta^{-1}$ and $\mathbf{b}' = \boldsymbol{\alpha}_i' \boldsymbol{\Sigma}_{\alpha_i}^{-1} - \boldsymbol{\mu}_\beta' \boldsymbol{\Sigma}_\beta^{-1}$, where $\boldsymbol{\alpha}_i$ is the vector sum $\sum_{j=1}^{J_i} \boldsymbol{\alpha}_{ij}$.

Mean of group-level regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta \mid \cdot] &\propto \prod_{i=1}^{n} [\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_{\mu_\beta}^2\right] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_{\mu_\beta}^2 \mathbf{I}\right) \\
&\propto \exp\left\{\sum_{i=1}^{n}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)'\left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n} \boldsymbol{\beta}_i' \boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(n \boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n} \boldsymbol{\beta}_i' \boldsymbol{\Sigma}_\beta^{-1}\right) \boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(n \boldsymbol{\Sigma}_\beta^{-1} + \left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}\right) \boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n \boldsymbol{\Sigma}_\beta^{-1} + \left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}' \boldsymbol{\Sigma}_\beta^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{i=1}^{n} \boldsymbol{\beta}_i$.

Variance-covariance of group-level regression coefficients ($\boldsymbol{\Sigma}_\beta$):

$$
[\boldsymbol{\Sigma}_\beta \mid \cdot] \propto \prod_{i=1}^{n} [\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta] [\boldsymbol{\Sigma}_\beta \mid \mathbf{S}_0, \nu]
$$

$$
\begin{aligned}
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \text{Wish}\left(\boldsymbol{\Sigma}_\beta \mid \mathbf{S}_0, \nu\right) \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \left|\mathbf{S}_0\right|^{-\frac{\nu}{2}} \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}_\beta^{-1}\right)\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{n+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}\text{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{n+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}\text{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1}\right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{n+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{i=1}^{n}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) \boldsymbol{\Sigma}_\beta^{-1} + \mathbf{S}_0 \boldsymbol{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{n+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{i=1}^{n}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right) \boldsymbol{\Sigma}_\beta^{-1}\right]\right\} \\
&= \text{Wish}\left(\left(\sum_{i=1}^{n}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, n+\nu\right).
\end{aligned}
$$

Variance-covariance of subgroup-level regression coefficients ($\boldsymbol{\Sigma}_{\alpha_i}$):

$$
\begin{aligned}
\left[\boldsymbol{\Sigma}_{\alpha_i} \mid \cdot\right] &\propto \prod_{j=1}^{J_i}\left[\alpha_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right]\left[\boldsymbol{\Sigma}_{\alpha_i} \mid \mathbf{S}_0, \nu\right] \\
&\propto \prod_{j=1}^{J_i} \mathcal{N}\left(\alpha_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \text{Wish}\left(\boldsymbol{\Sigma}_{\alpha_i} \mid \mathbf{S}_0, \nu\right) \\
&= \text{Wish}\left(\left(\sum_{j=1}^{J_i}\left(\left(\alpha_{ij} - \boldsymbol{\beta}_i\right)\left(\alpha_{ij} - \boldsymbol{\beta}_i\right)'\right) + \mathbf{S}_0\right)^{-1}, J_i + \nu\right).
\end{aligned}
$$