Generalized Linear Mixed Effects Model for Binary Data using the Probit Link

Brian M. Brost

08 March 2016

Description

A mixed effects regression model for binary data using the probit link.

Implementation

The file probit.varying.coef.sim.R simulates data according to the model statement presented below, and probit.varying.coef.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let y_{ij}, for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, denote observed data that takes on the values $\{0, 1\}$, where the index i denotes replicate observations within group j, and n_j is the number of observations in group j. Furthermore, let $\mathbf{x}_{ij}$ be a vector of p covariates (including the intercept) associated with y_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$
\begin{aligned}
y_{ij} &\sim \begin{cases} 0, & v_{ij} \leq 0 \\ 1, & v_{ij} > 1 \end{cases} \\
v_{ij} &\sim \mathcal{N}\left(\mathbf{x}'_{ij}\boldsymbol{\beta}_j, \mathbf{1}\right) \\
\boldsymbol{\beta}_j &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\Lambda}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Observation model auxiliary variable (v_{ij}):

$$
\begin{aligned}
[v_{ij} \mid \cdot] &\propto [y_{ij} \mid v_{ij}] \left[v_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j, \mathbf{1}\right] \\
&\propto \left(1_{\{y_{ij}=0\}} 1_{\{v_{ij} \leq 0\}} + 1_{\{y_{ij}=1\}} 1_{\{v_{ij} > 0\}}\right) \times \mathcal{N}\left(v_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j, \mathbf{1}\right) \\
&= \begin{cases} \mathcal{TN}\left(\mathbf{x}'_{ij}\boldsymbol{\beta}_j, \mathbf{1}\right)_{-\infty}^{0}, & y_{ij} = 0 \\ \mathcal{TN}\left(\mathbf{x}'_{ij}\boldsymbol{\beta}_j, \mathbf{1}\right)_{0}^{\infty}, & y_{ij} = 1 \end{cases}
\end{aligned}
$$

Regression coefficients ($\boldsymbol{\beta}_j$):

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \left[\mathbf{v}_j \mid \mathbf{X}_j\boldsymbol{\beta}_j, \mathbf{1}\right]\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right] \\
&\propto \mathcal{N}\left(\mathbf{v}_j \mid \mathbf{X}_j\boldsymbol{\beta}_j, \mathbf{1}\right)\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{v}_j - \mathbf{X}_j\boldsymbol{\beta}_j\right)'\left(\mathbf{v}_j - \mathbf{X}_j\boldsymbol{\beta}_j\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{v}_j'\mathbf{X}_j\right)\boldsymbol{\beta}_j + \boldsymbol{\beta}_j'\mathbf{X}_j'\mathbf{X}_j\boldsymbol{\beta}_j\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(-2\left(\boldsymbol{\mu}_\beta'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\beta}_j + \boldsymbol{\beta}_j'\boldsymbol{\Lambda}^{-1}\boldsymbol{\beta}_j\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{v}_j'\mathbf{X}_j + \boldsymbol{\mu}_\beta'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\beta}_j + \boldsymbol{\beta}_j'\left(\mathbf{X}_j'\mathbf{X}_j + \boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\beta}_j\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = \mathbf{X}_j{}'\mathbf{X}_j + \boldsymbol{\Lambda}^{-1}$, $\mathbf{b}' = \mathbf{v}_j'\mathbf{X}_j + \boldsymbol{\mu}_\beta'\boldsymbol{\Lambda}^{-1}$, $\mathbf{X}_j$ is an $n_j \times p$ matrix collecting the vectors $\mathbf{x}_{1,j}, \ldots \mathbf{x}_{n_j,j}$, and $\mathbf{v}_j' = \left\{v_{1,j}, \ldots, v_{n_j,j}\right\}$.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right]\left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right)\mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Lambda}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Lambda}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Lambda}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Lambda}$):

$$
\begin{aligned}
[\boldsymbol{\Lambda} \mid \cdot] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right] \left[\boldsymbol{\Lambda} \mid \mathbf{S}_0, \nu\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right) \text{Wish}\left(\boldsymbol{\Lambda} \mid \mathbf{S}_0, \nu\right) \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J}{2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^{J} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Lambda}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Lambda}|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \text{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Lambda}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \text{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Lambda}^{-1}\right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) \boldsymbol{\Lambda}^{-1} + \mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right) \boldsymbol{\Lambda}^{-1}\right]\right\} \\
&= \text{Wish}\left(\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, J + \nu\right).
\end{aligned}
$$