Generalized Linear Mixed Effects Model for Binary Data using the Probit Link

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Description

A mixed effects regression model for binary data using the probit link.

Implementation

The file probit.glmm.sim.R simulates data according to the model statement presented below, and probit.glmm.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let y_{ij}, for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, denote observed data that takes on the values $\{0, 1\}$, where the index i denotes replicate observations within group j, and n_j is the number of observations in group j. Furthermore, let $\mathbf{x}_{ij}$ be a vector of p covariates (including the intercept) associated with y_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$
\begin{aligned}
y_{ij} &\sim \begin{cases} 0, & v_{ij} \leq 0 \\ 1, & v_{ij} > 1 \end{cases} \\
v_{ij} &\sim \mathcal{N}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}_j, \mathbf{1}\right) \\
\boldsymbol{\beta}_j &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
\boldsymbol{\Lambda}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Observation model auxiliary variable (v_{ij}):

$$
\begin{aligned}
[v_{ij} \mid \cdot] &\propto [y_{ij} \mid v_{ij}]\left[v_{ij} \mid \mathbf{x}_{ij}'\boldsymbol{\beta}_j, \mathbf{1}\right] \\
&\propto \left(1_{\{y_{ij}=0\}}1_{\{v_{ij}\leq 0\}} + 1_{\{y_{ij}=1\}}1_{\{v_{ij}>0\}}\right) \times \mathcal{N}\left(v_{ij} \mid \mathbf{x}_{ij}'\boldsymbol{\beta}_j, \mathbf{1}\right) \\
&= \begin{cases} \mathcal{TN}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}_j, \mathbf{1}\right)_{-\infty}^{0}, & y_{ij} = 0 \\ \mathcal{TN}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}_j, \mathbf{1}\right)_{0}^{\infty}, & y_{ij} = 1 \end{cases}
\end{aligned}
$$

Regression coefficients ($\boldsymbol{\beta}_j$):

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \left[\mathbf{v}_j \mid \mathbf{X}_j\boldsymbol{\beta}_j, \mathbf{1}\right]\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right] \\
&\propto \mathcal{N}\left(\mathbf{v}_j \mid \mathbf{X}_j\boldsymbol{\beta}_j, \mathbf{1}\right)\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{v}_j - \mathbf{X}_j\boldsymbol{\beta}_j\right)'\left(\mathbf{v}_j - \mathbf{X}_j\boldsymbol{\beta}_j\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{v}_j'\mathbf{X}_j\right)\boldsymbol{\beta}_j + \boldsymbol{\beta}_j'\mathbf{X}_j'\mathbf{X}_j\boldsymbol{\beta}_j\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(-2\left(\boldsymbol{\mu}_\beta'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\beta}_j + \boldsymbol{\beta}_j'\boldsymbol{\Lambda}^{-1}\boldsymbol{\beta}_j\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{v}_j'\mathbf{X}_j + \boldsymbol{\mu}_\beta'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\beta}_j + \boldsymbol{\beta}_j'\left(\mathbf{X}_j'\mathbf{X}_j + \boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\beta}_j\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = \mathbf{X}_j{}'\mathbf{X}_j + \boldsymbol{\Lambda}^{-1}$, $\mathbf{b}' = \mathbf{v}_j'\mathbf{X}_j + \boldsymbol{\mu}_\beta'\boldsymbol{\Lambda}^{-1}$, $\mathbf{X}_j$ is an $n_j \times p$ matrix collecting the vectors $\mathbf{x}_{1,j}, \ldots \mathbf{x}_{n_j,j}$, and $\mathbf{v}_j' = \left\{v_{1,j}, \ldots, v_{n_j,j}\right\}$.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right]\left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right)\mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Lambda}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Lambda}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Lambda}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Lambda}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Lambda}$):

$$
\begin{aligned}
[\boldsymbol{\Lambda} \mid \cdot] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right] [\boldsymbol{\Lambda} \mid \mathbf{S}_0, \nu] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Lambda}\right) \mathrm{Wish}\left(\boldsymbol{\Lambda} \mid \mathbf{S}_0, \nu\right) \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J}{2}} \exp\left\{-\frac{1}{2} \sum_{j=1}^{J} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Lambda}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Lambda}|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2} \mathrm{tr}\left(\mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \mathrm{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Lambda}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right) + \mathrm{tr}\left(\mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \mathrm{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Lambda}^{-1}\right) + \mathrm{tr}\left(\mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) \boldsymbol{\Lambda}^{-1} + \mathbf{S}_0 \boldsymbol{\Lambda}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Lambda}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right) \boldsymbol{\Lambda}^{-1}\right]\right\} \\
&= \mathrm{Wish}\left(\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, J + \nu\right).
\end{aligned}
$$