# Poisson Generalized Linear Mixed Model for Count Data

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### Description

A generalized linear mixed model with varying-coefficients for grouped count data.

### Implementation

The file poisson.varying.coef.sim.R simulates data according to the model statement presented below, and poisson.varying.coef.mcmc.R contains the MCMC algorithm for model fitting.

### Model statement

Let $z_{ij}$, for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, denote observed count data (i.e., $z_{ij}$ are integers greater than or equal to 0), where the index $i$ denotes replicate observations within group $j$, and $n_j$ is the number of observations in group $j$. Furthermore, let $\mathbf{x}_{ij}$ be a vector of $p$ covariates (including the intercept) associated with $z_{ij}$ and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group $j$.

$$
\begin{aligned}
z_{ij} &\sim \text{Pois}\left(\lambda_{ij}\right) \\
\log\left(\lambda_{ij}\right) &= \mathbf{x}_{ij}'\boldsymbol{\beta}_j \\
\boldsymbol{\beta}_j &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

### Full conditional distributions

*Regression coefficients ($\boldsymbol{\beta}_j$):*

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \prod_{i=1}^{n_j}\left[z_{ij} \mid \mathbf{x}_{ij}'\boldsymbol{\beta}_j\right]\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{i=1}^{n_j}\text{Pois}\left(z_{ij} \mid \mathbf{x}_{ij}'\boldsymbol{\beta}_j\right)\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}
$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

*Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):*

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right]\left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right)\mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
\propto \quad & \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
& \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
\propto \quad & \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
= \quad & \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

*Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):*

$$
\begin{aligned}
[\boldsymbol{\Sigma} \mid \cdot] \quad \propto \quad & \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right]\left[\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right] \\
\propto \quad & \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right)\mathrm{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J}{2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{J}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
& \times |\mathbf{S}_0|^{-\frac{\nu}{2}}|\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\mathrm{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right) + \mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\mathrm{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\right) + \mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right)\boldsymbol{\Sigma}^{-1} + \mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)\boldsymbol{\Sigma}^{-1}\right]\right\} \\
= \quad & \mathrm{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, J+\nu\right).
\end{aligned}
$$