# Negative Binomial Model for Data Collected in Three Hierarchical Levels

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### Description

A negative binomial model for data collected in three hierarchical levels (data nested within subgroup nested within group), with varying coefficients at the second (subgroup) and third (group) levels.

### Implementation

The file nb.varying.coef.3.sim.R simulates data according to the model statement presented below, and nb.varying.coef.3.mcmc.R contains the MCMC algorithm for model fitting.

### Model statement

Let $y_{ijk}$ denote observed counts (i.e., $y_{ijk}$ are integers greater than or equal to 0) for groups $i = 1, \ldots, N$, subgroups $j = 1, \ldots, n_i$ nested within groups, and replicate observations $k = 1, \ldots, m_{ij}$ (level-1 units) within subgroup $j$ (level-2 units) and group $i$ (level-3 units). Furthermore, let $\mathbf{x}_{ijk}$ be a vector of $p$ covariates (including the intercept) associated with $y_{ijk}$ and $\boldsymbol{\alpha}_{ij}$ be the corresponding vector of coefficients for subgroup $j$ in group $i$. The vector $\boldsymbol{\beta}_i$ corresponds to group-level coefficients and $\boldsymbol{\mu}_\beta$ is a vector of population-level coefficients.

$$
\begin{aligned}
y_{ijk} &\sim \text{NB}\left(\lambda_{ijk}, \theta_i\right), \\
\log\left(\lambda_{ijk}\right) &= \mathbf{x}_{ijk}' \boldsymbol{\alpha}_{ij} \\
\boldsymbol{\alpha}_{ij} &\sim \mathcal{N}\left(\boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \\
\boldsymbol{\beta}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma_\beta^2 \mathbf{I}\right) \\
\theta_i &\sim \text{Gamma}\left(a, b\right) \\
\boldsymbol{\Sigma}_{\alpha_i}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right) \\
\boldsymbol{\Sigma}_\beta^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

### Full conditional distributions

*Subgroup-level regression coefficients ($\boldsymbol{\alpha}_{ij}$):*

$$
\begin{aligned}
[\boldsymbol{\alpha}_{ij} \mid \cdot] &\propto \prod_{k=1}^{m_{ij}} [y_{ijk} \mid \lambda_{ijk}, \theta_i] [\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}] \\
&\propto \prod_{k=1}^{m_{ij}} \text{NB}\left(y_{ijk} \mid \lambda_{ijk}, \theta_i\right) \mathcal{N}\left(\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right).
\end{aligned}
$$

The update for $\boldsymbol{\alpha}_{ij}$ proceeds using Metropolis-Hastings.

*Group-level regression coefficients ($\boldsymbol{\beta}_i$):*

$$
\begin{aligned}
[\boldsymbol{\beta}_i \mid \cdot] &\propto \prod_{j=1}^{n_i} [\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}] \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right] \\
&\propto \prod_{j=1}^{n_i} \mathcal{N}\left(\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
&\propto \exp\left\{\sum_{j=1}^{n_i}\left(-\frac{1}{2}\left(\boldsymbol{\alpha}_{ij}-\boldsymbol{\beta}_i\right)'\boldsymbol{\Sigma}_{\alpha_i}^{-1}\left(\boldsymbol{\alpha}_{ij}-\boldsymbol{\beta}_i\right)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{n_i}\boldsymbol{\alpha}_{ij}'\boldsymbol{\Sigma}_{\alpha_i}^{-1}\right)\boldsymbol{\beta}_i+\boldsymbol{\beta}_i'\left(n_i\boldsymbol{\Sigma}_{\alpha_i}^{-1}\right)\boldsymbol{\beta}_i\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(-2\left(\boldsymbol{\mu}_\beta'\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\beta}_i+\boldsymbol{\beta}_i'\left(\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\beta}_i\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{n_i}\boldsymbol{\alpha}_{ij}'\boldsymbol{\Sigma}_{\alpha_i}^{-1}-\boldsymbol{\mu}_\beta'\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\beta}_i+\boldsymbol{\beta}_i'\left(n_i\boldsymbol{\Sigma}_{\alpha_i}^{-1}+\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\beta}_i\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n_i\boldsymbol{\Sigma}_{\alpha_i}^{-1}+\boldsymbol{\Sigma}_\beta^{-1}$ and $\mathbf{b}' = \boldsymbol{\alpha}_i'\boldsymbol{\Sigma}_{\alpha_i}^{-1}-\boldsymbol{\mu}_\beta'\boldsymbol{\Sigma}_\beta^{-1}$, where $\boldsymbol{\alpha}_i$ is the vector sum $\sum_{j=1}^{n_i}\boldsymbol{\alpha}_{ij}$.

*Mean of group-level regression coefficients ($\boldsymbol{\mu}_\beta$):*

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta \mid \cdot] &\propto \prod_{i=1}^{N}\left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta,\boldsymbol{\Sigma}_\beta\right]\left[\boldsymbol{\mu}_\beta \mid \mathbf{0},\sigma_{\mu_\beta}^2\right] \\
&\propto \prod_{i=1}^{N}\mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta,\boldsymbol{\Sigma}_\beta\right)\mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0},\sigma_{\mu_\beta}^2\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{i=1}^{N}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)'\left(\sigma_{\mu_\beta}^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{N}\boldsymbol{\beta}_i'\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\mu}_\beta+\boldsymbol{\mu}_\beta'\left(N\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_{\mu_\beta}^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{N}\boldsymbol{\beta}_i'\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\mu}_\beta+\boldsymbol{\mu}_\beta'\left(N\boldsymbol{\Sigma}_\beta^{-1}+\left(\sigma_{\mu_\beta}^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = N\boldsymbol{\Sigma}_\beta^{-1}+\left(\sigma_{\mu_\beta}^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}_\beta^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{i=1}^{N}\boldsymbol{\beta}_i$.

*Negative binomial over-dispersion parameter ($\theta_i$):*

$$
\begin{aligned}
[\theta_i \mid \cdot] &\propto \prod_{k=1}^{m_{ij}}\left[y_{ijk} \mid \lambda_{ijk},\theta_i\right]\left[\theta_i \mid a,b\right] \\
&\propto \prod_{k=1}^{m_{ij}}\mathrm{NB}\left(y_{ijk} \mid \lambda_{ijk},\theta_i\right)\mathrm{Gamma}\left(\theta_i \mid a,b\right).
\end{aligned}
$$

The update for $\theta_i$ proceeds using Metropolis-Hastings.

*Variance-covariance of group-level regression coefficients ($\mathbf{\Sigma}_\beta$):*

$$
\begin{aligned}
[\mathbf{\Sigma}_\beta \mid \cdot] &\propto \prod_{i=1}^{N} \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \mathbf{\Sigma}_\beta\right] [\mathbf{\Sigma}_\beta \mid \mathbf{S}_0, \nu] \\
&\propto \prod_{i=1}^{N} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \mathbf{\Sigma}_\beta\right) \operatorname{Wish}\left(\mathbf{\Sigma}_\beta \mid \mathbf{S}_0, \nu\right) \\
&\propto |\mathbf{\Sigma}_\beta|^{-\frac{N}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \mathbf{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\mathbf{\Sigma}_\beta|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\operatorname{tr}\left(\mathbf{S}_0\mathbf{\Sigma}_\beta^{-1}\right)\right\} \\
&\propto |\mathbf{\Sigma}_\beta|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}\operatorname{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \mathbf{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right) + \operatorname{tr}\left(\mathbf{S}_0\mathbf{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto |\mathbf{\Sigma}_\beta|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}\operatorname{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \mathbf{\Sigma}_\beta^{-1}\right) + \operatorname{tr}\left(\mathbf{S}_0\mathbf{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto |\mathbf{\Sigma}_\beta|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right)\mathbf{\Sigma}_\beta^{-1} + \mathbf{S}_0\mathbf{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto |\mathbf{\Sigma}_\beta|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)\mathbf{\Sigma}_\beta^{-1}\right]\right\} \\
&= \operatorname{Wish}\left(\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, N+\nu\right).
\end{aligned}
$$

*Variance-covariance of subgroup-level regression coefficients ($\mathbf{\Sigma}_{\alpha_i}$):*

$$
\begin{aligned}
[\mathbf{\Sigma}_{\alpha_i} \mid \cdot] &\propto \prod_{j=1}^{n_i} \left[\alpha_{ij} \mid \boldsymbol{\beta}_i, \mathbf{\Sigma}_{\alpha_i}\right] [\mathbf{\Sigma}_{\alpha_i} \mid \mathbf{S}_0, \nu] \\
&\propto \prod_{j=1}^{n_i} \mathcal{N}\left(\alpha_{ij} \mid \boldsymbol{\beta}_i, \mathbf{\Sigma}_{\alpha_i}\right) \operatorname{Wish}\left(\mathbf{\Sigma}_{\alpha_i} \mid \mathbf{S}_0, \nu\right) \\
&= \operatorname{Wish}\left(\left(\sum_{j=1}^{n_i}\left(\left(\alpha_{ij} - \boldsymbol{\beta}_i\right)\left(\alpha_{ij} - \boldsymbol{\beta}_i\right)'\right) + \mathbf{S}_0\right)^{-1}, n_i+\nu\right).
\end{aligned}
$$