Binomial Generalized Linear Mixed Model

Brian M. Brost

02 April 2016

Description

A generalized linear mixed model with varying coefficients for binomially distributed data.

Implementation

The file binomial.varying.coef.sim.R simulates data according to the model statement presented below, and binomial.varying.coef.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let z_{ij} be the number of "successes" (i.e., z_{ij} are integers greater than or equal to 0) out of N_{ij} "trials", for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, where the index i denotes replicate observations within group j. Furthermore, let $\mathbf{x}_{ij}$ be a vector of covariates associated with z_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$
\begin{aligned}
z_{ij} &\sim \text{Binom}\left(N_{ij}, p_{ij}\right) \\
\text{logit}\left(p_{ij}\right) &= \mathbf{x}'_{ij}\boldsymbol{\beta}_j \\
\boldsymbol{\beta}_j &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}_j$):

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \prod_{i=1}^{n_j} \left[z_{ij} \mid N_{ij}, \mathbf{x}'_{ij}\boldsymbol{\beta}_j\right] \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{i=1}^{n_j} \text{Binom}\left(z_{ij} \mid N_{ij}, \mathbf{x}'_{ij}\boldsymbol{\beta}_j\right) \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}
$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_\beta\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J} \left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
\propto \quad & \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
& \qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
\propto \quad & \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
= \quad & \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$
\begin{aligned}
[\boldsymbol{\Sigma} \mid \cdot] \quad \propto \quad & \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right]\left[\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right] \\
\propto \quad & \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right)\text{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J}{2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{J}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
& \quad \times |\mathbf{S}_0|^{-\frac{\nu}{2}}|\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\text{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right) + \text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\text{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\right) + \text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right)\boldsymbol{\Sigma}^{-1} + \mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
\propto \quad & |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)\boldsymbol{\Sigma}^{-1}\right]\right\} \\
= \quad & \text{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, J+\nu\right).
\end{aligned}
$$