# N-mixture Model with Temporal Trend

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### Model implementation

The file N.mixture.trend.sim.R simulates data according to the model statement presented below, and N.mixture.trend.mcmc.R contains the MCMC algorithm for model fitting.

### Model statement

Let $y_{it}$ be the $i^{th}$ count of individuals during time period $t$, for $i = 1, \ldots, m$ and $t = 1, \ldots, T$, and $N_t$ be the true number of individuals at time $t$. Assuming the population is closed to mortality, recruitment, immigration, and emigration over the course of the $m$ surveys conducted during any given time period,

$$\begin{aligned}
y_{it} &\sim \text{Binom}(N_t, p_{it}) \\
N_1 &\sim \text{Pois}(\lambda_1) \\
N_t &\sim \text{Pois}(\lambda_t),\ t = 2, \ldots, T \\
\log(\lambda_t) &= \theta N_{t-1} \\
\theta &\sim \text{N}(\boldsymbol{\mu}_\theta, \sigma^2) \\
\lambda_1 &\sim \text{Gamma}(r, q) \\
\text{logit}(p_{it}) &\sim \mathbf{W}_{it}\boldsymbol{\alpha} \\
\boldsymbol{\alpha} &\sim \text{N}(\boldsymbol{\mu}_\alpha, \tau^2\mathbf{I}).
\end{aligned}$$

Note that an identity link for the model on $N_t$ could also be used, i.e., $N_t \sim \text{Pois}(\theta N_{t-1})$. The algorithm in N.mixture.trend.mcmc.R will also estimate the posterior predictive distribution for time periods in which no observations are available, i.e., $N_{\tilde{t}}$, $\tilde{t} = 2, \ldots, T-1$.

### Posterior distribution

$$[N_1, N_{\{t>2\}}, \boldsymbol{\alpha}, \lambda_1, \theta | \mathbf{Y}, \mathbf{W}] \propto \prod_{t=1}^{T}\prod_{i=1}^{m}[y_{it}|N_t, p_{it}][N_1|\lambda_1][N_{\{t>2\}}|\theta][\theta][\lambda_1][\boldsymbol{\alpha}]$$

### Full conditional distributions

*Coefficients describing the effect of covariates on detection probability ($\boldsymbol{\alpha}$):*

$$[\boldsymbol{\alpha}|\cdot] \propto \prod_{t=1}^{T}\prod_{i=1}^{m}[y_{it}|N_t, p_{it}][\boldsymbol{\alpha}].$$

This full-conditional distribution does not have a known analytical form; therefore, sample $\mathbf{p}$ using Metropolis-Hastings.

*The true number of individuals during $t = 1$ ($N_1$):*

$$\begin{aligned}
[N_1|\cdot] &\propto \prod_{i=1}^{m}[y_{i1}|N_1, p_{i1}][N_1|\lambda_1] \\
&\propto \prod_{i=1}^{m}\frac{(\lambda_1(1-p_{i1}))^{N_1 - y_{i1}}}{(N_1 - y_{i1})!}e^{-\lambda_1(1-p_{i1})}.
\end{aligned}$$

This full-conditional is a little strange because $[N_1 - y_{i1}|\cdot] \propto \text{Pois}(\lambda_1(1-p_{i1}))$, which suggests there is one true abundance per replicate count during $t = 1$. This is in contrast to the case in which only one observation exists per site, i.e., $[N_i - y_i|\cdot] \propto \text{Pois}(\lambda_i(1-p_i))$. Given that $[N_1|\cdot]$ lacks a clear analytical solution, sample $N_1$ using Metropolis-Hastings.

*The true number of individuals during $t = 2, \ldots, T-1$ ($N_t$):*

$$[N_t|\cdot] \quad \propto \quad \prod_{i=1}^{m} [y_{it}|N_t, p_{it}][N_t|\theta, N_{t-1}][N_{t+1}|\theta, N_t]$$

This full-conditional lacks an analytical solution; therefore, sample $N_t$ using Metropolis-Hastings. Note, for time periods $\tilde{t}$ in which observations are missing, the posterior predictive distribution for true abundance is:

$$[N_{\tilde{t}}|\cdot] \quad \propto \quad \prod_{i=1}^{m} [N_{\tilde{t}}|\theta, N_{\tilde{t}-1}][N_{\tilde{t}+1}|\theta, N_{\tilde{t}}].$$

*The true number of individuals during $t = T$ ($N_T$):*

$$[N_T|\cdot] \quad \propto \quad \prod_{i=1}^{m} [y_{iT}|N_T, p_{iT}][N_T|\theta, N_{T-1}]$$

This full-conditional lacks an analytical solution; therefore, sample $N_T$ using Metropolis-Hastings.

*Rate of the process model for $N_1$ ($\lambda_1$):*

$$\begin{aligned}
[\lambda_1|\cdot] &\propto [N_1|\lambda_1][\lambda_1] \\
&\propto \frac{\lambda_1^{N_1} e^{-\lambda_1}}{N_1!} \lambda_1^{r-1} e^{-\lambda_1 q} \\
&\propto \lambda_1^{N_1} e^{-\lambda_1} \lambda_1^{r-1} e^{-\lambda_1 q} \\
&\propto \lambda_1^{N_1+r-1} e^{-\lambda_1(1+q)} \\
&= \text{Gamma}\left(N_1 + r, 1 + q\right).
\end{aligned}$$

*Population growth rate ($\theta$):*

$$[\theta|\cdot] \quad \propto \quad [N_t|\theta, N_{t-1}][\theta]$$

This full-conditional lacks an analytical solution; therefore, sample $\theta$ using Metropolis-Hastings.