Community Occupancy Model with Correlated Occupancy and Detection

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1 Description

An occupancy model for multiple species where the detection and occupancy processes within a species are potentially correlated.

2 Implementation

The file occ.community.correlated.sim.R simulates data according to the model statement below, and occ.community.correlated.mcmc.R contains the MCMC algorithm for parameter estimation.

3 Model Statement

Let y_{ijt} represent the number of detections for species i ($i = 1, \ldots, n$) at site j ($j = 1, \ldots, R$) during sampling period t ($t = 1, \ldots, T_j$). Note that it is unnecessary to survey sites repeatedly through time and $T_j = 1$ is okay for some or all j.

$$
\begin{aligned}
y_{ijt} &\sim \begin{cases} \text{Binom}\left(J_{jt}, p_{ijt}\right), & z_{ijt} = 1 \\ 0, & z_{ijt} = 0 \end{cases} \\
z_{ijt} &\sim \begin{cases} 0, & v_{ijt} \leq 1 \\ 1, & v_{ijt} > 0 \end{cases} \\
v_{ijt} &\sim \mathcal{N}\left(\beta_{0i} + \mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right) \\
p_{ijt} &= \text{logit}^{-1}\left(\alpha_{0i} + \mathbf{w}'_{jt}\boldsymbol{\alpha}_i\right) \\
\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} &\sim \mathcal{N}\left(\begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix}, \boldsymbol{\Sigma}\right) \\
\boldsymbol{\beta}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\alpha}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\alpha, \sigma^2_\alpha \mathbf{I}\right) \\
\begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_0 \mathbf{I}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_{\mu_\beta} \mathbf{I}\right) \\
\boldsymbol{\mu}_\alpha &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_{\mu_\alpha} \mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1} &\sim \text{Wish}\left(S_0^{-1}, \nu\right) \\
\sigma^2_\beta &\sim \text{IG}\left(r, q\right) \\
\sigma^2_\alpha &\sim \text{IG}\left(r, q\right)
\end{aligned}
$$

4 Full-conditional Distributions

4.1 Occupancy state (z_{ijt})

$$\begin{aligned}
[z_{ijt} \mid \cdot] &\propto [y_{ijt} \mid p_{ijt}, z_{ijt}] [z_{ijt} \mid v_{ijt}] \\
&\propto \text{Binom}\left(y_{ijt} \mid J_{jt}, p_{ijt}\right)^{z_{ijt}} 1_{\{y_{ijt}=0\}}^{1-z_{ijt}} \text{Bern}\left(z_{ijt} \mid v_{ijt}\right) \\
&\propto \left(p_{ijt}^{y_{ijt}} \left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}}\right)^{z_{ijt}} \left(1_{\{y_{ijt}=0\}}^{1-z_{ijt}}\right) \psi_{ijt}^{z_{ijt}} \left(1-\psi_{ijt}\right)^{1-z_{ijt}} \\
&\propto \left(\psi_{ijt} p_{ijt}^{y_{ijt}} \left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}}\right)^{z_{ijt}} \left(\left(1-\psi_{ijt}\right) 1_{\{y_{ijt}=0\}}\right)^{1-z_{ijt}} \\
&= \text{Bern}\left(\tilde{\psi}_{ijt}\right),
\end{aligned}$$

where

$$\tilde{\psi}_{ijt} = \frac{\psi_{ijt} p_{ijt}^{y_{ijt}} \left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}}}{\psi_{ijt} p_{ijt}^{y_{ijt}} \left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}} + \left(1-\psi_{ijt}\right) 1_{\{y_{ijt}=0\}}}$$

and $\psi_{ijt} = \Phi\left(\mathbf{x}'_{jt}\boldsymbol{\beta}_i\right)$.

4.2 Occupancy state auxiliary variable (v_{ijt})

$$\begin{aligned}
[v_{ijt} \mid \cdot] &\propto [z_{ijt} \mid v_{ijt}] [v_{ijt} \mid \beta_{0i}, \boldsymbol{\beta}_i] \\
&\propto \left(1_{\{z_{ijt}=0\}} 1_{\{v_{ijt} \leq 0\}} + 1_{\{z_{ijt}=1\}} 1_{\{v_{ijt}>0\}}\right) \times \mathcal{N}\left(v_{ijt} \mid \beta_{0i} + \mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right) \\
&= \begin{cases} \mathcal{TN}\left(\beta_{0i} + \mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right)_{-\infty}^{0}, & z_{ijt} = 0 \\ \mathcal{TN}\left(\beta_{0i} + \mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right)_{0}^{\infty}, & z_{ijt} = 1 \end{cases}
\end{aligned}$$

4.3 Detection and occupancy intercepts (α_{0i}, β_{0i})

$$\begin{aligned}
\left[\left(\begin{array}{c} \alpha_{0i} \\ \beta_{0i} \end{array}\right) \mid \cdot\right] &\propto \prod_{j=1}^{R} \prod_{t=1}^{T_j} [y_{ijt} \mid p_{ijt}, z_{ijt}] [v_{ijt} \mid \beta_{0i}, \boldsymbol{\beta}_i] \left[\left(\begin{array}{c} \alpha_{0i} \\ \beta_{0i} \end{array}\right) \mid \left(\begin{array}{c} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{array}\right), \boldsymbol{\Sigma}\right] \\
&\propto \prod_{j=1}^{R} \prod_{t=1}^{T_j} \text{Binom}\left(y_{ijt} \mid J_{jt}, p_{ijt}\right)^{z_{ijt}} \mathcal{N}\left(v_{ijt} \mid \beta_{0i} + \mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right) \mathcal{N}\left(\left(\begin{array}{c} \alpha_{0i} \\ \beta_{0i} \end{array}\right) \mid \left(\begin{array}{c} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{array}\right), \boldsymbol{\Sigma}\right).
\end{aligned}$$

The updates for α_{0i} and β_{0i} proceed separately using Metropolis-Hastings. When updating α_{0i}, note that the product over j and t only includes instances of j and t such that $z_{ijt} = 1$.

4.4 Occupancy coefficients ($\boldsymbol{\beta}_i$)

$$\begin{aligned}
[\boldsymbol{\beta}_i \mid \cdot] &\propto [\mathbf{v}_i \mid \beta_{0i}, \boldsymbol{\beta}_i] \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2\right] \\
&\propto \mathcal{N}\left(\mathbf{v}_i \mid \beta_{0i} + \mathbf{X}\boldsymbol{\beta}_i, \mathbf{1}\right) \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2 \mathbf{I}\right) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{v}_i - \left(\beta_{0i} + \mathbf{X}\boldsymbol{\beta}_i\right)\right)' \left(\mathbf{v}_i - \left(\beta_{0i} + \mathbf{X}\boldsymbol{\beta}_i\right)\right)\right\} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\left(\mathbf{v}_i - \beta_{0i}\right) - \mathbf{X}\boldsymbol{\beta}_i\right)' \left(\left(\mathbf{v}_i - \beta_{0i}\right) - \mathbf{X}\boldsymbol{\beta}_i\right)\right\} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{v}_i - \beta_{0i}\right)' \mathbf{X}\boldsymbol{\beta}_i + \boldsymbol{\beta}'_i \mathbf{X}'\mathbf{X}\boldsymbol{\beta}_i\right)\right\} \exp\left\{-\frac{1}{2}\left(-2\left(\boldsymbol{\mu}'_\beta \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\right) \boldsymbol{\beta}_i + \boldsymbol{\beta}'_i \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \boldsymbol{\beta}_i\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\left(\mathbf{v}_i - \beta_{0i}\right)' \mathbf{X} + \boldsymbol{\mu}'_\beta \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\right) \boldsymbol{\beta}_i + \boldsymbol{\beta}'_i \left(\mathbf{X}'\mathbf{X} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\right) \boldsymbol{\beta}_i\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where $\mathbf{A}=\mathbf{X}'\mathbf{X}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$, $\mathbf{b}'=(\mathbf{v}_i-\beta_{0i})'\,\mathbf{X}+\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$, $\mathbf{v}_i'=\{v_{ijt},\forall j,t\}$, and $\mathbf{X}=\left\{\mathbf{x}_{jt}',\forall j,t\right\}$.

4.5 Detection coefficients ($\boldsymbol{\alpha}_i$)

$$
\begin{aligned}
[\boldsymbol{\alpha}_i \mid \cdot] &\propto \prod_{j=1}^{R}\prod_{t=1}^{T_j}[y_{ijt}\mid p_{ijk},z_{ijt}]\left[\boldsymbol{\alpha}_i\mid\boldsymbol{\mu}_\alpha,\sigma_\alpha^2\right]\\
&\propto \prod_{j=1}^{R}\prod_{t=1}^{T_j}\mathrm{Binom}\left(y_{ijt}\mid J_{jt},p_{ijt}\right)^{z_{ijt}}\mathcal{N}\left(\boldsymbol{\alpha}_i\mid\boldsymbol{\mu}_\alpha,\sigma_\alpha^2\mathbf{I}\right).
\end{aligned}
$$

The update for $\boldsymbol{\alpha}_i$ proceeds using Metropolis-Hastings. Note that the product over j and t only includes instances of j and t such that $z_{ijt}=1$.

4.6 Mean of occupancy and detection intercepts ($\mu_{\beta_0},\mu_{\alpha_0}$)

$$
\begin{aligned}
\left[\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\mid\cdot\right] &\propto \prod_{i=1}^{n}\left[\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}\mid\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix},\boldsymbol{\Sigma}\right]\left[\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\right]\\
&\propto \prod_{i=1}^{n}\mathcal{N}\left(\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}\mid\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix},\boldsymbol{\Sigma}\right)\mathcal{N}\left(\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\mid\mathbf{0},\sigma_0^2\mathbf{I}\right)\\
&\propto \exp\left\{\sum_{i=1}^{n}\left(-\frac{1}{2}\left(\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}-\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\right)'\boldsymbol{\Sigma}^{-1}\left(\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}-\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\right)\right)\right\}\times\\
&\qquad\exp\left\{-\frac{1}{2}\left(\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}-\mathbf{0}\right)'\left(\sigma_0^2\mathbf{I}\right)^{-1}\left(\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}-\mathbf{0}\right)\right\}\\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n}\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}'\boldsymbol{\Sigma}^{-1}\right)\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}+\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}'\left(n\boldsymbol{\Sigma}^{-1}\right)\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\right)\right\}\times\\
&\qquad\exp\left\{-\frac{1}{2}\left(\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}'\left(\sigma_0^2\mathbf{I}\right)^{-1}\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\right)\right\}\\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n}\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}'\boldsymbol{\Sigma}^{-1}\right)\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}+\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}'\left(n\boldsymbol{\Sigma}^{-1}+\left(\sigma_0^2\mathbf{I}\right)^{-1}\right)\begin{pmatrix}\mu_{\alpha_0}\\ \mu_{\beta_0}\end{pmatrix}\right)\right\}\\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A}=n\boldsymbol{\Sigma}^{-1}+\left(\sigma_0^2\mathbf{I}\right)^{-1}$, $\mathbf{b}'=\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}'\boldsymbol{\Sigma}^{-1}$, and $\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}$ is the vector sum $\sum_{i=1}^{n}\begin{pmatrix}\alpha_{0i}\\ \beta_{0i}\end{pmatrix}$.

4.7 Mean of occupancy coefficients ($\boldsymbol{\mu}_\beta$)

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta \mid \cdot] &\propto \prod_{i=1}^{n} [\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2] [\boldsymbol{\mu}_\beta] \\
&\propto \prod_{i=1}^{n} \mathcal{N}(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2 \mathbf{I}) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_{\mu_\beta}^2 \mathbf{I}\right) \\
&\propto \exp\left\{\sum_{i=1}^{n}\left(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)'(\sigma_\beta^2 \mathbf{I})^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)\right)\right\} \exp\left\{-\frac{1}{2}(\boldsymbol{\mu}_\beta - \mathbf{0})'\left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}(\boldsymbol{\mu}_\beta - \mathbf{0})\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n} \boldsymbol{\beta}_i'(\sigma_\beta^2 \mathbf{I})^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(n(\sigma_\beta^2 \mathbf{I})^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n} \boldsymbol{\beta}_i'(\sigma_\beta^2 \mathbf{I})^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(n(\sigma_\beta^2 \mathbf{I})^{-1} + \left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n\left(\sigma_\beta^2 \mathbf{I}\right)^{-1} + \left(\sigma_{\mu_\beta}^2 \mathbf{I}\right)^{-1}$, $\mathbf{b}' = \boldsymbol{\beta}'\left(\sigma_\beta^2 \mathbf{I}\right)^{-1}$, and $\boldsymbol{\beta}$ is the vector sum $\sum_{i=1}^{n} \boldsymbol{\beta}_i$.

4.8 Mean of detection coefficients ($\boldsymbol{\mu}_\alpha$)

$$
\begin{aligned}
[\boldsymbol{\mu}_\alpha \mid \cdot] &\propto \prod_{i=1}^{n} [\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2] [\boldsymbol{\mu}_\alpha] \\
&\propto \prod_{i=1}^{n} \mathcal{N}(\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2 \mathbf{I}) \mathcal{N}(\boldsymbol{\mu}_\alpha \mid \mathbf{0}, \sigma_{\mu_\alpha}^2 \mathbf{I}) \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n(\sigma_\alpha^2 \mathbf{I})^{-1} + (\sigma_{\mu_\alpha}^2 \mathbf{I})^{-1}$, $\mathbf{b}' = \boldsymbol{\alpha}'(\sigma_\alpha^2 \mathbf{I})^{-1}$, and $\boldsymbol{\alpha}$ is the vector sum $\sum_{i=1}^{n} \boldsymbol{\alpha}_i$.

4.9 Variance-covariance of occupancy and detection intercepts ($\boldsymbol{\Sigma}$)

$$
\begin{aligned}
[\boldsymbol{\Sigma} \mid \cdot] &\propto \prod_{i=1}^{n} \left[\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} \mid \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix}, \boldsymbol{\Sigma} \right] [\boldsymbol{\Sigma}] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} \mid \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix}, \boldsymbol{\Sigma} \right) \text{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right)' \boldsymbol{\Sigma}^{-1} \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right) \right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Sigma}|^{-\frac{\nu - qX - 1}{2}} \exp\left\{ -\frac{1}{2} \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right) \right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{n+\nu-qX-1}{2}} \exp\left\{ -\frac{1}{2} \left[\sum_{i=1}^{n} \text{tr}\left(\left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right)' \boldsymbol{\Sigma}^{-1} \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right) \right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right) \right] \right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{n+\nu-qX-1}{2}} \exp\left\{ -\frac{1}{2} \left[\sum_{i=1}^{n} \text{tr}\left(\left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right) \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right)' \boldsymbol{\Sigma}^{-1} \right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right) \right] \right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{n+\nu-qX-1}{2}} \exp\left\{ -\frac{1}{2} \left[\text{tr}\left(\sum_{i=1}^{n} \left(\left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right) \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right)' \right) \boldsymbol{\Sigma}^{-1} + \mathbf{S}_0 \boldsymbol{\Sigma}^{-1} \right) \right] \right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{n+\nu-qX-1}{2}} \exp\left\{ -\frac{1}{2} \left[\text{tr}\left(\sum_{i=1}^{n} \left(\left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right) \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right)' \right) + \mathbf{S}_0 \right) \boldsymbol{\Sigma}^{-1} \right] \right\} \\
&= \text{Wish}\left(\left(\sum_{i=1}^{n} \left(\left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right) \left(\begin{pmatrix} \alpha_{0i} \\ \beta_{0i} \end{pmatrix} - \begin{pmatrix} \mu_{\alpha_0} \\ \mu_{\beta_0} \end{pmatrix} \right)' \right) + \mathbf{S}_0 \right)^{-1}, n + \nu \right).
\end{aligned}
$$

4.10 Variance of occupancy coefficients (σ_β^2)

$$
\begin{aligned}
\left[\sigma_\beta^2 \mid \cdot\right] &\propto \prod_{i=1}^{n} \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2\right] \left[\sigma_\beta^2\right] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2 \mathbf{I}\right) \text{IG}\left(\sigma_\beta^2 \mid r, q\right) \\
&\propto \prod_{i=1}^{n} \left|\sigma_\beta^2 \mathbf{I}\right|^{-1/2} \exp\left\{ -\frac{1}{2} \left((\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)' \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta) \right) \right\} \left(\sigma_\beta^2\right)^{-(q+1)} \exp\left\{ -\frac{1}{r\sigma_\beta^2} \right\} \\
&\propto \left(\sigma_\beta^2\right)^{-(qX \times n)/2} \exp\left\{ \sum_{i=1}^{n} \left(-\frac{1}{2\sigma_\beta^2} (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)' (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta) \right) \right\} \left(\sigma_\beta^2\right)^{-(q+1)} \exp\left\{ -\frac{1}{r\sigma_\beta^2} \right\} \\
&\propto \left(\sigma_\beta^2\right)^{-((qX \times n)/2 + q + 1)} \exp\left\{ -\frac{1}{\sigma_\beta^2} \left(\frac{\sum_{i=1}^{n} (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)' (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)}{2} + \frac{1}{r} \right) \right\} \\
&= \text{IG}\left(\left(\frac{\sum_{i=1}^{n} (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)' (\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)}{2} + \frac{1}{r} \right)^{-1}, \frac{qX \times n}{2} + q \right),
\end{aligned}
$$

where qX is the column dimension of $\mathbf{X}$ (or length of $\boldsymbol{\beta}_i$).

4.11 Variance of detection coefficients (σ_α^2)

$$
\begin{aligned}
\left[\sigma_\alpha^2 \mid \cdot\right] &\propto \prod_{i=1}^{n} \left[\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2\right] \left[\sigma_\alpha^2\right] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2 \mathbf{I}\right) \mathrm{IG}\left(\sigma_\alpha^2 \mid r, q\right) \\
&= \mathrm{IG}\left(\left(\frac{\sum_{i=1}^{n} \left(\boldsymbol{\alpha}_i - \boldsymbol{\mu}_\alpha\right)' \left(\boldsymbol{\alpha}_i - \boldsymbol{\mu}_\alpha\right)}{2} + \frac{1}{r}\right)^{-1}, \frac{qW \times n}{2} + q\right),
\end{aligned}
$$

where qW is the column dimension of $\mathbf{W}$ (or length of $\boldsymbol{\alpha}_i$).