Basic Community Occupancy Model

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1 Description

An occupancy model for multiple species where the detection and occupancy processes within a species are assumed independent.

2 Implementation

The file occ.community.sim.R simulates data according to the model statement below, and occ.community.mcmc.R contains the MCMC algorithm for parameter estimation.

3 Model Statement

Let y_{ijt} represent the number of detections for species i ($i = 1, \ldots, n$) at site j ($j = 1, \ldots, R$) during sampling period t ($t = 1, \ldots, T_j$). Note that it is unnecessary to survey sites repeatedly through time and $T_j = 1$ is okay for some or all j.

$$
\begin{aligned}
y_{ijt} &\sim \begin{cases} \text{Binom}\left(J_{jt}, p_{ijt}\right), & z_{ijt} = 1 \\ 0, & z_{ijt} = 0 \end{cases} \\
z_{ijt} &\sim \begin{cases} 0, & v_{ijt} \leq 1 \\ 1, & v_{ijt} > 0 \end{cases} \\
v_{ijt} &\sim \mathcal{N}\left(\mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right) \\
p_{ijt} &= \text{logit}^{-1}\left(\mathbf{w}'_{jt}\boldsymbol{\alpha}_i\right) \\
\boldsymbol{\beta}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\alpha}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\alpha, \sigma^2_\alpha \mathbf{I}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_{\mu_\beta} \mathbf{I}\right) \\
\boldsymbol{\mu}_\alpha &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_{\mu_\alpha} \mathbf{I}\right) \\
\sigma^2_\beta &\sim \text{IG}\left(r, q\right) \\
\sigma^2_\alpha &\sim \text{IG}\left(r, q\right)
\end{aligned}
$$

4 Full-conditional Distributions

4.1 Occupancy state (z_{ijt})

$$
\begin{aligned}
[z_{ijt} \mid \cdot] &\propto [y_{ijt} \mid p_{ijt}, z_{ijt}]\,[z_{ijt} \mid v_{ijt}] \\
&\propto \text{Binom}\left(y_{ijt} \mid J_{jt}, p_{ijt}\right)^{z_{ijt}} 1^{1-z_{ijt}}_{\{y_{ijt}=0\}} \text{Bern}\left(z_{ijt} \mid v_{ijt}\right) \\
&\propto \left(p_{ijt}^{y_{ijt}}\left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}}\right)^{z_{ijt}} \left(1^{1-z_{ijt}}_{\{y_{ijt}=0\}}\right) \psi_{ijt}^{z_{ijt}} \left(1-\psi_{ijt}\right)^{1-z_{ijt}} \\
&\propto \left(\psi_{ijt} p_{ijt}^{y_{ijt}}\left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}}\right)^{z_{ijt}} \left(\left(1-\psi_{ijt}\right) 1_{\{y_{ijt}=0\}}\right)^{1-z_{ijt}} \\
&= \text{Bern}\left(\tilde{\psi}_{ijt}\right),
\end{aligned}
$$

where

$$
\tilde{\psi}_{ijt} = \frac{\psi_{ijt} p_{ijt}^{y_{ijt}}\left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}}}{\psi_{ijt} p_{ijt}^{y_{ijt}}\left(1-p_{ijt}\right)^{J_{jt}-y_{ijt}} + \left(1-\psi_{ijt}\right) 1_{\{y_{ijt}=0\}}}
$$

and $\psi_{ijt} = \Phi\left(\mathbf{x}'_{jt}\boldsymbol{\beta}_i\right)$.

4.2 *Occupancy state auxiliary variable (v_{ijt})*

$$
\begin{aligned}
[v_{ijt} \mid \cdot] &\propto [z_{ijt} \mid v_{ijt}]\,[v_{ijt} \mid \boldsymbol{\beta}_i] \\
&\propto \left(1_{\{z_{ijt}=0\}}1_{\{v_{ijt}\le 0\}} + 1_{\{z_{ijt}=1\}}1_{\{v_{ijt}>0\}}\right) \times \mathcal{N}\left(v_{ijt} \mid \mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right) \\
&= \begin{cases} \mathcal{TN}\left(\mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right)_{-\infty}^{0}, & z_{ijt}=0 \\ \mathcal{TN}\left(\mathbf{x}'_{jt}\boldsymbol{\beta}_i, 1\right)_{0}^{\infty}, & z_{ijt}=1 \end{cases}
\end{aligned}
$$

4.3 *Occupancy coefficients ($\boldsymbol{\beta}_i$)*

$$
\begin{aligned}
[\boldsymbol{\beta}_i \mid \cdot] &\propto [\mathbf{v}_i \mid \boldsymbol{\beta}_i]\left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2\right] \\
&\propto \mathcal{N}\left(\mathbf{v}_i \mid \mathbf{X}\boldsymbol{\beta}_i, \mathbf{1}\right)\mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma_\beta^2\mathbf{I}\right) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{v}_i - \mathbf{X}\boldsymbol{\beta}_i\right)'\left(\mathbf{v}_i - \mathbf{X}\boldsymbol{\beta}_i\right)\right\}\exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\mathbf{v}'_i\mathbf{X}\boldsymbol{\beta}_i + \boldsymbol{\beta}'_i\mathbf{X}'\mathbf{X}\boldsymbol{\beta}_i\right)\right\}\exp\left\{-\frac{1}{2}\left(-2\left(\boldsymbol{\mu}'_\beta\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}_i + \boldsymbol{\beta}'_i\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\beta}_i\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\mathbf{v}'_i\mathbf{X} + \boldsymbol{\mu}'_\beta\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}_i + \boldsymbol{\beta}'_i\left(\mathbf{X}'\mathbf{X} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}_i\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = \mathbf{X}'\mathbf{X} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}$, $\mathbf{b}' = \mathbf{v}'_i\mathbf{X} + \boldsymbol{\mu}'_\beta\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$, $\mathbf{v}'_i = \{v_{ijt}, \forall j, t\}$, and $\mathbf{X} = \left\{\mathbf{x}'_{jt}, \forall j, t\right\}$.

4.4 *Detection coefficients ($\boldsymbol{\alpha}_i$)*

$$
\begin{aligned}
[\boldsymbol{\alpha}_i \mid \cdot] &\propto \prod_{j=1}^{R}\prod_{t=1}^{T_j}[y_{ijt} \mid p_{ijk}, z_{ijt}]\left[\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2\right] \\
&\propto \prod_{j=1}^{R}\prod_{t=1}^{T_j}\text{Binom}\left(y_{ijt} \mid J_{jt}, p_{ijt}\right)^{z_{ijt}}\mathcal{N}\left(\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2\mathbf{I}\right).
\end{aligned}
$$

The update for $\boldsymbol{\alpha}_i$ proceeds using Metropolis-Hastings. Note that the product over j and t only includes instances of j and t such that $z_{ijt} = 1$.

4.5 Mean of occupancy coefficients ($\boldsymbol{\mu}_\beta$)

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta \mid \cdot] &\propto \prod_{i=1}^{n} [\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma^2_\beta] \, [\boldsymbol{\mu}_\beta] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma^2_\beta \mathbf{I}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_{\mu_\beta}\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{i=1}^{n}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\left(\sigma^2_\beta\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right)\right\} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n}\boldsymbol{\beta}'_i\left(\sigma^2_\beta\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}'_\beta\left(n\left(\sigma^2_\beta\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}'_\beta\left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{n}\boldsymbol{\beta}'_i\left(\sigma^2_\beta\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}'_\beta\left(n\left(\sigma^2_\beta\mathbf{I}\right)^{-1} + \left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n\left(\sigma^2_\beta\mathbf{I}\right)^{-1} + \left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}$, $\mathbf{b}' = \boldsymbol{\beta}'\left(\sigma^2_\beta\mathbf{I}\right)^{-1}$, and $\boldsymbol{\beta}$ is the vector sum $\sum_{i=1}^{n}\boldsymbol{\beta}_i$.

4.6 Mean of detection coefficients ($\boldsymbol{\mu}_\alpha$)

$$
\begin{aligned}
[\boldsymbol{\mu}_\alpha \mid \cdot] &\propto \prod_{i=1}^{n} [\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma^2_\alpha] \, [\boldsymbol{\mu}_\alpha] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma^2_\alpha \mathbf{I}\right) \mathcal{N}\left(\boldsymbol{\mu}_\alpha \mid \mathbf{0}, \sigma^2_{\mu_\alpha}\mathbf{I}\right) \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n\left(\sigma^2_\alpha\mathbf{I}\right)^{-1} + \left(\sigma^2_{\mu_\alpha}\mathbf{I}\right)^{-1}$, $\mathbf{b}' = \boldsymbol{\alpha}'\left(\sigma^2_\alpha\mathbf{I}\right)^{-1}$, and $\boldsymbol{\alpha}$ is the vector sum $\sum_{i=1}^{n}\boldsymbol{\alpha}_i$.

4.7 Variance of occupancy coefficients (σ^2_β)

$$
\begin{aligned}
\left[\sigma^2_\beta \mid \cdot\right] &\propto \prod_{i=1}^{n} [\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma^2_\beta] \, \left[\sigma^2_\beta\right] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \sigma^2_\beta \mathbf{I}\right) \mathrm{IG}\left(\sigma^2_\beta \mid r, q\right) \\
&\propto \prod_{i=1}^{n} \left|\sigma^2_\beta\mathbf{I}\right|^{-1/2} \exp\left\{-\frac{1}{2}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\left(\sigma^2_\beta\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right)\right\} \left(\sigma^2_\beta\right)^{-(q+1)} \exp\left\{-\frac{1}{r\sigma^2_\beta}\right\} \\
&\propto \left(\sigma^2_\beta\right)^{-(qX\times n)/2} \exp\left\{\sum_{i=1}^{n}\left(-\frac{1}{2\sigma^2_\beta}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right)\right\} \left(\sigma^2_\beta\right)^{-(q+1)} \exp\left\{-\frac{1}{r\sigma^2_\beta}\right\} \\
&\propto \left(\sigma^2_\beta\right)^{-((qX\times n)/2+q+1)} \exp\left\{-\frac{1}{\sigma^2_\beta}\left(\frac{\sum_{i=1}^{n}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)}{2} + \frac{1}{r}\right)\right\} \\
&= \mathrm{IG}\left(\left(\frac{\sum_{i=1}^{n}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)}{2} + \frac{1}{r}\right)^{-1}, \frac{qX\times n}{2} + q\right),
\end{aligned}
$$

where qX is the column dimension of $\mathbf{X}$ (or length of $\boldsymbol{\beta}_i$).

4.8 Variance of detection coefficients (σ_α^2)

$$\begin{aligned}
\left[\sigma_\alpha^2 \mid \cdot\right] &\propto \prod_{i=1}^{n} \left[\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2\right] \left[\sigma_\alpha^2\right] \\
&\propto \prod_{i=1}^{n} \mathcal{N}\left(\boldsymbol{\alpha}_i \mid \boldsymbol{\mu}_\alpha, \sigma_\alpha^2 \mathbf{I}\right) \mathrm{IG}\left(\sigma_\alpha^2 \mid r, q\right) \\
&= \mathrm{IG}\left(\left(\frac{\sum_{i=1}^{n} \left(\boldsymbol{\alpha}_i - \boldsymbol{\mu}_\alpha\right)' \left(\boldsymbol{\alpha}_i - \boldsymbol{\mu}_\alpha\right)}{2} + \frac{1}{r}\right)^{-1}, \frac{qW \times n}{2} + q\right),
\end{aligned}$$

where qW is the column dimension of $\mathbf{W}$ (or length of $\boldsymbol{\alpha}_i$).