Spatial, Inhomogenous Poisson Point Process Model for Grouped Data

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Description

An inhomogenous Poisson point process model for grouped spatial data.

Implementation

The file spatial.ppp.mixed.sim.R simulates data according to the model statement presented below, and spatial.ppp.mixed.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let $\mathbf{s}_j(t) = (s_{j,x}(t), s_{j,y}(t))'$, for $t \in \mathcal{T}$ and $j = 1, \ldots, J$, denote observed spatial locations, where the index t denotes replicate times within group j, and n_j is the number of observations in group j. Furthermore, let $\mathbf{x}_j(t)$ be a vector of p covariates (including the intercept) associated with the location $\mathbf{s}_j(t)$ and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$
\begin{aligned}
\mathbf{s}_j(t) &\sim \frac{\exp\left(\mathbf{x}\left(\mathbf{s}_j(t)\right)'\boldsymbol{\beta}_j\right)}{\int \exp\left(\mathbf{x}(\mathbf{s})'\boldsymbol{\beta}_j\right) d\mathbf{s}} \\
\boldsymbol{\beta}_j &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma_\beta^2 \mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}_j$):

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \prod_{t\in\mathcal{T}} \left[\mathbf{s}_j(t) \mid \boldsymbol{\beta}_j\right] \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{t\in\mathcal{T}} \left(\frac{\exp\left(\mathbf{x}\left(\mathbf{s}_j(t)\right)'\boldsymbol{\beta}_j\right)}{\int \exp\left(\mathbf{x}(\mathbf{s})'\boldsymbol{\beta}_j\right) d\mathbf{s}}\right) \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}
$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2 \mathbf{I}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\propto \exp\left\{\sum_{j=1}^{J}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times\exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta+\boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times\exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}_j'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta+\boldsymbol{\mu}_\beta'\left(J\boldsymbol{\Sigma}^{-1}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A}=J\boldsymbol{\Sigma}^{-1}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=\boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$
\begin{aligned}
[\boldsymbol{\Sigma}\mid\cdot] &\propto \prod_{j=1}^{J}\left[\boldsymbol{\beta}_j\mid\boldsymbol{\mu}_\beta,\boldsymbol{\Sigma}\right]\left[\boldsymbol{\Sigma}\mid\mathbf{S}_0,\nu\right] \\
&\propto \prod_{j=1}^{J}\mathcal{N}\left(\boldsymbol{\beta}_j\mid\boldsymbol{\mu}_\beta,\boldsymbol{\Sigma}\right)\mathrm{Wish}\left(\boldsymbol{\Sigma}\mid\mathbf{S}_0,\nu\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J}{2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{J}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times|\mathbf{S}_0|^{-\frac{\nu}{2}}|\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\mathrm{tr}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\right)+\mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J}\mathrm{tr}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\right)+\mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)\boldsymbol{\Sigma}^{-1}+\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right)\boldsymbol{\Sigma}^{-1}\right]\right\} \\
&= \mathrm{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right)^{-1},J+\nu\right).
\end{aligned}
$$