

# SPATIAL, INHOMOGENOUS POISSON POINT PROCESS MODEL

## FOR GROUPED DATA

Brian M. Brost

18 March 2016

### Description

An inhomogenous Poisson point process model for grouped spatial data.

### Implementation

The file `spatial.ppp.varying.coef.sim.R` simulates data according to the model statement presented below, and `spatial.ppp.varying.coef.mcmc.R` contains the MCMC algorithm for model fitting.

### Model statement

Let  $\mathbf{s}_j(t) = (s_{j,x}(t), s_{j,y}(t))'$ , for  $t \in \mathcal{T}$  and  $j = 1, \dots, J$ , denote observed spatial locations, where the index  $t$  denotes replicate times within group  $j$ , and  $n_j$  is the number of observations in group  $j$ . Furthermore, let  $\mathbf{x}_j(t)$  be a vector of  $p$  covariates (including the intercept) associated with the location  $\mathbf{s}_j(t)$  and  $\boldsymbol{\beta}_j$  be the corresponding vector of coefficients for group  $j$ .

$$\begin{aligned} \mathbf{s}_j(t) &\sim \frac{\exp(\mathbf{x}(\mathbf{s}_j(t))' \boldsymbol{\beta}_j)}{\int \exp(\mathbf{x}(\mathbf{s})' \boldsymbol{\beta}_j) d\mathbf{s}} \\ \boldsymbol{\beta}_j &\sim \mathcal{N}(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \\ \boldsymbol{\mu}_\beta &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\ \boldsymbol{\Sigma}^{-1} &\sim \text{Wish}(\mathbf{S}_0^{-1}, \nu) \end{aligned}$$

### Full conditional distributions

*Regression coefficients ( $\boldsymbol{\beta}_j$ ):*

$$\begin{aligned} [\boldsymbol{\beta}_j | \cdot] &\propto \prod_{t \in \mathcal{T}} [\mathbf{s}_j(t) | \boldsymbol{\beta}_j] [\boldsymbol{\beta}_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] \\ &\propto \prod_{t \in \mathcal{T}} \left( \frac{\exp(\mathbf{x}(\mathbf{s}_j(t))' \boldsymbol{\beta}_j)}{\int \exp(\mathbf{x}(\mathbf{s})' \boldsymbol{\beta}_j) d\mathbf{s}} \right) \mathcal{N}(\boldsymbol{\beta}_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}). \end{aligned}$$

The update for  $\boldsymbol{\beta}_j$  proceeds using Metropolis-Hastings.

*Mean of regression coefficients ( $\boldsymbol{\mu}_\beta$ ):*

$$\begin{aligned} [\boldsymbol{\mu}_\beta | \cdot] &\propto \prod_{j=1}^J [\boldsymbol{\beta}_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] [\boldsymbol{\mu}_\beta | \mathbf{0}, \sigma_\beta^2 \mathbf{I}] \\ &\propto \prod_{j=1}^J \mathcal{N}(\boldsymbol{\beta}_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \mathcal{N}(\boldsymbol{\mu}_\beta | \mathbf{0}, \sigma_\beta^2 \mathbf{I}) \end{aligned}$$

$$\begin{aligned}
& \propto \exp \left\{ \sum_{j=1}^J \left( -\frac{1}{2} (\beta_j - \mu_\beta)' \Sigma^{-1} (\beta_j - \mu_\beta) \right) \right\} \\
& \quad \times \exp \left\{ -\frac{1}{2} (\mu_\beta - \mathbf{0})' (\sigma_\beta^2 \mathbf{I})^{-1} (\mu_\beta - \mathbf{0}) \right\} \\
& \propto \exp \left\{ -\frac{1}{2} \left( -2 \left( \sum_{j=1}^J \beta_j' \Sigma^{-1} \right) \mu_\beta + \mu_\beta' (J \Sigma^{-1}) \mu_\beta \right) \right\} \\
& \quad \times \exp \left\{ -\frac{1}{2} (\mu_\beta' (\sigma_\beta^2 \mathbf{I})^{-1} \mu_\beta) \right\} \\
& \propto \exp \left\{ -\frac{1}{2} \left( -2 \left( \sum_{j=1}^J \beta_j' \Sigma^{-1} \right) \mu_\beta + \mu_\beta' (J \Sigma^{-1} + (\sigma_\beta^2 \mathbf{I})^{-1}) \mu_\beta \right) \right\} \\
& = \mathcal{N}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where  $\mathbf{A} = J \Sigma^{-1} + (\sigma_\beta^2 \mathbf{I})^{-1}$  and  $\mathbf{b}' = \beta' \Sigma^{-1}$ , where  $\beta$  is the vector sum  $\sum_{j=1}^J \beta_j$ .

*Variance-covariance of regression coefficients ( $\Sigma$ ):*

$$\begin{aligned}
[\Sigma \mid \cdot] & \propto \prod_{j=1}^J [\beta_j \mid \mu_\beta, \Sigma] [\Sigma \mid \mathbf{S}_0, \nu] \\
& \propto \prod_{j=1}^J \mathcal{N}(\beta_j \mid \mu_\beta, \Sigma) \text{Wish}(\Sigma \mid \mathbf{S}_0, \nu) \\
& \propto |\Sigma|^{-\frac{J}{2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^J (\beta_j - \mu_\beta)' \Sigma^{-1} (\beta_j - \mu_\beta) \right\} \\
& \quad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\Sigma|^{-\frac{\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \text{tr}(\mathbf{S}_0 \Sigma^{-1}) \right\} \\
& \propto |\Sigma|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[ \sum_{j=1}^J \text{tr}((\beta_j - \mu_\beta)' \Sigma^{-1} (\beta_j - \mu_\beta)) + \text{tr}(\mathbf{S}_0 \Sigma^{-1}) \right] \right\} \\
& \propto |\Sigma|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[ \sum_{j=1}^J \text{tr}((\beta_j - \mu_\beta) (\beta_j - \mu_\beta)' \Sigma^{-1}) + \text{tr}(\mathbf{S}_0 \Sigma^{-1}) \right] \right\} \\
& \propto |\Sigma|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[ \text{tr} \left( \sum_{j=1}^J ((\beta_j - \mu_\beta) (\beta_j - \mu_\beta)') \Sigma^{-1} + \mathbf{S}_0 \Sigma^{-1} \right) \right] \right\} \\
& \propto |\Sigma|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[ \text{tr} \left( \sum_{j=1}^J ((\beta_j - \mu_\beta) (\beta_j - \mu_\beta)') + \mathbf{S}_0 \right) \Sigma^{-1} \right] \right\} \\
& = \text{Wish} \left( \left( \sum_{j=1}^J ((\beta_j - \mu_\beta) (\beta_j - \mu_\beta)') + \mathbf{S}_0 \right)^{-1}, J + \nu \right).
\end{aligned}$$