Semiparametric Regression for Binomially Distributed Data

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Description

A semiparametric, probit regression model for binomially distributed data.

Implementation

The file probit.semipar.sim.R simulates data according to the model statement presented below, and probit.semipar.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let y_t, for $t = 1, \ldots, T$, be observed data that take on the values {0,1}. Also let $\mathbf{X}$ be a design matrix containing covariates for which inference is desired, and $\mathbf{Z}$ be a design matrix containing some basis expansion. The vectors $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are the corresponding 'fixed' and 'random' effects, respectively. Note that $\mathbf{Z}\boldsymbol{\alpha}$ models non-linear patterns or dependence non-parametrically.

$$
\begin{aligned}
y_t &\sim \begin{cases} 0, & u_t \leq 0 \\ 1, & u_t > 1 \end{cases} \\
u_t &\sim \mathcal{N}(\mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1}) \\
\boldsymbol{\beta} &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\
\boldsymbol{\alpha} &\sim \mathcal{N}(\mathbf{0}, \sigma_\alpha^2 \mathbf{I})
\end{aligned}
$$

Models of this type are typically fit using a large number of basis vectors, more than necessary to approximate non-linear trends or dependence. Regularization (e.g., a ridge penalty) is subsequently conducted to shrink the coefficients $\boldsymbol{\alpha}$ toward 0 where appropriate. Therefore, the parameter σ_α^2 must be selected using cross-validation or some model selection criterion. Model-based estimation of σ_α^2 , i.e., $\sigma_\alpha^2 \sim \text{IG}(r, q)$, results in a mixed effects model similar to that implemented by the function `glmer` (???) in the R package `lme4`.

Full conditional distributions

Observation model auxiliary variable (u_t):

$$
\begin{aligned}
[u_t|\cdot] &\propto [y_t|u_t][u_t] \\
&\propto \left(1_{\{y_t=0\}}1_{\{u_t\leq 0\}} + 1_{\{y_t=1\}}1_{\{u_t>0\}}\right) \times \mathcal{N}(u_t \mid \mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1}) \\
&= \begin{cases} \mathcal{TN}(\mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1})_{-\infty}^{0}, & y_t = 0 \\ \mathcal{TN}(\mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1})_{0}^{\infty}, & y_t = 1 \end{cases}
\end{aligned}
$$

Fixed effects ($\boldsymbol{\beta}$):

$$
\begin{aligned}
[\boldsymbol{\beta}|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta},\boldsymbol{\alpha},\sigma^2][\boldsymbol{\beta}] \\
&\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha},\mathbf{1})\mathcal{N}(\boldsymbol{\beta}|\mathbf{0},\sigma_\beta^2\mathbf{I}) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{y}-(\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha})\right)'\left(\mathbf{y}-(\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha})\right)\right\} \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left((\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})-\mathbf{X}\boldsymbol{\beta}\right)'\left((\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})-\mathbf{X}\boldsymbol{\beta}\right)\right\} \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2(\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})'\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\right)\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\beta}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left((\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})'\mathbf{X}\right)\boldsymbol{\beta}+\boldsymbol{\beta}'\left(\mathbf{X}'\mathbf{X}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}\right)\right\}\times \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1})
\end{aligned}
$$

where $\mathbf{A}=\mathbf{X}'\mathbf{X}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=(\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})'\mathbf{X}$.

Random effects ($\boldsymbol{\alpha}$):

$$
\begin{aligned}
[\boldsymbol{\alpha}|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta},\boldsymbol{\alpha},\sigma^2][\boldsymbol{\alpha}] \\
&\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha},\mathbf{1})\mathcal{N}(\boldsymbol{\alpha}|\mathbf{0},\sigma_\alpha^2\mathbf{I}) \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1})
\end{aligned}
$$

where $\mathbf{A}=\mathbf{Z}'\mathbf{Z}+\left(\sigma_\alpha^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'\mathbf{Z}$.