Semiparametric Probit Regression Model

Brian M. Brost

07 August 2015

Description

A semiparametric, probit regression model for data that take the values {0,1}.

Implementation

The file probit.semipar.sim.R simulates data according to the model statement presented below, and probit.semipar.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let y_t, for $t = 1, \ldots, T$, be observed data that take on the values {0,1}. Also let $\mathbf{X}$ be a design matrix containing covariates for which inference is desired, and $\mathbf{Z}$ be a design matrix containing some basis expansion. The vectors $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are the corresponding 'fixed' and 'random' effects, respectively. Note that $\mathbf{Z}\boldsymbol{\alpha}$ models non-linear patterns or dependence non-parametrically.

$$
\begin{aligned}
y_t &\sim \begin{cases} 0, & u_t \leq 0 \\ 1, & u_t > 1 \end{cases} \\
u_t &\sim \mathcal{N}(\mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1}) \\
\boldsymbol{\beta} &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\
\boldsymbol{\alpha} &\sim \mathcal{N}(\mathbf{0}, \sigma_\alpha^2 \mathbf{I})
\end{aligned}
$$

Models of this type are typically fit using a large number of basis vectors, more than necessary to approximate non-linear trends or dependence. Regularization (e.g., a ridge penalty) is subsequently conducted to shrink the coefficients $\boldsymbol{\alpha}$ toward 0 where appropriate. Therefore, the parameter σ_α^2 must be selected using cross-validation or some model selection criterion. Model-based estimation of σ_α^2 , i.e., $\sigma_\alpha^2 \sim \text{IG}(r, q)$, results in a mixed effects model similar to that implemented by the function `glmer` (???) in the R package `lme4`.

Full conditional distributions

Observation model auxiliary variable (u_t):

$$
\begin{aligned}
[u_t|\cdot] &\propto [y_t|u_t][u_t] \\
&\propto \left(1_{\{y_t=0\}}1_{\{u_t\leq 0\}} + 1_{\{y_t=1\}}1_{\{u_t>0\}}\right) \times \mathcal{N}(u_t \mid \mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1}) \\
&= \begin{cases} \mathcal{TN}(\mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1})_{-\infty}^{0}, & y_t = 0 \\ \mathcal{TN}(\mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\alpha}, \mathbf{1})_{0}^{\infty}, & y_t = 1 \end{cases}
\end{aligned}
$$

Fixed effects ($\boldsymbol{\beta}$):

$$
\begin{aligned}
[\boldsymbol{\beta}|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta},\boldsymbol{\alpha},\sigma^2][\boldsymbol{\beta}] \\
&\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha},\mathbf{1})\mathcal{N}(\boldsymbol{\beta}|\mathbf{0},\sigma_\beta^2\mathbf{I}) \\
&\propto \exp\left\{-\frac{1}{2}\left(\mathbf{y}-(\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha})\right)'\left(\mathbf{y}-(\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha})\right)\right\} \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left((\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})-\mathbf{X}\boldsymbol{\beta}\right)'\left((\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})-\mathbf{X}\boldsymbol{\beta}\right)\right\} \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\beta}-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2(\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})'\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\right)\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\beta}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left((\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})'\mathbf{X}\right)\boldsymbol{\beta}+\boldsymbol{\beta}'\left(\mathbf{X}'\mathbf{X}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\beta}\right)\right\}\times \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1})
\end{aligned}
$$

where $\mathbf{A}=\mathbf{X}'\mathbf{X}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=(\mathbf{y}-\mathbf{Z}\boldsymbol{\alpha})'\mathbf{X}$.

Random effects ($\boldsymbol{\alpha}$):

$$
\begin{aligned}
[\boldsymbol{\alpha}|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta},\boldsymbol{\alpha},\sigma^2][\boldsymbol{\alpha}] \\
&\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\alpha},\mathbf{1})\mathcal{N}(\boldsymbol{\alpha}|\mathbf{0},\sigma_\alpha^2\mathbf{I}) \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b},\mathbf{A}^{-1})
\end{aligned}
$$

where $\mathbf{A}=\mathbf{Z}'\mathbf{Z}+\left(\sigma_\alpha^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}'=(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'\mathbf{Z}$.