

SEMIPARAMETRIC REGRESSION FOR NORMALLY DISTRIBUTED DATA

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Implementation

The file `normal.semipar.sim.R` simulates data according to the model statement presented below, and `normal.semipar.mcmc.R` contains the MCMC algorithm for model fitting.

Model statement

Let $\mathbf{y} = (y_1, \dots, y_T)'$ be a vector of observations. Also let \mathbf{X} be a design matrix containing covariates for which inference is desired, and \mathbf{Z} be a design matrix containing some basis expansion. The vectors $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are the corresponding 'fixed' and 'random' effects, respectively. Note that $\mathbf{Z}\boldsymbol{\alpha}$ models non-linear patterns or dependence non-parametrically.

$$\begin{aligned}\mathbf{y} &\sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha}, \sigma^2 \mathbf{I}) \\ \boldsymbol{\beta} &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\ \boldsymbol{\alpha} &\sim \mathcal{N}(\mathbf{0}, \sigma_\alpha^2 \mathbf{I}) \\ \sigma^2 &\sim \text{IG}(r, q)\end{aligned}$$

Models of this type are typically fit using a large number of basis vectors, more than necessary to approximate non-linear trends or dependence. Regularization (e.g., a ridge penalty) is subsequently conducted to shrink the coefficients toward 0 where appropriate. Therefore, the parameter σ_α^2 must be selected using cross-validation or some model selection criterion. Model-based estimation of σ_α^2 , i.e., $\sigma_\alpha^2 \sim \text{IG}(r, q)$, results in a mixed effects model similar to that implemented by the function `lme` in the R package `nlme`.

Full conditional distributions

Fixed effects ($\boldsymbol{\beta}$):

$$\begin{aligned}[\boldsymbol{\beta}|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta}, \boldsymbol{\alpha}, \sigma^2][\boldsymbol{\beta}] \\ &\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha}, \sigma^2 \mathbf{I}) \mathcal{N}(\boldsymbol{\beta}|\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\ &\propto \exp \left\{ -\frac{1}{2} (\mathbf{y} - (\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha}))' (\sigma^2 \mathbf{I})^{-1} (\mathbf{y} - (\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha})) \right\} \\ &\quad \exp \left\{ -\frac{1}{2} (\boldsymbol{\beta} - \mathbf{0})' (\sigma_\beta^2 \mathbf{I})^{-1} (\boldsymbol{\beta} - \mathbf{0}) \right\} \\ &\propto \exp \left\{ -\frac{1}{2} ((\mathbf{y} - \mathbf{Z}\boldsymbol{\alpha}) - \mathbf{X}\boldsymbol{\beta})' (\sigma^2 \mathbf{I})^{-1} ((\mathbf{y} - \mathbf{Z}\boldsymbol{\alpha}) - \mathbf{X}\boldsymbol{\beta}) \right\} \\ &\quad \exp \left\{ -\frac{1}{2} (\boldsymbol{\beta} - \mathbf{0})' (\sigma_\beta^2 \mathbf{I})^{-1} (\boldsymbol{\beta} - \mathbf{0}) \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \left(-2(\mathbf{y} - \mathbf{Z}\boldsymbol{\alpha})' (\sigma^2 \mathbf{I})^{-1} \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}' \mathbf{X}' (\sigma^2 \mathbf{I})^{-1} \mathbf{X}\boldsymbol{\beta} \right) \right\} \times \\ &\quad \exp \left\{ -\frac{1}{2} \left(\boldsymbol{\beta}' (\sigma_\beta^2 \mathbf{I})^{-1} \boldsymbol{\beta} \right) \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \left(-2 \left((\mathbf{y} - \mathbf{Z}\boldsymbol{\alpha})' (\sigma^2 \mathbf{I})^{-1} \mathbf{X} \right) \boldsymbol{\beta} + \boldsymbol{\beta}' \left(\mathbf{X}' (\sigma^2 \mathbf{I})^{-1} \mathbf{X} + (\sigma_\beta^2 \mathbf{I})^{-1} \right) \boldsymbol{\beta} \right) \right\} \times \\ &= \mathcal{N}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})\end{aligned}$$

where $\mathbf{A} = (\mathbf{y} - \mathbf{Z}\boldsymbol{\alpha})' (\sigma^2 \mathbf{I})^{-1} \mathbf{X}$ and $\mathbf{b}' = \mathbf{X}' (\sigma^2 \mathbf{I})^{-1} \mathbf{X} + (\sigma_\beta^2 \mathbf{I})^{-1}$.

Random effects ($\boldsymbol{\alpha}$):

$$\begin{aligned} [\boldsymbol{\alpha}|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta}, \boldsymbol{\alpha}, \sigma^2][\boldsymbol{\alpha}] \\ &\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha}, \sigma^2 \mathbf{I}) \mathcal{N}(\boldsymbol{\alpha}|\mathbf{0}, \sigma_\alpha^2 \mathbf{I}) \\ &= \mathcal{N}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1}) \end{aligned}$$

where $\mathbf{A} = \mathbf{Z}' (\sigma^2 \mathbf{I})^{-1} \mathbf{Z} + (\sigma_\alpha^2 \mathbf{I})^{-1}$ and $\mathbf{b}' = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' (\sigma^2 \mathbf{I})^{-1} \mathbf{Z}$.

Observation error (σ^2):

$$\begin{aligned} [\sigma^2|\cdot] &\propto [\mathbf{y}|\boldsymbol{\beta}, \boldsymbol{\alpha}, \sigma^2][\sigma^2] \\ &\propto \mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha}, \sigma^2 \mathbf{I}) \text{IG}(r, q) \\ &\propto |\sigma^2 \mathbf{I}|^{-1/2} \exp \left\{ -\frac{1}{2} \left((\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})' (\sigma^2 \mathbf{I})^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha}) \right) \right\} \times \\ &\quad (\sigma^2)^{-(q+1)} \exp \left\{ -\frac{1}{r\sigma^2} \right\} \\ &\propto (\sigma^2)^{-T/2} \exp \left\{ -\frac{1}{2\sigma^2} ((\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})) \right\} \\ &\quad (\sigma^2)^{-(q+1)} \exp \left\{ -\frac{1}{r\sigma^2} \right\} \\ &\propto (\sigma^2)^{-(T/2+q+1)} \exp \left\{ -\frac{1}{\sigma^2} \left(\frac{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})}{2} + \frac{1}{r} \right) \right\} \\ &= \text{IG} \left(\left(\frac{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\boldsymbol{\alpha})}{2} + \frac{1}{r} \right)^{-1}, \frac{T}{2} + q \right) \end{aligned}$$