Zero-Inflated Poisson Model for Data Collected in Three Hierarchical Levels

Brian M. Brost
07 April 2016

Description

A zero-inflated Poisson model for data collected in three hierarchical levels (data nested within subgroup nested within group), with varying coefficients at the second (subgroup) and third (group) levels.

Implementation

The file zi.poisson.varying.coef.3.sim.R simulates data according to the model statement presented below, and zi.poisson.varying.coef.3.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let y_{ijk} denote observed counts (i.e., y_{ijk} are integers greater than or equal to 0) for groups $i = 1, \ldots, N$, subgroups $j = 1, \ldots, n_i$ nested within groups, and replicate observations $k = 1, \ldots, m_{ij}$ (level-1 units) within subgroup j (level-2 units) and group i (level-3 units). Furthermore, let $\mathbf{x}_{ijk}$ be a vector of p covariates (including the intercept) associated with y_{ijk} and $\boldsymbol{\alpha}_{ij}$ be the corresponding vector of coefficients for subgroup j in group i. The vector $\boldsymbol{\beta}_i$ corresponds to group-level coefficients and $\boldsymbol{\mu}_\beta$ is a vector of population-level coefficients.

$$
\begin{aligned}
y_{ijk} &\sim \begin{cases} \text{Pois}\left(\lambda_{ijk}\right), & z_{ijk} = 1 \\ 0, & z_{ijk} = 0 \end{cases} \\
z_{ijk} &\sim \text{Bern}\left(p_{ij}\right) \\
\log\left(\lambda_{ijk}\right) &= \mathbf{x}'_{ijk}\boldsymbol{\alpha}_{ij} \\
\boldsymbol{\alpha}_{ij} &\sim \mathcal{N}\left(\boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \\
\boldsymbol{\beta}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
p_{ij} &\sim \text{Beta}\left(\alpha_1, \alpha_2\right) \\
\boldsymbol{\Sigma}^{-1}_{\alpha_i} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right) \\
\boldsymbol{\Sigma}^{-1}_{\beta} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right)
\end{aligned}
$$

Full conditional distributions

Latent mixture component indicator variable (z_{ijk}):

$$
\begin{aligned}
[z_{ijk} \mid \cdot] &\propto [y_{ijk} \mid \lambda_{ijk}, z_{ijk}]\,[z_{ijk} \mid p_{ij}] \\
&\propto \text{Pois}\left(y_{ijk} \mid \lambda_{ijk}\right)^{z_{ijk}} 1^{1-z_{ijk}}_{\{y_{ijk}=0\}} p_{ij}^{z_{ijk}} \left(1 - p_{ij}\right)^{1-z_{ijk}} \\
&\propto \left(\frac{\lambda_{ijk}^{y_{ijk}} \exp\left(-\lambda_{ijk}\right)}{y_{ijk}!}\right)^{z_{ijk}} p_{ij}^{z_{ijk}} \left(1 - p_{ij}\right)^{1-z_{ijk}} \\
&\propto \left(\exp\left(-\lambda_{ijk}\right)\right)^{z_{ijk}} p_{ij}^{z_{ijk}} \left(1 - p_{ij}\right)^{1-z_{ijk}} \\
&\propto \left(p_{ij} \times \exp\left(-\lambda_{ijk}\right)\right)^{z_{ijk}} \left(1 - p_{ij}\right)^{1-z_{ijk}} \\
&= \text{Bern}\left(\tilde{p}\right),
\end{aligned}
$$

where

$$
\tilde{p} = \frac{p_{ij} \times \exp\left(-\lambda_{ijk}\right)}{p_i \times \exp\left(-\lambda_{ijk}\right) + 1 - p_{ij}}.
$$

Note that z_{ijk} is only estimated for instances where $y_{ijk} = 0$ ($z_{ijk} = 1$ when $y_{ijk} > 0$).

Probability associated with the mixture component indicator variables (p_{ij}):

$$
\begin{aligned}
[p_{ij} \mid \cdot] &\propto \prod_{j=1}^{n_i} [z_{ijk} \mid p_{ij}] \, [p_{ij} \mid \alpha_1, \alpha_2] \\
&\propto \prod_{j=1}^{n_i} p_{ij}^{z_{ijk}} \, (1 - p_{ij})^{1 - z_{ijk}} \, p_{ij}^{\alpha_1 - 1} \, (1 - p_{ij})^{\alpha_2 - 1} \\
&\propto p_{ij}^{\sum_{j=1}^{n_i} z_{ijk}} \, (1 - p_{ij})^{n_i - \sum_{j=1}^{n_i} z_{ijk}} \, p_{ij}^{\alpha_1 - 1} \, (1 - p_{ij})^{\alpha_2 - 1} \\
&= \text{Beta}\left(\sum_{j=1}^{n_i} z_{ijk} + \alpha_1, n_i - \sum_{j=1}^{n_i} z_{ijk} + \alpha_2 \right)
\end{aligned}
$$

Subgroup-level regression coefficients ($\boldsymbol{\alpha}_{ij}$):

$$
\begin{aligned}
[\boldsymbol{\alpha}_{ij} \mid \cdot] &\propto \prod_{k=1}^{m_{ij}} [y_{ijk} \mid \lambda_{ijk}, z_{ijk}] \, [\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}] \\
&\propto \prod_{k=1}^{m_{ij}} \text{Pois}\left(y_{ijk} \mid \lambda_{ijk}\right)^{z_{ijk}} \mathcal{N}\left(\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right).
\end{aligned}
$$

The update for $\boldsymbol{\alpha}_{ij}$ proceeds using Metropolis-Hastings.

Group-level regression coefficients ($\boldsymbol{\beta}_i$):

$$
\begin{aligned}
[\boldsymbol{\beta}_i \mid \cdot] &\propto \prod_{j=1}^{n_i} [\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}] \, [\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta] \\
&\propto \prod_{j=1}^{n_i} \mathcal{N}\left(\boldsymbol{\alpha}_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \\
&\propto \exp\left\{ \sum_{j=1}^{n_i} \left(-\frac{1}{2} \left(\boldsymbol{\alpha}_{ij} - \boldsymbol{\beta}_i\right)' \boldsymbol{\Sigma}_{\alpha_i}^{-1} \left(\boldsymbol{\alpha}_{ij} - \boldsymbol{\beta}_i\right) \right) \right\} \\
&\qquad \times \exp\left\{ -\frac{1}{2} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right) \right\} \\
&\propto \exp\left\{ -\frac{1}{2} \left(-2 \left(\sum_{j=1}^{n_i} \boldsymbol{\alpha}_{ij}' \boldsymbol{\Sigma}_{\alpha_i}^{-1} \right) \boldsymbol{\beta}_i + \boldsymbol{\beta}_i' \left(n_i \boldsymbol{\Sigma}_{\alpha_i}^{-1}\right) \boldsymbol{\beta}_i \right) \right\} \\
&\qquad \times \exp\left\{ -\frac{1}{2} \left(-2 \left(\boldsymbol{\mu}_\beta' \boldsymbol{\Sigma}_\beta^{-1} \right) \boldsymbol{\beta}_i + \boldsymbol{\beta}_i' \left(\boldsymbol{\Sigma}_\beta^{-1} \right) \boldsymbol{\beta}_i \right) \right\} \\
&\propto \exp\left\{ -\frac{1}{2} \left(-2 \left(\sum_{j=1}^{n_i} \boldsymbol{\alpha}_{ij}' \boldsymbol{\Sigma}_{\alpha_i}^{-1} - \boldsymbol{\mu}_\beta' \boldsymbol{\Sigma}_\beta^{-1} \right) \boldsymbol{\beta}_i + \boldsymbol{\beta}_i' \left(n_i \boldsymbol{\Sigma}_{\alpha_i}^{-1} + \boldsymbol{\Sigma}_\beta^{-1} \right) \boldsymbol{\beta}_i \right) \right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = n_i \boldsymbol{\Sigma}_{\alpha_i}^{-1} + \boldsymbol{\Sigma}_\beta^{-1}$ and $\mathbf{b}' = \boldsymbol{\alpha}_i' \boldsymbol{\Sigma}_{\alpha_i}^{-1} - \boldsymbol{\mu}_\beta' \boldsymbol{\Sigma}_\beta^{-1}$, where $\boldsymbol{\alpha}_i$ is the vector sum $\sum_{j=1}^{n_i} \boldsymbol{\alpha}_{ij}$.

Mean of group-level regression coefficients ($\boldsymbol{\mu}_\beta$):

$$\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{i=1}^{N} \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_{\mu_\beta}\right] \\
&\propto \prod_{i=1}^{N} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma^2_{\mu_\beta}\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{i=1}^{N}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{N}\boldsymbol{\beta}_i'\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(N\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta'\left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{N}\boldsymbol{\beta}_i'\boldsymbol{\Sigma}_\beta^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta'\left(N\boldsymbol{\Sigma}_\beta^{-1} + \left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where $\mathbf{A} = N\boldsymbol{\Sigma}_\beta^{-1} + \left(\sigma^2_{\mu_\beta}\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}_\beta^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{i=1}^{N}\boldsymbol{\beta}_i$.

Variance-covariance of group-level regression coefficients ($\boldsymbol{\Sigma}_\beta$):

$$\begin{aligned}
\left[\boldsymbol{\Sigma}_\beta \mid \cdot\right] &\propto \prod_{i=1}^{N}\left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right]\left[\boldsymbol{\Sigma}_\beta \mid \mathbf{S}_0, \nu\right] \\
&\propto \prod_{i=1}^{N}\mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right)\mathrm{Wish}\left(\boldsymbol{\Sigma}_\beta \mid \mathbf{S}_0, \nu\right) \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{N}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \left|\mathbf{S}_0\right|^{-\frac{\nu}{2}}\left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}_\beta^{-1}\right)\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{N+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}\mathrm{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}_\beta^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right) + \mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{N+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}\mathrm{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}_\beta^{-1}\right) + \mathrm{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{N+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right)\boldsymbol{\Sigma}_\beta^{-1} + \mathbf{S}_0\boldsymbol{\Sigma}_\beta^{-1}\right)\right]\right\} \\
&\propto \left|\boldsymbol{\Sigma}_\beta\right|^{-\frac{N+\nu-p-1}{2}}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)\boldsymbol{\Sigma}_\beta^{-1}\right]\right\} \\
&= \mathrm{Wish}\left(\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, N+\nu\right).
\end{aligned}$$

Variance-covariance of subgroup-level regression coefficients ($\boldsymbol{\Sigma}_{\alpha_i}$):

$$
\begin{aligned}
[\boldsymbol{\Sigma}_{\alpha_i} \mid \cdot] &\propto \prod_{j=1}^{n_i} [\alpha_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}] [\boldsymbol{\Sigma}_{\alpha_i} \mid \mathbf{S}_0, \nu] \\
&\propto \prod_{j=1}^{n_i} \mathcal{N}\left(\alpha_{ij} \mid \boldsymbol{\beta}_i, \boldsymbol{\Sigma}_{\alpha_i}\right) \mathrm{Wish}\left(\boldsymbol{\Sigma}_{\alpha_i} \mid \mathbf{S}_0, \nu\right) \\
&= \mathrm{Wish}\left(\left(\sum_{j=1}^{n_i} \left(\left(\alpha_{ij} - \boldsymbol{\beta}_i\right)\left(\alpha_{ij} - \boldsymbol{\beta}_i\right)'\right) + \mathbf{S}_0\right)^{-1}, n_i + \nu\right).
\end{aligned}
$$