Poisson Generalized Linear Model for Zero-truncated Count Data

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Description

A generalized linear model for zero-truncated count data.

Implementation

The file zt.poisson.glm.sim.R simulates data according to the model statement presented below, and zt.poisson.glm.mcmc.R contains the MCMC algorithm for model fitting.

Derivation of zero-truncated Poisson distribution

The probability mass function of the (non-truncated) Poisson distribution is:

$$[z] = \frac{\lambda^z \exp(-\lambda)}{z!}. \tag{1}$$

It follows that the probability that $z = 0$ is

$$\begin{aligned} [z \mid z = 0] &= \frac{\lambda^0 \exp(-\lambda)}{0!} && (2) \\ &= \exp(-\lambda), && (3) \end{aligned}$$

and thus the probability that $z > 0$ is $1 - [z \mid z = 0] = 1 - \exp(-\lambda)$. We arrive at the density function of the zero-truncated Poisson distribution by excluding the probability that $z = 0$ from the standard Poisson distribution (Eq. 1). This is accomplished by dividing Eq. 1 by $[z \mid z = 0]$:

$$[z \mid z > 0] = \frac{\lambda^z \exp(-\lambda)}{(1 - \exp(-\lambda))\, z!}. \tag{4}$$

We abbreviate the probability mass function for the zero-truncated Poisson distribution as $\text{ZTP}(\lambda_i)$.

Model statement

Let z_i, for $i = 1, \ldots, n$, be observed, non-zero count data (i.e., z_i are integers greater than 0). Also let $\mathbf{x}_i$ be a vector of covariates associated with z_i for which inference is desired, and the vector $\boldsymbol{\beta}$ be the corresponding coefficients.

$$\begin{aligned} z_i \mid z_i > 0 &\sim \text{ZTP}(\lambda_i) \\ \log(\lambda_i) &= \mathbf{x}_i' \boldsymbol{\beta} \\ \boldsymbol{\beta} &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \end{aligned}$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}$):

$$\begin{aligned} [\boldsymbol{\beta} \mid \cdot] &\propto \prod_{i=1}^{n} [z_i \mid \boldsymbol{\beta}] [\boldsymbol{\beta}] \\ &\propto \prod_{i=1}^{n} \text{ZTP}(z_i \mid \mathbf{x}_i' \boldsymbol{\beta}) \mathcal{N}(\boldsymbol{\beta} \mid \mathbf{0}, \sigma_\beta^2 \mathbf{I}). \end{aligned}$$

The update for $\boldsymbol{\beta}$ proceeds using Metropolis-Hastings.