Zero-Truncated Poisson Model for Count Data

Brian M. Brost
06 April 2016

Description

A zero-truncated Poisson model for count data.

Implementation

The file zt.poisson.sim.R simulates data according to the model statement presented below, and zt.poisson.mcmc.R contains the MCMC algorithm for model fitting.

Derivation of zero-truncated Poisson distribution

The probability mass function of the (non-truncated) Poisson distribution is:

$$[z] = \frac{\lambda^z \exp(-\lambda)}{z!}. \tag{1}$$

It follows that the probability that $z = 0$ is

$$\begin{aligned} [z \mid z = 0] &= \frac{\lambda^0 \exp(-\lambda)}{0!} && (2) \\ &= \exp(-\lambda), && (3) \end{aligned}$$

and thus the probability that $z > 0$ is $1 - [z \mid z = 0] = 1 - \exp(-\lambda)$. We arrive at the density function of the zero-truncated Poisson distribution by excluding the probability that $z = 0$ from the standard Poisson distribution (Eq. 1). This is accomplished by dividing Eq. 1 by $[z \mid z = 0]$:

$$[z \mid z > 0] = \frac{\lambda^z \exp(-\lambda)}{(1 - \exp(-\lambda))\, z!}. \tag{4}$$

We abbreviate the probability mass function for the zero-truncated Poisson distribution as $\text{ZTP}(\lambda_i)$.

Model statement

Let z_i, for $i = 1, \ldots, n$, be observed, non-zero count data (i.e., z_i are integers greater than 0). Also let $\mathbf{x}_i$ be a vector of covariates associated with z_i for which inference is desired, and the vector $\boldsymbol{\beta}$ be the corresponding coefficients.

$$\begin{aligned} z_i \mid z_i > 0 &\sim \text{ZTP}(\lambda_i) \\ \log(\lambda_i) &= \mathbf{x}_i' \boldsymbol{\beta} \\ \boldsymbol{\beta} &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \end{aligned}$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}$):

$$\begin{aligned} [\boldsymbol{\beta} \mid \cdot] &\propto \prod_{i=1}^{n} [z_i \mid \boldsymbol{\beta}]\, [\boldsymbol{\beta}] \\ &\propto \prod_{i=1}^{n} \text{ZTP}(z_i \mid \mathbf{x}_i' \boldsymbol{\beta})\, \mathcal{N}\left(\boldsymbol{\beta} \mid \mathbf{0}, \sigma_\beta^2 \mathbf{I}\right). \end{aligned}$$

The update for $\boldsymbol{\beta}$ proceeds using Metropolis-Hastings.