Poisson Generalized Linear Mixed Model for Zero-truncated Count Data

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Description

A generalized linear mixed model with varying-coefficients for zero-truncated, grouped count data.

Implementation

The file zt.poisson.varying.coef.sim.R simulates data according to the model statement presented below, and zt.poisson.varying.coef.mcmc.R contains the MCMC algorithm for model fitting.

Derivation of zero-truncated Poisson distribution

The probability mass function of the (non-truncated) Poisson distribution is:

$$[z] = \frac{\lambda^z \exp(-\lambda)}{z!}. \tag{1}$$

It follows that the probability that $z = 0$ is

$$\begin{aligned} [z \mid z = 0] &= \frac{\lambda^0 \exp(-\lambda)}{0!} && (2) \\ &= \exp(-\lambda), && (3) \end{aligned}$$

and thus the probability that $z > 0$ is $1 - [z \mid z = 0] = 1 - \exp(-\lambda)$. We arrive at the density function of the zero-truncated Poisson distribution by excluding the probability that $z = 0$ from the standard Poisson distribution (Eq. 1). This is accomplished by dividing Eq. 1 by $[z \mid z = 0]$:

$$[z \mid z > 0] = \frac{\lambda^z \exp(-\lambda)}{(1 - \exp(-\lambda))\, z!}. \tag{4}$$

We abbreviate the probability mass function for the zero-truncated Poisson distribution as $\text{ZTP}(\lambda_i)$.

Model statement

Let z_{ij}, for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, denote observed, non-zero count data (i.e., z_{ij} are integers greater than 0), where the index i denotes replicate observations within group j, and n_j is the number of observations in group j. Furthermore, let $\mathbf{x}_{ij}$ be a vector of p covariates (including the intercept) associated with z_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$\begin{aligned} z_{ij} \mid z_{ij} > 0 &\sim \text{ZTP}(\lambda_{ij}) \\ \log(\lambda_{ij}) &= \mathbf{x}'_{ij}\boldsymbol{\beta}_j \\ \boldsymbol{\beta}_j &\sim \mathcal{N}(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \\ \boldsymbol{\mu}_\beta &\sim \mathcal{N}(\mathbf{0}, \sigma^2_\beta \mathbf{I}) \\ \boldsymbol{\Sigma}^{-1} &\sim \text{Wish}(\mathbf{S}_0^{-1}, \nu) \end{aligned}$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}_j$):

$$
\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \prod_{i=1}^{n_j} \left[z_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j\right] \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{i=1}^{n_j} \text{ZTP}\left(z_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j\right) \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}
$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2 \mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J} \left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)' \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J} \boldsymbol{\beta}'_j \boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}'_\beta \left(J\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}'_\beta \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J} \boldsymbol{\beta}'_j \boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}'_\beta \left(J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J} \boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$
\begin{aligned}
\left[\boldsymbol{\Sigma} \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \text{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J}{2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^{J} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
&\quad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \operatorname{tr}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\right)+\operatorname{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \operatorname{tr}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\right)+\operatorname{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right) \boldsymbol{\Sigma}^{-1}+\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right) \boldsymbol{\Sigma}^{-1}\right]\right\} \\
&= \quad \operatorname{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right)^{-1}, J+\nu\right).
\end{aligned}
$$