

Description

A zero-truncated negative binomial model for count data collected in three hierarchical levels (data, group, and population levels), with varying coefficients at the group level.

Implementation

The file `zt.nb.varying.coef.sim.3.R` simulates data according to the model statement presented below, and `zt.nb.varying.coef.mcmc.3.R` contains the MCMC algorithm for model fitting.

Derivation of zero-truncated negative binomial distribution

The probability mass function of the (non-truncated) negative binomial distribution is:

$$[z] = \left(\frac{\Gamma(z + \alpha)}{\Gamma(\alpha) \Gamma(z + 1)} \right) \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha \left(1 - \frac{\alpha}{\alpha + \lambda} \right)^z \quad (1)$$

It follows that the probability that $z = 0$ is

$$[z | z = 0] = \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha) \Gamma(1)} \right) \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha \left(1 - \frac{\alpha}{\alpha + \lambda} \right)^0 \quad (2)$$

$$= \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha, \quad (3)$$

and thus the probability that $z > 0$ is $1 - [z | z = 0] = 1 - \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha$. We arrive at the density function of the zero-truncated negative binomial distribution by excluding the probability that $z = 0$ from the standard negative binomial distribution (Eq. 1). This is accomplished by dividing Eq. 1 by $[z | z = 0]$:

$$[z | z > 0] = \left(\frac{\Gamma(z + \alpha)}{\Gamma(\alpha) \Gamma(z + 1)} \right) \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha \left(1 - \frac{\alpha}{\alpha + \lambda} \right)^z \left(1 - \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha \right)^{-1} \quad (4)$$

$$= \text{NB}(z | \lambda, \alpha) \left(1 - \left(\frac{\alpha}{\alpha + \lambda} \right)^\alpha \right)^{-1}. \quad (5)$$

We abbreviate the density function for the zero-truncated negative binomial distribution as $\text{ZTNB}(\lambda, \alpha)$.

Model statement

Let z_{ij} , for $i = 1, \dots, n_j$ and $j = 1, \dots, J$, denote observed, non-zero count data (i.e., z_{ij} are integers greater than 0), where the index i denotes replicate observations within group j , and n_j is the number of observations in group j . Furthermore, let \mathbf{x}_{ij} be a vector of p covariates (including the intercept) associated with z_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j .

$$\begin{aligned} z_{ij} | z_{ij} > 0 &\sim \text{ZTNB}(\lambda_{ij}, \alpha) \\ \log(\lambda_{ij}) &= \mathbf{x}_{ij}' \boldsymbol{\beta}_j \\ \boldsymbol{\beta}_j &\sim \mathcal{N}(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \\ \boldsymbol{\mu}_\beta &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\ \boldsymbol{\Sigma}^{-1} &\sim \text{Wish}(\mathbf{S}_0^{-1}, \nu) \\ \alpha &\sim \text{Gamma}(a, b) \end{aligned}$$

Full conditional distributions

Regression coefficients (β_j):

$$\begin{aligned} [\beta_j | \cdot] &\propto \prod_{i=1}^{n_j} [z_{ij} | \mathbf{x}'_{ij}\beta_j, \alpha] [\beta_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] \\ &\propto \prod_{i=1}^{n_j} \text{ZTNB}(z_{ij} | \mathbf{x}'_{ij}\beta_j, \alpha) \mathcal{N}(\beta_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}). \end{aligned}$$

The update for β_j proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$\begin{aligned} [\boldsymbol{\mu}_\beta | \cdot] &\propto \prod_{j=1}^J [\beta_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] [\boldsymbol{\mu}_\beta | \mathbf{0}, \sigma_\beta^2 \mathbf{I}] \\ &\propto \prod_{j=1}^J \mathcal{N}(\beta_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \mathcal{N}(\boldsymbol{\mu}_\beta | \mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\ &\propto \exp \left\{ \sum_{j=1}^J \left(-\frac{1}{2} (\beta_j - \boldsymbol{\mu}_\beta)' \boldsymbol{\Sigma}^{-1} (\beta_j - \boldsymbol{\mu}_\beta) \right) \right\} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\boldsymbol{\mu}_\beta - \mathbf{0})' (\sigma_\beta^2 \mathbf{I})^{-1} (\boldsymbol{\mu}_\beta - \mathbf{0}) \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \left(-2 \left(\sum_{j=1}^J \beta_j' \boldsymbol{\Sigma}^{-1} \right) \boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta' (J \boldsymbol{\Sigma}^{-1}) \boldsymbol{\mu}_\beta \right) \right\} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\boldsymbol{\mu}_\beta' (\sigma_\beta^2 \mathbf{I})^{-1} \boldsymbol{\mu}_\beta) \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \left(-2 \left(\sum_{j=1}^J \beta_j' \boldsymbol{\Sigma}^{-1} \right) \boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta' (J \boldsymbol{\Sigma}^{-1} + (\sigma_\beta^2 \mathbf{I})^{-1}) \boldsymbol{\mu}_\beta \right) \right\} \\ &= \mathcal{N}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1}), \end{aligned}$$

where $\mathbf{A} = J \boldsymbol{\Sigma}^{-1} + (\sigma_\beta^2 \mathbf{I})^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}' \boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^J \beta_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$\begin{aligned} [\boldsymbol{\Sigma} | \cdot] &\propto \prod_{j=1}^J [\beta_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] [\boldsymbol{\Sigma} | \mathbf{S}_0, \nu] \\ &\propto \prod_{j=1}^J \mathcal{N}(\beta_j | \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \text{Wish}(\boldsymbol{\Sigma} | \mathbf{S}_0, \nu) \\ &\propto |\boldsymbol{\Sigma}|^{-\frac{J}{2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^J (\beta_j - \boldsymbol{\mu}_\beta)' \boldsymbol{\Sigma}^{-1} (\beta_j - \boldsymbol{\mu}_\beta) \right\} \\ &\quad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \text{tr}(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}) \right\} \end{aligned}$$

$$\begin{aligned}
&\propto |\mathbf{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[\sum_{j=1}^J \text{tr} \left((\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta) \right) + \text{tr} (\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}) \right] \right\} \\
&\propto |\mathbf{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[\sum_{j=1}^J \text{tr} \left((\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta) (\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta)' \boldsymbol{\Sigma}^{-1} \right) + \text{tr} (\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}) \right] \right\} \\
&\propto |\mathbf{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[\text{tr} \left(\sum_{j=1}^J \left((\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta) (\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta)' \right) \boldsymbol{\Sigma}^{-1} + \mathbf{S}_0 \boldsymbol{\Sigma}^{-1} \right) \right] \right\} \\
&\propto |\mathbf{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp \left\{ -\frac{1}{2} \left[\text{tr} \left(\sum_{j=1}^J \left((\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta) (\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta)' \right) + \mathbf{S}_0 \right) \boldsymbol{\Sigma}^{-1} \right] \right\} \\
&= \text{Wish} \left(\left(\sum_{j=1}^J \left((\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta) (\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta)' \right) + \mathbf{S}_0 \right)^{-1}, J + \nu \right).
\end{aligned}$$

Dispersion (i.e., size) parameter (α):

$$\begin{aligned}
[\alpha \mid \cdot] &\propto \prod_{j=1}^J \prod_{i=1}^{n_j} [z_{ij} \mid \boldsymbol{\beta}_j, \alpha] [\alpha] \\
&\propto \prod_{j=1}^J \prod_{i=1}^{n_j} \text{ZTNB}(z_{ij} \mid \mathbf{x}_{ij}' \boldsymbol{\beta}_j, \alpha) \text{Gamma}(\alpha \mid a, b).
\end{aligned}$$

The update for α proceeds using Metropolis-Hastings.