Three-Level Hierarchical Zero-Inflated Poisson Model for Count Data

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Description

A zero-inflated Poisson model for data collected in three hierarchical levels (data, group, and population levels), with varying coefficients at the group level.

Implementation

The file zi.poisson.varying.coef.3.sim.R simulates data according to the model statement presented below, and zi.poisson.varying.coef.3.mcmc.R contains the MCMC algorithm for model fitting.

Model statement

Let z_{ij}, for $i = 1, \ldots, N$ and $j = 1, \ldots, n_i$, denote observed count data (i.e., z_{ij} are integers greater than or equal to 0), where the index j denotes replicate observations (level-1 units) within group i (level-2 units), and n_i is the number of observations in group i. Furthermore, let $\mathbf{x}_{ij}$ be a vector of p covariates (including the intercept) associated with y_{ij} and $\boldsymbol{\beta}_i$ be the corresponding vector of coefficients for group i.

$$
\begin{aligned}
y_{ij} &\sim \begin{cases} \text{Pois}\left(\lambda_{ij}\right), & z_{ij} = 1 \\ 0, & z_{ij} = 0 \end{cases} \\
z_{ij} &\sim \text{Bern}\left(p_i\right) \\
\log\left(\lambda_{ij}\right) &= \mathbf{x}'_{ij}\boldsymbol{\beta}_i \\
\boldsymbol{\beta}_i &\sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2_\beta \mathbf{I}\right) \\
\boldsymbol{\Sigma}^{-1} &\sim \text{Wish}\left(\mathbf{S}_0^{-1}, \nu\right) \\
p_i &\sim \text{Beta}\left(\alpha_1, \alpha_2\right)
\end{aligned}
$$

Full conditional distributions

Latent mixture component indicator variable (z_{ij}):

$$
\begin{aligned}
[z_{ij} \mid \cdot] &\propto [y_{ij} \mid \lambda_{ij}, z_{ij}]\,[z_{ij} \mid p_i] \\
&\propto \text{Pois}\left(y_{ij} \mid \lambda_{ij}\right)^{z_{ij}} 1_{\{y_{ij}=0\}}^{1-z_{ij}} p_i^{z_{ij}} \left(1 - p_i\right)^{1-z_{ij}} \\
&\propto \left(\frac{\lambda_{ij}^{y_{ij}} \exp\left(-\lambda_{ij}\right)}{y_{ij}!}\right)^{z_{ij}} p_i^{z_{ij}} \left(1 - p_i\right)^{1-z_{ij}} \\
&\propto \left(\exp\left(-\lambda_{ij}\right)\right)^{z_{ij}} p_i^{z_{ij}} \left(1 - p_i\right)^{1-z_{ij}} \\
&\propto \left(p_i \times \exp\left(-\lambda_{ij}\right)\right)^{z_{ij}} \left(1 - p_i\right)^{1-z_{ij}} \\
&= \text{Bern}\left(\tilde{p}\right),
\end{aligned}
$$

where

$$
\tilde{p} = \frac{p_i \times \exp\left(-\lambda_{ij}\right)}{p_i \times \exp\left(-\lambda_{ij}\right) + 1 - p_i}.
$$

Note that z_{ij} is only updated for instances where $y_{ij} = 0$ ($z_{ij} = 1$ when $y_{ij} > 0$).

Probability associated with the mixture component indicator variables (p_i):

$$\begin{aligned}
[p_i \mid \cdot] &\propto \prod_{j=1}^{n_i} [z_{ij} \mid p_i]\,[p_i] \\
&\propto \prod_{j=1}^{n_i} p_i^{z_{ij}}\,(1-p_i)^{1-z_{ij}}\; p_i^{\alpha_1-1}\,(1-p_i)^{\alpha_2-1} \\
&\propto p_i^{\sum_{j=1}^{n_i} z_{ij}}\,(1-p_i)^{n_i-\sum_{j=1}^{n_i} z_{ij}}\; p_i^{\alpha_1-1}\,(1-p_i)^{\alpha_2-1} \\
&= \text{Beta}\left(\sum_{j=1}^{n_i} z_{ij}+\alpha_1, n_i-\sum_{j=1}^{n_i} z_{ij}+\alpha_2\right)
\end{aligned}$$

Regression coefficients ($\boldsymbol{\beta}_i$):

$$\begin{aligned}
[\boldsymbol{\beta}_i \mid \cdot] &\propto \prod_{j=1}^{n_i} \left[y_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_i, z_{ij}\right]\left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{j=1}^{n_i} \text{Pois}\left(y_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_i\right)^{z_{ij}} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}$$

The update for $\boldsymbol{\beta}_i$ proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{i=1}^{N} \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right]\left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{i=1}^{N} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right)\mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{i=1}^{N}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_i-\boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta-\mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{N}\boldsymbol{\beta}'_i\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta+\boldsymbol{\mu}'_\beta\left(N\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}'_\beta\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{i=1}^{N}\boldsymbol{\beta}'_i\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta+\boldsymbol{\mu}'_\beta\left(N\boldsymbol{\Sigma}^{-1}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where $\mathbf{A} = N\boldsymbol{\Sigma}^{-1}+\left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{i=1}^{N}\boldsymbol{\beta}_i$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$
\begin{aligned}
[\boldsymbol{\Sigma} \mid \cdot] &\propto \prod_{i=1}^{N} \left[\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] [\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu] \\
&\propto \prod_{i=1}^{N} \mathcal{N}\left(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \text{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{N}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}\text{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\right) + \text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N}\text{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\right) + \text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right)\boldsymbol{\Sigma}^{-1} + \mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{N+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\text{tr}\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)\boldsymbol{\Sigma}^{-1}\right]\right\} \\
&= \text{Wish}\left(\left(\sum_{i=1}^{N}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta\right)'\right) + \mathbf{S}_0\right)^{-1}, N+\nu\right).
\end{aligned}
$$