Three-Level Hierarchical Zero-Truncated Poisson Model for Count Data

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Description

A zero-truncated Poisson model for data collected in three hierarchical levels (data, group, and population levels), with varying coefficients at the group level.

Implementation

The file zt.poisson.varying.coef.sim.3.R simulates data according to the model statement presented below, and zt.poisson.varying.coef.3.mcmc.R contains the MCMC algorithm for model fitting.

Derivation of zero-truncated Poisson distribution

The probability mass function of the (non-truncated) Poisson distribution is:

$$[z] = \frac{\lambda^z \exp(-\lambda)}{z!}. \tag{1}$$

It follows that the probability that $z = 0$ is

$$\begin{aligned} [z \mid z = 0] &= \frac{\lambda^0 \exp(-\lambda)}{0!} \qquad (2) \\ &= \exp(-\lambda), \qquad (3) \end{aligned}$$

and thus the probability that $z > 0$ is $1 - [z \mid z = 0] = 1 - \exp(-\lambda)$. We arrive at the density function of the zero-truncated Poisson distribution by excluding the probability that $z = 0$ from the standard Poisson distribution (Eq. 1). This is accomplished by dividing Eq. 1 by $[z \mid z = 0]$:

$$[z \mid z > 0] = \frac{\lambda^z \exp(-\lambda)}{(1 - \exp(-\lambda))\, z!}. \tag{4}$$

We abbreviate the probability mass function for the zero-truncated Poisson distribution as $\text{ZTP}(\lambda_i)$.

Model statement

Let z_{ij}, for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$, denote observed, non-zero count data (i.e., z_{ij} are integers greater than 0), where the index i denotes replicate observations within group j, and n_j is the number of observations in group j. Furthermore, let $\mathbf{x}_{ij}$ be a vector of p covariates (including the intercept) associated with z_{ij} and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group j.

$$\begin{aligned} z_{ij} \mid z_{ij} > 0 &\sim \text{ZTP}(\lambda_{ij}) \\ \log(\lambda_{ij}) &= \mathbf{x}_{ij}' \boldsymbol{\beta}_j \\ \boldsymbol{\beta}_j &\sim \mathcal{N}(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}) \\ \boldsymbol{\mu}_\beta &\sim \mathcal{N}(\mathbf{0}, \sigma_\beta^2 \mathbf{I}) \\ \boldsymbol{\Sigma}^{-1} &\sim \text{Wish}(\mathbf{S}_0^{-1}, \nu) \end{aligned}$$

Full conditional distributions

Regression coefficients ($\boldsymbol{\beta}_j$):

$$\begin{aligned} [\boldsymbol{\beta}_j \mid \cdot] &\propto \prod_{i=1}^{n_j} [z_{ij} \mid \mathbf{x}_{ij}'\boldsymbol{\beta}_j] [\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] \\ &\propto \prod_{i=1}^{n_j} \text{ZTP}(z_{ij} \mid \mathbf{x}_{ij}'\boldsymbol{\beta}_j)\, \mathcal{N}(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}). \end{aligned}$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta \mid \cdot] \quad &\propto \quad \prod_{j=1}^{J} [\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] \, [\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2] \\
&\propto \quad \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2 \mathbf{I}\right) \\
&\propto \quad \exp\left\{ \sum_{j=1}^{J} \left(-\frac{1}{2} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \right) \right\} \\
&\qquad \times \exp\left\{ -\frac{1}{2} \left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)' \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \left(\boldsymbol{\mu}_\beta - \mathbf{0}\right) \right\} \\
&\propto \quad \exp\left\{ -\frac{1}{2} \left(-2 \left(\sum_{j=1}^{J} \boldsymbol{\beta}_j' \boldsymbol{\Sigma}^{-1} \right) \boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta' \left(J \boldsymbol{\Sigma}^{-1}\right) \boldsymbol{\mu}_\beta \right) \right\} \\
&\qquad \times \exp\left\{ -\frac{1}{2} \left(\boldsymbol{\mu}_\beta' \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \boldsymbol{\mu}_\beta \right) \right\} \\
&\propto \quad \exp\left\{ -\frac{1}{2} \left(-2 \left(\sum_{j=1}^{J} \boldsymbol{\beta}_j' \boldsymbol{\Sigma}^{-1} \right) \boldsymbol{\mu}_\beta + \boldsymbol{\mu}_\beta' \left(J \boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1} \right) \boldsymbol{\mu}_\beta \right) \right\} \\
&= \quad \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}
$$

where $\mathbf{A} = J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2 \mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J} \boldsymbol{\beta}_j$.

Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):

$$
\begin{aligned}
[\boldsymbol{\Sigma} \mid \cdot] \quad &\propto \quad \prod_{j=1}^{J} [\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}] \, [\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu] \\
&\propto \quad \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \text{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J}{2}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{J} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}} \exp\left\{ -\frac{1}{2} \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right) \right\} \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{ -\frac{1}{2} \left[\sum_{j=1}^{J} \text{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right) \right] \right\} \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{ -\frac{1}{2} \left[\sum_{j=1}^{J} \text{tr}\left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1} \right) + \text{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right) \right] \right\} \\
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{ -\frac{1}{2} \left[\text{tr}\left(\sum_{j=1}^{J} \left(\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right) \left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \right) \boldsymbol{\Sigma}^{-1} + \mathbf{S}_0 \boldsymbol{\Sigma}^{-1} \right) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\propto \quad |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right)\boldsymbol{\Sigma}^{-1}\right]\right\} \\
&= \quad \operatorname{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right)^{-1}, J+\nu\right).
\end{aligned}
$$