# Negative Binomial Generalized Linear Mixed Model for Zero-truncated Count Data

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### Description

A generalized linear mixed model with varying-coefficients for zero-truncated, grouped count data.

### Implementation

The file zt.nb.varying.coef.sim.R simulates data according to the model statement presented below, and zt.nb.varying.coef.mcmc.R contains the MCMC algorithm for model fitting.

### Derivation of zero-truncated negative binomial distribution

The probability mass function of the (non-truncated) negative binomial distribution is:

$$[z] = \left(\frac{\Gamma(z+\alpha)}{\Gamma(a)\,\Gamma(z+1)}\right)\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\left(1-\frac{\alpha}{\lambda+\alpha}\right)^{z} \tag{1}$$

It follows that the probability that $z=0$ is

$$\begin{aligned}
[z \mid z=0] &= \left(\frac{\Gamma(\alpha)}{\Gamma(a)\,\Gamma(1)}\right)\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\left(1-\frac{\alpha}{\lambda+\alpha}\right)^{0} && (2)\\
&= \left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}, && (3)
\end{aligned}$$

and thus the probability that $z>0$ is $1-[z \mid z=0] = 1-\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}$. We arrive at the density function of the zero-truncated negative binomial distribution by excluding the probability that $z=0$ from the standard negative binomial distribution (Eq. 1). This is accomplished by dividing Eq. 1 by $[z \mid z=0]$:

$$\begin{aligned}
[z \mid z>0] &= \left(\frac{\Gamma(z+\alpha)}{\Gamma(a)\,\Gamma(z+1)}\right)\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\left(1-\frac{\alpha}{\lambda+\alpha}\right)^{z}\left(1-\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\right)^{-1} && (4)\\
&= \mathrm{NB}(z \mid \lambda,\alpha)\left(1-\left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha}\right)^{-1}. && (5)
\end{aligned}$$

We abbreviate the density function for the zero-truncated negative binomial distribution as $\mathrm{ZTNB}(\lambda,\alpha)$.

### Model statement

Let $z_{ij}$, for $i=1,\ldots,n_j$ and $j=1,\ldots,J$, denote observed, non-zero count data (i.e., $z_{ij}$ are integers greater than 0), where the index $i$ denotes replicate observations within group $j$, and $n_j$ is the number of observations in group $j$. Furthermore, let $\mathbf{x}_{ij}$ be a vector of $p$ covariates (including the intercept) associated with $z_{ij}$ and $\boldsymbol{\beta}_j$ be the corresponding vector of coefficients for group $j$.

$$\begin{aligned}
z_{ij} \mid z_{ij}>0 &\sim \mathrm{ZTNB}(\lambda_{ij},\alpha)\\
\log(\lambda_{ij}) &= \mathbf{x}_{ij}'\boldsymbol{\beta}_j\\
\boldsymbol{\beta}_j &\sim \mathcal{N}(\boldsymbol{\mu}_\beta,\boldsymbol{\Sigma})\\
\boldsymbol{\mu}_\beta &\sim \mathcal{N}(\mathbf{0},\sigma_\beta^2\mathbf{I})\\
\boldsymbol{\Sigma}^{-1} &\sim \mathrm{Wish}(\mathbf{S}_0^{-1},\nu)\\
\alpha &\sim \mathrm{Gamma}(a,b)
\end{aligned}$$

**Full conditional distributions**

*Regression coefficients ($\boldsymbol{\beta}_j$):*

$$\begin{aligned}
\left[\boldsymbol{\beta}_j \mid \cdot\right] &\propto \prod_{i=1}^{n_j} \left[z_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j, \alpha\right] \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \\
&\propto \prod_{i=1}^{n_j} \text{ZTNB}\left(z_{ij} \mid \mathbf{x}'_{ij}\boldsymbol{\beta}_j, \alpha\right) \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right).
\end{aligned}$$

The update for $\boldsymbol{\beta}_j$ proceeds using Metropolis-Hastings.

*Mean of regression coefficients ($\boldsymbol{\mu}_\beta$):*

$$\begin{aligned}
\left[\boldsymbol{\mu}_\beta \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \mathcal{N}\left(\boldsymbol{\mu}_\beta \mid \mathbf{0}, \sigma_\beta^2\mathbf{I}\right) \\
&\propto \exp\left\{\sum_{j=1}^{J}\left(-\frac{1}{2}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)'\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\left(\boldsymbol{\mu}_\beta - \mathbf{0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}'_j\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}'_\beta\left(J\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}'_\beta\left(\sigma_\beta^2\mathbf{I}\right)^{-1}\boldsymbol{\mu}_\beta\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(-2\left(\sum_{j=1}^{J}\boldsymbol{\beta}'_j\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}_\beta + \boldsymbol{\mu}'_\beta\left(J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}\right)\boldsymbol{\mu}_\beta\right)\right\} \\
&= \mathcal{N}(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1}),
\end{aligned}$$

where $\mathbf{A} = J\boldsymbol{\Sigma}^{-1} + \left(\sigma_\beta^2\mathbf{I}\right)^{-1}$ and $\mathbf{b}' = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\beta}$ is the vector sum $\sum_{j=1}^{J}\boldsymbol{\beta}_j$.

*Variance-covariance of regression coefficients ($\boldsymbol{\Sigma}$):*

$$\begin{aligned}
\left[\boldsymbol{\Sigma} \mid \cdot\right] &\propto \prod_{j=1}^{J} \left[\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right] \left[\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right] \\
&\propto \prod_{j=1}^{J} \mathcal{N}\left(\boldsymbol{\beta}_j \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}\right) \text{Wish}\left(\boldsymbol{\Sigma} \mid \mathbf{S}_0, \nu\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J}{2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^{J}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j - \boldsymbol{\mu}_\beta\right)\right\} \\
&\qquad \times |\mathbf{S}_0|^{-\frac{\nu}{2}} |\boldsymbol{\Sigma}|^{-\frac{\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\mathbf{S}_0\boldsymbol{\Sigma}^{-1}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \operatorname{tr}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\right)+\operatorname{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{J} \operatorname{tr}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)' \boldsymbol{\Sigma}^{-1}\right)+\operatorname{tr}\left(\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right) \boldsymbol{\Sigma}^{-1}+\mathbf{S}_0 \boldsymbol{\Sigma}^{-1}\right)\right]\right\} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{J+\nu-p-1}{2}} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right) \boldsymbol{\Sigma}^{-1}\right]\right\} \\
&= \operatorname{Wish}\left(\left(\sum_{j=1}^{J}\left(\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)\left(\boldsymbol{\beta}_j-\boldsymbol{\mu}_\beta\right)'\right)+\mathbf{S}_0\right)^{-1}, J+\nu\right).
\end{aligned}
$$

*Dispersion (i.e., size) parameter ($\alpha$):*

$$
\begin{aligned}
[\alpha \mid \cdot] &\propto \prod_{j=1}^{J} \prod_{i=1}^{n_j}\left[z_{ij} \mid \boldsymbol{\beta}_j, \alpha\right][\alpha] \\
&\propto \prod_{j=1}^{J} \prod_{i=1}^{n_j} \operatorname{ZTNB}\left(z_{ij} \mid \mathbf{x}_{ij}' \boldsymbol{\beta}_j, \alpha\right) \operatorname{Gamma}(\alpha \mid a, b).
\end{aligned}
$$

The update for $\alpha$ proceeds using Metropolis-Hastings.