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CS3343 Section 01
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CS3343 Analysis of Algorithms Fall 2017

Homework 1

Due 9/8/17 before 11:59pm

Justify all of your answers with comments/text in order to receive full credit. Completing the assignment in Latex will earn you extra credit on Midterm 1.

1. Sorting (8 points)

Consider the sorting algorithm below which sorts the array $A[1 \dots n]$ into increasing order by repeatedly bubbling the minimum element of the remaining array to the left.

Algorithm 1 mysterysort(int $A[1 \dots n]$)

$i = 1$;
while $i <= n$ **do**
    //(I) The elements in $A[1 \dots (i-1)]$ of the array are in sorted order and are all smaller than the elements in $A[i \dots n]$
    $k = n$;
    while $(k >= i + 1)$ **do**
        //Inner while loop moves the smallest element in $A[i \dots n]$ to $A[i]$
        if $A[k-1] > A[k]$ **then**
            swap $A[k]$ with $A[k-1]$
        end if
        $k - -$;
    end while
    $i + +$;
end while

(1) (2 points) Consider running the above sort on the array $[5, 3, 1, 4, 2]$. Show the sequence of changes which the algorithm makes to the array.
$[5, 3, 1, \mathbf{4}, \mathbf{2}]$ → Swap
$[5, 3, \mathbf{1}, \mathbf{2}, 4]$ → Continue
$[5, \mathbf{3}, \mathbf{1}, 2, 4]$ → Swap
$[\mathbf{5}, \mathbf{1}, 3, 2, 4]$ → Swap
$[1, 5, 3, 2, 4]$ → End of first iteration
$[1, 5, 3, \mathbf{2}, \mathbf{4}]$ → Continue
$[1, 5, \mathbf{3}, \mathbf{2}, 4]$ → Swap
$[1, \mathbf{5}, \mathbf{2}, 3, 4]$ → Swap
$[1, 2, 5, 3, 4]$ → End of second iteration

$[1, 2, 5, \mathbf{3}, \mathbf{4}] \rightarrow$ Continue
$[1, 2, \mathbf{5}, \mathbf{3}, 4] \rightarrow$ Swap
$[1, 2, 3, 5, 4] \rightarrow$ End of third iteration
$[1, 2, 3, \mathbf{5}, \mathbf{4}] \rightarrow$ Swap
$[1, 2, 3, 4, 5] \rightarrow$ Sorted

(2) (4 points) Use induction to prove the loop invariant (I) is true and then use this to prove the correctness of the algorithm. Specifically complete the following:

(a) **Base case:**

Solution:
$i = 1$
$A[1 \ldots (i-1)] = A[1 \ldots 0] = [\,]$
$A[i \ldots n] = [5, 3, 1, 4, 2]$
Base case holds true since $A[1 \ldots (i-1)] = [\,]$ is sorted and
$A[1 \ldots (i-1)] = [\,] < A[i \ldots n] = [5, 3, 1, 4, 2]$

(b) **Inductive step:**
(**Hint:** you can assume that the inner loop moves the smallest element in $A[i \ldots n]$ to $A[i]$)

Solution:
Assume: The elements in $A[1 \ldots (i_{old} - 1)]$ of the array are in sorted order and are all smaller than the elements in $A[i \ldots n]$

Prove: The elements in $A[1 \ldots (i_{new} - 1)]$ of the array are in sorted order and are all smaller than the elements in $A[i \ldots n]$

$i_{new} = i_{old} + 1$
$A[1 \ldots (i_{new} - 1)]$
$= A[1 \ldots (i_{old} + 1 - 1)]$
$= A[1 \ldots (i_{old})]$

Therefore, because the inner loop moves the smallest element in $A[i_{old} \ldots n]$ to $A[i_{old}], A[1 \ldots i_{old}]$ will always be sorted and smaller than elements in $A[i \ldots n]$. Because the loop invariant states the elements in the sorted part of the array are less than the elements in the unsorted part, i_{new} will always be greater than i_{old}.

(c) **Termination step:**

Solution:
The loop terminates when $i <= n$ is false. Therefore $i > n$ when we fall out of the loop. $i = n + 1$ when we leave the loop. If we plug i into the loop invariant $A[1 \ldots (i-1)]$ which is sorted we get $A[1 \ldots (n+1)-1] = A[1 \ldots n]$ which is sorted by the loop invariant. Thus, the algorithm is correct.

(3) (2 points) Give the best-case and worst-case runtimes of this sort in asymptotic (i.e., O) notation.

2. Asymptotic Notation (8 points)
Show the following using the definitions of O, Ω, and Θ.

(1) (2 points) $2n^3 + n^2 + 4 \in \Theta(n^3)$
Solution:

- Show $O : 2n^3 + n^2 + 4 <= cn^3$
  $2n^3 + n^2 + 4 <= 2n^3 + (\mathbf{n})n^2 + 4$ $\quad n >= 1$
  $3n^3 + 4 <= 3n^3 + (\mathbf{n^3})$ $\quad n >= \sqrt[3]{4}$
  $4n^3 <= cn^3$ $\quad c = 4, n >= \sqrt[3]{4}$

- Show $\Omega : 2n^3 + n^2 + 4 >= cn^3$
  $2n^3 + n^2 + 4(-\mathbf{4}) >= 2n^3 + n^2$
  $2n^3 + n^2(-\mathbf{n^2}) >= 2n^3$
  $2n^3 >= cn^3$ $\quad c = 2, n >= 0$

  Therefore $2n^3 + n^2 + 4 \in \Theta(n^3)$

(2) (2 points) $3n^4 - 9n^2 + 4n \in \Theta(n^4)$
(**Hint:** careful with the negative number)
Solution:

- Show $O : 3n^4 - 9n^2 + 4n <= cn^4$
  $3n^4 - 9n^2 + 4n <= 3n^4 - 9n^2 + (\mathbf{n^3})n$ $\quad n >= \sqrt[3]{4}$
  $4n^4 - 9n^2 <= 4n^4 - 9n^2(+\mathbf{9n^2})$
  $4n^4 <= cn^4$ $\quad c = 4, n >= \sqrt[3]{4}$

- Show $\Omega : 3n^4 - 9n^2 + 4n >= cn^4$
  $3n^4 - 9n^2 + 4n >= 3n^4 - 9n^2 + 4n(-\mathbf{4n})$
  $3n^4 - 9n^2 >= 3n^4 - (\mathbf{n^2})n^2$ $\quad n >= 3$
  $2n^4 >= cn^4$ $\quad c = 2, n >= 3$

  Therefore $3n^4 - 9n^2 + 4n \in \Theta(n^4)$

(3) (4 points) Suppose $f(n) \in O(g_1(n))$ and $f(n) \in O(g_2(n))$. Which of the following are true? Justify your answers using the definition of O. Give a counter example if it is false.

(a) $f(n) \in O(\ 5 * g_1(n) + 100\ )$

Solution:

Assume: $f(n) \in O(g_1(n)) \Leftrightarrow f(n) <= c_1 g_1(n)$ for all $n >= n_1$

Prove: $f(n) \in O(5g_1(n) + 100) \Leftrightarrow f(n) <= c_3(5g_1(n) + 100)$

$\rightarrow LHS = f(n) <= c_1 g_1(n)$

$\rightarrow c_1 g_1(n) <= c_1 5 g_1(n)$

$\rightarrow c_1 5 g_1(n) <= c_1 5 g_1(n) + 100 c_1$

$\rightarrow c_1 5 g_1(n) + 100 c_1 <= c_1(5g_1(n) + 100) = RHS$

$\boxed{c_1(5g_1(n) + 100)}$ $\boxed{c_3 = c_1, n_3 = n_1}$

Therefore $f(n) \in O(\ 5 * g_1(n) + 100\ )$

(b) $f(n) \in O(\ g_1(n) + g_2(n)\ )$

Solution:

Assume: $f(n) \in O(g_1(n)$ and $f(n) \in O(g_2(n)) \Leftrightarrow$

$f(n) <= c_1 g_1(n)$ for all $n >= n_1$ and

$f(n) <= c_2 g_2(n)$ for all $n >= n_2$

Prove: $f(n) \in O(g_1(n) + g_2(n)) \Leftrightarrow f(n) <= c_3(g_1(n) + g_2(n))$

$\rightarrow LHS = f(n) <= c_1 g_1(n)$

$\rightarrow c_1 g_1(n) <= c_1 g_1(n) + c_2 g_2(n)$

$\rightarrow c_1 g_1(n) + c_2 g_2(n) <= c_1 g_1(n) c_2 + c_2 g_2(n) c_1$

$\rightarrow c_1 g_1(n) c_2 + c_2 g_2(n) c_1 = c_1 c_2 (g_1(n) + g_2(n))$

$\rightarrow c_1 c_2 (g_1(n) + g_2(n)) <= c_3(g_1(n) + g_2(n)) = RHS$

$\rightarrow \boxed{c_3(g_1(n) + g_2(n))}$ $\boxed{c_3 = c_1 c_2, n_3 = \max(n_1, n_2)}$

Therefore $f(n) \in O(\ g_1(n) + g_2(n)\ )$

(c) $f(n) \in O(\ \frac{g_1(n)}{g_2(n)}\ )$

Solution:

False. Counterexample:

$f(n) = n$

$g_1(n) = n$

$g_2(n) = n^2$

$f(n) \in g_1(n)$ and $f(n) \in g_2(n)$

$$\boxed{f(n) \notin \frac{g_1(n)}{g_2(n)}}$$

(d) $f(n) \in O(\ \max(g_1(n), g_2(n))\ )$

Solution:

Assume: $f(n) \in O(g_1(n)$ and $f(n) \in O(g_2(n)) \Leftrightarrow$
$f(n) <= c_1 g_1(n)$ for all $n >= n_1$ and
$f(n) <= c_2 g_2(n)$ for all $n >= n_2$

Prove: $f(n) \in O(\ \max(g_1(n), g_2(n))\ ) \Leftrightarrow f(n) <= c_3(\ \max(g_1(n), g_2(n))\ )$

$\rightarrow LHS = f(n) <= c_1 g_1(n)$
$\rightarrow c_1 g_1(n) <= \max(c_1 g_1(n), c_2 g_2(n))$
$\rightarrow \max(c_1 g_1(n), c_2 g_2(n)) <= \max(c_1 g_1(n) c_2, c_2 g_2(n) c_1)$
$\rightarrow \max(c_1 g_1(n) c_2, c_2 g_2(n) c_1) <= c_1 c_2 * \max(g_1(n), g_2(n))$
$\rightarrow c_1 c_2 * \max(g_1(n), g_2(n)) <= c_3 * \max(g_1(n), g_2(n)) = RHS$
$\rightarrow \boxed{c_3(\ \max(g_1(n), g_2(n))\ )}$ $\qquad \boxed{c_3 = c_1 c_2, n_3 = \max(n_1, n_2)}$

Therefore $f(n) \in O(\ \max(g_1(n), g_2(n))\ )$

3. Summations (4 points)

Find the order of growth of the following sums.

(1) $\sum_{i=5}^{n} (4i + 1)$.

Solution:

$$\sum_{i=5}^{n} 4i + 1$$
$$= \sum_{i=5}^{n} 4i + \sum_{i=5}^{n} 1$$
$$= 4 \sum_{i=5}^{n} i + (n - 4)$$
$$= 4\frac{(n+5)(n-4)}{2} + (n - 4)$$
$$= 2(n^2 + n - 20) + (n - 4)$$
$$= 2n^2 + 3n - 44$$

Therefore $\sum_{i=5}^{n} (4i + 1) \in \Theta(n^2)$

(2) $\sum\limits_{i=0}^{\log_2(n)} 2^i$ (for simplicity you can assume n is a power of 2)

Solution:

$\sum\limits_{i=0}^{\log_2(n)} 2^i = 2^0 + 2^1 + 2^2 + \ldots + 2^{\log_2(n-1)} + 2^{\log_2(n)}$

$= 1 + 2 + 4 + 8 + \ldots + \frac{n}{2} + n$

$= 2n - 1$

Therefore $\sum\limits_{i=0}^{\log_2(n)} 2^i \in \Theta(n)$