ECE:5450 - Homework 2

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Problem 1

We know that

$$r = \begin{bmatrix} x \\ y \end{bmatrix}, \mu = E[r] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \sum = E\left[(r-\mu)(r-\mu)^T\right] = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$$

and that $\sum$ is diagonal because of the placement of 0, this means that x and y are independent. This means that we can split the value $E[x^2y]$ into

$$E[x^2y] = E[x^2] * E[y]$$

and solve them one at a time. For $E[x^2]$, we have x distributed as $P(x) = \mathcal{N}$:

$$E[x^2] = Var(x) + E[x]^2$$

We know that $E[x] = 1$ because we derive that from μ and that the $Var(x) = 1$ because the covariance matrix is diagonal and thus independent we can pull the 1 directly from it. So we get

$$E[x^2] = Var(x) + E[x]^2 = 1 + 1^2 = 2$$

For $E[y]$, we have y distributed as $P(y) = \mathcal{N}$ and we can see that

$$E[y] = 2$$

Thus,

$$E[x^2y] = E[x^2] * E[y] = 2 * 2 = \boxed{4}$$

Problem 2

a. Since $p(x)$ is exponential we just need to maximize $-\frac{1}{2}(x-\mu)^T \sum^{-1}(x-\mu)$. To do this we need $(x-\mu) = 0$. So this exponential then becomes:

$$-\frac{1}{2}(x-\mu)^T \sum^{-1}(x-\mu) = 0$$

$$(x-\mu)^T \sum^{-1}(x-\mu) = 0$$

$$(x-\mu)^T(x-\mu) = 0$$

$$(x - \mu) = 0$$

Thus we get $exp(0) = 1$ so,

$$x = \mu$$

TODO: come back to this part - unsure about this

b. part 2

c. part 3

Problem 3

1. We know $i = 1...N$ and the sample mean $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$
2. Then we can find the expected value using the sample mean $E[\hat{\mu}]$.
$$E[\hat{\mu}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i]$$
3. Since each x_i comes from a Gaussian distribution with mean μ we know $E[x_i] = \mu$ for all i thus,
$$E[\hat{\mu}] = \frac{1}{N}\sum_{i=1}^{N} \mu = \frac{1}{N} * N\mu = \mu$$
4. Thus $E[\hat{\mu}] = \mu$ proving $\hat{\mu}$ is an unbiased estimator of the population mean μ.

Problem 4

Problem statement: Sample Mean of Gaussian Random Variables

Solution:

1. We know $i = 1...N$ and the sample mean $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$
2. Then we can find the expected value using the sample mean $E[\hat{\mu}]$.
$$E[\hat{\mu}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i]$$
3. Since each x_i comes from a Gaussian distribution with mean μ we know $E[x_i] = \mu$ for all i thus,
$$E[\hat{\mu}] = \frac{1}{N}\sum_{i=1}^{N} \mu = \frac{1}{N} * N\mu = \mu$$
4. Thus $E[\hat{\mu}] = \mu$ proving $\hat{\mu}$ is an unbiased estimator of the population mean μ.