ECE:5450 - Homework 2

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Problem 1

We know that

$$r = \begin{bmatrix} x \\ y \end{bmatrix}, \mu = E[r] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \sum = E\left[(r-\mu)(r-\mu)^T\right] = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$$

and that $\sum$ is diagonal because of the placement of 0, this means that x and y are independent. This means that we can split the value $E[x^2y]$ into

$$E[x^2y] = E[x^2] * E[y]$$

and solve them one at a time. For $E[x^2]$, we have x distributed as $P(x) = \mathcal{N}$:

$$E[x^2] = Var(x) + E[x]^2$$

We know that $E[x] = 1$ because we derive that from μ and that the $Var(x) = 1$ because the covariance matrix is diagonal and thus independent we can pull the 1 directly from it. So we get

$$E[x^2] = Var(x) + E[x]^2 = 1 + 1^2 = 2$$

For $E[y]$, we have y distributed as $P(y) = \mathcal{N}$ and we can see that

$$E[y] = 2$$

Thus,

$$E[x^2y] = E[x^2] * E[y] = 2 * 2 = \boxed{4}$$

Problem 2

a. Since $p(x)$ is exponential we just need to maximize $-\frac{1}{2}(x-\mu)^T \sum^{-1}(x-\mu)$. To do this we need $(x-\mu) = 0$. So this exponential then becomes:

$$-\frac{1}{2}(x-\mu)^T \sum^{-1}(x-\mu) = 0$$

$$(x-\mu)^T \sum^{-1}(x-\mu) = 0$$

$$(x-\mu)^T(x-\mu) = 0$$

$$(x-\mu) = 0$$

Thus we get $exp(0) = 1$ so,

$$x = \mu$$

b. Since we know that $x \sim \mathcal{N}(\mu, \sum)$ we can use the second-moment formula which gives us

$$E[xx^T] = Cov(x) + E[x]E[x]^T$$

The $Cov(x)$ is given by $x \sim \mathcal{N}(\mu, \sum)$ so $Cov(x) = \sum$. As well as we know that the expectation of x is μ. Thus we get

$$E[xx^T] = \sum + \mu\mu^T$$

c. To solve this we need to separate the parts into 2 cases. For the case $m = n$, the expectation is the same as in part **b.** so that means we get

$$E[x_m x_n^T] = \sum + \mu\mu^T$$

For the case $m \neq n$, since they are independent the covariance is 0, which means we get

$$E[x_m x_n^T] = \mu\mu^T$$

For combining the two parts, we bring in the delta function, since we know in the case of $m = n$ the value is 1 and 0 otherwise, we can add it to the $\sum$ value. So by combining these parts we get

$$\boxed{E[x_m x_n^T] = \sum \delta(m - n) + \mu\mu^T}$$

Problem 3

We know that $\mathcal{L}\ (x, y, \lambda) = f(x, y) + \lambda g(x, y)$ and from the problem we know $f(x, y) = x^2 + 2y^2$ and $g(x, y) = y - 3x - 2$. Then we start with the min-max problem but we will flip it to start by solving for the min values first so we get

$$\underset{\lambda}{max}\ \underset{x,y}{min}\ \mathcal{L}(x, y, \lambda)$$

and now we first start solving

$$\underset{x,y}{min}\ x^2 + 2y^2 + \lambda(y - 3x - 2)$$

First we take the partial derivatives with respect to x and y.

$$D_{x,y}\mathcal{L}(x, y) = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x - 3\lambda \\ 4y + \lambda \end{pmatrix}$$

Then we solve for x and y

$$2x - 3\lambda = 0 \qquad\qquad 4y + \lambda = 0$$

$$2x = 3\lambda \qquad\qquad 4y = -\lambda$$

$$x = \frac{3}{2}\lambda \qquad\qquad y = \frac{-\lambda}{4}$$

Then we plug these back into the original equation, take its' derivative with respect to λ then solve for λ.

$$\mathcal{L}(x,y,\lambda) = (\frac{3}{2}\lambda)^2 + 2(\frac{-\lambda}{4})^2 + \lambda((\frac{-\lambda}{4}) - 3(\frac{3}{2}\lambda) - 2) = \frac{-19}{8}\lambda^2 - 2\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{-19}{4}\lambda - 2 = 0$$

$$\lambda = \frac{-8}{19}$$

Next we take this value and plug it back into the x and y we found earlier

$$x = \frac{3}{2} * \frac{-8}{19} = \frac{-12}{19}$$

$$y = \frac{-1}{4} * \frac{-8}{19} = \frac{2}{19}$$

and we get

$$\boxed{(x,y) = \left(\frac{-12}{19}, \frac{2}{19}\right)}$$

Problem 4

We know that $R(h,s) = 4hs$ and the constraint is $20h + 10s = 100$ so we get $\mathcal{L}\ (x,y,\lambda) = 4hs + \lambda(100 - 20h - 10s)$. Then we start with the min-max problem but we will flip it to start by solving for the min values first so we get

$$\underset{\lambda}{max}\ \underset{x,y}{min}\ \mathcal{L}(h,s,\lambda)$$

and now we first start solving

$$\underset{x,y}{min}\ 4hs + \lambda(100 - 20h - 10s)$$

First we take the partial derivatives with respect to h and s.

$$D_{h,s}\mathcal{L}(h,s) = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial h} \\ \frac{\partial \mathcal{L}}{\partial s} \end{pmatrix} = \begin{pmatrix} 4s - 20\lambda \\ 4h - 10\lambda \end{pmatrix}$$

Then we solve for h and s

$$4s - 20\lambda = 0 \qquad\qquad 4h - 10\lambda = 0$$

$$4s = 20\lambda \qquad\qquad 4h = 10\lambda$$

$$s = 5\lambda \qquad\qquad h = \frac{5\lambda}{2}$$

Then we plug these back into the original equation, take its' derivative with respect to λ then solve for λ.

$$\mathcal{L}(h,s,\lambda) = 4(\frac{5}{2}\lambda)(5\lambda) + \lambda(100 - 20(\frac{5}{2}\lambda) - 10(5\lambda)) = 100\lambda - 50\lambda^2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - 100\lambda = 0$$

$$\lambda = 1$$

Next we take this value and plug it back into the x and y we found earlier

$$h = \frac{5}{2}(1) = \frac{5}{2}$$

$$s = 5(1) = 5$$

and we get

$$\boxed{R(\frac{5}{2}, 5) = 4 * \frac{5}{2} * 5 = 50}$$

and we can see that

$$20 * \frac{5}{2} + 10(5) = \frac{100}{2} + 50 = 100$$