ECE:5450 - Homework 1

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Problem 1

Problem statement: Independent Random Variables

Solution:

1. We know that
$$E[x] = \int_{-\infty}^{\infty} xp(x)\,dx$$
and
$$E[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)p(x,y)\,dx\,dy$$

2. So we can plug in $x+y$ and we get
$$\begin{aligned} E[x+y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)p(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xp(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yp(x,y)\,dx\,dy \\ E[x+y] &= E_x + E_y \end{aligned}$$
where
$$E_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xp(x,y)\,dx\,dy,\ E_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yp(x,y)\,dx\,dy$$

3. We know that a joint PDF is $p(x,y) = p(x)p(y)$, so we can sub this in for both E_x and E_y.
$$E_x = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} xp(x)\,dx\right)p(y)\,dy$$
where
$$E[x] = \int_{-\infty}^{\infty} xp(x)\,dx$$
meaning we can get
$$E_x = E[x] * \int_{-\infty}^{\infty} p(y)\,dy = E[x]$$
since
$$\int_{-\infty}^{\infty} p(y)\,dy = 1$$

4. We can then follow this same set of steps for E_y which will simply to

$$E_y = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} yp(y)\, dy \right) p(x)\, dx$$

$$E[y] = \int_{-\infty}^{\infty} yp(y)\, dy$$

$$E_y = E[y] * \int_{-\infty}^{\infty} p(x)\, dx = E[y]$$

5. Thus this means that $E[x+y] = E[x] + E[y]$

Problem 2

Problem statement: Bayes Theorem

Solution:

1. $c =$ event a women has breast cancer, and $m =$ event that the Mammogram test is positive.

2. We know: $P(c) = 0.12, P(m|c) = 0.80, P(\neg m|c) = 0.20, P(m|\neg c) = 0.08, P(\neg m|\neg c) = 0.92$

3. (a) - We need $P(c|m) = \frac{P(m|c)P(c)}{P(m)}$

$$P(m) = P(m|c)P(c) + P(m|\neg c)P(\neg c)$$

$$P(\neg c) = 1 - P(c) = 1 - 0.12 = 0.88$$

$$P(m) = (0.80)(0.12) + (0.08)(0.88) = 0.096 + 0.0704 = 0.1664$$

$$P(c|m) = \frac{(0.8)(0.12)}{(0.1664)} = 0.576923 \approx \boxed{57.7\%}$$

4. (b) - We need $P(c|\neg m) = \frac{P(\neg m|c)P(c)}{P(\neg m)}$

$$P(\neg m) = P(\neg m|c)P(c) + P(m|\neg c)P(\neg c)$$

$$P(\neg m) = (0.20)(0.12) + (0.92)(0.88) = 0.024 + 0.8096 = 0.8336$$

$$P(c|\neg m) = \frac{(0.20)(0.12)}{(0.8336)} = 0.028791 \approx \boxed{2.88\%}$$

Problem 3

Problem statement: Sample Mean of Gaussian Random Variables

Solution:

1. We know $i = 1...N$ and the sample mean $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$
2. Then we can find the expected value using the sample mean $E[\hat{\mu}]$.

$$E[\hat{\mu}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i]$$

3. Since each x_i comes from a Gaussian distribution with mean μ we know $E[x_i] = \mu$ for all i thus,

$$E[\hat{\mu}] = \frac{1}{N}\sum_{i=1}^{N} \mu = \frac{1}{N} * N\mu = \mu$$

4. Thus $E[\hat{\mu}] = \mu$ proving $\hat{\mu}$ is an unbiased estimator of the population mean μ.