Exploring the self capacitance of single-layer solenoid inductors with the TSSP's VSD

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https://bartmcguyer.com/notes/note-14-SelfCapTSSP.pdf

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TL;DR: Tests of simple models for the self capacitance of single-layer solenoid inductors based on low-frequency capacitance or unloaded resonances using numerical results from the Tesla Secondary Simulation Project's Virtual Secondary Database.

CONTENTS

I. SETUP

Consider a solenoid arranged as shown in Fig. 1(a). Let the solenoid be a single layer of wire reasonably close-wound around a cylindrical dielectric form that's insulating and non-magnetic. Let the solenoid be grounded at one end and connected in parallel with an ideal lumped capacitive load C_{load}. What's the lowest resonant frequency f_1 of this setup, say, if you were to excite it inductively or with an ideal signal generator?

The traditional approach to estimate f_1 is to model the setup with the lumped-element equivalent circuit shown in Fig. 1(b), which is known as the "classic" model of an inductor.[1,2] For simplicity, let $R_s = 0$ and ignore any other sources of loss going forward. This series RLC circuit then predicts resonance at a frequency

$$f_1 = \frac{1}{2\pi\sqrt{L_s(C_{\text{self}} + C_{\text{load}})}} \tag{1}$$

given by an inductance L_s and a total capacitance that is the sum of C_{load} and a correction C_{self} from the solenoid. By convention, L_s is the low-frequency inductance of the solenoid.

Empirically, it was discovered at least as early as 1911 that the self-capacitances C_{self} of single-layer solenoids are practically constant for reasonably large C_{load}.[3] Surprisingly, there is still no accepted analytical model for this empirical C_{self}, even after more than a century of attention in the scientific and engineering literature.[4] Today, this unresolved subject is mostly an intriguingly stubborn curiosity, because numerical modeling can estimate f_1 in any important modern application. Nevertheless, it's compelling to explore for fun.

This Note attempts to "demystify" C_{self} by exploring a few simple analytical models for it using numerical results available in the Tesla Secondary Simulation Project's Virtual Secondary Database (TSSP's VSD).[5] The approach here does not attempt to derive C_{self} from first principles, but instead connects it to other observables that can be measured or numerically estimated. This approach circumvents a potentially important issue, which is that the usual problem setup as outlined above isn't fully specified: In practice, the solenoid in Fig. 1(a) is perturbed by an undefined environment, including any leads. In a transmission-line perspective, the solenoid is only one wire of a fundamental pair, and there

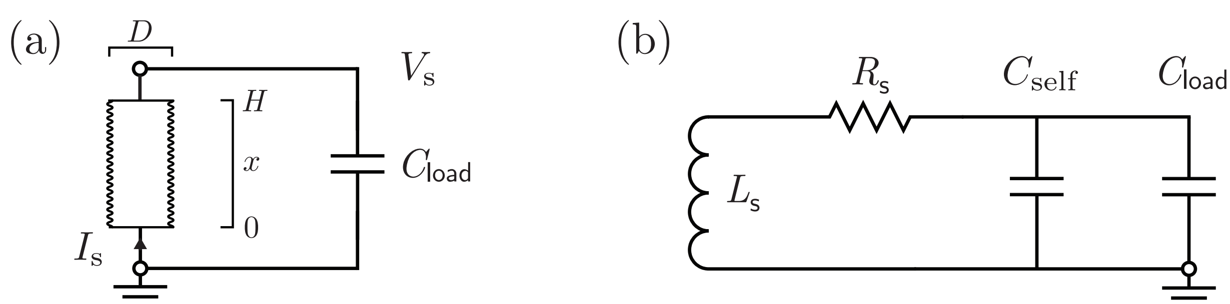

FIG. 1. Solenoid self-capacitance. (a) Physical setup of a grounded solenoid inductor with a lumped capacitive load C_{load}. (b) Equivalent circuit intended only to determine the lowest resonant frequency $\omega_1 = 2\pi f_1$, which includes a "self-capacitance" C_{self} for the solenoid in addition to resistance R_s and self-inductance L_s.

are no explicit constraints for the second wire. While a full derivation would have to address these details directly, the approach here approximately includes them indirectly through their influence on the observables used as inputs.

Additionally, this Note explores the most widely used estimate for C_{self}, the Medhurst empirical formula from 1947,[6]

$$C_{\text{Medhurst}} = \left(0.1126\,\frac{H}{D} + 0.08 + 0.27\sqrt{\frac{D}{H}}\right) D \quad \text{pF/cm}, \tag{2}$$

which is essentially synonymous with C_{self}. What's particularly remarkable about this formula and its longevity is that it implies the only important variables in practice are the solenoid's length (or height) H and diameter D. While its accuracy is not known (see Appendix A), it seems to be widely regarded as trustworthy. There are no standard corrections for the environment, dielectrics (including the coil form and wire insulation), or other potentially relevant details, though estimates are available.[7]

II. SIMPLE MODELS

This section presents the two simplest models I'm aware of for C_{self}, along with some variations. Each assumes we can compute C_{self} as the quasistatic capacitive energy stored by the solenoid when it has an approximately linear voltage distribution. The first model uses the uniform-voltage capacitance C_0 as an observable, and the second model uses the unloaded ($C_{\text{load}} = 0$) resonant frequencies f'_ν along with L_s as observables. Both need a correction to approximately include energy stored directly by the voltage gradient.

The justification for this assumption is that the value of C_{self} is supposed to be constant for $C_{\text{load}} \gg C_{\text{self}}$. In the lumped regime of C_{load} approaching infinity, the current in the solenoid becomes very nearly spatially uniform, so the voltage $V(x,t)$ at any position $x \in [0, H]$ along its length should vary approximately linearly, $V(x,t) \approx (x/H)V_s(t)$, at least for solenoids that are not short ($H \gtrsim D$). Additionally, in this regime, some subtleties with nonuniqueness of circuit parameters and misrepresentation of energy should be negligible, so we shouldn't have to worry about competing definitions of capacitance.[8,13]

However, the TSSP's VSD entries do <u>not</u> satisfy $C_{\text{load}} \gg C_{\text{self}}$. Nevertheless, the approach here still applies because both its extracted values of C_{self} represent energy well, as we'll see below, and its entries are said to have approximately linear voltage profiles.[5]

A. Approach

Following the notation and conventions of Ref. 9, the stored energy can be modeled using the distributed capacitances $c(x_n, x_j) \approx C_{nj}/h^2 = -\mathbb{C}_{nj}/h^2$ and $c(x_n) = -\int_0^H c(x_n, y)dy \approx C_n/h = \sum_{j=1}^{N} \mathbb{C}_{nj}/h$, where C_{nj} is a mutual capacitance between the n-th and j-th turns in a discrete network model of N turns, $h = H/N$ is a uniform turn spacing, and $\mathbb{C}$ is the capacitance matrix of electrostatics. This leads to a theory definition of

$$C_{\text{self}} = -\int_0^H \int_0^H c(x,y)\langle V(x,t)V(y,t)\rangle dxdy \Big/ \langle V(H,t)^2\rangle \tag{3}$$

(c.f., C_U on p. 9 of Ref. 9), where brackets denote a time average. Approximating the voltage as varying linearly gives an expression that separates into a sum of two terms,

$$C_{\text{self}} \approx C_{\text{linear}} = -\int_0^H \int_0^H c(x,y)\left(\frac{x}{H}\right)\left(\frac{y}{H}\right) dxdy \tag{4}$$

$$= \frac{1}{H^2}\int_0^H c(x)x^2 dx \tag{5}$$

$$+ \frac{1}{2H^2}\int_0^H \int_0^H c(x,y)(x-y)^2 dxdy. \tag{6}$$

As discussed in Ref. 9, the first term (5) provides an approximate lower bound for C_{self} and the second term (6) is a positive correction for excluded gradient energy that is expected to be small for long solenoids.

The next two sections present estimates for the lower-bound term (5). To simplify things as much as possible, both models assume a uniformly distributed capacitance

$$c(x) \approx c = \frac{1}{H}\int_0^H c(x)dx. \tag{7}$$

The third section then presents an estimate for the gradient-energy term (6).

Before we continue, note that we could also estimate an upper bound for C_{self} as follows. At first, it's tempting to consider the uniform-voltage capacitance

$$C_0 = -\int_0^H \int_0^H c(x,y)dxdy = \int_0^H c(x)dx = cH \tag{8}$$

as an upper bound, but this fails because $c(x,y)$ is negative near the diagonal $x \approx y$ and positive elsewhere.[9] However, keeping only the diagonal contributions produces an upper bound. This is clear rewriting (4) in a discrete network form following Ref. 9:

$$C_{\text{linear}} \approx -\sum_{n=1}^{N}\sum_{j=1}^{N} C_{nj}\left(\frac{x_n}{H}\right)\left(\frac{x_j}{H}\right) \tag{9}$$

where $x_n = nh$. Noting that C_{nj} is positive for $n \neq j$ but negative for $n = j$ gives

$$C_{\text{linear}} \approx -\sum_{n=1}^{N}\sum_{j=1}^{N} C_{nj}\left(\frac{nj}{N^2}\right) \leq -\sum_{n=1}^{N} C_{nn}\left(\frac{n}{N}\right)^2 \leq -\sum_{n=1}^{N} C_{nn} = \sum_{n=1}^{N} \mathbb{C}_{nn}. \tag{10}$$

Each coefficient of capacity $\mathbb{C}_{nn}$ is the positive self-capacitance of the n-th turn when all other turns are grounded. These capacitances could be estimated numerically[9] or analytically[10] to construct an approximate upper bound for C_{self}.

B. Model using uniform capacitance (Miller self-capacitance)

The first model uses the uniform-voltage capacitance C_0 of (8) as an observable, which is experimentally and numerically accessible. Multiplying (5) by C_0/C_0 and using (8) to rewrite C_0 in the denominator gives the lower bound

$$C_{\text{self}} \geq \int_0^H c(x)(x/H)^2 dx = \frac{\int_0^H c(x)(x/H)^2 dx}{\int_0^H c(x) dx} C_0. \tag{11}$$

For uniform $c(x) = c$, this lower bound evaluates to the remarkably simply result

$$C_{\text{Miller}} = \frac{1}{3} C_0. \tag{12}$$

This model was presented at least as early as 1918 by J. M. Miller,[11,12] the famous electrical engineer after whom the "Miller capacitance" in amplifiers is named. I've called this the Miller self-capacitance of a solenoid in three previous notes: Ref. 9 provides background and an alternate version of the above derivation; Ref. 13 provides a detailed transmission-line derivation of it; and Ref. 14 experimentally demonstrated it in controlled conditions.

For real solenoids, the shape of $c(x)$ controls whether the Miller self-capacitance (12) overestimate or underestimates the lower bound (11). This is because $c(x)$ tends to resemble a bathtub shape with sharp peaks near the edges at $x = 0$ and H, whose details depend sensitively on the environment. Exploring shapes numerically, it seems reasonable to expect C_{Miller} to be a slight underestimate for a typical free solenoid with symmetric peaks $[c(x) = c(H - x)]$. However, if the bottom end $(x = 0)$ is near a ground plane, then the bottom peak likely contributes significantly more (c.f. Fig. 3 of Ref. 15), which seems to make C_{Miller} an overestimate. Conversely, a dominant contribution from the top peak seems likely to make C_{Miller} an underestimate.

Note that the factor of 1/3 estimates how much energy is stored by a linear voltage distribution compared to a uniform distribution. In principle, this factor could be improved by replacing it with a function that gives the exact ratio of these energies, say as a function of H and D for a free solenoid approximated as a cylinder with conductive endcaps. However, you would still need to account for the environment, directly or indirectly. This refinement is beyond the scope of this Note and its simple models, but might be worth future exploration.

Later, we'll compare C_{Miller} and C_{Medhurst} using two standard theoretical C_0 values,[16]

$$C_{\text{Butler}} = \begin{cases} 2\pi^2 D\epsilon_0 / \ln(16D/H) & H/D \lesssim 4 \\ -2\pi H\epsilon_0 / [1 + \ln(D/(4H))] & H/D \gtrsim 4 \end{cases} \tag{13}$$

$$C_{\text{Smythe}} = \epsilon_0 D \left[4.00 + 3.475(H/D)^{0.76}\right]. \tag{14}$$

These assume no dielectrics and no nearby conductors. C_{Butler} is the capacitance of a hollow tube and should be widely applicable (Ref. 16 examined it for $1/4 < H/D < 200$ and found a worst error of about 4%). C_{Smythe} is the capacitance of a solid right cylinder with conductive endcaps and is meant only for $1/8 < H/D < 8$.

C. Model using unloaded resonant frequencies and L_s

The second model uses the unloaded ($C_{\text{load}} = 0$) resonant frequencies f'_ν and L_s as observables. Following the approach of Ref. 8 and assuming $c(x) = c$, the lower-bound term (5) is equivalent to

$$C_{\text{freq}} = \frac{2}{\pi^4 L_s} \sum_{\nu=1}^{\infty} \frac{1}{[(2\nu-1)f'_\nu]^2}, \tag{15}$$

where ν indexes the unloaded quarter-wave resonances with increasing frequency. I don't think I've seen this model, or its variations below, presented elsewhere.

The derivation of (15) follows from Ref. 8 after expanding a linear voltage spatial distribution with a quarter-wave Fourier series,

$$V(x,t) = \left(\frac{x}{H}\right) V_s(t) = V_s(t) \sum_{\nu=1}^{\infty} a_\nu \sin(k_\nu x), \tag{16}$$

where $k_\nu = (2\nu-1)\pi/(2H)$. The coefficients evaluate to $a_\nu = (-1)^{\nu+1}\, 8/[\pi(2\nu-1)]^2$. Using this series, its orthogonality, and $c(x) \approx c$, (5) becomes

$$\int_0^H c(x)(x/H)^2 dx = \sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty} a_\nu a_\mu \int_0^H c(x)\sin(k_\nu x)\sin(k_\mu x)dx \approx \sum_{\nu=1}^{\infty} a_\nu^2 C_\nu^U \tag{17}$$

where $C_\nu^U = cH/2$. Then, rewriting $C_\nu^U = (C_\nu^U/C_\nu)(L_s/L_\nu)(L_\nu C_\nu)/L_s$ and using $C_\nu/c = L_\nu/l = 1/k_\nu$, $L_s = lH$, and $L_\nu C_\nu = 1/(2\pi f'_\nu)^2$ gives (15). Intuitively, this follows from the unloaded resonances sampling $c(x) \approx c$ with different orthogonal basis functions for the voltage, which lets us reconstruct a linear voltage distribution. It might be possible to extend this approach to handle nonuniform $c(x)$ using some knowledge of the actual voltage and current spatial profiles during the resonances.

In practice, only the first or the first few frequencies f_ν are usually known. Partial sums including only a few known terms, such as

$$C_{f1} = \frac{2}{\pi^4 L_s}\left(\frac{1}{f_1'^2}\right) \tag{18}$$

$$C_{f123} = \frac{2}{\pi^4 L_s}\left(\frac{1}{f_1'^2} + \frac{1}{9f_2'^2} + \frac{1}{25f_3'^2}\right) \tag{19}$$

that will be used later, are variations of (15) and also lower bounds for it. Roughly, these should be decent models because the significance of the terms decrease rapidly: For a uniform transmission line, $f'_\nu = (2\nu-1)f'_1$. Using this to complete the sum, the fractional contributions of the first three terms are $\{0.9855, 0.0122, 0.0016, \ldots\}$. Correcting for excluded terms gives the additional variations

$$C_{f1\text{inf}} = 1.015\, C_{f1} \tag{20}$$

$$C_{f123\text{inf}} = 1.007\, C_{f123}, \tag{21}$$

which show how little correction is expected for an ideal line. Solenoids are not ideal, of course, so we'll return to examine experimental data for f_ν near the end of this Note.

For real solenoids, and as discussed for C_{Miller} above, the shape of $c(x)$ controls whether C_{freq} or its variations overestimate or underestimate the lower bound (5). Roughly speaking, though, C_{freq} should likely be less sensitive than C_{Miller} to this shape, because it uses a quarter-wave sine series instead of uniform and parabolic weights to sample $c(x)$. In particular, the dominant contribution comes from the fundamental resonance ($\nu = 1$) that closely resembles a linear voltage distribution.

Note that the factor of L_s in (15) cancels when using this model for C_{self} in (1), giving

$$f_1 = \frac{1}{2\pi\sqrt{L_s C_{\text{load}} + 2\,\pi^{-4}\sum_{\nu=1}^{\infty}[(2\nu-1)f'_\nu]^{-2}}} \approx \frac{1}{2\pi\sqrt{L_s C_{\text{load}} + 2(\pi^2 f'_1)^{-2}}}. \tag{22}$$

D. Correcting for missing gradient energy

The first term (5) and the models given above that come from it neglect some energy stored directly by voltage gradients captured in the second term (6). This is because they are based on the distributed capacitance $c(x)$ that assumes no gradient by construction. For example, for a long solenoid with conductive endcaps, $c(x)$ assumes there is no energy stored by electric fields inside the solenoid interior. However, for a nearly linear voltage distribution, the field in the interior would resemble that of an ideal parallel-plate capacitor, clearly storing energy. Similarly, the fields outside the solenoid would be different than assumed by $c(x)$. All of these effects are captured by $c(x, y)$ in (6).

Therefore, a crude underestimate for the missing gradient energy (6) is the energy stored by an ideal parallel-plate capacitor filling the solenoid interior,

$$C_{\text{pp}}(\epsilon_r) = \frac{\epsilon_r \epsilon_0 \pi D^2}{4H}, \tag{23}$$

where the relative permittivity ϵ_r accounts for the coil form or any other dielectrics in the interior and ϵ_0 is the vacuum permittivity. This capacitance can be added as a correction to the models given above. ("Plus" indicates this in some plots.) As a correction, it should be reasonable for long coils ($H \geq D$) or coils with endcaps. It's an underestimate because it does not attempt to correct for gradient energy in the exterior. For more on this correction, please see Ref. 9. Ref. 17 uses it to estimate a dispersion-relation parameter.

Later below we'll explore this correction in comparisons of C_{Miller} and C_{Medhurst} using theoretical values for C_0. We'll see that such gradient energy likely helps create the local minimum observed in C_{Medhurst}/D versus H/D near $H/D \approx 1$. While the approach here is crude, the detailed modeling in Ref. 18 adds a similar correction, the capacitance between two rings representing the solenoid ends, to a sheath-helix model to explain this local minimum.

III. TSSP'S VIRTUAL SECONDARY DATABASE

The TSSP's Virtual Secondary Database (VSD) contains the results of electromagnetic simulations of a broad range of typical "secondary coil" arrangements for Tesla transformers (or Tesla coils). Each simulation modeled a single-layer solenoid suspended vertically above a ground plane, with or without a toroidal top electrode (or topload). The database and its documentation are available online in Ref. 5.

The VSD contains 13,578 entries formed by varying seven parameters: solenoid length $H \in [0.1, 3.179]$ m (16 values), solenoid diameter D (6 values of $H/D \in [1, 10]$), solenoid base height B above the ground plane (units of H, 3 values), wire size (4 values), wire spacing (3 values), toroid outer diameter TD (units of H, 2 values or no toroid), and height TB of the toroid central plane above the solenoid's top (units of H, 2 values or no toroid). I've kept the notation the same except for capitalization.

As best as I can tell, the VSD entries did not model any dielectrics such as coil forms.

A. Extracting C_0, C_{load}, and C_{self}

The VSD provides nearly all parameters of interest, but does not explicitly provide the main capacitances we need. This is the approach used here to to extract them:

For each "loaded" entry that includes a toroid electrode there is a corresponding "unloaded" entry that is identical except for having no electrode. Both entries contain a bulk low-frequency capacitance C_{dc}. The value of C_{dc} from the unloaded entry provides an estimate of

$$C_0 \approx C_{\text{dc}}(\text{unloaded}). \tag{24}$$

The value of C_{dc} from the loaded entry is the capacitance of both the solenoid and electrode as a single conductor. Together, these two values provide an estimate of

$$C_{\text{load}} \approx C_{\text{top}} = C_{\text{dc}}(\text{loaded}) - C_{\text{dc}}(\text{unloaded}) \tag{25}$$

for the electrode of the loaded entry. (Many thanks to Paul Nicholson for suggesting this approach.) Let's call this quantity C_{top} to flag that the separation of C_0 and C_{top} isn't ideal. That is, the proximity of the electrode to the solenoid likely perturbs the true value of C_0 for the loaded entry away from the value of the unloaded entry. Thus, there is some slight error expected in both C_0 and C_{top} when extracted this way. Relatedly, the VSD includes capacitive coupling between the electrode and solenoid that's assumed negligible in Fig. 1.

Note that the extracted values of C_{top} generally differ significantly from the typical formulas used for the electrostatic self-capacitance of isolated toroidal electrodes. Fig. 2 shows that this is likely due to the ground plane and shielding by the solenoid.

Finally, we can estimate an effective C_{self} satisfying (1) for the loaded entry as

$$C_{\text{self}} \approx \frac{1}{L_{\text{dc}}[2\pi f_1(\text{loaded})]^2} - C_{\text{top}} \tag{26}$$

where f_1 is the calculated fundamental resonant frequency from the loaded entry and $L_{\text{dc}} = L_s$ is the low-frequency inductance from either entry. Again, this extracted value likely inherits error from C_{top} as discussed above.

This extraction succeeded for 10,827 pairs of loaded and unloaded entries in the VSD, providing a dataset for the analysis below. For each solenoid and ground plane arrangement, this dataset has four C_{top} variations from the different combinations of TB and TD.

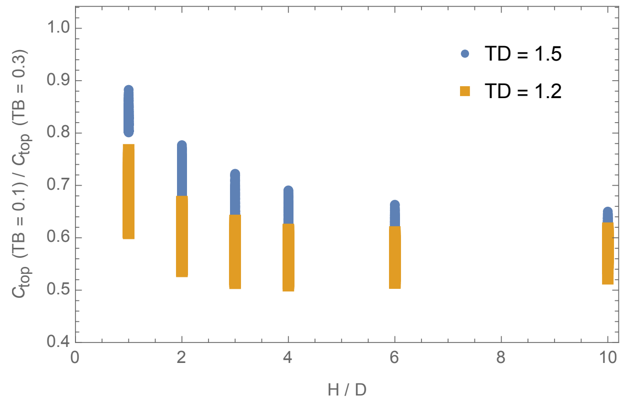

FIG. 2. Ratio of the extracted capacitance C_{top} of (25) for identical toroid electrodes at two different heights TB above the solenoid. Raising the height generally decreases C_{top} by up to almost half, depending on H/D and the toroid diameter TD. The overlap of thousands of data points covering all parametric variations gives the appearance of lines at each discrete value of H/D used in the VSD. Data points for different TD overlap significantly at large H/D.

B. Applicability of C_{self} models

As warned above, at first glance it doesn't seem like we can use the VSD to study C_{self} because we're not in the "Howe" regime: Fig. 3 shows that the loaded VSD entries do <u>not</u> satisfy $C_{\text{load}} \approx C_{\text{top}} \gg C_{\text{self}}$. Instead, $C_{\text{top}} < C_{\text{self}}$. Additionally, the typical current spatial profiles appear to be decently nonuniform.[5]

Fortunately, we may still proceed to test the simple models because the VSD satisfies their assumptions outlined above. Fig. 3 shows that the extracted C_{Self} represents energy reasonably well, and the typical voltage spatial profiles are said to be approximately linear.[5] In other words, the VSD satisfies the assumptions capturing the "Howe" regime despite not being in it, so the extracted C_{self} can be interpreted as if they were in that regime.

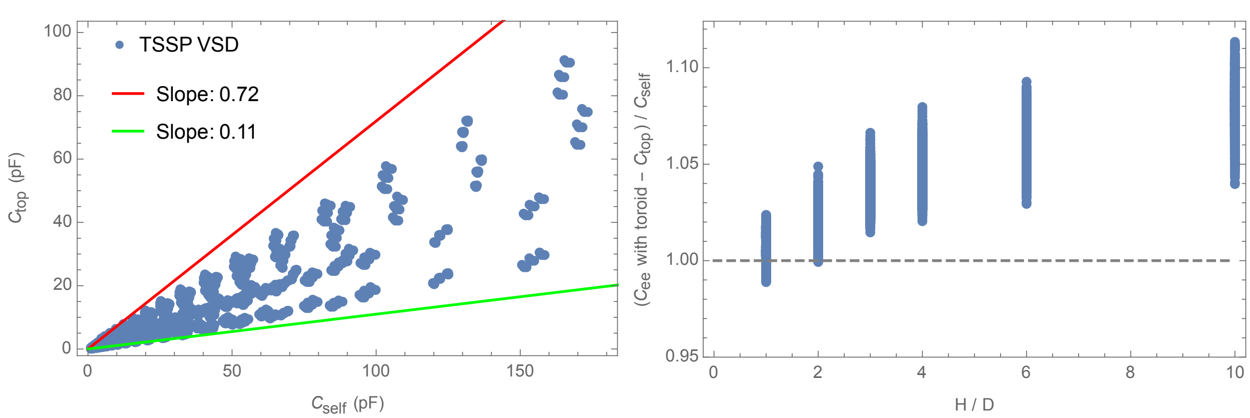

FIG. 3. Exploring the applicability of C_{self}. (Top) Extracted C_{top} versus C_{self}, showing $C_{\text{top}} < C_{\text{self}}$ for all entries. (Left) Comparison of C_{self} and VSD energy-equivalent capacitance C_{ee} showing that C_{self} represents energy correctly for $V_s(t)$ to within about 12% error.

C. Comparing C_{self} with Medhurst

Remarkably, Fig. 4 shows that the extracted C_{self} are quite close to the Medhurst empirical formula (2) – wow! Even with all its parametric variation, the spread in VSD data points at each allowed value of H/D is rather small. This provides support for the Medhurst formula's implication that the most important variables in practice are H and D.

Such good agreement is a bit surprising because of differing environments: Medhurst measured small solenoids of insulated wire wound around solid polystyrene rods and oriented horizontally above a metal box (Twin-T impedance bridge). The VSD simulated ideal solenoids with no dielectrics oriented vertically above a ground plane and with top electrodes.

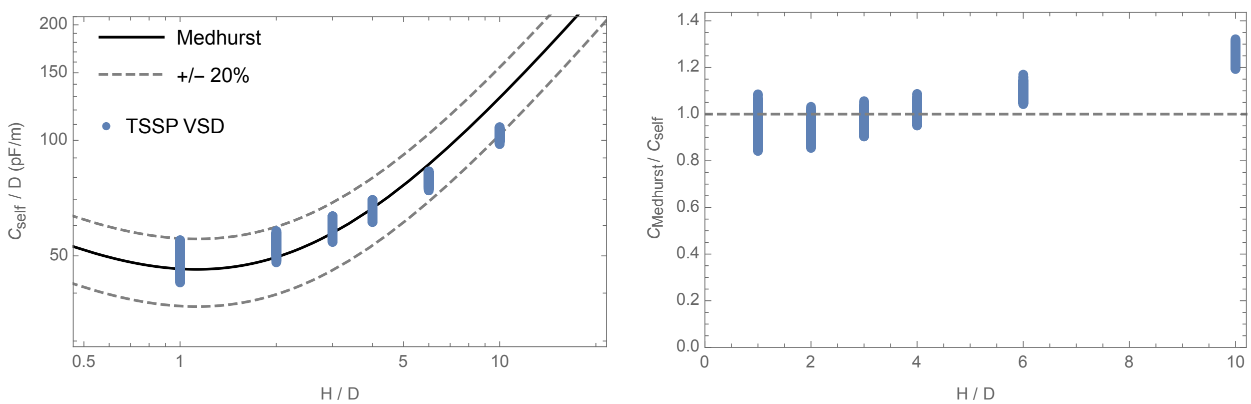

FIG. 4. Comparison of C_{self} with C_{Medhurst} of (2). (Left) Conventional log-log plot of capacitance/D versus H/D. The VSD values of C_{self} are within roughly $[-30\%, 20\%]$ of C_{Medhurst}. (Right) Alternative plot of the ratio of values versus H/D, for direct comparison with later plots.

D. Comparing C_{self} with simple models

Fig. 5 compares the Miller self-capacitance (12) using C_0 with C_{self}. For the two largest solenoid heights B above the ground plane, C_{Miller} is a decent underestimate of C_{self}. However, for the smallest height B, C_{Miller} increases and switches between being a slight underestimate at large H/D to being an overestimate at small H/D. This behavior seems consistent with the ground plane leading (12) to overestimate (5), as discussed above. Including a gradient correction (23) slightly improves the agreement for small H/D for the two larger heights B. Using a theoretical value (14) instead of the extracted C_0 gives a slightly worse underestimate that's visually similar to the results with the two larger heights B.

Fig. 6 compares the frequency-based model (15) and its variations with C_{self}. Overall, this model is a decent underestimate of C_{self} with less sensitivity to the environment than C_{Miller}, as forecast earlier. The difference between the variations are minimal, suggesting that really only the first unloaded resonant frequency is needed for this method. Adding a gradient correction improves the agreement for small H/D.

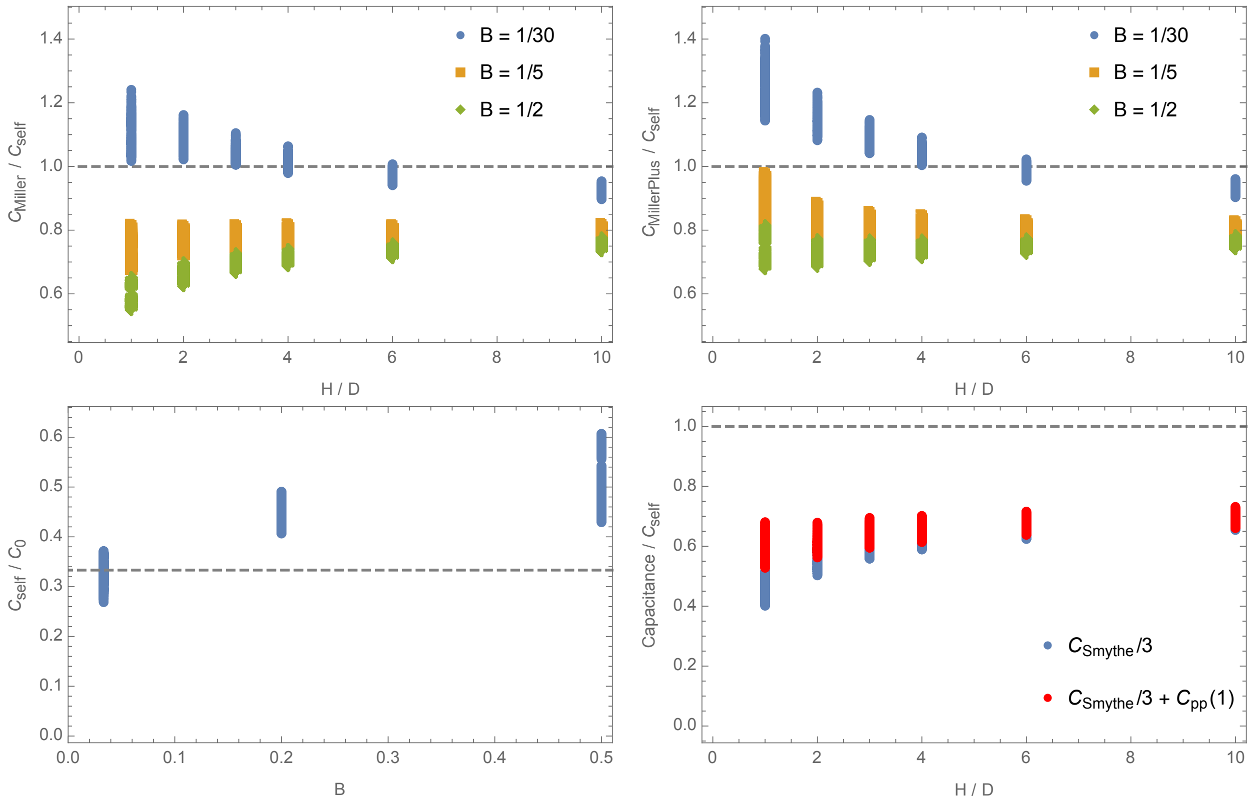

FIG. 5. Tests of the Miller self-capacitance (12). (Top left) C_{Miller} versus H/D for different solenoid heights B above the ground plane. (Top right) Adding a gradient correction (23): $C_{\text{MillerPlus}} = C_{\text{Miller}} + C_{\text{pp}}(1)$. (Bottom left) Alternate plot of C_{self}/C_0 versus B to compare with the ideal Miller value of 1/3. (Bottom right) Comparison using a theoretical estimate (14) for C_0.

IV. ADDITIONAL TESTS

A. Using theoretical capacitances

Fig. 7 compares the Miller self-capacitance using theoretical values of isolated tubes or solid cylinders for C_0 given above with C_{Medhurst}. Generally, this leads to an underestimate by almost a factor of 2. Adding a gradient correction (23) improves the agreement a bit, in a way that suggests it contributes to the observed minimum, though note that this correction isn't intended for $H < D$. (The choice of $\epsilon_r \approx 2.5$ follows from Medhurst using solenoids wound on polystyrene rods.) Some rough tests with a model from Ref. 15 for a tube above a ground plane (not shown) suggest Medhurst's Twin-T impedance bridge may also have contributed to the minimum in the log-log plot.

B. Using experimental measurements

Fig. 8(left) shows partial sums of the frequency-based model (15) using experimental resonance frequencies and inductances of typical Tesla transformer secondary coils (magnet

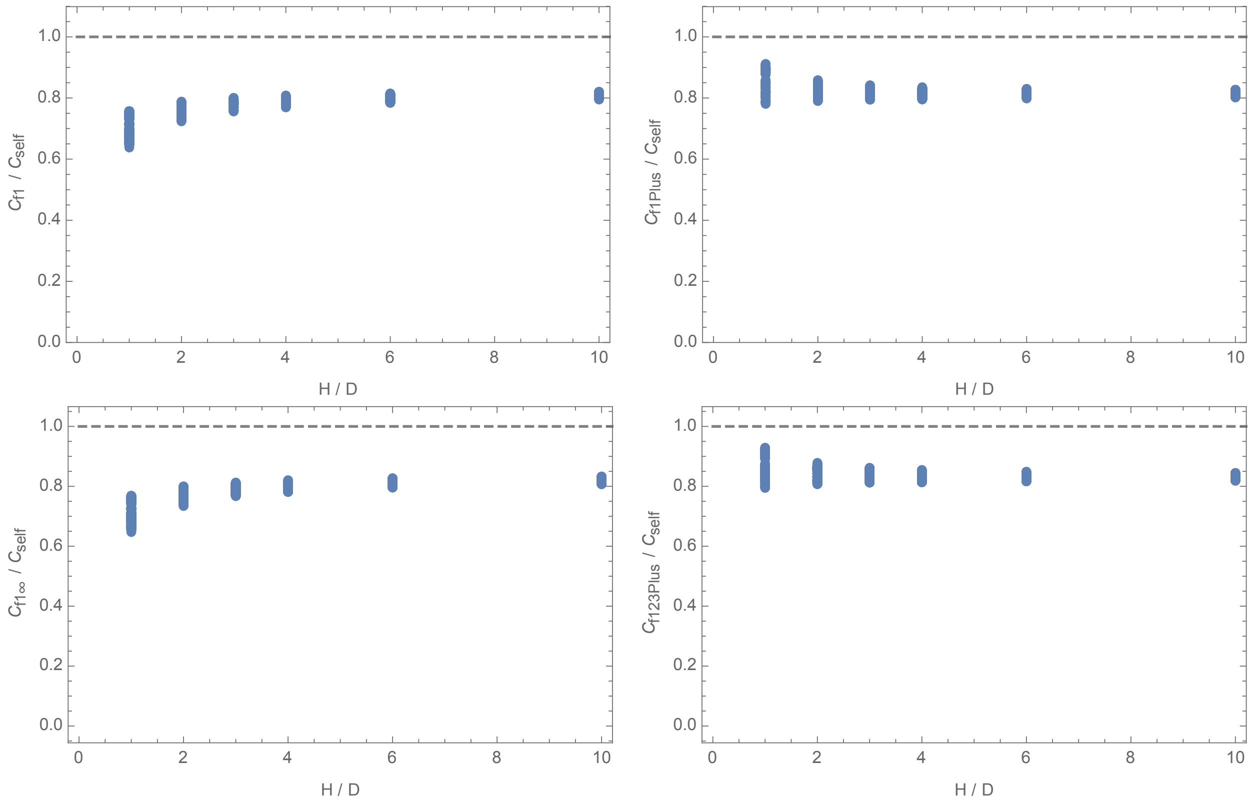

FIG. 6. Tests of the frequency-based model (15) and its variations. (Top left) Using the first resonant frequency with (18). (Top right) Adding the gradient correction $C_{\rm pp}(1)$ of (23). (Bottom left) Using the first resonant frequency with (20), which has a slight correction. (Bottom right) Using the first three resonant frequencies with (19), and adding the gradient correction $C_{\rm pp}(1)$.

wire wound on PVC pipe forms). One data set comes from Ref. 19, and the rest come from values the TSSP used to test its software.[5] Comparing with Fig. 4(left), this test slightly underestimates both $C_{\rm Medhurst}$ and the extracted VSD $C_{\rm self}$.

Fig. 8(right) illustrates the convergence of the frequency-based model (15) using measured resonances from Ref. 19. Note that this reference contains quarter- and half-wave resonances, but only the quarter-wave resonances are used in (15). The sum converges to a slightly lower value than the estimate (20) because of the typical curvature in solenoid dispersion relations. $C_{\rm Medhurst}$ evaluates to 9.66 ± 0.07 pF for this solenoid, propagating uncertainty in H and D, so the model (18) underestimates $C_{\rm Medhurst}$ by about 13%. Unfortunately, I do not have a useful measurement of $C_{\rm self}$ from Ref. 14 that used the same solenoid.

Ref. 9 reports a measurement of $C_0 \approx 22 \pm 2$ pF for the same solenoid as Refs. 14 and 19, which leads to a $C_{\rm Miller} = 7.3 \pm 0.7$ pF. This underestimates $C_{\rm Medhurst}$ by about 24%. A gradient correction of $C_{\rm pp}(1)$ would add only about 0.12 pF.

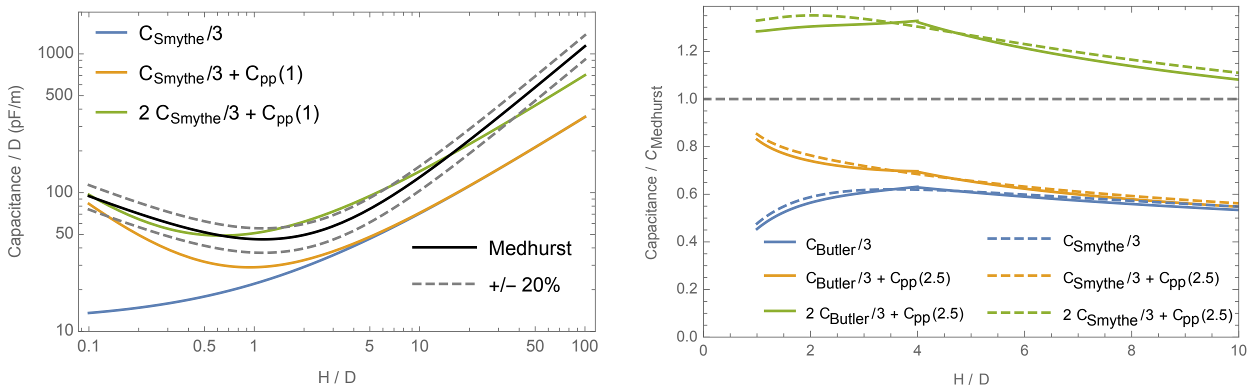

FIG. 7. Comparison of the Miller self-capacitance (12) and C_{Medhurst} using the theoretical values (13) or (14) for C_0. (Left) Conventional log-log comparison using (14). The results are similar for (13). (Right) Alternative linear plot of ratio.

V. DISCUSSION

All together, the simple models presented here worked surprisingly well with the TSSP's VSD. For as crude as they are, the models came reasonably close to the extracted values of C_{self}, well within a factor of 2. In particular, the frequency-based model with a gradient correction generally reproduced roughly 80% of C_{self}. The remaining discrepancy is likely a combination of both model limitations (oversimplifications, neglecting gradient energy) and error from extracting C_{top}.

Given that $C_{\mathrm{top}} < C_{\mathrm{self}}$ for the VSD, it's tempting to use the unloaded limit of a uniform transmission-line model for C_{self} discussed in Refs. 13 and 14. This limit gives $C_{\mathrm{self}} = (4/\pi^2)C_0$, so naively predicts that both the Miller and frequency-based models should reproduce C_{self} only up to $(1/3)/(4/\pi^2) \approx 82\%$, rather close to what's observed in Figs. 5

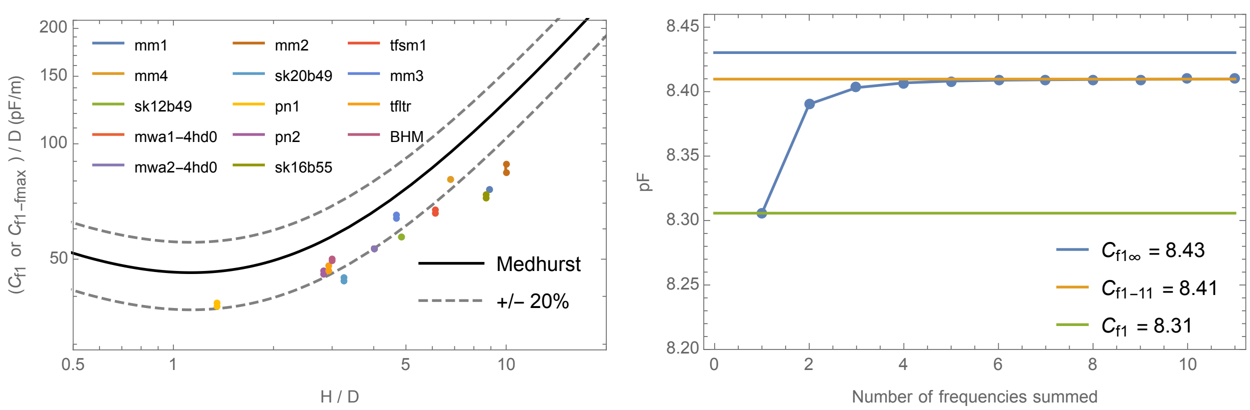

FIG. 8. Tests of the frequency-based model (15) and its variations using experimental values. (Left) Using measurements provided in the TSSP's online documentation[5] and from Ref. 19 ("BHM"). For each solenoid, there are points for both (18) and a partial sum of (15) up to the highest reported resonance. (Right) Sum convergence using measured frequencies from Ref. 19.

and 6. However, the derivation of this unloaded limit isn't fully justified for the VSD, so this is likely just an interesting coincidence.

All together, I think these results paint the following rough picture: Figs. 5 and 6 suggest that the majority of $C_{\rm self}$ comes from "external" capacitance [meaning, modeled by $c(x)$ in (5)] with a decent contribution (maybe, $\sim 20\%$?) from "internal" or "gradient" capacitance [meaning, modeled by $c(x, y)$ in (6)]. The additional tests with theoretical C_0 in Fig. 7 seem to suggest that, in practice, typical VSD environmental perturbations are a decent portion (maybe, $\sim 25\%$?) of the external part of $C_{\rm self}$. (Note that the VSD excluded dielectrics.)

The agreement of the VSD and the Medhurst formula adds support to that formula's implication that the most important variables in practice are H and D. It also provides support for that formula's longevity, including its long use by Tesla coil enthusiasts. However, the VSD seems to shows that $C_{\rm top}$ generally doesn't follow typical formulas for isolated toroids. Therefore, there's an opportunity to generate improved formulas using the VSD.

Please take all of this discussion with a grain of salt (or a healthy factor of 2, give or take). To truly make confident assertions would require additional simulation and experiments. Simulations with commercial or TSSP software would need to be in the right regime (Cload $\gg C_{\rm self}$), to cleanly separate $C_{\rm load}$ from C_0, and include typical dielectrics. And, given how little data seems available, direct verification through experiments with real solenoids or, perhaps, substitutes (e.g., resistive pipes that simulate linear voltage profiles) in representative environments would be ideal.

Potential areas for future work include exploring the refinement mentioned in Section II B, the upper bound in Section II A, or better formulas (or measurement techniques) for $C_{\rm top}$. I'd be very interested to learn whether the models presented here are practically useful with real solenoids, such as Tesla secondary coils. For example, Fig. 7 suggests that C_{f1} of (18) might be a decent practical way to estimate $C_{\rm self}$, and that it could be improved by artificially inflating its value by about 22% or so. Or, perhaps $C_{\rm top}$ and C_0 could be carefully measured in practice?

Appendix A: Medhurst accuracy notes

I've been unable to find any documented tests of the accuracy of the Medhurst empirical formula (2), which is a bit odd given its longevity and popularity.[a] The original work by Medhurst[6] in 1947 claims a fit error of about 5% for (2), but does not provide data.[b] There is also lore about Medhurst providing alternate formulas (c.f., Eq. (2) of Ref. 21). If you know of any such documentation, please share!

Inaccuracy could have escaped notice because (1) is rather insensitive to $C_{\rm self}$ errors in the applicable regime of $C_{\rm load} \gg C_{\rm self}$. For example, if initially $C_{\rm load} = 5C_{\rm self}$ and we double the value of $C_{\rm self}$, then f_1 only changes by a factor of $\sqrt{6/7} \approx 0.93$ (-7%). For the same reason, it's difficult to measure $C_{\rm self}$ accurately.[14]

[a] The closest I've found is an assertion in Ref. 20 without data.

[b] Table V in Medhurst[6] looks like data, but seems to be a table of values for the convenience of a 1940's era audience. In particular, the values of Length / Diameter don't seem to match the scatter in Figure 9. Interestingly, Table V's values have up to 8% error compared to (2). If there was a typographic error where the values given for {3.0,3.5,4.0,4.5} were meant for {3.5,4.0,4.5,5.0}, that would reduce this error.

Appendix B: Literature quotes

Here are some fun quotes about the stubborn nature of this topic from its vast literature:

Hubbard (1917):[22] "The problem of a coil oscillating in its own free period has proved one of the most difficult in mathematical physics."

Breit (1921):[23] "This paper is intended to call the attention of physicists and mathematicians to some interesting aspects of the subject of distributed capacity of coils. . . . The subject has been largely neglected by mathematical physicists." (Gregory Breit's Ph.D. dissertation was on C_{self}.)

Rhea (1997):[24] "This author is unaware of any worker who mathematically attacked either the resistance or capacitance problem and was rewarded for doing so."

Lee p. 142 (2004):[4] "This latter capacitance is somewhat difficult to compute analytically. To the best of the author's knowledge, no correct general analytical solution has ever been published. . . . Many have been offered, but close inspection reveals gross errors."

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* Typical voltage and current profiles for entries with toroids: `http://abelian.org/tssp/pn1710/#EX8`

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