Lecture 12

Bayesian Regression

and

pymc3

AM 207

Last time: Bayes

- Gibbs Sampling samples from conditionals
- Hierarchical models have a graph structure
- Makes conditional sampling easy
- Best to use log posteriors
- Gibbs can have strong correlations

Today

- the normal-normal model with MCMC
- then with pymc3
- bayesian regression and updating
- regularization and the ridge
- from the normal model to regression using pymc
- posterior vs predictive in regression problems

The levels of Bayesian analysis (from last time)

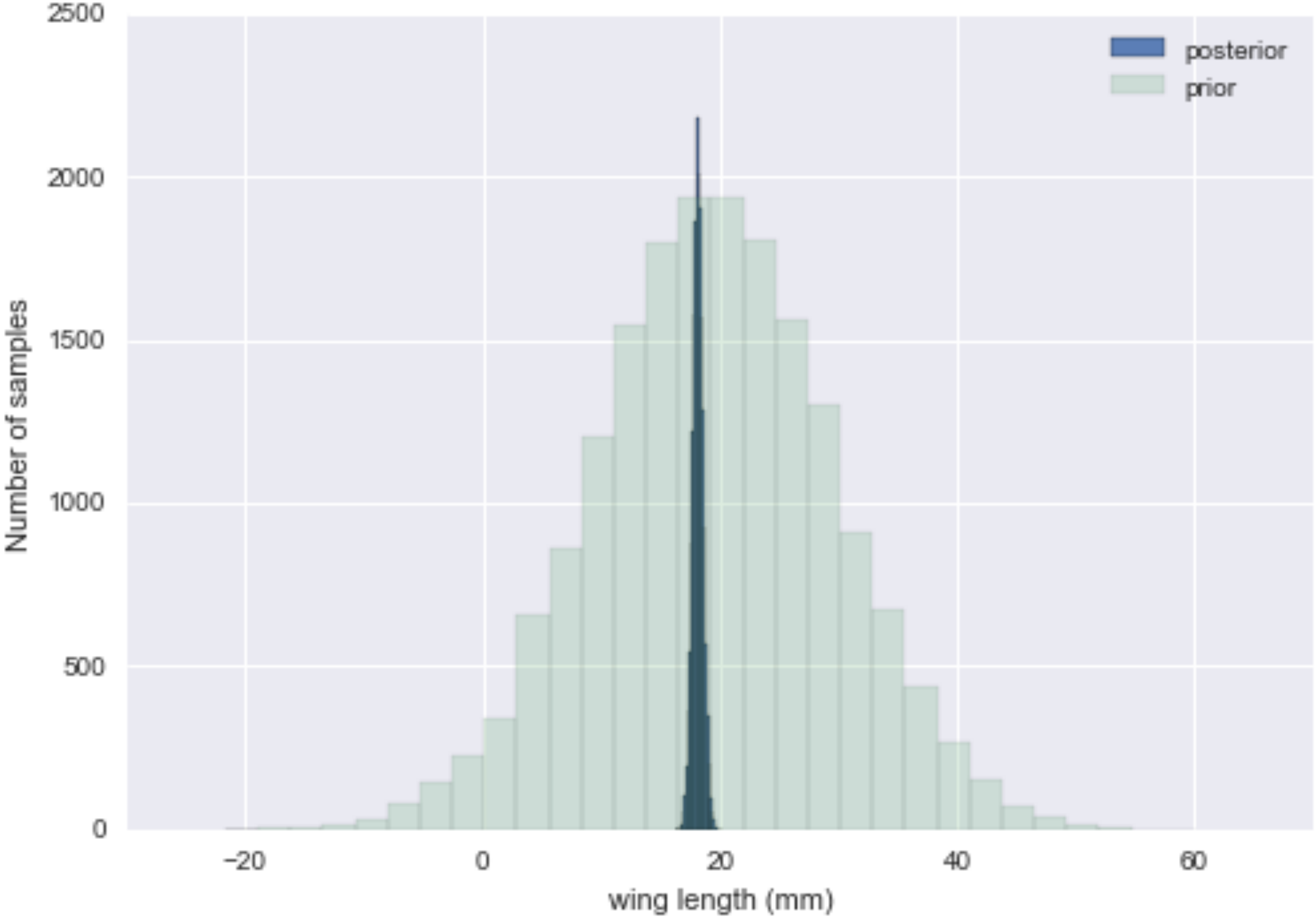
Method	Definition
Maximum Likelihood	$\hat{\theta} = argmax_\theta p(D\|\theta)$
MAP estimation	$\hat{\theta} = argmax_\theta p(D\|\theta)p(\theta\|\eta)$
ML-2 (Empirical Bayes)	$\hat{\eta} = argmax_\eta \int d\theta\, p(D\|\theta)p(\theta\|\eta) = argmax_\eta p(D\|\eta)$
MAP-2	$\hat{\eta} = argmax_\eta \int d\theta\, p(D\|\theta)p(\theta\|\eta)p(\eta) = argmax_\eta p(D\|\eta)p(\eta)$
Full Bayes	$p(\theta, \eta\|D) \propto p(D\|\theta)p(\theta\|\eta)p(\eta)$

Normal-normal model

We have data on the wing length in millimeters of a nine members of a particular species of moth. We wish to make inferences from those measurements on the population mean μ.

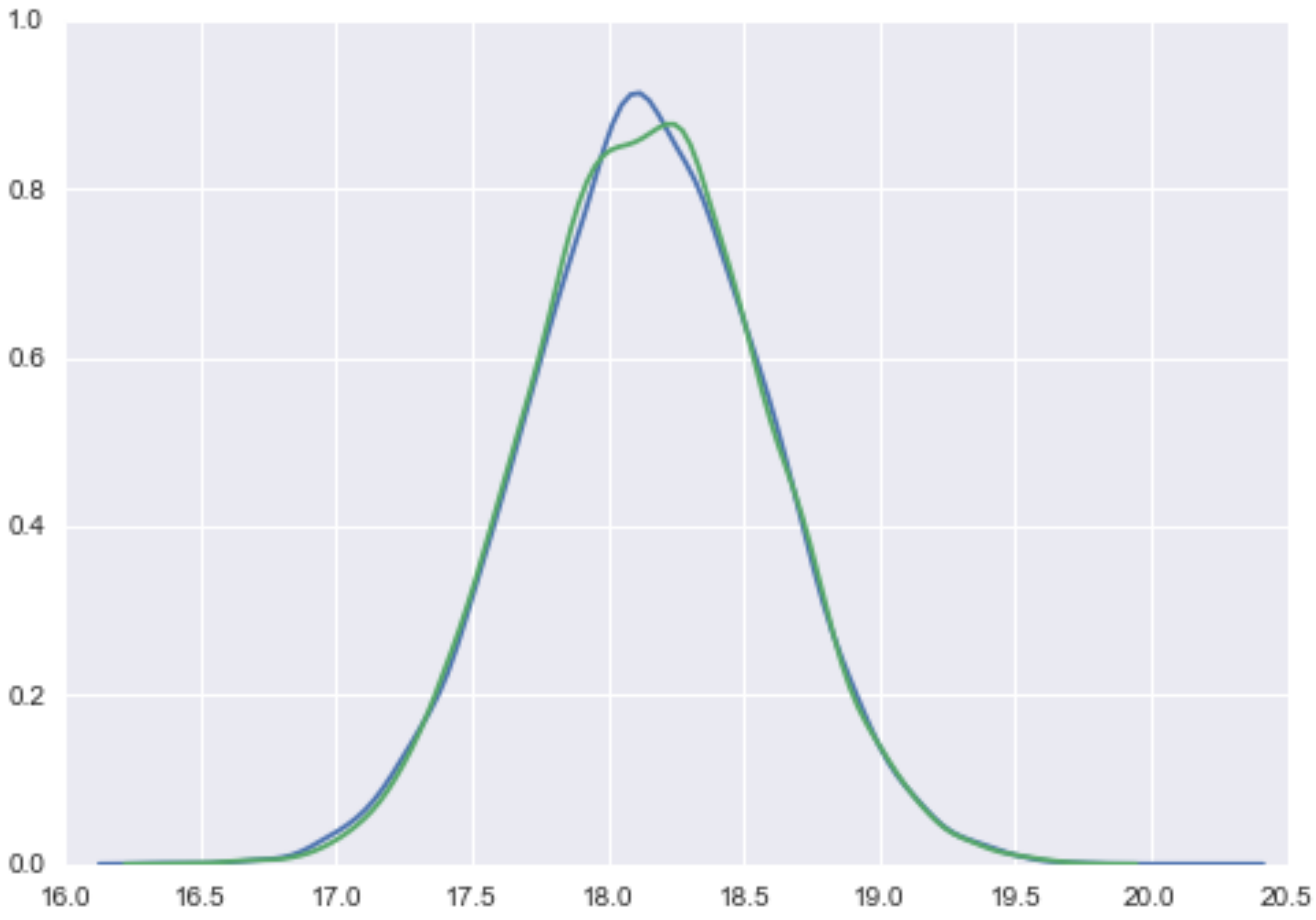
Other studies show the wing length to be around 19 mm. We also know that the length must be positive. We can choose a prior that is normal and most of the density is above zero ($\mu=19.5, \tau=10$).

```
Y = [16.4, 17.0, 17.2, 17.4, 18.2, 18.2, 18.2,
19.9, 20.8]
```



posterior
prior
Number of samples
wing length (mm)
2500
2000
1500
1000
500
0
−20
0
20
40
60

```python
def metropolis(logp, qdraw, stepsize, nsamp, xinit):
    samples=np.empty(nsamp)
    x_prev = xinit
    accepted = 0
    for i in range(nsamp):
        x_star = qdraw(x_prev, stepsize)
        logp_star = logp(x_star)
        logp_prev = logp(x_prev)
        logpdfratio = logp_star -logp_prev
        u = np.random.uniform()
        if np.log(u) <= logpdfratio:
            samples[i] = x_star
            x_prev = x_star
            accepted += 1
        else:#we always get a sample
            samples[i]= x_prev

    return samples, accepted

logprior = lambda mu: norm.logpdf(mu, loc=19.5, scale=10)
loglike = lambda mu: np.sum(norm.logpdf(Y, loc=mu, scale=np.std(Y)))
logpost = lambda mu: loglike(mu) + logprior(mu)
```



Sampling with pymc

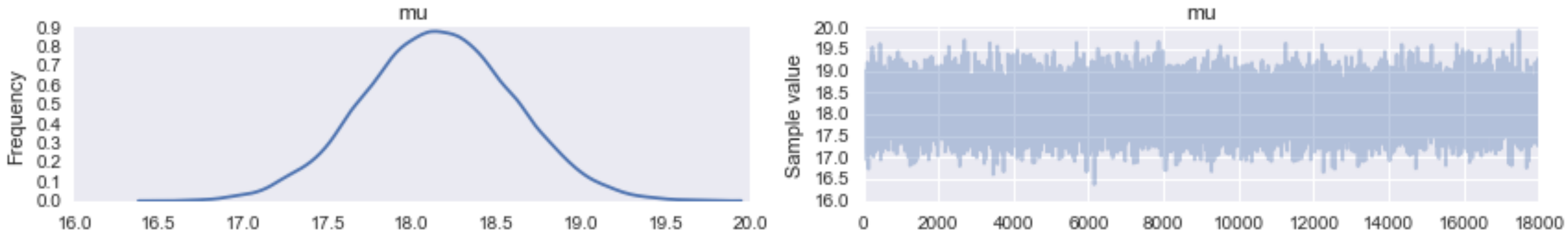
```
conda install pymc3 patsy
```

Installed `3.0rc4` for me.

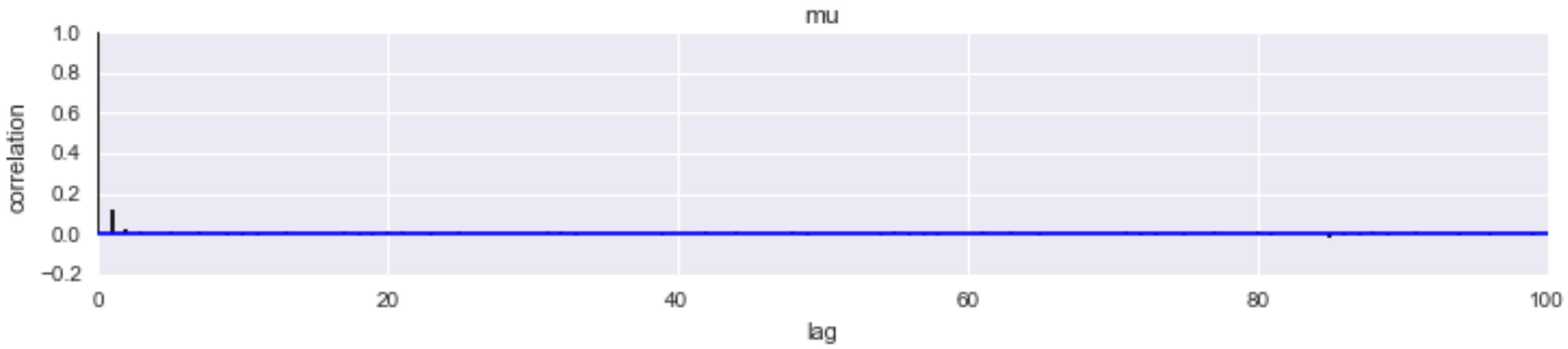
```python
import pymc3 as pm
with pm.Model() as model1:
    mu = pm.Normal('mu', mu=19.5, sd=10)#parameter's prior
    wingspan = pm.Normal('wingspan', mu=mu, sd=np.std(Y), observed=Y)#likelihood
    stepper=pm.Metropolis()
    tracemodel1=pm.sample(100000, step=stepper)
```

```
100%|██████████| 100000/100000 [00:10<00:00, 9878.33it/s]| 528/100000 [00:00<00:18, 5279.00it/s]
```

```
pm.traceplot(tracemodel1[10000::5]);
```



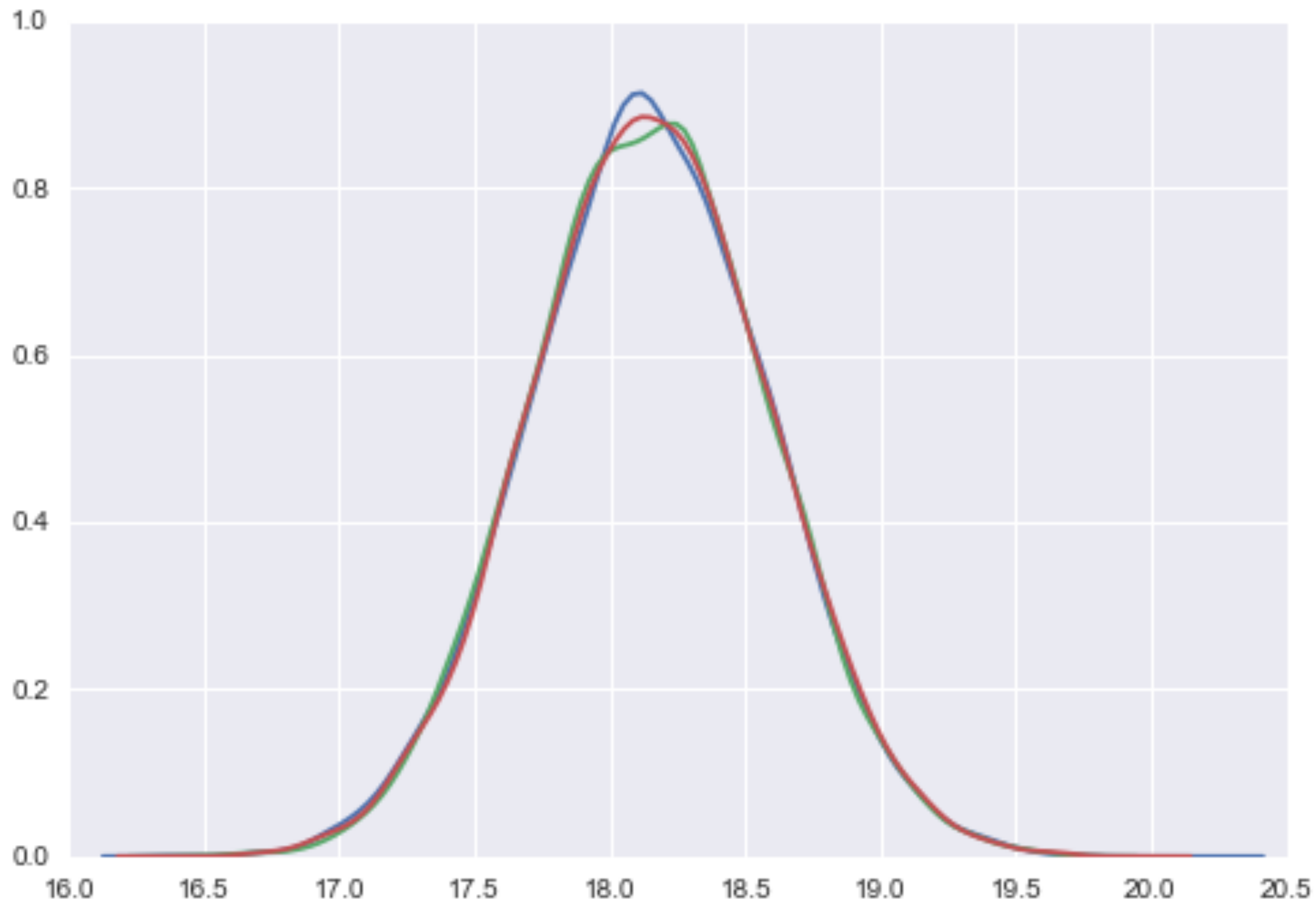

```
pm.autocorrplot(tracemodel1[10000::5]);
```



HPD: The highest-posterior-density smallest width Bayesian Credible Interval.

```
pm.summary(tracemodel1)
mu:

  Mean             SD               MC Error         95% HPD interval
  -------------------------------------------------------------------

  18.148           0.443            0.003            [17.285, 19.019]

  Posterior quantiles:
  2.5            25             50             75             97.5
  |--------------|==============|==============|--------------|

  17.277         17.849         18.146         18.446         19.012
```

Posteriors all match up!



Posterior predictives

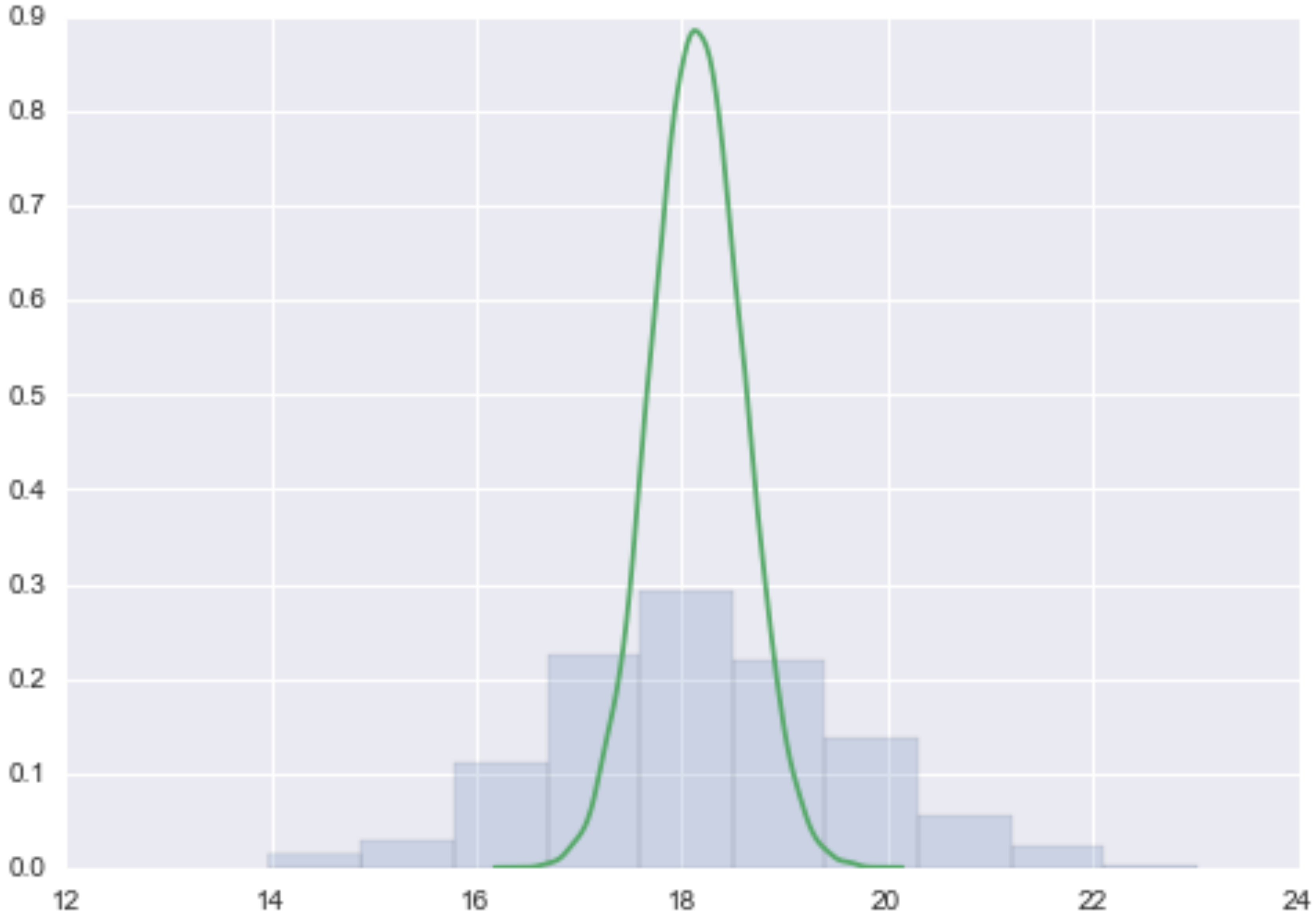
The posterior predictive is accessed via the `sample_ppc` function:

`model1.observed_RVs`

`[wingspan]`

```
tr1 = tracemodel1[10000::5]
postpred = pm.sample_ppc(tr1, 1000, model1)
```

```
100%|██████████| 1000/1000 [00:01<00:00, 510.20it/s]   | 25/1000 [00:00<00:03, 244.20it/s]
```



Bayesian Formulation of Regression

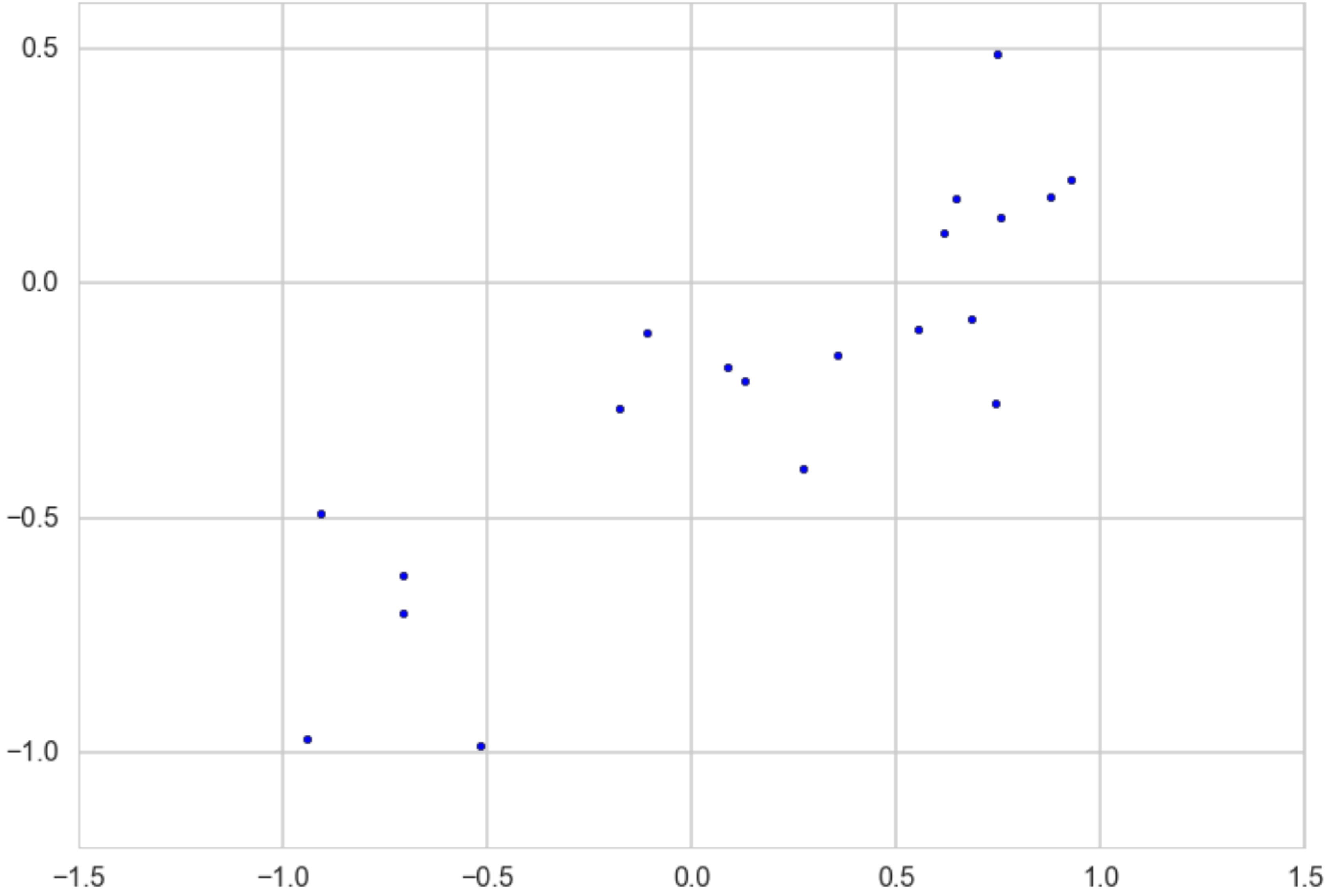
Data
$D = \{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_n, y_n)\}$

All data points are combined into a $D \times n$ matrix X.

Model:

$$y = \mathbf{x}^{\mathbf{T}}\mathbf{w} + \epsilon$$

$$\epsilon \sim N(0, \sigma_n^2)$$

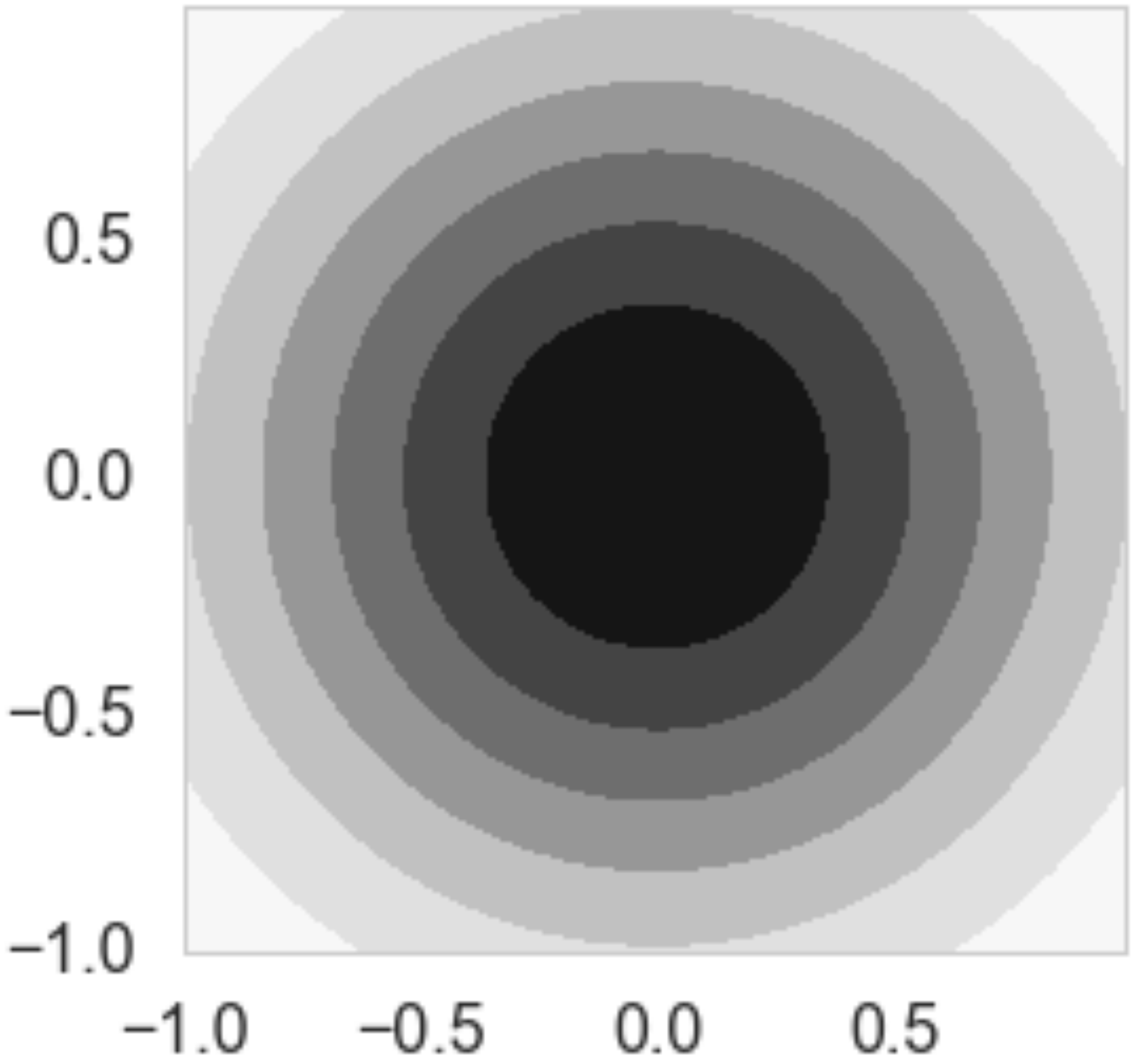


Likelihood

The likelihood is, because we assume independency, the product

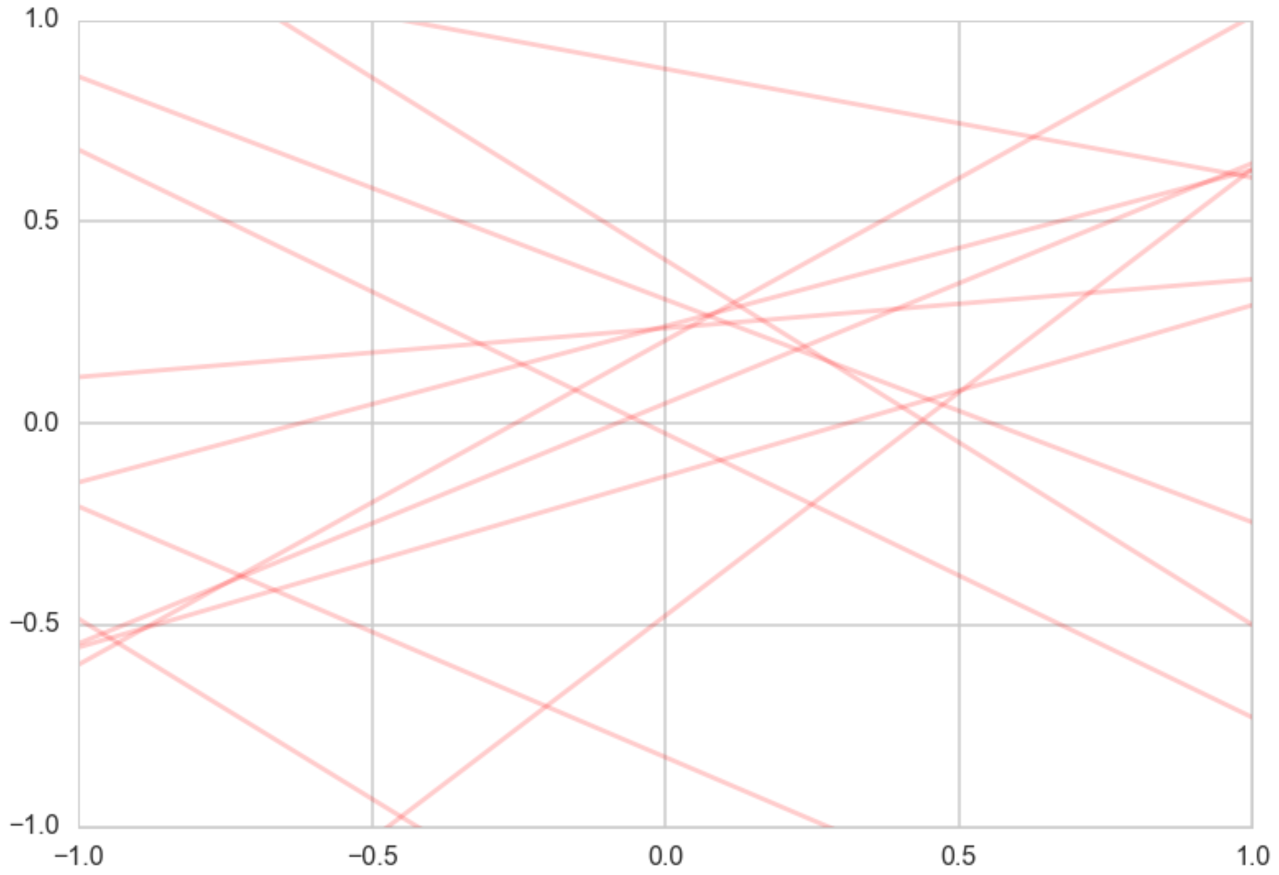
$$\mathcal{L} = p(\mathbf{y}|\mathbf{X}, \mathbf{w}) = \prod_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{p}(\mathbf{y_i}|\mathbf{X_i}, \mathbf{w}) = \prod_{\mathbf{i=1}}^{\mathbf{n}} \frac{\mathbf{1}}{\sqrt{\mathbf{2}\pi}\sigma_{\mathbf{n}}} \exp\left(-\frac{(\mathbf{y_i} - \mathbf{X_i^T w})^{\mathbf{2}}}{\mathbf{2}\sigma_{\mathbf{n}}^{\mathbf{2}}}\right)$$

$$\propto \exp\left(-\frac{|\mathbf{y} - \mathbf{X^T w}|^{\mathbf{2}}}{2\sigma_n^2}\right) \propto N(X^T \mathbf{w}, \sigma_{\mathbf{n}}^{\mathbf{2}} \mathbf{I})$$



Prior $\mathbf{w} \sim \mathbf{N}(\mathbf{w_0}, \mathbf{\Sigma})$

$$\mathbf{w} \sim \mathbf{N}(\mathbf{w_0}, \tau^2 \mathbf{I})$$



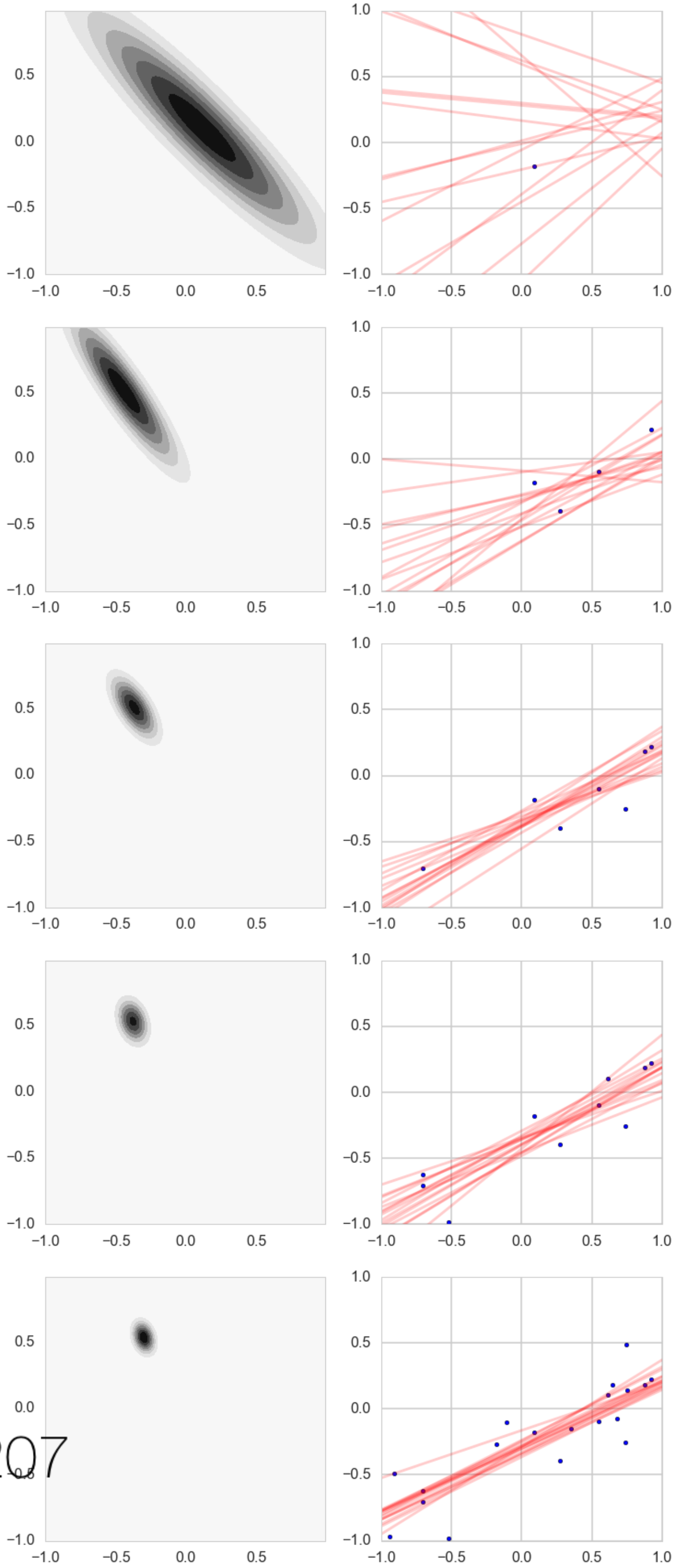
Posterior

$$p(\mathbf{w}|\mathbf{y},\mathbf{X}) \propto p(\mathbf{y}|\mathbf{X},\mathbf{w})\,\mathbf{p(w)}$$
$$\propto \exp\left(-\frac{1}{2\sigma_n^2}(\mathbf{y}-\mathbf{X^T w})^\mathbf{T}(\mathbf{y}-\mathbf{X^T w})\right)\exp\left(-\frac{1}{2}\mathbf{w^T}\boldsymbol{\Sigma}^{-1}\mathbf{w}\right)$$

$$p(\mathbf{w}|\mathbf{y},\mathbf{X}) \propto \exp\left(-\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{w}-\bar{\mathbf{w}})^\mathbf{T}(\frac{\mathbf{1}}{\sigma_\mathbf{n}^\mathbf{2}}\mathbf{XX^T}+\boldsymbol{\Sigma}^{-1})(\mathbf{w}-\bar{\mathbf{w}})\right)$$

Inverse covariance $A = \sigma_n^{-2} X X^T + \Sigma^{-1}$

where the new mean is $\bar{\mathbf{w}} = A^{-1}\Sigma^{-1}\mathbf{w_0} + \sigma_n^{-2}(A^{-1}X^T\mathbf{y})$



Bayesian updating

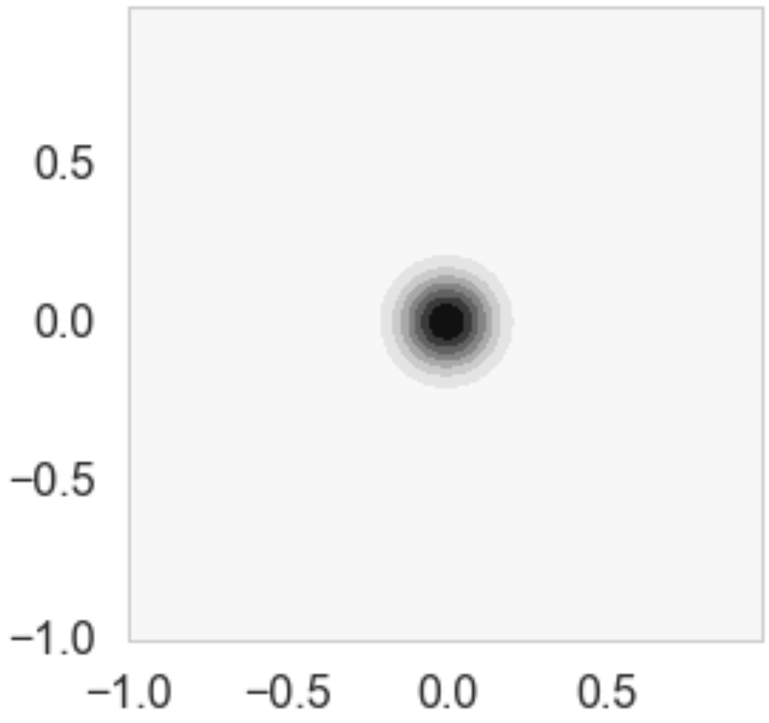
```
def update(x,y,likelihoodPrecision,priorMu,priorCovariance):
    postCovInv  = np.linalg.inv(priorCovariance) + likelihoodPrecision*np.outer(x.T,x)
    #The outer product looks wrong but when updating we need a 2x1 matrix while x is 1x2
    postCovariance = np.linalg.inv(postCovInv)
    postMu =
        np.dot(np.dot(postCovariance,np.linalg.inv(priorCovariance)),
            priorMu) +likelihoodPrecision*
            np.dot(postCovariance,np.outer(x.T,y)).flatten()
    postW = lambda w:multivariate_normal.pdf(w,postMu,postCovariance)
    return postW, postMu, postCovariance
```

Posterior predictive

$$p(y^*|x^*,\mathbf{x},\mathbf{y}) = \int p(\mathbf{y}^*|\mathbf{x}^*,\mathbf{w})p(\mathbf{w}|\mathbf{X},\mathbf{y})\mathbf{dw}$$

$$= \mathcal{N}\left(y|\bar{\mathbf{w}}^T x^*, \sigma_n^2 + {x^*}^T A^{-1} x^*\right),$$

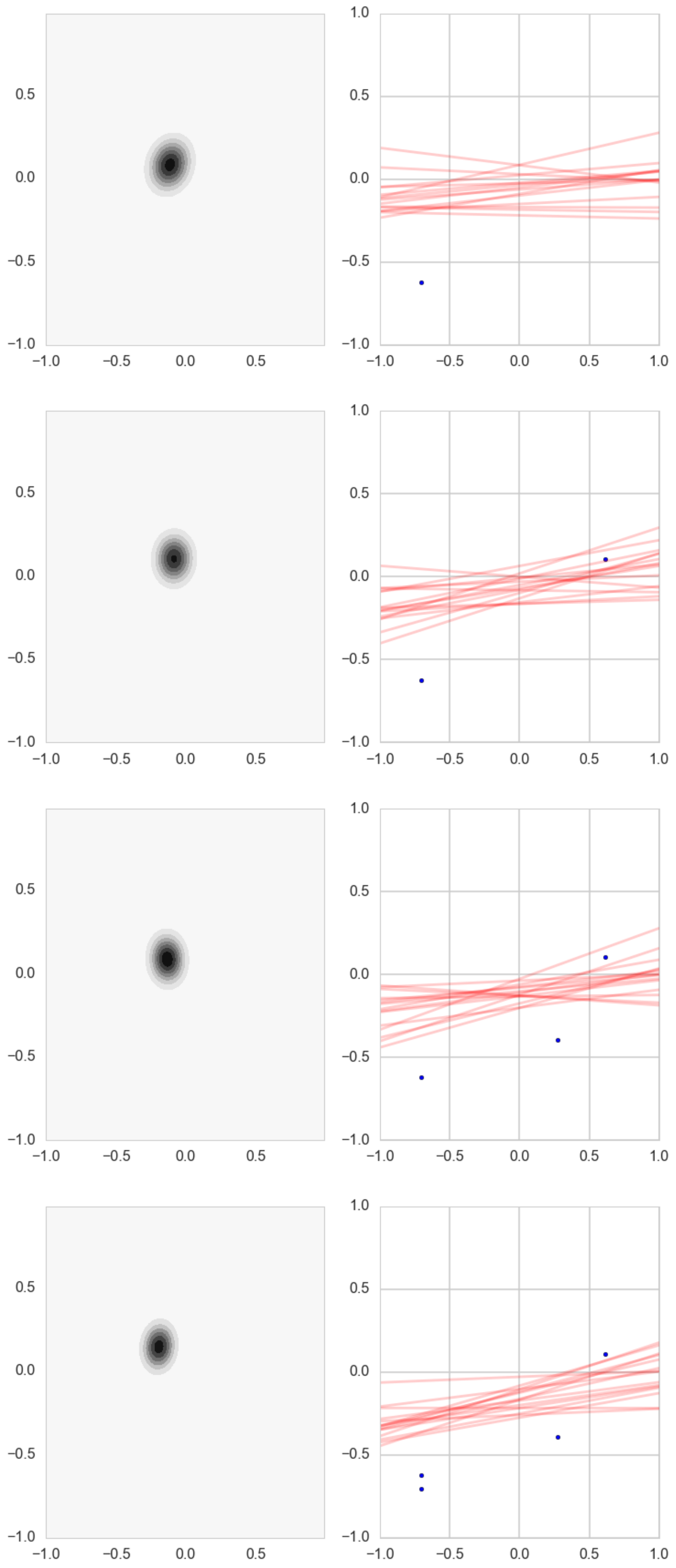
Regularization




```
priorPrecision/likelihoodPrecision
```

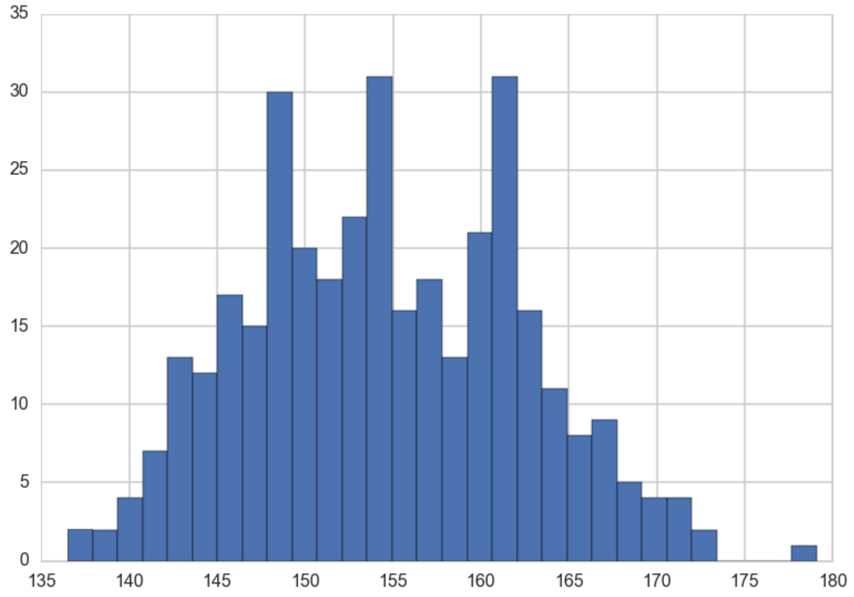
```
4.0
```

This ratio is the ridge α.



Howell's data

- These are census data for the Dobe area !Kung San people
- Nancy Howell conducted detailed quantitative studies of this Kalahari foraging population in the 1960s.



	height	weight	age	male
0	151.765	47.825606	63.0	1
1	139.700	36.485807	63.0	0
2	136.525	31.864838	65.0	0
3	156.845	53.041915	41.0	1
4	145.415	41.276872	51.0	0

Model

$$h \sim N(\mu, \sigma)$$
$$\mu \sim Normal(148, 20)$$
$$\sigma \sim Unif(0, 50)$$

```
with pm.Model() as hm1:
    mu = pm.Normal('mu', mu=148, sd=20)#parameter
    sigma = pm.Uniform('sigma', lower=0, upper=20)#testval=df2.height.mean()
    height = pm.Normal('height', mu=mu, sd=sigma, observed=df2.height)

with hm1:
    stepper=pm.Metropolis()
    tracehm1=pm.sample(10000, step=stepper)# a start argument could be used here
    #as well

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```
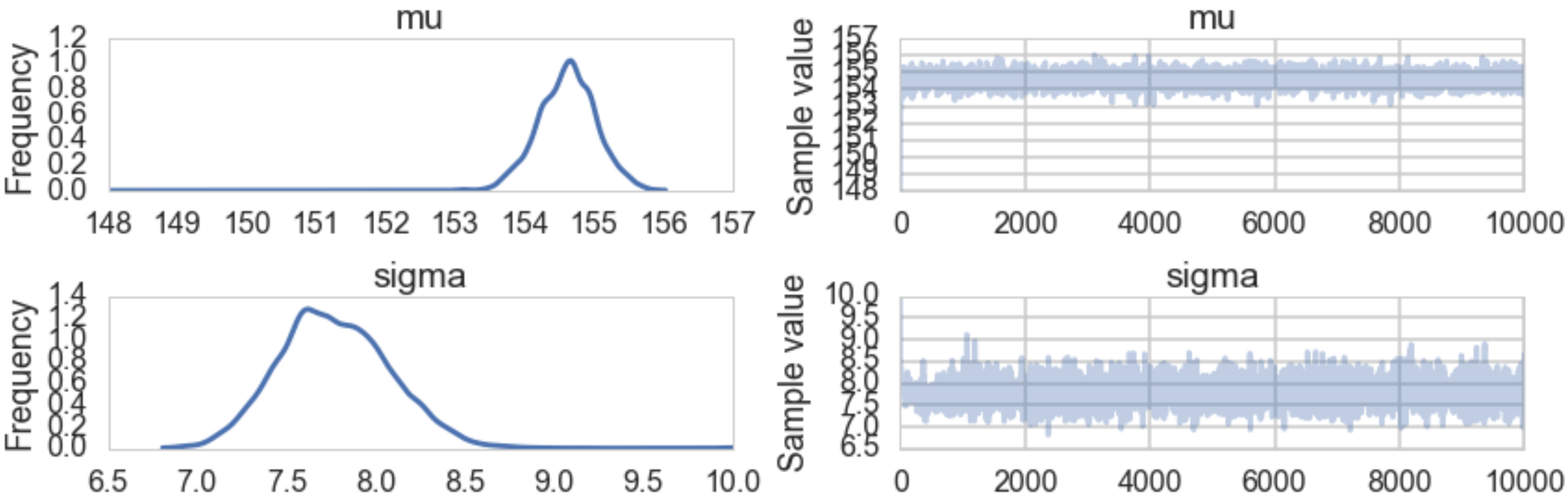
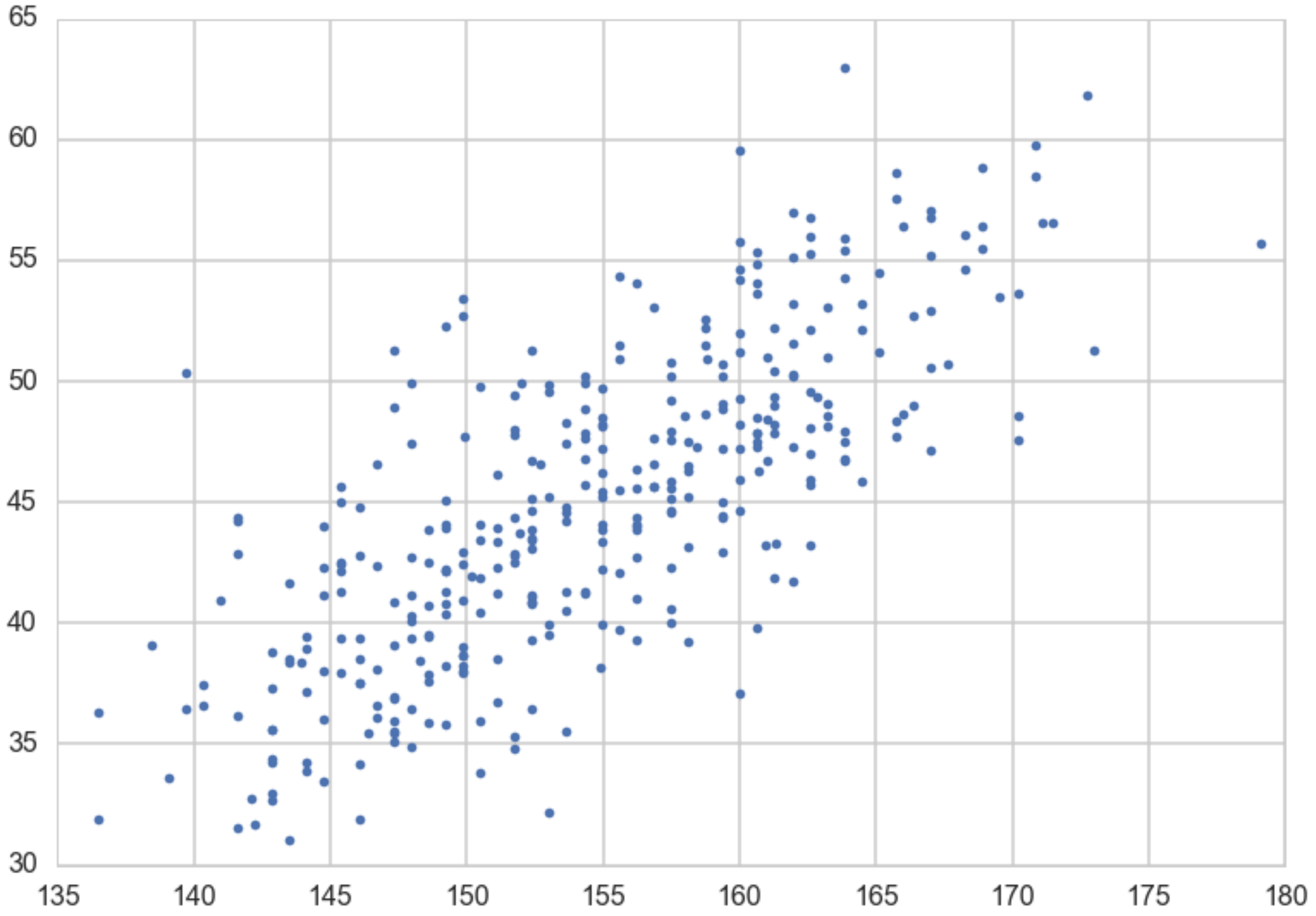




```
def acceptance(trace, paramname):
    accept = np.sum(trace[paramname][1:] != trace[paramname][:-1])
    return accept/trace[paramname].shape[0]

acceptance(tracehm1, 'mu'), acceptance(tracehm1, 'sigma')
(0.3896, 0.30009999999999998)
```

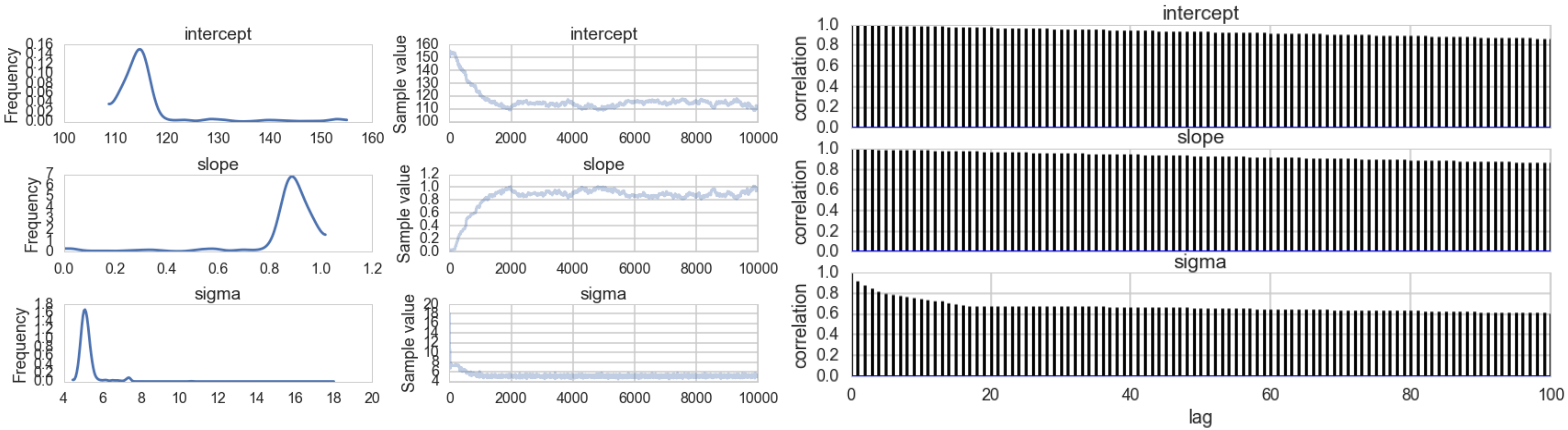


Regression, adding a predictor, weight

$$
\begin{gathered}
h \sim N(\mu, \sigma) \\
\mu = intercept + slope \times weight \\
intercept \sim N(150, 100) \\
slope \sim N(0, 10) \\
\sigma \sim Unif(0, 50)
\end{gathered}
$$

```
with pm.Model() as hm2:
    intercept = pm.Normal('intercept', mu=150, sd=100)
    slope = pm.Normal('slope', mu=0, sd=10)
    sigma = pm.Uniform('sigma', lower=0, upper=50)
    # below is a deterministic
    mu = intercept + slope * df2.weight
    height = pm.Normal('height', mu=mu, sd=sigma, observed=df2.height)
    stepper=pm.Metropolis()
    tracehm2 = pm.sample(10000, step=stepper)
```

Traces are awful

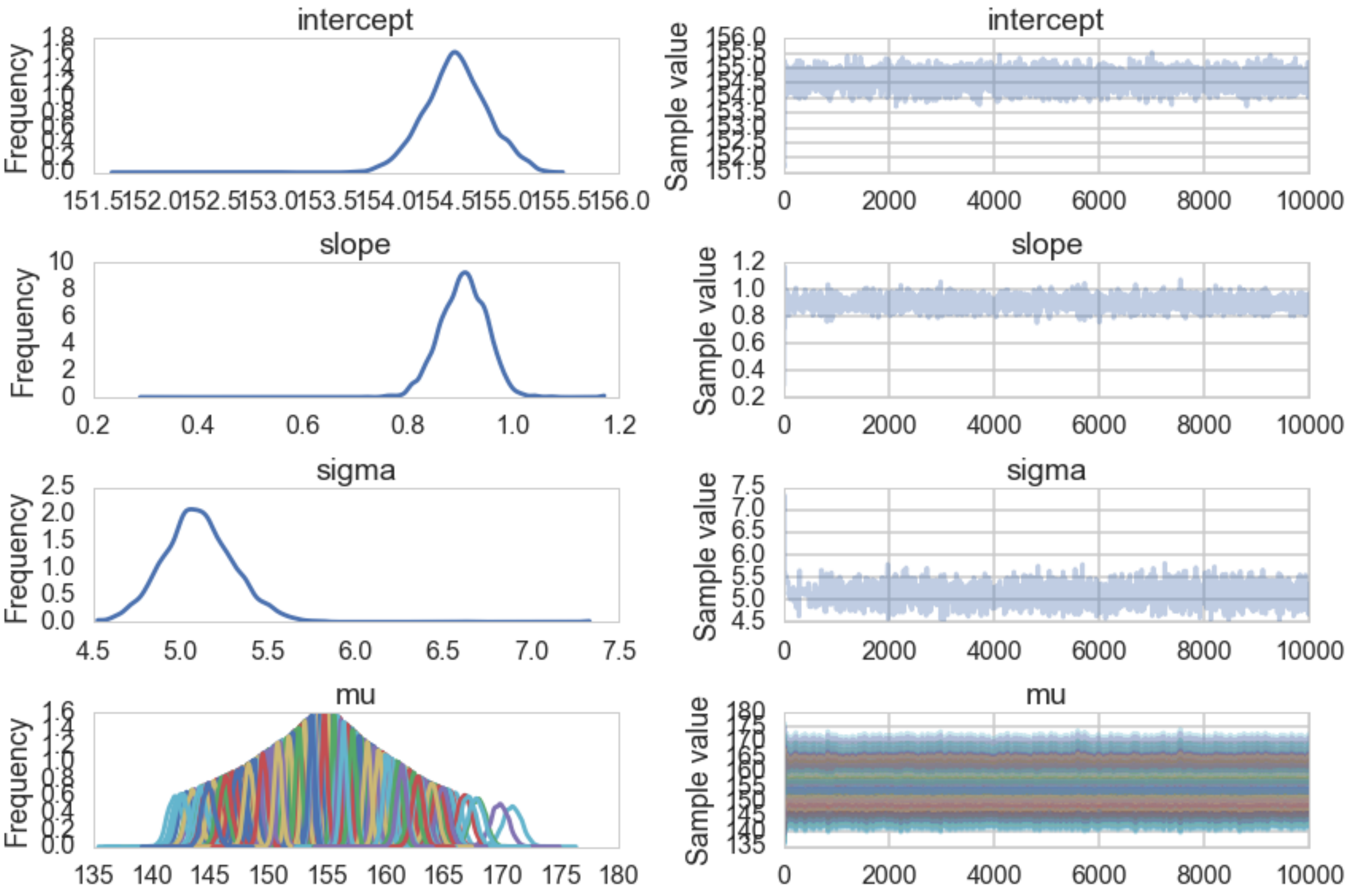


The slope and intercept are very highly correlated: -0.99!

Sympton of shared information and identifiability

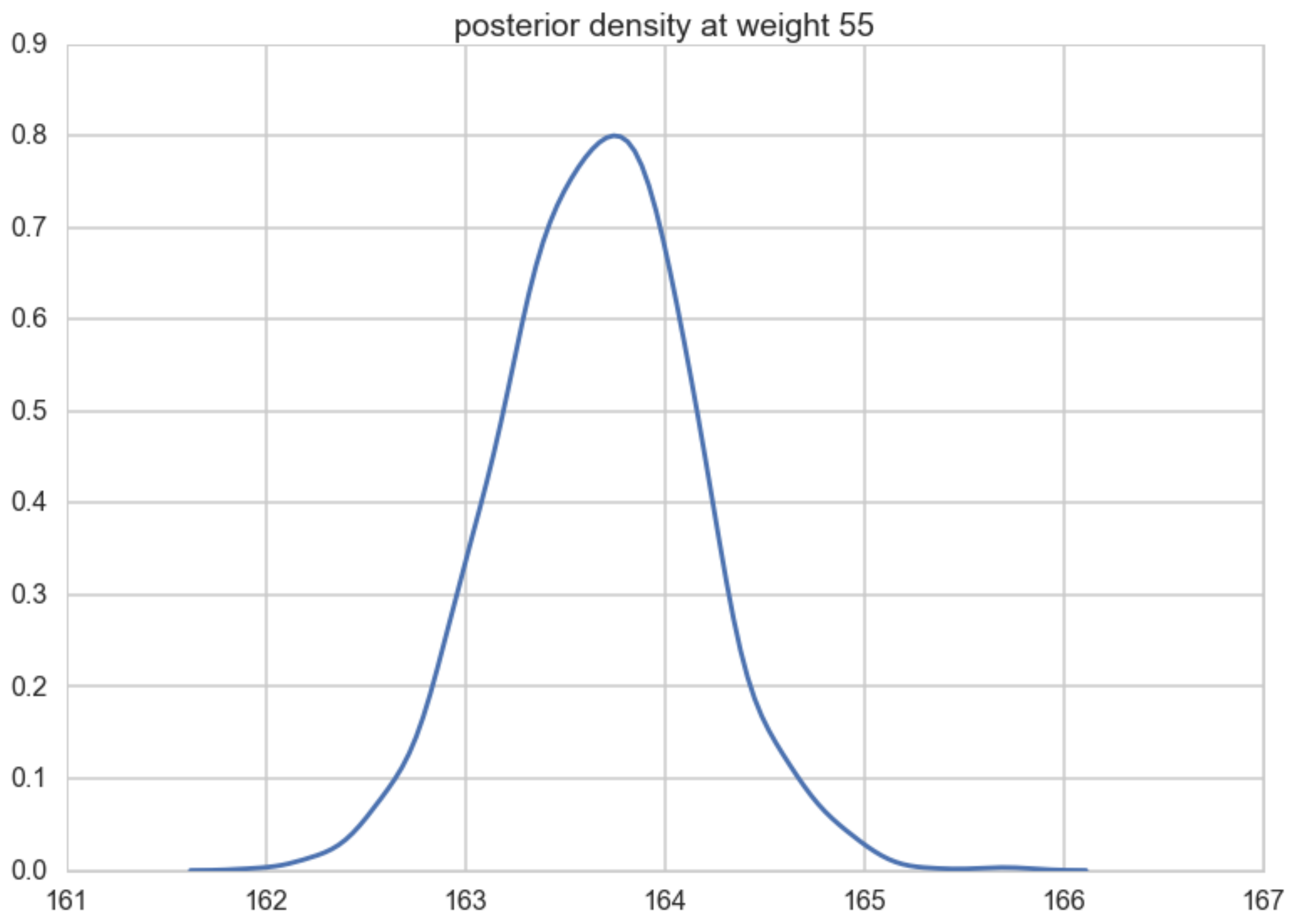
fix by centering. intercept then gives response when predictor=mean.

```
with pm.Model() as hm2c:
    intercept = pm.Normal('intercept', mu=150, sd=100)
    slope = pm.Normal('slope', mu=0, sd=10)
    sigma = pm.Uniform('sigma', lower=0, upper=50)
    # below is a deterministic
    #mu = intercept + slope * (df2.weight -df2.weight.mean())
    mu = pm.Deterministic('mu', intercept + slope * (df2.weight -df2.weight.mean()))
    height = pm.Normal('height', mu=mu, sd=sigma, observed=df2.height)
    stepper=pm.Metropolis()
    tracehm2c = pm.sample(10000, step=stepper)
```

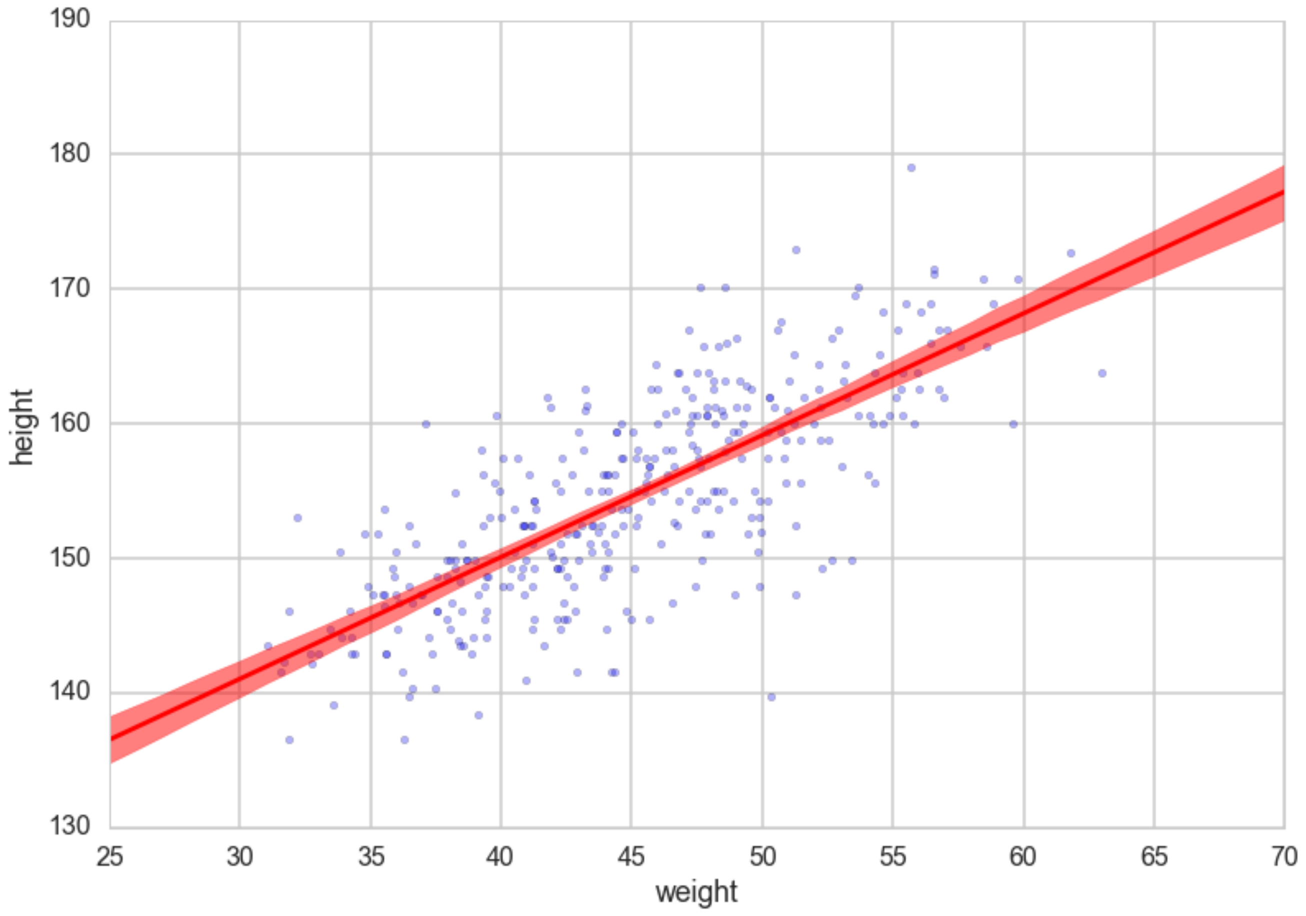

intercept
Frequency
151.5152.0152.5153.0153.5154.0154.5155.0155.5156.0
intercept
Sample value
0
2000
4000
6000
8000
10000
slope
Frequency
10
8
6
4
2
0
0.2
0.4
0.6
0.8
1.0
1.2
slope
Sample value
1.2
1.0
0.8
0.6
0.4
0.2
sigma
Frequency
2.5
2.0
1.5
1.0
0.5
0.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
sigma
Sample value
7.5
7.0
6.5
6.0
5.5
5.0
4.5
mu
Frequency
135
140
145
150
155
160
165
170
175
180
mu
Sample value

Posteriors

```
meanweight = df2.weight.mean()
weightgrid = np.arange(25, 71)
mu_pred = np.zeros((len(weightgrid), len(tr2c)))
for i, w in enumerate(weightgrid):
    mu_pred[i] = tr2c['intercept'] + tr2c['slope'] * (w - meanweight)

mu_mean = mu_pred.mean(axis=1)
mu_hpd = pm.hpd(mu_pred.T)
```



Posteriors on a grid



Why so tight?

Posterior predictive

At data:

```
postpred = pm.sample_ppc(tr2c, 1000, hm2c)
100%|██████████| 1000/1000 [00:19<00:00, 57.56it/s]    | 1/1000 [00:00<08:17,  2.01it/s]
```

On a full grid:

```
n_ppredsamps=1000
weightgrid = np.arange(25, 71)
meanweight = df2.weight.mean()
ppc_samples=np.zeros((len(weightgrid), n_ppredsamps))

for j in range(n_ppredsamps):
    k=np.random.randint(len(tr2c))#samples with replacement
    musamps = tr2c['intercept'][k] + tr2c['slope'][k] * (weightgrid - meanweight)
    sigmasamp = tr2c['sigma'][k]
    ppc_samples[:,j] = np.random.normal(musamps, sigmasamp)
```

Predictives at data and on grid