Discrete Mathematics 118 (1993) 45–56
North-Holland

The $3x+1$ problem: new lower bounds on nontrivial cycle lengths

Shalom Eliahou

Section de Mathématiques, Université de Genève, C.P. 240, 1211 Genève 24, Switzerland

Received 20 August 1991
Revised 19 November 1991

Abstract

Eliahou, S., The $3x+1$ problem: new lower bounds on nontrivial cycle lengths, Discrete Mathematics 118 (1993) 45–56.

Let $T: \mathbb{N} \to \mathbb{N}$ be the function defined by $T(n)=n/2$ if n is even, $T(n)=(3n+1)/2$ if n is odd. We show, among other things, that any nontrivial cyclic orbit under iteration of T must contain at least 17 087 915 elements.

1. Introduction

Let $T: \mathbb{N} \to \mathbb{N}$ be the function defined by

$$T(n)=\begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

where $\mathbb{N}$ denotes the set of positive integers $1, 2, \ldots$

For every $k \in \mathbb{N}$, we will denote by $T^{(k)}$ the kth iterate of the function T. A well-known conjecture, *the* $3x+1$ *problem*, states that, given any starting value $n \in \mathbb{N}$, there is a positive integer k such that $T^{(k)}(n)=1$. See [3] for a complete survey on this problem. As stated in [3], the conjecture has been verified with a computer up to $n=2^{40} \simeq 1.1 \times 10^{12}$ by N. Yoneda. (And later on, up to $3 \cdot 10^{12}$ by K. Ishihata [5].)

Given $n \in \mathbb{N}$, we will call *trajectory* of n the set of iterates

$$\Omega(n)=\{n, T(n), T^{(2)}(n), \ldots\}.$$

A trajectory Ω will be called a *cycle* (of length k) if $T^{(k)}(x)=x$ for all $x \in \Omega$, where $k=\operatorname{Card}\Omega$. For example, $\Omega(1)=\{1,2\}$ is a cycle of length 2, called the *trivial cycle*.

The $3x+1$ conjecture ($1 \in \Omega(n)$ for all $n \in \mathbb{N}$) implies in particular that $\Omega(1)$ should be the *only cycle* of T. Various authors have observed that nontrivial cycles of T (if any)

Correspondence to: Shalom Eliahou, Section de Mathématiques, Universite dé Genève, C.P. 240, 1211 Genève 24, Switzerland.

must be very big. For example, a result of Crandall [2] leads to the estimate $\operatorname{Card}\Omega \geqslant 275\,000$ if Ω is a nontrivial cycle of T, provided $\min\Omega > 2^{40}$ [3]. In this paper, we obtain a somewhat stronger result.

Theorem 1.1. *Let Ω be a nontrivial cycle of T. Provided* $\min\Omega > 2^{40}$, *we have*

$$\operatorname{Card}\Omega = 301\,994a + 17\,087\,915b + 85\,137\,581c,$$

where a, b, c are nonnegative integers, $b > 0$, and $ac = 0$. In particular, the smallest admissible values for $\operatorname{Card}\Omega$ *are* 17 087 915, 17 389 909, 17 691 903, *and so on.*

A basic constraint on cycle lengths is proved in Section 2. In Section 3, we recall elementary facts about continued fraction expansions, and derive a classical result about one-sided Diophantine approximation. The tools gathered enable us, in Section 4, to express numerically the results of Section 2, and thus prove the theorem above, and related statements.

Throughout the paper, we will denote by $[x]$ the integral part of a real number x.

2. Constraints on cycle lengths

The present paper is based on the following observation, which shows that the inverse proportion of odd elements in a cycle is very close to $\log_2(3)$, in a precise sense.

Theorem 2.1. *Let Ω be a cycle of T, and let $\Omega_1 \subset \Omega$ denote the subset of its odd elements. Then*

$$\log_2(3 + M^{-1}) < \frac{\operatorname{Card}\Omega}{\operatorname{Card}\Omega_1} \leqslant \log_2(3 + m^{-1}),$$

where $M = \max\Omega$, $m = \min\Omega$. *In fact, a sharper right inequality holds*:

$$\frac{\operatorname{Card}\Omega}{\operatorname{Card}\Omega_1} \leqslant \log_2(3 + \mu),$$

where $\mu = (1/\operatorname{Card}\Omega_1)\,(\sum_{n\in\Omega_1} n^{-1})$.

The proof is based on a Diophantine equation associated with any cycle of T.

Lemma 2.2. *Let Ω, Ω_1 be as above, and let* $k = \operatorname{Card}\Omega$. *Then*

$$\prod_{n\in\Omega_1} (3 + n^{-1}) = 2^k.$$

Proof. Since Ω is a cycle, we have

$$\prod_{n\in\Omega} n = \prod_{n\in\Omega} T(n),$$

and, therefore,

$$\prod_{n\in\Omega}\frac{T(n)}{n}=1.$$

Substituting into the above the value

$$\frac{T(n)}{n}=\begin{cases}1/2 & \text{if } n \text{ is even,}\\ (3+n^{-1})/2 & \text{if } n \text{ is odd,}\end{cases}$$

we obtain

$$\prod_{n\in\Omega_1}(3+n^{-1})=2^k,$$

as claimed. □

Proof of Theorem 2.1. Let us set $k=\operatorname{Card}\Omega$, $k_1=\operatorname{Card}\Omega_1$. By Lemma 2.2, we have

$$(3+M^{-1})^{k_1}<2^k\leqslant(3+m^{-1})^{k_1},$$

where $M=\max\Omega$, $m=\min\Omega$. (The strict inequality on the left is due to the fact that M is even and, therefore, larger than $\max\Omega_1$.) Taking logarithms in base 2, we obtain

$$\log_2(3+M^{-1})<\frac{k}{k_1}\leqslant\log_2(3+m^{-1}),$$

as claimed. For the sharper right inequality, we invoke the relationship between the geometric and arithmetic means of any nonnegative real numbers $\alpha_1,\ldots,\alpha_r$, namely:

$$\sqrt[r]{\prod_i\alpha_i}\leqslant\frac{1}{r}\sum_i\alpha_i.$$

In our situation, this gives

$$\begin{aligned}2^k=\prod_{n\in\Omega_1}(3+n^{-1})&\leqslant\left(\frac{1}{k_1}\sum_{n\in\Omega_1}(3+n^{-1})\right)^{k_1}\\&=\left(3+\frac{1}{k_1}\sum_{n\in\Omega_1}n^{-1}\right)^{k_1}.\end{aligned}$$

Taking logarithms in base 2 again, we obtain the claimed inequality

$$\frac{k}{k_1}\leqslant\log_2(3+\mu). \qquad \square$$

Remark. In the situation and notation of Theorem 2.1, we have the stronger bound

$$\mu\leqslant\tfrac{8}{9}\,m^{-1}$$

if $m=\min\Omega>1$, as observed by Roland Bacher after reading a preliminary version of this paper.

Indeed, it is easy to see that if $n\in\Omega_1$ and $n<\frac{9}{7}m$, then $T(n)\in\Omega_1$ and $T(n)\geqslant\frac{9}{7}m$, so that at least half of Ω_1 lies in the interval $[\frac{9}{7}m,\infty)$. It follows that

$$\mu\leqslant\frac{1}{k_1}\left(\frac{k_1}{2}m^{-1}+\frac{k_1}{2}\frac{7}{9}m^{-1}\right)$$
$$=\frac{8}{9}m^{-1}.$$

For simplicity, we will only use the inequalities

$$\log_2(3)<\frac{\operatorname{Card}\Omega}{\operatorname{Card}\Omega_1}\leqslant\log_2(3+m^{-1})$$

in the sequel.

We obtain a reformulation of Theorem 2.1 that is easier to work with by defining functions

$$K:\mathbb{N}\to\mathbb{N}\quad\text{and}\quad L:\mathbb{N}\to\mathbb{N}$$

as follows: for each $m\in\mathbb{N}$, $K(m)$ is the smallest positive integer k such that

$$\log_2(3)<\frac{k}{l}\leqslant\log_2(3+m^{-1}) \tag{2.1}$$

for some positive integer l. Similarly, $L(m)$ is the smallest positive integer l such that (2.1) holds, for some positive integer k. Observe that K and L are *nondecreasing functions* of m. Theorem 2.1 yields the following corollary.

Corollary 2.3. *Let Ω be a cycle of T, and let $\Omega_1\subset\Omega$ denote the subset of its odd elements. Then*

$$\operatorname{Card}\Omega\geqslant K(\min\Omega),$$
$$\operatorname{Card}\Omega_1\geqslant L(\min\Omega).\qquad\square$$

In Section 4, we will see that $K(2^{40})=17\,087\,915$, implying (together with Yoneda's computer result) that no nontrivial cycle of T can have elements fewer than this.

Remark. Some authors consider, instead of T, a slightly different function, namely

$$T'(n)=\frac{3n+1}{2^{r(n)}}\quad(n\text{ odd}),$$

where $2^{r(n)}$ is the highest power of 2 dividing $3n+1$. It is clear that the trajectory of an odd m under T' coincides exactly with $\Omega_1(m)$, the subset of odd elements in the T-trajectory $\Omega(m)$. Thus, the statement about card Ω_1 in Corollary 2.3 yields a lower bound for the lengths of T'-cycles, as well.

Now, given $m \in \mathbb{N}$, how do we determine $K(m)$ and $L(m)$? A complete answer can be given in terms of the continued fraction expansion of $\log_2(3)$. This is explained in the next section, in a more general context.

3. Continued fractions and rational approximations

Prompted by Theorem 2.1 and its corollary, we will examine here the following problem.

Problem. Given an irrational number $\theta > 0$ and a real number θ' such that $\theta < \theta' < [\theta] + 1$, what is the smallest positive numerator $k = k(\theta, \theta')$ such that

$$\theta < \frac{k}{l} \leqslant \theta' \tag{3.1}$$

for some $l \in \mathbb{N}$? Similarly, what is the smallest positive denominator $l = l(\theta, \theta')$ such that (3.1) holds, for some $k \in \mathbb{N}$?

The numerator version of the problem does not seem to be explicitly mentioned in the literature, but, of course, it is equivalent to the classical denominator version. A complete answer, which goes back to Wallis (1685), can be given in terms of the continued fraction expansion of θ. See [1, Ch. 32], from which our presentation below is (loosely) inspired.

We start with a key notion in the theory.

Definition. Two fractions (p/q), (p'/q') (with p, p', q, q' nonnegative integers) are said to form a *Farey pair* [4] if

$$pq' - p'q = \pm 1.$$

A crucial property of Farey pairs is that any intermediate fraction must have a denominator strictly larger than either denominator of the pair. More precisely, we have the following result.

Lemma 3.1. *Let $p/q < p'/q'$ be a Farey pair. Then, any intermediate fraction $p/q < x/y < p'/q'$ with $y > 0$ is of the form*

$$\frac{x}{y} = \frac{ap + bp'}{aq + bq'}$$

where a, b are positive integers. In particular, $x \geqslant p + p'$ and $y \geqslant q + q'$.

Proof. The lemma is trivial if the Farey pair is just $0/1 < 1/1$. For the general case, consider the matrix

$$F = \begin{pmatrix} p' - p & p \\ q' - q & q \end{pmatrix} \in SL_2(\mathbb{Z}),$$

whose inverse matrix is

$$F^{-1}=\begin{pmatrix} q & -p \\ q-q' & p'-p \end{pmatrix}.$$

Then, F provides a bijection

$$f:\left[\frac{0}{1},\frac{1}{1}\right]\cap\mathbb{Q}\to\left[\frac{p}{q},\frac{p'}{q'}\right]\cap\mathbb{Q}$$

defined by

$$f\left(\frac{u}{v}\right)=\frac{(p'-p)u+pv}{(q'-q)u+qv},$$

whose inverse is

$$f^{-1}\left(\frac{u}{v}\right)=\frac{qu-pv}{(q-q')u-(p-p')v}.$$

Suppose now that $p/q<x/y<p'/q'$. Defining $a=p'y-q'x$ and $b=qx-py$, then a,b are positive integers by hypothesis, and

$$f^{-1}\left(\frac{x}{y}\right)=\frac{qx-py}{(q-q')x-(p-p')y}=\frac{b}{a+b}$$

by construction. Thus,

$$\frac{x}{y}=ff^{-1}\left(\frac{x}{y}\right)=f\left(\frac{b}{a+b}\right)=\frac{ap+bp'}{aq+bq'},$$

as claimed. $\square$

This lemma will be important for the proof of Theorem 1.1. (See Section 4.)

Let us now return to our irrational number $\theta>0$. We will denote by $[a_0,a_1,a_2,\ldots]$ the simple continued fraction expansion of θ, that is,

$$\theta=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots}}}$$

where the a_i are integers for all $i\geqslant 0$, and $a_i\geqslant 1$ for all $i\geqslant 1$. Recall that the a_i are uniquely determined as follows: $a_i=[\theta_i]$, where $\theta_0=\theta$ and $\theta_{i+1}=1/(\theta_i-a_i)$.

Denoting by $[a_0,a_1,\ldots,a_n]$ the truncated fraction

$$a_0+\cfrac{1}{a_1+\cfrac{1}{\cdots+\cfrac{1}{a_n}}}$$

we have $[a_0, a_1, \dots, a_n] = p_n/q_n$, where $\gcd(p_n, q_n) = 1$ and p_n, q_n are obtained recursively as follows:

$$p_n = a_n p_{n-1} + p_{n-2},$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n \geqslant 0), \tag{3.2}$$

with the initial values $p_{-2} = 0$, $p_{-1} = 1$ and $q_{-2} = 1$, $q_{-1} = 0$.

The fractions p_n/q_n will be called here the *principal convergents* to θ. It is classical, and not very difficult, to prove that

any two consecutive fractions (p_n/q_n), (p_{n+1}/q_{n+1}) *form a Farey pair.*

More precisely,

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}; \tag{3.3}$$

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \theta < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}; \tag{3.4}$$

$$\lim \frac{p_n}{q_n} = \theta, \quad \text{as } n \text{ goes to infinity.} \tag{3.5}$$

We will only be interested in the principal convergents which are larger than θ, namely those p_n/q_n with n odd. Two such consecutive fractions $p_{n+2}/q_{n+2} < p_n/q_n$ do not always form a Farey pair, in general. However, we can describe an intermediate chain of Farey pairs, as follows. Let us set

$$p_{n,i} = p_n + i p_{n+1}, \qquad q_{n,i} = q_n + i q_{n+1},$$

where $n \geqslant -2$ and i is a nonnegative integer. We have $p_{n,0} = p_n$. Furthermore, since $p_n + a_{n+2} p_{n+1} = p_{n+2}$ by (3.2), we will always suppose $i \leqslant a_{n+2} - 1$.

A fraction of the form

$$\frac{p_{n,i}}{q_{n,i}} = \frac{p_n + i p_{n+1}}{q_n + i q_{n+1}}$$

with $n \geqslant -2$ and $0 \leqslant i \leqslant a_{n+2} - 1$ will be called a *convergent* to θ. (Some authors say 'intermediate convergent'.) If n is odd, we say that $p_{n,i}/q_{n,i}$ is an *upper convergent* to θ, because it is larger than θ. More generally, the following is easy to prove, using (3.3) and (3.4):

If n is odd and $a = a_{n+2}$, then

$$\frac{p_{n+2}}{q_{n+2}} = \frac{p_{n,a}}{q_{n,a}} < \frac{p_{n,a-1}}{q_{n,a-1}} < \cdots < \frac{p_{n,1}}{q_{n,1}} < \frac{p_n}{q_n}; \tag{3.6}$$

any two consecutive upper convergents form a Farey pair. (3.7)

Note that the numerator sequence $p_n, p_{n,1}, \dots, p_{n,a-1}, p_{n+2}$ is strictly increasing, and the same is true of the denominator sequence. We agree to start the sequence of upper convergents with $p_{-1,1}/q_{-1,1} = (a_0 + 1)/1$.

We are now ready to solve the problem stated at the beginning of this section.

Theorem 3.2. *Let $\theta > 0$ be an irrational number, and let θ' be any number satisfying $\theta < \theta' < [\theta] + 1$. If k, l are positive integers such that*

$$\theta < \frac{k}{l} \leqslant \theta',$$

and if either k or l is minimal for this property, then k/l is one of the upper convergents to θ. More precisely, k/l is the largest upper convergent to θ smaller than or equal to θ'. In particular, in the notation above,

$$k(\theta, \theta') = p_{n,i}, \qquad l(\theta, \theta') = q_{n,i}$$

for some odd positive integer n and for some integer $i = 0, 1, \ldots, a_{n+2} - 1$.

Proof. If k/l is not an upper convergent to θ, then it must lie between two consecutive upper convergents to θ, say

$$\frac{p}{q} < \frac{k}{l} < \frac{p'}{q'}.$$

(Recall that the sequence of upper convergents starts with $(p_{-1,1}/q_{-1,1}) = a_0 + 1 = [\theta] + 1$, and that $(k/l) \leqslant \theta' < [\theta] + 1$.) But (p/q), (p'/q') form a Farey pair, by (3.7). Thus, Lemma 3.1 implies $p < k$ and $q < l$, contradicting the minimality of either k or l. □

4. Back to $3x+1$

In Section 2, we proved that if Ω is a cycle of T, then

$$\operatorname{Card} \Omega \geqslant K(\min \Omega),$$

where $K(m)$ is defined as the smallest positive integer k such that

$$\log_2(3) < \frac{k}{l} \leqslant \log_2(3 + m^{-1})$$

for some positive integer l. (We defined $L(m)$ similarly.)

The tools in Section 3 enable us to determine $K(m)$ and $L(m)$ for any given $m \in \mathbb{N}$. What we need are the continued fraction expansion

$$\theta = [a_0, a_1, a_2, \ldots]$$

of $\theta = \log_2(3)$, and the integers p_n, q_n defined recursively by (3.2). In Table 1, we list a_n, p_n, q_n and $(p_n/q_n) - \theta$ for all $n = 0, 1, \ldots, 35$.

Table 1
Principal convergents to $\theta=\log_2(3)$

n	a_n	p_n	q_n	$(p_n/q_n)-\theta$
0	1	1	1	-5.84×10^{-1}
1	1	2	1	4.15×10^{-1}
2	1	3	2	-8.49×10^{-2}
3	2	8	5	1.50×10^{-2}
4	2	19	12	-1.62×10^{-3}
5	3	65	41	4.03×10^{-4}
6	1	84	53	-5.68×10^{-5}
7	5	485	306	4.81×10^{-6}
8	2	1 054	665	-9.47×10^{-8}
9	23	24 727	15 601	1.68×10^{-9}
10	2	50 508	31 867	-3.28×10^{-10}
11	2	125 743	79 335	6.66×10^{-11}
12	1	176 251	111 202	-4.67×10^{-11}
13	1	301 994	190 537	4.88×10^{-13}
14	55	16 785 921	10 590 737	-7.12×10^{-15}
15	1	17 087 915	10 781 274	1.63×10^{-15}
16	4	85 137 581	53 715 833	-9.32×10^{-17}
17	3	272 500 658	171 928 773	1.50×10^{-17}
18	1	357 638 239	225 644 606	-1.07×10^{-17}
19	1	630 138 897	397 573 379	3.84×10^{-19}
20	15	9 809 721 694	6 189 245 291	-2.23×10^{-20}
21	1	10 439 860 591	6 586 818 670	2.21×10^{-21}
22	9	103 768 467 013	65 470 613 321	-1.01×10^{-22}
23	2	217 976 794 617	137 528 045 312	9.42×10^{-24}
24	5	1 193 652 440 098	753 110 839 881	-2.27×10^{-25}
25	7	8 573 543 875 303	5 409 303 924 479	1.75×10^{-26}
26	1	9 767 196 315 401	6 162 414 764 360	-1.24×10^{-26}
27	1	18 340 740 190 704	11 571 718 688 839	1.60×10^{-27}
28	4	83 130 157 078 217	52 449 289 519 716	-3.97×10^{-29}
29	8	683 381 996 816 440	431 166 034 846 567	4.46×10^{-30}
30	1	766 512 153 894 657	483 615 324 366 283	-3.32×10^{-31}
31	11	9 115 015 689 657 667	5 750 934 602 875 680	2.66×10^{-32}
32	1	9 881 527 843 552 324	6 234 549 927 241 963	-1.20×10^{-33}
33	20	206 745 572 560 704 147	130 441 933 147 714 940	1.98×10^{-35}
34	2	423 372 672 964 960 618	267 118 416 222 671 843	-8.86×10^{-36}
35	1	630 118 245 525 664 765	397 560 349 370 386 783	5.50×10^{-37}

Next, we need the sequence of upper convergents to θ. Recall that these are the fractions of the form

$$\frac{p_{n,i}}{q_{n,i}}=\frac{p_n+ip_{n+1}}{q_n+iq_{n+1}}$$

with n odd and $i=0,1,\ldots,a_{n+2}-1$. The numerators and denominators of all upper convergents to θ between p_{29}/q_{29} and p_9/q_9 are listed, with other information, in Table 2.

Table 2
Upper convergents to θ, and transition points of K

(n, i)	$p_{n,i}$	$q_{n,i}$	$(p_{n,i}/q_{n,i})-\theta$	$\mathrm{tr}(n, i)$
(9, 0)	24 727	15 601	1.68×10^{-9}	1.06×2^{28}
(9, 1)	75 235	47 468	3.32×10^{-10}	1.34×2^{30}
(11, 0)	125 743	79 335	6.66×10^{-11}	1.68×2^{32}
(13, 0)	301 994	190 537	4.88×10^{-13}	1.79×2^{39}
(15, 0)	17 087 915	10 781 274	1.63×10^{-15}	1.04×2^{48}
(15, 1)	102 225 496	64 497 107	1.95×10^{-16}	1.09×2^{51}
(15, 2)	187 363 077	118 212 940	6.42×10^{-17}	1.66×2^{52}
(17, 0)	272 500 658	171 928 773	1.50×10^{-17}	1.77×2^{54}
(19, 0)	630 138 897	397 573 379	3.84×10^{-19}	1.08×2^{60}
(21, 0)	10 439 860 591	6 586 818 670	2.21×10^{-21}	1.46×2^{67}
(21, 1)	114 208 327 604	72 057 431 991	1.10×10^{-22}	1.84×2^{71}
(23, 0)	217 976 794 617	137 528 045 312	9.42×10^{-24}	1.35×2^{75}
(23, 1)	1 411 629 234 715	890 638 885 193	1.26×10^{-24}	1.25×2^{78}
(23, 2)	2 605 281 674 813	1 643 749 725 074	5.79×10^{-25}	1.37×2^{79}
(23, 3)	3 798 934 114 911	2 396 860 564 955	3.26×10^{-25}	1.21×2^{80}
(23, 4)	4 992 586 555 009	3 149 971 404 836	1.93×10^{-25}	1.02×2^{81}
(23, 5)	6 186 238 995 107	3 903 082 244 717	1.12×10^{-25}	1.77×2^{81}
(23, 6)	7 379 891 435 205	4 656 193 084 598	5.72×10^{-26}	1.73×2^{82}
(25, 0)	8 573 543 875 303	5 409 303 924 479	1.75×10^{-26}	1.41×2^{84}
(27, 0)	18 340 740 190 704	11 571 718 688 839	1.60×10^{-27}	1.93×2^{87}
(27, 1)	101 470 897 268 921	64 021 008 208 555	2.58×10^{-28}	1.50×2^{90}
(27, 2)	184 601 054 347 138	116 470 297 728 271	1.23×10^{-28}	1.56×2^{91}
(27, 3)	267 731 211 425 355	168 919 587 247 987	7.31×10^{-29}	1.32×2^{92}
(27, 4)	350 861 368 503 572	221 368 876 767 703	4.63×10^{-29}	1.04×2^{93}
(27, 5)	433 991 525 581 789	273 818 166 287 419	2.98×10^{-29}	1.62×2^{93}
(27, 6)	517 121 682 660 006	326 267 455 807 135	1.86×10^{-29}	1.29×2^{94}
(27, 7)	600 251 839 738 223	378 716 745 326 851	1.05×10^{-29}	1.14×2^{95}
(29, 0)	683 381 996 816 440	431 166 034 846 567	4.46×10^{-30}	1.36×2^{96}

Finally, for the problem at hand, it will be convenient to determine the *transition points* of the function K, namely, those rare integers m for which

$$K(m) > K(m-1).$$

(Indeed, the function K tends to be constant on long intervals; for example, $K(m) = 301\,994$ for all $2^{33} \leqslant m \leqslant 2^{39}$.) By Theorem 3.2, the range of the function K is exactly the set of numerators of upper convergents to θ, namely, the set $\{p_{n,i} \mid n \geqslant -1$ odd, $0 \leqslant i \leqslant a_{n+2}-1\}$. Thus, there is one transition point of K per relevant pair (n, i) and so we define

$$\mathrm{tr}(n, i) = \text{the least integer } m \text{ such that } K(m) \text{ equals } p_{n,i}.$$

(n odd, $0 \leqslant i \leqslant a_{n+2}-1$.)

How do we determine these transition points? Well, let us define

$$\delta(n, i) = \frac{p_{n,i}}{q_{n,i}} - \theta$$

and

$$\Delta(m)=\log_2(3+m^{-1})-\theta.$$

By definition, $\mathrm{tr}(n,i)$ is the least integer m for which $\Delta(m)\leqslant\delta(n,i)$. Now, given $\delta=\Delta(m)$, we can recover m by the formula

$$m=\frac{1}{3(2^{\delta}-1)}.$$

Setting $\delta=\delta(n,i)$, it follows from the above that

$$\mathrm{tr}(n,i)=\left[\frac{1}{3(2^{\delta(n,i)}-1)}\right]+1.$$

(n odd, $0\leqslant i\leqslant a_{n+2}-1$.) The transition points of K are listed in Table 2, whose content is as follows. First, we list all pairs (n,i) with n odd, $9\leqslant n\leqslant 27$, and $i=0,1,\ldots,a_{n+2}-1$, plus the pair $(29,0)$. Then, we list $p_{n,i}, q_{n,i}$, and approximate values of $\delta(n,i)$ and $\mathrm{tr}(n,i)$. Note that the list of $\mathrm{tr}(n,i)$ in the range considered gives the *complete set of transition points of K in the interval* $[2^{28},2^{97}]$.

To show how to use the tables, let us take $m=2^{40}$. Looking at Table 2, we see that there is a transition point of K near 2^{39}, namely $\mathrm{tr}(13,0)$, the next one being $\mathrm{tr}(15,0)$ which is near 2^{48}. Thus, $K(2^{39})=p_{13,0}=301\,994$, and $K(2^{40})=K(2^{48})=p_{15,0}=17\,087\,915$. By Corollary 2.3, this proves that $\mathrm{Card}\,\Omega\geqslant 17\,087\,915$ for any cycle Ω, provided $\min\Omega>2^{40}$.

The fact that $K(2^{39})=301\,994$ is rather striking, because it means that, had Yoneda verified the conjecture only up to 2^{39}, we could only have claimed 301 994 as a lower bound for $\mathrm{Card}\,\Omega$!

The fact that the next transition point occurs near 2^{48} means that we have to push the computer verification up to 2^{49} if we want to improve, by the same arguments, the lower bound 17 087 915. If we can push the verification up to 2^{49}, then the lower bound on cycle lengths will jump to 102 225 496. Up to 2^{52}, the lower bound jumps to 187 363 077. And so on.

It remains to prove the more precise statement of Theorem 1.1, namely,

$$\mathrm{Card}\,\Omega=301\,994a+17\,087\,915b+85\,137\,581c,$$

where Ω is a nontrivial cycle, $\min\Omega>2^{40}$, a,b,c are nonnegative integers, $b>0$, $ac=0$.

Proof of Theorem 1.1. Let us write $k=\mathrm{Card}\,\Omega$, $l=\mathrm{Card}\,\Omega_1$. Looking at Tables 1 and 2, we observe the following situation:

$$\frac{p_{16}}{q_{16}}<\log_2(3)<\frac{p_{15}}{q_{15}}<\log_2(3+2^{-40})<\frac{p_{13}}{q_{13}},$$

and, of course, by Theorem 2.1,

$$\frac{k}{l}\in[\log_2(3),\log_2(3+2^{-40})].$$

Observe that $p_{13} = 301\,994$, $p_{15} = 17\,087\,915$, and $p_{16} = 85\,137\,581$. Furthermore, the pair (p_{15}/q_{15}), (p_{13}/q_{13}) is of Farey type, by (3.6), (3.7) and the fact that $a_{15} = 1$; as is the pair (p_{16}/q_{16}), (p_{15}/q_{15}), by (3.3).

There are three possibilities for k/l. Either k/l lies in the open interval $((p_{16}/q_{16}), (p_{15}/q_{15}))$, or $k/l = (p_{15}/q_{15})$, or k/l lies in the open interval $((p_{15}/q_{15}), (p_{13}/q_{13}))$. In the first case, Lemma 3.1 implies that $k = p_{15}b + p_{16}c$, where $b, c \in \mathbb{N}$. In the second case, $k = p_{15}b$, for some $b \in \mathbb{N}$. Finally, in the last case, Lemma 3.1 again implies that $k = p_{13}a + p_{15}b$, where $a, b \in \mathbb{N}$. This completes the proof of Theorem 1.1. $\square$

Finally, for the pleasure of it, let us determine a lower bound for the length of any cycle whose minimum is greater than 2^{1000}: we find that

$$K(2^{1000}) = p_{293} = 3\,455\,855\,834\,088\,510\,383\,194\,418\,949\,502\,834\,449\,170\,762$$
$$556\,421\,140\,329\,434\,475\,955\,563\,654\,995\,933\,855\,896\,907\,118\,422\,521\,674$$
$$356\,842\,952\,193\,311\,231\,504\,604\,271\,559\,270\,666\,467\,099\,370\,224\,072\,891,$$

which is approximately equal to 1.05×2^{500}, or 3.45×10^{150}.

Note added in proof. Lemma 2.2 can also be found in: B. Thwaites, My conjecture, Bull. Inst. Math. Appl. 21; 35–41.

Acknowledgements

The author gratefully acknowledges support from the Fonds National Suisse de la Recherche Scientifique during the preparation of this paper.

I would like to thank Roland Bacher, Daniel Coray, Michel Kervaire, Y.-F.S. Pétermann, John Steinig and Pablo del Val each of whom had some positive influence on this paper. The numerical results were obtained on a Toshiba 1600 portable computer, using the lovely programming language muLISP from Soft Warehouse, Inc.

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