Procesamiento Digital de Imágenes Biomédicas

Morfología matemática en imágenes binarias

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- Language of mathematical morphology: set theory
- Sets $\equiv$ objects in an image
- Binary images: sets $\in Z^2$
- Gray-scale images: sets $\in Z^3$

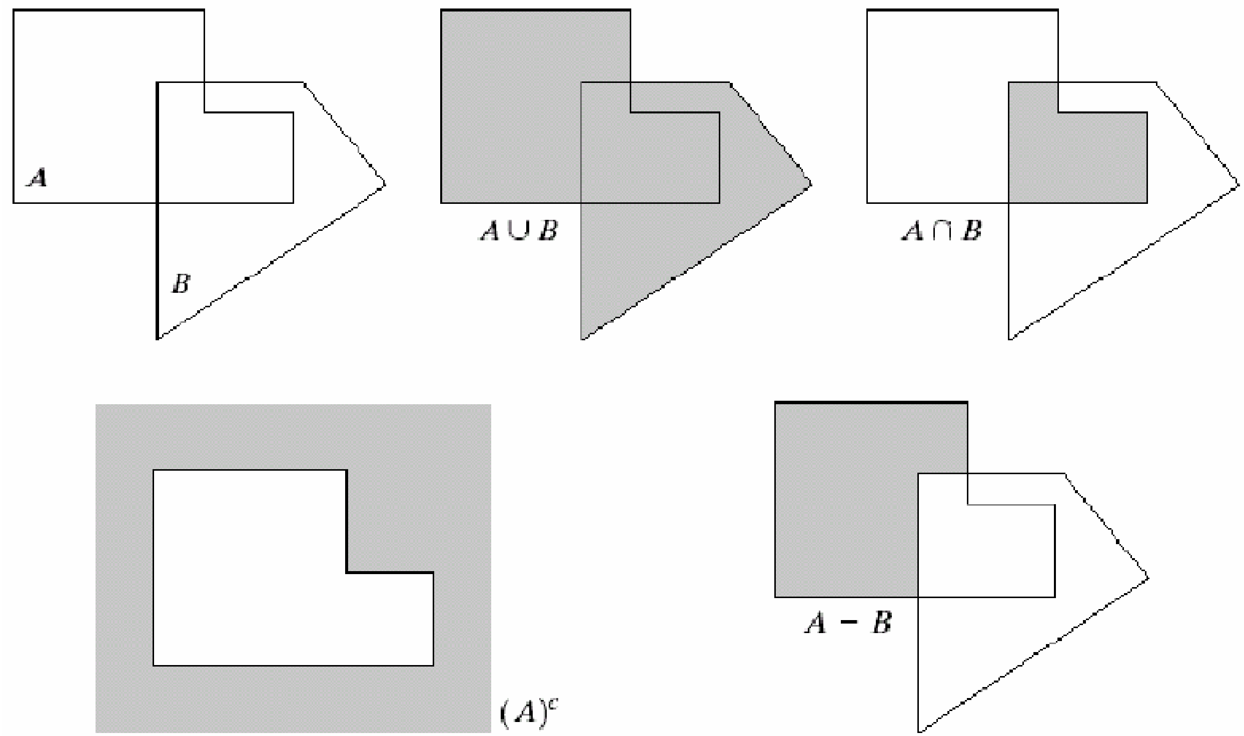
$\boxed{\text{9.1 Preliminaries}}$: 9.1.1 Set theory

- Let A be a set in Z^2. If $a = (a_1, a_2)$ is an element of A, then we write $a \in A$
- Subset, union, intersection:

$$A \subseteq B,\ C = A \cup B,\ D = A \cap B$$

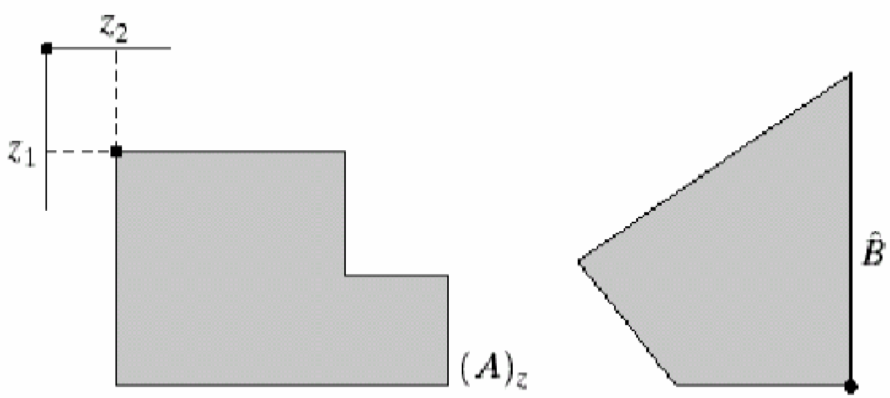
- Disjoint or mutually exclusive: $A \cap B = \emptyset$
- Complement: $A^c = \{w | w \notin A\}$
- Difference: $A - B = \{w | w \in A, w \notin B\} = A \cap B^c$
- Reflection: $\hat{B} = \{w | w = -b, \text{ for } b \in B\}$
- Translation of set A by point $z = (z_1, z_2)$:

$$(A)_z = \{c | c = a + z, \text{ for } a \in A\}$$



a b c
d e

FIGURE 9.1
(a) Two sets A and B. (b) The union of A and B. (c) The intersection of A and B. (d) The complement of A. (e) The difference between A and B.



a b

FIGURE 9.2
(a) Translation of A by z.
(b) Reflection of B. The sets A and B are from Fig. 9.1.

9.1.2 Logic operations involving binary images

TABLE 9.1
The three basic logical operations.

p	q	p AND q (also $p \cdot q$)	p OR q (also $p + q$)	NOT (p) (also $\bar{p}$)
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

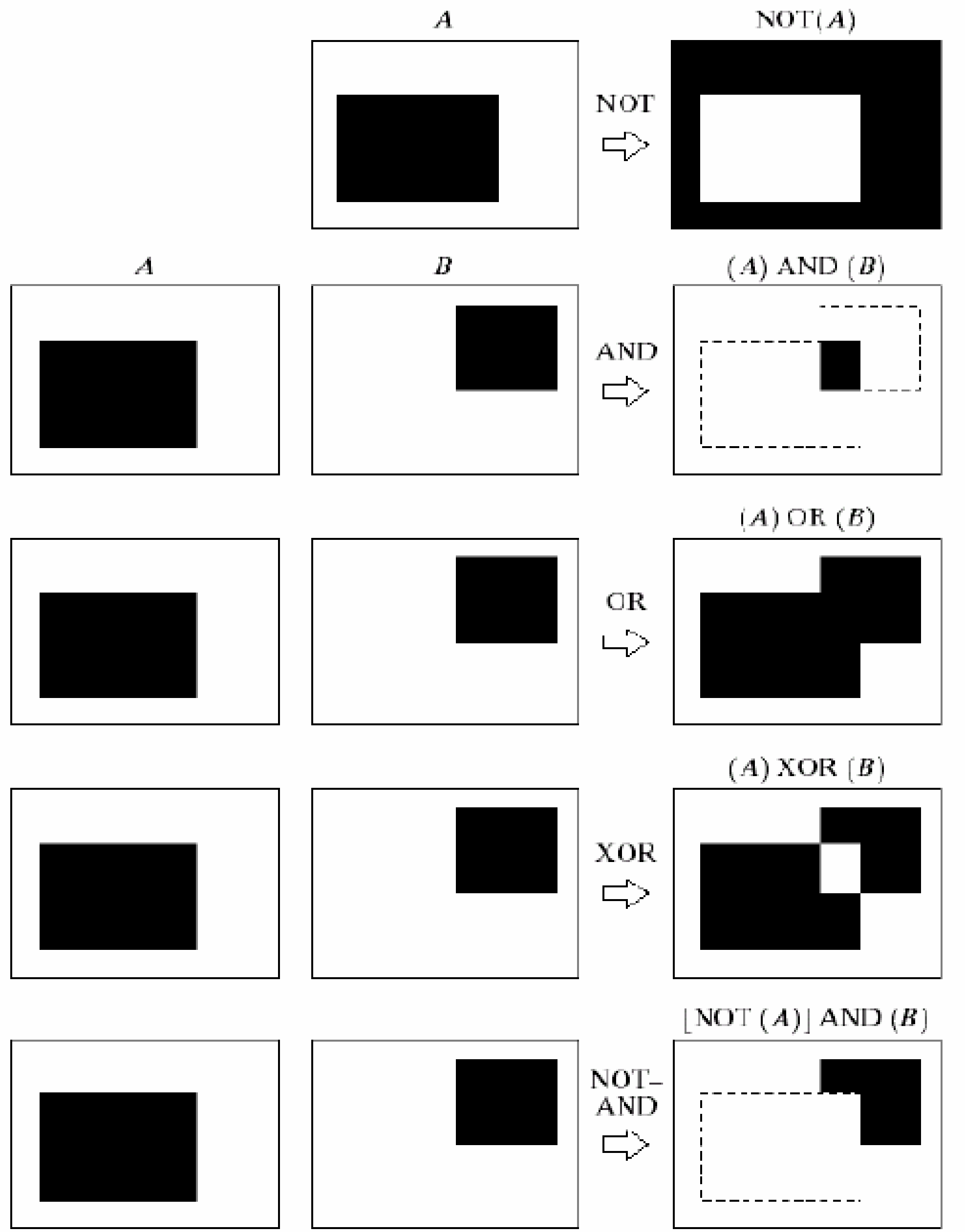


FIGURE 9.3 Some logic operations between binary images. Black represents binary 1s and white binary 0s in this example.

9.2 Dilation and erosion

These operations are fundamental to morphological processing

9.2.1 Dilation

With A and B sets in Z^2, the dilation of A by B, is defined as

$$A \oplus B = \left\{ z | (\hat{B})_z \cap A \neq \emptyset \right\}$$

$$A \oplus B = \left\{ z | \left[(\hat{B})_z \cap A \right] \subseteq A \right\}$$

- B is the **structuring element**

- Note that dilation is a convolution process

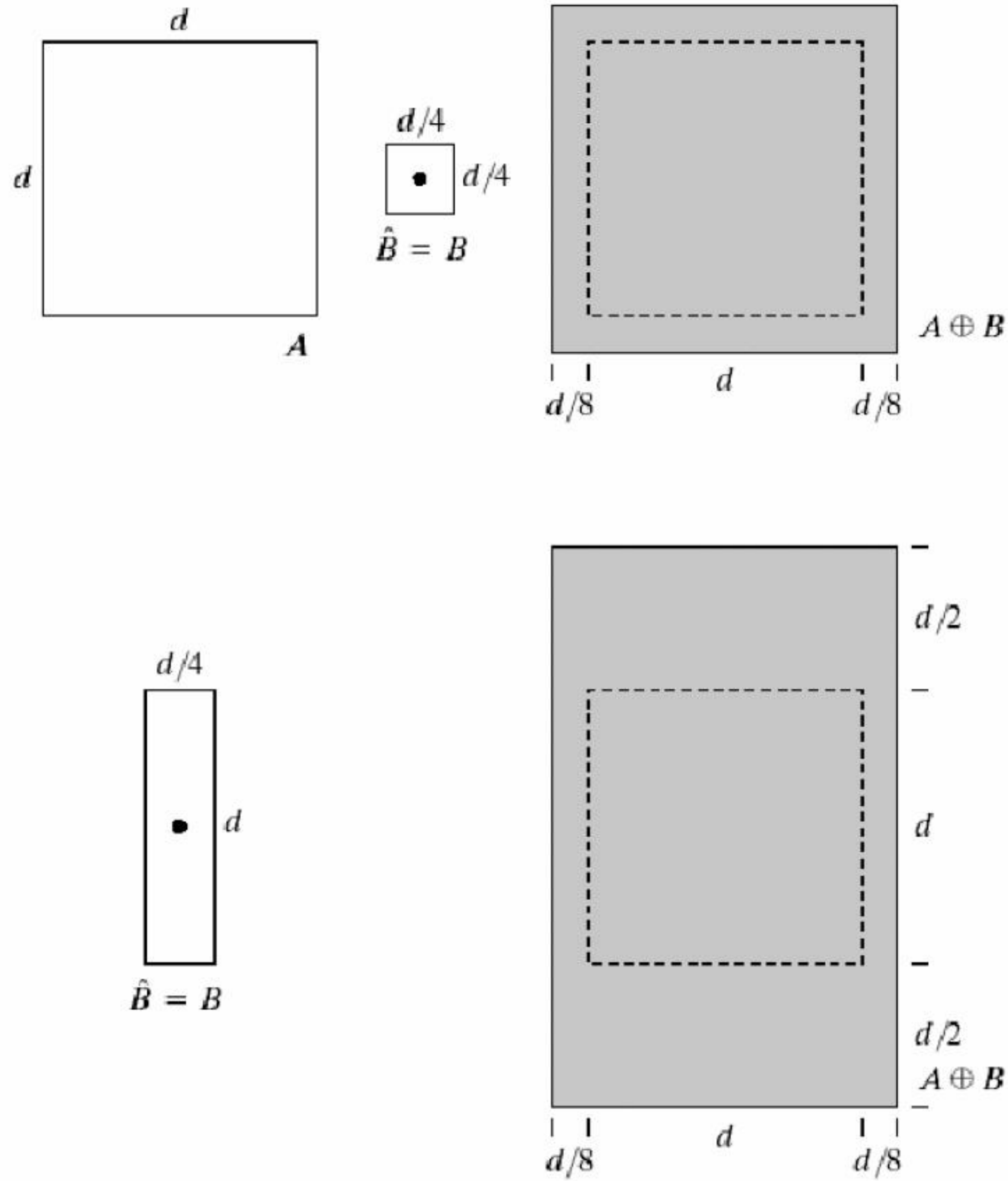
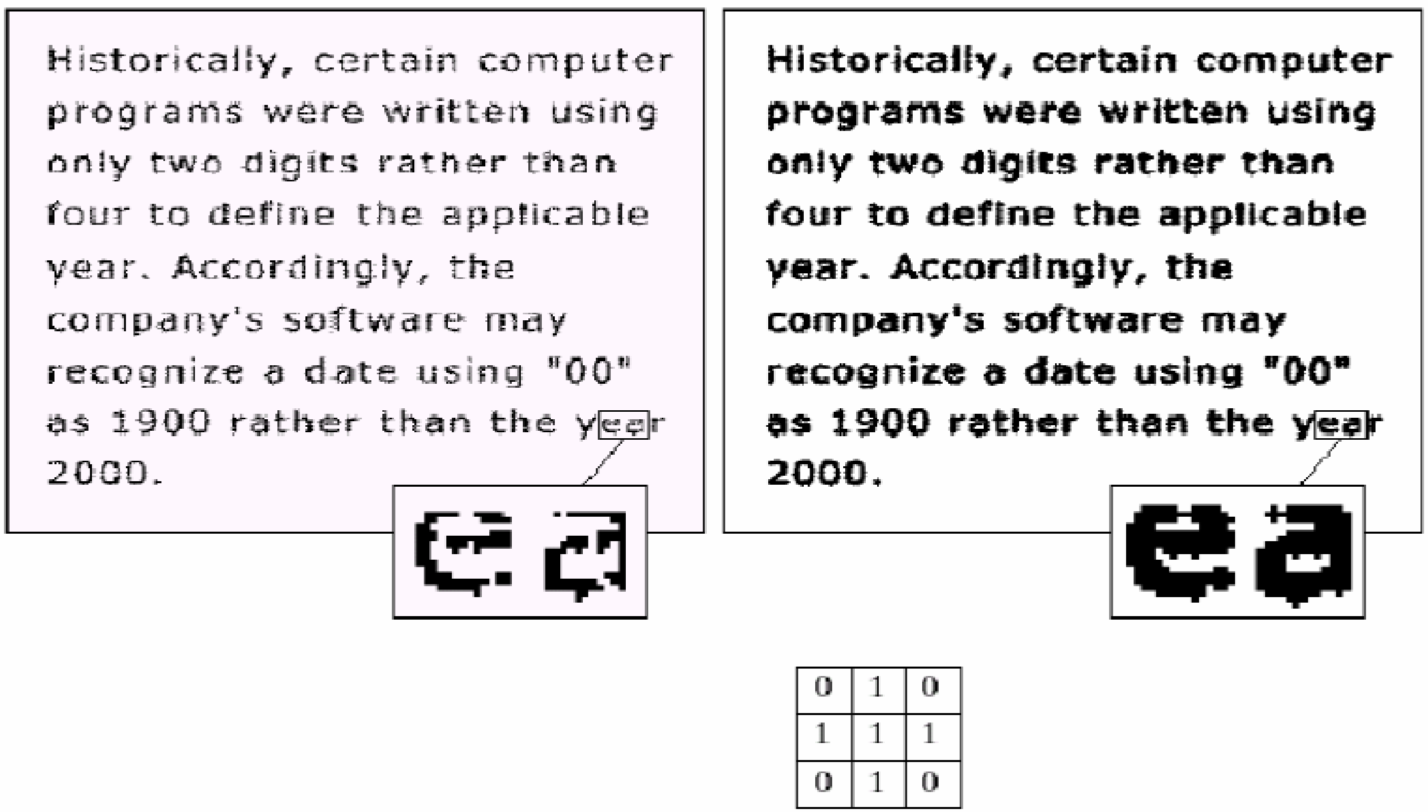


FIGURE 9.4
(a) Set A.
(b) Square structuring element (dot is the center).
(c) Dilation of $\boldsymbol{A}$ by $\boldsymbol{B}$, shown shaded.
(d) Elongated structuring element.
(e) Dilation of $\boldsymbol{A}$ using this element.

Example 9.1 Using dilation to bridge gaps



a c
b

FIGURE 9.5
(a) Sample text of poor resolution with broken characters (magnified view). (b) Structuring element. (c) Dilation of (a) by (b). Broken segments were joined.

9.2.2 Erosion

With A and B sets in Z^2, the erosion of A by B, is defined as

$$A \ominus B = \{z | (B)_z \subseteq A\}$$

Dilation and erosion are duals of each other with respect to set complementation and reflection, that is

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

Proof:

$$\begin{aligned}
(A \ominus B)^c &= \{z | (B)_z \subseteq A\}^c \\
&= \{z | (B)_z \cap A^c = \emptyset\}^c \\
&= \{z | (B)_z \cap A^c \neq \emptyset\} \\
&= A^c \oplus \hat{B}
\end{aligned}$$

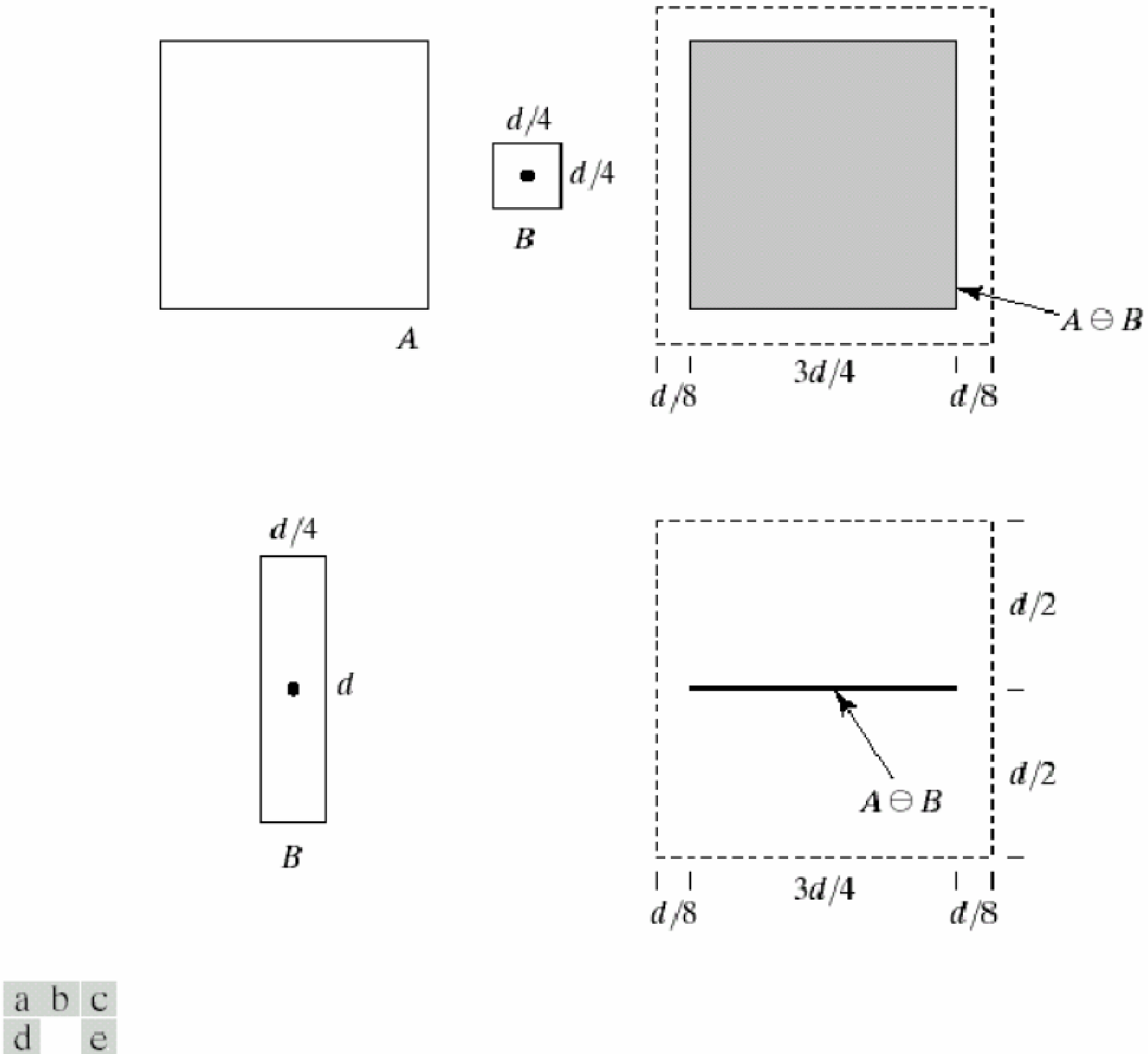
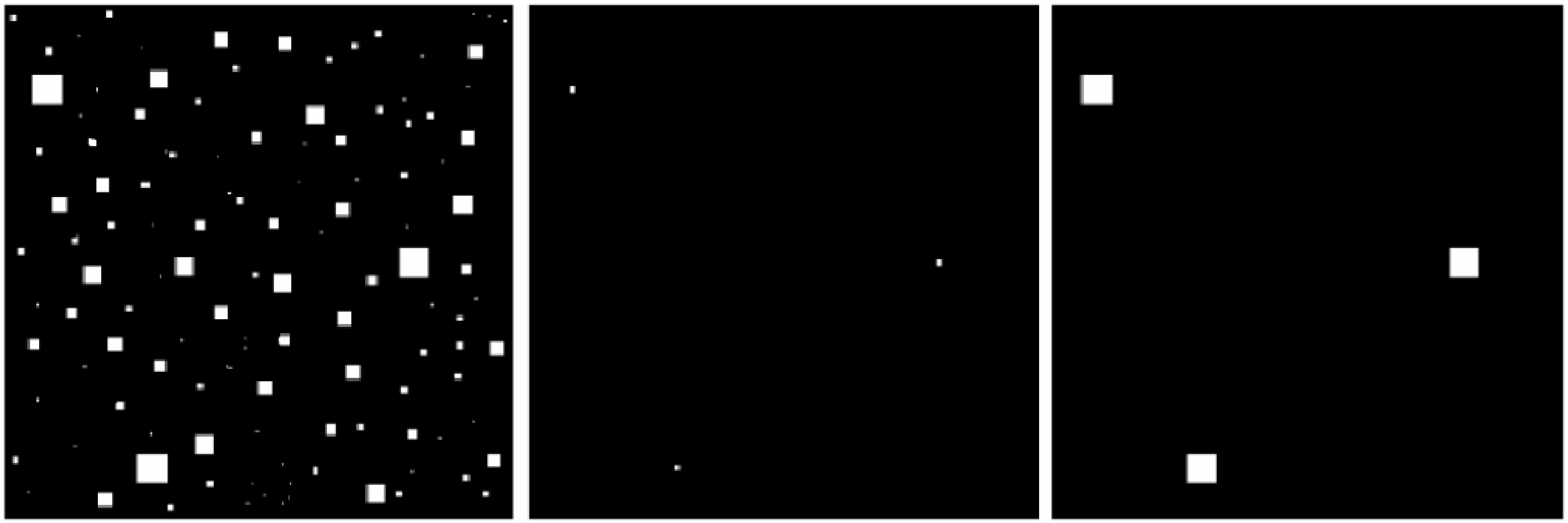


FIGURE 9.6 (a) Set A. (b) Square structuring element. (c) Erosion of A by B, shown shaded. (d) Elongated structuring element. (e) Erosion of A using this element.

Example 9.2: Using erosion to remove image components



a b c

FIGURE 9.7 (a) Image of squares of size 1, 3, 5, 7, 9, and 15 pixels on the side. (b) Erosion of (a) with a square structuring element of 1's, 13 pixels on the side. (c) Dilation of (b) with the same structuring element.

9.3 Opening and closing

Uses...

Opening: Smoothes the contour of an object
Breaks narrow isthmuses ("bridges")
Eliminates thin protrusions

Closing: Smoothes sections of contours
Fuses narrow breaks and long thin gulfs
Eliminates small holes in contours
Fills gaps in contours

Definitions...

The **opening** of set A by structuring element B:

$$A \circ B = (A \ominus B) \oplus B$$

The **closing** of set A by structuring element B:

$$A \bullet B = (A \oplus B) \ominus B$$

Illustration of opening...

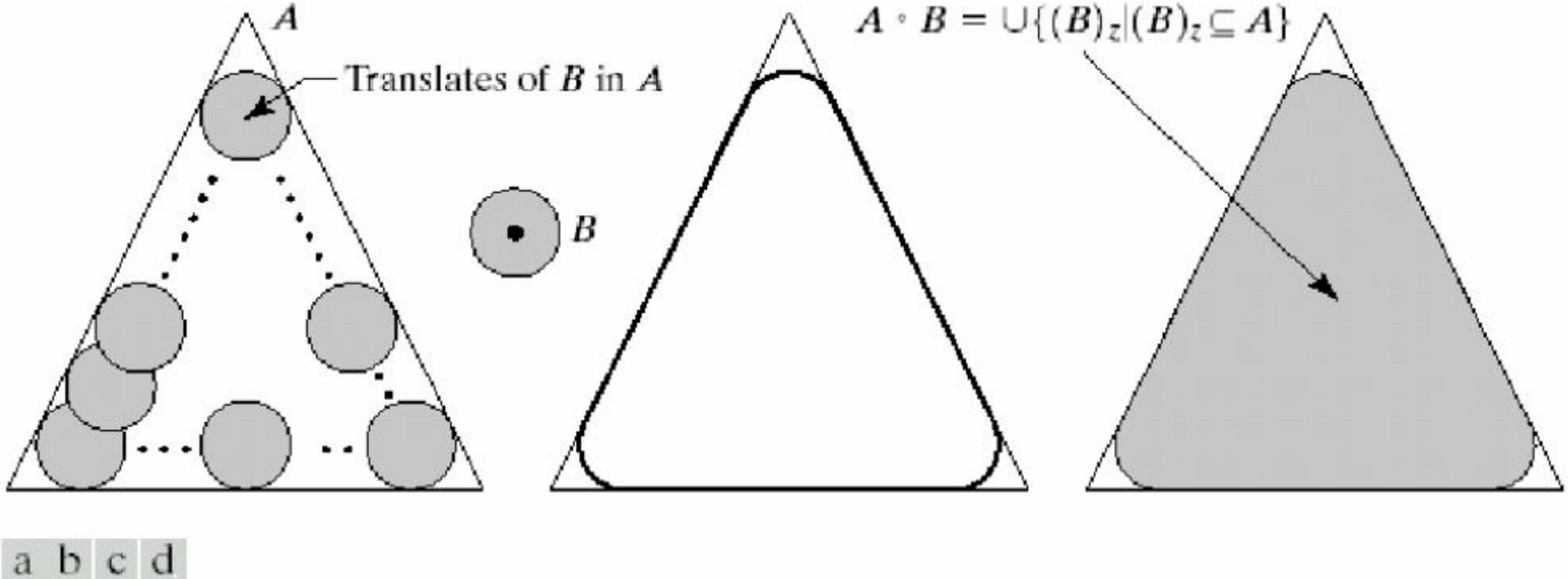


FIGURE 9.8 (a) Structuring element B "rolling" along the inner boundary of A (the dot indicates the origin of B). (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded).

Alternative definition for opening:

$$A \circ B = \cup \{(B)_z | (B)_z \subseteq A\}$$

Illustration of closing...

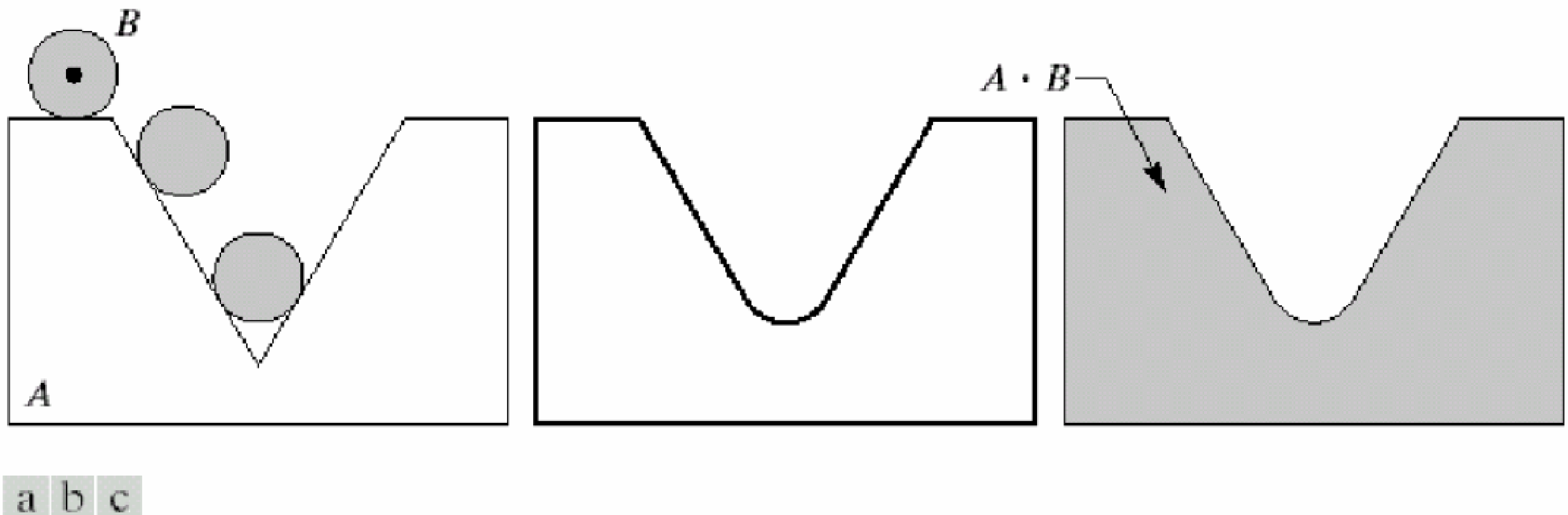


FIGURE 9.9 (a) Structuring element B "rolling" on the outer boundary of set A. (b) Heavy line is the outer boundary of the closing. (c) Complete closing (shaded).

Alternative definition for closing:

A point w is an element of $A \bullet B$ if and only if $(B)_z \cap A \neq \emptyset$ for any translate of $(B)_z$ that contains w

Opening and closing are also duals of each other with respect to set complementation and reflection, that is

$$\boxed{(A \bullet B)^c = A^c \circ \hat{B}}$$

Verificar

The opening operation satisfies the following properties:

(i) $A \circ B \subseteq A$ (ii) If $C \subseteq D$, then $C \circ B \subseteq D \circ B$

(iii) $(A \circ B) \circ B = A \circ B$

The closing operation satisfies the following properties:

(i) $A \subseteq A \bullet B$ (ii) If $C \subseteq D$, then $C \bullet B \subseteq D \bullet B$

(iii) $(A \bullet B) \bullet B = A \bullet B$

Example 9.3: Illustration of opening and closing

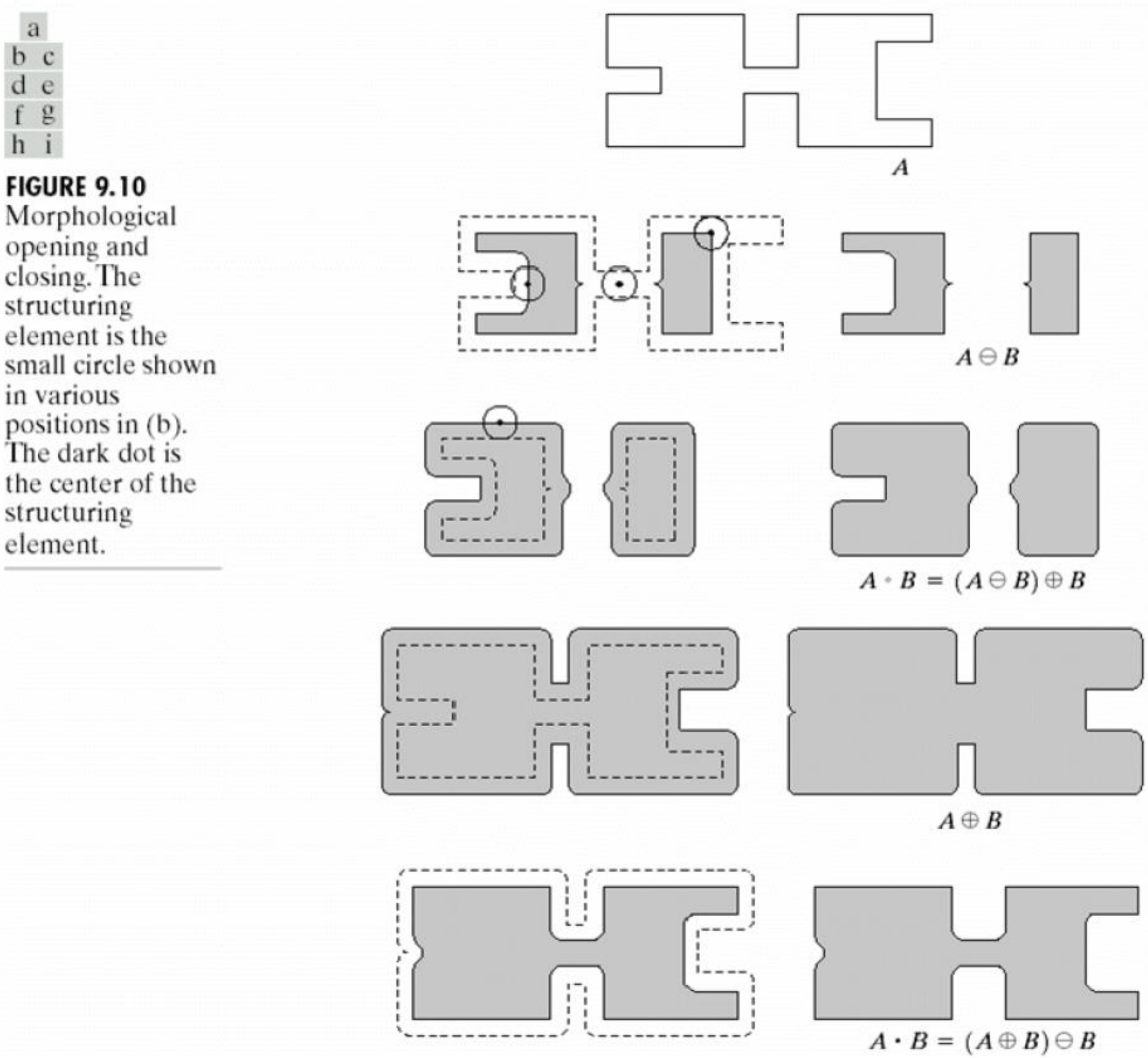
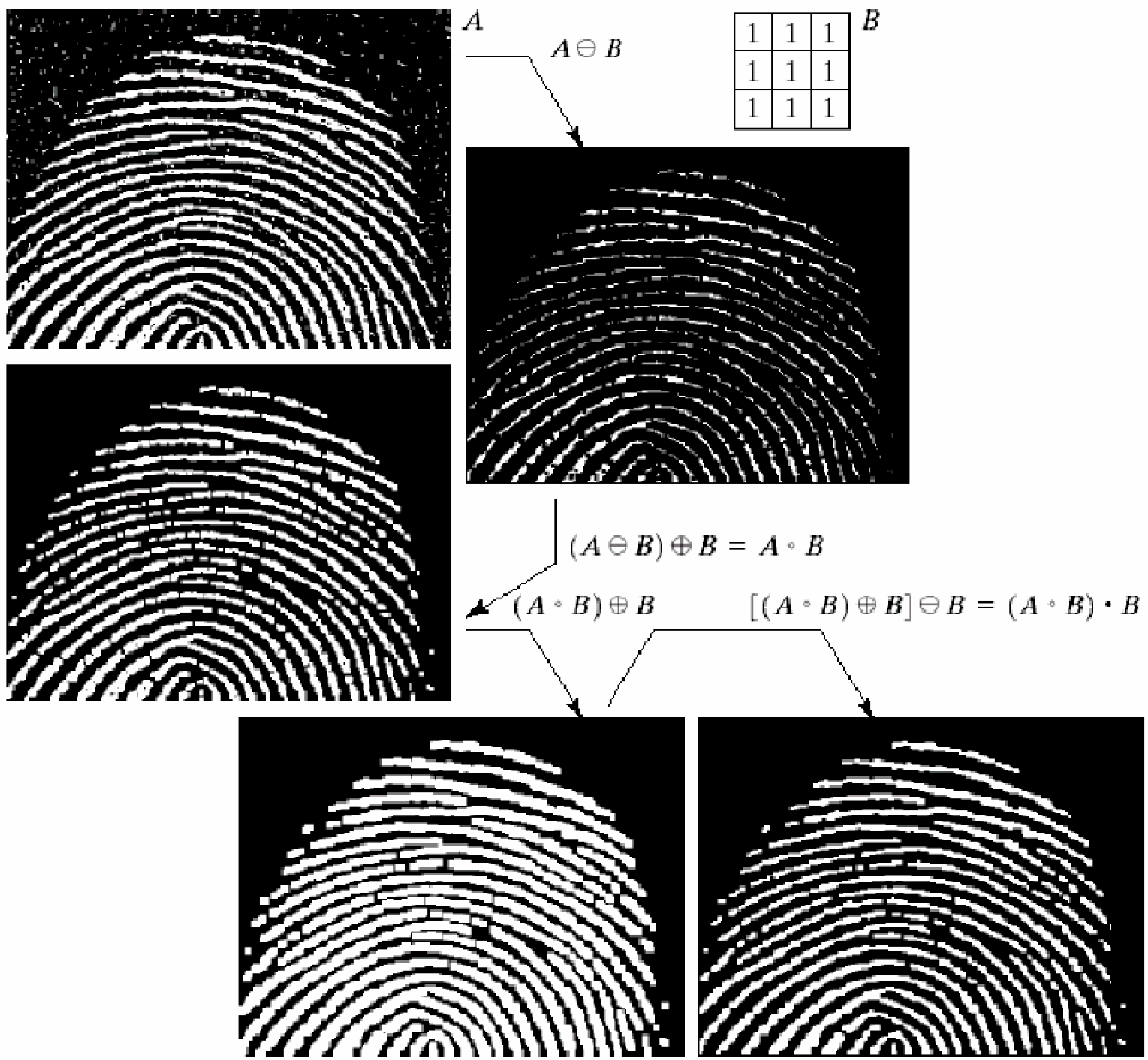


FIGURE 9.10 Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The dark dot is the center of the structuring element.

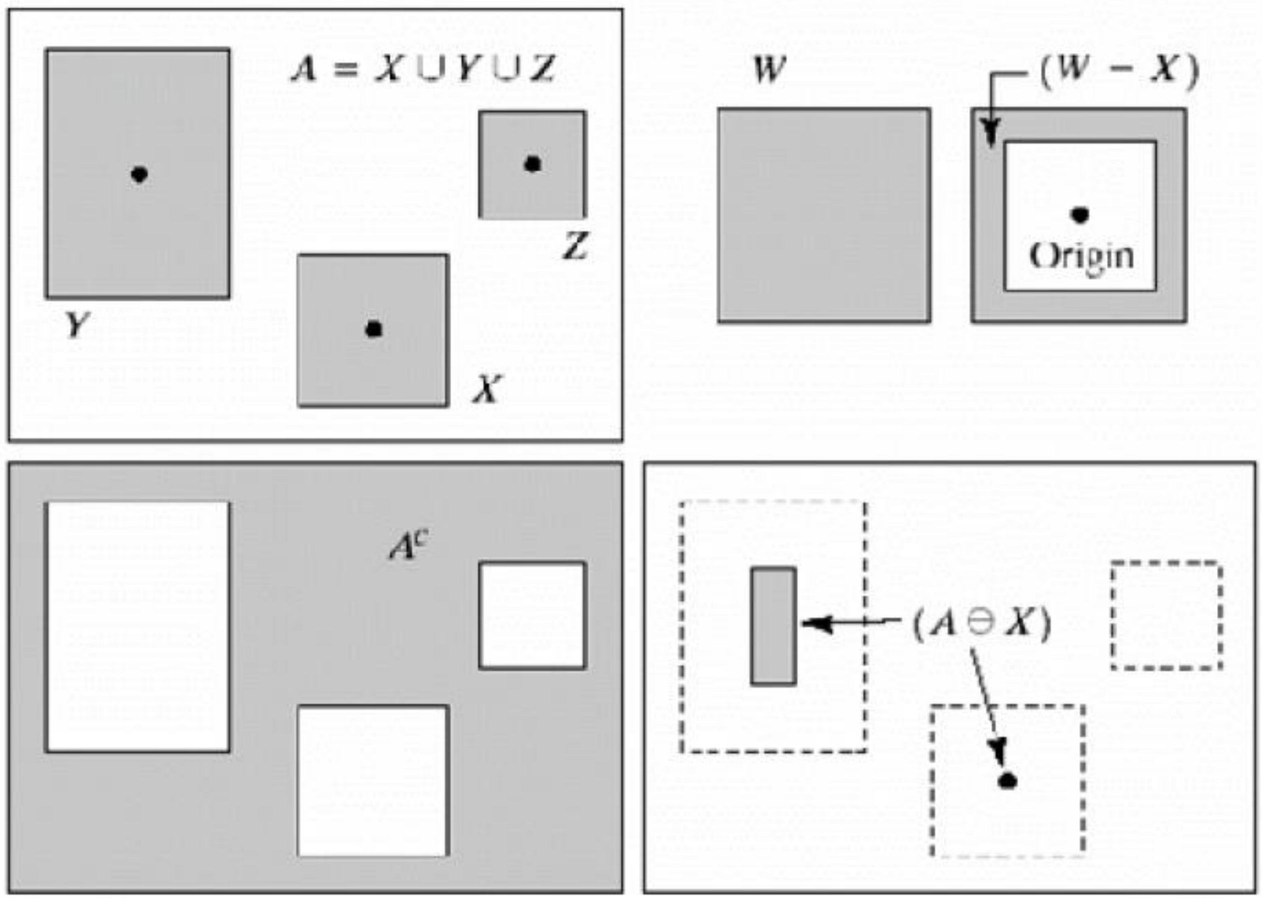
Example 9.4: Use of opening and closing

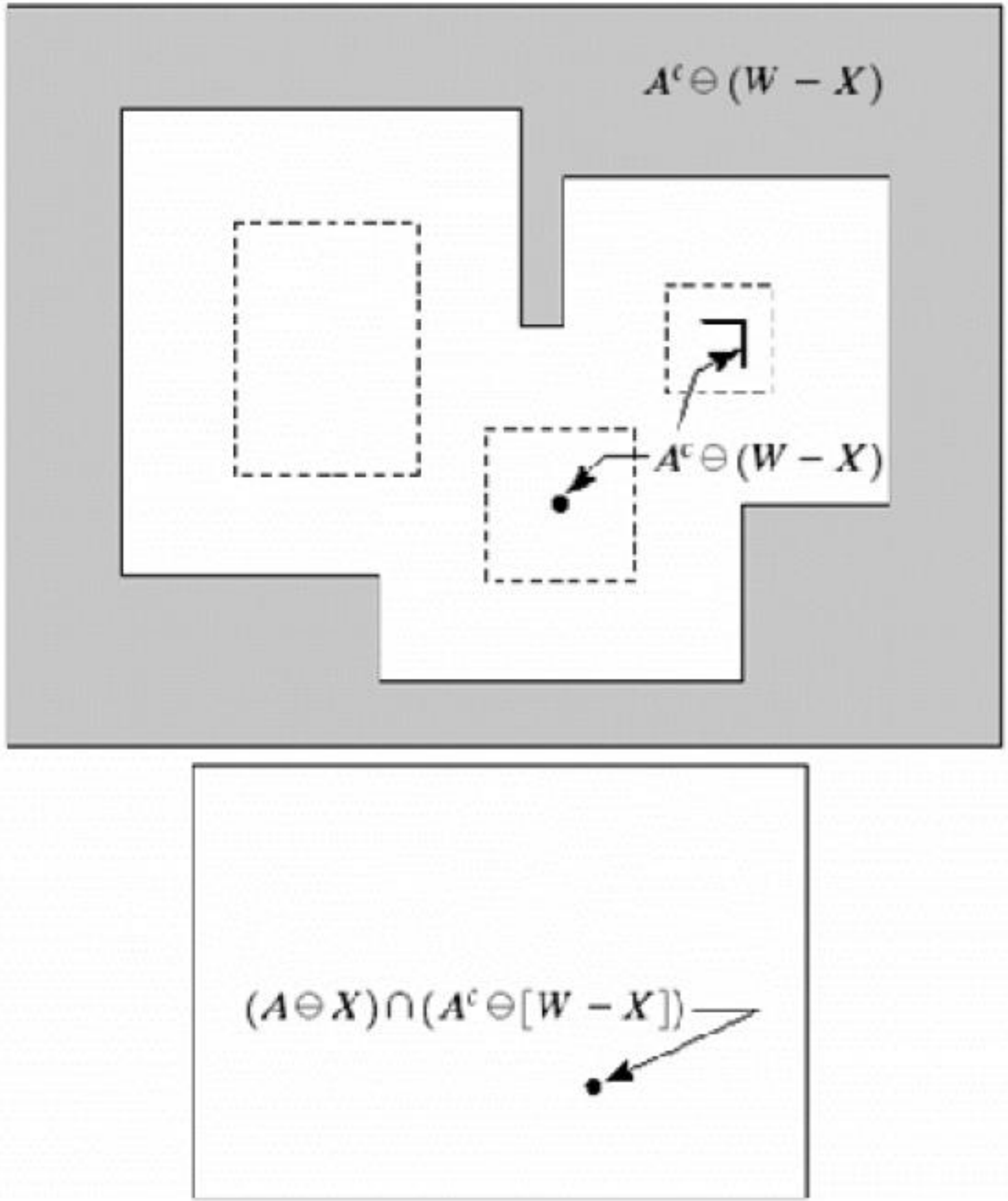


a b
d c
e f

FIGURE 9.11
(a) Noisy image.
(c) Eroded image.
(d) Opening of $\boldsymbol{A}$.
(d) Dilation of the opening.
(e) Closing of the opening. (Original image for this example courtesy of the National Institute of Standards and Technology.)

9.4 The hit-or-miss transformation Illustration...





Aᶜ ⊖ (W − X)
Aᶜ ⊖ (W − X)
(A ⊖ X) ∩ (Aᶜ ⊖ [W − X])

- Objective is to find a disjoint region (set) in an image
- If B denotes the set composed of X and its background, the match/hit (or set of matches/hits) of B in A, is

$$A \circledast B = (A \ominus X) \cap [A^c \ominus (W - X)]$$

- Generalized notation: $B = (B_1, B_2)$
  - B_1: Set formed from elements of B associated with an object
  - B_2: Set formed from elements of B associated with the corresponding background

[Preceeding discussion: $B_1 = X$ and $B_2 = (W - X)$]

- More general definition:

$$A \circledast B = (A \ominus B_1) \cap [A^c \ominus B_2]$$

- $A \circledast B$ contains all the origin points at which, simultaneously, B_1 found a hit in A and B_2 found a hit in A^c

- Alternative definition:

$$A \circledast B = (A \ominus B_1) - (A \oplus \hat{B}_2)$$

- A background is necessary to detect disjoint sets

- When we only aim to detect certain patterns within a set, a background is not required, and simple erosion is sufficient

9.5 Some basic morphological algorithms

When dealing with **binary images**, the principle application of morphology is extracting image components that are useful in the representation and description of shape

9.5.1 Boundary extraction

The boundary $\beta(A)$ of a set A is

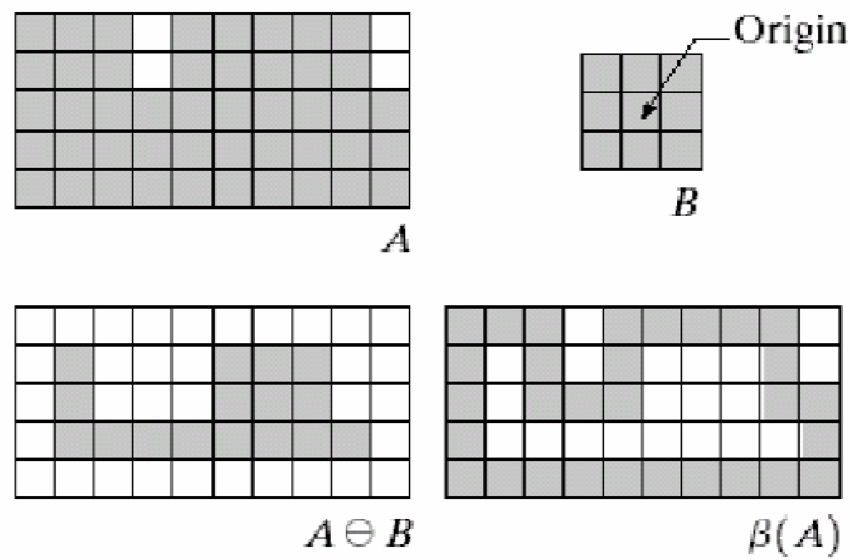
$$\beta(A) = A - (A \ominus B),$$

where B is a suitable structuring element

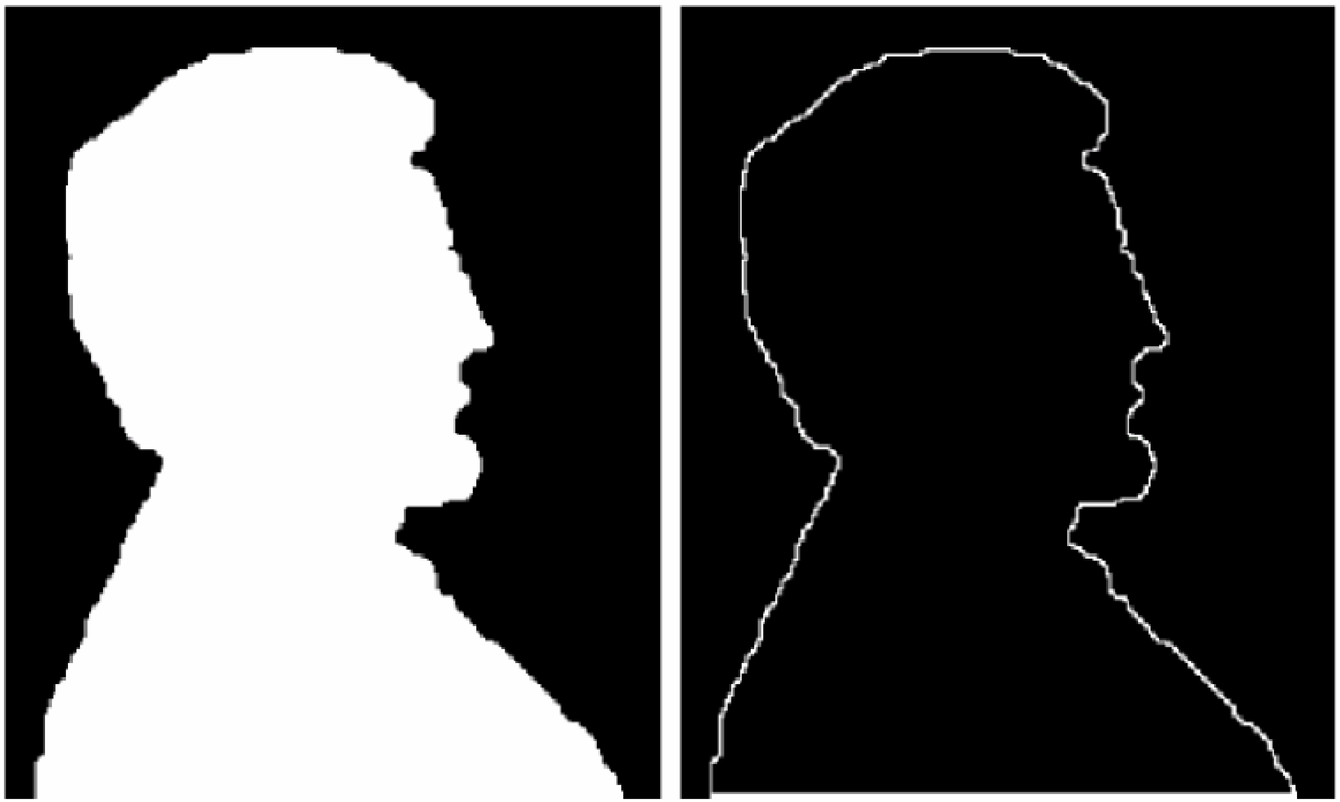
Illustration...

a b
c d

FIGURE 9.13 (a) Set *A*. (b) Structuring element *B*. (c) *A* eroded by *B*. (d) Boundary, given by the set difference between *A* and its erosion.



Example 9.5: Morphological boundary extraction



a b

FIGURE 9.14 (a) A simple binary image, with 1's represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

9.5.2 Region filling

- Begin with a point p inside the boundary, and then fill the entire region with 1's

- All non-boundary (background) points are labeled 0

- Assign a value of 1 to p to begin...

- The following procedure fills the region with 1's,

$$X_k = (X_{k-1} \oplus B) \cap A^c, \quad k = 1, 2, 3, \ldots,$$

where $X_0 = p$, and B is the symmetric structuring element in figure 9.15 (c)

- The algorithm terminates at iteration step k if $X_k = X_{k-1}$

- The set union of X_k and A contains the filled set and its boundary

Note that the intersection at each step with A^c limits the dilation result to inside the region of interest

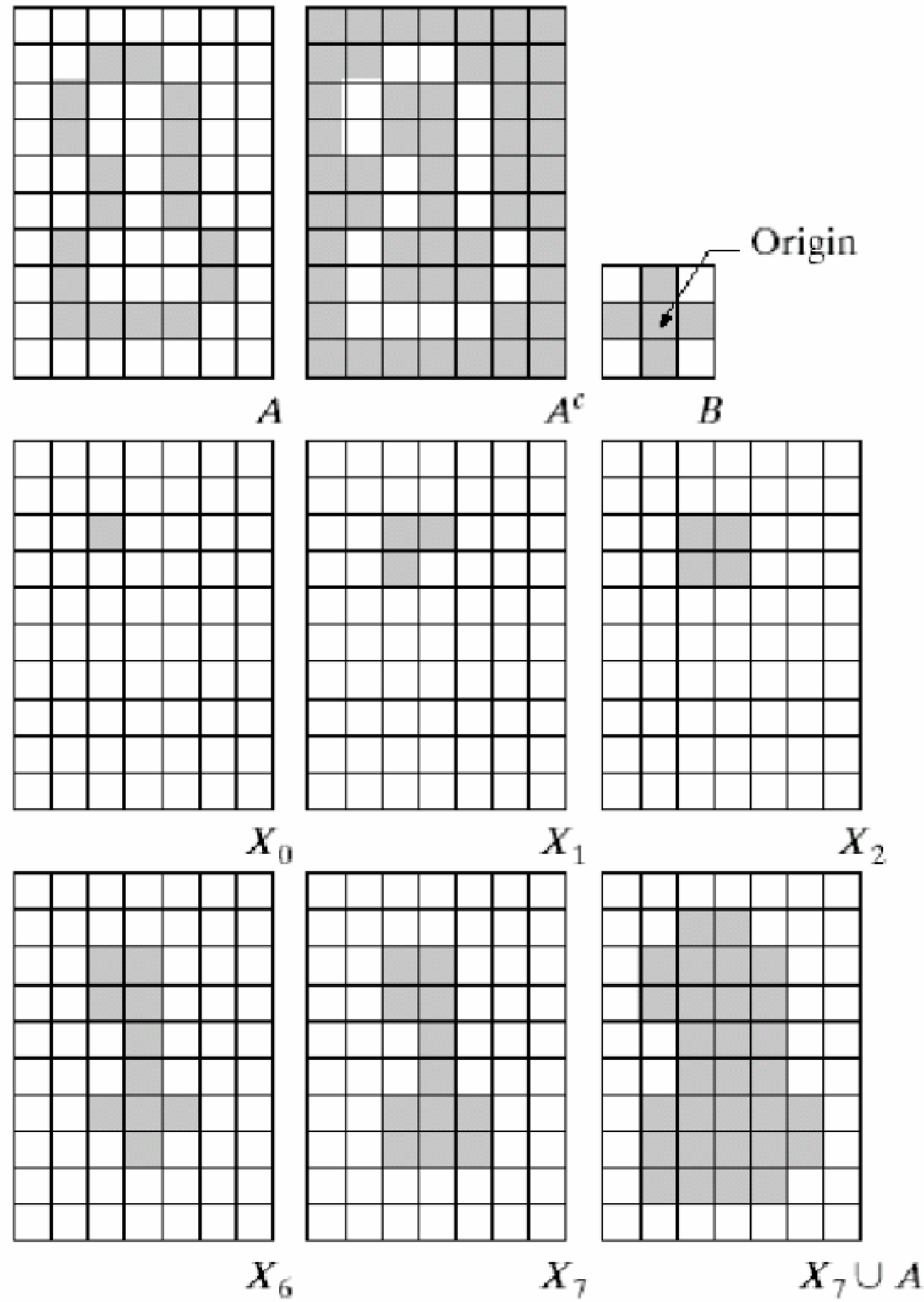
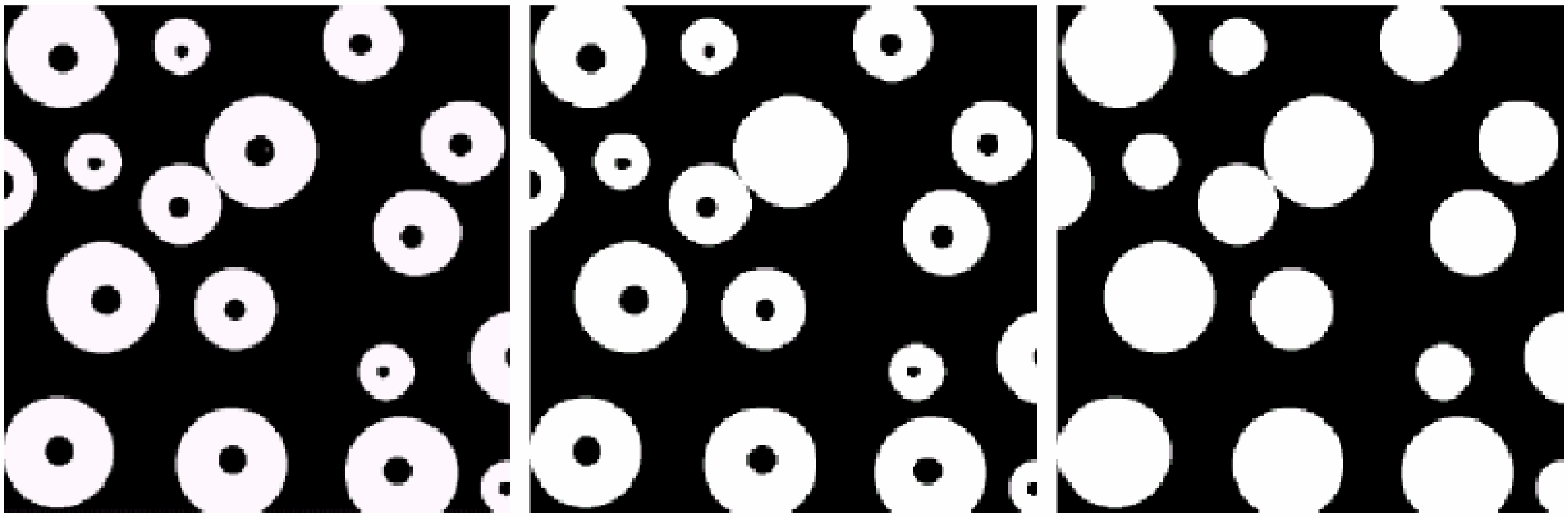


FIGURE 9.15 Region filling. (a) Set A. (b) Complement of A. (c) Structuring element B. (d) Initial point inside the boundary. (e)–(h) Various steps of Eq. (9.5-2). (i) Final result [union of (a) and (h)].

Example 9.6: Morphological region filling



a b c

FIGURE 9.16 (a) Binary image (the white dot inside one of the regions is the starting point for the region-filling algorithm). (b) Result of filling that region (c) Result of filling all regions.

9.5.3 Extraction of connected components

Let Y represent a connected component contained in a set A and assume that a point p of Y is known. Then the following iterative expression yields all the elements of Y:

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \dots,$$

where $X_0 = p$, and B is a suitable structuring element. If $X_k = X_{k-1}$, the algorithm has converged and we let $Y = X_k$.

This algorithm is applicable to any finite number of sets of connected components contained in A, assuming that a point is known in **each** connected component

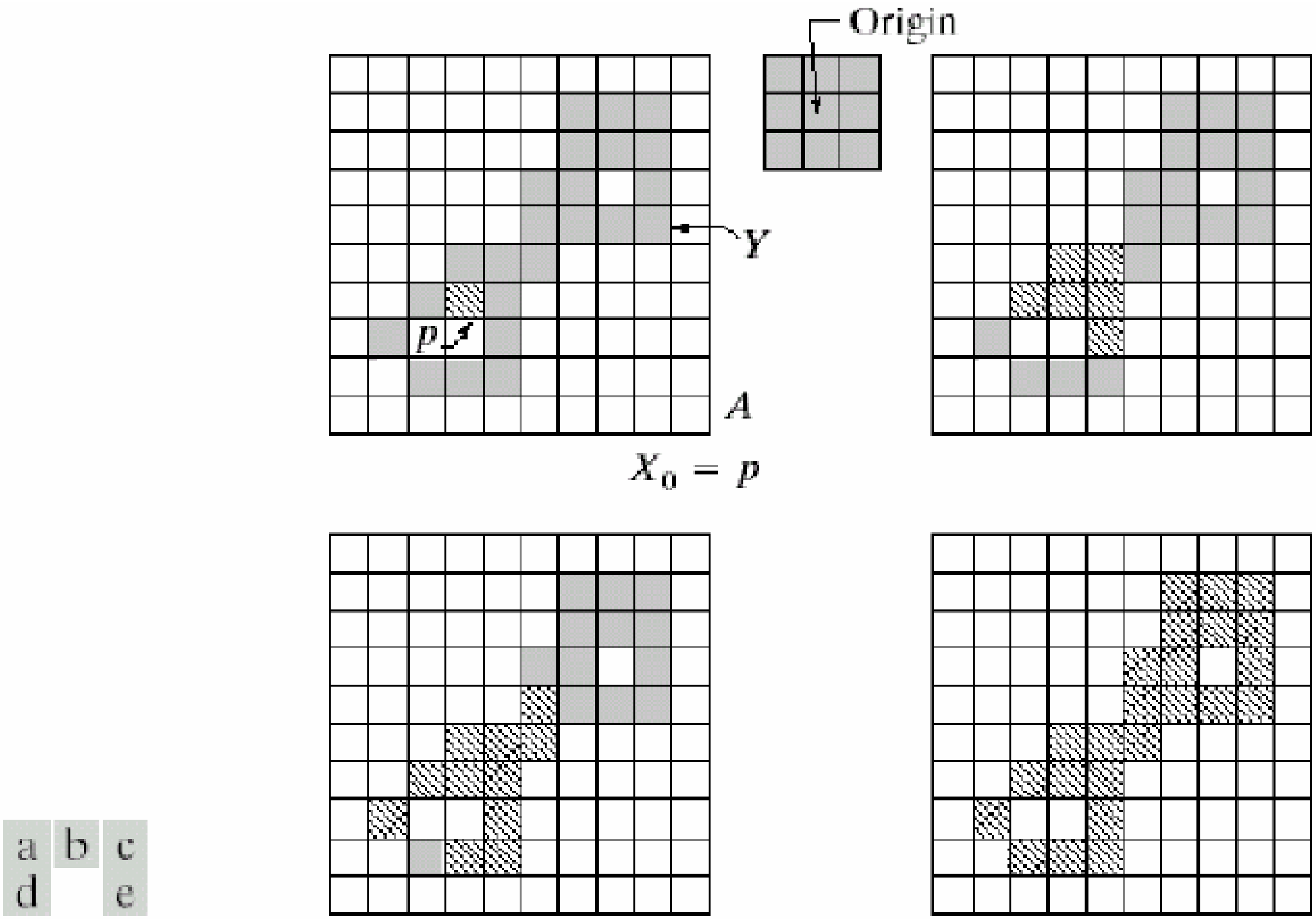
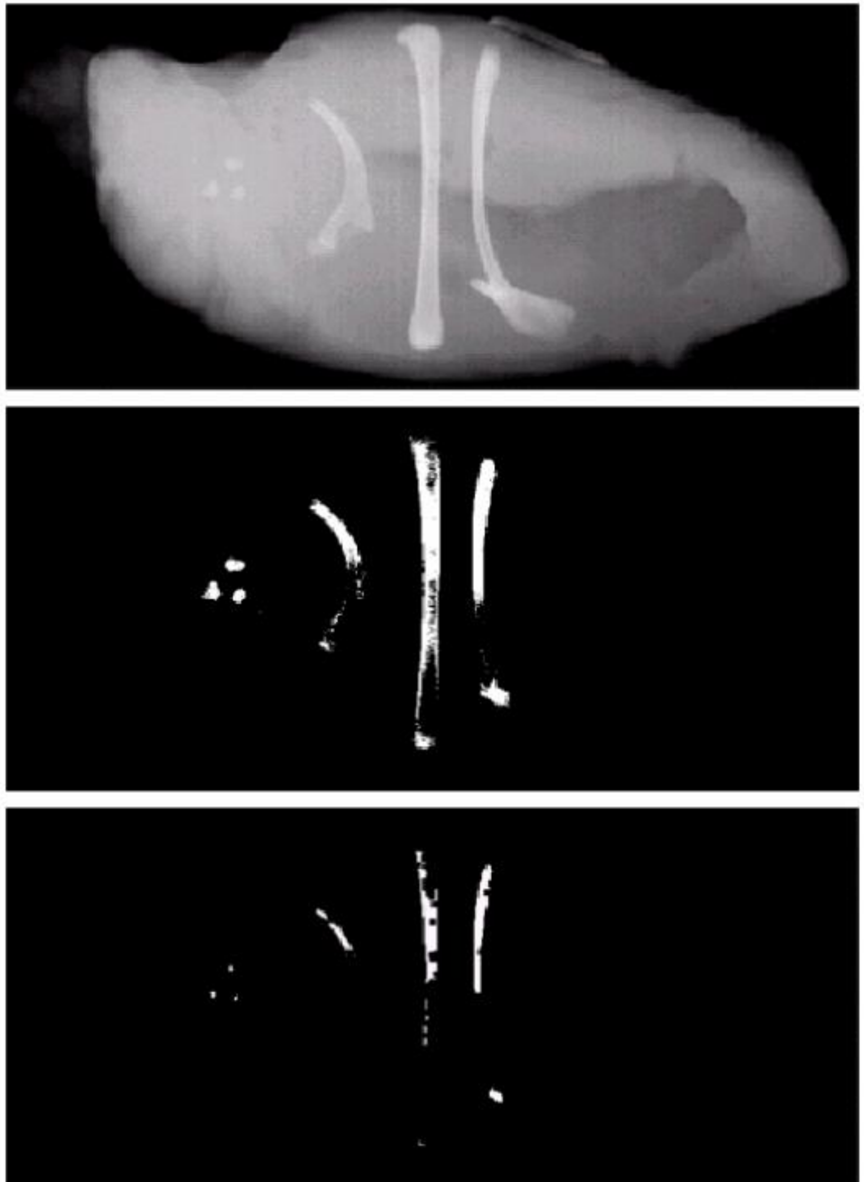


FIGURE 9.17 (a) Set A showing initial point p (all shaded points are valued 1, but are shown different from p to indicate that they have not yet been found by the algorithm). (b) Structuring element. (c) Result of first iterative step. (d) Result of second step. (e) Final result.

Example 9.7:



a
b
c d

FIGURE 9.18
(a) X-ray image of chicken filet with bone fragments. (b) Thresholded image. (c) Image eroded with a 5×5 structuring element of 1's. (d) Number of pixels in the connected components of (c). (Image courtesy of NTB Elektronische Geraete GmbH, Diepholz, Germany, www.ntbxray.com.)

Connected component	No. of pixels in connected comp
01	11
02	9
03	9
04	39
05	133
06	1
07	1
08	743
09	7
10	11
11	11
12	9
13	9
14	674
15	85

9.5.4 Convex hull

Morphological algorithm for obtaining the convex hull, $C(A)$, of a set A...

Let B_1, B_2, B_3 and B_4 represent the four structuring elements in figure 9.19 (a), and then implement the equation ...

$$X_k^i = (X_{k-1} \circledast B^i) \cup A,\ i = 1, 2, 3, 4,\ k = 1, 2, \ldots,\ X_0^i = A$$

Now let $D^i = X_{\mathrm{conv}}^i$, where "conv" indicates convergence in the sense that $X_k^i = X_{k-1}^i$. Then the convex hull of A is

$$C(A) = \cup_{i=1}^4 D^i$$

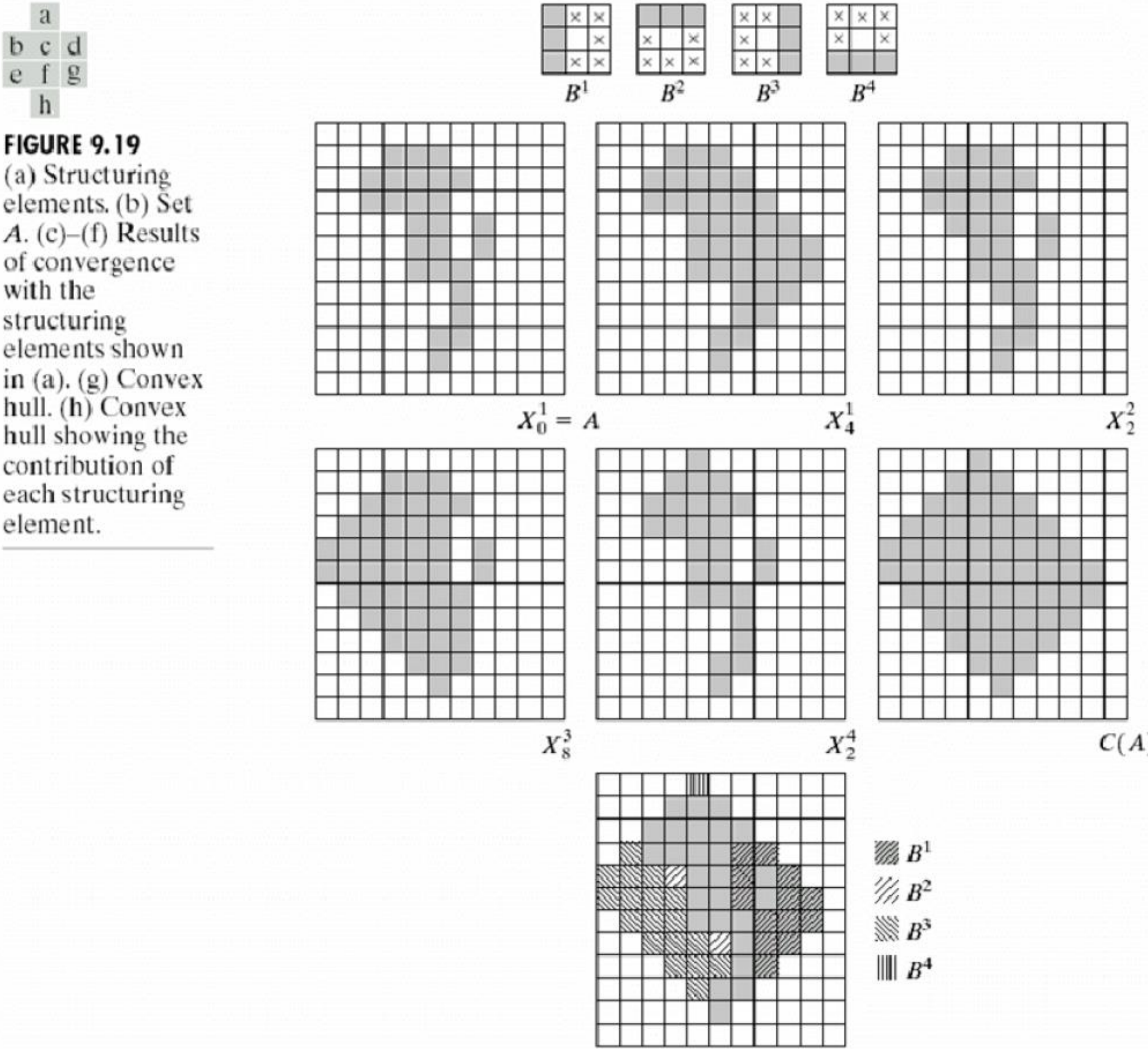


FIGURE 9.19 (a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

Shortcoming of above algorithm: convex hull can grow beyond the minimum dimensions required to guarantee convexity

Possible solution: Limit growth so that it does not extend past the vertical and horizontal dimensions of the original set of points

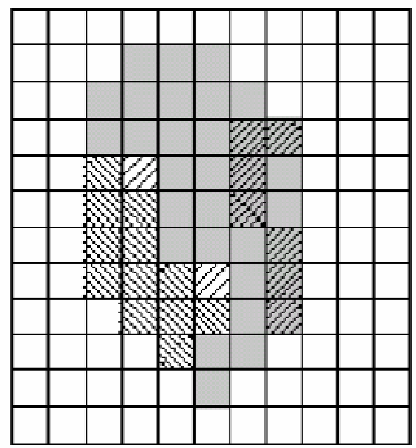


FIGURE 9.20 Result of limiting growth of convex hull algorithm to the maximum dimensions of the original set of points along the vertical and horizontal directions.

Boundaries of greater complexity can be used to limit growth even further in images with more detail

9.5.5 Thinning

The thinning of a set A by a structuring element B:

$$A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c$$

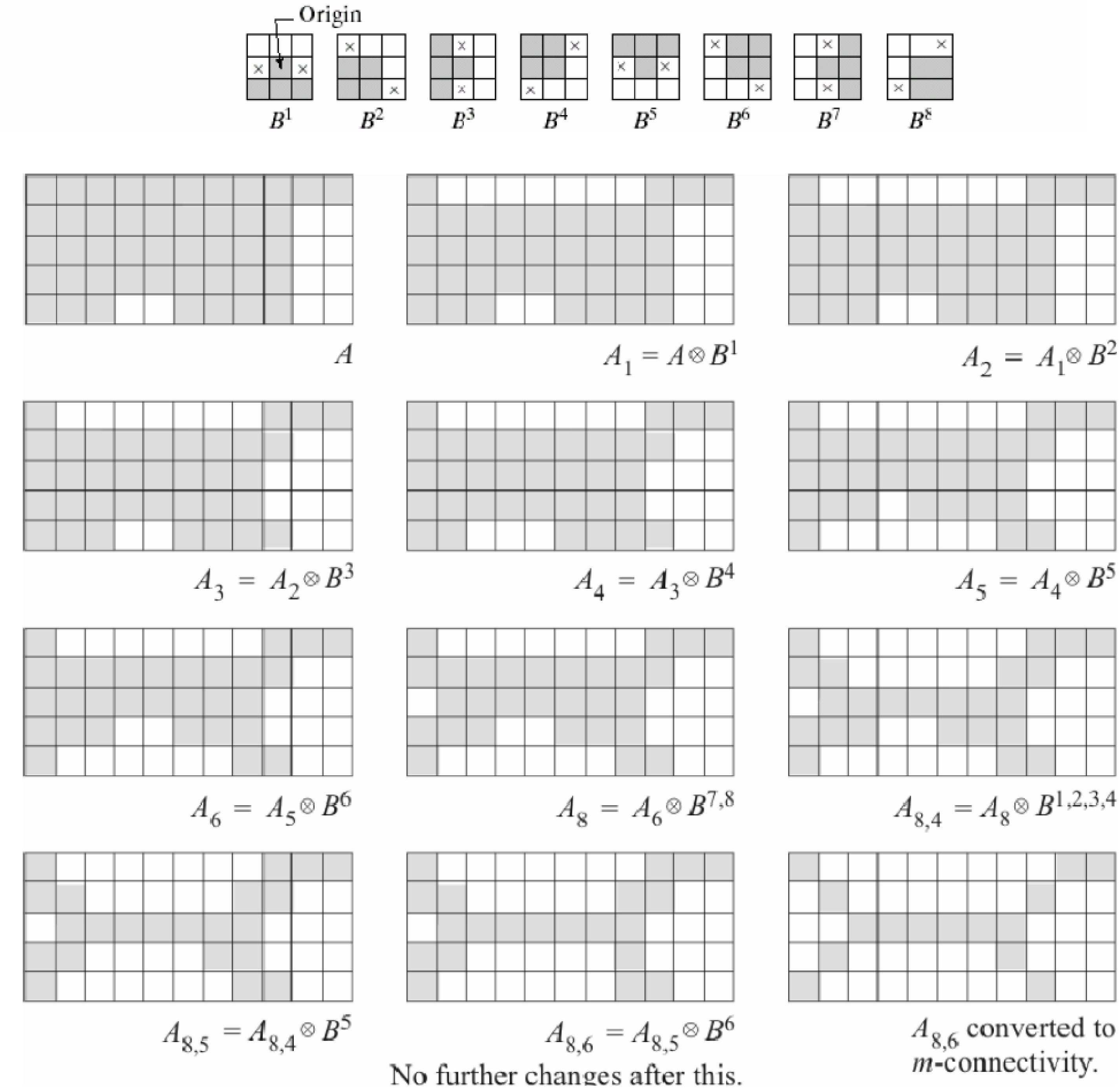
Symmetric thinning: sequence of structuring elements,

$$\{B\} = \{B^1, B^2, B^3, \ldots, B^n\},$$

where B^i is a rotated version of B^{i-1}

$$A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$$

Illustration: Note that figure 9.21 (in the handbook) has many errors – this one is correct...


Origin
B^1 B^2 B^3 B^4 B^5 B^6 B^7 B^8
A
$A_1 = A \otimes B^1$
$A_2 = A_1 \otimes B^2$
$A_3 = A_2 \otimes B^3$
$A_4 = A_3 \otimes B^4$
$A_5 = A_4 \otimes B^5$
$A_6 = A_5 \otimes B^6$
$A_8 = A_6 \otimes B^{7,8}$
$A_{8,4} = A_8 \otimes B^{1,2,3,4}$
$A_{8,5} = A_{8,4} \otimes B^5$
$A_{8,6} = A_{8,5} \otimes B^6$
No further changes after this.
$A_{8,6}$ converted to
m-connectivity.

9.5.6 Thickening

Thickening is the morphological dual of thinning and is defined by

$$A \odot B = A \cup (A \circledast B),$$

where B is a structuring element

Similar to thinning...

$$A \odot \{B\} = ((\ldots((A \odot B^1) \odot B^2) \ldots) \odot B^n)$$

Structuring elements for thickening are similar to those of figure 9.21 (a), but with all 1's and 0's interchanged

A separate algorithm for thickening is seldom used in practice – we thin the background instead, and then complement the result

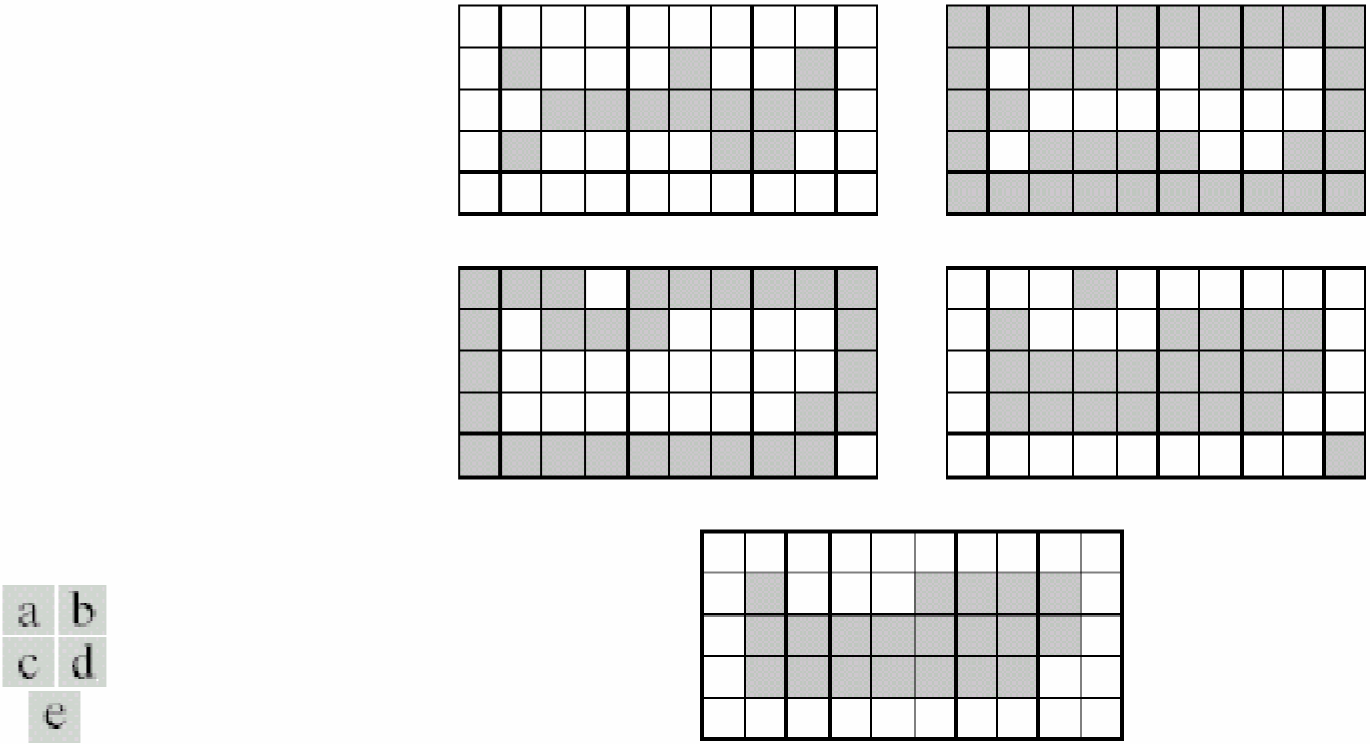


FIGURE 9.22 (a) Set A. (b) Complement of A. (c) Result of thinning the complement of A. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.

9.5.7 Skeletons

The algorithm proposed in this section is similar to the medial axis transformation (MAT). The MAT transformation is discussed in section 11.1.5 and is far inferior to the skeletonization algorithm introduced in section 11.1.5. The skeletonization algorithm proposed in this section also does not guarantee connectivity. We therefore do not discuss this algorithm.

Illustration of the above comments...

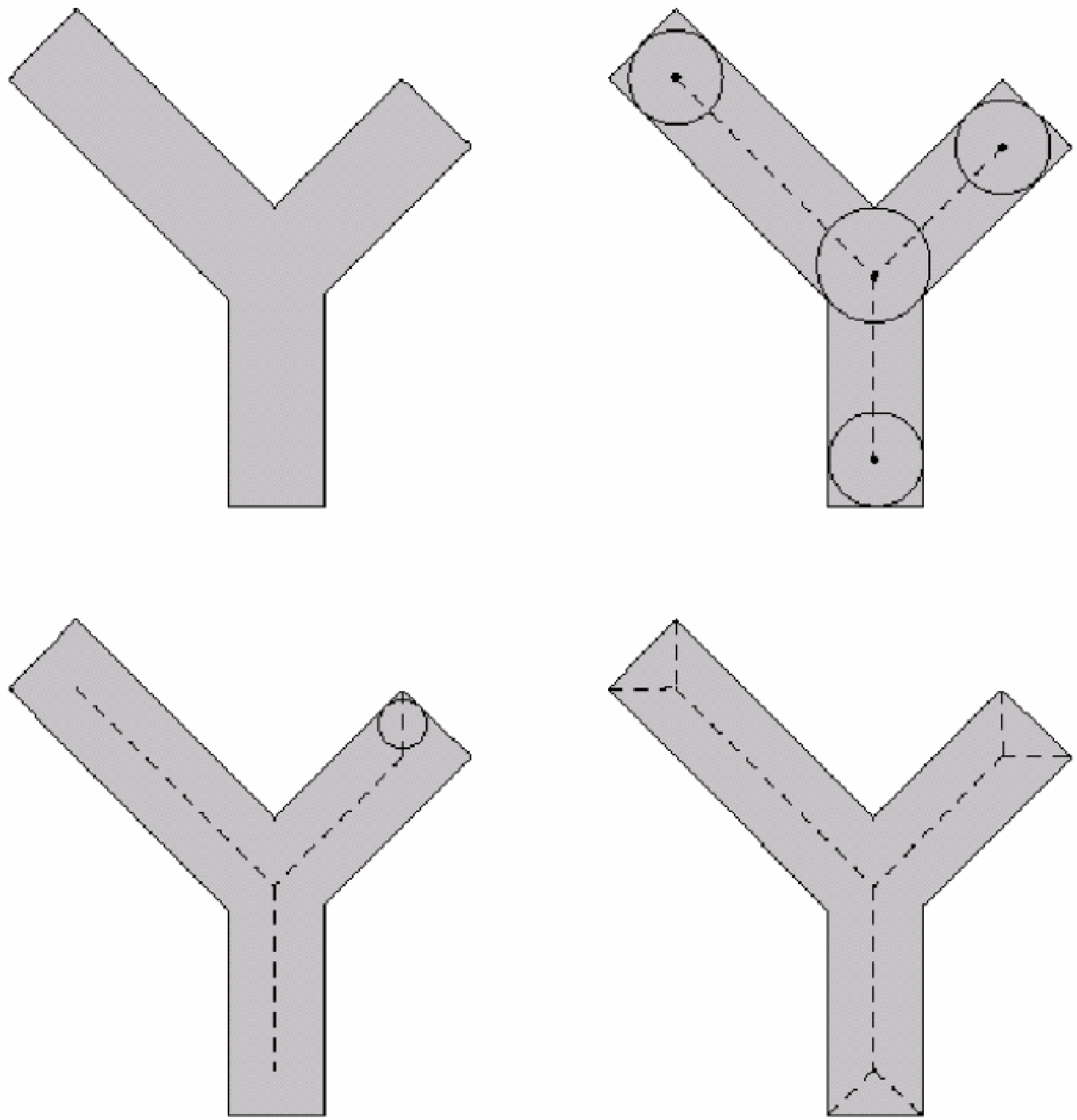


FIGURE 9.23
(a) Set A.
(b) Various positions of maximum disks with centers on the skeleton of A.
(c) Another maximum disk on a different segment of the skeleton of A.
(d) Complete skeleton.

A further illustration...

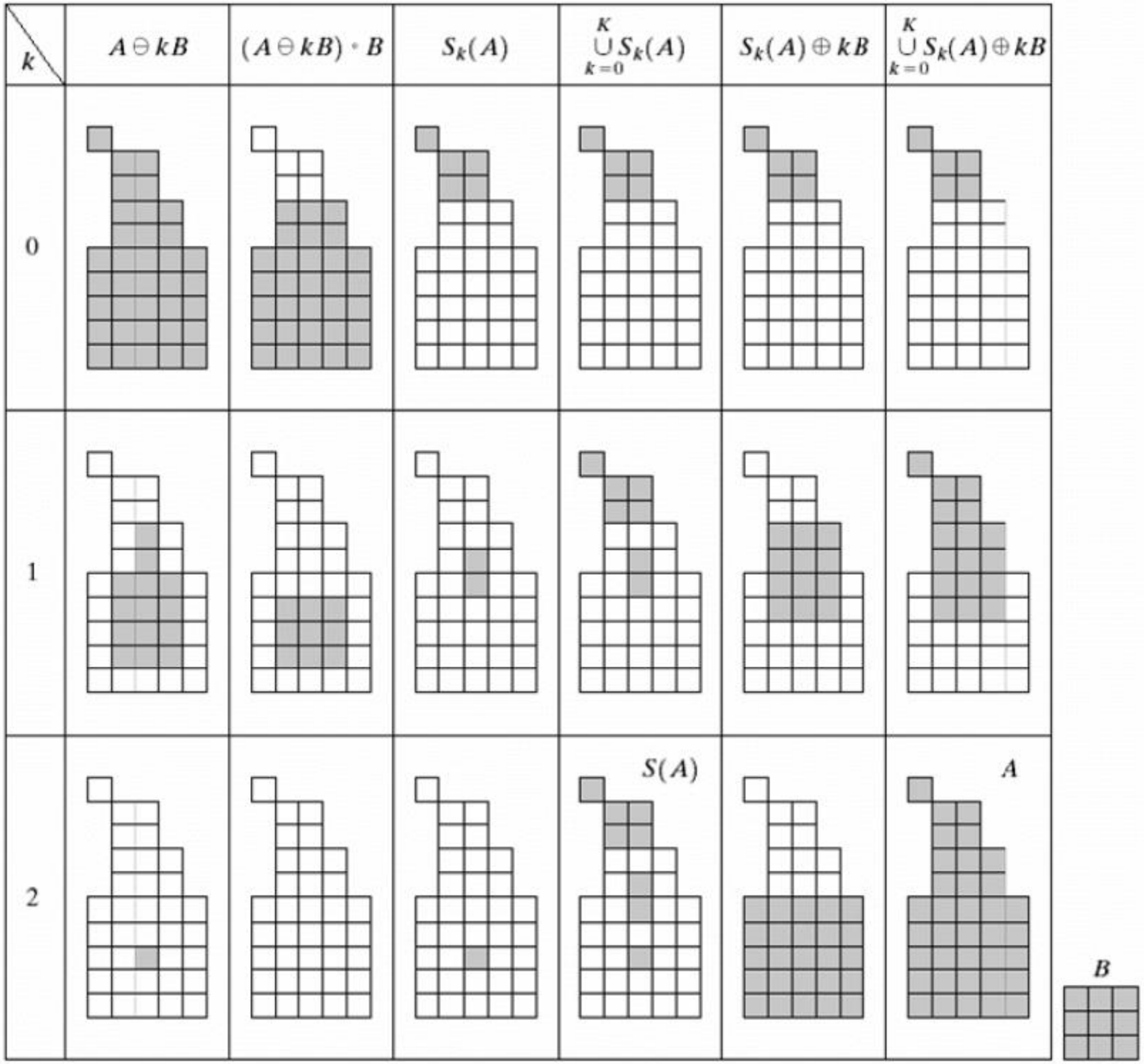


FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

9.5.8 Pruning

- Cleans up "parasitic" components left by thinning and skeletonization

- Use combination of morphological techniques

Illustrative problem: hand-printed character recognition

- Analyze shape of skeleton of character

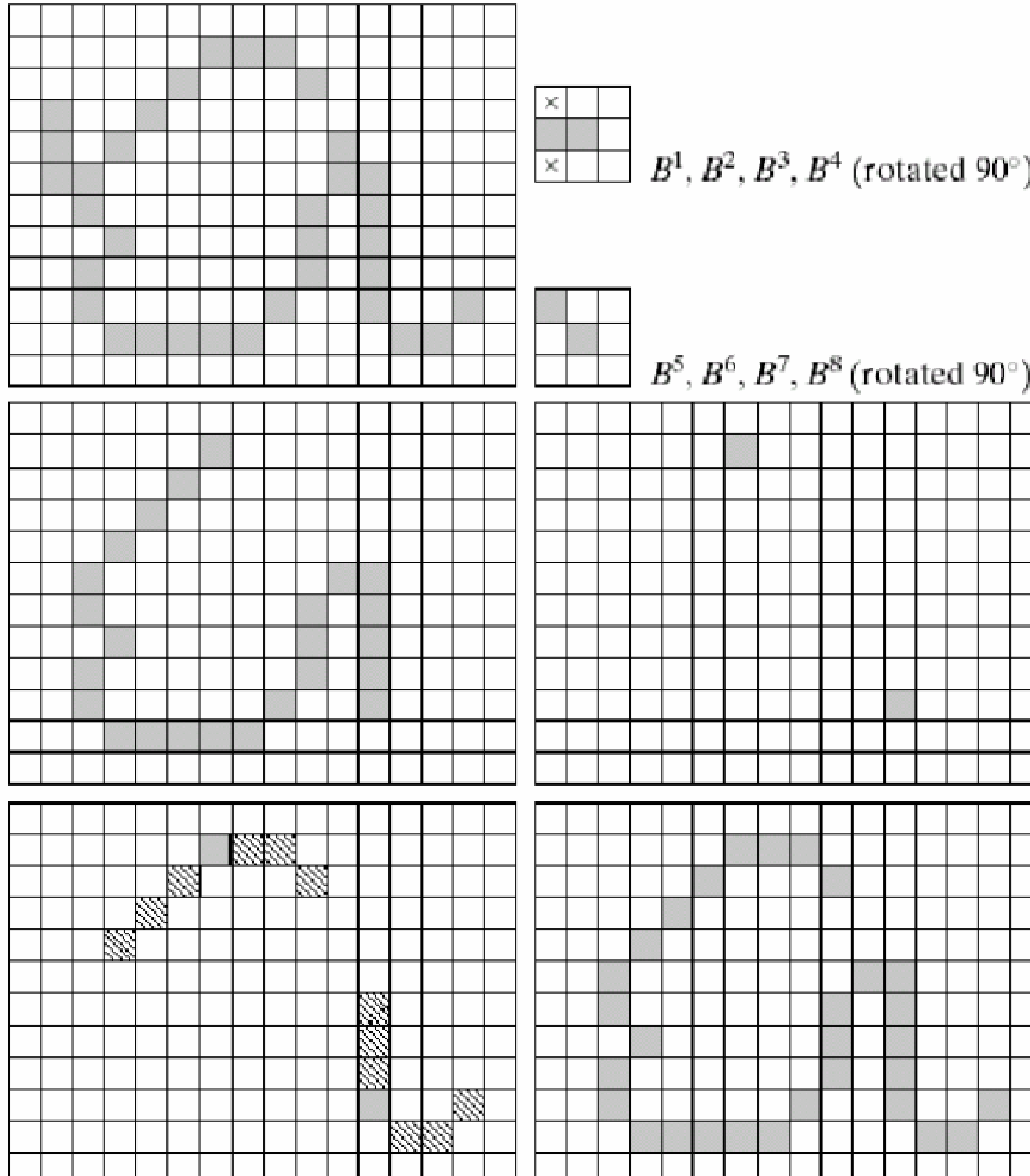
- Skeletons characterized by spurs ("parasitic" components)

- Spurs caused during erosion of non-uniformities in strokes

- We assume that the length of a parasitic component does not exceed a specified number of pixels

a b
c
d e
f g

FIGURE 9.25 (a) Original image. (b) and (c) Structuring elements used for deleting end points. (d) Result of three cycles of thinning. (e) End points of (d). (f) Dilation of end points conditioned on (a). (g) Pruned image.



Any branch with three or less pixels is to be eliminated

(1) Three iterations of:

$$X_1 = A \otimes \{B\}$$

(2) Find all the end points in X_1:

$$X_2 = \cup_{k=1}^{8} (X_1 \circledast B^k)$$

(3) Dilate end points three times, using A as a delimiter:

$$X_3 = (X_2 \oplus H) \cap A, \quad H = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

(4) Finally:

$$X_4 = X_1 \cup X_3$$