In Situ Spatial RNA Detection of Type II Endometrial Carcinoma Tumor Microenvironment within FFPE Uterine Tissue

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Abstract

This paper outlines the process of qualifying and optimizing Formalin-Fixed Paraffin-Embedded (FFPE) uterus tissue for in situ spatial RNA detection.(**?**.

Keywords spatial transcriptomics, immunoncology, endometrial cancer, FFPE, RNA amplification, lock-and-roll, rolling circle

JEL codes D81, D91, E21

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1 Introduction

In the never-ceasing fight against cancer, the immune microenvironment has become front and center. As B-cells and T-cells have became phenotyped, and immune population clusters divided into macrophage high and natural killer cell low groups, the spatial distribution of these immune soldiers become more and more relevant. A prime example being tertiary lymphoid structures - distinct immune structures of varying immune cell composition, frequently found in tumor microenvironments. In the quest for higher resolution data, spatial transcriptomics has emerged as a worthwhile contender.

2 The Growing ECMA Caseload Problem

2.1 Setup

Endometrial cancer is increasing in incident rate, due to a number of factors. Some of these factors include obesity (sedentary lifestyle instead?) and delaying or never having children. While Type I EC has an relatively high recovery and survival rate, the same is not true for Type II EC. Early detection and early resection are key to a high chance of survival. It is anticipated that Type II case with continue to climb as the time marches on. Developing better tools for early, non-invasive diagnosis, as well as deeper investigative tools for disease pathology, therapy development, screening, and therapy qualifying.

Here is some stuff about ECMA.

2.2 Comparison to Existing iTME Knowledge

In the never-ceasing fight against cancer, the immune microenvironment has become front and center. As B-cells and T-cells have became phenotyped, and immune population clusters divided into macrophage high and natural killer cell low groups, the spatial distribution of these immune soldiers become more and more relevant. A prime example being tertiary lymphoid structures - distinct immune structures of varying immune cell composition, frequently found in tumor microenvironments. In the quest for higher resolution data, spatial transcriptomics has emerged as a worthwhile contender.

Within the immune landscape of EC, there are many important players. The lineage of these cells is an important consideration as mutations within the surrounding tumor microenvironment are common. Myeloid-derived suppressor cells (MDSCs) originate from bone marrow stem cells, and are often upregulated in chronically inflamed regions, a result of altered hematopoiesis. Regions with high MDSC populations also exhibit T-cell suppression. The investigation of this mechanism throughout early stage endometrioid development would provide further pathological insight into EC.

2.3 Recent ECMA iTME Discoveries

Powerful therapy tools exist to combat cancer; among the many options, immune checkpoint inhibitors, CAR-T cell therapy, among many other treatments. When these therapies work, tumor rapidly shrink, metastasis is stopped in its tracks, and the risk of recurrence is low. However, often these expensive and taxing treatments results in no improvement. A growing consensus is the importance of the immune microenvironment in the treatment of malignancies in the body. With increased focus on the immune players of the tumor milieu, research has focused on categorizing immune populations, phenotyping T-cells, B-Cells, NKs, and macrophages, and single cell sequencing. Alongside immunohistology and the identification of tertiary immune structures within chronically inflammation tissue, high resolution, spatial RNA detection and distribution of immune cells and biomarkers are the newest frontiers.

Endometrial cancer is a sneaky beast - she can lie in wait and slowly grow, enduring a steady assault of T-cell and NK cells at her perimeter. But EC is clever and plays the long game, recruiting a sympathetic agent within the immune camp. MDSCs hold significant influence over T-cells, and, drunk on power from EC, MDSCs convince the T-cells that EC is no threat and that they really needn't do anything much at all about her. Our main character is MDSC - and understanding her motivations, we must also know where she goes and to whom she talks.

2.4 An Overview of Spatial Transcriptomics

High spatial resolution within FFPE is a challenge, but a challenge worth undertaking. Endless bio-banks of data could be used if only the biomaterial were accessible. Both immunostaining and fluorescent in situ hybridization (FISH) are techniques currently used and commercially available for single-cell (is this true?) resolution,; however, these techniques perform considerably better on freshly frozen tissue, with no exposure to paraffin. Paraffin was not only creates difficulties in permeabilizing the tissue, but also with autofluorescence and background during fluorescence imaging.

2.5 In Situ RNA Detection

Lock'n'Roll, an in situ nucleic acid amplification technique (patent; how is this referenced?) paired with nucleic acid tags, has had early success on FFPE tissue. Lock'n'Roll is a form of rolling circle amplification, occurring in situ, "locked" onto the target RNA. This results in a " ball" of amplified targets with a nucleic acid tag sequence embedded repeatedly within the Lock'N'Roll ball. Using a 4 sequence barcode, multiple fluorescent tags can be imaged and analyzed to call out specific targets whenever the correct colocalization is detected. The size of the Lock'n'Roll ball is important, allowing colocalization during imaging analysis.

2.5.1 Review of RNA Detection

2.5.2 The Formalin-Fixed, Paraffin-Embedded Situation

High spatial resolution within FFPE is a challenge, but a challenge worth undertaking. Endless bio-banks of data could be used if only the biomaterial were accessible. Both immunostaining and fluorescent in situ hybridization (FISH) are techniques currently used and commercially available for single-cell (is this true?) resolution,; however, these techniques perform considerably better on freshly frozen tissue, with no exposure to paraffin. Paraffin was not only creates difficulties in permeabilizing the tissue, but also with autofluorescence and background during fluorescence imaging.

2.5.3 Nondegenerate PF Unconstrained Solution Requires FHWC

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting upper bound consumption function will be

$$\bar{c}(m) = (m + h - 1)\underline{\kappa}, \tag{1}$$

and so in order to rule out the degenerate limiting solution $\bar{c}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (FHWC), eq. (**??**).

2.5.4 The Growth Impatience Condition (GIC)

By analogy to the RPF, we define a 'growth patience factor' as

$$\text{GPF:} \quad \text{Þ}_{\mathbf{\Phi}} = \text{Þ}/\mathbf{\Phi}, \tag{2}$$

which defines a 'growth impatience condition':

$$\text{GIC:} \quad \text{Þ}_{\mathbf{\Phi}} < 1. \tag{3}$$

2.5.5 Perfect Foresight Finite Value of Autarky Condition (PFFVAC)

Under 'autarky,' capital markets do not exist; the consumer has no choice but to spend permanent noncapital income $\mathbf{p}$ in every period. Because $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, the value the consumer would achieve is

$$\begin{aligned}\mathbf{v}_t^{\text{autarky}} &= \mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\mathbf{\Phi}) + \beta^2\mathrm{u}(\mathbf{p}_t\mathbf{\Phi}^2) + \ldots \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\mathbf{\Phi}^{1-\rho})^{T-t+1}}{1-\beta\mathbf{\Phi}^{1-\rho}}\right)\end{aligned}$$

which (for $\mathbf{\Phi} > 0$) asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if any of these equivalent conditions holds:

$$\text{PF-FVAC:} \quad \begin{aligned} \overbrace{\beta\mathbf{\Phi}^{1-\rho}}^{\equiv \beth} &< 1 \\ \beta\mathsf{R}\mathbf{\Phi}^{-\rho} &< \mathsf{R}/\mathbf{\Phi} \equiv \mathcal{R} \\ \text{Þ}_{\mathsf{R}} &< \mathcal{R}^{1-1/\rho}, \end{aligned} \tag{4}$$

where we call $\beth$[1] the 'Perfect Foresight Value Of Autarky Factor' (PF-VAF), and the variants of (4) constitute alternative versions of the Perfect Foresight Finite Value of Autarky Condition; they guarantee that a perfect-foresight consumer who always spends all permanent income has 'finite autarky value.'[2]

If the FHWC is satisfied, the PF-FVAC implies that the RIC is satisfied.[3] Likewise, if the FHWC and the GIC are both satisfied, PF-FVAC follows:

$$\begin{aligned} \text{Þ} &< \mathbf{\Phi} < \mathsf{R} \\ \text{Þ}_{\mathsf{R}} &< \mathbf{\Phi}/\mathsf{R} < (\mathbf{\Phi}/\mathsf{R})^{1-1/\rho} < 1 \end{aligned} \tag{6}$$

(the last line holds because FHWC $\Rightarrow 0 \leq (\mathbf{\Phi}/\mathsf{R}) < 1$ and $\rho > 1 \Rightarrow 0 < 1 - 1/\rho < 1$).

The first panel of Table **??** summarizes: The PF-Unconstrained model has a non-degenerate limiting solution if we impose the RIC and FHWC (these conditions are necessary as well as sufficient). Together the PF-FVAC and the FHWC imply the RIC. If we impose the GIC and the FHWC, both the PF-FVAC and the RIC follow, so GIC+FHWC are also sufficient. But there are circumstances under which the RIC and FHWC can hold while the PF-FVAC fails ('~~PF-FVAC~~'). For example, if $\mathbf{\Phi} = 0$, the problem is a standard 'cake-eating' problem with a nondegenerate solution under the RIC (when the consumer has access to capital markets).

The easiest way to grasp the relations among these conditions is by studying Figure **??**. Each node represents a quantity defined above. The arrow associated with each inequality imposes that condition. For example, one way we wrote the PF-FVAC in equation (4) is $\text{Þ} < \mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho}$, so imposition of the PF-FVAC is captured by the diagonal arrow connecting Þ and $\mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho}$. Traversing the boundary of the diagram clockwise starting at Þ involves imposing first the GIC then the FHWC, and the consequent arrival at the bottom right node tells us that these two conditions jointly imply the PF-FVAC. Reversal of a condition reverses the arrow's direction; so, for example, the bottom-most arrow going to $\mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho}$ imposes ~~FHWC~~; but we can cancel the cancellation and reverse the arrow. This would allow us to traverse the diagram clockwise from Þ through $\mathbf{\Phi}$ to $\mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho}$ to R, revealing that imposition of GIC and FHWC (and, redundantly, FHWC again) let us conclude that the RIC holds because the starting point is Þ and the endpoint is R. (Consult Appendix I for an exposition of diagrams of this type, which are a simple application of Category Theory (**?**)).

2.5.6 PF Constrained Solution Exists Under RIC or {~~RIC~~, GIC}

We next sketch the perfect foresight constrained solution because it defines a benchmark (and limit) for the unconstrained problem with uncertainty (our ultimate interest).

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, defined by the lower bound,

[1] This is another kind of discount factor, so we use the Hebrew 'bet' which is a cognate of the Greek 'beta'.

[2] This is related to the key impatience condition in **?**.

[3] Divide both sides of the second inequality in (4) by R:

$$\text{Þ}/\mathsf{R} < (\mathbf{\Phi}/\mathsf{R})^{1-1/\rho} \tag{5}$$

and FHWC $\Rightarrow$ the RHS is < 1 because $(\mathbf{\Phi}/\mathsf{R}) < 1$ (and the RHS is raised to a positive power (because $\rho > 1$)).

$b_t = 0$ (if it were relevant at any higher level of m, it would certainly be relevant here: $\mathrm{u}' > 0$. The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ it is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (**??**):

$$1^{-\rho} > \mathsf{R}\beta\mathbf{\Phi}^{-\rho}1^{-\rho}, \tag{7}$$

which is just a restatement of the GIC.

We now examine implications of possible configurations of the conditions. (Tables **??** and **??** (and the table in appendix E) codify.)

~~GIC~~ and RIC. If the GIC fails but the RIC (**??**) holds, Appendix E shows that, for some $0 < m_{\#} < 1$, an unconstrained consumer behaving according to the perfect foresight solution (1) would choose $c < m$ for all $m > m_{\#}$. In this case the solution to the constrained consumer's problem is simple: For any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future); for such m the constrained consumption function is identical to the unconstrained one. If the consumer were somehow[4] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would consume $c = m$. Using $\grave{\bullet}$ for the version of a function $\bullet$ in the presence of constraints (and recalling that $\bar{\mathrm{c}}(m)$ is the unconstrained perfect foresight solution):

$$\grave{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#}. \end{cases}$$

GIC and RIC. When the RIC and GIC both hold, Appendix E shows that the limiting constrained consumption function is piecewise linear, with $\grave{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m_{\#}^{0} > 1$, and with discrete declines in the MPC at a set of kink points $\{m_{\#}^{1}, m_{\#}^{2}, \ldots\}$. As $m \uparrow \infty$ the constrained consumption function $\grave{\mathrm{c}}(m)$ becomes arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume function $\grave{\boldsymbol{\kappa}}(m) \equiv \grave{\mathrm{c}}'(m)$ limits to $\underline{\kappa}$.[5] Similarly, the value function $\grave{\mathrm{v}}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer.

This logic holds even when the finite human wealth condition fails (~~FHWC~~), because the constraint prevents the (limiting) consumer[6] from borrowing against unbounded human wealth to finance unbounded current consumption. Under these circumstances, the consumer who starts with any $b_t > 0$ will, over time, run those resources down so that after some finite number of periods τ the consumer will reach $b_{t+\tau} = 0$, and thereafter will set $\mathbf{c} = \mathbf{p}$ for eternity (which the PF-FVAC says yields finite value). Using the same steps as for equation (4), value of the interim program is also finite:

$$\mathbf{v}_{t+\tau} = \mathbf{\Phi}^{\tau(1-\rho)}\mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\mathbf{\Phi}^{1-\rho})^{T-(t+\tau)+1}}{1-\beta\mathbf{\Phi}^{1-\rho}}\right).$$

So, even under ~~FHWC~~, the limiting consumer's value for any finite m will be the sum

[4] "Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

[5] See **?** for details.

[6] That is, one obeying $\mathrm{c}(m) = \lim_{n \uparrow \infty} \mathrm{c}_{T-n}(m)$.

of two finite numbers: One due to the unconstrained choice made over the finite horizon leading up to $b_{t+\tau} = 0$, and one reflecting the value of consuming $\mathbf{p}_{t+\tau}$ thereafter.

GIC and ~~RIC~~. The most peculiar possibility occurs when the RIC fails. Under these circumstances the FHWC must also fail (Appendix E), and the constrained consumption function is nondegenerate. (See appendix Figure 2 for a numerical example). While $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\grave{c}(m)$ is finite, strictly positive, and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where ~~RIC~~ would suggest a degenerate limit of $\grave{c}(m) = 0$ while ~~FHWC~~ would suggest a degenerate limit of $\grave{c}(m) = \infty$.

We now examine the case with uncertainty but without constraints, which turns out to be a close parallel to the model with constraints but without uncertainty.

3 Results

3.1 Limits as m Approaches Infinity

3.2 Limits as m Approaches Zero

3.3 Unique 'Stable' Points

3.3.1 'Individual Target Wealth' $\hat{m}$

3.4 Example With Balanced-Growth $\check{m}$ But No Target $\hat{m}$

4 Discussion

Assume a continuum of *ex ante* identical buffer-stock households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$.

Szeidl (?) proved that such a population will be characterized by invariant distributions of m, c, and a under the condition[7]

$$\text{GIC-Szeidl:} \quad \log\ \text{Þ}_{\boldsymbol{\Phi}} < \mathbb{E}[\log \boldsymbol{\Psi}] \tag{8}$$

which is stronger than our GIC ($\text{Þ}_{\boldsymbol{\Phi}} < 1$), but weaker than our GIC-Mod ($\text{Þ}_{\underline{\boldsymbol{\Phi}}} < 1$).[8]

? substitutes a clever change of probability-measure into Szeidl's proof, with the implication that if the GIC holds, invariant *permanent-income-weighted* distributions exist. This allows him to prove a conjecture from an earlier draft of this paper (?) that under the GIC, aggregate consumption grows at the same rate $\boldsymbol{\Phi}$ as aggregate noncapital

[7] ?'s equation (9), in our notation, is:

$$\begin{aligned} \mathbb{E} \log \mathsf{R}(1-\kappa) &< \mathbb{E} \log \boldsymbol{\Phi}\boldsymbol{\Psi} \\ \mathbb{E} \log \mathsf{R}\text{Þ}_{\mathsf{R}} &< \mathbb{E} \log \boldsymbol{\Phi}\boldsymbol{\Psi} \\ \log \text{Þ}_{\boldsymbol{\Phi}} &< \mathbb{E} \log \boldsymbol{\Psi} \end{aligned}$$

which, exponentiated, yields (8).

[8] Under our default (though not required) assumption that $\log \boldsymbol{\Psi} \sim \mathcal{N}(-\sigma^2_{\boldsymbol{\Psi}}/2, \sigma^2_{\boldsymbol{\Psi}})$; the GIC-Mod in this case, is $\text{Þ}_{\boldsymbol{\Phi}} < \exp(-\sigma^2)$, so if the GIC-Mod holds then Szeidl's condition will hold.

income in the long run (with the corollary that aggregate assets and market resources grow at that same rate). **?** shows that his reformulation of the problem can reduce costs of calculation enormously.[9] In confirmation, this notebook finds that the Harmenberg method reduces the simulation size required for a given degree of accuracy by roughly a factor of 100 (!) under the baseline parameter values defined above.

The remainder of this section briefly draws out some implications of these points.

4.1 Individual Balanced Growth of Income, Consumption, and Wealth

Define $\mathbb{M}[\bullet]$ to yield the mean of its argument in the population (as distinct from the expectations operator $\mathbb{E}[\bullet]$ which represents beliefs about the future). Using boldface capitals for aggregates, the growth factor for aggregate noncapital income is:

$$\begin{aligned}\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}\left[\boldsymbol{\xi}_{t+1}\mathbf{\Phi}\mathbf{\Psi}_{t+1}\mathbf{p}_t\right]/\mathbb{M}\left[\mathbf{p}_t\boldsymbol{\xi}_t\right] \\ &= \mathbf{\Phi}\end{aligned}$$

because of the independence assumptions we have made about $\boldsymbol{\xi}$ and $\mathbf{\Psi}$.

Consider an economy that satisfies the Szeidl impatience condition (8) and has existed for long enough by date t that we can consider it as Szeidl-converged. In such an economy a microeconomist with a population-representative panel dataset could calculate the growth rate of consumption for each individual household, and take the average:

$$\begin{aligned}\mathbb{M}\left[\Delta \log \mathbf{c}_{t+1}\right] &= \mathbb{M}\left[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t\right] \\ &= \mathbb{M}\left[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t\right] + \mathbb{M}\left[\log c_{t+1} - \log c_t\right].\end{aligned} \tag{9}$$

Because this economy is Szeidl-converged, distributions of c_t and c_{t+1} will be identical, so that the second term in (9) disappears; hence, mean cross-sectional growth rates of consumption and permanent income are the same:

$$\mathbb{M}\left[\Delta \log \mathbf{c}_{t+1}\right] = \mathbb{M}\left[\Delta \log \mathbf{p}_{t+1}\right] = \log \mathbf{\Phi}. \tag{10}$$

In a Harmenberg-invariant economy (and therefore also any Szeidl-invariant economy), a similar proposition holds in the cross-section as a direct implication of the fact that a constant proportion of total permanent income is accounted for by the successive sets of consumers with any particular m. This fact is one way of interpreting Harmenberg's definition of the density of the permanent-income-invariant distribution of m; call this density $p(m)$.[10] Call $\mathbf{c}_t(m)$ the total amount of consumption at date t by persons with market resources m, and note that in the invariant economy this is given by the converged consumption function $\mathrm{c}(m)$ multiplied by the amount of permanent income accruing to such people $p(m)\mathbf{p}_t$. Since $p(m)$ is invariant and aggregate permanent income grows according to $\mathrm{P}_{t+1} = \mathbf{\Phi}\mathrm{P}_t$, for any m:

$$\begin{aligned}\log \mathbf{c}_{t+1}(m) - \log \mathbf{c}_t(m) &= \log \mathrm{c}(m)p(m)\mathrm{P}_{t+1} - \log \mathrm{c}(m)p(m)\mathrm{P}_t \\ &= \log \mathbf{\Phi}.\end{aligned}$$

[9]The Harmenberg method is implemented in the Econ-ARK; see the last part of `test_Harmenbergs_method.sh`.

[10]In his notation, this is $\tilde{\psi}^m$.

4.2 Aggregate Balanced Growth and Idiosyncratic Covariances

Harmenberg shows that the covariance between the individual consumption ratio c and the idiosyncratic component of permanent income $\mathbf{p}$ does not shrink to zero; thus, covariances are another potential measurement for construction of microfoundations.

Consider a date-t Harmenberg-converged economy, and define the mean value of the consumption ratio as $\mathfrak{c}_{t+n} \equiv \mathbb{M}[c_{t+n}]$. Normalizing period-$t$ aggregate permanent income to $\mathrm{P}_t = 1$, total consumption at $t+1$ and $t+2$ are

$$\begin{aligned} \mathbf{C}_{t+1} &= \mathbb{M}[c_{t+1}\mathbf{p}_{t+1}] = \mathfrak{c}_{t+1}\mathbf{\Phi}^1 + \mathrm{cov}_{t+1}(c_{t+1}, \mathbf{p}_{t+1}) \\ \mathbf{C}_{t+2} &= \mathbb{M}[c_{t+2}\mathbf{p}_{t+2}] = \mathfrak{c}_{t+2}\mathbf{\Phi}^2 + \mathrm{cov}_{t+2}(c_{t+2}, \mathbf{p}_{t+2}) \end{aligned} \tag{11}$$

and Harmenberg's proof that $\mathbf{C}_{t+2} - \mathbf{\Phi}\mathbf{C}_{t+1} = 0$ allows us to obtain:

$$(\mathfrak{c}_{t+2} - \mathfrak{c}_{t+1})\,\mathbf{\Phi}^2 = \mathbf{\Phi}\mathrm{cov}_{t+1} - \mathrm{cov}_{t+2}. \tag{12}$$

In a Szeidl-invariant economy, $\mathfrak{c}_{t+2} = \mathfrak{c}_{t+1}$, so the economy exhibits balanced growth in the covariance:

$$\mathrm{cov}_{t+2} = \mathbf{\Phi}\mathrm{cov}_{t+1}. \tag{13}$$

The more interesting case is when the economy is Harmenberg- but not Szeidl-invariant. In that case, if the cov and the $\mathfrak{c}$ terms have constant growth factors Ω_{cov} and $\Omega_{\mathfrak{c}}$,[11] an equation corresponding to (12) will hold in $t+n$:

$$\begin{aligned} (\overbrace{\Omega_{\mathfrak{c}}^n \mathfrak{c}_t}^{\mathfrak{c}_{t+n}} - \Omega_c^{n-1}\mathfrak{c}_t)\mathbf{\Phi}^n &= \left(\mathbf{\Phi}\Omega_{\mathrm{cov}}^{n-1} - \Omega_{\mathrm{cov}}^n\right)\mathrm{cov}_t \\ (\Omega_{\mathfrak{c}}\mathbf{\Phi})^{n-1}(\Omega_{\mathfrak{c}} - 1)\mathfrak{c}_t\mathbf{\Phi} &= \Omega_{\mathrm{cov}}^{n-1}(\mathbf{\Phi} - \Omega_{\mathrm{cov}})\mathrm{cov}_t \end{aligned} \tag{14}$$

so for the LHS and RHS to grow at the same rates we need

$$\Omega_{\mathrm{cov}} = \Omega_{\mathfrak{c}}\mathbf{\Phi}. \tag{15}$$

This is intuitive: In the Szeidl-invariant economy, it just reproduces our result above that the covariance exhibits balanced growth because $\Omega_{\mathfrak{c}} = 1$. The revised result just says that in the Harmenberg case where the mean value $\mathfrak{c}$ of the consumption ratio c can grow, the covariance must rise in proportion to any ongoing expansion of $\mathfrak{c}$ (as well as in proportion to the growth in $\mathbf{p}$).

4.3 Implications for Microfoundations

Thus we have microeconomic propositions, for both growth rates and for covariances of observable variables,[12] that can be tested in either cross-section or panel microdata to judge (and calibrate) the microfoundations that should hold for any macroeconomic analysis that requires balanced growth for its conclusions.

At first blush, these points are reassuring; one of the most persuasive arguments for the agenda of building microfoundations of macroeconomics is that newly available 'big

[11] This 'if' is a conjecture, not something proven by Harmenberg (or anyone else). But see appendix J for an example of a Harmenberg-invariant economy in which simulations suggest this proposition holds.

[12] Parallel results to those for consumption can be obtained for other measures like market assets.

data' allow us to measure cross-sectional covariances with great precision, so that we can use microeconomic natural experiments to disentangle questions that are hopelessly entangled in aggregate time-series data. Knowing that such covariances ought to be a stable feature of a stably growing economy is therefore encouraging.

But this discussion also highlights an uncomfortable point: In the model as specified, permanent income does not have a limiting distribution; it becomes ever more dispersed as the economy with infinite-horizon consumers continues to exist indefinitely.

A few microeconomic data sources attempt direct measurement of 'permanent income'; **?**, for example, show that their assumptions about the magnitude of permanent shocks (and mortality; see below) yield a simulated distribution of permanent income that roughly matches answers in the U.S. *Survey of Consumer Finances* ('SCF') to a survey question designed to elicit a direct measure of permanent income. They use those results to calibrate a model that matches empirical facts about the distribution of permanent income and wealth, and to show that the model also does a reasonable job of matching empirical facts about the marginal propensity to consume. The quantitative credibility of the argument depends on the model's match to the distribution of permanent income inequality, which would not be possible in a model without a nondegenerate steady-state distribution of permanent income.

For macroeconomists who want to build microfoundations by comparing the microeconomic implications of their models to micro data (directly – not in ratios to difficult-to-meaure 'permanent income'), it would be something of a challenge to determine how to construct empirical-data-comparable simulated results from a model with no limiting distribution of permanent income.

Death can solve this problem.

4.4 Mortality Yields Invariance

Most heterogeneous-agent models incorporate a constant positive probability of death, following **?** and **?**. In the Blanchardian model, if the probability of death exceeds a threshold that depends on the size of the permanent shocks, **?** show that the limiting distribution of permanent income has a finite variance. **?** assumes a universal annuitization scheme in which estates of dying consumers are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has considerable analytical advantages, most notably that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor. That is, if the 'pure' time preference factor is β and probability of remaining alive (not dead) is $\mathcal{L}$, then the assumption that no utility accrues after death makes the effective discount factor $\underline{\beta} = \beta\mathcal{L}$ while the enhancement to the rate of return from the annuity scheme yields an effective interest factor $\bar{\mathsf{R}} = \mathsf{R}/\mathcal{L}$ (recall that because of white-noise mortality, the average wealth of the two groups is identical). Combining these, the effective patience factor in the new economy $\underline{\beta}\bar{\mathsf{R}}$ is unchanged from its value in the infinite horizon model:

$$\underline{\beta}\bar{\mathsf{R}} = (\beta\mathcal{L}\mathsf{R}/\mathcal{L})^{1/\rho} = (\mathsf{R}\beta)^{1/\rho} = \text{Þ}. \tag{16}$$

The only adjustments this requires to the analysis above are therefore to the few elements that involve a role for the interest factor distinct from its contribution to Þ (principally, the RIC, which becomes $\text{Þ}/\bar{\mathsf{R}}$).

?'s innovation was valuable not only for the insight it provided but also because when he wrote, the principal alternative, the Life Cycle model of ?, was computationally challenging given then-available technologies. Despite its (considerable) conceptual value, Blanchard's analytical solution is now rarely used because essentially all modern modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway.

The simplest alternative to Blanchard is to follow Modigliani in constructing a realistic description of income over the life cycle and assuming that any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, ??).

Even if bequests are accidental, a macroeconomic model must make some assumption about how they are disposed of: As windfalls to heirs, estate tax proceeds, etc. We again consider the simplest choice, because it represents something of a polar alternative to Blanchard. Without a bequest motive, there are no behavioral effects of a 100 percent estate tax; we assume such a tax is imposed and that the revenues are effectively thrown in the ocean: The estate-related wealth effectively vanishes from the economy.

The chief appeal of this approach is the simplicity of the change it makes in the condition required for the economy to exhibit a balanced growth equilibrium (for consumers without a life cycle income profile). If $\mathcal{L}$ is the probability of remaining alive, the condition changes from the plain GIC to a looser mortality-adjusted GIC:

$$\mathcal{L}\text{Þ}_{\mathbf{\Phi}} < 1. \tag{17}$$

With no income growth, what is required to prohibit unbounded growth in aggregate wealth is the condition that prevents the per-capita wealth-to-permanent-income ratio of surviving consumers from growing faster than the rate at which mortality diminishes their collective population. With income growth, the aggregate wealth-to-income ratio will head to infinity only if a cohort of consumers is patient enough to make the desired rate of growth of wealth fast enough to counteract combined erosive forces of mortality and productivity.

5 Conclusions

Numerical solutions to optimal consumption problems, in both life cycle and infinite horizon contexts, have become standard tools since the first reasonably realistic models were constructed in the late 1980s. One contribution of this paper is to show that finite horizon ('life cycle') versions of the simplest such models, with assumptions about income shocks (transitory and permanent) dating back to ? and standard specifications of preferences — and without plausible (but computationally and mathematically inconvenient) complications like liquidity constraints — have attractive properties (like continuous differentiability of the consumption function, and analytical limiting MPC's as resources approach their minimum and maximum possible values).

The main focus of the paper, though, is on the limiting solution of the finite horizon model as the horizon extends to infinity. This simple model has other appealing features: A 'Finite Value of Autarky' condition guarantees convergence of the consumption function, under the mild additional requirement of a 'Weak Return Impatience Condition' that will never bind for plausible parameterizations, but provides intuition for the bridge between this model and models with explicit liquidity constraints. The paper also provides a roadmap for the model's relationships to the perfect foresight model without and with constraints. The constrained perfect foresight model provides an upper bound to the consumption function (and value function) for the model with uncertainty, which explains why the conditions for the model to have a nondegenerate solution closely parallel those required for the perfect foresight constrained model to have a nondegenerate solution.

The main use of infinite horizon versions of such models is in heterogeneous agent macroeconomics. The paper articulates intuitive 'Growth Impatience Conditions' under which populations of such agents, with Blanchardian (tighter) or Modiglianian (looser) mortality will exhibit balanced growth. Finally, the paper provides the analytical basis for many results about buffer-stock saving models that are so well understood that even without analytical foundations researchers uncontroversially use them as explanations of real-world phenomena like the cross-sectional pattern of consumption dynamics in the Great Recession.

Appendices

A c Functions Exist, are Concave, and Differentible

To show that (??) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, \ldots, \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$
5. $\mathrm{n}(z) < 0$
6. $\lim_{z \downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \boldsymbol{\xi}_{t+1})\right].$$

Since there is a positive probability that $\boldsymbol{\xi}_{t+1}$ will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a \downarrow 0} \mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (?).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as

$$\underline{\mathrm{v}}_t(m, c) = \mathrm{u}(c) + \mathfrak{v}_t(m - c) \tag{18}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (??) can be written as

$$\mathrm{v}_t(m) = \max_c\ \underline{\mathrm{v}}_t(m, c). \tag{19}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c \downarrow 0} \underline{\mathrm{v}}_t(m, c) = \lim_{c \uparrow m} \underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c \downarrow 0} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c \uparrow m} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \underset{0<c<m}{\arg\max}\ \underline{\mathrm{v}}_t(m, c) \tag{20}$$

exists and is unique, and (??) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{21}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $c_t(m)$ and $a_t(m) = m - c_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (21) we can conclude that $c_t(m)$ is continuously differentiable and

$$c_t'(m) = \frac{\mathfrak{v}_t''(a_t(m))}{u''(c_t(m)) + \mathfrak{v}_t''(a_t(m))}. \tag{22}$$

Similarly we can easily show that $c_t(m)$ is twice continuously differentiable (as is $a_t(m)$) (See Appendix B.) This implies that $v_t(m)$ is nice, since $v_t(m) = u(c_t(m)) + \mathfrak{v}_t(a_t(m))$.

B $c_t(m)$ is Twice Continuously Differentiable

First we show that $c_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $u'(c_t(y)) - u'(c_t(m)) = \mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))$ and $\frac{a_t(y)-a_t(m)}{dm} = 1 - \frac{c_t(y)-c_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)} = \left(\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)}\right)\frac{c_t(y) - c_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm \to +0} \frac{u'(c_t(y))-u'(c_t(m))}{c_t(y)-c_t(m)} < 0$ and $\lim_{dm \to +0} \frac{\mathfrak{v}_t'(a_t(y))-\mathfrak{v}_t'(a_t(m))}{a_t(y)-a_t(m)} < 0$ are satisfied. Then $\frac{u'(c_t(y))-u'(c_t(m))}{c_t(y)-c_t(m)} + \frac{\mathfrak{v}_t'(a_t(y))-\mathfrak{v}_t'(a_t(m))}{a_t(y)-a_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{c_t(y) - c_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(a_t(y))-\mathfrak{v}_t'(a_t(m))}{a_t(y)-a_t(m)}}{\frac{u'(c_t(y))-u'(c_t(m))}{c_t(y)-c_t(m)} + \frac{\mathfrak{v}_t'(a_t(y))-\mathfrak{v}_t'(a_t(m))}{a_t(y)-a_t(m)}}.$$

This implies that the right-derivative, $c_t'^{+}(m)$ is well-defined and

$$c_t'^{+}(m) = \frac{\mathfrak{v}_t''(a_t(m))}{u''(c_t(m)) + \mathfrak{v}_t''(a_t(m))}.$$

Similarly we can show that $c_t'^{+}(m) = c_t'^{-}(m)$, which means $c_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $c_t'(m)$ exists and is continuous. $c_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $c_t(m)$ is $\mathbf{C}^1$ and $u''(c_t(m)) + \mathfrak{v}_t''(a_t(m)) < 0$. $c_t''(m)$ is given by

$$c_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[u''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t' u'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[u''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}. \tag{23}$$

Since $\mathfrak{v}_t''(a_t(m))$ is continuous, $c_t''(m)$ is also continuous.

C $\mathcal{T}$ Is a Contraction Mapping

We must show that our operator $\mathcal{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathcal{T}z\}$ be continuous for any $F-$bounded z, $\{\mathcal{T}z\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchi (?).

Consider condition (1). For this problem,

$$\{\mathcal{T}\mathrm{x}\}(m_t) \text{ is } \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\}$$
$$\{\mathcal{T}\mathrm{y}\}(m_t) \text{ is } \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\},$$

so $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)$ implies $\{\mathcal{T}\mathrm{x}\}(m_t) \leq \{\mathcal{T}\mathrm{y}\}(m_t)$ by inspection.[13]

Condition (2) requires that $\{\mathcal{T}\mathbf{0}\} \in \mathcal{C}_{F}\left(\mathcal{A}, \mathcal{B}\right)$. By definition,

$$\{\mathcal{T}\mathbf{0}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right) + \beta 0\right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa} m_t)$. Thus, condition (2) will hold if $\left(\bar{\kappa} m_t\right)^{1-\rho}$ is F-bounded, which it is if we use the bounding function

$$F(m) = \eta + m^{1-\rho}, \tag{24}$$

defined in the main text.

Finally, we turn to condition (3), $\{\mathcal{T}(\mathrm{z}+\zeta F)\}(m_t) \leq \{\mathcal{T}\mathrm{z}\}(m_t) + \zeta\alpha F(m_t)$. The proof will be more compact if we define $\breve{\mathrm{c}}$ and $\breve{\mathrm{a}}$ as the consumption and assets functions[14] associated with $\mathcal{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathcal{T}(\mathrm{z}+\zeta F)$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\breve{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}) + \zeta\alpha F.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\breve{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha F \\
\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha F \\
\beta\zeta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \zeta\alpha F \\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \alpha F \\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &< F.
\end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.[15]

Using $F(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\mathbf{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \boldsymbol{\xi}_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta\,\mathbb{E}_t\,\mathbf{\Phi}_{t+1}^{1-\rho}}_{=\beth})$$

[13] For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

[14] Section **??** proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[15] The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

which by imposing PF-FVAC (equation (4), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\boldsymbol{\xi}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}}{1-\beth}. \tag{25}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (25) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\le&(1-\wp)\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]\\
&+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]\\
&+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)\\
<&(1-\wp)\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}(\underline{\boldsymbol{\theta}}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\boldsymbol{\theta}}^{1-\rho}.
\end{aligned}$$

We can thus conclude that equation (25) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1-\wp)^{\rho}\underline{\boldsymbol{\theta}}^{1-\rho}}{1-\beth} \tag{26}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (**??**) and (**??**) is now complete.

C.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \le c_t \le \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (**??**)) that $\underline{\kappa}_t m_t \le \mathrm{c}_t(m_t) \le \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (**??**) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (26). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} &< 1\\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) &> 1
\end{aligned}$$

$$(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1}) > 1$$

where we have used (**??**) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1} \approx 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ so that it becomes

$$\begin{aligned}(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}} &> 1\\ (\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1\\ \beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1.\end{aligned}$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping in $F-$bounded space with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathcal{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (**??**) converge, they converge to the same unique v defined by $\mathcal{T}$.[16]

D Convergence in Euclidian Space

D.1 Convergence of v_t

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in an F-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathcal{T}$, since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_{F} \leq \alpha^{n-1}\|\mathrm{v}_T - \mathrm{v}^*\|_{F}\,. \tag{27}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{F}(\mathcal{A},\mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_{F} < \infty$ because v_T and v^* are in $\mathcal{C}_{F}(\mathcal{A},\mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1}|F(m)|\,. \tag{28}$$

Then we obtain

$$\lim_{n\to\infty}\mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{29}$$

[16]It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathfrak{T}\mathrm{v}_{T-1} \leq \mathfrak{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

D.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\left\{\mathrm{c}_{T-n(i)}(m)\right\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\begin{aligned} \mathrm{u}(c_{T-n(i)}(m)) + &\beta\,\mathbb{E}_{T-n(i)}[\boldsymbol{\Phi}_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)] \\ &\geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_{T-n(i)}[\boldsymbol{\Phi}_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)], \end{aligned} \tag{30}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\boldsymbol{\Phi}_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\boldsymbol{\Phi}_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\boldsymbol{\Phi}_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\boldsymbol{\Phi}_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{31}$$

Hence, $c^* \in \underset{c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\boldsymbol{\Phi}_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

E Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions.

Results are summarized in table 1.

E.1 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails (~~GIC~~, $1 < \text{Þ}/\boldsymbol{\Phi}$). Under ~~GIC~~ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\boldsymbol{\Phi}$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[17]

$$
\begin{aligned}
1 &< (\mathsf{R}\beta)^{1/\rho}\mathbf{\Phi}^{-1} \\
1 &< \mathsf{R}\beta\mathbf{\Phi}^{-\rho} \\
\mathrm{u}'(1) &< \mathsf{R}\beta\mathrm{u}'(\mathbf{\Phi}).
\end{aligned}
$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHWC holds. If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$
\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{32}
$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHWC fails. ~~FHWC~~ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[18]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from Equation (1):

$$
\begin{aligned}
m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\
m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\
m_{\#} &= (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right)
\end{aligned} \tag{33}
$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[19] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[20] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

(**?** obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

[17]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\mathbf{\Phi})$ which is $c_{\#} = \mathbf{\Phi}/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[18]The knife-edge case is where $\text{Þ} = \mathbf{\Phi}$, in which case the two quantities counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

[19]Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~GIC~~.

[20]As an illustration, consider a consumer for whom $\text{Þ} = 1$, $\mathsf{R} = 1.01$ and $\mathbf{\Phi} = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\mathbf{\Phi} < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

E.2 If GIC Holds

Imposition of the GIC reverses the inequality in (32), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b^n_{\#}$ as the b_t such that an unconstrained consumer holding $b_t = b^n_{\#}$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b^0_{\#}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b^1_{\#}$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m^0_{\#} = 1 + b^0_{\#} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c^0_{\#} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\text{\textbf{Þ}}_{\mathbf{\Phi}}$, so consumption n periods in the past must have been

$$c^n_{\#} = \text{\textbf{Þ}}_{\mathbf{\Phi}}^{-n} c_t = \text{\textbf{Þ}}_{\mathbf{\Phi}}^{-n}. \tag{34}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}^t_{t-n} &= c_{t-n}(1 + \text{\textbf{Þ}}/\mathsf{R} + \cdots + (\text{\textbf{Þ}}/\mathsf{R})^n) \\
&= c^n_{\#}(1 + \text{\textbf{Þ}}_{\mathsf{R}} + \cdots + \text{\textbf{Þ}}_{\mathsf{R}}^n) \\
&= \text{\textbf{Þ}}_{\mathbf{\Phi}}^{-n} \left(\frac{1 - \text{\textbf{Þ}}_{\mathsf{R}}^{n+1}}{1 - \text{\textbf{Þ}}_{\mathsf{R}}} \right) \\
&= \left(\frac{\text{\textbf{Þ}}_{\mathbf{\Phi}}^{-n} - \text{\textbf{Þ}}_{\mathsf{R}}}{1 - \text{\textbf{Þ}}_{\mathsf{R}}} \right)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h^n_{\#} = 1 + \cdots + \mathcal{R}^{-n} \tag{35}$$

while the intertemporal budget constraint says

$$\mathbb{C}^t_{t-n} \; = b^n_{\#} + h^n_{\#}$$

from which we can solve for the $b^n_{\#}$ such that the consumer with $b_{t-n} = b^n_{\#}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b^n_{\#} = \mathbb{C}^t_{t-n} - \overbrace{\left(\frac{1 - \mathcal{R}^{-(n+1)}}{1 - \mathcal{R}^{-1}} \right)}^{h^n_{\#}}. \tag{36}$$

Defining $m^n_{\#} = b^n_{\#} + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m^n_{\#}, c^n_{\#}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^{n} - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^{n} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (36) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{37}$$

E.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (36) limits to a constant as $n \uparrow \infty$, and (37) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathbf{\Phi}}^{-n} - (\text{Þ}_{\mathsf{R}}/\text{Þ}_{\mathbf{\Phi}})^{n}\text{Þ}_{\mathsf{R}}}{1 - \text{Þ}_{\mathsf{R}}} \right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathbf{\Phi}}^{-n} - \mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1 - \text{Þ}_{\mathsf{R}}} \right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathbf{\Phi}}^{-n}}{1 - \text{Þ}_{\mathsf{R}}} \right) &= \infty.
\end{aligned}$$

Given the GIC $\text{Þ}_{\mathbf{\Phi}}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \mathbf{\Phi}$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{38}$$

E.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (37) requires

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}} - \text{Þ}_{\mathbf{\Phi}}^{-n}}{\text{Þ}_{\mathsf{R}} - 1} \right) + \left(\frac{1 - \mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1} - 1} \right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}} - 1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1} - 1} \right) \mathcal{R}^{-n} - \left(\frac{\text{Þ}_{\mathbf{\Phi}}^{-n}}{\text{Þ}_{\mathsf{R}} - 1} \right) &= \infty
\end{aligned}$$

If RIC Holds. When the RIC holds, rearranging (39) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right)-\mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}+\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\quad=\infty$$

and for this to be true we need

$$\begin{aligned}\text{Þ}_{\boldsymbol{\Phi}}^{-1}&>\mathcal{R}^{-1}\\ \boldsymbol{\Phi}/\text{Þ}&>\boldsymbol{\Phi}/\mathsf{R}\\ 1&>\text{Þ}/\mathsf{R}\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n}=\lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right)\tag{39}$$

which with a bit of algebra[21] can be shown to asymptote to the MPC in the perfect foresight model:[22]

$$\lim_{m\to\infty}\grave{\boldsymbol{\kappa}}(m)=1-\text{Þ}_{\mathsf{R}}.\tag{41}$$

If RIC Fails. Consider now the ~~RIC~~ case, $\text{Þ}_{\mathsf{R}}>1$. We can rearrange (39)as

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}-\frac{\mathcal{R}^{-1}(\text{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)\quad=\infty.\tag{42}$$

which makes clear that with ~~FHWC~~ $\Rightarrow\mathcal{R}^{-1}>1$ and ~~RIC~~ $\Rightarrow\text{Þ}_{\mathsf{R}}>1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (39) will be positive if

$$\begin{aligned}\text{Þ}_{\mathsf{R}}\mathcal{R}^{-1}-\text{Þ}_{\mathsf{R}}&>\mathcal{R}^{-1}\text{Þ}_{\mathsf{R}}-\mathcal{R}^{-1}\\ \mathcal{R}^{-1}&>\text{Þ}_{\mathsf{R}}\\ \boldsymbol{\Phi}&>\text{Þ}\end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\boldsymbol{\Phi}}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned}\mathcal{R}^{-1}&>\text{Þ}_{\boldsymbol{\Phi}}^{-1}\\ \boldsymbol{\Phi}/\mathsf{R}&>\boldsymbol{\Phi}/\text{Þ}\\ \text{Þ}/\mathsf{R}&>1\end{aligned}$$

[21]Calculate the limit of

$$\left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{\text{Þ}_{\boldsymbol{\Phi}}^{-n}/(1-\text{Þ}_{\mathsf{R}})-(1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right)=\left(\frac{1}{1/(1-\text{Þ}_{\mathsf{R}})+\mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right)\tag{40}$$

[22]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

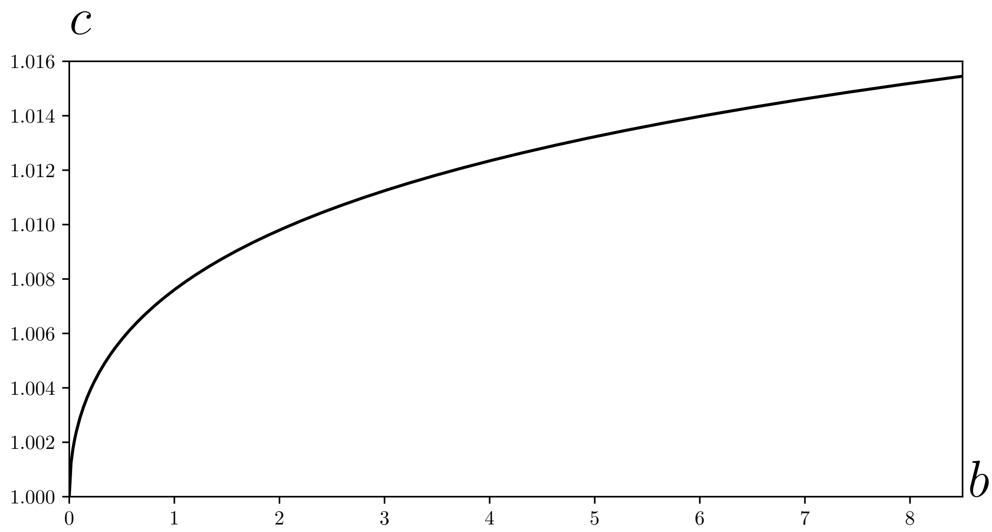

Figure 1 Appendix: Nondegenerate c Function with ~~FHWC~~ and ~~RIC~~

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c^n_{\#}$ term is increasing backward in time at rate dominated in the limit by $\boldsymbol{\Phi}/\mathbf{Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by $\boldsymbol{\Phi}/\mathsf{R}$ term and

$$\boldsymbol{\Phi}/\mathsf{R} \;>\; \boldsymbol{\Phi}/\mathbf{Þ} \tag{43}$$

because ~~RIC~~ $\Rightarrow \mathbf{Þ} > \mathsf{R}$.

Consequently, while $\lim_{n\uparrow\infty} b^n_{\#} = \infty$, the limit of the *ratio* $c^n_{\#}/b^n_{\#}$ in (39) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that ~~RIC~~ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{44}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 2 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\boldsymbol{\Phi} = 0.99$; note that the horizontal axis is bank balances $b = m-1$; the part of the consumption function below the depicted points is uninteresting — $c = m$ — so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

? characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

F The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m). \tag{45}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \mathbf{\Phi} = 1$, and there are no permanent shocks, $\mathbf{\Psi} = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\boldsymbol{\theta}}}^{\bar{\boldsymbol{\theta}}} \mathrm{v}_T(a+\boldsymbol{\theta}) d\mathcal{F}_{\boldsymbol{\theta}} \right\}. \tag{46}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\boldsymbol{\theta}}}^{\bar{\boldsymbol{\theta}}} \mathrm{u}'(\mathrm{c}_T(a+\boldsymbol{\theta})) d\mathcal{F}_{\boldsymbol{\theta}},$$

and the solution to (46) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{47}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (**?**)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (47), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp)\int_{\underline{\boldsymbol{\theta}}}^{\bar{\boldsymbol{\theta}}} (\mathrm{c}_T(a+\boldsymbol{\theta}/(1-\wp)))^{-\rho} d\mathcal{F}_{\boldsymbol{\theta}}\right], \tag{48}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{49}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (47) and (49) are identical, this means that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (48) is $\lim_{a\downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (45). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (51) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp \downarrow 0} \bar{\kappa} \;\; = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

G The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (**??**) can be rewritten

$$\begin{aligned}
\mathrm{e}_t(m_t)^{-\rho} &= \beta \mathsf{R}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(m_{t+1}) \left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \mathbf{\Phi}_{t+1}\boldsymbol{\xi}_{t+1}}^{=m_{t+1}\mathbf{\Phi}_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \\
&= \;\; (1-\wp)\beta \mathsf{R} m_t^{\rho}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(m_{t+1}) m_{t+1} \mathbf{\Phi}_{t+1})^{-\rho} \vert\, \boldsymbol{\xi}_{t+1} > 0 \right] \\
&\quad + \wp \beta \mathsf{R}^{1-\rho}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t)) \frac{m_t - \mathrm{c}_t(m_t)}{m_t} \right)^{-\rho} \vert\, \boldsymbol{\xi}_{t+1} = 0 \right].
\end{aligned}$$

Consider the first conditional expectation in (**??**), recalling that if $\boldsymbol{\xi}_{t+1} > 0$ then $\boldsymbol{\xi}_{t+1} \equiv \boldsymbol{\theta}_{t+1}/(1-\wp)$. Since $\lim_{m \downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1}) m_{t+1} \mathbf{\Phi}_{t+1})^{-\rho} \mid \boldsymbol{\xi}_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\boldsymbol{\theta}}/(1-\wp)) \mathbf{\Phi} \underline{\mathbf{\Psi}} \underline{\boldsymbol{\theta}}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\boldsymbol{\theta}}/(1-\wp)) \mathbf{\Phi} \bar{\mathbf{\Psi}} \bar{\boldsymbol{\theta}}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta \wp \mathsf{R}^{1-\rho} \bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta \mathsf{R})^{-1/\rho} \mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho} \overbrace{\mathsf{R}^{-1}(\beta \mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv \wp^{1/\rho}\text{Þ}_{\mathsf{R}}} \bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned}
(\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1} &= 1 + \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}.
\end{aligned}$$

As noted in the main text, we need the WRIC (**??**) for this to be a convergent sequence:

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{50}$$

Since $\bar{\kappa}_T = 1$, iterating (50) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{51}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving $\boldsymbol{\xi}_{t+1}$ in (50) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} = \beta\mathsf{R}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\,(\mathsf{R}\mathrm{a}_t(m_t)))^{-\rho}\right]$$

and using L'Hôpital's rule $\lim_{m_t\to\infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t\underline{\kappa}_t)^{-\rho} &= \beta\mathsf{R}\big(\underline{\kappa}_{t+1}\mathsf{R}(1-\underline{\kappa}_t)m_t\big)^{-\rho} \\ \underbrace{\mathsf{R}^{-1}\text{Þ}}_{\equiv\text{Þ}_{\mathsf{R}}=(1-\underline{\kappa})}\ \underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1}\text{Þ}_{\mathsf{R}} \tag{52}$$

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n\uparrow\infty} \kappa_{T-n}^{-1} \tag{53}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\to\infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + \cdots\right)}_{=1+\text{Þ}_{\mathsf{R}}(1+\text{Þ}_{\mathsf{R}}\underline{\kappa}_{t+2}^{-1})\ldots} = c_t\underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \tag{54}$$

H Unique and Stable Target and Steady State Points

This appendix proves Theorems 1-2 and:

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

H.1 Proof of Theorem 1

Theorem 1. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC-Mod all hold, there exists a unique cash-on-hand-to-permanent-income ratio $\hat{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{55}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{56}$$

The elements of the proof of Theorem 1 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

H.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem **??**).

Section **??** shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \boldsymbol{\xi}_t$, even if $\boldsymbol{\xi}_t$ takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

H.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

H.3.1 Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (**??**), but dropping the $\mathbf{\Phi}_{t+1}$ from the RHS, establishes that

$$\begin{aligned}\lim_{m_t\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t}\right] \\ &= \mathbb{E}_t[(\mathsf{R}/\mathbf{\Phi}_{t+1})\text{Þ}_{\mathsf{R}}] \\ &= \mathbb{E}_t[\text{Þ}/\mathbf{\Phi}_{t+1}] \\ &< 1\end{aligned} \tag{57}$$

where the inequality reflects imposition of the GIC-Mod (**??**).

If RIC fails. When the RIC fails, the fact that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$ (see equation (**??**)) means that the limit of the RHS of (57) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Mod and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

H.3.2 Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

H.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,\end{aligned} \tag{58}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t\right)\right] \\ &= \bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1.\end{aligned} \tag{59}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1 < \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1$$

$$
\begin{aligned}
&= \bar{\mathcal{R}} \text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t \left[\frac{\mathsf{R}}{\mathbf{\Phi} \mathbf{\Psi}} \frac{\text{Þ}}{\mathsf{R}} \right] - 1 \\
&= \underbrace{\mathbb{E}_t \left[\frac{\text{Þ}}{\mathbf{\Phi} \mathbf{\Psi}} \right]}_{= \text{Þ}_{\underline{\mathbf{\Phi}}}} - 1
\end{aligned}
$$

which is negative because the GIC-Mod says $\text{Þ}_{\underline{\mathbf{\Phi}}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}} (1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (59) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[\frac{\mathsf{R}}{\mathbf{\Phi} \mathbf{\Psi}} \right] < 1. \tag{60}$$

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t \left[\frac{\text{Þ}}{\mathbf{\Phi} \mathbf{\Psi}} \right]}^{\text{Þ}_{\underline{\mathbf{\Phi}}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t \left[\frac{\mathsf{R}}{\mathbf{\Phi} \mathbf{\Psi}} \right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (60).

H.4 Proof of Theorem 2

Theorem 2. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{61}$$

Moreover, $\check{m}$ is a point of stability in the sense that

$$
\begin{aligned}
&\forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t > \mathbf{\Phi} \\
&\forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t < \mathbf{\Phi}.
\end{aligned}
\tag{62}
$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1} - m_t]$ is monotonically decreasing

H.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\boldsymbol{\Psi}} \leq \boldsymbol{\Psi}_{t+1} \leq \bar{\boldsymbol{\Psi}} < \infty$, our proof in H.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t]$.

H.4.2 Existence of a stable point

Since by assumption $0 < \underline{\boldsymbol{\Psi}} \leq \boldsymbol{\Psi}_{t+1} \leq \bar{\boldsymbol{\Psi}} < \infty$, our proof in Subsection H.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$
\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\boldsymbol{\Phi}_{t+1}\left((\mathsf{R}/\boldsymbol{\Phi}_{t+1})\mathrm{a}(m_t)+\boldsymbol{\xi}_{t+1}\right)/\boldsymbol{\Phi}}{m_t}\right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\boldsymbol{\Phi})\mathrm{a}(m_t)+\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t}\right] \\
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\boldsymbol{\Phi})\mathrm{a}(m_t)+1}{m_t}\right] \\
&= (\mathsf{R}/\boldsymbol{\Phi})\text{Þ}_{\mathsf{R}} \qquad (63) \\
&= \text{Þ}_{\boldsymbol{\Phi}} \\
&< 1
\end{aligned}
$$

where the last two lines are merely a restatement of the GIC (3).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

H.4.3 $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}] - m_t$ and note that

$$
\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] = 1 \qquad (64) \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] > 1,
\end{aligned}
$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$
\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - m_t\right)\right] \qquad (65) \\
&= (\mathsf{R}/\boldsymbol{\Phi})\left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned}
$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \mathcal{R}\,(1 - \mathrm{c}'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\boldsymbol{\Phi})\text{Þ}_{\mathsf{R}} - 1 \end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\boldsymbol{\Phi}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\,(1 - \mathrm{c}'(m_t)) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (65) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\boldsymbol{\Phi}) < 1. \tag{66}$$

But we showed in Section **??** that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (66) holds).

H.5 A Third Measure

A footnote in Section 3 mentions reasons why it may be useful to calculate $\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\log \mathbf{m}_t)]$. Here we show that one way of doing that is to calculate a nonlinear adjustment factor for the expectation of the growth factor.

$$\begin{aligned} \log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right) &= \log(\boldsymbol{\Phi}\boldsymbol{\Psi}_{t+1} m_{t+1}) - \log m_t \\ &= \log \boldsymbol{\Phi}(a_t\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}) - \log m_t \\ &= \log \boldsymbol{\Phi}(a_t\mathcal{R} + 1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)) - \log m_t \end{aligned}$$

Now define $\breve{m}_{t+1} = a_t\mathcal{R} + 1$, and compute the expectation:

$$\begin{aligned} \mathbb{E}_t[\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right)] &= \mathbb{E}_t\left[\log \boldsymbol{\Phi}(\breve{m}_{t+1} + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1))\right] - \log m_t \\ &= \log \boldsymbol{\Phi} + \mathbb{E}_t\left[\log \breve{m}_{t+1}\left(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1}\right)\right] - \log m_t \\ &= \underbrace{\log \boldsymbol{\Phi} + \log \breve{m}_{t+1} - \log m_t}_{\equiv \log \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]} + \mathbb{E}_t\left[\log(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right] \end{aligned}$$

and exponentiating tells us that

$$\exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \exp(\mathbb{E}_t\left[\log(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \tag{67}$$

and this latter factor is a number that approaches 1 from below as m_t rises. Thus the expected growth rate of the log is smaller than the log of the growth rate of the expected growth factor. This implies that the m at which 'balanced growth' can be expected in the log, $\tilde{m}$, exceeds the corresponding point for the ratio, $\breve{m}$.

Furthermore, in the limit as $\mathbf{m}_t$ gets arbitrarily large, if the RIC holds and thus $\underline{\kappa} > 0$, a_{t+1} rises without bound, as does $\breve{m}_{t+1} = a_{t+1}\mathcal{R} + 1$, so the approximation $\log(1+\epsilon) \approx \epsilon$ becomes arbitrarily good. Consequently, the last term on the RHS of (67) can be approximated as

$$\begin{aligned}\mathbb{E}_t\left[\log(1+(\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}-1)\check{m}_{t+1}^{-1})\right]\big) &\approx \mathbb{E}_t\left[(\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}-1)\check{m}_{t+1}^{-1})\right]\big) \\ &= 0\end{aligned}$$

This demonstrates that

$$\lim_{\mathbf{m}_t\uparrow\infty}\exp(\mathbb{E}_t[\log\mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \tag{68}$$

H.6 Proof of Lemma

H.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$\begin{aligned}\check{\mathrm{m}}_{t+1}(a) &= 1 + a\mathcal{R} \\ \hat{\mathrm{m}}_{t+1}(a) &= 1 + a\underbrace{\mathcal{R}/\underline{\boldsymbol{\Psi}}}_{\bar{\mathcal{R}}>\mathcal{R}}\end{aligned} \tag{69}$$

so that we can implicitly define the target and pseudo-steady-state points as

$$\begin{aligned}\hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m} - \mathrm{c}(\hat{m})) \\ \check{m} &= \check{\mathrm{m}}_{t+1}(\check{m} - \mathrm{c}(\check{m}))\end{aligned} \tag{70}$$

Then subtract:

$$\begin{aligned}\hat{m} - \check{m} &= \left(\hat{a}\underline{\boldsymbol{\Psi}}^{-1} - \check{a}\right)\mathcal{R} \\ &= \left(a(\hat{m})\underline{\boldsymbol{\Psi}}^{-1} - a(\check{m})\right)\mathcal{R} \\ &= \left(a(\hat{m})\underline{\boldsymbol{\Psi}}^{-1} - (a(\hat{m} + \check{m} - \hat{m}))\right)\mathcal{R} \\ &\approx \left(a(\hat{m})\underline{\boldsymbol{\Psi}}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\ (\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{<\text{Þ}_{\boldsymbol{\Phi}}<1}) &= (\underline{\boldsymbol{\Psi}}^{-1} - 1)\hat{a}\mathcal{R}\end{aligned} \tag{71}$$

The RHS of this equation is strictly positive because $\underline{\boldsymbol{\Psi}}^{-1} > 1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[23]

[23]The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a′ over the interval to remove the approximation in the derivations above.

H.6.2 The m Achieving Individual Expected-Log-Balanced-Growth Is Smaller than the Individual Pseudo-Steady-State m

Expected log balanced growth occurs when

$$\begin{aligned}
\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \boldsymbol{\Phi}\mathbf{m}_t \\
\mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \boldsymbol{\Phi}\mathbf{p}_t m_t \\
\mathbb{E}_t[\log \boldsymbol{\Psi}_{t+1} m_{t+1}] &= \log \boldsymbol{\Phi} m_t \\
\mathbb{E}_t[\log (a(m_t)\mathsf{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\boldsymbol{\Phi})] &= \log \boldsymbol{\Phi} m_t \\
\mathbb{E}_t[\log (a(m_t)\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})] &= \log m_t
\end{aligned} \tag{72}$$

and we call the m that satisfies this equation $\tilde{m}$.

Subtract the definition of $\check{m}$ from that of $\tilde{m}$:

$$\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})]) - (a(\check{m})\mathcal{R} + 1) = \tilde{m} - \check{m} \tag{73}$$

Now we use the fact that the expectation of the log is less than the log of the expectation,

$$\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})]) < (a(\tilde{m})\mathcal{R} + 1) \tag{74}$$

so

$$\begin{aligned}
\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + 1)]) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m})\mathcal{R} + 1) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - a(\tilde{m} + \check{m} - \tilde{m}))\mathcal{R} &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - (a(\tilde{m}) + (\check{m} - \tilde{m})\bar{\mathrm{a}}')\mathcal{R} &< \tilde{m} - \check{m} \\
(\tilde{m} - \check{m})\bar{\mathrm{a}}'\mathcal{R} &< \tilde{m} - \check{m} \\
\underbrace{\bar{\mathrm{a}}'\mathcal{R}}_{<\text{Þ}_{\boldsymbol{\Phi}}} &< 1
\end{aligned} \tag{75}$$

where we are interpreting $\bar{\mathrm{a}}'$ as the mean of the value of a' over the interval between $\tilde{m}$ and $\check{m}$.

I Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures **??**, **??**).

I.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 3, whose three nodes represent values of the absolute patience factor **Þ**, the permanent-income growth factor $\boldsymbol{\Phi}$, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from **Þ** to $\boldsymbol{\Phi}$, reflects imposition of the GIC condition (clicking GIC should take you to its definition; definitions of other conditions are also

$$\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \textbf{Þ} \arrow[rr, "\text{GIC}(\equiv \textbf{Þ} < \Gamma)"] \arrow[dr, "(\textbf{Þ} < \mathsf{R} \equiv)\ \text{RIC}"'] & \arrow[d, phantom, "\neq"] & \Gamma \arrow[dl, "\text{FHWC}\ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}$$

Figure 2 Appendix: Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

linked below)).[24] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the **Þ** node and impose the GIC and then the FHWC, and see that imposition of these conditions allows us to conclude that $\textbf{Þ} < \mathsf{R}$.

One could also impose $\textbf{Þ} < \mathsf{R}$ directly (without imposing GIC and FHWC) by following the downward-sloping diagonal arrow exiting **Þ**. Although alternate routes from one node to another all justify the same core conclusion ($\textbf{Þ} < \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[25] to indicate that the diagram is not commutative.[26]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, ~~RIC~~ $\equiv \textbf{Þ} > \mathsf{R}$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting **Þ** and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and $\mathbf{\Phi}$, but the $\mathbf{\Phi}$ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed GIC (that is, if we imposed ~~GIC~~), that would take us to the **Þ** node, and we could deduce $\mathsf{R} > \textbf{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the **Þ** node points back to our starting point, which (if valid) would lead us to the conclusion

[24] For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[25] For a popular introduction to category theory, see **?**.

[26] But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~GIC~~) requires us also to reverse the final condition, giving us ~~RIC~~.[27]

Under these conventions, Figure **??** in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{Þ} \arrow[r, "\text{GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}"] & \Gamma \arrow[d, "\text{FHWC}"] \arrow[dl, phantom, "\neq" near start] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] \arrow[ur, phantom, "\neq" near start] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 3 Appendix: Relation of GIC, FHWC, RIC, and PFVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{Þ} < \mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (4)

This diagram can be interpreted, for example, as saying that, starting at the $\mathbf{Þ}$ node, it is possible to derive the PF-FVAC[28] by imposing both the GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the $\mathbf{\Phi}$ node, we can follow the imposition of the FHWC (twice — reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\mathbf{Þ} < \mathbf{\Phi}$. Algebraically,

$$\begin{aligned} \text{FHWC}: &\quad \mathbf{\Phi} < \mathsf{R} \\ \cancel{\text{RIC}}: &\quad \mathsf{R} < \mathbf{Þ} \\ &\quad \mathbf{\Phi} < \mathbf{Þ} \end{aligned} \tag{76}$$

which leads to the negation of both of the conditions leading into $\mathbf{Þ}$. ~~GIC~~ is obtained directly as the last line in (76) and ~~PF-FVAC~~ follows if we start by multiplying the

[27]The corresponding algebra is

$$\begin{aligned} \cancel{\text{FHWC}}: &\quad \mathsf{R} < \mathbf{\Phi} \\ \cancel{\text{GIC}}: &\quad \mathbf{\Phi} < \mathbf{Þ} \\ \Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \mathbf{Þ}, \end{aligned}$$

.

[28]in the form $\mathbf{Þ} < (\mathsf{R}/\mathbf{\Phi})^{1/\rho}\mathbf{\Phi}$

Return Patience Factor (RPF=$\text{Þ}/\mathsf{R}$) by the FHWF ($=\mathbf{\Phi}/\mathsf{R}$) raised to the power $1/\rho-1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$
\begin{aligned}
1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\mathbf{\Phi}/\mathsf{R})^{1/\rho-1}}^{>1 \text{ from FHWC}} \\
1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\mathbf{\Phi})^{1/\rho}\mathsf{R}\mathbf{\Phi}/\mathsf{R}}\right) \\
\mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho} = (\mathsf{R}/\mathbf{\Phi})^{1/\rho}\mathbf{\Phi} &< \text{Þ}
\end{aligned}
$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure **??** in the paper (reproduced here).

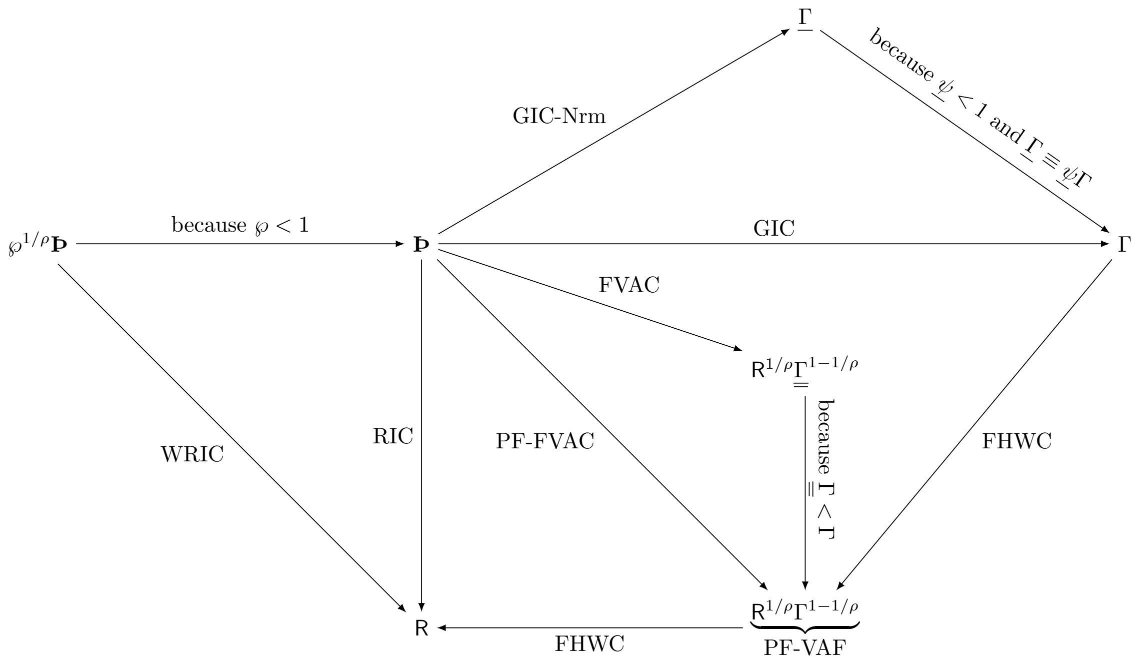

Figure 4 Appendix: Relation of All Inequality Conditions

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

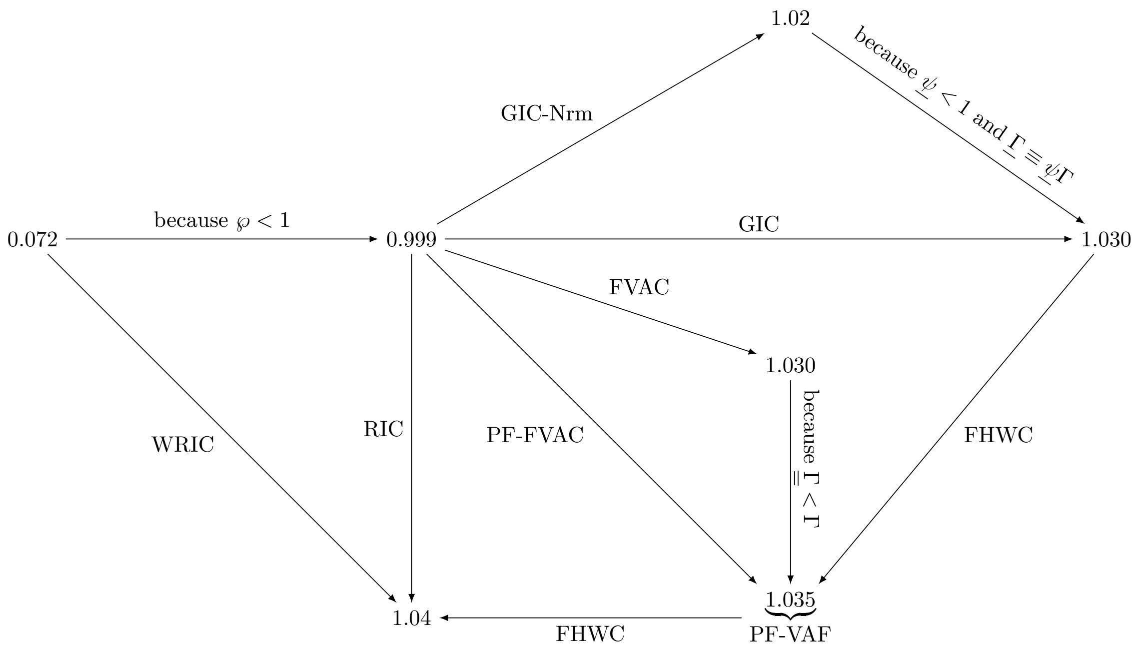

Figure 5 Appendix: Numerical Relation of All Inequality Conditions

J Apparent Balanced Growth in $\mathfrak{c}$ and cov$(c, \mathbf{p})$

Section 4.2 demonstrates some propositions under the assumption that, when an economy satisfies the GIC, there will be constant growth factors $\Omega_{\mathfrak{c}}$ and Ω_{cov} respectively for $\mathfrak{c}$ (the average value of the consumption ratio) and $\text{cov}(c, \mathbf{p})$. In the case of a Szeidl-invariant economy, the main text shows that these are $\Omega_{\mathfrak{c}} = 1$ and $\Omega_{\text{cov}} = \mathbf{\Phi}$. If the economy is Harmenberg- but not Szeidl-invariant, no proof is offered that these growth factors will be constant.

J.1 $\log c$ and $\log(\text{cov}(c, \mathbf{p}))$ Grow Linearly

Figures 7 and 8 plot the results of simulations of an economy that satisfies Harmenberg- but not Szeidl-invariance with a population of 4 million agents over the last 1000 periods (of a 2000 period simulation).[29] The first figure shows that $\log \mathfrak{c}$ increases apparently linearly. The second figure shows that $\log(-\text{cov}(c, \mathbf{p}))$ also increases apparently linearly. (These results are produced by the notebook `ApndxBalancedGrowthcNrmAndCov.ipynb`).

[29]For an exposition of our implementation of Harmenberg's method, see this supplemental appendix.

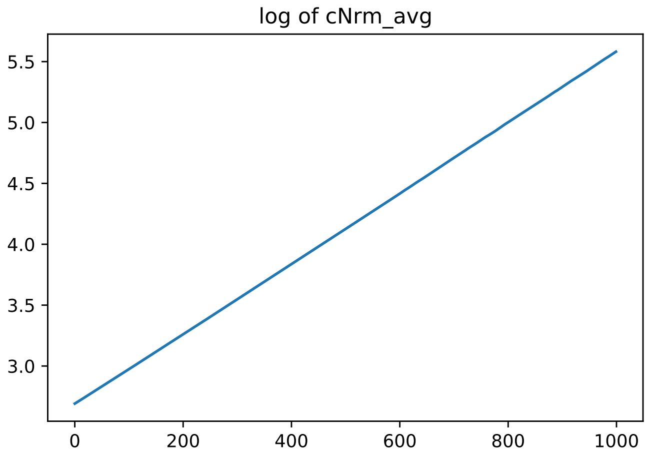

Figure 6 Appendix: log $\mathfrak{c}$ Appears to Grow Linearly

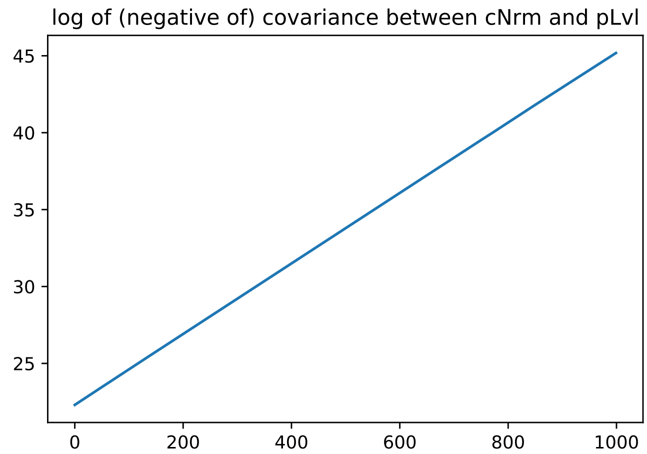

Figure 7 Appendix: log $(-\text{cov}(c, \mathbf{p}))$ Appears to Grow Linearly

Table 1 Appendix: Perfect Foresight Liquidity Constrained Taxonomy

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
~~GIC~~	$1 < \mathbf{Þ}/\boldsymbol{\Phi}$	Constraint never binds for $m \geq 1$
and RIC	$\mathbf{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \boldsymbol{\Phi}$);
		$\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
and ~~RIC~~	$1 < \mathbf{Þ}/\mathsf{R}$	$\grave{\mathrm{c}}(m)$ is degenerate: $\grave{\mathrm{c}}(m) = 0$
GIC	$\mathbf{Þ}/\boldsymbol{\Phi} < 1$	Constraint binds in finite time $\forall\ m$
and RIC	$\mathbf{Þ}/\mathsf{R} < 1$	FHWC may or may not hold
		$\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \grave{\mathrm{c}}(m) = 0$
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~	$1 < \mathbf{Þ}/\mathsf{R}$	~~FHWC~~
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GICholds, the constraint will bind in finite time.