Web Webfalse
bigPrint bigPrinttrue bigPrintfalse
wideMargins wideMarginstrue wideMarginsfalse
BigAndWide BigAndWidetrue BigAndWidefalse
bigPrintfalse wideMarginsfalse BigAndWidefalse
KoppaOn KoppaOntrue
KoppaOn
wideMargins
bigPrint
paperwidthpaperheightpreamble

1 Unique and Stable Target and Steady State Points

This appendix proves Theorems 1.1-1.4 and:
If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.

1.1 Proof of Theorem 1.1

For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC-Mod all hold, there exists a unique cash-on-hand-to-permanent-income ratio $\hat{m} > 0$ such that

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{1}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{2}$$

The elements of the proof of Theorem 1.1 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \boldsymbol{\xi}_t$, even if $\boldsymbol{\xi}_t$ takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 *Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$*

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned} \boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1, \end{aligned} \tag{4}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right) \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t \left[\left(\frac{d}{dm_t}\right) \left(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t\right) \right] \\ &= \bar{\mathcal{R}} \left(1 - \mathrm{c}'(m_t)\right) - 1. \end{aligned} \tag{5}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \bar{\mathcal{R}} \left(1 - \mathrm{c}'(m_t)\right) - 1 < \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ = \bar{\mathcal{R}} \text{Þ}_{\mathsf{R}} - 1 \\ = \mathbb{E}_t \left[\frac{\mathsf{R}}{\boldsymbol{\Phi}\boldsymbol{\Psi}} \frac{\text{Þ}}{\mathsf{R}} \right] - 1 \\ = \underbrace{\mathbb{E}_t \left[\frac{\text{Þ}}{\boldsymbol{\Phi}\boldsymbol{\Psi}} \right]}_{=\text{Þ}_{\underline{\boldsymbol{\Phi}}}} - 1 \end{aligned}$$

which is negative because the GIC-Mod says $\text{Þ}_{\underline{\boldsymbol{\Phi}}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}} \left(1 - \mathrm{c}'(m_t)\right) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (5) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathbf{\Phi}\mathbf{\Psi}}\right] < 1. \tag{6}$$

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathbf{\Phi}\mathbf{\Psi}}\right]}^{\text{Þ}_{\underline{\mathbf{\Phi}}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\mathbf{\Phi}\mathbf{\Psi}}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (6).

1.4 Proof of Theorem 1.4

For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-to-income ratio $\check{m} > 0$ such that

$$\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{7}$$

Moreover, $\check{m}$ is a point of stability in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t > \mathbf{\Phi} \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t < \mathbf{\Phi}. \end{aligned} \tag{8}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\mathbf{\Psi}} \leq \mathbf{\Psi}_{t+1} \leq \bar{\mathbf{\Psi}} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\mathbf{\Psi}} \leq \mathbf{\Psi}_{t+1} \leq \bar{\mathbf{\Psi}} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\boldsymbol{\Phi}_{t+1} \left((\mathsf{R}/\boldsymbol{\Phi}_{t+1}) \mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1} \right) / \boldsymbol{\Phi}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathsf{R}/\boldsymbol{\Phi}) \mathrm{a}(m_t) + \boldsymbol{\Psi}_{t+1} \boldsymbol{\xi}_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\boldsymbol{\Phi}) \mathrm{a}(m_t) + 1}{m_t} \right] \\
&= (\mathsf{R}/\boldsymbol{\Phi}) \text{Þ}_{\mathsf{R}} \qquad (9) \\
&= \text{Þ}_{\boldsymbol{\Phi}} \\
&< 1
\end{aligned}$$

where the last two lines are merely a restatement of the GIC (19).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] = 1 \qquad (10) \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] > 1,
\end{aligned}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t} \right) \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) \left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\Psi}_{t+1} \boldsymbol{\xi}_{t+1} - m_t \right) \right] \qquad (11) \\
&= (\mathsf{R}/\boldsymbol{\Phi}) \left(1 - \mathrm{c}'(m_t) \right) - 1.
\end{aligned}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\mathcal{R} \left(1 - \mathrm{c}'(m_t) \right) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= (\mathsf{R}/\boldsymbol{\Phi}) \text{Þ}_{\mathsf{R}} - 1
\end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\boldsymbol{\Phi}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R} \left(1 - \mathrm{c}'(m_t) \right) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (11) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\boldsymbol{\Phi}) < 1. \tag{12}$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (12) holds).

1.5 A Third Measure

A footnote in Section 3 mentions reasons why it may be useful to calculate $\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\log \mathbf{m}_t)]$. Here we show that one way of doing that is to calculate a nonlinear adjustment factor for the expectation of the growth factor.

$$\begin{aligned}
\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right) &= \log(\boldsymbol{\Phi}\boldsymbol{\Psi}_{t+1}m_{t+1}) - \log m_t \\
&= \log \boldsymbol{\Phi}(a_t\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}) - \log m_t \\
&= \log \boldsymbol{\Phi}(a_t\mathcal{R} + 1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)) - \log m_t
\end{aligned}$$

Now define $\breve{m}_{t+1} = a_t\mathcal{R} + 1$, and compute the expectation:

$$\begin{aligned}
\mathbb{E}_t[\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right)] &= \mathbb{E}_t\left[\log \boldsymbol{\Phi}(\breve{m}_{t+1} + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1))\right] - \log m_t \\
&= \log \boldsymbol{\Phi} + \mathbb{E}_t\left[\log \breve{m}_{t+1}\left(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1}\right)\right] - \log m_t \\
&= \underbrace{\log \boldsymbol{\Phi} + \log \breve{m}_{t+1} - \log m_t}_{\equiv \log \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]} + \mathbb{E}_t\left[\log(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]
\end{aligned}$$

and exponentiating tells us that

$$\exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]\exp(\mathbb{E}_t\left[\log(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \tag{13}$$

and this latter factor is a number that approaches 1 from below as m_t rises. Thus the expected growth rate of the log is smaller than the log of the growth rate of the expected growth factor. This implies that the m at which 'balanced growth' can be expected in the log, $\tilde{m}$, exceeds the corresponding point for the ratio, $\breve{m}$.

Furthermore, in the limit as $\mathbf{m}_t$ gets arbitrarily large, if the RIC holds and thus $\underline{\kappa} > 0$, a_{t+1} rises without bound, as does $\breve{m}_{t+1} = a_{t+1}\mathcal{R} + 1$, so the approximation $\log(1+\epsilon) \approx \epsilon$ becomes arbitrarily good. Consequently, the last term on the RHS of (13) can be approximated as

$$\begin{aligned}
\mathbb{E}_t\left[\log(1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) &\approx \mathbb{E}_t\left[(\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \\
&= 0
\end{aligned}$$

This demonstrates that

$$\lim_{\mathbf{m}_t \uparrow \infty} \exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \tag{14}$$

1.6 Proof of Lemma

1.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$
\begin{aligned}
\check{\mathrm{m}}_{t+1}(a) &= 1 + a\mathcal{R} \\
\hat{\mathrm{m}}_{t+1}(a) &= 1 + a\,\underbrace{\mathcal{R}/\underline{\mathbf{\Psi}}}_{\bar{\mathcal{R}}>\mathcal{R}}
\end{aligned}
\tag{15}
$$

so that we can implicitly define the target and pseudo-steady-state points as

$$
\begin{aligned}
\hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m} - \mathrm{c}(\hat{m})) \\
\check{m} &= \check{\mathrm{m}}_{t+1}(\check{m} - \mathrm{c}(\check{m}))
\end{aligned}
\tag{16}
$$

Then subtract:

$$
\begin{aligned}
\hat{m} - \check{m} &= \left(\hat{a}\underline{\mathbf{\Psi}}^{-1} - \check{a}\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - a(\check{m})\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - (a(\hat{m} + \check{m} - \hat{m}))\right)\mathcal{R} \\
&\approx \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\
(\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{<\text{\textbf{Þ}}_{\mathbf{\Phi}}<1}) &= (\underline{\mathbf{\Psi}}^{-1} - 1)\hat{a}\mathcal{R}
\end{aligned}
\tag{17}
$$

The RHS of this equation is strictly positive because $\underline{\mathbf{\Psi}}^{-1} > 1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[1]

1.6.2 The m Achieving Individual Expected-Log-Balanced-Growth Is Smaller than the Individual Pseudo-Steady-State m

Expected log balanced growth occurs when

$$
\begin{aligned}
\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \mathbf{\Phi}\mathbf{m}_t \\
\mathbb{E}_t[\log \mathbf{p}_{t+1}m_{t+1}] &= \log \mathbf{\Phi}\mathbf{p}_t m_t \\
\mathbb{E}_t[\log \mathbf{\Psi}_{t+1}m_{t+1}] &= \log \mathbf{\Phi}m_t \\
\mathbb{E}_t[\log\left(a(m_t)\mathsf{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\mathbf{\Phi}\right)] &= \log \mathbf{\Phi}m_t \\
\mathbb{E}_t[\log\left(a(m_t)\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\right)] &= \log m_t
\end{aligned}
\tag{18}
$$

and we call the m that satisfies this equation $\tilde{m}$.

Subtract the definition of $\check{m}$ from that of $\tilde{m}$:

$$
\exp(\mathbb{E}_t[\log\left(a(\tilde{m})\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\right)]) - (a(\check{m})\mathcal{R} + 1) = \tilde{m} - \check{m}
\tag{19}
$$

Now we use the fact that the expectation of the log is less than the log of the

[1] The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a' over the interval to remove the approximation in the derivations above.

expectation,

$$\exp(\mathbb{E}_t[\log\left(a(\tilde{m})\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\right)]) < (a(\tilde{m})\mathcal{R} + 1) \tag{20}$$

so

$$\begin{aligned}
\exp(\mathbb{E}_t[\log\left(a(\tilde{m})\mathcal{R} + 1\right)]) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m})\mathcal{R} + 1) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - a(\tilde{m} + \check{m} - \tilde{m}))\mathcal{R} &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - (a(\tilde{m}) + (\check{m} - \tilde{m})\bar{\mathrm{a}}')\mathcal{R} &< \tilde{m} - \check{m} \\
(\tilde{m} - \check{m})\bar{\mathrm{a}}'\mathcal{R} &< \tilde{m} - \check{m} \\
\underbrace{\bar{\mathrm{a}}'\mathcal{R}}_{<\textbf{Þ}_{\boldsymbol{\Phi}}} &< 1
\end{aligned} \tag{21}$$

where we are interpreting $\bar{\mathrm{a}}'$ as the mean of the value of a' over the interval between $\tilde{m}$ and $\check{m}$.