What Are the Odds?—Constructing Competition Probabilities

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Odds-making can be a lucrative business—witness Jimmy the Greek. For most, predicting winners and making odds are part of the enjoyment of getting ready for the "big game" (there always seems to be one a few days away). This article will outline a technique of using past performances of competitors against common opponents to set probabilities of particular outcomes of future competition. Common opponent analysis has a basic weakness, of course. Competition is not transitive, so anomalies like "Slippery Rock is thirty points better than Ohio State" can arise. To the extent that competition is transitive, the technique presented here will be valid.

A Model of Competition. Before predictions about a competition can be made, a model for the competition itself must be created. One way of doing this is to put marbles of two different colors into a jar and then represent the competition by drawing one marble at random from the jar. By pre-assigning a color to each of the two teams, the winner would be the team whose colored marble was drawn. If team A is supposed to win over team B 70% of the time, then seven red and three white marbles would be placed in the jar, with A having been assigned the color red and B the color white.

This is a very natural model, and yet when an attempt is made to apply it to the problem of predicting future outcomes from past results some difficulties arise. Consider the situation in which A beats B 70% of the time and B beats C 60% of the time. Can the probability of A winning over C be found? How can the single jars associated with the first two competitions be used to construct a jar representing the third competition? Since each competition is represented by a single jar of marbles, there appears to be no way to find the proper proportion of colors to use for the jar representing the competition between A and C. What is needed is some device that represents each *team* instead of each *competition*.

A Second Model. To accomplish this, let's represent each team by a jar of marbles, some white and some black. As an example, let A's jar contain seven white and three black marbles and B's jar contain six white and four black marbles.

A competition will consist of drawing one marble at random from each jar. If the pair of marbles is such that one marble is white and the other black, then the team from whose jar the white marble was drawn wins. If the marbles are of the same color, then there is no contest and the marbles are returned for another draw. Table 1 indicates the four possible outcomes and their probabilities, where $p = .7$ and $q = .6$ in this case. The conditional probability that A would win, given that there is a winner, is

$$\frac{(.7)(.4)}{(.7)(.4) + (.3)(.6)} = \frac{14}{23} \approx .61.$$

Table 1

Possible Outcomes and Their Probabilities

Draw	Probability	Result
A white & B black	$p \cdot (1-q)$	A wins
A black & B white	$(1-p) \cdot q$	B wins
A white & B white	$p \cdot q$	No contest
A black & B black	$(1-p) \cdot (1-q)$	No contest

In general this is

$$\frac{p(1-q)}{p(1-q) + (1-p)q}.$$

Some questions are, what is the unconditional probability of A winning? Can the "no contest" outcomes be ignored? The probability that A eventually wins is the following infinite sum: P (A wins on the first draw) + P (A wins on the second *and* there is no contest on the first) + P (A wins on the third *and* there is no contest on the first and second) + $\cdots$. Since the probability of no contest is $p \cdot q + (1-p)(1-q)$, in our particular example the infinite sum becomes $(.28) + (.28)(.54) + (.28)(.54)^2 + \cdots = \sum_{n=0}^{\infty} (.28)(.54)^n$. This is a geometric series and the sum can be found. So the probability that A eventually wins is

$$\frac{(.28)}{1-(.54)} = \frac{28}{46} = \frac{14}{23} \approx .61,$$

precisely as before. In general, the probability that A eventually wins would be

$$\sum_{n=0}^{\infty} [p(1-q)][pq + (1-p)(1-q)]^n = \frac{p(1-q)}{1-[pq+(1-p)(1-q)]}$$
$$= \frac{p(1-q)}{p+q-2pq}$$
$$= \frac{p(1-q)}{p(1-q)+(1-p)q}.$$

This is exactly the unconditional probability computed before. In other words, the no contest outcomes could have been ignored.

Can two jars of marbles be constructed, using this model, so that the probability that A wins over B is a particular predetermined value, say t? Again, let the probability of drawing a white marble from A's jar be p and the probability of drawing a white marble from B's jar be q. Since the no contest outcome can be ignored, the probability that A wins over B is as before,

$$\frac{p(1-q)}{p(1-q)+(1-p)q}\,. \tag{1}$$

So if p and q are chosen so that (1) is equal to t, then the model will represent the competition. For example, if t is supposed to be .7, for convenience choose $q=.5$ and solve for p.

$$\frac{p(1-.5)}{p(1-.5)+(1-p)(.5)}=.7$$

$$\frac{(.5)p}{(.5)}=.7$$

$$p=.7.$$

It follows that B's jar would have an equal number of white and black marbles and A's 70% white and 30% black marbles. Many different compositions are possible, depending upon the choice of q.

This model could be used under appropriate assumptions to predict the outcome of a meeting between A and C given the probabilities of A winning over B and B winning over C. Let the probability that A beats B be t and the probability that B beats C be s. As Figure 1 indicates, the problem is to find x, the probability that A beats C.

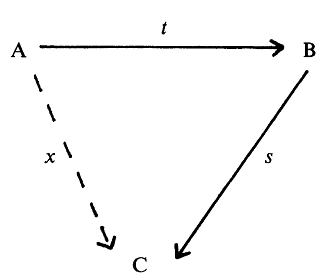

Figure 1.

Let the probabilities of drawing a white marble from the jars of A, B, and C be p, q, r, respectively. Again, for convenience let's choose $q = \frac{1}{2}$. Then,

$$t = \frac{p(1-q)}{p(1-q) + (1-p)q},$$

$$t = \frac{p(\frac{1}{2})}{p(\frac{1}{2}) + (1-p)(\frac{1}{2})},$$

$$t = p,$$

and

$$s = \frac{q(1-r)}{q(1-r) + (1-q)r},$$

$$s = \frac{(\frac{1}{2})(1-r)}{(\frac{1}{2})(1-r) + (\frac{1}{2})r},$$

$$s = 1 - r, \quad \text{and}$$

$$r = 1 - s.$$

All the proportions of colors in each jar are now known and the probability of A winning over C can be computed by following equation (1), with r replacing q:

$$x = \frac{p(1-r)}{p(1-r) + (1-p)r} \quad \text{and} \quad x = \frac{ts}{ts + (1-t)(1-s)}. \tag{2}$$

(As an exercise, the reader might leave q arbitrary and show that (2) holds in general.)

In the case where A beats B 70% of the time and B beats C 60% of the time, then the model predicts that A would beat C with the probability

$$x = \frac{(.7)(.6)}{(.7)(.6) + (.3)(.4)},$$

$$x = \frac{42}{54} \approx .77.$$

The model implies transitivity of the competition. For if $t > \frac{1}{2}$ and $s > \frac{1}{2}$, $ts > \frac{1}{4}$ and $(1-t)(1-s) < \frac{1}{4}$. This means that $1/ts < 4$ and $(1-t)(1-s)/ts < 1$. By equation (2),

$$x = \frac{ts}{ts + (1-t)(1-s)},$$

$$x = \frac{1}{1 + \dfrac{(1-t)(1-s)}{ts}},$$

$$x > \tfrac{1}{2}, \quad \text{as is expected.}$$

Limitations. Some special cases are of interest. If $t = 1$ and $s = 0$, then formula (2) leaves x undefined. In this situation both A and C have a certain win over B. In other situations where A and C would be equally effective against B ($t = 1 - s$) then

$$x = \frac{(1-s)s}{(1-s)s + s(1-s)},$$

$$x = \tfrac{1}{2}.$$

It seems reasonable to circumvent the awkwardness of the case $t = 1$ and $s = 0$ (and, by symmetry, the case $t = 0$, $s = 1$) by defining $x = \frac{1}{2}$ in such a situation.

Of more practical concern is the case in which $t = 0$ and s takes on any value not equal to one. In this situation, $x = 0$—an unrealistic prediction. The difficulty arises because to represent A's certain loss to B, it is necessary to place all black marbles in A's jar. This removes all flexibility from the model and reduces A to ultimate defeat by C no matter how large t might be. Such an unrealistic prediction is due to unrealistic entry probabilities—there are no certain outcomes in athletics, if anywhere else. If past performances indicate that A has always lost to B, then some probability close to zero should be assigned to A's victory over B, say $t = .01$ or $t = .001$. For example, let $t = .01$ and $s = .95$, then

$$x = \frac{(.01)(.95)}{(.01)(.95) + (.99)(.05)},$$

$$x = \frac{95}{590} \approx .16.$$

Though arbitrary, this technique seems to make the model more realistic. Similarly, by symmetry, the case of $s = 1$ and t arbitrary ($t \neq 0$) can be salvaged by making s close to unity, say $s = .99$.

Some Applications. In 1975, Billie Jean King and Martina Navratilova, two of the highest-ranked women tennis players in the world, did not play each other. Can the probability of King beating Navratilova in 1975 be computed by using the model constructed above? The two players had five common top-ranked oppo-

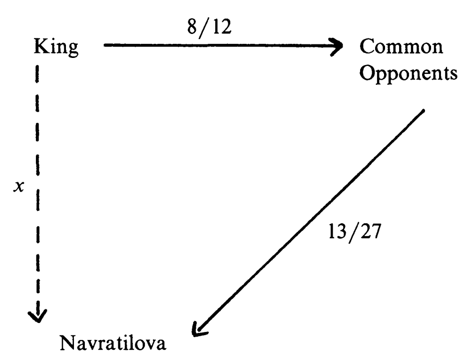

Figure 2.

nents. King's record against those five was eight wins and four losses; Navratilova's was fourteen wins and thirteen losses. If the common opponents would be lumped together to form a single team, then Figure 2 represents the situation in terms of the model that has been used previously.

Applying the formula (2), $x = 104/160 \approx .65$. The official ratings of 1975 listed King second and Navratilova third—an order justified by their records against common opponents.

In 1978, the Los Angeles Dodgers and the Philadelphia Phillies met in a five-game playoff for the National League championship. Against common opponents the Dodgers had a record of 88–62 and the Phillies had a record of 85–65. The model would construct the probability of a Dodger victory in a single game as

$$x = \frac{(88/150)(65/150)}{(88/150)(65/150) + (62/150)(85/150)} \approx .52.$$

Of course, the two teams met in the regular season with the Dodgers winning 7 of 12 games. Using competition against each other to predict probabilities gives .58 as the probability of a Dodger victory in a single game. The two estimates differ, but not by very much (the common opponent model would predict 6.2 Dodger victories in 12 games). Neither one reflects particularly well the ease of the Dodger victory in the playoff, however.

Conclusion. The model can be used in similar situations. The results are only as valid as the entry probabilities are accurate representations of the relative strengths of the two competitors against past opponents. Since transitivity is implied, the model does not take into account improvement over time and all the other intangibles in an athletic competition. Anyone finding a model that takes those factors into account probably would not tell the world anyway, now would they?

When asked what it was like to set about proving something, the mathematician likened proving a theorem to seeing the peak of a mountain and trying to climb to the top. One establishes a base camp and begins scaling the mountain's sheer face, encountering obstacles at every turn, often retracing one's steps and struggling every foot of the journey. Finally, when the top is reached, one stands examining the peak, taking in the view of the surrounding countryside—and then noting the automobile road up the other side!

—Robert J. Kleinhenz