Delaunay Triangulations

Peter Schröder

Discretization

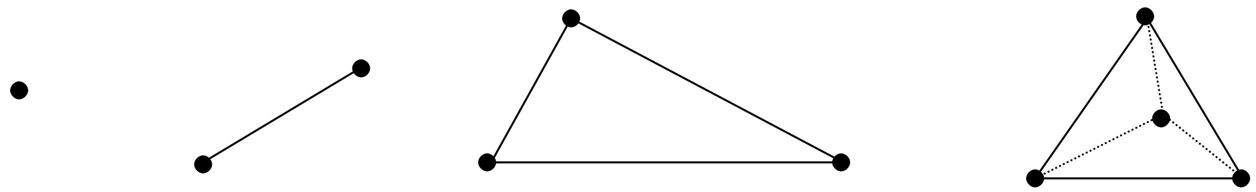
Describe geometry

- 1D, 2D, 3D,...
- need to distinguish
  - topology: who is neighbor to whom
    - datastructures
  - geometry: domain and range
    - embedding, color, texture,...

Setting

Abstract simplicial complex

- 0-dim simplex: vertex $\{i\}$
- 1-dim simplex: edge $\{i,j\}$
- 2-dim simplex: triangle, $\{i,j,k\}$
- 3-dim simplex: tetrahedron, $\{i,j,k,l\}$



Simplicial Complex

Definition

- set K of subsets of $\{1,\ldots,n\}$ each of which is a simplex
- such that every non-empty subset of an element of K is also in K
  - e.g., if a triangle is present, so are its edges and vertices

Triangulation

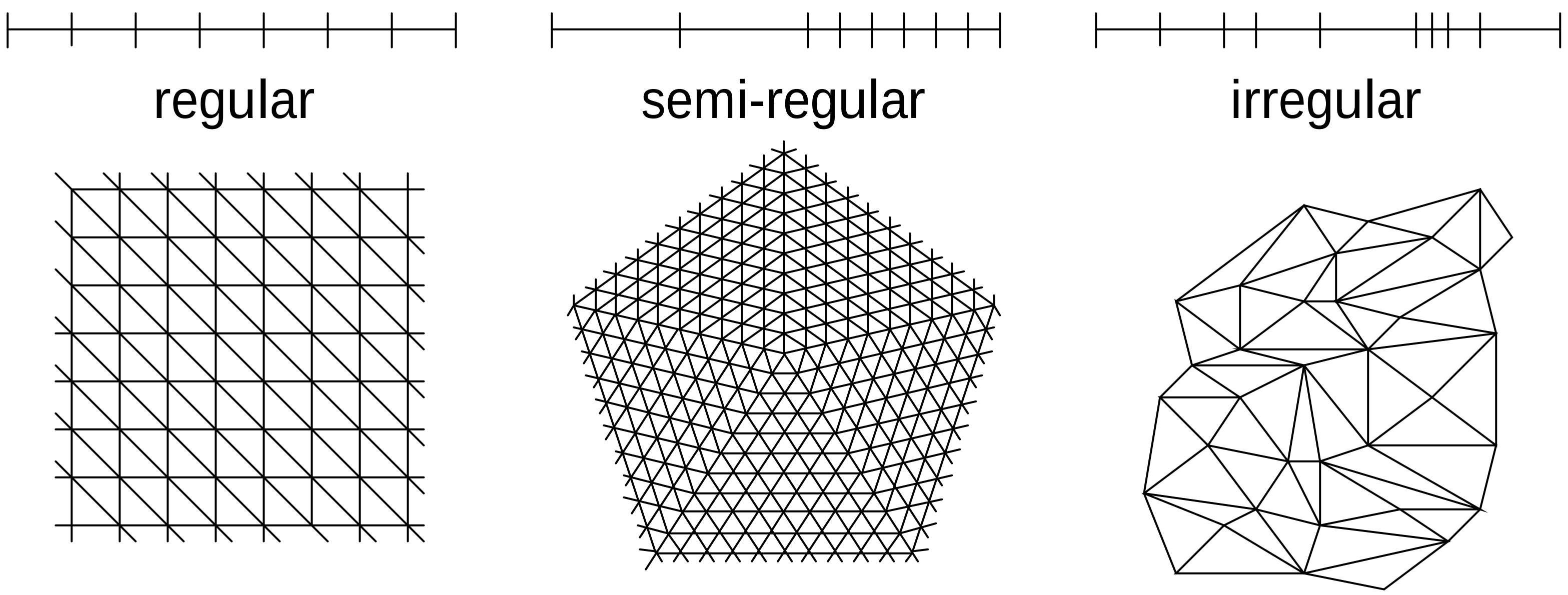
Generic term

- set of point positions P together with a complex K
- 1D straightforward, $\{\{1\},\{2\},\ldots,\{n\},\{1,2\},\{2,3\},\ldots,\{n\text{-}1,n\}\}$
- 2D more interesting... (even more so for 3D and beyond)

Triangulations

Different types

- based on geometry



Triangulation in 2D

What is a good triangulation?

- question of geometry
  - aspect ratio, minimum angle, size
- in animation (simulation) these will impact error bounds

Today:

- planar triangulations (“terrain”)

Triangulation

Definition for $\mathbf{R}^2$

- given a point set $P=\{p_1,p_2,\ldots,p_n\}$
- a triangulation of P is a maximal planar subdivision
  - no edge connecting two vertices can be added without crossing an existing edge

Triangulation

All triangulations have the same number of triangles

- n points (not all collinear)
- k points on the boundary of the convex hull

Then

- $2n$-2-k triangles and $3n$-3-k edges

Triangulation

Proof

- number of triangles: m
- $n_f = m+1$ (# of faces)
- $n_e = (3m+k)/2$ (# of edges)
- Euler characteristic: $n - n_e + n_f = 2$
  - $m = 2n - 2 - k$
  - $n_e = 3n - 3 - k$

Angle-Vector

Sort all angles in a triangulation in lexicographic order

- well defined since all triangulations have the same number of triangles

$$(\alpha_1, \ldots, \alpha_{3m}) = A(T)$$

- triangulation T is angle optimal if

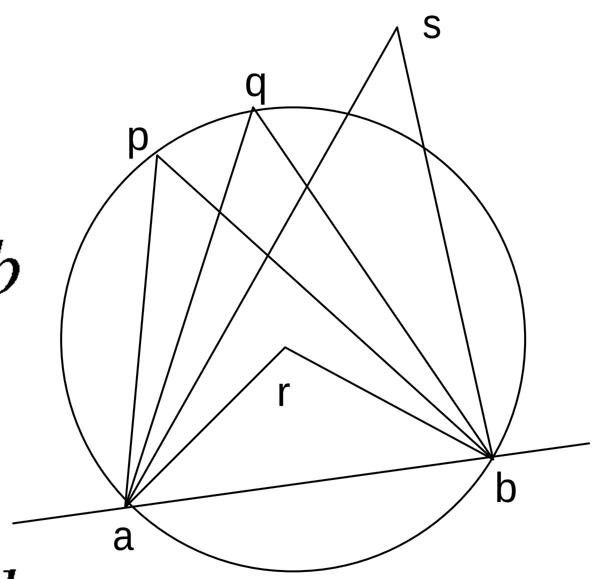
$$\forall T' : A(T) \geq A(T')$$

Angle Optimality

Preliminaries

- Thale's theorem

$$\angle arb > \angle apb = \angle aqb > \angle asb$$



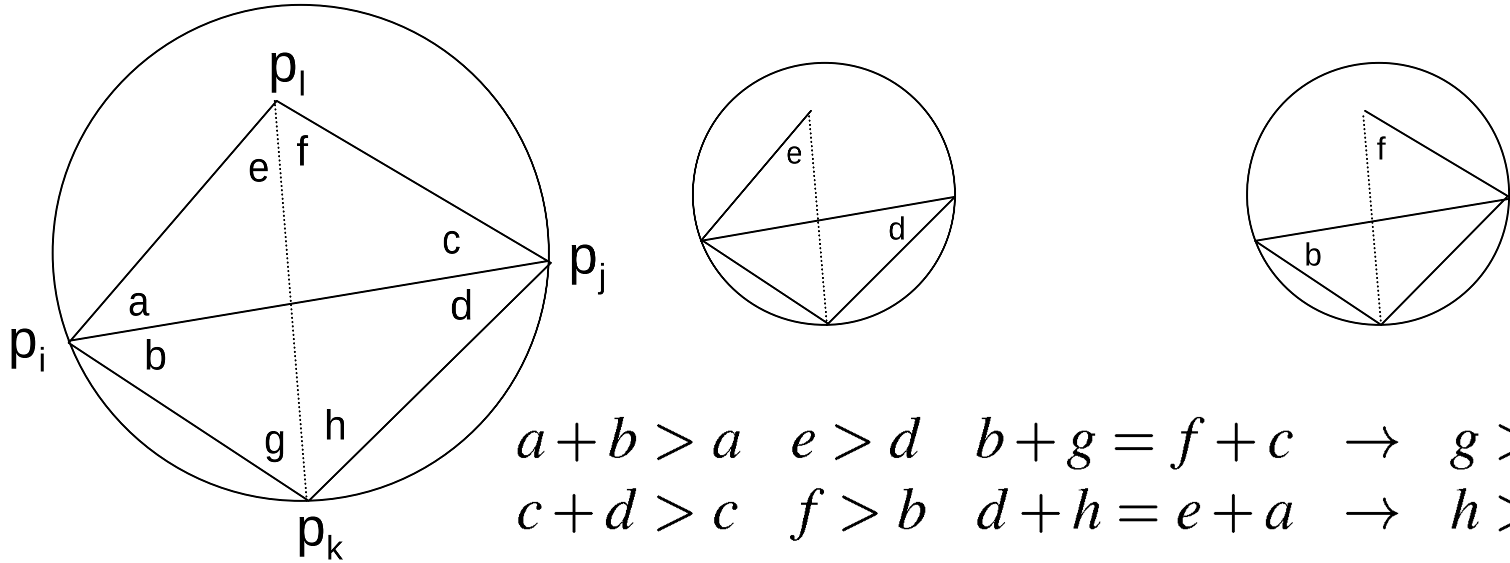
- proof

$$\begin{aligned} 2\angle apb &= \angle arb \\ \angle arb &= 2\pi - \angle arp - \angle brp \\ &= 2\pi - (\pi - 2\angle apr) - (\pi - 2\angle rpb) \\ &= 2(\angle apr + \angle rpb) \end{aligned}$$

Illegal Edges

How to increase the angle-vector?

- edge flip changes 6 angles



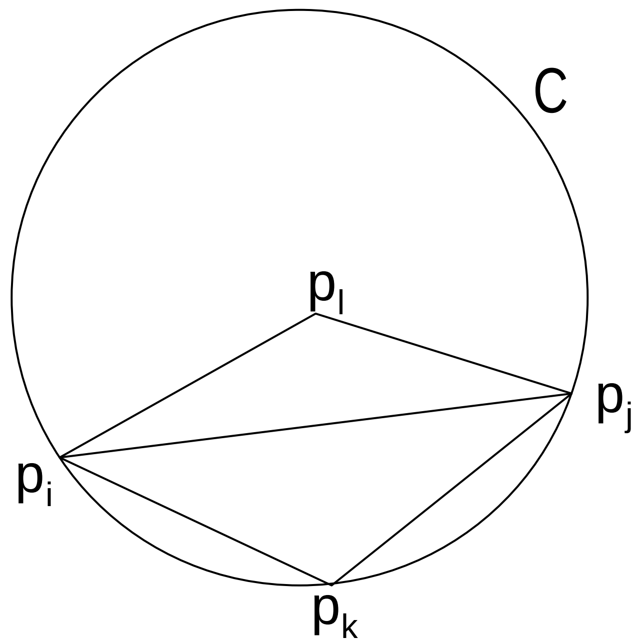
$$a+b>a \quad e>d \quad b+g=f+c \;\rightarrow\; g>c$$
$$c+d>c \quad f>b \quad d+h=e+a \;\rightarrow\; h>a$$

$$A=\{a,b,c,d,e+f,g+h\}<A'=\{a+b,c+d,e,f,g,h\}$$

Illegal Edges

Theorem

- empty circumcircle property
- $p_i p_j$ is illegal
  - iff p_l in C
- triangles incident on illegal edge form convex quad (why?)



Algorithm

Remove all illegal edges for an angle-optimal triangulation

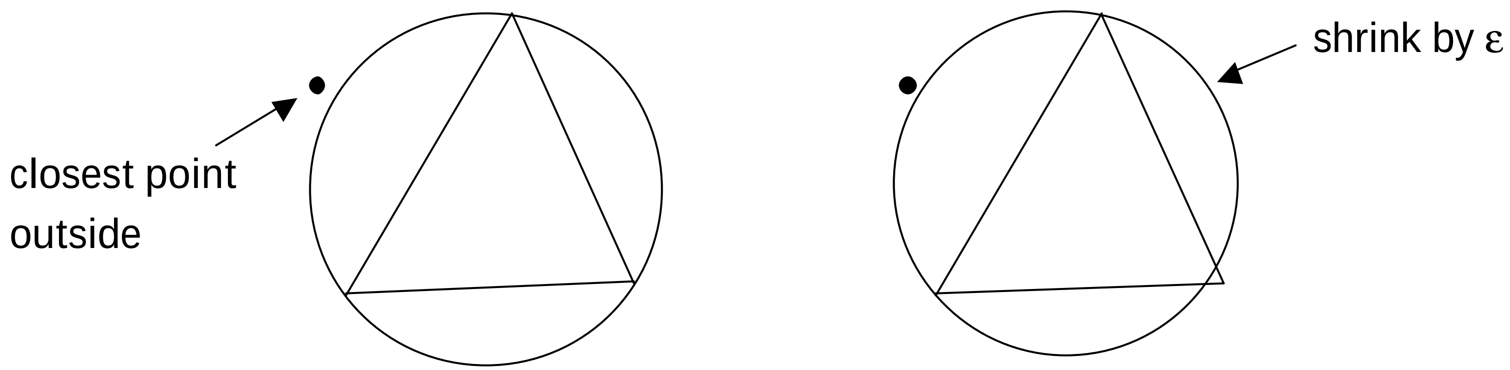
```
LegalTriangulation(T)
while( T contains illegal edge pipj )
     flip( pipj );
return T
```

- termination? yes!
- too slow

Properties

Delaunay triangulation

- empty circumcircle for every face
- every edge possesses disk not containing any other point

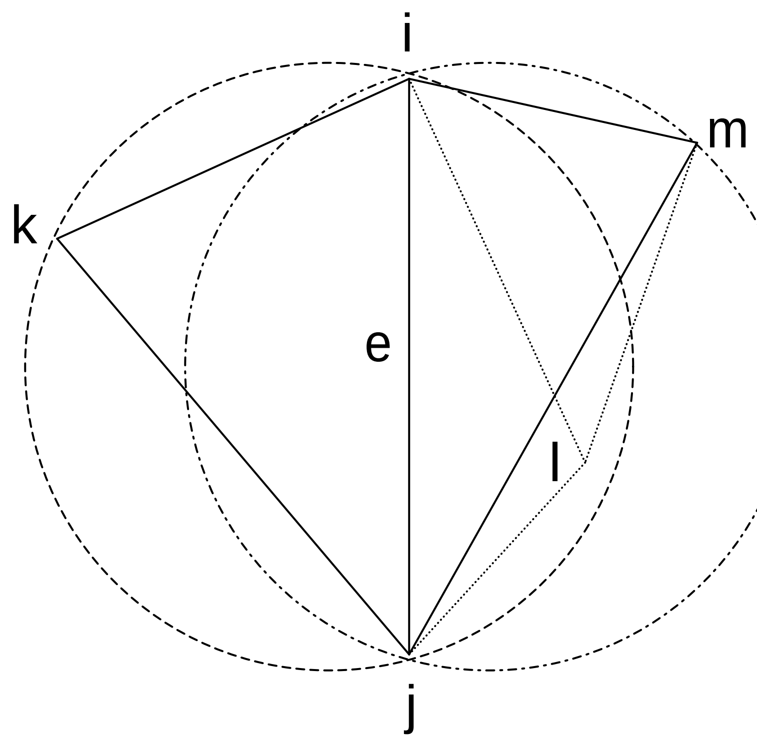


Theorem

Triangulation legal iff Delaunay

- Delaunay > legal (by definition)
- legal > Delaunay
  - by contraction: assume there is a triangle $\{i,j,k\}$ with p_l in $C(i,j,k)$
  - of all such quads pick the one with largest angle at l

Proof



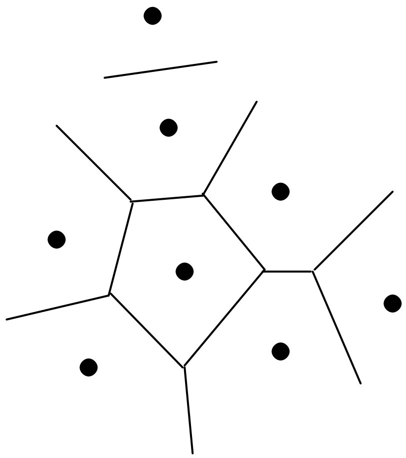
e is legal > m not in C(i,j,k)

C(i,j,m) contains part of C(i,j,k) on other side of e > l lies in C(i,j,m) > angle at (j,l,m) is larger than (j,l,i) > contradiction with assumption that triangle (i,j,k) was chosen with l having maximum angle between all quads (i,j,k,l)

Voronoi Diagram

Dual of Delaunay

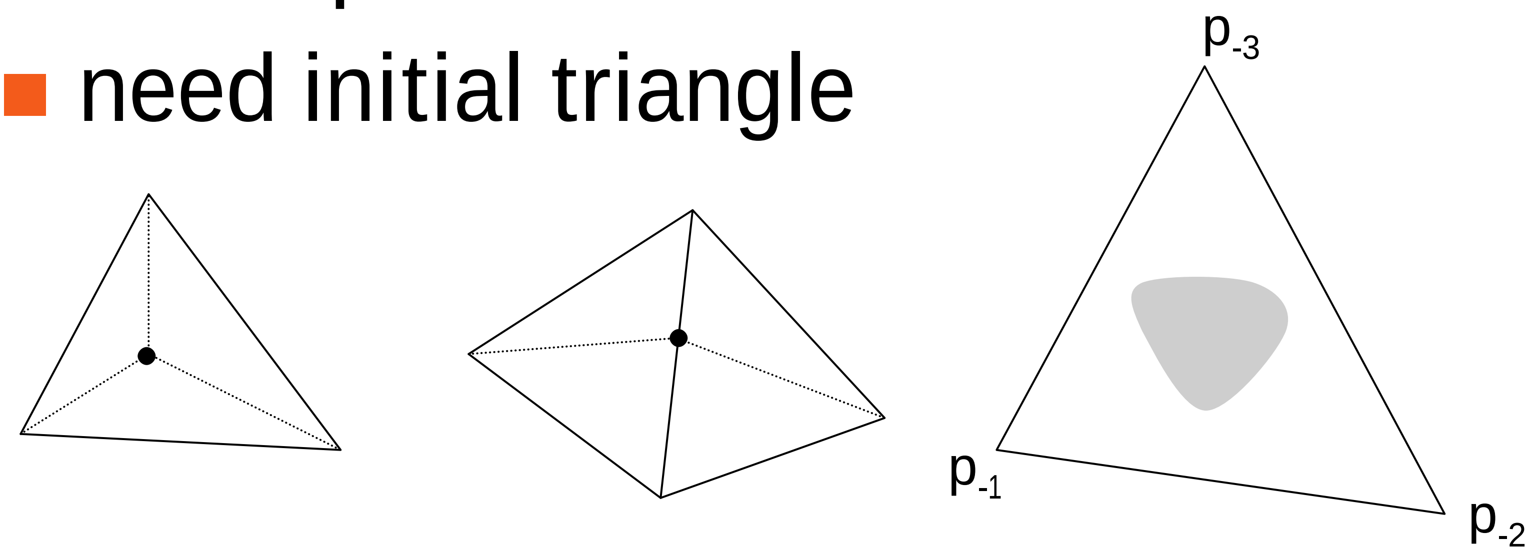
- regions which are closest to a given site
- need to argue
  - planar graph
- general position



Fast Algorithm

Randomized incremental construction

- insert points one at a time
- need initial triangle



Algorithm

```
DelaunayTriangulation(P)
Initialize T to {p-1,p-2,p-3}
for( r = 1; r <= n; ++r ){
        Find triangle containing pr
        if( pr in interior of pipjpk ){
                Add three edges
                LegalizeEdge( pr, pipj, T )
                LegalizeEdge( pr, pjpk, T )
                LegalizeEdge( pr, pkpi, T )
        }else{
                Add two edges
                Call LegalizeEdge for the 4 possibilities
        }
}
remove p-1, p-2, p-3 and all incident edges
return T
```

Algorithm

Test for legality

- p_{-1}, p_{-2}, p_{-3} are “infinitely” far away
- modify legality test accordingly

```
LegalizeEdge( p_r, p_i p_j, T )
if( !legal( p_i p_j ) ){
        let p_i p_j p_k be adjacent to p_r p_i p_j
        flip( p_i p_j )
        LegalizeEdge( p_r, p_i p_k, T )
        LegalizeEdge( p_r, p_k p_j, T )
}
```

Correctness

Show no illegal edges left

- any edge created is incident on p_r
- any new edge is legal (needs proof)
- edge can only become illegal when one of its incident triangles changes > algorithm tests all edges which may become illegal

Termination & Legaility

Follows from

- increasing angle vector

Any new edge is legal

- initial new edges are legal
  - each edge has empty circle
- any flipped edge has empty circle as well

Analysis

The expected number of triangles created in DelaunayTriangulation is at most $9n+1$

- insertion of p_r causes at most $2(k\text{-}3)+3=2k\text{-}3$ new triangles
- expected valence is 6
- expected number of triangles $9n+1$

Analysis

Complexity

- the Delaunay triangulation of a set P of n points can be computed in O(n log n) expected time and O(n) expected storage
- search for new point: O(log n)

Minimum Roughness

Remarkable property

- define roughness of a piecewise linear interpolating surface

$$R(g,T) = |g|^2_{T,1} = \sum_{i=1}^{n} |g|^2_{T_i,1}$$

$$|g|^2_{T_i,1} = \int_{T_i} \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 dx\,dy$$

Minimum Roughness

Interpolating surface

- given locations in the plane and sample values $\{(x_i, y_i, g_i)\}$
  - sample values arbitrary but fixed
  - no four coplanar

Theorem

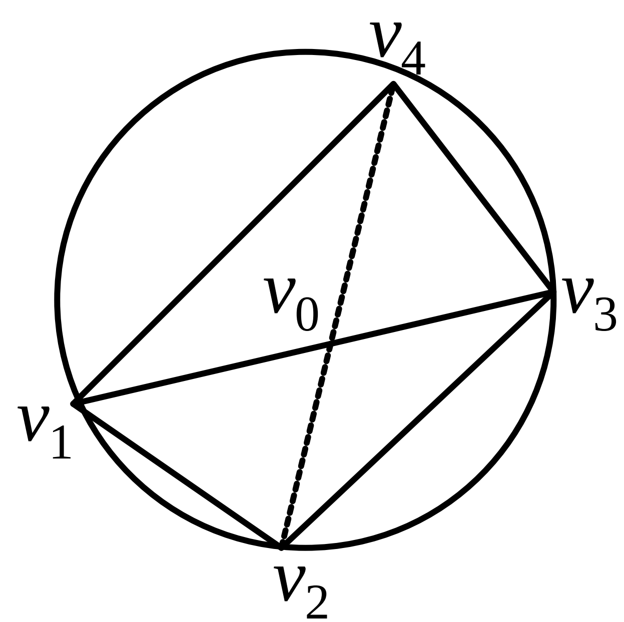
- T^* Delaunay iff $\forall T : R(g, T^*) \leq R(g, T)$

Roughness Reduction

Edge flip

- illegal edges are locally more rough
- flip v_1,v_3 iff $r_2 r_4 < r_1 r_3$

$$r_i = d(v_0, v_i)$$



Roughness Minimization

Rippa, CAGD 1990

- Lemma:

$$|g_T|_{T,1}^2 - |g_{T'}|_{T',1}^2 = A(r_1 r_3 - r_2 r_4)$$

$$A = (g_T(x_0,y_0) - g_{T'}(x_0,y_0))^2 \left(\frac{(r_1+r_3)(r_2+r_4)}{2 r_1 r_2 r_3 r_4 \sin\angle v_1 v_0, v_2} \right)$$

Implementation

Geometric conditionals

- great care required
  - arbitrary precision...
  - when are points collinear...
  - when are 4 points on a circle...
- exact methods
- symbolic perturbation methods

Higher D?

Ouch

- may need Steiner points
  - additional points not in original set
- slivers
  - in 3D can have thin tets with large circumsphere
- active area of research

Other Methods

Edge split

- Rivara's algorithm
  - works in 2D and 3D
  - bi-section
- unrefine
  - edge collapse (Hoppe)