

From Mathematics to Generic Programming

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4.4

Solution.

This is a *constructive* proof. We are tasked with proving that for any number that conforms to a particular form, there exists a number such that a particular statement is true. In this proof, we will provide a way to obtain the number that exists, thereby proving its existence.

We have an odd square number x . We might imagine that $x = 2m + 1$, but we can actually go further and consider $\sqrt{x} = 2l + 1$. This is because an odd square number must be obtained by multiplying two odd numbers together. Similarly, we can consider the number y that we would like to show exists as a number expressed by $2n$. But of course $y = \sqrt{2k}^2$, since an even square number implies that two even numbers were multiplied together to produce it.

We are told that with an appropriate y , which we must show exists, the following true:

$$x + y = a^2.$$

We can also express this fact that $x + y$ yields a square number as:

$$(2l + 1)^2 + (2k)^2 = a^2.$$

Given an appropriate choice for k , we might be able to show that the left hand side (LHS) factors into two identical factors, which would imply that indeed there exists some y such that $x + y = a^2$.

$$\begin{aligned}(2l + 1)^2 + (2k)^2 &= a^2 \\ 4l^2 + 4l + 1 + 4k^2 &= a^2 \\ 4k^2 + 4l^2 + 4l + 1 &= a^2\end{aligned}$$

This looks promising. We might intuitively realize the potential to factorize a quadratic. Here is where we will attempt to construct an appropriate y by choosing a value for k .

We can create a system of equations that reflects the form of the coefficients we would like to see so that we can factor a quadratic into two equal factors.

$$\begin{aligned}4 + 4\lambda^2 &= q^2 \\ q &= 4\end{aligned}$$

We have that:

$$\begin{aligned}4 + 4\lambda^2 &= 4^2 \\ 4\lambda^2 &= 12 \\ \lambda^2 &= 3 \\ \lambda &= \sqrt{3}.\end{aligned}$$

Let $k = \sqrt{3}$. Then we have,

$$\begin{aligned}4(\sqrt{3}l)^2 + 4l^2 + 4l + 1 &= a^2 \\ 4(\sqrt{3}l)^2 + 4l^2 + 4l + 1 &= a^2 \\ 16l^2 + 4l + 1 &= a^2 \\ (4l + 1)(4l + 1) &= a^2.\end{aligned}$$

Letting $y = (2k)^2 = (2\sqrt{3})^2$ gives us $x + y = a^2$. It is clear that the y we have constructed is both a square number, and even.