InFoMM – Optimisation
Lecture 2

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LP Duality

Let us again consider the LP instance we studied previously,

$$
\begin{aligned}
\text{(P)} \quad & \max\ 5x_1 + 4x_2 + 3x_3 \\
& \text{s.t. } 2x_1 + 3x_2 + x_3 \leq 5 \\
& \quad 4x_1 + x_2 + 2x_3 \leq 11 \\
& \quad 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

We saw that the optimal value is 13.

In integer programming, instead of solving an LP relaxation to optimality one is often interested in finding merely upper and lower bounds on the optimal value.

A lower bound is provided by any feasible solution. For example, $x_1, x_2 = 1$, $x_3 = 0$ is feasible with objective value 9.

How can we obtain upper bounds?

Multiplying the first constraint by 3 we obtain

$$6x_1 + 9x_2 + 3x_3 \leq 15,$$

and since $x_1, x_2, x_3 \geq 0$, this yields an upper bound on the objective function:

$$z = 5x_1 + 4x_2 + 3x_3 \leq 6x_1 + 9x_2 + 3x_3 \leq 15,$$

Likewise, taking the sum of the first two constraints yields the valid upper bound

$$z = 5x_1 + 4x_2 + 3x_3 \leq 6x_1 + 4x_2 + 3x_3 \leq 16.$$

More generally, such bounds can be obtained from any sum of positive multiples of the constraints for which the resulting coefficients are no smaller than the corresponding coefficients of the objective function:

$$\begin{bmatrix}5 & 4 & 3\end{bmatrix} \leq \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix} \begin{bmatrix}2 & 3 & 1\\ 4 & 1 & 2\\ 3 & 4 & 2\end{bmatrix}, \quad y_1, y_2, y_3 \geq 0$$

$$\Rightarrow z \leq \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix} \begin{bmatrix}5\\ 11\\ 8\end{bmatrix}.$$

The best such upper bound is obtained by solving the LP instance

$$
\begin{aligned}
\text{(D)} \quad & \min_{y} 5y_1 + 11y_2 + 8y_3 \\
& \text{s.t. } 2y_1 + 4y_2 + 3y_3 \geq 5, \\
& \qquad 3y_1 + y_2 + 4y_3 \geq 4, \\
& \qquad y_1 + 2y_2 + 2y_3 \geq 3, \\
& \qquad y_1, y_2, y_3 \geq 0.
\end{aligned}
$$

This is called the *dual* of the LP instance (P), the latter being called the *primal*.

More generally, an LP of the form

$$
\begin{aligned}
\text{(P)} \quad z^* = \max_{(x,s)} \; & c^{\mathsf{T}} x + d^{\mathsf{T}} s \\
\text{s.t. } & Ax + Cs \leq a, \\
& Bx + Ds = b, \\
& x \geq 0, \\
& s \text{ arbitrary}
\end{aligned}
$$

is associated with a dual

$$
\begin{aligned}
\text{(D)} \quad w^* = \min_{(y,t)} \; & a^{\mathsf{T}} y + b^{\mathsf{T}} t \\
\text{s.t. } & A^{\mathsf{T}} y + B^{\mathsf{T}} t \geq c \\
& C^{\mathsf{T}} y + D^{\mathsf{T}} t = d \\
& y \geq 0, \\
& t \text{ arbitrary.}
\end{aligned}
$$

(D) has itself a dual: casting (D) in primal form,

$$
\begin{aligned}
\text{(D')} \quad \max_{(y,t)} \; & -a^{\mathsf{T}} y - b^{\mathsf{T}} t \\
\text{s.t.} \; & -A^{\mathsf{T}} y - B^{\mathsf{T}} t \leq -c \\
& -C^{\mathsf{T}} y - D^{\mathsf{T}} t = -d \\
& y \geq 0,
\end{aligned}
$$

the bi-dual is found to be

$$
\begin{aligned}
\text{(P')} \quad \min_{(x,s)} \; & -c^{\mathsf{T}} x - d^{\mathsf{T}} s \\
\text{s.t.} \; & -Ax - Cs \geq -a, \\
& -Bx - Ds = -b, \\
& x \geq 0,
\end{aligned}
$$

which is just the primal cast in dual form.

Duality Theorems

To analyse the relationship between the primal-dual pair (P), (D), we will henceforth consider LPs in the following *standard form* into which any LP may be cast under an appropriate reformulation,

$$
\begin{aligned}
\text{(P)} \quad & \max_x \sum_{j=1}^{n} c_j x_j \\
& \text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \ (i = 1, \dots, m),
\end{aligned}
$$

$$
\begin{aligned}
\text{(D)} \quad & \min_y \sum_{i=1}^{m} y_i b_i \\
& \text{s.t.} \sum_{i=1}^{m} y_i a_{ij} = c_j, \ (j = 1, \dots, n) \\
& y_i \ge 0, \ (i = 1, \dots, m).
\end{aligned}
$$

Theorem (Weak Duality Theorem)

i) *If x is primal feasible and y is dual feasible (feasible for (P), (D) respectively), then*

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} y_i b_i. \tag{1}$$

ii) *If equality holds in* (1)*, then x is primal optimal and y is dual optimal.*

iii) *If either (P) or (D) is unbounded, then the other programme is infeasible (has no feasible solutions).*

Proof. i) By assumption, we have $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for all i and $\sum_{i=1}^{m} y_i a_{ij} = c_j$, $y_i \geq 0$ for all j. Therefore,

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j = \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) \leq \sum_{i=1}^{m} y_i b_i.$$

ii) and iii) are immediate consequences of i).

Theorem (Theorem of Alternatives for Linear Inequalities)

Consider the following two systems of linear inequalities,

(2)
$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i = 1, \dots, m)$$

and

(3)
$$\begin{aligned} \sum_{i=1}^{m} y_i a_{ij} &= 0, \quad (j = 1, \dots, n) \\ y_i &\geq 0, \quad (i = 1, \dots, m) \\ \sum_{i=1}^{m} y_i b_i &< 0. \end{aligned}$$

Then system (2) *has a solution if and only if* (3) *has no solution.*

Proof. See problem sheets.

Theorem (Strong Duality Theorem)

i) *If (P) and (D) both have feasible solutions, then they have optimal solutions x and y such that* $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} y_i b_i$.

ii) *If either (P) or (D) is infeasible, then the other programme is either unbounded or infeasible.*

Proof. Expanding each equality of the dual constraints as two inequalities, the claim of part i) may be written as the following system,

$$
\begin{aligned}
\sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad (i = 1, \dots, m) \text{ (primal feasibility)} \\
\sum_{i=1}^{m} y_i a_{ij} &\leq c_j, \quad (j = 1, \dots, n) \text{ (dual feasibility)} \\
-\sum_{i=1}^{m} y_i a_{ij} &\leq -c_j, \quad (j = 1, \dots, n) \text{ (dual feasibility)} \\
-y_i &\leq 0, \quad (i = 1, \dots, m) \text{ (dual feasibility)} \\
-\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} b_i y_i &\leq 0 \text{ (optimality)}
\end{aligned}
\tag{4}
$$

By the theorem of alternatives (4) is feasible if and only if the following system is infeasible,

$$\sum_{i=1}^{m} u_i a_{ij} - c_j \tau = 0, \ (j = 1, \ldots, n) \text{ (accounting for } x_j)$$

$$\sum_{j=1}^{n} a_{ij} v_j - \sum_{j=1}^{n} a_{ij} w_j - s_i + b_i \tau = 0, \ (i = 1, \ldots, m) \text{ (accounting for } y_i)$$

$$u_i, v_j, w_j, s_i, \tau \geq 0, \ (i = 1, \ldots, m; j = 1, \ldots, n)$$

$$\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} c_j (v_j - w_j) < 0,$$

which can be rewritten as follows, using $h_j = w_j - v_j$,

$$\begin{aligned} &\sum_{i=1}^{m} u_i a_{ij} - c_j \tau = 0, \ (j = 1, \ldots, n) \\ &-\sum_{j=1}^{n} a_{ij} h_j - s_i + b_i \tau = 0, \ (i = 1, \ldots, m) \\ &u_i, s_i, \tau \geq 0, \ (i = 1, \ldots, m; j = 1, \ldots, n) \\ &\sum_{i=1}^{m} b_i u_i - \sum_{j=1}^{n} c_j h_j < 0, \end{aligned} \tag{5}$$

To prove that (4) is feasible, we must show that assuming the feasibility of (5) leads to a contradiction.

Case 1: (5) has solution with $\tau > 0$. Let $x_j = \frac{1}{\tau} h_j \; \forall j$ and $y_i = \frac{1}{\tau} u_i \; \forall i$. Then x and y are primal and dual feasible, but

$$\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} c_j h_j < 0$$

implies $\sum_{i=1}^{m} y_i b_i < \sum_{j=1}^{n} c_j x_j$, violating weak duality.

Case 2: (5) has solution with $\tau = 0$ and $\sum_{i=1}^{m} u_i b_i < 0$. Take $\tilde{y}$ dual feasible. Then $y = \tilde{y} + \lambda u$ is dual feasible for all $\lambda \geq 0$, and

$$\sum_{i=1}^{m} y_i b_i = \sum_{i=1}^{m} \tilde{y}_i b_i + \lambda \sum_{i=1}^{m} u_i b_i$$

becomes arbitrarily negative for large λ. By weak duality, (P) is infeasible.

Case 3: (5) has solution with $\tau = 0$ and $\sum_{j=1}^{n} c_j h_j > 0$: similar construction to Case 2 with $x = \tilde{x} + \lambda h$ and $\tilde{x}$ primal feasible.

For parts ii) proceed similarly (see problem sheets).

Linear Complementarity

Definition (Complementarity)

$x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are complementary solutions relative to (P) and (D) if

$$y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad (i = 1, \dots, m),$$

that is, either $y_i = 0$ or the primal constraint $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ is active (holds as equality).

Theorem (Complementary Slackness)

x and y are complementary solutions relative to (P) and (D) if and only if they are optimal solutions of (P) and (D).

Proof. x and y are primal and dual feasible. Therefore,

$$\sum_{i=\mathbf{1}}^{m} y_i \left(b_i - \sum_{j=\mathbf{1}}^{n} a_{ij} x_j \right) = \sum_{i=\mathbf{1}}^{m} y_i b_i - \sum_{j=\mathbf{1}}^{n} \left(\sum_{i=\mathbf{1}}^{m} y_i a_{ij} \right) x_j = \sum_{i=\mathbf{1}}^{m} y_i b_i - \sum_{j=\mathbf{1}}^{n} c_j x_j . \tag{6}$$

If x and y are complementary, then the l.h.s. of (6) equals zero, hence by weak duality, the r.h.s. shows that x and y are optimal.

If x and y are optimal, then by strong duality, the r.h.s. of (6) equals zero, and since the l.h.s. consists of non-negative summands, we have $y_i \left(b_i - \sum_{j=\mathbf{1}}^{n} a_{ij} x_j \right)$ for all i.

Definition (Linear Complementarity)

More generally, solutions (x, s) and (y, t) are complementary relative to the primal-dual pair of LP problems

$$
\begin{aligned}
\text{(P)} \quad z^* = \max_{(x,s)} \; & c^{\mathsf{T}} x + d^{\mathsf{T}} s \\
\text{s.t.} \; & Ax + Cs \leq a, \\
& Bx + Ds = b, \\
& x \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{(D)} \quad w^* = \min_{(y,t)} \; & a^{\mathsf{T}} y + b^{\mathsf{T}} t \\
\text{s.t.} \; & A^{\mathsf{T}} y + B^{\mathsf{T}} t \geq c \\
& C^{\mathsf{T}} y + D^{\mathsf{T}} t = d \\
& y \geq 0.
\end{aligned}
$$

if the following conditions are satisfied,

$$
\begin{aligned}
y_i \left(a_i - \sum_j a_{ij} x_j - \sum_k c_{ik} s_k \right) = 0, \quad & \forall i, \\
\left(\sum_i y_i a_{ij} + \sum_\ell t_\ell b_{\ell j} - c_j \right) x_j = 0, \quad & \forall j.
\end{aligned}
$$

Corollary (Complementarity and duality)

Complementary solutions relative to a primal-dual pair of LPs remain complementary when the roles of primal and dual are interchanged.

Proof. Follows from the the Complementary Slackness Theorem and the fact that the bi-dual equals the primal.

Example (Check optimality by complementarity)

Consider the LP instance

$$
\begin{aligned}
\text{(P)}\quad & \max\, 5x_1 + 4x_2 + 3x_3 \\
& \text{s.t. } 2x_1 + 3x_2 + x_3 \leq 5 \\
& \qquad 4x_1 + x_2 + 2x_3 \leq 11 \\
& \qquad 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& \qquad x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Someone gives us the solution $x_1^* = 2$, $x_2^* = 0$, $x_3^* = 1$ and claims it is optimal for (P), a claim we want to verify by constructing a dual optimal y^* via complementary slackness.

We check that x^* is feasible for (P) by substitution into the constraint inequalities.

If x^* is optimal, then by complementary slackness, $x_1^*, x_3^* > 0$ implies

$$2y_1^* + 4y_2^* + 3y_3^* = 5 \tag{7}$$
$$y_1^* + 2y_2^* + 2y_3^* = 3. \tag{8}$$

Furthermore, $4x_1^* + x_2^* + 2x_3^* = 10 < 11$ implies $y_2^* = 0$.

Substituting into (7) and (8), we obtain

$$\begin{aligned} 2y_1^* + 3y_3^* &= 5 \\ y_1^* + 2y_3^* &= 3 \end{aligned}$$

Solving this linear system, we find $y_1^* = 1$, $y_3^* = 1$.

By construction, y^* satisfies the constraints $A^{\mathbf{T}} y^* \geq c$, and we also see that $y^* \geq 0$. Furthermore,

$$c^{\mathbf{T}} x^* = 5 \cdot 2 + 4 \cdot 0 + 3 \cdot 1 = 13 = 5 \cdot 1 + 11 \cdot 0 + 8 \cdot 1 = b^{\mathbf{T}} y^*,$$

confirming the optimality of both x^* and y^*.