InFoMM – Optimisation, MT19 Assessment

Prof. Raphael Hauser

Problem 1. The generalisation of Gauss-Jordan Elimination to systems of linear inequalities is called Fourier-Motzkin Elimination. It works as follows. Consider a system of linear inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i = 1, \dots, m),$$

and let us select a variable x_k to eliminate. We partition the set $M = \{1, \dots, m\}$ into

$$\begin{aligned} M_+ &:= \{i : a_{ik} > 0\}, \\ M_- &:= \{i : a_{ik} < 0\}, \\ M_0 &:= \{i : a_{ik} = 0\}. \end{aligned}$$

The new system consists of the following inequalities,

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i \in M_0),$$

$$\sum_{j=1}^{n} (a_{ik}a_{\ell j} - a_{\ell k}a_{ij})x_j \leq a_{ik}b_\ell - a_{\ell k}b_i, \quad ((i, \ell) \in M_+ \times M_-)\,.$$

Prove that the new system of linear inequalities does not involve x_k and is equivalent to the original system in the following sense: $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ satisfies the new system if and only if there exists a value of x_k for which $(x_1, \dots, x_k, \dots, x_n)$ satisfies the original system. [*Hint: the set of values x_k can take is an interval you should determine. Note that the procedure can be applied repetitively, and if $M_+ \cup M_0 = \emptyset$ or $M_- \cup M_0 = \emptyset$, then the new system is empty and is satisfied by all values of $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$.*]

Problem 2. Prove the Theorem of the Alternative for Linear Inequalities by breaking it down into the following steps:

i) Show that both systems cannot simultaneously have solutions.

ii) Suppose that the first system has no solution, and eliminate all of its n variables via Fourier-Motzkin Elimination. This yields an inconsistent system (a system with no solution) of the form

$$\sum_{j=1}^{n} 0 \cdot x_j \leq d_k, \quad (k = 1, \dots, p),$$

Show that there exists k^* such that $d_{k^*} < 0$, and $y_1, \dots, y_m \geq 0$ such that the k^*-th inequality is obtained as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} x_j \leq \sum_{i=1}^{m} y_i b_i.$$

iii) Show that y is a solution of the alternative system.

Problem 3. A scheduling model in which a machine can be switched on at most $k < n$ times is modelled by the following constraints, where y_0 can be considered as zero,

$$
\begin{aligned}
\sum_{t=1}^{n} z_t &\le k, \\
z_t - y_t + y_{t-1} &\ge 0, \quad (t = 1, \dots, n), \\
z_t &\le y_t, \quad (t = 1, \dots, n), \\
0 \le y_t, z_t &\le 1, \quad (t = 1, \dots, n), \\
y_t, z_t &\in \mathbb{Z}, \quad (t = 1, \dots, n).
\end{aligned}
$$

i) Give an economic interpretation of the decision variables y_t,z_t.

ii) It can be shown that the following are sufficient conditions for a matrix $A = (a_{ij})$ to be totally unimodular,

  a) $a_{ij} \in \{0, +1, -1\}$ for all i, j,

  b) for any subset M of the rows of A, there exists a partition (M_1, M_2) of M such that each column j satisfies

$$\left| \sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} \right| \le 1.$$

  (M_1 and M_2 are the same for each column.)

  Use this criterion to prove that the constraint matrix of the above described scheduling problem is totally unimodular.

Problem 4. A *matching* of a graph G is a set of edges meeting each node of G at most once. *König's Theorem* says that in a bipartite graph $G = (V_1, V_2, E)$ the number of edges in a matching of maximum cardinality is equal to the minimal cardinality needed for a set of vertices to be incident to all edges of E (covering by nodes). For any $v \in V$ let $E(v)$ be the set of edges incident to v. The maximum cardinality matching problem is thus given by

$$
\begin{aligned}
\text{(MaxMatch)} \quad & \max \sum_{e \in E} x_e \\
& \text{s.t.} \sum_{e \in E(v)} x_e \le 1, \quad (v \in V_1) \\
& \sum_{e \in E(w)} x_e \le 1, \quad (w \in V_2) \\
& x_e \ge 0, x_e \in \mathbb{Z}, \quad (e \in E).
\end{aligned}
$$

i) Introduce slack variables and show that the constraint matrix is totally unimodular.

ii) Set up the dual of the LP relaxation of (M) and interpret it as the LP relaxation of the minimum cardinality node covering problem.

iii) Using the Strong LP Duality Theorem, prove König's Theorem.

Problem 5. Consider the 0-1 knapsack problem $\max\{\sum_{j=1}^n c_j x_j : \sum_{j=1}^n a_j x_j \leq b,\ x \in \{0,1\}^n\}$, where $a_j, c_j > 0$ for $j = 1, \ldots, n$.

i) Show that if $\frac{c_1}{a_1} \geq \cdots \geq \frac{c_n}{a_n} > 0$, $\sum_{j=1}^{r-1} a_j \leq b$ and $\sum_{j=1}^{r} a_j > b$, the solution of the LP relaxation is $x_j = 1$ for $j = 1, \ldots, r-1$, $x_r = (b - \sum_{j=1}^{r-1} a_j)/a_r$ and $x_j = 0$ for $j > r$. [*Hint: First assume $x_r > 0$ and use complementary slackness to generate a certificate of optimality. Then extend the proof to the case $x_r = 0$.*]

ii) Solve the following instance by branch-and-bound,

$$\begin{aligned}\max\ & 17x_1 + 10x_2 + 25x_3 + 17x_4\\ \text{s.t.}\ & 5x_1 + 3x_2 + 8x_3 + 7x_4 \leq 12\\ & x \in \{0,1\}^4.\end{aligned}$$

Problem 6. A relaxation of an integer programming problem (IP) $z = \max\{c^{\mathrm{T}}x : x \in \mathscr{F}\}$ is any optimisation problem (R) $w = \max\{g(x) : x \in \mathscr{R}\}$ with feasible set $\mathscr{R} \supseteq \mathscr{F}$ and an objective function $g(x)$ that satisfies $g(x) \geq c^{\mathrm{T}}x$ for all $x \in \mathscr{F}$.

i) Show that if (R) is a relaxation of (IP), then $w \geq z$.

ii) Consider the *equality knapsack problem*

$$\begin{aligned}\text{(EKP)}\quad \max_x\ & \sum_{j=1}^n c_j x_j\\ \text{s.t.}\ & \sum_{j=1}^n a_j x_j = b,\\ & x_j \in \mathbb{Z}_+, \quad (j = 1, \ldots, n),\end{aligned}$$

where $b > 0$ is a positive integer, $a_j > 0$ are positive integers for all j, and $c_j > 0$ are positive reals. Let $k \in \arg\max\{c_j/a_j : j \in [1,n]\}$. Show that the following problem (called *group relaxation*) is a relaxation of (EKP),

$$\begin{aligned}\text{(GR)}\quad \frac{c_k}{a_k}b + \max_x\ & \sum_{j\neq k}\left(c_j - \frac{c_k}{a_k}a_j\right)x_j\\ \text{s.t.}\ & \sum_{j\neq k} a_j x_j \equiv b \,(\mathrm{mod}\, a_k),\\ & x_j \in \mathbb{Z}_+, \quad (j \neq k).\end{aligned}$$

iii) Now consider a network consisting of a digraph $G = (V, E)$ with vertices $V = \{0, 1, \ldots, a_k - 1\}$ and edges

$$E = \{e_{i,j} := (i, s_{ij}) : i \in V,\ j \neq k,\ s_{ij} = i + a_j \,(\mathrm{mod}\, a_k)\}$$

and edge weights $C_{i,j} := -c_j + \frac{c_k}{a_k}a_j \geq 0$ associated with edge $e_{i,j}$, the non-negativity being guaranteed by our choice of k. Show that (GR) can be solved as a shortest path problem in this network from vertex 0 to vertex $b \,(\mathrm{mod}\, a_k)$. Work out the complexity when Dijkstra's Algorithm (the shortest path algorithm from Problem Sheet 2, Problem 5.ii) is applied for this purpose.

iv) Group relaxation can be used in a branch & bound algorithm to solve (EKP), with subproblems of the following form, where $\ell = (\ell_1, \dots, \ell_n)$ is a vector of non-negative integers,

$$\begin{aligned} (\mathrm{EKP}(\ell)) \quad \max_x \ & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \ & \sum_{j=1}^n a_j x_j = b, \\ & x_j \geq \ell_j, \quad (j = 1, \dots, n), \\ & x_j \in \mathbb{Z}, \quad (j = 1, \dots, n). \end{aligned}$$

Show how group relaxation can be applied to (EKP(ℓ)) and, starting with $\ell = 0$ for the root problem, derive a branching rule that allows one to branch each problem (EKP(ℓ)) into n subproblems of the same type (EKP($\ell^{[s]}$)), $(s = 1, \dots, n)$.

v) Give a termination criterion and an upper bound on the number of subproblems that have to be solved before the criterion applies.

Problem 7. Give a proof of correctness for Algorithm (Minimal cover separation) from Lecture 8, that is, prove that its output is a minimal cover whose associated cover inequality is a cut for the point x^*.

Problem 8. Apply the cutting plane algorithm with Gomoroy cuts to the following IP,

$$\min_{x_1, x_2} \{x_1 + x_2 : 6x_1 + x_2 \leq 4, \, 3x_1 \geq 1, \, x_1, x_2 \geq 0, \, x_1, x_2 \in \mathbb{Z}\}$$

Problem 9. Consider problem

$$(\mathrm{GAP}) \quad \max_x \sum_{i=1}^m \sum_{j=1}^n p_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^n c_{ij} x_{ij} \leq b_i, \quad (i = 1, \dots, m), \tag{1}$$

$$\sum_{i=1}^m x_{ij} = 1, \quad (j = 1, \dots, n), \tag{2}$$

$$x_{ij} \in \{0, 1\}, \quad (i = 1, \dots, m; j = 1, \dots, n), \tag{3}$$

where $p_{ij}, c_{ij}, b_i \in \mathbb{Q}$ are fixed problem parameters for all i, j.

i) Suppose that the parameters of problem (GAP) are such that there exists an index i for which $c_{ij} > 0$ for all j and $b_i > 0$, and a set $C_i \subseteq \{1, \dots, n\}$ such that $\sum_{j \in C_i} c_{ij} > b_i$. Prove that all feasible solutions x of (GAP) must satisfy the inequality

$$\sum_{j \in C_i} x_{ij} \leq |C_i| - 1. \tag{4}$$

ii) Denote the feasible set of problem (GAP) by $\mathscr{F}$, and assume that we are given an optimal solution x^* of the LP-relaxation of (GAP), and let (i^*, j^*) be the indices of its most fractional component x_{i^*, j^*}. Generalised Upper Bound branching (GUB) splits $\mathscr{F}$ into two disjoint sets according to the rule

$$\begin{aligned} \mathscr{F}_1 &= \mathscr{F} \cap \{x : x_{ij^*} = 0, \ i = 1, \dots, p\} \\ \mathscr{F}_2 &= \mathscr{F} \cap \{x : x_{ij^*} = 0, \ i = p+1, \dots, m\}, \end{aligned}$$

where $p := \min\{t : \sum_{i=1}^t x^*_{ij^*} \geq 1/2\}$. Discuss why this is not a good branching rule when $x^*_{mj^*} > 1/2$. Propose an algorithm to reorder the indices x_{ij^*} $(i = 1, \dots, m)$ so that GUB branching yields balanced sets $\mathscr{F}_1$ and $\mathscr{F}_2$, that is, sets of nearly equal cardinality.