Assignment Wk 1: The pendulum

1. Consider the familiar simple pendulum, released from rest at an angle from the vertical of $-\theta_0$ $(0 < \theta_0 < \pi)$, and whose angle θ at time t can be modelled by the second order ODE

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0, \quad \theta(0) = -\theta_0, \dot{\theta}(0) = 0. \tag{1}$$

   Use `ode45` to solve the ODE numerically by setting $\omega_0 = 1$. Plot the solution $\theta(t)$ for six different values of θ_0 between 0 and π. For each case overlay a plot of a cosine function of matching amplitude and with a frequency

$$\omega(\theta_0) = \frac{\pi\omega_0}{2K(\sin^2(\theta_0/2))}, \tag{2}$$

   where $K(m)$ is the complete elliptic integral of the first kind which can be evaluated using MATLAB's `ellipticK` command. Verify if Eq. (2) gives the correct frequency, but that for large amplitudes the motion of the pendulum is not sinusoidal.

2. Now consider the simple pendulum with quadratic damping, modelled by the equation

$$\ddot{\theta} + \omega_0^2 \sin\theta + \gamma|\dot{\theta}|\,\dot{\theta} = 0, \quad \theta(0) = \theta_0,\ \dot{\theta}(0) = 0. \tag{3}$$

   Try to solve this model numerically using the ODE solver by forming coupled equations with $\omega_0 = 1$ and $\gamma = 1/2$ using three different initial conditions for θ_0. Run the solution for a total time of 20 seconds. Plot the solutions as a function of time and also plot the trajectories of the solutions in a phase space diagram (i.e. θ vs v). Identify each of the three initial conditions on your plots, and explain what the pendulum does along each trajectory.

3. Use `ode45` to solve the equation for the motion of a rapidly driven pendulum

$$\ddot{\theta} + \omega_0^2 \sin\theta + \frac{b\omega^2}{\ell}\sin\theta\sin\omega t = 0, \tag{4}$$

   and the slow time-averaged equation of motion

$$\ddot{\Theta} + \omega_0^2 \sin\Theta + \frac{b^2\omega^2}{2\ell^2}\sin\Theta\cos\Theta = 0. \tag{5}$$

   Set $\omega_0 = 1$, $\ell = 1$, $b = 0.02$ and $\omega = 30$ with initial conditions $\theta(0) = 1$, $\dot{\theta}(0) = 0$ and run for a total time of 30 seconds. Overlay the plots of $\theta(t)$ and $\Theta(t)$ to see that the averaged solution approximates the un-averaged equation. Now run it again with initial conditions $\theta(0) = 3.1, \dot{\theta}(0) = 0$ and check the agreement again.[1]

4. Now redo part 4 but only change $b = 0.05$ this time. For the case with $\theta(0) = 3.1$ you should get a stable pendulum in the straight-up position! This is not a mistake as you can observe from the inverted pendulum demonstration on YouTube https://www.youtube.com/watch?v=5oGYCxkgnHQ.

5. Alter `pendulum_animation.m` to animate the solution of Eqs. (1), and (3). Call your new codes `pendulum_animation#.m` and replace # with the respective number of the equation you animate.

[1] Note: The averaged solution won't go through the middle of the wiggles of the full solution unless you adjust the averaged initial conditions as shown in the lectures.

Guidelines

You should produce your reports using the MATLAB's `publish` facility and make sure your codes are well documented. Please upload your reports along with your MATLAB scripts as a `.zip` or `.tar` file to the course's online portal (https://courses.maths.ox.ac.uk/node/322/assignments) using your name as the filename.

Lecturer: Dr. Vassilios Dallas
Contact: dallas@maths.ox.ac.uk