

# Homogenisation

Jon Chapman

September 5, 2019

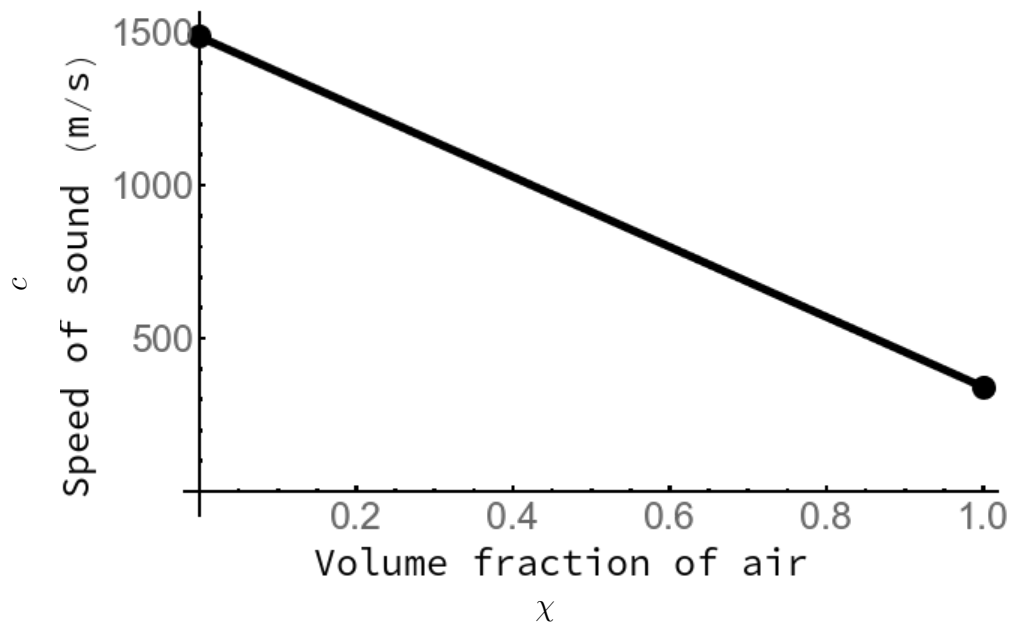


Figure 1: Naïve interpolation of sound speed in bubbly water

# 1 Introduction

## 1.1 Example

Mixing different materials can lead to surprising results. Speed of sound in air is  $343 \text{ ms}^{-1}$ . Speed of sound in water is  $1498 \text{ ms}^{-1}$ . What is the speed of sound in bubbly water? Suppose the volume fraction of air is  $\chi$ , and that of water is  $1 - \chi$ .

**Options.** Perhaps a straight average:

$$c = \chi c_a + (1 - \chi) c_w? \quad \Rightarrow \quad c(50 : 50) = 920 \text{ ms}^{-1}$$

But the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

So perhaps we should average  $c^2$ :

$$c^2 = \chi c_a^2 + (1 - \chi) c_w^2? \quad \Rightarrow \quad c(50 : 50) = 1087 \text{ ms}^{-1}$$

But we usually write the wave equation as

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

So perhaps we should average  $1/c^2$ :

$$\frac{1}{c^2} = \frac{\chi}{c_a^2} + \frac{(1 - \chi)}{c_w^2}? \quad \Rightarrow \quad c(50 : 50) = 472 \text{ ms}^{-1}$$

The sound speed is given in terms of physical parameters by

$$c^2 = \frac{K}{\rho},$$

where  $K$  is the bulk modulus and  $\rho$  is the density. Perhaps we should average these individually?

$$c^2 = \frac{\chi K_a + (1 - \chi) K_w}{\chi \rho_a + (1 - \chi) \rho_w}? \quad \Rightarrow \quad c(50 : 50) = 1498 \text{ ms}^{-1}.$$

In fact none of these is correct. The correct 50:50 sound speed is  $24 \text{ ms}^{-1}$ . This is much lower than the sound speed in either air or water!

The correct way to average is to average the compressibility  $1/K$  and the density  $\rho$ :

$$\frac{1}{c^2} = (\chi \rho_a + (1 - \chi) \rho_w) \left( \frac{\chi}{K_a} + \frac{1 - \chi}{K_w} \right).$$

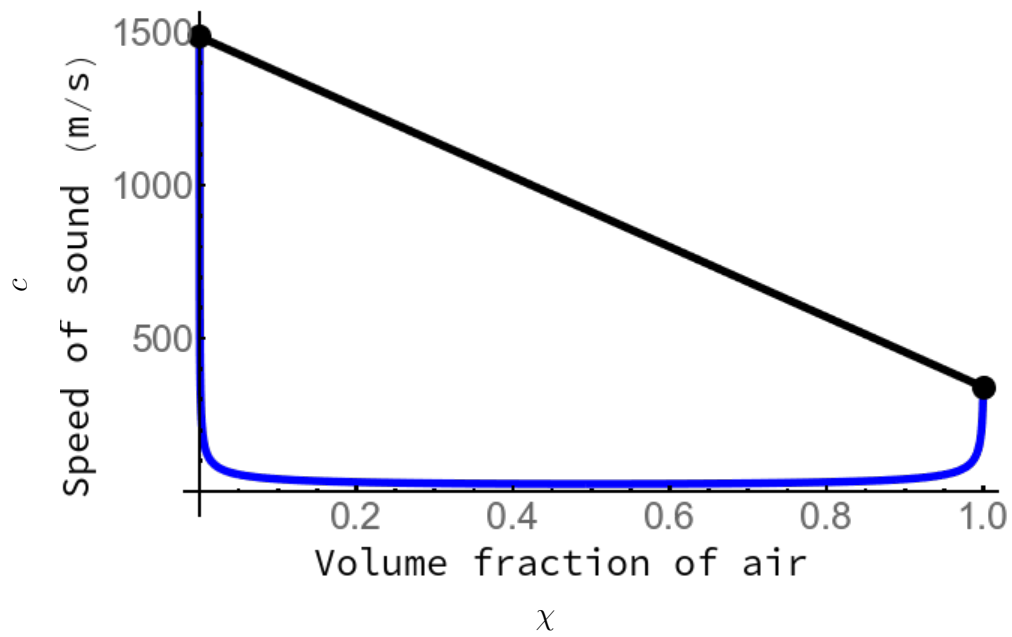


Figure 2: Actual sound speed in bubbly water

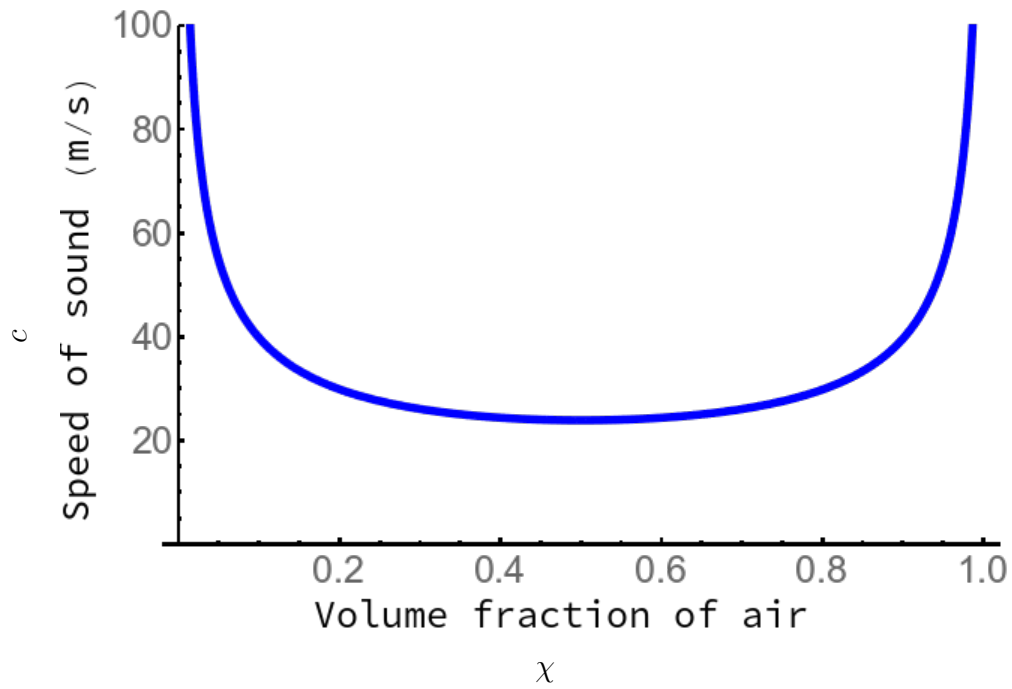
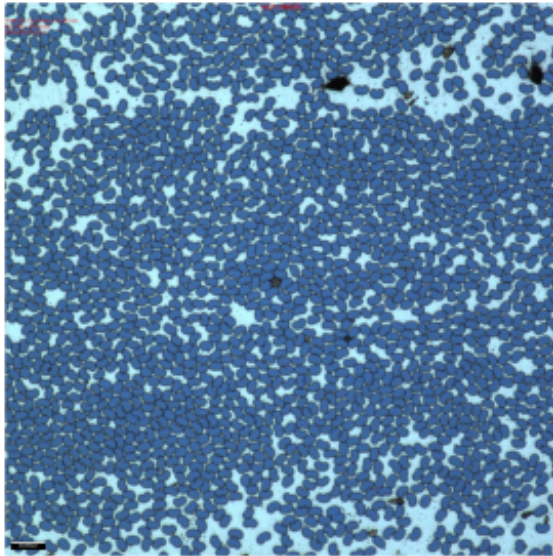
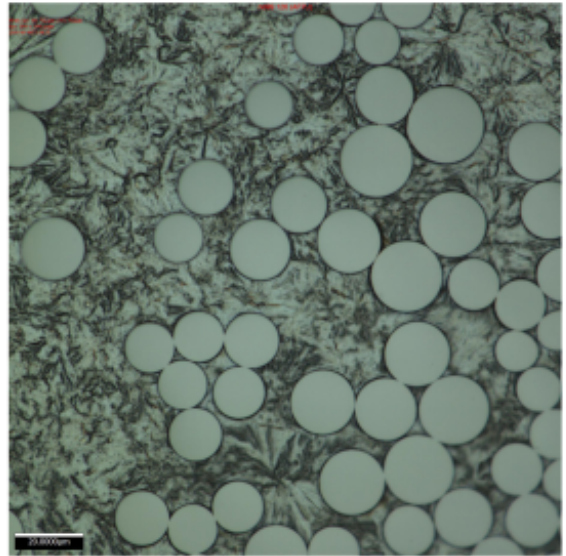


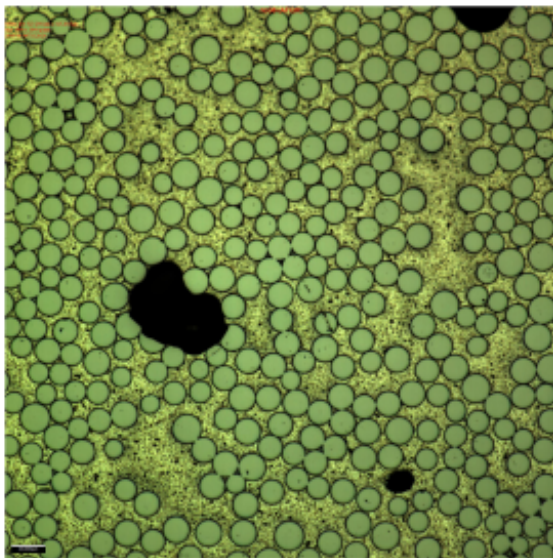
Figure 3: Actual sound speed in bubbly water



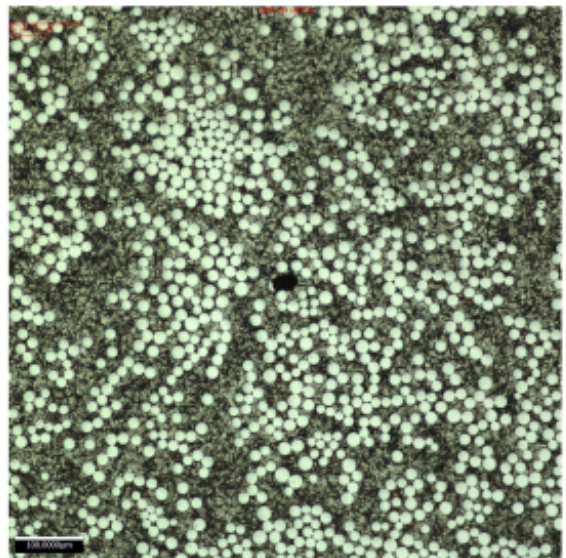
(a)



(b)



(c)



(d)

Figure 4: Examples of microstructure

## 2 Multiple scales

### 2.1 A one-dimensional problem

We consider the following one-dimensional paradigm problem:

$$\frac{d}{dx} \left( a \left( \frac{x}{\epsilon} \right) \frac{dy}{dx} \right) = f \left( \frac{x}{\epsilon} \right), \quad (2.1)$$

where  $a(X)$  and the source term  $f(X)$  are assumed 1-periodic in  $X$ .

Introducing the fast scale  $X = x/\epsilon$ , writing  $y = y(x, X)$  with  $y$  1-periodic in  $X$  and treating  $x$  and  $X$  as independent as is usual in multiple scales gives

$$\left( \epsilon \frac{\partial}{\partial x} + \frac{\partial}{\partial X} \right) \left( a(X) \left( \epsilon \frac{\partial y}{\partial x} + \frac{\partial y}{\partial X} \right) \right) = \epsilon^2 f(X). \quad (2.2)$$

Expanding  $y = y_0 + \epsilon y_1 + \dots$  gives at leading order

$$\frac{\partial}{\partial X} \left( a(X) \left( \frac{\partial y_0}{\partial X} \right) \right) = 0$$

for which the relevant solution is  $y_0 = y_0(x)$ . At next order in (2.2) we find

$$\frac{\partial}{\partial X} \left( a(X) \left( \frac{dy_0}{dx} + \frac{\partial y_1}{\partial X} \right) \right) = 0,$$

so that

$$\frac{dy_0}{dx} + \frac{\partial y_1}{\partial X} = \frac{Q(x)}{a(X)},$$

say. Integrating over  $X$  and using the periodicity of  $y_1$  gives

$$\frac{dy_0}{dx} = Q(x) \int_0^1 \frac{dX}{a(X)}.$$

At next order in (2.2) we find

$$\frac{\partial}{\partial x} \left( a(X) \left( \frac{dy_0}{dx} + \frac{\partial y_1}{\partial X} \right) \right) + \frac{\partial}{\partial X} \left( a(X) \left( \frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial X} \right) \right) = f(X),$$

Figure

Figure 5: The idea behind multiple scales

i.e.

$$\frac{dQ}{dx} + \frac{\partial}{\partial X} \left( a(X) \left( \frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial X} \right) \right) = f(X),$$

Integrating over  $X$  (and using the continuity of  $a(X)(\partial y_1/\partial x + \partial y_2/\partial X)$ ) gives

$$\frac{dQ}{dx} = \int_0^1 f(X) dX.$$

Thus the leading-order homogenised problem is

$$\frac{d}{dx} \left( \bar{a} \frac{dy_0}{dx} \right) = \bar{f}.$$

where

$$\frac{1}{\bar{a}} = \int_0^1 \frac{dX}{a(X)}, \quad \bar{f} = \int_0^1 f(X) dX.$$

## 2.2 Three dimensional example with continuous coefficients

We consider the equation

$$\nabla \cdot (A(\mathbf{x}/\epsilon) \nabla u) = f(\mathbf{x}/\epsilon).$$

We suppose that  $A$  and  $f$  are smooth periodic functions with unit period (so that  $A(\mathbf{x}/\epsilon)$  and  $f(\mathbf{x}/\epsilon)$  have period  $\epsilon$ ).

We introduce the microscale variable  $\mathbf{X} = \mathbf{x}/\epsilon$  and write  $u$  as a function of both the slow and fast scales:  $u = u(\mathbf{x}, \mathbf{X})$ . We treat  $\mathbf{x}$  and  $\mathbf{X}$  as independent variables. This gives a lot of freedom in how to write  $u$ ; we use this freedom to impose that  $u$  is exactly periodic in  $\mathbf{X}$  with unit period—the non-periodic part of  $u$  becomes the slow variation with  $\mathbf{x}$ . The chain rule gives

$$\nabla \rightarrow \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}},$$

and the equation becomes

$$\left( \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}} \right) \cdot \left( A(\mathbf{X}) \left( \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}} \right) \right) u = f(\mathbf{X}).$$

Expanding  $u = u_0 + \epsilon u_1 + \dots$  gives at leading order

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} u_0) = 0,$$

with  $u_0$  1-periodic in  $\mathbf{X}$ . The only solution is for  $u_0$  to be constant in  $\mathbf{X}$ , so that  $u_0 = u_0(\mathbf{x})$ .

At next order

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} u_1) + \nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{x}} u_0) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{X}} u_0) = 0,$$

with  $u_1$  1-periodic in  $\mathbf{X}$ , which simplifies to

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} u_1) = -\nabla_{\mathbf{X}} A \cdot \nabla_{\mathbf{x}} u_0, \quad (2.3)$$

since  $u_0$  is independent of  $\mathbf{X}$ . This is a linear equation for  $u_1$  with an as-yet-unknown right-hand side. Let us write out exactly what the right-hand side is:

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} u_1) = -\frac{\partial A}{\partial X_1} \frac{\partial u_0}{\partial x_1} - \frac{\partial A}{\partial X_2} \frac{\partial u_0}{\partial x_2} - \frac{\partial A}{\partial X_3} \frac{\partial u_0}{\partial x_3}. \quad (2.4)$$

Suppose we solve the following auxilliary problems:

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} \psi_1) = -\frac{\partial A}{\partial X_1} \quad (2.5)$$

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} \psi_2) = -\frac{\partial A}{\partial X_2}, \quad (2.6)$$

$$\nabla_{\mathbf{X}} \cdot (A \nabla_{\mathbf{X}} \psi_3) = -\frac{\partial A}{\partial X_3}, \quad (2.7)$$

with  $\psi_1, \psi_2$ , and  $\psi_3$  1-periodic in  $\mathbf{X}$ . Then the solution to (16.6) is just a linear combination of these solutions:

$$u_1 = \psi_1 \frac{\partial u_0}{\partial x_1} + \psi_2 \frac{\partial u_0}{\partial x_2} + \psi_3 \frac{\partial u_0}{\partial x_3} + \bar{u}_1(\mathbf{x}), \quad (2.8)$$

where  $\bar{u}_1$  is a solution of the homogeneous problem, which we have already seen does not depend on  $\mathbf{X}$ . Since we could also add any such function to the  $\psi_i$ , at the moment they are not defined uniquely. If we want to specify them uniquely we can fix the constant by imposing the integral constraint

$$\int_{\Omega} \psi_i \, d\mathbf{X} = 0,$$

where  $\Omega = [0, 1]^3$  is the unit cell.

The advantage of writing the solution in this way is that  $\psi_1, \psi_2$  and  $\psi_3$  do not depend on the solution  $u_0$ , only on the microstructure  $A(\mathbf{X})$ . They can be solved for once and



for all. This form of solution would arise naturally if you solved (16.6) using a Green's function, but it is easy to spot. The functions  $\psi_i$  are known as cell functions, and the problems (2.5)-(2.7) cell problems. There will be one cell function for each unknown variable on the right-hand side of (16.6). Usually (2.8) is written in the compact notation

$$u_1 = \mathbf{\Psi} \cdot \nabla_{\mathbf{x}} u_0 + \bar{u}_1(\mathbf{x}),$$

where  $\mathbf{\Psi} = (\psi_1, \psi_2, \psi_3)$ .

At next order

$$\nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_2) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_1) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_1) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_0) = f,$$

with  $u_2$  1-periodic in  $\mathbf{X}$ . Integrating over the unit cell  $\Omega = [0, 1]^3$  gives

$$\begin{aligned} & \int_{\Omega} (\nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_2) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_1) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_1) + \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u_0)) \, d\mathbf{X} \\ &= \int_{\partial\Omega} (A \nabla_{\mathbf{x}} u_2 + A \nabla_{\mathbf{x}} u_1) \cdot \mathbf{n} \, dS + \nabla_{\mathbf{x}} \cdot \int_{\Omega} (A \nabla_{\mathbf{x}} u_1 + A \nabla_{\mathbf{x}} u_0) \, d\mathbf{X} \\ &= \nabla_{\mathbf{x}} \cdot \int_{\Omega} A \left( \nabla_{\mathbf{x}} \psi_1 \frac{\partial u_0}{\partial x_1} + \nabla_{\mathbf{x}} \psi_2 \frac{\partial u_0}{\partial x_2} + \nabla_{\mathbf{x}} \psi_3 \frac{\partial u_0}{\partial x_3} + \nabla_{\mathbf{x}} u_0 \right) \, d\mathbf{X} \\ &= \int_{\Omega} f \, d\mathbf{X}, \end{aligned}$$

where the integral over  $\partial\Omega$  vanishes due to periodicity. Thus

$$\nabla_{\mathbf{x}} \cdot (A^{\text{eff}} \nabla_{\mathbf{x}} u) = f^{\text{eff}},$$

where

$$f^{\text{eff}} = \int_{\Omega} f(\mathbf{X}) \, d\mathbf{X},$$

and the components of the effective conductivity tensor  $A^{\text{eff}}$  are

$$A_{ij}^{\text{eff}} = \int_{\Omega} A \left( \delta_{ij} + \frac{\partial \psi_j}{\partial x_i} \right) \, d\mathbf{X}. \quad (2.9)$$

This can be written

$$A^{\text{eff}} = \int_{\Omega} A (I + \nabla_{\mathbf{x}} \mathbf{\Psi}) \, d\mathbf{X}, \quad (2.10)$$

where  $I$  is the identity tensor, but I would advise sticking to index notation since it is easy to get confused between  $(\nabla_{\mathbf{x}} \mathbf{\Psi})^T$  and  $\nabla_{\mathbf{x}} \mathbf{\Psi}$ . The definition here takes  $\mathbf{\Psi}$  to be a row vector with the gradient acting to produce the columns, i.e. the first column is  $\nabla_{\mathbf{x}} \psi_1$ .

Another common definition is to consider  $\mathbf{\Psi}$  as a column vector and the gradient acting on rows:

$$(\nabla_{\mathbf{x}} \mathbf{\Psi})_{ij} = \frac{\partial \Psi_i}{\partial x_j}.$$

## 2.3 Three dimensional example with discontinuous coefficients

We suppose that  $A$  and  $f$  are periodic with unit period and are both piecewise constant, so that

$$A = A_i, \quad f = f_i \quad \text{in } \Omega_i, \quad (2.11)$$

$$A = A_e, \quad f = f_e, \quad \text{in } \Omega_e, \quad (2.12)$$

where  $A_i, A_e, f_i, f_e$  are constant, where  $\Omega = \Omega_i \cup \Omega_e$ . At the boundary  $\Gamma$  between  $\Omega_i$  and  $\Omega_e$  we have the continuity conditions

$$\begin{aligned} [u]_i^e &= 0, \\ \left[ A \frac{\partial u}{\partial n} \right]_i^e &= [A \mathbf{n} \cdot \nabla u]_i^e = 0. \end{aligned}$$

As usual, we write  $u = u(\mathbf{x}, \mathbf{X})$  and suppose that  $u$  is periodic in  $\mathbf{X}$  with unit period, treating  $\mathbf{x}$  and  $\mathbf{X}$  as independent. The equation then becomes

$$\left( \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}} \right) \cdot \left( A(\mathbf{X}) \left( \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}} \right) \right) u = f(\mathbf{X}),$$

with boundary conditions

$$\begin{aligned} [u]_i^e &= 0, \\ \left[ A \mathbf{n} \cdot \left( \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}} \right) u \right]_i^e &= 0, \end{aligned}$$

Expanding  $u = u_0 + \epsilon u_1 + \dots$  gives at leading order (because  $A$  is piecewise constant)

$$\nabla_{\mathbf{X}}^2 u_0 = 0 \quad \text{in } \Omega \equiv \Omega_i \cup \Omega_e,$$

with boundary conditions

$$\begin{aligned} [u_0]_i^e &= 0, \\ [A \mathbf{n} \cdot \nabla_{\mathbf{X}} u_0]_i^e &= 0, \end{aligned}$$

and  $u_0$  1-periodic in  $\mathbf{X}$ . The only solution is for  $u_0$  to be constant in  $\mathbf{X}$ , so that  $u_0 = u_0(\mathbf{x})$ .

At next order

$$\nabla_{\mathbf{X}}^2 u_1 = 0 \quad \text{in } \Omega,$$

with boundary conditions

$$\begin{aligned} [u_1]_i^e &= 0, \\ [A\mathbf{n} \cdot \nabla_{\mathbf{X}} u_1]_i^e &= -[A]_i^e \mathbf{n} \cdot \nabla_{\mathbf{x}} u_0, \end{aligned}$$

and  $u_1$  1-periodic in  $\mathbf{X}$ . The solution is

$$u_1 = \Psi \cdot \nabla_{\mathbf{x}} u_0 + \bar{u}_1(\mathbf{x}),$$

where  $\Psi = (\psi_1, \psi_2, \psi_3)$  satisfies the *cell problem*

$$\nabla_{\mathbf{X}}^2 \psi_j = 0 \quad \text{in } \Omega, \quad (2.13)$$

with boundary conditions

$$[\psi_j]_i^e = 0, \quad (2.14)$$

$$[A\mathbf{n} \cdot \nabla_{\mathbf{X}} \psi_j]_i^e = -[A]_i^e \mathbf{n} \cdot \mathbf{e}_j, \quad (2.15)$$

with  $\mathbf{e}_j$  the unit vector in the  $j$ -direction, along with

$$\int_{\Omega} \psi_j \, d\mathbf{X} = 0. \quad (2.16)$$

At next order

$$\begin{aligned} A_i \nabla_{\mathbf{X}} \cdot (\nabla_{\mathbf{X}} u_2 + \nabla_{\mathbf{x}} u_1) + A_i \nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{X}} u_1 + \nabla_{\mathbf{x}} u_0) &= f(\mathbf{X}) & \text{in } \Omega_i, \\ A_e \nabla_{\mathbf{X}} \cdot (\nabla_{\mathbf{X}} u_2 + \nabla_{\mathbf{x}} u_1) + A_e \nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{X}} u_1 + \nabla_{\mathbf{x}} u_0) &= f(\mathbf{X}) & \text{in } \Omega_e, \end{aligned}$$

with boundary conditions

$$\begin{aligned} [u_2]_i^e &= 0, \\ [A\mathbf{n} \cdot \nabla_{\mathbf{X}} u_2]_i^e &= -[A\mathbf{n} \cdot \nabla_{\mathbf{x}} u_1]_i^e. \end{aligned}$$

and  $u_2$  1-periodic in  $\mathbf{X}$ . Integrating over the unit cell  $\Omega$  gives

$$\nabla_{\mathbf{x}} \cdot (A^{\text{eff}} \nabla_{\mathbf{x}} u) = f^{\text{eff}},$$

where

$$f^{\text{eff}} = \int_{\Omega} f(\mathbf{X}) \, d\mathbf{X},$$

and the components of the effective conductivity tensor  $A^{\text{eff}}$  are

$$A_{ij}^{\text{eff}} = \int_{\Omega} A \left( \delta_{ij} + \frac{\partial \psi_j}{\partial X_i} \right) \, d\mathbf{X}. \quad (2.17)$$

## 2.4 Dispersion in porous media

# 15 Problems

## 15.1 One dimensional acoustic waves in a bubbly fluid

The equations of acoustics in one dimension are

$$\begin{aligned}\frac{\partial p}{\partial t} &= -K \frac{\partial u}{\partial x}, \\ \rho \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x}\end{aligned}$$

where

$u$  = velocity,  $p$  = pressure,  $K$  = bulk modulus,  $\rho$  = density.

Eliminating  $p$  gives

$$\frac{\partial}{\partial t} \left( \rho \frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial x} \left( K \frac{\partial u}{\partial x} \right).$$

If the bulk modulus  $K$  and the density  $\rho$  are constant then

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

where the wavespeed

$$c = \sqrt{\frac{K}{\rho}}.$$

Suppose now that  $K(x/\epsilon)$  and  $\rho(x/\epsilon)$  are periodic functions of position with period  $\epsilon$  (so that  $K(\xi)$  and  $\rho(\xi)$  are 1-periodic in  $\xi$ ). Find the effective wavespeed.

## 15.2 Darcy from Stokes

The incompressible Stokes equations are

$$\begin{aligned}\epsilon^2 \nabla^2 \mathbf{u} &= -\nabla p, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

with boundary conditions

$$\mathbf{u} = \mathbf{0}$$

on any fixed boundary. Consider a porous material with a periodic microstructure, of period  $\epsilon$ . Show that the homogenised model is Darcy flow, and determine the permeability.

### 15.3 Reaction-diffusion

Suppose a chemical diffuses in the space between a bed of microspheres, and reacts with the surface of the spheres. If  $c$  is the concentration of this chemical, then  $c$  satisfies

$$\begin{aligned}\nabla^2 c &= \frac{\partial c}{\partial t} && \text{in } \Omega, \\ \frac{\partial c}{\partial n} &= \epsilon \gamma c && \text{on } \partial\Omega.\end{aligned}$$

If the centres of the spheres are arranged on a rectangular lattice with period  $\epsilon$ , find the homogenised equations.

### 15.4 Elasticity

The steady Navier equations of elasticity may be written

$$\begin{aligned}\nabla \cdot \boldsymbol{\sigma} &= \mathbf{0}, \\ \boldsymbol{\sigma} &= \mathbb{C} : \nabla \mathbf{u},\end{aligned}$$

where

$$\mathbb{C} = \{c_{ijkl}\}$$

is the (4th order) stiffness tensor. For an isotropic material

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{jk}$$

where  $\lambda$  and  $\mu$  are the Lamé coefficients;  $\mu$  is the bulk modulus, and

$$\lambda = \frac{2\nu}{1-2\nu}$$

where  $\nu$  is Poisson's ratio. Suppose  $\mathbb{C}$  is periodic with period  $\epsilon$ . Find the homogenised equations.

# 16 Answers

## 16.1 Darcy from Stokes

In multiple scales form

$$\begin{aligned}(\epsilon \nabla_{\mathbf{x}} + \nabla_{\mathbf{X}})^2 \mathbf{u} &= \nabla_{\mathbf{x}} p + \frac{1}{\epsilon} \nabla_{\mathbf{X}} p, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

with boundary conditions

$$\mathbf{u} = \mathbf{0} \tag{16.1}$$

on any fixed boundary. Expanding

$$\mathbf{u} = \mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + \cdots, \quad p = p^{(0)} + \epsilon p^{(1)} + \cdots,$$

and equating coefficients of  $\epsilon$  gives at leading order

$$\nabla_{\mathbf{X}} p^{(0)} = 0, \tag{16.2}$$

$$\nabla_{\mathbf{X}} \cdot \mathbf{u}^{(0)} = 0. \tag{16.3}$$

Thus  $p^{(0)} = p^{(0)}(\mathbf{x})$ . At next order we find

$$\nabla_{\mathbf{X}}^2 \mathbf{u}^{(0)} = \nabla_{\mathbf{x}} p^{(0)} + \nabla_{\mathbf{X}} p^{(1)}, \tag{16.4}$$

$$\nabla_{\mathbf{X}} \cdot \mathbf{u}^{(1)} + \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(0)} = 0. \tag{16.5}$$

equations (16.3), (16.4) constitute the cell problem. We write

$$p^{(1)} = \Phi^i \frac{\partial p^{(0)}}{\partial x_i}, \quad \mathbf{u}^{(0)} = \mathbf{w}^i \frac{\partial p^{(0)}}{\partial x_i} \tag{16.6}$$

where

$$\nabla_{\mathbf{X}}^2 \mathbf{w}^i - \nabla_{\mathbf{X}} \Phi^i = \mathbf{e}_i, \tag{16.7}$$

$$\nabla_{\mathbf{X}} \cdot \mathbf{w}^i = 0, \tag{16.8}$$

where  $\mathbf{e}_i$  is the unit vector in the  $i$ th direction. Thus we have  $p^{(1)}$  and  $\mathbf{u}^{(0)}$  in terms of  $\nabla_{\mathbf{x}} p^{(0)}$ . To close the problem we integrate (16.5) over the fluid region to give

$$\begin{aligned} 0 &= \int_{\Omega_f} \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(1)} dV + \int_{\Omega_f} \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(0)} dV \\ &= \int_{\partial\Omega_f} \mathbf{u}^{(1)} \cdot \mathbf{n} dS + \nabla_{\mathbf{x}} \cdot \int_{\Omega_f} \mathbf{u}^{(0)} dV \\ &= \nabla_{\mathbf{x}} \cdot \int_{\Omega_f} \mathbf{u}^{(0)} dV \end{aligned}$$

since the surface integral is zero by (16.1) and periodicity. Let us define the phase-average velocity

$$\bar{\mathbf{u}} = \frac{1}{|\Omega_f|} \int_{\Omega_f} \mathbf{u}^{(0)} dV.$$

Then

$$\nabla_{\mathbf{x}} \cdot \bar{\mathbf{u}} = 0.$$

Integrating (16.6) gives

$$\bar{\mathbf{u}} = K \nabla p^{(0)}$$

where

$$K_{ij} = \frac{1}{|\Omega_f|} \int_{\Omega_f} w_i^j dV.$$

## 16.2 Reaction-diffusion

In multiple-scales form

$$\begin{aligned} \left( \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{X}} \right)^2 c &= \frac{\partial c}{\partial t} && \text{in } \Omega, \\ \mathbf{n} \cdot \left( \nabla_{\mathbf{x}} c + \frac{1}{\epsilon} \nabla_{\mathbf{X}} c \right) &= \epsilon \gamma c && \text{on } \partial\Omega. \end{aligned}$$

Expanding  $c = c_0 + \epsilon c_1 + \dots$ , and equating coefficients of powers of  $\epsilon$  gives, at leading order,

$$\nabla_{\mathbf{X}}^2 c_0 = 0 \quad \text{in } \Omega$$

with

$$\mathbf{n} \cdot \nabla_{\mathbf{X}} c_0 = 0 \quad \text{on } \partial\Omega.$$

The only solution is for  $c$  to be independent of the fast scale so that  $c = c_0(\mathbf{x}, t)$ . At next order

$$\begin{aligned}\nabla_{\mathbf{x}}^2 c_1 &= 0 && \text{in } \Omega, \\ \mathbf{n} \cdot (\nabla_{\mathbf{x}} c_0 + \nabla_{\mathbf{x}} c_1) &= 0 && \text{on } \partial\Omega.\end{aligned}$$

The solution may be written as

$$c_1 = \psi_i \frac{\partial c_0}{\partial x_i}$$

where

$$\begin{aligned}\nabla_{\mathbf{x}}^2 \psi_i &= 0 && \text{in } \Omega, \\ \mathbf{n} \cdot \nabla_{\mathbf{x}} \psi_i &= -n_i && \text{on } \partial\Omega,\end{aligned}$$

where  $\mathbf{n} = (n_1, n_2, n_3)$ . At next order

$$\nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{x}} c_2 + \nabla_x c_1) + \nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{x}} c_1 + \nabla_x c_0) = \frac{\partial c_0}{\partial t} \quad \text{in } \Omega, \quad (16.9)$$

$$\mathbf{n} \cdot (\nabla_{\mathbf{x}} c_1 + \nabla_{\mathbf{x}} c_2) = \gamma c_0 \quad \text{on } \partial\Omega. \quad (16.10)$$

Integrating (16.9) over  $\Omega$  gives (remember  $-\mathbf{n}$  is the outward normal to  $\Omega$ )

$$\begin{aligned}|\Omega| \frac{\partial c_0}{\partial t} &= \int_{\Omega} \nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{x}} c_2 + \nabla_x c_1) dV + \int_{\Omega} \nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{x}} c_1 + \nabla_x c_0) dV \\ &= - \int_{\partial\Omega} \mathbf{n} \cdot (\nabla_{\mathbf{x}} c_2 + \nabla_x c_1) dS + \nabla_{\mathbf{x}} \cdot \int_{\Omega} (\nabla_{\mathbf{x}} c_1 + \nabla_x c_0) dV \\ &= - \int_{\partial\Omega} \gamma c_0 dS + \frac{\partial}{\partial x_i} \int_{\Omega} \left( \frac{\partial \psi_j}{\partial X_i} \frac{\partial c_0}{\partial x_j} + \frac{\partial c_0}{\partial x_i} \right) dV \\ &= -|\partial\Omega| \gamma c_0 + \frac{\partial}{\partial x_i} \int_{\Omega} \left( \frac{\partial \psi_j}{\partial X_i} + \delta_{ij} \right) dV \frac{\partial c_0}{\partial x_j}\end{aligned}$$

Thus

$$\frac{\partial c_0}{\partial t} = -\frac{|\partial\Omega| \gamma}{|\Omega|} c_0 + \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial c_0}{\partial x_j} \right) = -\frac{|\partial\Omega| \gamma}{|\Omega|} c_0 + \nabla_{\mathbf{x}} \cdot (K \nabla_{\mathbf{x}} c_0),$$

where

$$K_{ij} = \frac{1}{|\Omega|} \int_{\Omega} \left( \frac{\partial \psi_j}{\partial X_i} + \delta_{ij} \right) dV.$$