(very brief intro to) Spectral methods

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Spectral methods

We have seen that FD and FEM can be understood as employing piecewise polynomial approximation:

- Finite difference: piecewise linear or low-degree polynomial
- Finite elements: piecewise (low-degree) polynomial

Can we take this idea further? In particular, we saw that

- Approximate $u \approx p$ by polynomial interpolation $u(x_i) = p(x_i)$, $i = 1, \ldots, \ell + 1$,
- $u'(x) \approx p'(x)$. Gives $O(h^{\ell-1})$ accuracy.

What if we do this everywhere? This leads to
Spectral methods: global (high-degree) polynomial

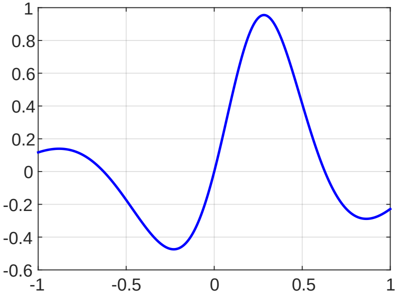
Polynomial approximation

Goal is given (linear ordinary)DE $Lu = f$,

1. Find polynomial $u(x) \approx p(x) = \sum_{i=0}^{n} c_i T_i(x)$.

1st step commonly done by **interpolation**:

$$f(x_i) = p(x_i), \quad i = 1, \ldots, n+1$$



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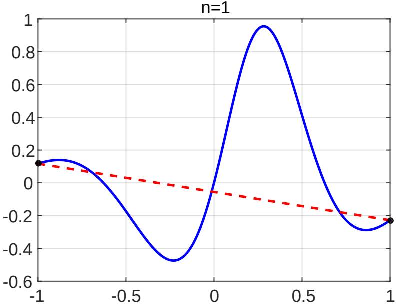

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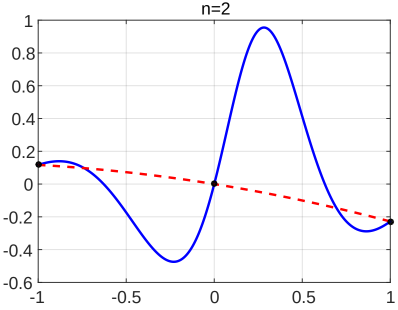

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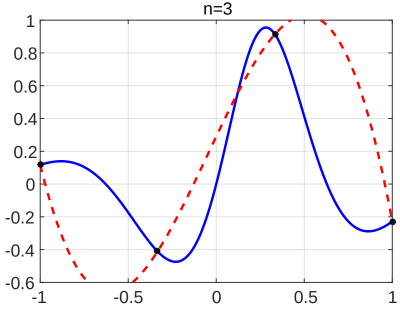

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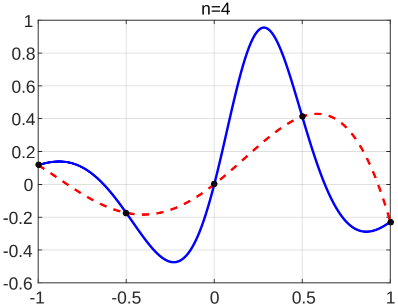

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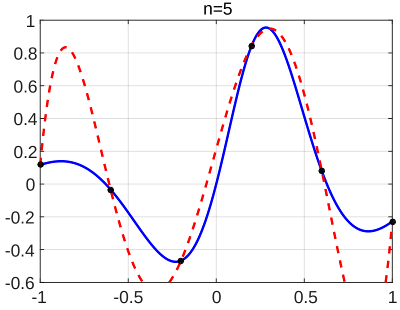

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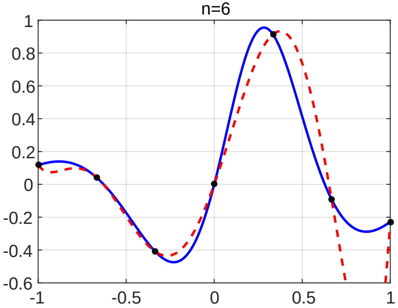

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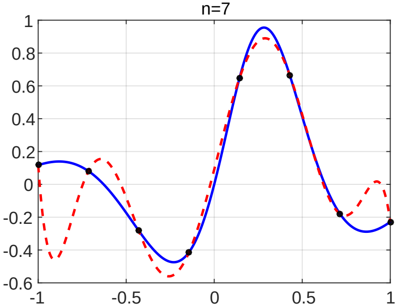

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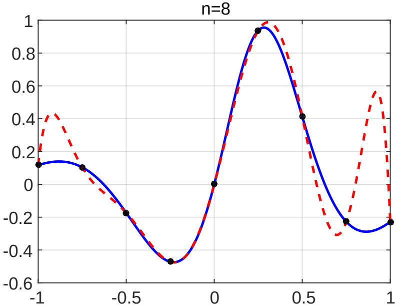

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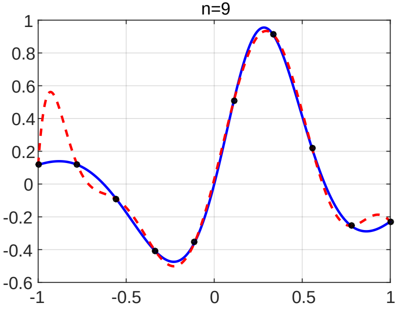

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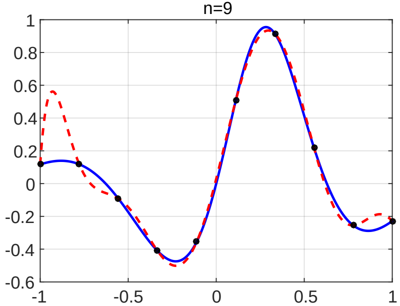

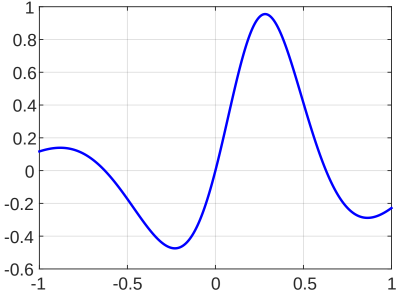

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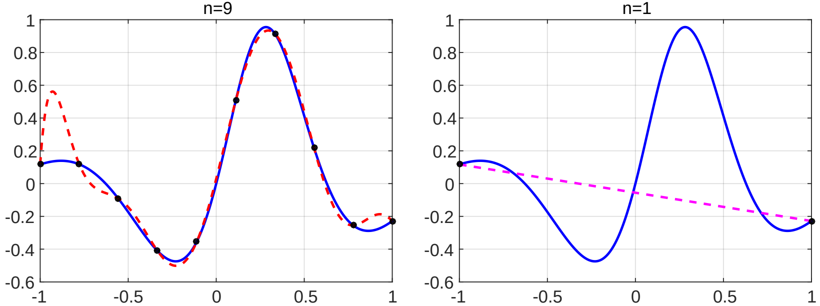

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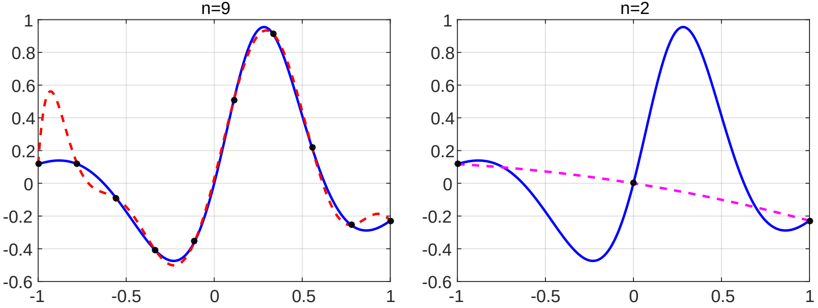

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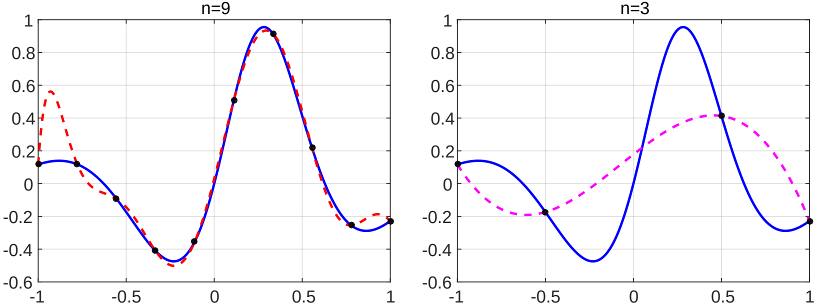

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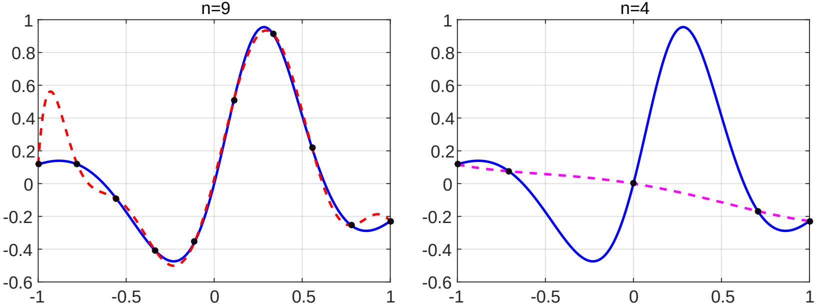

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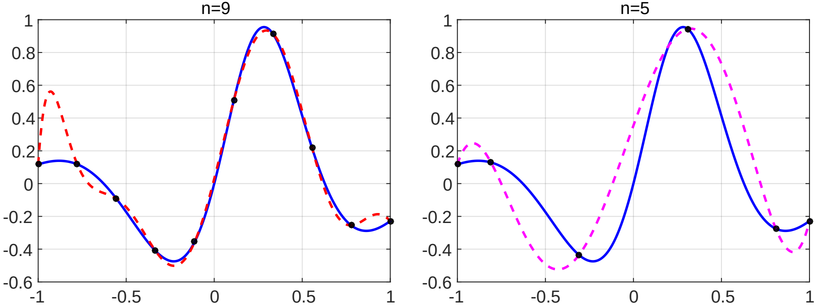

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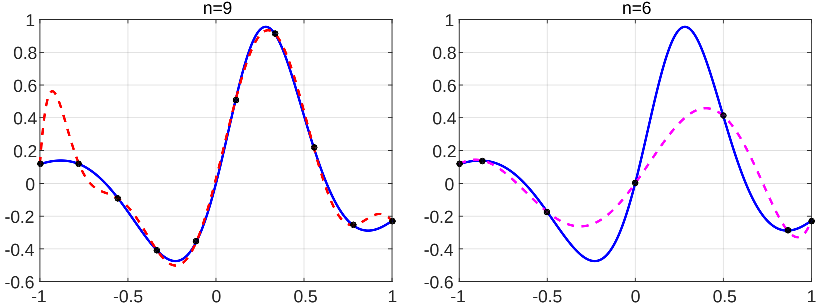

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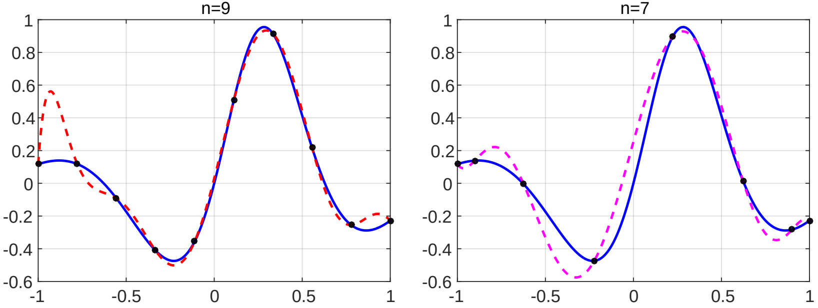

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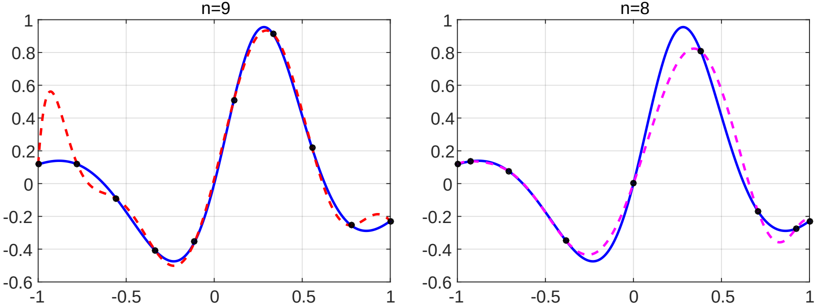

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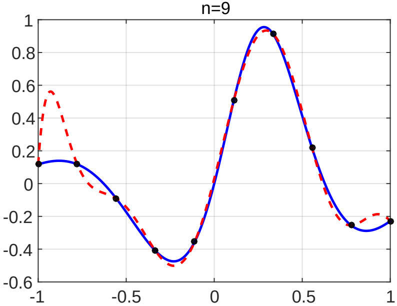

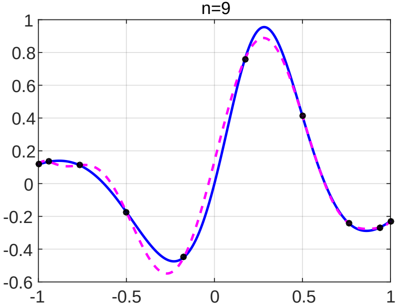

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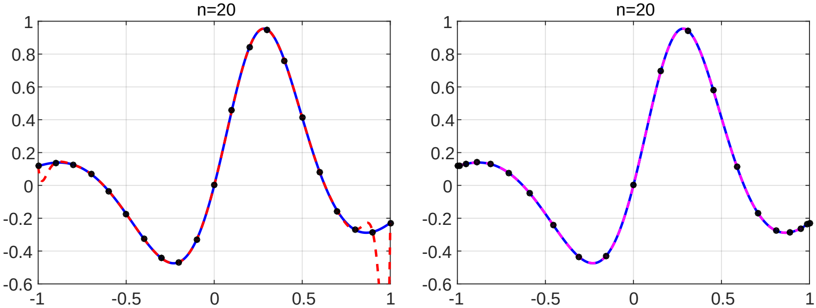

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Principles of spectral method

Recall in finite difference,

- use a (small number of) sample points and find low-degree polynomial interpolant $p(x_{i+\ell}) = u(x_{i+\ell})$, $\ell = -1, 0, 1$ (e.g.)
- use $p'(x_i)$ to approximate $u'(x_i)$

In spectral methods,

- use all $N+1$ sample points and find degree-N polynomial interpolant $p(x_j) = u(x_j)$, $j = 0, 1, \ldots, N$
- use $p'(x_i)$ to approximate $u'(x_i)$ for all i.

Spectral vs. FEM (, FD)

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Spectral methods converge very fast for smooth u: spectral accuracy

- if u analytic, **exponential** w.r.t. # points $O(\exp(-cn))$
- if u is k-times differentiable, $O(n^{-k})$ convergence

Brief comparison

- Spectral converges (much) faster when applicable
- Spectral methods (conventionally) lead to dense matrices
- Spectral requires orthogonal polynomial basis: often restriced to simple domains (rectangles, spheres, disks etc)
- FEM much more amenable to complicated domains
- 'best of both worlds': Spectral elements