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Finite Element Method for the Tricomi Equation

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Tricomi Equation

The Tricomi equation [1],

$$g(y)u_{xx} + u_{yy} = f, \tag{1}$$

arises when studying transonic flow in fluid mechanics. The function g is assumed to be

- smooth in the domain.
- monotonic increasing with y.
- of the same sign as y, that is, $\operatorname{sgn} g = \operatorname{sgn} y$.

The most common form is $g(y) = y$, which is analyzed in [2].

Tricomi Equation

We rewrite (1) as

$$Lu = f \tag{2}$$

within the domain Ω bounded by Γ_0, Γ_1, and Γ_2. Here Γ_0 is a curve in the upper half-plane connecting $(-1, 0)$ and $(1, 0)$, and Γ_1 and Γ_2 are the characteristics passing through $(-1, 0)$ and $(1, 0)$, respectively. The boundary value problem can now be stated as

$$Lu = f \qquad \text{in } \Omega, \tag{3}$$

$$u = 0 \qquad \text{on } \Gamma_0 \cup \Gamma_1. \tag{4}$$

Tricomi Domain

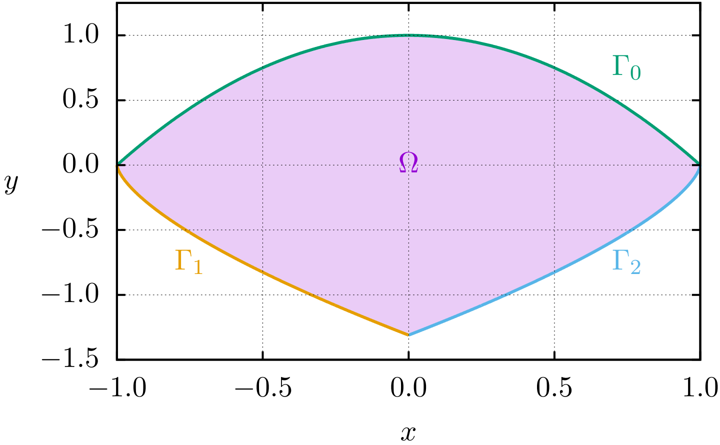

Figure 1: Example domain of $yu_{xx} + u_{yy} = 0$.

Weak Formulation

We define the function space, W, as

$$W = \{\phi \in C(\bar{\Omega}) \cap H^2(\Omega) \cap H^1(\partial\Omega) : \\ L\phi \in L^2(\Omega) \text{ and } \phi|_{\Gamma_0 \cup \Gamma_1} = 0\}.$$

Then the weak formulation can be phrased as: given $f \in L^2(\Omega)$, we wish to find $u \in W$ such that

$$(Lu, lv)_{L^2(\Omega)} = (f, lv)_{L^2(\Omega)} \tag{5}$$

for all $v \in W$. Here $l : W \to L^2(\Omega)$ is defined as

$$l\phi = \alpha_1 \phi_x + \alpha_2 \phi_y \tag{6}$$

for all $\phi \in W$, where α_1 and α_2 are known functions.

Finite Elements

We let $\{V_h\}$ be a family of finite dimensional subspaces of W, and define an approximate solution, $u_h \in V_h$, to be one where given $f \in L^2(\Omega)$

$$(Lu_h, lv_h)_{L^2(\Omega)} = (f, lv_h)_{L^2(\Omega)} \tag{7}$$

for all $v_h \in V_h$.
Let $\{\phi_j\}$ be a basis for V_h, then (7) can be cast into the linear system

$$\mathbf{Au} = \mathbf{b}, \tag{8}$$

where $A_{ij} = (L\phi_j, l\phi_i)_{L^2(\Omega)}$, $b_i = (f, l\phi_i)_{L^2(\Omega)}$, and $u_h = \sum_i u_i \phi_i$.

Lemma

The matrix $\mathbf{A}$ is invertible.

Proof.

Suppose there exists a vector $\mathbf{z}$ such that $\mathbf{Az} = \mathbf{0}$, and let $z_h = \sum_i z_i \phi_i$. Then

$$0 = \mathbf{z}^T \mathbf{Az}, \quad (9)$$

$$= (Lz_h, lz_h)_{L^2(\Omega)}, \quad (10)$$

$$\geq C\|z_h\|^2_{L^2(\Omega)} \quad (11)$$

by coercivity. Thus, $z_h \equiv 0$ and $\mathbf{z} = \mathbf{0}$, and so $\mathbf{A}$ is invertible. $\square$

Finite Elements

Theorem
There is a unique $u_h \in V_h$ satisfying the weak formulation

$$(Lu_h, lv_h)_{L^2(\Omega)} = (f, lv_h)_{L^2(\Omega)}. \tag{12}$$

Proof.
This follows immediately from the previous lemma since $\mathbf{A}$ has a non-zero determinant. $\square$

Error Analysis

Theorem

For $u \in W \cap H^s(\Omega)$ and with Lagrange elements of degree k where $k+1 \leq s$, there exists a positive constant, C, such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^{k-1}|u|_{H^{k+1}(\Omega)}. \tag{13}$$

Thus, the method will converge for $k \geq 2$ and $s \geq 3$. This error estimate requires quite a lot of smoothness, as is typical with mixed problems.

References

[1] A. K. Aziz, M. Schneider, and A. Werschulz, "A Finite Element Method for the Tricomi Problem," *Numerische Mathematik*, vol. 35, pp. 13–20, 1980.

[2] J. A. Trangenstein, "A Finite Element Method for the Tricomi Problem in the Elliptic Region," *SIAM Journal on Numerical Analysis*, vol. 14, no. 6, pp. 1066–1077, 1977.