Optimization Assignment

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Problem 1. The new system consist of

$$\sum_j^n a_{ij}x_j \le b_i, \qquad i \in M_0, \tag{1}$$

$$\sum_j^n (a_{ik}a_{lj} - a_{lk}a_{ij})x_j, \le a_{ik}b_l - a_{lk}b_i \qquad (i,l) \in M_+ \times M_-. \tag{2}$$

The coefficient of x_k in (1) and (2) is 0 since $i \in M_0$ and $a_{ik}a_{lk} - a_{lk}a_{ik} = 0$, respectively. Therefore, x_k is not in this new system.

By taking positive linear combinations of the original system we have

$$a_{ik}\left(\sum_j^n a_{lj}x_j\right) - a_{lk}\left(\sum_j^n a_{ij}x_j\right) \le a_{ik}(b_l) - a_{lk}(b_i),$$

because $a_{ik} > 0$ and $-a_{lk} > 0$ since $(i,l) \in M_+^k \times M_-^k$. We can now expand to yield (2). Note, from the original system we have

$$\begin{cases} a_{ik}x_k \le b_i - \sum\limits_{j \ne k}^n a_{ij}x_j & i \in M_+^k, \\ a_{lk}x_k \ge b_l - \sum\limits_{j \ne k}^n a_{lj}x_j & l \in M_-^k. \end{cases}$$

It must then be that

$$\frac{1}{a_{lk}}\left(b_l - \sum_{j \ne k}^n a_{lj}x_j\right) \le x_k \le \frac{1}{a_{ik}}\left(b_i - \sum_{j \ne k}^n a_{ij}x_j\right)$$

for all $(i,l) \in M_+^k \times M_-^k$. Therefore,

$$\max_{l \in M_-^k} \frac{1}{a_{lk}}\left(b_l - \sum_{j \ne k}^n a_{lj}x_j\right) \le x_k \le \min_{i \in M_+^k} \frac{1}{a_{ik}}\left(b_i - \sum_{j \ne k}^n a_{ij}x_j\right).$$

Problem 2.

i) If both systems had a solution, that would imply

$$\begin{aligned} \mathbf{0}^T &= yA, \\ 0 &= (yA)x, \\ &= y(Ax), \\ &\le yb, \\ &< 0, \end{aligned}$$

which is a contradiction. Thus, both systems cannot have solutions.

ii) There must exist a k_* such that $d_{k_*} < 0$, otherwise, our new system is in fact consistent.

Problem 3.

i) Both y_t and z_t are binary variables. If production occurs within period t, then $y_t = 1$. Furthermore, if production is switched on within period t, then $z_t = 1$.

ii) We can express the system in matrix form as

$$A\begin{pmatrix} y \\ z \end{pmatrix} \leq \begin{pmatrix} k \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} \mathbf{0}^T & \mathbf{1}^T \\ I - L & -I \\ -I & I \\ I & 0 \\ 0 & I \end{pmatrix}.$$

We wish to show that A is totally unimodular, to start, we notice that A is totally unimodular if and only if

$$\begin{pmatrix} \mathbf{0} & I - L^T & -I \\ \mathbf{1} & -I & I \end{pmatrix}$$

is totally unimodular, as this is the transpose of A after removing the identity in the lower portion.

Problem 4.

i) The constraint matrix is the vertex-edge incidence matrix, A. Therefore, each column contains exactly two 1s. We can then choose the partitions to be $M_1 = V_1$ and $M_2 = V_2$, and the constraint matrix is totally unimodular.

ii) The LP relaxation is

$$\begin{aligned} &\max_x \sum_{e \in E} x_e, \\ \text{s.t.} \quad &Ax \leq 1, \\ &x_e \geq 0. \end{aligned} \tag{3}$$

The dual is then given by

$$\begin{aligned} &\min_y \sum_{v \in V} y_v, \\ \text{s.t} \quad &A^T y \geq 1, \\ &y_v \geq 0. \end{aligned} \tag{4}$$

Since A^T is totally unimodular as well, the solution to the LP is identical to the IP. Additionally, for the minimum cardinality node covering, we do not require y_v to be greater than one, and so $y_v \in \{0, 1\}$.

iii) We can find trivial feasible solutions to (3) and (4) is the empty matching and the full graph. Then by the Strong Duality Theorem and since A is totally unimodular, König's Theorem holds.

Problem 5.

i) The LP relaxation of the binary knapsack problem is

$$\max_x \, c^T x, \tag{5}$$

$$\text{s.t.} \quad \begin{pmatrix} a^T \\ I \end{pmatrix} x \leq \begin{pmatrix} b \\ \mathbf{1} \end{pmatrix}. \tag{6}$$

With associated dual

$$\max_y \, (b, \; \mathbf{1}^T) y,$$
$$\text{s.t.} \quad \begin{pmatrix} a & I \end{pmatrix} y \geq c.$$

We find that (6) is an equality for the first r rows. Expanding the constraint yields

$$a_i y_1 + y_{i+1} \geq c_i. \tag{7}$$

Now, by complementary slackness, (7) must be active for $i \leq r$, and $y_i = 0$ for $i = r+1, \cdots, n+1$. Evaluating (7) at $i = r$ gives

$$y_1 = \frac{c_r}{a_r},$$

and so,

$$y_{i+1} = c_i - a_i \frac{c_r}{a_r}.$$

We can now show that the claimed solution is indeed optimal by showing the values of the objective functions are equal:

$$\begin{aligned}
(b, \; \mathbf{1}^T) y &= b\frac{c_r}{a_r} + \sum_j^{r-1} \left(c_j - a_j \frac{c_r}{a_r} \right), \\
&= b\frac{c_r}{a_r} + \sum_j^{r-1} c_j - \frac{c_r}{a_r} \sum_j^{r-1} a_j, \\
&= \frac{c_r}{a_r} \left(b - \sum_j^{r-1} a_j \right) + \sum_j^{r-1} c_j, \\
&= \sum_j^{r-1} c_j + c_r \frac{b - \sum_j^{r-1} a_j}{a_r}, \\
&= c^T x.
\end{aligned}$$

In the case that $x_r = 0$, our analysis remains the same, and the claimed solution is still optimal.

ii) We seek the solution to

$$\max_x \, (17, \, 10, \, 25, \, 17) x,$$
$$\text{s.t.} \quad (5, \, 3, \, 8, \, 7) x \leq 12,$$
$$x \in \mathbb{Z}_2^4,$$

using branch and bound, as depicted in Figure 1. As the system is well ordered already, we may use the result of the previous question to find the relaxed greedy solution $(1, \, 1, \, 1/2, \, 0)^T$. This gives us the upper bound $\bar{z} = \lfloor 39.5 \rfloor$, lower bound $\underline{z} = 27$, and incumbent $(1, \, 1, \, 0, \, 0)^T$. As $x_3 \in \mathbb{Q}$, we branch $\mathcal{F}$ into $\mathcal{F}_0$ with $x_3 = 0$, and $\mathcal{F}_1$ with $x_3 = 1$.

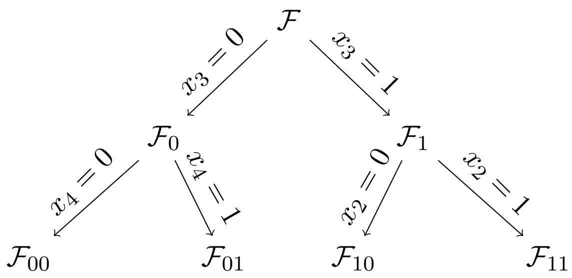

Figure 1: Binary tree of branch and bound.

Considering first $\mathcal{F}_1$, as we wish to maximize, we find the simplified problem

$$\max_x \,(17,\ 10,\ 17)\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix},$$

$$\text{s.t.} \quad (5,\ 3,\ 7)\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \leq 4.$$

Clearly, from the constraints $x_1 = 0$ and $x_4 = 0$ as they must be integers. This allows us to branch into $\mathcal{F}_{10}$ and $\mathcal{F}_{11}$ for $x_2 = 0$ and $x_2 = 1$, respectively. The node $\mathcal{F}_{10}$ contains a single feasible point which yields 25 as the objective, as this is less than $\underline{z}$, and therefore, may be pruned. Similarly, $\mathcal{F}_{11}$ gives the improved lower bound $\underline{z} = 35$ with incumbent .

Moving back to $\mathcal{F}_0$, we have the problem

$$\max_x \,(17,\ 10,\ 17)\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix},$$

$$\text{s.t.} \quad (5,\ 3,\ 7)\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \leq 12.$$

Solving the LP relaxation we find $(1,\ 1,\ 0,\ 4/7)^T$, and so we branch to $\mathcal{F}_{00}$ and $\mathcal{F}_{00}$ based on the value of x_4. In the case $x_4 = 0$ ($\mathcal{F}_{00}$), we find $(1,\ 1,\ 0,\ 0)^T$, which is a former incumbent, and can be pruned. On the other hand, $x_4 = 1$ and $\mathcal{F}_{01}$, we find $(1,\ 0,\ 0,\ 1)^T$ which produces an objective of 34. This branch may also be pruned as this is worse than our current incumbent.

Now all of the branches have been exhausted, and therefore, we have found the optimal solution of $(0,\ 1,\ 1,\ 0)^T$.

Problem 6.

i) We can show that $w \geq z$ since

$$\begin{aligned} w &\geq g(x) && \forall x \in \mathscr{R}, \\ &\geq g(x) && \forall x \in \mathscr{F}, \\ &\geq c^T x && \forall x \in \mathscr{F}, \\ &\geq z. \end{aligned}$$

ii) We can simplify the objective function in (GR) by

$$
\begin{aligned}
\frac{c_k}{a_k} b + \max_x \sum_{j \neq k} \left(c_j - \frac{c_k}{a_k} a_j \right) x_j &= \max_x \left(\frac{c_k}{a_k} b + \sum_{j \neq k} c_j x_j + c_k x_k - c_k x_j - \frac{c_k}{a_k} \sum_{j \neq k} a_j x_j \right), \\
&= \max_x \left(\frac{c_k}{a_k} \sum_j a_j x_j + \sum_j c_j x_j - c_k \left(\sum_{j \neq k} \frac{a_j}{a_k} x_j + \frac{a_k}{a_k} x_k \right) \right), \\
&= \max_x \left(\frac{c_k}{a_k} \sum_j a_j x_j + \sum_j c_j x_j - c_k \sum_j \frac{a_j}{a_k} x_j \right), \\
&= \max_x c^T x,
\end{aligned}
$$

to show that it is equivalent to the objective in (EKP). Additionally, any feasible solution to (EKP) must also be a feasible solution to (GR):

$$
\begin{aligned}
0 &= \sum_j a_j x_j - b, \\
&= \sum_{j \neq k} a_j x_j + a_k x_k - b, \\
&\equiv \sum_{j \neq k} a_j x_j - b \pmod{a_k},
\end{aligned}
$$

since $x \in \mathbb{Z}^n$.

iii) this is part 3 right

Problem 7. By assumption

$$
\begin{aligned}
a^T x^* &= \sum_{j \in \{i : x_i^* = 1\}} a_j, \\
&> r.
\end{aligned}
$$

Therefore, there exists a cover, C, such that

$$
\sum_{j \in C} a_j x_j \leq r
$$

is a cut for x^*. Since the algorithm first reorders a_j to be in descending order, it must be that it finds the minimal cover.

Problem 8. Consider the system

$$
\begin{aligned}
&\min_x \mathbf{1}^T x \\
\text{such that } &\begin{pmatrix} 6 & 1 \\ -3 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

By examining the first two rows, we find that

$$
\begin{aligned}
1 &\leq 3x_1, \\
2 &\leq 6x_1, \\
2 \leq 6x_1 &\leq 6x_1 + x_2 \leq 4.
\end{aligned}
$$

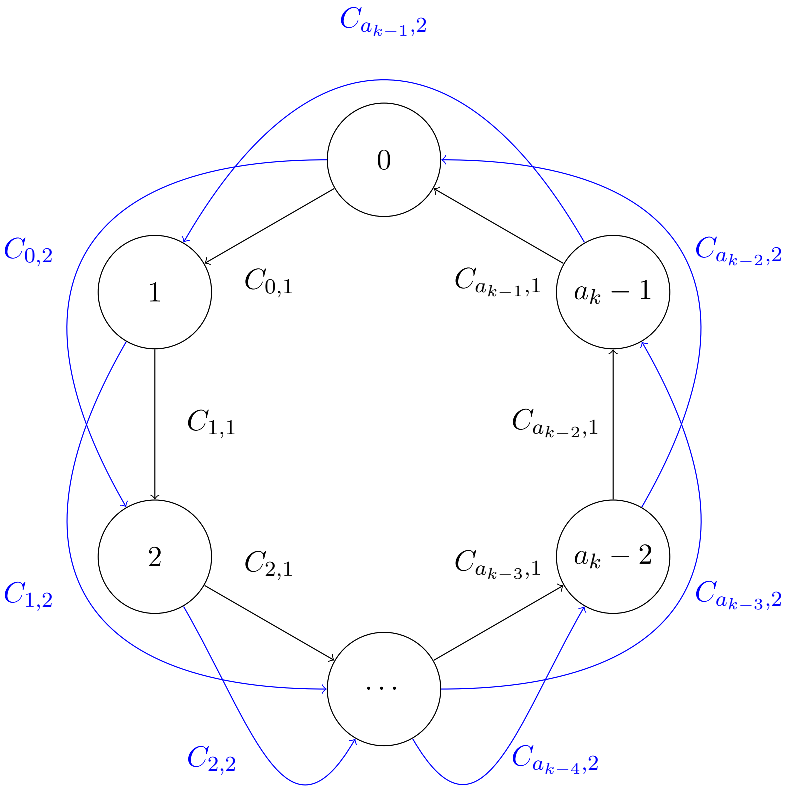

Figure 2: System (GR) depicted as a shortest path problem.

We can then simplify this to show

$$2 \leq 6x_1 \leq 4,$$
$$\frac{1}{3} \leq x_1 \leq \frac{2}{3}$$

which has no feasible solutions for $x_1 \in \mathbb{Z}$, and therefore the IP is infeasible.

Problem 9.

i) Take the cover, C_i, to be $\{j : 1 \leq j \leq n\}$. Then, by assumption,

$$\sum_j c_{ij} > b_i,$$

therefore, for x to be feasible there must be at least one 0 entry. Thus,

$$\sum_j x_{ij} \leq n - 1,$$
$$\leq |C_i| - 1,$$

as desired.

ii) If $x^*_{mj^*} > 1/2$ then GUB is not a good branching rule. As $\sum_i x_{ij} = 1$, then $p = m$. Thus, $\mathscr{F}_1 = \emptyset$ and $\mathscr{F}_2 = \mathscr{F}$. A better rule is follows:

- Reorder the elements such that $\tilde{x}_{i,j^*} \leq \tilde{x}_{i+1,j^*}$.
- Reorder the elements once again such that (without loss of generality)

$$\tilde{x}_{1,j^*}, \tilde{x}_{3,j^*}, \cdots, \tilde{x}_{m,j^*}, \tilde{x}_{(m-1),j^*}, \tilde{x}_{(m-3),j^*}, \cdots, \tilde{x}_{2,j^*}.$$

- Apply GUB on this new arrangement.