InFoMM – Optimisation
Lecture 3

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Michaelmas Term 2019

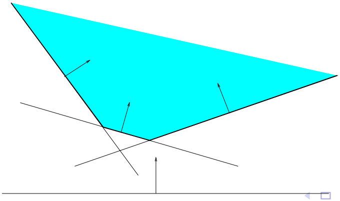
Polyhedra and Polytopes

Definition (Polyhedron)

A *polyhedron* is a set $\mathcal{P} \subset \mathbb{R}^n$ described as an intersection of finitely many affine half spaces

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \, (i = 1, \dots, m) \right\},$$

for some $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

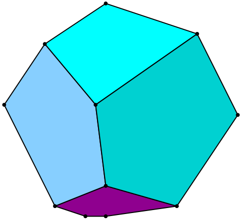


Definition (Polytope)

A *polytope* is a set $\mathcal{P}' \subset \mathbb{R}^n$ described as the convex hull of finitely many points

$$\mathcal{P}' = \operatorname{conv}\left\{x^k : \; k \in [1,p]\right\} := \left\{\sum_{k=1}^{p} \lambda_k x^k : \; \sum_{k=1}^{p} \lambda_k = 1, \; \lambda_k \geq 0, \; \forall k\right\}$$

for some $x^1, \ldots, x^p \in \mathbb{R}^n$.



Theorem (Weyl's Theorem)

If $\mathcal{P}'$ is a polytope, then $\mathcal{P}'$ is also a polyhedron.

Proof. $x \in \mathcal{P}'$ iff $\exists\, \lambda$ such that

$$x_j - \sum_{k=1}^{p} \lambda_k x_j^k = 0, \quad (j = 1, \dots, n)$$

$$\sum_{k=1}^{p} \lambda_k = 1,$$

$$\lambda_k \geq 0, \quad (k = 1, \dots, p),$$

where x_j and λ_k are variables but x_k^j are constants. Equivalently, the requirement can be written as a system of linear inequalities,

$$x_j - \sum_{k=1}^{p} \lambda_k x_j^k \leq 0, \quad (j = 1, \dots, n)$$

$$-x_j + \sum_{k=1}^{p} \lambda_k x_j^k \leq 0, \quad (j = 1, \dots, n)$$

$$\sum_{k=1}^{p} \lambda_k \leq 1,$$

$$-\sum_{k=1}^{p} \lambda_k \leq -1,$$

$$-\lambda_k \leq 0, \quad (k = 1, \dots, p).$$

Apply Fourier-Motzkin Elimination (see problem sheet) to eliminate all the variables λ_k.

This yields a new system of inequalities

$$\sum_{j=1}^{n} a_{ij}x_j + \sum_{k=1}^{N} 0 \cdot \lambda_k \leq b_i, \quad (i = 1, \dots, m)$$

for some $A = (a_{ij})$, b and m, each of which is a positive linear combination of inequalities of the original system.

By the properties of Fourier-Motzkin elimination, $\exists\, \lambda$ such that (x, λ) satisfy the orginal system iff x satisfies the new system, which now doesn't involve λ.

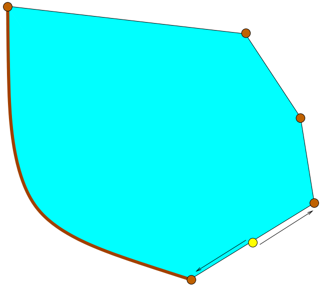
Hence,

$$\mathcal{P}' = \mathcal{P} = \left\{x \in \mathbb{R}^n : Ax \leq b\right\}.$$

Definition (Extreme points)

Let $C \subset \mathbb{R}^n$ be a convex set. A point $x \in C$ is an *extreme point* of C if x is not a convex combination of two points in C distinct from x

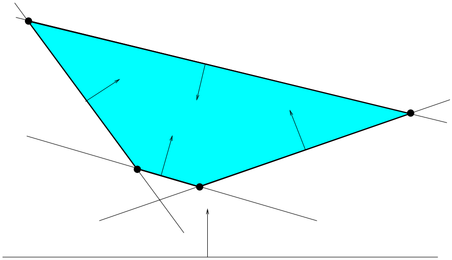
$$\nexists\, x_1, x_2 \in C \setminus \{x\}, \lambda \in (0,1) \text{ s.t. } x = \lambda x_1 + (1-\lambda)x_2.$$



Theorem (Minkowski's Theorem)

If $\mathcal{P} \subset \mathbb{R}^n$ is a polyhedron and bounded, then $\mathcal{P}$ is a polytope, that is, $\mathcal{P}$ has a finite set X of extreme points, and $\mathcal{P} = \mathrm{conv}(X)$.

(Proof uses the Theorem of Alternatives for Linear Inequalities.)



Formulations of IP Problems

Let us now consider the mixed integer programming problem

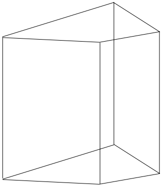
$$(\text{MIP}) \qquad \max_{(x,y)\in\mathbb{R}^{n+p}} \; c_x^{\mathrm{T}} x + c_y^{\mathrm{T}} y$$
$$\text{s.t.} \qquad Ax + By \leq b$$
$$x \in \mathbb{Z}^n,$$

where A and B are matrices, and let us denote the set of feasible solutions of (MIP) by

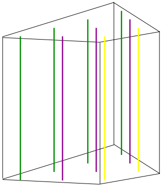
$$\mathscr{F} := \left\{(x, y) \in \mathbb{R}^{n+p} : \; Ax + By \leq b, \, x \in \mathbb{Z}^n\right\}.$$

If we drop the integrality constraints $x \in \mathbb{Z}^n$, the set of points that satisfy the remaining constraints is a polyhedron:

$$\mathcal{P} := \left\{(x, y) \in \mathbb{R}^{n+p} : Ax + By \leq b\right\}.$$



Furthermore, we have $\mathscr{F} = \mathcal{P} \cap (\mathbb{Z}^n \times \mathbb{R}^p)$.

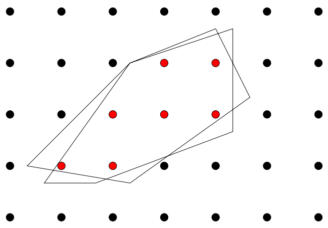


Definition (Formulation of a MIP-feasible set)

A polyhedron $\mathcal{P} \subset \mathbb{R}^{n+p}$ is called a *formulation* of a set $\mathscr{F} \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if

$$\mathcal{P} \cap \left(\mathbb{Z}^n \times \mathbb{R}^p\right) = \mathscr{F}.$$

A formulation of Problem (MIP) is any formulation of its feasible set $\mathscr{F}$.



Example (Alternative formulations)

Consider the polyhedra

$$\mathcal{P}_1 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1,\ 83x_1 + 61x_2 + 49x_3 + 20x_4 \leq 100\},$$
$$\mathcal{P}_2 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1,\ 4x_1 + 3x_2 + 2x_3 + x_4 \leq 4\},$$
$$\mathcal{P}_3 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1,\ 4x_1 + 3x_2 + 2x_3 + x_4 \leq 4,\ x_1 + x_2 + x_3 \leq 1,\ x_1 + x_4 \leq 1\}.$$

Then the IPs

$$(\text{IP}_i) \quad \max_{x \in \mathbb{Z}^4} c^{\mathrm{T}} x \quad \text{s.t. } x \in \mathcal{P}_i$$

all describe the same mathematical optimisation problem for $i = 1, 2, 3$, because all three polyhedra are formulations of the same feasible set

$$\mathscr{F} = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,1,0,1), (0,0,1,1)\}$$

Algorithmically, however, Formulation 3 is easier to solve than Formulation 2, and the latter is easier to solve than Formulation 1, because $\mathcal{P}_3 \subset \mathcal{P}_2 \subset \mathcal{P}_1$.

Let us verify these claims:

If $x \in \mathcal{P}_{\mathbf{1}} \cap \{0,1\}^{\mathbf{4}}$, then dividing $83x_{\mathbf{1}} + 61x_{\mathbf{2}} + 49x_{\mathbf{3}} + 20x_{\mathbf{4}} \leq 100$ by $20\frac{\mathbf{1}}{\mathbf{3}}$ yields

$$4.08x_{\mathbf{1}} + 3x_{\mathbf{2}} + 2.41x_{\mathbf{3}} + 0.98x_{\mathbf{4}} \leq 4.92,$$

and since $x_{\mathbf{4}} \leq 1$, adding $0.02x_{\mathbf{4}}$ yields

$$4.08x_{\mathbf{1}} + 3x_{\mathbf{2}} + 2.41x_{\mathbf{3}} + x_{\mathbf{4}} \leq 4.94.$$

Since all $x_i \geq 0$, rounding down the coefficients in the l.h.s. can only render the inequality more satisfied,

$$4x_{\mathbf{1}} + 3x_{\mathbf{2}} + 2x_{\mathbf{3}} + x_{\mathbf{4}} \leq 4.94.$$

Seen as all x_i are integers, the l.h.s must be integer, hence we can round down the r.h.s.,

$$4x_{\mathbf{1}} + 3x_{\mathbf{2}} + 2x_{\mathbf{3}} + x_{\mathbf{4}} \leq 4.$$

This shows $\mathcal{P}_{\mathbf{1}} \cap \{0,1\}^{\mathbf{4}} \subseteq \mathcal{P}_{\mathbf{2}} \cap \{0,1\}^{\mathbf{4}}$.

Moreover, $x \in \mathcal{P}_{\mathbf{2}}$ implies (multiplying $4x_{\mathbf{1}} + 3x_{\mathbf{2}} + 2x_{\mathbf{3}} + x_{\mathbf{4}} \leq 4$ by 25),

$$100x_{\mathbf{1}} + 75x_{\mathbf{2}} + 50x_{\mathbf{3}} + 25x_{\mathbf{4}} \leq 100,$$

and using $x_i \geq 0$,

$$83x_{\mathbf{1}} + 61x_{\mathbf{2}} + 49x_{\mathbf{3}} + 20x_{\mathbf{4}} \leq 100,$$

whence $x \in \mathcal{P}_{\mathbf{1}}$.

Therefore, $\mathcal{P}_2 \subset \mathcal{P}_1$ and $\mathcal{P}_2 \cap \{0,1\}^4 \subseteq \mathcal{P}_1 \cap \{0,1\}^4$, which implies

$$\mathcal{P}_1 \cap \{0,1\}^4 = \mathcal{P}_2 \cap \{0,1\}^4.$$

Next, If $x \in \mathcal{P}_2 \cap \{0,1\}^4$, then $4x_1 + 3x_2 + 2x_3 + x_4 \leq 4$, thus at most one of x_1, x_2, x_3 can be equal to 1, and at most one of x_1, x_4 can be equal to 1. Therefore,

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 1, \\ x_1 + x_4 &\leq 1, \end{aligned}$$

so that $x \in \mathcal{P}_3$. This shows that $\mathcal{P}_2 \cap \{0,1\}^4 \subseteq \mathcal{P}_3 \cap \{0,1\}^4$.

It is also immediate that $\mathcal{P}_3 \subset \mathcal{P}_2$, and hence, $\mathcal{P}_3 \cap \{0,1\}^4 \subseteq \mathcal{P}_2 \cap \{0,1\}^4$. Therefore,

$$\mathcal{P}_2 \cap \{0,1\}^4 = \mathcal{P}_3 \cap \{0,1\}^4.$$

Definition (Tigher formulation)

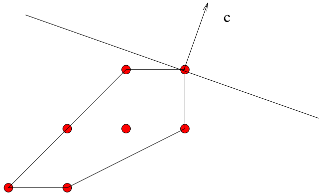
If two formulations $\mathcal{P}_1$, $\mathcal{P}_2$ of the feasible set $\mathscr{F}$ of an IP or MIP satisfy $\mathcal{P}_2 \subset \mathcal{P}_1$, we say that $\mathcal{P}_2$ is a *tighter* formulation than $\mathcal{P}_1$.

Definition (Ideal formulation)

If a formulation $\mathcal{P}$ of the feasible set $\mathscr{F}$ satisfies $\mathcal{P} = \mathrm{conv}(\mathscr{F})$, then $\mathcal{P}$ is called an *ideal* formulation.

Example (Ideal formulation)

Formulation P_3 of the previous example is an ideal formulation of $\mathscr{F}$.



Theorem (Ideal formulations are solved by LP relaxation)

Let

$$\begin{aligned}(IP)\quad z = \max_x \ & c^{\mathrm{T}} x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \\ & x_j \in \mathbb{Z}, \quad (j = 1, \dots, n)\end{aligned}$$

be an IP given with an ideal formulation $\mathcal{P} = \{x : Ax = b, x \geq 0\}$, *and let* x^* *be an optimal basic solution (one found by application of the simplex algorithm) of the LP relaxation*

$$\begin{aligned}(LP)\quad w = \max_x \ & c^{\mathrm{T}} x \\ \text{s.t. } & Ax = b \\ & x \geq 0.\end{aligned}$$

Then x^* *is also an optimal solution of (IP).*

Proof. Since (LP) is a relaxation of (IP), we already know that $z \leq w$, so all we need to establish is that x^* is feasible for (IP), that is, it takes integer values, and optimality follows automatically.

Since by assumption $\mathcal{P} = \text{conv}(\mathscr{F})$, each extreme point $\tilde{x}$ of $\mathcal{P}$ is in $\mathscr{F}$, and thus $\tilde{x}$ is integer valued. It therefore suffices to prove that x^* is an extreme point of $\mathcal{P}$.

Since x^* is an optimal basic solution, it is a basic feasible solution, and it thus suffices to prove that any basic feasilbe solution $\tilde{x}$ of $\mathcal{P}$ is an extreme point.

Let $\tilde{x}$ be a basic feasible solution with basis B and non-basis N, that is, $\tilde{x}_B = A_B^{-1} b$ and $\tilde{x}_N = 0$. Suppose that

$$\tilde{x} = \lambda x^{\mathbf{1}} + (1 - \lambda) x^{\mathbf{2}} \tag{1}$$

for some $x^{\mathbf{1}}, x^{\mathbf{2}} \in \mathcal{P}$ and $\lambda \in (0, 1)$.

We have $x_N^{\mathbf{1}}, x_N^{\mathbf{2}} \geq 0$ and $0 = \tilde{x}_N = \lambda x_N^{\mathbf{1}} + (1 - \lambda) x_N^{\mathbf{2}}$, whence $x_N^{\mathbf{1}} = x_N^{\mathbf{2}} = 0$. It follows that $b = Ax^i = A_B x_B^i$, so that $x_B^i = A_B^{-1} b = \tilde{x}_B$ for $(i = 1, 2)$.

We have shown that $x^{\mathbf{1}} = x^{\mathbf{2}} = \tilde{x}$ for all representations of the form (1), which shows that $\tilde{x}$ is an extreme point.

Totally Unimodular Matrices

We consider the IP

$$\text{(IP)} \quad \max_x \{c^{\mathsf{T}}x : Ax = b,\ x \geq 0,\ x \in \mathbb{Z}^n\}$$

and ask the question whether we have a chance to recognise if $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ is an ideal formulation, in which case we are in the fortuitous situation that (IP) is solved by its LP relaxation.

While this is difficult to decide in general, there exists at least an important family of cases where this is possible:

Definition (Totally unimodular matrix)

A matrix $A \in \mathbb{R}^{m \times n}$ is called *totally unimodular* if every square non-singular submatrix of A has determinant ± 1.

Lemma (Extreme point implies basic feasible)

Each extreme point of $\mathcal{P}$ is a basic feasible solution.

Proof. See problem sheet.

Theorem (Total unimodularity implies integrality)

If the constraint matrix A of problem (IP) is totally unimodular and b is integer valued, then every extreme point of the formulation $\mathcal{P}$ is integer valued.

Proof. W.l.o.g., we assume that the rows of A are linearly independent. By the Lemma, every extreme point $\tilde{x}$ is a basic feasible solution, and thus there exists a basis $B = \{j_{\mathbf{1}}, \ldots, j_m\}$ and non-basis N such that $\tilde{x}_N = 0$ and

$$A_B \tilde{x}_B = b.$$

The nonbasic components $\tilde{x}_N$ are clearly integer valued, and using Cramer's Rule, we have

$$\tilde{x}_{j_k} = \frac{\det(A_B^k)}{\det(A_B)},$$

where $A_B^k = \left[A_{j_{\mathbf{1}}} \; \cdots \; A_{j_{k-\mathbf{1}}} \; b \; A_{j_{k+\mathbf{1}}} \; \cdots \; A_{j_m} \right]$.

Since A_B^k has integer components, we have $\det(A_B^k) \in \mathbb{Z}$, and since $\det(A_B) = \pm 1$, it follows that $\tilde{x}_B$ is integer valued.