Optimization Assignment

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16 December 2019

Problem 1. The new system consist of

$$\sum_{j}^{n} a_{ij}x_j \leq b_i, \qquad\qquad i \in M_0, \tag{1}$$

$$\sum_{j}^{n} (a_{ik}a_{lj} - a_{lk}a_{ij})x_j, \leq a_{ik}b_l - a_{lk}b_i \qquad\qquad (i,l) \in M_+ \times M_-. \tag{2}$$

The coefficient of x_k in (1) and (2) is 0 since $i \in M_0$ and $a_{ik}a_{lk} - a_{lk}a_{ik} = 0$, respectively. Therefore, x_k is not in this new system.

By taking positive linear combinations of the original system we have

$$a_{ik}\left(\sum_{j}^{n} a_{lj}x_j\right) - a_{lk}\left(\sum_{j}^{n} a_{ij}x_j\right) \leq a_{ik}(b_l) - a_{lk}(b_i), \tag{3}$$

because $a_{ik} > 0$ and $-a_{lk} > 0$ since $(i,l) \in M_+^k \times M_-^k$. We can now expand to yield (2).

Note, from the original system we have

$$\begin{cases} a_{ik}x_k \leq b_i - \sum_{j\neq k}^{n} a_{ij}x_j & i \in M_+^k, \\ a_{lk}x_k \geq b_l - \sum_{j\neq k}^{n} a_{lj}x_j & l \in M_-^k. \end{cases} \tag{4}$$

It must then be that

$$\frac{1}{a_{lk}}\left(b_l - \sum_{j\neq k}^{n} a_{lj}x_j\right) \leq x_k \leq \frac{1}{a_{ik}}\left(b_i - \sum_{j\neq k}^{n} a_{ij}x_j\right) \tag{5}$$

for all $(i,l) \in M_+^k \times M_-^k$. Therefore,

$$\max_{l\in M_-^k} \frac{1}{a_{lk}}\left(b_l - \sum_{j\neq k}^{n} a_{lj}x_j\right) \leq x_k \leq \min_{i\in M_+^k} \frac{1}{a_{ik}}\left(b_i - \sum_{j\neq k}^{n} a_{ij}x_j\right). \tag{6}$$

Problem 2.

i) If both systems had a solution, that would imply

$$\begin{aligned} \mathbf{0}^T &= yA, \\ 0 &= (yA)x, \\ &= y(Ax), \\ &\leq yb, \\ &< 0, \end{aligned}$$

which is a contradiction. Thus, both systems cannot have solutions.

ii) There must exist a k_* such that $d_{k_*} < 0$, otherwise, our new system is in fact consistent.

Problem 3.

i) Both y_t and z_t are binary variables. If production occurs within period t, then $y_t = 1$. Furthermore, if production is switched on within period t, then $z_t = 1$.

ii)

$$\begin{pmatrix} \mathbf{0}^T & \mathbf{1}^T \\ I-L & -I \\ -I & I \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \leq \begin{pmatrix} k \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} \mathbf{0} & I-L^T & -I \\ \mathbf{1} & -I & I \end{pmatrix} \tag{8}$$

Problem 4.

i) The constraint matrix is the vertex-edge incidence matrix, A. Therefore, each column contains exactly two 1s. We can then choose the partitions to be $M_1 = V_1$ and $M_2 = V_2$, and the constraint matrix is totally unimodular.

ii) The LP relaxation is

$$\begin{aligned} \max_x \sum_{e \in E} x_e&, \\ \text{s.t.} \quad Ax &\leq 1, \\ x_e &\geq 0. \end{aligned} \tag{9}$$

The dual is then given by

$$\begin{aligned} \min_y \sum_{v \in V} y_v&, \\ \text{s.t} \quad A^T y &\geq 1, \\ y_v &\geq 0. \end{aligned} \tag{10}$$

Since A^T is totally unimodular as well, the solution to the LP is identical to the IP. Additionally, for the minimum cardinality node covering, we do not require y_v to be greater than one, and so $y_v \in \{0, 1\}$.

iii) We can find trivial feasible solutions to (9) and (10) is the empty matching and the full graph. Then by the Strong Duality Theorem and since A is totally unimodular, König's Theorem holds.

Problem 5.
Problem 6.
Problem 7.
Problem 8.
Problem 9.