Continuous Optimization
InFoMM Assignment

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We explore numerically solving unconstrained and constrained optimization problems with the solver KNITRO [1] and the Julia interface [2, 3]. First we examine the Rosenbrock function, the well-known test problem. We then investigate the discrete catenary problem and how the solution relates to the continuous case.

1 Rosenbrock Function

We begin by using KNITRO to solve a well-known test problem in optimization—minimizing the 2-d Rosenbrock function:

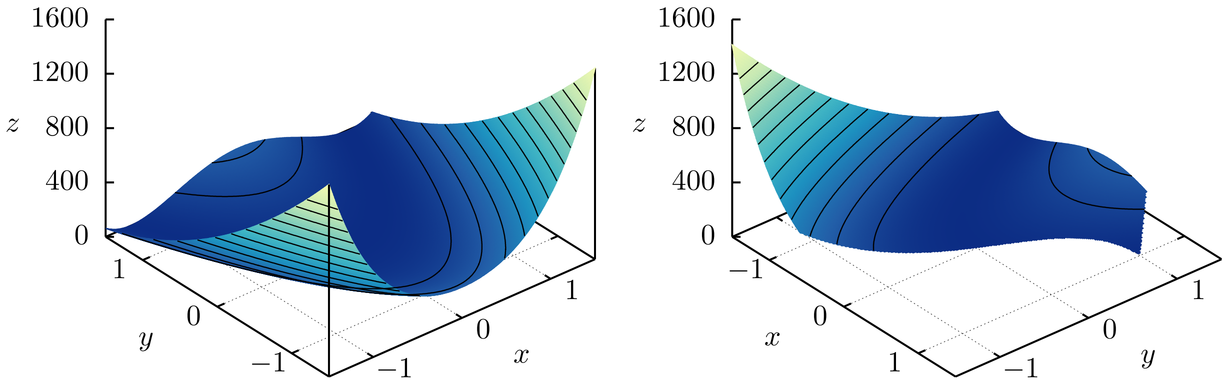
$$\min_{x,y\in\mathbb{R}} 100(y-x^2)^2+(1-x)^2. \tag{1}$$

This function is useful to test minimization algorithms because it has a unique minimum, $(1,1)$, but has a very small gradient within a valley. The Rosenbrock function is plotted in Figure 1a. Additionally, we impose the constraints

$$\begin{aligned}(x-1)^2+y&\geq 1,\\ x+y&\leq 2,\end{aligned} \tag{2}$$

to further test KNITRO. With these constraints the minimum $(1,1)$ is still feasible, but lays on the boundary (Figure 1b).

Both the unconstrained and constrained minimization problems are solved with KNITRO for a variety of initial points and a range of derivative information given. The number of iterations required to find the solution $(1,1)$ are shown in Table 1. Generally, providing the analytic gradient does not reduce the number of iterations required to converge—especially for the starting values of $(0,0)$ and $(2,4)$ where the solver does not converge at all. It is suspected this is because these points lay on the line $y=x^2$, and so one of the partial derivatives vanish. However, by providing the Hessian



(a) Unconstrained.

(b) Constrained by (2).

Figure 1: Rosenbrock function given by (1).

Starting point	$(-2,-3)$	$(2,1)$	$(-3,4)$	$(\pi,-e)$	$(0,0)$	$(2,4)$
Unconstrained						
No derivatives	15	43	28	35	22	35
Gradient	15	41	28	35	DNC[1]	DNC
Hessian	25	13	28	20	DNC	DNC
Constrained						
No derivatives	36	16	40	22	17	18
Gradient	36	16	41	22	DNC	DNC
Hessian	20	12	23	12	DNC	DNC

[1]Did not converge.

Table 1: Iterations needed to converge to the minimum of the constrained and unconstrained Rosenbrock function.

the number of iterations decreases substantially most of the time. Interestingly, the unconstrained problem requires more iterations than the constrained one.

With an understanding of how to use KNITRO, we move on to a more exciting problem.

2 Catenary Problem

We now turn our attention to the catenary problem. The catenary problem considers the shape of a freely hanging rope with fixed endpoints neglecting bending stiffness. The solution is the shape that minimizes the gravitational potential energy, and thus this problem is a staple in calculus of variations and mechanics courses. In this section we focus our efforts on the discrete analogue of this problem. We instead consider a series of n rigid beams attached together. By assuming the gravitational potential energy is focused entirely at the centre of each beam, we obtain the expression

$$U = mg\left(\frac{1}{2}y_0 + \frac{1}{2}y_n + \sum_{i=1}^{n-1} y_i\right) \tag{3}$$

for the total potential energy, as the midpoint is the average of the endpoints of each beam. Additionally, there are multiple constraints of the system. We assume both endpoints are fixed at the same height, and that each beam is the same length, L. Combining these constraints with the expression for the potential energy yields the optimization problem

$$\min_{\mathbf{x},\mathbf{y}\in\mathbb{R}^{n+1}} mg\left(\frac{1}{2}y_0 + \frac{1}{2}y_n + \sum_{i=1}^{n-1} y_i\right), \tag{4}$$

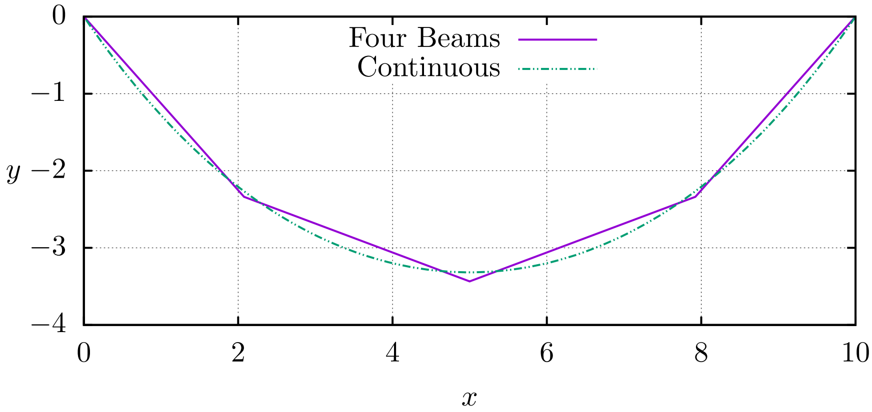
$$\begin{aligned} x_0 &= 0, & x_n &= \gamma n L, \\ y_0 &= 0, & y_n &= 0, \\ & (x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2 = L^2, \end{aligned} \tag{5}$$

where γ is the ratio of the span to the total length of the beams. While solving the problem we use the the following numerical values:

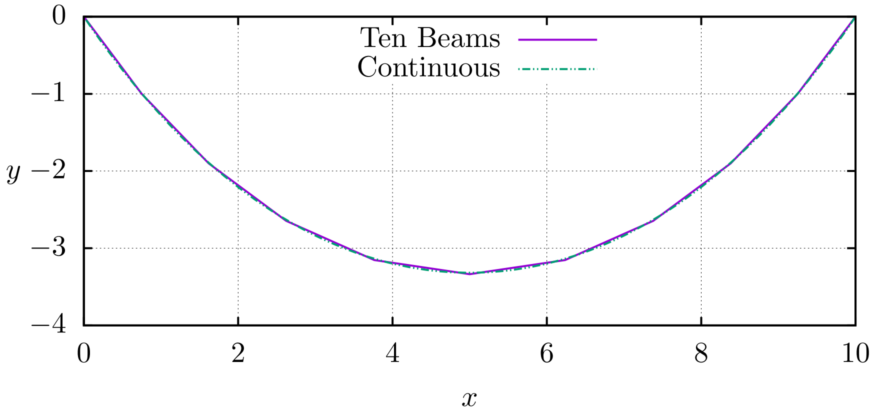
$$x_n = 10, \qquad g = 1, \qquad m = 1, \qquad \gamma = \frac{4}{5}, \qquad L = \frac{x_n}{\gamma n}. \tag{6}$$

The solutions from KNITRO for two values of n can be found in Figure 2. We shall compare our results to the solution of the continuous case, given by

$$f(x) = a\left(\cosh\left(\frac{2x - x_n}{2a}\right) - \cosh\left(\frac{x_n}{2a}\right)\right), \tag{7}$$



(a) $n = 4$.



(b) $n = 10$.

Figure 2: Solution of the discrete catenary problem.

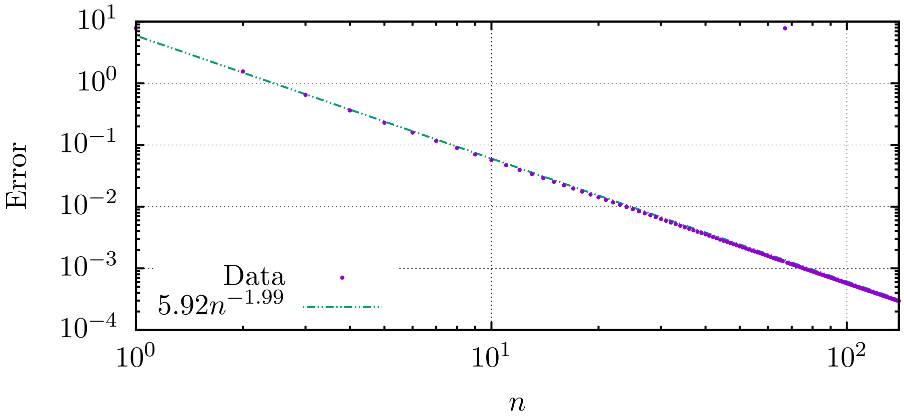


Figure 3: The L^2 norm of the difference between the discrete solution and the continuous solution.

where a is the positive solution to

$$\frac{x_n}{2a} = \gamma \sinh\left(\frac{x_n}{2a}\right). \tag{8}$$

We see that for $n = 4$ (Figure 2a) the discrete solution closely approximates the continuous solution. Moreover, it is evident from $n = 10$ (Figure 2b) that as $n \to \infty$ the continuous solution will be obtained. This observation is justified by examining the L^2 norm of the difference of the two solutions for a range of number of beams—the result of this can be seen in Figure 3. We find that the discrete solution converges to the continuous solution approximately quadratically. However, when $n \gtrsim 140$ KNITRO is no longer able to solve the optimization problem since the number of variables becomes too large.

3 Conclusion

We investigated using the solver KNITRO to solve both unconstrained and constrained optimization problems. We first explored the number of iterations required to converge to the minimum of the Rosenbrock function. We then solved the discrete catenary problem for a wide range of number of beams. From this we compared the discrete solutions to the continuous solution and found that the discrete solution converged quadratically.

References

[1] Artelys, "KNITRO." Online, 2020. `artelys.com/solvers/knitro/`.

[2] JuliaOpt, "KNITRO.jl Documentation." Online, 2020. `juliaopt.org/KNITRO.jl/latest/`.

[3] JuliaOpt, "KNTIRO.jl," *GitHub Repository*, Last commit February 2020. `github.com/JuliaOpt/KNITRO.jl`.