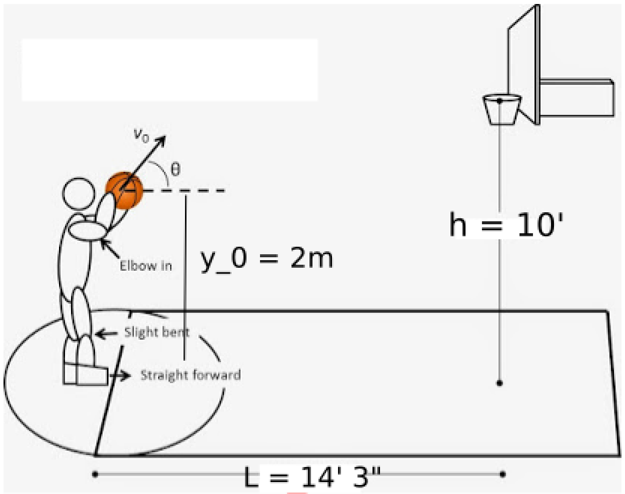

Figure 1: Diagram of a free throw.

1.5 Scoring a Free Throw

Starting from $F = ma$ we have

$$m\ddot{y} = -gm, \qquad y(0) = y_0, \qquad y(t_{crit}) = h, \qquad \dot{y} = v_0 \sin\theta$$

in the y direction, and

$$m\ddot{x} = 0, \qquad x(0) = 0, \qquad x(t_{crit}) = L, \qquad \dot{x} = v_0 \cos\theta$$

in the x direction. Integrating these expressions yields

$$\begin{aligned} y &= -\frac{1}{2}gt^2 + v_0 \sin\theta t + y_0, \\ x &= v_0 \cos\theta t. \end{aligned} \tag{1}$$

Since $x(t_{crit}) = L$, it must be by (1) that

$$t_{crit} = \frac{L}{v_0 \cos\theta}.$$

From this expression and the fact that $y(t_{crit}) = h$ we obtain

$$h - y_0 = -\frac{1}{2}g\left(\frac{L}{v_0\cos\theta}\right)^2 + v_0 \sin\theta \frac{L}{v_0\cos\theta} + y_0,$$

or,

$$v_0 = \left(\frac{2\cos^2\theta}{gL^2}\left(y_0 - h + L\tan\theta\right)\right)^{-1/2}.$$

Additionally, as the ball enters the hoop $\dot{y} < 0$, and therefore,

$$\begin{aligned}\dot{y}(t_{crit}) &< 0\\ -g\frac{L}{v_0\cos\theta} + v_0\sin\theta &< 0\\ v_0^2\sin(2\theta) &< 2gL\end{aligned}$$

Now, in order to maximize the likelyhood of scoring the free throw we wish to minimize the final velocity. We shall accomplish this by first finding an expression for v_f:

$$\begin{aligned}v_f^2 &= \dot{x}^2 + \dot{y}^2,\\ &= \left(-gt_{crit} + v_0\sin\theta\right)^2 + \left(v_0\cos\theta\right)^2,\\ &= \frac{g^2L^2}{v_0^2\cos^2\theta} + v_0^2 - 2gL\tan\theta,\\ &= 2g\left(y_0 - h + L\tan\theta\right) + \frac{gL^2}{2\cos^2\theta}\frac{1}{y_0 - h + L\tan\theta} - 2gL\tan\theta,\\ &= 2g\left(y_0 - h\right) + \frac{gL^2}{2\cos^2\theta}\frac{1}{y_0 - h + L\tan\theta}.\end{aligned}$$

We can now proceed by differentiating with respect to θ to yield

$$2v_f\frac{dv_f}{d\theta} = \frac{gL^2}{2}\left[2\left(\cos\theta\right)^{-3}\sin\theta\left(y_0 - h + L\tan\theta\right)^{-1} + \left(\cos\theta\right)^{-2}\left(-\left(y_0 - h + L\tan\theta\right)^{-2}\right)L\sec^2\theta\right].$$

Which we will now set to 0 and simplify to find[1]

$$\begin{aligned}0 &= \frac{gL^2}{2}\left[\frac{2\sin\theta}{\left(\cos\theta\right)^3}\frac{1}{y_0 - h + L\tan\theta} - \frac{L}{\left(\cos\theta\right)^4}\frac{1}{\left(y_0 - h + L\tan\theta\right)^2}\right],\\ &= -\frac{gL}{2\cos^4\theta}\frac{\cos(2\theta) + \frac{y_0 - h}{L}\sin(2\theta)}{\tan\theta - \frac{y_0 - h}{L}}\end{aligned}$$

or,

$$\begin{aligned}\tan(2\theta) &= \frac{-L}{h - y_0}\\ \theta &= \frac{1}{2}\left(\arctan\left(\frac{-L}{h - y_0}\right) + \pi\right).\end{aligned}$$

Evaluating with typical values, we find $\theta \sim 51.78°$.

[1] With the aid of Mathematica.

3.5 Boating

We assume the area of the wetted area, A, is a function of the number of people, N, the volume per person, V, and the power generated per person, P, then

$$A = f(N, V, P).$$

By comparing the units we find

$$L^2 = p^\alpha \left(\frac{L^3}{p}\right)^\beta \left(\frac{ML^2}{T^3 p}\right)^\gamma,$$

where L is our length unit, p is our persons unit, M is our mass unit, and T is our time unit. This gives us the system

$$\begin{aligned} 2 &= 3\beta + 2\gamma, \\ 0 &= \alpha - \beta - \gamma, \\ 0 &= \gamma, \\ 0 &= -3\gamma. \end{aligned}$$

Solving this system we find $\alpha = \beta = \frac{2}{3}$, and thus,

$$A \propto (NV)^{2/3}.$$

We can use a similar proceedure for the drag force experienced by the boat by considering the area, as well as the speed of boat, U, and the density of the water, ρ. By again considering the units, we have

$$\frac{ML}{T^2} = \left(L^2\right)^\alpha \left(\frac{L}{T}\right)^\beta \left(\frac{M}{L^3}\right)^\gamma.$$

Once again we obtain a system from this equation:

$$\begin{aligned} 1 &= \gamma, \\ 1 &= 2\alpha + \beta - 3\gamma, \\ -2 &= -2\beta, \end{aligned}$$

which has the solution $\alpha = 1$, $\beta = 2$, $\gamma = 1$. Therefore,

$$D \propto AU^2\rho.$$

The total power of the rowers, NP, must be proportional to rate at which work is done. The work done is the force multiplied by the velocity. That is, $NP \propto DU$. We then find

$$\begin{aligned} NP &\propto (NV)^{2/3} \rho U^3, \\ U^3 &\propto \frac{NP}{(NV)^{2/3}\rho}, \\ &\propto \left(\frac{NP^3}{V^2\rho^3}\right)^{1/3}, \end{aligned}$$

and finally,

$$U \propto \left(\frac{NP^3}{V^2\rho^3}\right)^{1/9}.$$

If V and P are both proportional to mass then it is advantageous to have a larger mass since

$$\begin{aligned} U &\propto \left(\frac{NM^3}{M^2\rho^3}\right)^{1/9} \\ &\propto \left(\frac{N}{\rho^3}\right)^{1/9} M^{1/9}. \end{aligned}$$

However, since it is raised to the power of one ninth, this is relatively negligible.