A Ring Isomorphism and corresponding Pseudoinverses

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Abstract

This paper studies the set of $n \times n$ matrices for which all row and column sums equal zero. By representing these matrices in a lower dimensional space, it is shown that this set is isomorphic to the set of arbitrary $(n-1) \times (n-1)$ matrices. The Moore-Penrose pseudoinverse corresponds with the true inverse (when it exists) in this lower dimension and an explicit representation of this pseudoinverse in terms of the lower dimensional space is given. This analysis is then extended to non-square matrices with all row or all column sums equal to zero.

1   Introduction

In various areas of mathematics, one comes across matrices where all row and column sums are constrained to equal zero. Examples arise in coding theory, stochastic processes, statistics and many other areas where linear algebra is used. Particular examples are the rate matrices of doubly stochastic, continuous time, finite-state Markov chains, the quadratic covariation matrices of any finite-state Markov chain (see for example [2]) and the 'Laplacian matrices' of graph theory. One problem which arises when studying these matrices is that their inverses do not exist: the vector with all elements one is an eigenvector with corresponding eigenvalue zero, so the determinant of these matrices will be zero. This supports reference to alternative definitions of the inverse, the most common being the Moore-Penrose (pseudo-)inverse.

The Moore-Penrose inverse has interesting properties in this context, which are derived in this paper. The primary result is that the set of $n \times n$ matrices for which all row and column sums equal zero is isomorphic to the set of arbitrary $(n-1) \times (n-1)$ matrices, with a modified product operation. Under this

isomorphism, the Moore-Penrose inverse maps to the true inverse when it exists, and to a modified Moore-Penrose type inverse when it does not. Using this, explicit formulae can be given for the Moore-Penrose inverse, and the range of the projection operator X^+X explicitly described, for X a rank $n-1$, $n \times n$ matrix with all row and column sums equal to zero.

2   Preliminary Ideas

We denote by M_n the set of all $n \times n$ matrices with components lying in a field F, which we shall assume to be of characteristic zero. The canonical example which we shall discuss is the real matrices; however, the analysis can easily be extended to the complex matrices. We denote by S_n the subset of M_n for which all row and column sums are equal to zero.

Definition 2.1. *For any matrix A, a pseudoinverse is any matrix A^+ which satisfies*
$$AA^+A = A.$$

If furthermore A^+ satisfies

$$A^+AA^+ = A^+$$
$$A^+A = (A^+A)^*$$
$$AA^+ = (AA^+)^*,$$

where $[.]^*$ denotes the conjugate transpose, it is said to be the Moore-Penrose (pseudo)inverse of A.

It is well known that the Moore-Penrose inverse of any matrix A exists and is unique (see, for example, [4]). If A is invertible then $A^+ = A^{-1}$ is the only pseudoinverse, and if A has a left- or a right-inverse, then this will be the Moore-Penrose inverse. A good overview of generalised inverses can be found in [1].

3   Representation

Theorem 3.1. *There is a bijection between S_{n+1} and M_n, given by the map*

$$\phi : M_n \to S_{n+1}, X \mapsto J_n^* X J_n$$

where

$$J_n := [I_n | - \mathbf{1}],$$

an identity matrix with a column of -1's added to the right hand side.

Proof. Consider an element $\tilde{X}$ of S_{n+1}. Let e_i denote the ith standard basis vector in $\mathbb{R}^{n+1}$. As all the row sums are zero, that is, for any i, $\sum_j e_j^* \tilde{X} e_i = 0$, we know that

$$e_{n+1}^* \tilde{X} e_i = -\sum_{j=1}^n e_j^* \tilde{X} e_i.$$

Similarly

$$e_i^* \tilde{X} e_{n+1} = -\sum_{j=1}^{n} e_i^* \tilde{X} e_j$$

for all i. Therefore, if $X \in M_n$ is the submatrix of $\tilde{X}$ excluding the $(n+1)$'th row and column, we know that

$$e_i^* \tilde{X} e_{n+1} = -\sum_{j=1}^{n} e_i^* \tilde{X} e_j = -e_i^* X \mathbf{1},$$

and similarly for the rows of $\tilde{X}$.

Therefore, for any $\tilde{X} \in S_n$, we can write

$$\tilde{X} = \left[\begin{array}{c|c} X & -X\mathbf{1} \\ \hline -\mathbf{1}^* X & \mathbf{1}X\mathbf{1} \end{array} \right] = J_n^* X J_n = \phi(X).$$

This establishes the surjectivity of ϕ. Injectivity follows immediately from the fact J_n is right-invertible (see Lemma 3.5 below). $\qquad\square$

Corollary 3.2. X and $\phi(X)$ have the same rank.

Proof. J_n is right invertible, and so no degrees of freedom are lost under this transformation. $\qquad\square$

Corollary 3.3. $X = X^*$ if and only if $\phi(X) = \phi(X)^*$.

Proof. It is clear that

$$\phi(X)^* = (J_n^* X J_n)^* = J_n^* X^* J_n = \phi(X^*),$$

the result follows as ϕ is a bijection. $\qquad\square$

Remark 3.4. It is clear that the choice of placing the column of -1's on the right hand side of J_n is arbitrary, and that this column can equivalently be placed elsewhere. This will not change the value of $J_n J_n^*$, and so the following analysis will be unchanged. Similarly a set of matrices with other linear dependencies could be considered, with the following holding *mutatis mutandis*.

Definition 3.1. *Let $\times$ denote standard matrix multiplication and $\circ$ denote the 'twisted' product $X \circ Y = X K_n Y$, $K_n := J_n J_n^*$, where X and Y are in M_n.*

Note that K_n has a simple form, namely

$$K_n = J_n J_n^* = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix} \in M_n$$

with inverse

$$K_n^{-1} = \frac{1}{n+1} \begin{bmatrix} n & -1 & \cdots & -1 \\ -1 & n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{bmatrix}.$$

3

Lemma 3.5. *The Moore-Penrose inverse of J_n is $J_n^+ = J_n^* K_n^{-1}$. This is also a right inverse.*

Proof. It is clear $J_n J_n^* K_n^{-1} = I_n$, and that $J_n^* K_n^{-1} J_n$ is symmetric. This establishes all the requirements of Definition 2.1. $\qquad\square$

Theorem 3.6. $(M_n, +, \circ)$ *is an algebraic ring with identity K_n^{-1}.*

Proof. As $\circ$ is distributive and associative, and all the properties for a ring follow as in the classical case. It is also clear by inspection that K_n^{-1} is an identity element, and is unique. $\qquad\square$

Theorem 3.7. ϕ *is an isomorphism between $(S_{n+1}, +, \times)$ and $(M_n, +, \circ)$, hence $(S_{n+1}, +, \times)$ is a ring with identity*

$$\phi(K_n^{-1}) = J_n^* K_n^{-1} J_n = I_{n+1} - \frac{1}{n+1} \mathbf{1}_{(n+1)\times(n+1)},$$

where $\mathbf{1}_{(n+1)\times(n+1)}$ here denotes the matrix with all entries 1.

Proof. We have

$$\phi(X) + \phi(Y) = J_n^*[X + Y]J_n = \phi(X + Y)$$

and

$$\phi(X) \times \phi(Y) = J_n^* X J_n J_n^* Y J_n = J_n^* X K_n Y J_n = \phi(X \circ Y).$$

ϕ is therefore a bijection preserving addition and multiplication, and hence is an isomorphism between these rings. $\qquad\square$

Remark 3.8. This identity differs from the usual matrix identity $I_{n+1} \notin S_{n+1}$. We could also have observed this directly, as the vector with all entries one is the only vector in the left and right null spaces of every matrix in S_{n+1}.

We now extend our definition of the Moore-Penrose inverse to the ring $(M_n, +, \circ)$ by replacing each instance of matrix multiplication by the $\circ$ operation. For notational clarity, we shall denote a classical inverse X^{-1}, a classical Moore-Penrose inverse X^+, and a Moore-Penrose inverse under $\circ$ by $X^\oplus$.

Definition 3.2. $X^\oplus$ *is the (unique) matrix satisfying*

$$X \circ X^\oplus \circ X = X, \qquad X^\oplus \circ X \circ X^\oplus = X^\oplus$$
$$(X^\oplus \circ X)^* = X^\oplus \circ X, \qquad (X \circ X^\oplus)^* = X \circ X^\oplus$$

Theorem 3.9. ϕ *maps all pseudoinverses under $\times$ to pseudoinverses under $\circ$ and vice versa. In particular it preserves the Moore-Penrose inverse.*

Proof. We simply check that $\phi(X^\oplus)$ satisfies the required properties for $\phi(X)^+$ whenever $X^\oplus$ satisfies Definition 3.2, and vice versa. By Corollary 3.3, ϕ preserves symmetry, and therefore must preserve the Moore-Penrose inverse. This also establishes the uniqueness of $X^\oplus$, as ϕ is a bijection. $\qquad\square$

Corollary 3.10. *For any matrix $X \in S_n$, its Moore-Penrose inverse $X^+ \in S_n$.*

Proof. From the above, $X^+ = \phi([\phi^{-1}(X)]^\oplus) \in S_n$. $\qquad\square$

4 Pseudoinverses

We now move to the key result of this analysis – a representation of the Moore-Penrose pseudoinverses of matrices in S_{n+1}.

Theorem 4.1. *For any rank n matrix $\tilde{X} = \phi(X)$ in S_{n+1}, the Moore-Penrose inverse of $\tilde{X}$ is the unique pseudoinverse of $\tilde{X}$ in S_{n+1}, and is given by*

$$\tilde{X}^+ = J_n^* K_n^{-1} X^{-1} K_n^{-1} J_n = \phi(K_n^{-1} X^{-1} K_n^{-1}) = (J_n^+)^* X^{-1} J_n^+.$$

Proof. We know $\tilde{X} = \phi(X)$ and X is of full rank. Hence X has a unique inverse in $(M_n, +, \circ)$, that is, a matrix $X^\ominus$ such that

$$X \circ X^\ominus = X^\ominus \circ X = K_n^{-1}.$$

Inspection shows that this corresponds to the regular inverse of X through the formula

$$X^\ominus = K_n^{-1} X^{-1} K_n^{-1}.$$

As $(M_n, +, \circ)$ and $(S_{n+1}, +, \times)$ are isomorphic (by Theorem 3.7), and $X^\ominus$ is the (Moore-Penrose) inverse of X in M_n, this leads to the Moore-Penrose inverse of $\tilde{X}$ having a representation

$$\tilde{X}^+ = \phi(X^\ominus) = J_n^* K_n^{-1} X^{-1} K_n^{-1} J_n$$

by Theorem 3.9. $\qquad\square$

Corollary 4.2. *For any rank n matrix $\tilde{X}$ in S_{n+1}, we have*

$$\tilde{X}\tilde{X}^+ = \tilde{X}^+\tilde{X} = \phi(K_n^{-1}) = I_{n+1} - \frac{1}{n+1}\mathbf{1}_{(n+1)\times(n+1)}$$

and hence $\tilde{X}^+$ acts as a true inverse in S_{n+1}.

Now suppose that $\tilde{X} = \phi(X) \in S_{n+1}$ is of rank $< n$. Clearly, X will have multiple pseudoinverses in $(M_n, +, \circ)$, as it is of incomplete rank. To find the Moore-Penrose inverse of $\tilde{X}$, we wish to find the Moore-Penrose inverse $X^\oplus$ under $\circ$. Expanding the conditions of Definition 3.2, along with our earlier result for rank n matrices, suggests an inverse of the form $K_n^{-1} X^+ K_n^{-1}$. Unfortunately this does not satisfy the symmetry requirements under $\circ$ (and will give a different pseudoinverse).

Provided we are dealing with real or complex matrices, we know that K_n is symmetric and positive definite, and therefore has a Cholesky decomposition $K_n = k_n k_n^*$, where k_n is an invertible, lower triangular $n \times n$ matrix (in fact, we can use any square matrix k_n with $K_n = k_n k_n^*$).

Theorem 4.3. *For any real or complex matrix $\tilde{X} = \phi(X) \in S_{n+1}$, the Moore-Penrose inverse of $\tilde{X}$ is given by the equations*

$$\tilde{X}^+ = J_n^* (k_n^{-1})^* (k_n^* X k_n)^+ k_n^{-1} J_n$$

and

$$\tilde{X}^+ = J_n^* \lim_{\delta\downarrow 0}(X^* K_n X K_n + \delta I)^{-1} X^* J_n.$$

5

Proof. Expanding K_n^{-1} and K_n using the decomposition $K_n = k_n k_n^*$, we find that the matrix

$$(k_n^{-1})^*(k_n^* X k_n)^+ k_n^{-1}$$

will satisfy the requirements for the Moore-Penrose inverse. This is done in full in the more general non-square case below.

As a Moore-Penrose inverse can be expressed as the limit of a perturbed solution (see, for example, [3]), we have also the useful formula

$$
\begin{aligned}
(k_n^* X k_n)^+ &= \lim_{\delta \downarrow 0} (k_n^* X^* K_n X k_n + \delta I)^{-1} k_n^* X^* k_n \\
&= \lim_{\delta \downarrow 0} (k_n^* X^* K_n X k_n + \delta k_n^* (k_n^{-1})^* k_n^{-1} k_n)^{-1} k_n^* X^* k_n \\
&= \lim_{\delta \downarrow 0} k_n^{-1} (X^* K_n X + \delta K_n^{-1})^{-1} X^* k_n
\end{aligned}
$$

and so

$$
\begin{aligned}
(k_n^{-1})^*(k_n^* X k_n)^+ k_n^{-1} &= \lim_{\delta \downarrow 0} K_n^{-1} (X^* K_n X + \delta K_n^{-1})^{-1} X^* \\
&= \lim_{\delta \downarrow 0} (X^* K_n X K_n + \delta I)^{-1} X^*.
\end{aligned}
$$

$\square$

5   Non-square matrices

This methodology can be applied to give explicit results for Moore-Penrose inverses for other types of matrices. We now consider the case for real or complex $\tilde{X}$, where $\tilde{X}$ has all row and column sums equal to zero but we do not assume that $\tilde{X}$ is square.

Using the same logic as above, we can write $\tilde{X}$ as

$$\tilde{X} = J_m^* X J_n$$

for some m and n, where X will now be an $m \times n$ matrix. We again define $K_m = J_m J_m^* = k_m k_m^*$ for some invertible matrix k_m.

Theorem 5.1. *For* $\tilde{X} = J_m^* X J_n$, $\tilde{X}^+$ *is given by*

$$\tilde{X}^+ = J_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} J_m.$$

Proof. Inspection verifies that all these matrices are conformable, so we need only check that this formula satisfies the assumptions for the Moore-Penrose

inverse.

$$\tilde{X}\tilde{X}^{+}\tilde{X} = J_m^* X J_n J_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} J_m J_m^* X J_n$$
$$= J_m^* X k_n k_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} k_m k_m^* X J_n$$
$$= J_m^* (k_m^{-1})^* (k_m^* X k_n)(k_m^* X k_n)^+ (k_m^* X k_n) k_n^{-1} J_n$$
$$= J_m^* X J_n$$
$$\tilde{X}^{+}\tilde{X}\tilde{X}^{+} = J_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} J_m J_m^* X J_n J_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} J_m$$
$$= J_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} J_m$$
$$(\tilde{X}\tilde{X}^{+})^* = (J_m^* X J_n J_n^* (k_n^{-1})^* (k_m^* X k_n)^+ k_m^{-1} J_m)^*$$
$$= (J_m^* X k_n (k_m^* X k_n)^+ k_m^{-1} J_m)^*$$
$$= (J_m^* (k_m^{-1})^* (k_m^* X k_n)(k_m^* X k_n)^+ k_m^{-1} J_m)^*$$
$$= J_m^* (k_m^{-1})^* (k_m^* X k_n)(k_m^* X k_n)^+ k_m^{-1} J_m$$

and the final condition follows as for the third. Therefore, the conditions for the Moore-Penrose inverse are satisfied. □

We can also deal with (possibly non-square) matrices where only the row sums are equal to zero (or equivalently, where only the column sums are equal to zero). In this case we have the representation

$$\tilde{X} = X J_n$$

with J_n as before. This gives the following.

Theorem 5.2. *Let $\tilde{X} = X J_n$, with X a left-invertible matrix. Then the Moore-Penrose inverse of $\tilde{X}$ is given by*

$$\tilde{X}^{+} = J_n^* K_n^{-1} X^{+} = J_n^+ X^{+}.$$

Proof. Inspection shows that this will satisfy all the requirements of Definition 2.1, as $J_n J_n^+ = X^+ X = I$. □

Theorem 5.3. *Let $\tilde{X} = J_n^* X$, with X a right-invertible matrix. Then the Moore-Penrose inverse of $\tilde{X}$ is given by*

$$\tilde{X}^{+} = X^{+} K_n^{-1} J_n = X^{+} (J_n^+)^*.$$

Proof. Again clear by inspection. □

6 Extra Zero Rows

A particular example of interest is for $\tilde{X} \in S_{n+1}$, where $\tilde{X}$ is of rank m and contains $n-m$ rows and columns which are invariably zero. An example of such a matrix with $n = 3$, $m = 2$ is

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 3 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 3 \end{bmatrix}.$$

This can be written through the linear transformation

$$
\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}
= J^*_{2,\{3\}}
\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} J_{2,\{2\}},
$$

where $J_{2,\{3\}}$ refers to the J_2 matrix as defined above, with an additional column of zeros inserted into the third place.

For any $J_{m,\mathbf{a}}$, where $\mathbf{a}$ lists the columns where zeros are inserted,

$$
J_{m,\mathbf{a}} J^*_{m,\mathbf{a}} = K_m.
$$

Theorem 6.1. *For a fixed $\mathbf{a}$ with m entries, $m < n$, let $S^{\mathbf{a}}_{n+1}$ be the set of $(n+1) \times (n+1)$ matrices with all row and column sums zero, and with zeros in those rows and columns listed in $\mathbf{a}$. Then there exists an isomorphism*

$$
\phi : (M_{n-m}, +, \circ) \to (S^{\mathbf{a}}_{n+1}, +, \times), X \mapsto J^*_{m,\mathbf{a}} X J_{m,\mathbf{a}}
$$

where $X \circ Y = X K_m Y$ as before.

Proof. The proof of this follows exactly as for Theorem 3.7. $\qquad\square$

Theorem 6.2. *The Moore-Penrose inverse of $\tilde{X} = J^*_{m,\mathbf{b}} X J_{m,\mathbf{a}}$, where $\tilde{X}$ is of rank m, is given by*

$$
\tilde{X}^+ = (J^*_{m,\mathbf{b}} X J_{m,\mathbf{a}})^+ = J^*_{m,\mathbf{a}} K_m^{-1} X^{-1} K_m^{-1} J_{m,\mathbf{b}} = (J^+_{m,\mathbf{a}})^* X^+ J^+_{m,\mathbf{b}}.
$$

Proof. If

$$
\tilde{X} = J^*_{m,\mathbf{b}} X J_{m,\mathbf{a}}
$$

then

$$
\tilde{X}^* = J^*_{m,\mathbf{a}} X^* J_{m,\mathbf{b}}.
$$

We can see that if $\tilde{X}$ is of rank m, so is its equivalent $m \times m$ matrix X. Therefore, X has a true inverse. Using the fact that $J_{m,\mathbf{a}} J^*_{m,\mathbf{a}} = J_{m,\mathbf{b}} J^*_{m,\mathbf{b}} = K_m$, we can then expand the definition of the Moore-Penrose inverse to give the desired result. $\qquad\square$

Remark 6.3. Note that this theorem does not require that $\tilde{X}$ is square, as $\mathbf{a}$ and $\mathbf{b}$ can list different numbers of positions where zeros are inserted.

We can then establish the following general property for all matrices of this form.

Theorem 6.4. *For $\tilde{X} = J^*_{m,\mathbf{b}} X J_{m,\mathbf{a}}$ of rank m,*

$$
\begin{aligned}
\tilde{X}^+ \tilde{X} &= J^*_{m,\mathbf{a}} K_m^{-1} X^{-1} K_m^{-1} J_{m,\mathbf{b}} J^*_{m,\mathbf{b}} X J_{n,\mathbf{a}} \\
&= J^*_{m,\mathbf{a}} K_m^{-1} J_{m,\mathbf{a}}.
\end{aligned}
$$

7   A final fact

A final useful result is that, for matrices of this form, if $\tilde{X} = J^*_{m,\mathbf{b}} X J_{m,\mathbf{a}}$ is of rank m, then

$$\tilde{X}^+ \tilde{X} = J^*_{m,\mathbf{a}} K_m^{-1} J_{m,\mathbf{a}}$$
$$= J^*_{m,\mathbf{a}} \left[I_m - \frac{1}{m+1} \mathbf{1}_{m \times m} \right] J_{m,\mathbf{a}}$$

where $\mathbf{1}_{m \times m}$ again refers to a matrix with all entries one.

Theorem 7.1. *Let M be any matrix with m rows, and $\tilde{X}$ any rank m square matrix of the form $\tilde{X} = J^*_{m,\mathbf{b}} X J_{m,\mathbf{a}}$. If $\tilde{M} = J^*_{m,\mathbf{a}} M$, then*

$$\tilde{X}^+ \tilde{X} \tilde{M} = \tilde{M}.$$

Proof. We have

$$\tilde{X}^+ \tilde{X} \tilde{M} = J^*_{m,\mathbf{a}} K_m^{-1} J_{m,\mathbf{a}} J^*_{m,\mathbf{a}} M = J^*_{m,\mathbf{a}} K_m^{-1} K_m M = \tilde{M}.$$

$\square$

Considering this form of $\tilde{M}$, it is clear that these are the matrices with column sums of zero and with zeros in the rows denoted by $\mathbf{a}$, that is, corresponding to the columns of zeros in $\tilde{X}$.

Theorem 7.2. *The vectors of the form $\tilde{M} = J_{m,\mathbf{a}} M$ form an m-dimensional subspace spanning the range of the projection operator $\tilde{X}^+ \tilde{X}$.*

8   Conclusion

We have examined some of the properties of matrices with all row and column sums equal to zero, particularly in relation to their Moore-Penrose inverses. The square matrices of this form are isomorphic to the ring of matrices of smaller dimension, which easily gives general results for the calculation of the pseudoinverses within this ring. We have shown that the true inverse in this smaller ring, if it exists, corresponds with the Moore-Penrose inverse for the original matrices. If we consider the subset of matrices with the extra restriction that a certain number of rows and columns are zero, then a further explicit formula for the projection operator $X^+ X$ can be obtained, which characterises its range. These results have applications in the theory of Markov chains and other areas.

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