InFoMM – Optimisation
Lecture 4

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Total Unimodularity Theory

In the last lecture we proved the following theorem relating to the formulation of the integer programming problem

$$\text{(IP)} \quad \max_x \{c^{\mathsf{T}} x : Ax = b,\ x \geq 0,\ x \in \mathbb{Z}^n\},$$

where A is assumed to have full row rank m.

Theorem (Total unimodularity implies integrality I)

If A is TU, then for all $b \in \mathbb{Z}^m$, all extreme points of the polyhedron $\mathcal{P}'(b) := \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ are integer valued.

We can easily extend this result to polyhedra in inequality constrained form:

Theorem (Total unimodularity implies integrality II)

If A is TU, then for all $b \in \mathbb{Z}^m$, all extreme points of the polyhedron $\mathcal{P}(b) := \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ are integer valued.

Proof. If A is TU, then $\hat{A} = [\,A\ \mathrm{I}\,]$ is TU, so that by Theorem I, the extreme points of

$$\mathcal{P}''(b) := \left\{z \in \mathbb{R}^{n+m} : \hat{A}z = b,\ z \geq 0\right\}$$

are integer valued.

Let $\Pi_{\mathbb{R}^n} : z = (x, s) \in \mathbb{R}^{n+m} \mapsto x$ be the projection onto the first n components of the variables z. Then $\mathcal{P}(b) = \Pi_{\mathbb{R}^n}\mathcal{P}''(b)$, and all extreme points of $\mathcal{P}(b)$ are projections of extreme points of $\mathcal{P}''(b)$. Therefore, the extreme points of $\mathcal{P}(b)$ are also integer valued.

The following is a near-converse result:

Theorem (Integrality implies total unimodularity)

If $A \in \mathbb{Z}^{m \times n}$ is such that all extreme points of the polyhedron $\mathcal{P}(b) := \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ are integer valued for all $b \in \mathbb{Z}^m$, then A is TU.

Proof. Let $\hat{A} = [\, A \; \mathbf{I} \,]$. Claim: all extreme points of

$$\mathcal{P}''(b) = \left\{ z \in \mathbb{R}^{n+m} : \hat{A}z = b, \; z \geq 0 \right\}$$

are integer valued. (See problem sheet.) Thus, x is an extreme point of $\mathcal{P}$, as claimed.

Let $A_{I,J}$ be an arbitrary invertible square submatrix of A, corresponding to row indices I and column indices J.

Let $B = J \cup \{n + i : i \notin I\}$, then $\hat{A}_B$ is an invertible $m \times m$ submatrix of $\hat{A}$ such that if rows I are permuted into the top position by left multiplication with an appropriate permutation matrices $P_{\mathbf{1}}, P_{\mathbf{2}}$, it is of the form

$$P_{\mathbf{1}} \hat{A}_B P_{\mathbf{2}} = \begin{bmatrix} A_{I,J} & 0 \\ \star & \mathbf{I} \end{bmatrix},$$

hence $\det(\hat{A}_B) = \pm \det(A_{I,J})$, so that $\det(A_{I,J}) = \pm 1$ if and only if $\det(\hat{A}_B) = \pm 1$.

Let $\mathbf{1} := [\, \mathbf{1} \; \ldots \; \mathbf{1} \,]^{\mathbf{T}}$ a m-dimensional vector of ones, and

$$\delta = \left\lceil \max_{k,\ell} \left| \left(\hat{A}_B^{-\mathbf{1}} \right)_{k,\ell} \right| \right\rceil .$$

For each $i = 1, \dots, m$, let

$$b^i = \delta \cdot \hat{A}_B \mathbf{1} + \mathbf{e}^i,$$

where e^i is the i-th canonical unit vector in $\mathbb{R}^m$. Then b^i is an integer vector.

The basic solution associated with the basis B and the r.h.s. is b^i is

$$x_B = \hat{A}_B^{-\mathbf{1}} b^i = \delta \cdot \mathbf{1} + \hat{A}_B^{-\mathbf{1}} e_i \geq 0, \quad x_N = 0,$$

so it is basic feasible and hence an extreme point of $\mathcal{P}''$.

Therefore, x_B is integer valued, and so is the i-th column of $\hat{A}_B^{-\mathbf{1}}$,

$$\left(\hat{A}_B^{-\mathbf{1}}\right)_i = \hat{A}_B^{-\mathbf{1}} \mathbf{e}^i = x_B - \delta \cdot \mathbf{1}.$$

Since this is true for all i, $\hat{A}_B^{-\mathbf{1}}$ is integer valued, and thus both $\det(\hat{A}_B)$ and its reciprocal $\det(\hat{A}_B^{-\mathbf{1}})$ are integers, which is only possible if $\det(\hat{A}_B) = \pm 1$.

Practical tools to recognise TU matrices

Verifying that a given matrix is TU seems a task of complexity exponential in the size of the matrix.

There are two categories of simple tools to recognise special cases:

- Rules by which small TU matrices can be assembled into larger ones. By applying the inverse of these rules, we may be able to recognise how to decompose a matrix into smaller parts whose total unimodularity is computationally cheaper to verify.
- Sufficient criteria that can easily be checked may allow us to identify some important families of TU matrices.

The following rules are easy to prove: $A \in \mathbb{R}^{m \times n}$ is TU if and only if any of the following matrices are TU,

i) A^{T},

ii) $\left[A \; -A \right]$,

iii) $A \cdot P$, where P is a $n \times n$ permutation matrix

iv) $P \cdot A$, where P is a $m \times m$ permutation matrix,

v) $\left[\begin{smallmatrix} A & J_1 \\ J_2 & 0 \end{smallmatrix}\right]$, with $J_i = P_i \left[\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{smallmatrix}\right] Q_i$, I an identity matrix, 0 a block of zeros, and P_i, Q_i permutation matrices of appropriate size.

Example (TU matrix)

The following matrix is TU,

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Indeed, it is trivial to check that $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ is TU. By application of ii),

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

is TU, and by application of v),

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is TU. By permuting the last two columns we find that A is TU.

Definition (Consecutive ones)

A $0,1$-valued matrix A has the *consecutive ones property* if the rows can be ordered so that the 1s in each column appear consecutively.

Theorem (Consecutive ones implies TU)

If $A \in \{0,1\}^{m\times n}$ has the consecutive ones property, then A is TU.

Proof. See problem sheet.

Example (Consecutive ones)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Example (Workforce planning)

- Workers are assigned to shifts consisting of consecutive time periods in periods $i = 1, \dots, m$. There are thus at most $\binom{m+1}{2}$ possible shifts $j = 1, \dots, n$.
- Hiring for shift j costs c_j per worker.
- In period i at last d_i workers are needed to operate the machinery.
- How many workers x_j to hire for each shift so as to minimise the total cost?

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq d_i, \quad (i = 1, \dots, m) \\
& x_j \geq 0, \quad (j = 1, \dots, n) \\
& x_j \in \mathbb{Z}, \quad (j = 1, \dots, n).
\end{aligned}
$$

Each column of $A = (a_{ij})$ is of the form $[0 \dots 0\, 1 \dots 1\, 0 \dots 0]^T$ because all shifts must consist of a set of consecutive time periods. Therefore, the matrix A is TU.

Theorem (Sufficient condition)

Let $A = [a_{ij}]$ be a matrix such that

i) $a_{ij} \in \{+1, -1, 0\}$ *for all* i, j.

ii) *Each column contains at most two nonzero coefficients,*

$$\sum_{i=1}^{m} |a_{ij}| \leq 2 \quad (j \in [1, n]).$$

iii) *The set M of rows can be partitioned into (M_1, M_2) such that each column j containing* two *nonzero coefficients satisfies*

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0.$$

Then A is totally unimodular.

Proof. The proof is by contradiction, assuming that A is not TU.

Let B be a smallest submatrix of A such that $\det(B) \notin \{0, +1, -1\}$. Then all columns of B contain exactly two nonzero coefficients, for else there exist permutation matrices P_1, P_2 such that

$$P_1 B P_2 = \begin{bmatrix} \pm 1 & * \\ 0 & C \end{bmatrix},$$

and then $\det(C) = \pm \det(B) \notin \{0, +1, -1\}$, and C is the row permutation of a strict submatrix of B, contradicting the choice of B. $\lightning$

Because of iii), adding the rows of B with indices in M_1 and subtracting the rows with indices in M_2 yields the zero vector, showing that the rows of B are linearly dependent and $\det(B) = 0$, in contradiction to the choice of B. $\lightning$

Application to graph problems

Definition (Graph)

A graph $G = (V, E)$ consists of a finite set of *vertices* (or *nodes*) V and a finite collection of *edges* $E \subset \{\{v, w\} : v, w \in V\}$ consisting of unordered pairs of vertices, referred to as the *heads* or *endpoints* of the edge.

If v is a head of e, we say that e and v are *incident* to one another.

An edge $e \in E$ is called a *loop* at $v \in V$ if both heads of e equal v.

The *vertex-edge incidence matrix* of G is the matrix $0, \pm 1$-valued matrix

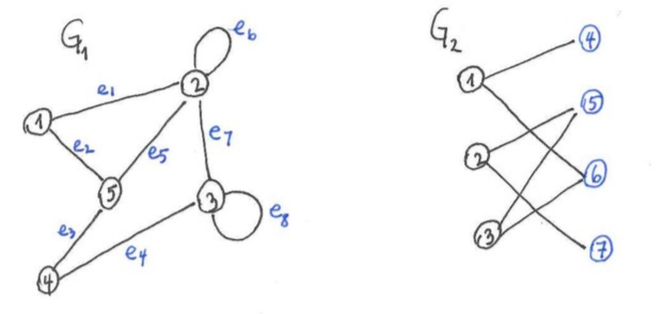
$$A(G) = \left(A_{v,e}(G)\right)_{v \in V, e \in E},$$

$$A_{v,e}(G) = \begin{cases} 1 & \text{if } v \text{ is one of two distinct heads of } e, \\ 2 & \text{if } e \text{ is a loop at } v, \\ 0 & \text{otherwise.} \end{cases}$$

Definition (Bipartite graph)

A graph G is *bipartite* if $V = V_1 \dot\cup V_2$ is a partition and $E \subset \{\{v, w\} : v \in V_1, w \in V_2\}$.

Example (Graph and bipartite graph)



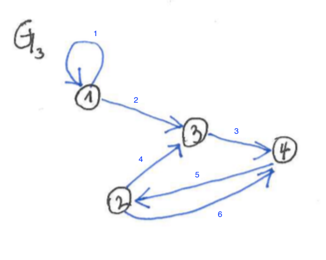
$$A(G_1) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad A(G_2) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Definition (Digraph)

A graph G is a *digraph* (directed graph) if $E \subset \{(v, w) : v, w \in V\}$ consists of *ordered* pairs, giving each edge (or *arc*) a direction from its *tail* v to its *head* w. The vertex-edge incidence matrix is then defined as

$$A_{v,e}(G) = \begin{cases} 1 & \text{if } v \text{ is the head of } e, \\ -1 & \text{if } v \text{ is the tail of } e, \\ 0 & \text{if } e \text{ is a loop at } v, \\ 0 & \text{otherwise.} \end{cases}$$

Example (Digraph)



$$A(G_3) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

Theorem (Incidence matrix of bipartite graph implies TU)

The vertex-edge incidence matrix of any bipartite graph is TU.

Proof. Each column of $A(G)$ contains exactly two nonzero components, a 1 for some $v \in V_1$, and a 1 for some $w \in V_2$.

Therefore, the sufficient criterion of the above theorem applies for the choice $M_1 = V_1$, $M_2 = V_2$.

Theorem (Incidence matrix of digraph implies TU)

The vertex-edge incidence matrix of any digraph is TU.

Proof. Each column $A_{:,e}(G)$ corresponding to a loop is a zero vector.

If e is not a loop, then $A_{:,e}(G)$ contains exactly two nonzero components, a $+1$ for the head, and a -1 for the tail.

Therefore, the sufficiency theorem applies with $M_1 = M$, $M_2 = \emptyset$.

Example (Shortest Path Problem)

- Given is a digraph $G = (V, E)$ with nonnegative arc lengths c_e for all $e \in E$.
- Two nodes $s, t \in V$ are marked.
- Find a shortest path from s to t in G.

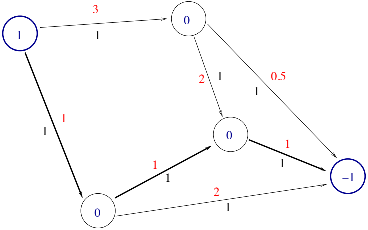
For each $e \in E$, let $x_e = 1$ if e lies along the path taken, and $x_e = 0$ otherwise.

For each $v \in V$, let $b_v = 1$ if $v = s$, $b_v = -1$ if $v = t$, and $b_v = 0$ otherwise.

$$
\begin{aligned}
\text{(SP)} \quad & \min \sum_{e \in E} c_e x_e \\
& \text{s.t. } A(G)x = b, \\
& \qquad 0 \leq x_e \leq 1, \quad (e \in E), \\
& \qquad x_e \in \mathbb{Z}, \quad (e \in E).
\end{aligned}
$$

The constraint matrix of (SP) (reformulated in inequality constrained form) is $[A^{\mathbf{T}}, -A^{\mathbf{T}}, \mathrm{I}]^{\mathbf{T}}$, in which $A = A(G)$ is the vertex-edge incidence matrix of a digraph, hence the model is TU and may be solved via LP relaxation.

Note: (SP) has an interpretation as an s-t flow problem with capacities 1 on each edge and integrality constraints on the x_e, with flow conservation constraints at each vertex. We write $V^{+}(v)$, $V^{-1}(v)$ for the successor and predecessor nodes of v.



$$
\begin{aligned}
\text{(SP)} \quad z = \min \sum_{(i,j)\in E} & c_{ij}x_{ij} \\
\text{s.t.} \quad \sum_{j\in V^{+}(s)} x_{sj} - \sum_{j\in V^{-}(s)} x_{js} &= 1 \\
\sum_{j\in V^{+}(t)} x_{tj} - \sum_{j\in V^{-}(t)} x_{jt} &= -1 \\
\sum_{j\in V^{+}(i)} x_{ij} - \sum_{j\in V^{-}(i)} x_{ji} &= 0 \quad (i \in V \setminus \{s,t\}) \\
0 \leq x_{ij} \leq 1 & \quad ((i,j) \in E) \\
x \in \mathbb{Z}^{|E|}&.
\end{aligned}
$$

Example (Assignment Problem)

The problem lives in a bipartite graph $G = (V_1, V_2, E)$ with $V_1 = \{i_1, \ldots, i_n\}$ workers, $V_2 = \{j_1, \ldots, j_n\}$ jobs, $E = V_1 \times V_2$.

$$\begin{aligned} \min_{x \in \mathbb{R}^{n \times n}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, \ldots, n, \\ & \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, \ldots, n, \\ & x_{ij} \in \{0, 1\} \quad \text{for } i, j = 1, \ldots, n. \end{aligned}$$

Reformulating the problem in terms of the vertex-edge incidence matrix $A(G)$,

$$\begin{aligned} \min_{x \in \mathbb{R}^{n \times n}} \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & A(G)x = 1, \\ & 0 \leq x_e \leq 1, \quad (e \in E), \\ & x_e \in \mathbb{Z}, \quad (e \in E), \end{aligned}$$

we recognise the problem as totally unimodular.