InFoMM – Optimisation
Lecture 7

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The Cutting Plane Algorithm

Our discussion of preprocessing steps shows that it is possible to derive valid inequalities for the integer feasible solutions of an IP from its polyhedral formulation. This idea can be generalised and systematically exploited in the design of algorithms.

Consider the problem

$$\begin{aligned}(\mathrm{IP})\quad & \max\ c^{\mathsf{T}}x \\ & \text{s.t. } x \in \mathscr{X} = \mathscr{P} \cap \mathbb{Z}^n,\end{aligned}$$

where $\mathscr{P} = \{x \in \mathbb{R}^n : a_i^{\mathsf{T}}x = b_i,\ (i = 1, \dots, m),\ x \geq 0\}$.

Definition (Valid inequalities and cuts)

A *valid inequality* for $\mathscr{X}$ is an inequality of the form

$$\alpha^{\mathsf{T}}x \leq \alpha_0$$

that is satisfied for all $x \in \mathscr{X}$ (but not necessarily for all $x \in \mathscr{P}$).

A *cut* for $x^* \in \mathscr{P} \setminus \mathscr{X}$ is a valid inequality for $\mathscr{X}$ such that

$$\alpha^{\mathsf{T}}x^* > \alpha_0,$$

that is, $\mathscr{P} \cap \{x : \alpha^{\mathsf{T}}x \leq \alpha_0\}$ is a tighter formulation of $\mathscr{X}$ that excludes x^*.

Algorithm (Cutting Plane Algorithm)

solve LP relaxation $x^* = \arg\max_x \{c^{\mathrm{T}} x : x \in \mathscr{P}\}$; `// initialisation`
while x^* *fractional* **do**
  find a cut $\alpha^{\mathrm{T}} x \leq \alpha_0$ for x^*;
  $\mathscr{P} \leftarrow \mathscr{P} \cap \{x : \alpha^{\mathrm{T}} x \leq \alpha_0\}$;
  solve LP relaxation $x^* = \arg\max_x \{c^{\mathrm{T}} x : x \in \mathscr{P}\}$;
end

Notes:

- The algorithm relies on systematic methods to generate cuts, an issue we will discuss further.
- In contrast to the Branch & Bound Algorithm, the convergence of the Cutting Plance Algorithm is not guaranteed but depends on the nature of the cuts that are applied.
- Ideally, one would like to apply cuts that are easily computed and cut off a large part of $\mathscr{P}$, but the two goals are often contradictory.
- The Cutting Plane Algorithm can be combined with the Branch-and-Bound Method, which yields the most powerful black-box solvers for IPs (Branch-and-Cut Algorithm).

Chvàtal Cuts

Example (Chvàtal cut)

Consider the IP $\min_x\{-x_1 - x_2 - x_3 : x \in \mathscr{X}\}$ with $\mathscr{X} = \mathscr{P} \cap \mathbb{Z}^3$ and

$$\mathscr{P} = \{x \in \mathbb{R}^3 : x_1 + x_2 \leq 1,\ x_2 + x_3 \leq 1,\ x_1 + x_3 \leq 1,\ x \geq 0\}.$$

Using slack variables, $x^* = (0.5, 0.5, 0.5, 0, 0)$ is optimal for the LP relaxation of (IP),

$$(\mathrm{LP}) \quad \min_{x \geq 0} -x_1 - x_2 - x_3 \quad \text{s.t.} \begin{cases} x_1 + x_2 + x_4 = 1 \\ x_2 + x_3 + x_5 = 1 \\ x_1 + x_3 + x_6 = 1 \end{cases}$$

Multiplying the three equality constraints with 0.5 and adding them yields

$$x_1 + x_2 + x_3 + 0.5x_4 + 0.5x_5 + 0.5x_6 = 1.5.$$

Using the non-negativity of x_4, x_5, x_6, this implies $x_1 + x_2 + x_3 \leq 1.5$, which is a valid inequality not only for $\mathscr{X}$ but also for $\mathscr{P}$. Now using the integrality of x_1, x_2, x_3, we obtain the valid inequality

$$x_1 + x_2 + x_3 \leq 1,$$

which is a cut for x^* because $x_1^* + x_2^* + x_3^* = 1.5$.

We now generalise the approach described above: For any $r \in \mathbb{R}^n$, let $\lfloor r \rfloor = [\lfloor r_1 \rfloor, \dots, \lfloor r_n \rfloor]$.

Definition (Chvàtal cuts)

Let $\mathscr{P}^{(0)} = \{x \geq 0 : Ax = b\}$ be a polyhedron given by a system of m equations and n non-negativity constraints, and let $u \in \mathbb{R}^m$. The Chvàtal cut associated with u is given by

$$\alpha^{\mathsf{T}} x \leq \alpha_0,$$

where $\alpha^{\mathsf{T}} := \lfloor u^{\mathsf{T}} A \rfloor$ and $\alpha_0 := \lfloor u^{\mathsf{T}} b \rfloor$.

Lemma (Chvàtal cuts are valid inequalities)

All Chvàtal cuts are valid inequalities for the set $\mathscr{X} = \mathscr{P}^{(0)} \cap \mathbb{Z}^n$.

Proof. $x \in \mathscr{X}$ implies $Ax = b$, and hence, $u^{\mathsf{T}} Ax = u^{\mathsf{T}} b$. Using $x \geq 0$ this implies $\lfloor u^{\mathsf{T}} A \rfloor x \leq u^{\mathsf{T}} b$, and using integrality of x_i, $(i = 1, \dots, n)$, this implies $\lfloor u^{\mathsf{T}} A \rfloor x \leq \lfloor u^{\mathsf{T}} b \rfloor$.

We state the next result without proof.

Theorem (Separation of non-integral vertices)

Given a vertex x^* *of* $\mathscr{P}^{(0)} \setminus \mathbb{Z}^n$, *there exists a vector* $u \in \mathbb{R}^m$ *such that* $\lfloor u^{\mathsf{T}} A \rfloor x \leq \lfloor u^{\mathsf{T}} b \rfloor$ *is a cut for* x^*.

Chvàtal Closures

In general, not all valid inequalities for $\mathscr{X}$ can be obtained as Chvàtal cuts of a given formulation $\mathscr{P}^{(0)}$:

Let us write $A^{(1)}x \leq b^{(1)}$ for the set of all Chvàtal cuts that can be derived from the formulation $\mathscr{P}^{(0)}$ of $\mathscr{X}$, and let us define the new set

$$\mathscr{P}^{(1)} := \{x \geq 0 : \ Ax = b, \ A^{(1)}x \leq b^{(1)}\}.$$

The next result is given without proof.

Lemma (Polyhedrality of Chvàtal closure)

Only finitely many of the inequalities $A^{(1)}x \leq b^{(1)}$ are essential (non-redundant), that is, $\mathscr{P}^{(1)}$ is a polyhedron, and we assume without loss of generality that the system $A^{(1)}x \leq b^{(1)}$ lists only essential Chvàtal cuts, and that there are m_1 of these.

We can now repeat this construction an apply new Chvàtal cuts

$$\lfloor u^{\mathrm{T}}A + w_1^{\mathrm{T}}A^{(1)} \rfloor x \leq \lfloor u^{\mathrm{T}}b + w_1^{\mathrm{T}}b^{(1)} \rfloor$$

with $u \in \mathbb{R}^m$ and $w_1 \in \mathbb{R}^{m_1}_+$ to $\mathscr{P}^{(1)}$. Denote the essential inequalities of this form by $A^{(2)}x \leq b^{(2)}$. By Lemma (Polyhedrality of Chvàtal closure), there are a finite number m_2 of new inequalities, hence

$$\mathscr{P}^{(2)} := \{x \geq 0 : \ Ax = b, \ A^{(1)}x \leq b^{(1)}, \ A^{(2)}x \leq b^{(2)}\}$$

is a polyhedron.

Applying this process iteratively, for $k \in \mathbb{N}$ we obtain polyhedra

$$\mathscr{P}^{(k)} = \{x \geq \mathbf{0} : \ Ax = b, \ A^{(\mathbf{1})}x \leq b^{(\mathbf{1})}, \dots, \ A^{(k-\mathbf{1})}x \leq b^{(k-\mathbf{1})}\}$$

that generate a new set of m_k inequalities $A^{(k)}x \leq b^{(k)}$ defined as the set of essential Chvàtal cuts of the form

$$\lfloor u^{\mathbf{T}}A + \sum_{j=\mathbf{1}}^{k} w_j^{\mathbf{T}} A^{(j)} \rfloor x \leq \lfloor u^{\mathbf{T}} b + \sum_{j=\mathbf{1}}^{k} w_j^{\mathbf{T}} b^{(j)} \rfloor$$

with $u \in \mathbb{R}^m$ and $w_j \in \mathbb{R}_+^{m_j}$, $(j = \mathbf{1}, \dots, k)$.

It is clear from the above construction that

$$\mathscr{P}^{(\mathbf{0})} \supseteq \mathscr{P}^{(\mathbf{1})} \supseteq \cdots \supseteq \mathscr{P}^{(k)} \supset \mathrm{conv}(\mathscr{X})$$

is a nesting of ever tighter formulations of $\mathscr{X}$. The next result, given without proof, shows that this process is finite.

Theorem (Finiteness of Chvàtal rank)

For any polyhedron $\mathscr{P}^{(\mathbf{0})}$ and $\mathscr{X} = \mathscr{P}^{(\mathbf{0})} \cap \mathbb{Z}^n$ there exists a finite k for which $\mathscr{P}^{(k)} = \mathrm{conv}(\mathscr{X})$.

Definition (Chvàtal closures)

$\mathscr{P}^{(k)}$ is called the *k-th Chvàtal closure* of $\mathscr{P}^{(\mathbf{0})}$, and $\min\{k : \mathscr{P}^{(k)} = \mathrm{conv}(\mathscr{X})\}$ is called the *Chvàtal rank* of $\mathscr{P}^{(\mathbf{0})}$. In particular, $\mathscr{P}^{(\mathbf{0})}$ is an ideal formulation if and only if its Chvàtal rank is zero.

Notes:

- We started with an equality constrained IP $\max\{c^{\mathbf{T}}x : Ax = b\}$, which is why the vectors u are unconstrained, but the vectors w_k used to take linear combinations of added inequalities $A^{(k)}x \leq b^{(k)}$ must be chosen $w_{\mathbf{1}} \geq 0$. Alternatively, note that $A^{(k)}, b^{(k)}$ are integer valued, so that if the IP is brought into equality constrained form by adding slack variables $s^{(k)} = b^{(k)} - A^{(k)}x$, these are forced to be integer valued variables too.
- The use of an essential Chvàtal cut in each iteration of the cutting plane algorithm guarantees finite termination, because an ideal formulation of $\mathscr{X}$ is found after applying finitely many iterations. However, there exist examples for which the Chvàtal rank is larger than the number n of decision variables.
- In general even the number of essential inequalities $A^{(\mathbf{1})}x \leq b^{(\mathbf{1})}$ in the first Chvàtal closure is of order $m_{\mathbf{1}} = O\left(\binom{n}{m}\right)$. Applying the cutting plane algorithm with Chvàtal cuts is thus very inefficient in practice.
- However, to solve an IP it is not necessary to find an ideal formulation. A more economical approach is required that only applies cuts that make the optimal solution emerge.

Gomoroy Cuts

Let $\mathscr{P} = \{x \in \mathbb{R}^n : \ a_i^{\mathbf{T}} x = b_i, \ (i = 1, \dots, m), \ x \geq 0\}$ and consider the IP problem

$$\text{(IP)} \quad \max \ c^{\mathbf{T}} x \quad \text{s.t.} \ x \in \mathscr{X} = \mathscr{P} \cap \mathbb{Z}^n.$$

The LP relaxation is given by

$$\text{(LP)} \quad \max\{c^{\mathbf{T}} x : \ Ax = b, \ x \geq 0\}$$

with $A = [\, a_1 \ \cdots \ a_m \,]^{\mathbf{T}}$ and $b = [\, b_1 \ \dots \ b_m \,]^{\mathbf{T}}$. Let x^* be obtained by solving (LP) with the simplex algorithm, and assume x_h^* fractional for some $h \in [1, n]$.

By permuting decision variables we may assume w.l.o.g. that $B = [1, m]$, i.e., the columns so that the basic variables associated with x^* appear in the first $m \times m$ block. The optimal tableau looks as follows,

$$\begin{array}{cc|c} \mathbf{I}_m & \overline{A}_N & \bar{b} \\ \hline 0 & \bar{c}^{\mathbf{T}} & -z^* \end{array}$$

where $z^* = c^{\mathbf{T}} x^*$, $\overline{A} = A_B^{-1} A$, $\bar{b} = A_B^{-1} b = x_B^*$ and $\bar{c}_N = c_N - A_N^{\mathbf{T}} A_B^{-\mathbf{T}} c_B$ (see Lecture 2).

If $\bar{b} \in \mathbb{Z}^n$, x^* solves (IP), else there exists $t \in B$ s.t. x_t^* is fractional and the t-th row of the tableau reads

$$x_t + \sum_{j \in N} \bar{a}_{tj} x_j = \bar{b}_t,$$

Definition (Gomoroy Cut)

The *Gomoroy cut* associated with variable x_t^* is the valid inequality

$$x_t + \sum_{j \in N} \lfloor \bar{a}_{tj} \rfloor x_j \leq \lfloor \bar{b}_t \rfloor.$$

Lemma

The Gomoroy cut associated with variable x_t^* is a cut for x^* and a special case of a Chvàtal cut.

Proof. Let e_t be the t-th canonical unit vector. The Gomoroy cut is then obtained as the Chvàtal cut associated with vector $u = A_B^{-T} e_t$, and it is a cut for x^* since

$$x_t^* + \sum_{j \in N} \lfloor \bar{a}_{tj} \rfloor x_j^* = x_t^* > \lfloor \bar{b}_t \rfloor.$$

Writing $\varphi(a) = a - \lfloor a \rfloor$ for the fractional part of any real number a, and introducing a slack variable s, the Gomoroy cut can be equivalently expressed in *fractional form*

$$\sum_{j \in N} -\varphi(\bar{a}_{tj})x_j + s = -\varphi(\bar{b}_t), \quad s \in \mathbb{N}_{\mathbf{0}}$$

and added to the formulation of our IP,

$$\begin{aligned} \text{(IP)} \quad & \max \left[\begin{smallmatrix} c \\ \mathbf{0} \end{smallmatrix}\right]^{\mathbf{T}} \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \\ & \text{s.t. } \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \in \mathscr{X}^+ := \mathscr{P}^+ \cap \mathbb{Z}^{n+\mathbf{1}}, \end{aligned}$$

where

$$\mathscr{P}^+ = \{\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] : x \in \mathscr{P}\} \cap \left\{ \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] : \sum_{j \in N} -\varphi(\bar{a}_{tj})x_j + s = -\varphi(\bar{b}_t) \right\}.$$

Writing $\varphi(\bar{a}_{t,:})$ for the vector $[\varphi(\bar{a}_{t,j})]_{j=\mathbf{1}}^n$, the LP relaxation of this new formulation is given by

$$\begin{aligned} \text{(LP')} \quad & \max_{x,s} c^{\mathbf{T}} x \\ & \text{s.t. } \begin{bmatrix} \bar{A} & 0 \\ -\varphi(\bar{a}_{t,:}) & 1 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} \bar{b} \\ -\varphi(\bar{b}_t) \end{bmatrix}, \\ & x, s \geq 0. \end{aligned}$$

The Gomoroy cut can be added to the tableau corresponding to x^*, which now has one extra row (for the cut) and column (for the new basic variable s),

$$\begin{array}{cccc|c} \mathbf{I}_m & \overline{A}_N & 0 & & \bar{b} \\ 0 & -\varphi(\bar{a}_{t,N}) & 1 & & -\varphi(\bar{b}_t) \\ \hline 0 & \bar{c}^{\mathbf{T}} & 0 & & -z^* \end{array}$$

which now represents a basic (but infeasible) solution of (LP').

However, the dual of (LP') is given by

$$\begin{aligned} \text{(D')} \quad & \min_{y,w} b^{\mathbf{T}} y - \varphi(\bar{b}_t) w \\ & \text{s.t. } \overline{A}^{\mathbf{T}} y - \varphi(\bar{a}_{t,:}) w \geq c \\ & \qquad w \geq 0. \end{aligned}$$

We can extend the basic optimal solution y^* of the dual of (LP), obtained by complementarity, to a dual basic feasible solution (y^*, w^*) by setting $w^* = 0$ and adding w to the set of basic variables.

Since $\varphi(\bar{b}_t) > 0$, we can pivot on column w of the dual tableau, and this can be done directly in the primal tableau using so-called *dual simplex* pivoting operations.

Example (IP by cutting plane algorithm with Gomoroy cuts)

$$
\begin{aligned}
\text{(IP)} \quad & \max_x x_2 \\
& \text{s.t.} \quad 3x_1 + 2x_2 + x_3 = 6 \\
& \qquad -3x_1 + 2x_2 + x_4 = 0 \\
& \qquad x_j \geq 0,\ x_j \in \mathbb{Z},\ (j = 1, \dots, 4).
\end{aligned}
$$

The initial tableau is found as

3	2	1	0	6
-3	2	0	1	0
0	1	0	0	0

and after a few pivoting steps, the optimal tableau is found:

1	0	1/6	-1/6	1
0	1	1/4	1/4	3/2
0	0	-1/4	-1/4	-3/2

from which we read off the LP-optimal solution $x^* = [1, 3/2, 0, 0]$. Since x_2^* is fractional, row 2 yields

$$
\begin{aligned}
x_2 + 0.25x_3 + 0.25x_4 &= 1.5, \\
\Rightarrow x_2 + 0 \times x_3 + 0 \times x_4 &\leq \lfloor 1.5 \rfloor, \\
\Rightarrow x_2 &\leq 1.
\end{aligned}
$$

Example (IP by Gomoroy cuts continued)

Introduction of a slack variable x_5 and subtraction of the two equations yields

$$\begin{aligned} x_2 + x_5 &= 1, \quad x_5 \geq 0,\ x_5 \in \mathbb{Z} \\ -(x_2 + 0.25x_3 + 0.25x_4 &= 1.5) \\ \Rightarrow -0.25x_3 - 0.25x_4 + x_5 &= -0.5, \quad x_5 \geq 0,\ x_5 \in \mathbb{Z}. \end{aligned}$$

Add this equation to the tableau,

$$\begin{array}{ccccc|c} 1 & 0 & 1/6 & -1/6 & 0 & 1 \\ 0 & 1 & 1/4 & 1/4 & 0 & 3/2 \\ 0 & 0 & -1/4 & -1/4 & 1 & -1/2 \\ \hline 0 & 0 & -1/4 & -1/4 & 0 & -3/2 \end{array}$$

This tableau describes a basic solution (the intersection of m active constraints) by setting $x_N = 0$, but the solution is not feasible, because $x_5 = -1/2$. To render this variable non-negative, we must use row 3 as a pivot row in which to eliminate a different variable.

To decide on which column to pivot, note that if the pivot row t were to read

$$\sum_{j=1}^{n} \bar{a}_{tj} x_j = \bar{b}_t$$

with $\bar{a}_{tj} \geq 0$ ($j = 1, \ldots, n$) and $\bar{b}_t < 0$, then no matter how $x \geq 0$ is chosen, constraint t cannot be satisfied. In that case, we would have to conclude that the primal problem is infeasible.

Example (IP by Gomoroy cuts continued)

Luckily, in our case, this is not so, and in pivoting on column h with $\bar{a}_{th} < 0$, the last row changes as follows,

$$\bar{c}_j \leftarrow \bar{c}_j - \frac{\bar{c}_h}{\bar{a}_{th}} \bar{a}_{tj}.$$

To ensure that dual feasibility is not destroyed, we must not allow any $\bar{c}_j$ to become positive, that is,

$$\begin{aligned}
&\bar{c}_j - \frac{\bar{c}_h}{\bar{a}_{th}} \bar{a}_{tj} \leq 0, \quad (j = 1, \dots, n) \quad \text{(not a problem if } \bar{a}_{tj} \geq 0\text{, since } \bar{c}_h \leq 0 \text{ and } \bar{a}_{th} < 0) \\
\Leftrightarrow\ &\frac{\bar{c}_j}{|\bar{a}_{tj}|} - \frac{\bar{c}_h}{|\bar{a}_{th}|} \leq 0, \quad (j \in [1, n],\ \bar{a}_{tj} < 0) \\
&\Leftrightarrow h \in \arg\max \left\{ \frac{\bar{c}_j}{|\bar{a}_{tj}|} : j \in [1, n],\ \bar{a}_{tj} < 0 \right\}.
\end{aligned}$$

For example in our case, $t = 3$, $h \in \{3, 4\}$. Eliminating x_3 in row 3 reoptimises the tableau,

$$\begin{array}{ccccc|c}
1 & 0 & 1/6 & -1/6 & 0 & 1 \\
0 & 1 & 1/4 & 1/4 & 0 & 3/2 \\
0 & 0 & 1 & 1 & -4 & 2 \\
\hline
0 & 0 & -1/4 & -1/4 & 0 & 3/2
\end{array}
\rightarrow
\begin{array}{ccccc|c}
1 & 0 & 0 & -1/3 & 2/3 & 2/3 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -4 & 2 \\
\hline
0 & 0 & 0 & 0 & -1 & -1
\end{array}$$

Row 1 yields the Gomoroy cut $x_1 - x_4 \leq 0$, or in fractional form,

$$\begin{aligned}
x_1 - x_4 + x_6 &= 0, \quad x_6 \geq 0,\ x_6 \in \mathbb{Z} \\
-(x_1 - 1/3x_4 + 2/3x_5 &= 2/3) \\
\Rightarrow -2/3x_4 - 2/3x_5 + x_6 &= -2/3, \quad x_6 \geq 0,\ x_6 \in \mathbb{Z}.
\end{aligned}$$

Example (IP by Gomoroy cuts continued)

Add this equation to the tableau,

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & -1/3 & 2/3 & 0 & 2/3 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & -4 & 0 & 2 \\
0 & 0 & 0 & -2/3 & -2/3 & 1 & -2/3 \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & -1
\end{array}
$$

Now $t = 4$, $h = 4$. Eliminating x_4 in row 4 reoptimises the tableau,

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & -1/3 & 2/3 & 0 & 2/3 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & -4 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & -3/2 & 1 \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & -1
\end{array}
\rightarrow
\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 1 & -1/2 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & -5 & 3/2 & 1 \\
0 & 0 & 0 & 1 & 1 & -3/2 & 1 \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & -1
\end{array}
$$

The new optimal solution is $x^* = [1, 1, 1, 1, 0, 0]$, and the IP is now solved by LP relaxation.

Note: Alternatively, the second Gomoroy cut $x_1 - x_4 \leq 0$ could have also been reformulated using the substitution $x_4 = 3x_1 - 2x_2$, yielding

$$x_2 \leq x_1 .$$

Notes:

- The use of Gomoroy cuts is easy to understand and apply, and it guarantees that the cutting plane algorithm converges. However, the technique is not particularly effective, because the cuts become very shallow very quickly.
- To help the method select deeper cuts, it is advised to generate the Gomoroy cut from the *most fractional* variable x_j^*, that is from the row corresponding to $\arg\max_j \varphi(x_j^*)$.
- If the objective vector c is integer, one can also use the last row

$$-z + \sum_{j \in N} \bar{c}_j x_j = \bar{c}_{\mathbf{0}}$$

of the optimised tableau to generate a Gomoroy cut when $\bar{c}_{\mathbf{0}}$ is fractional, yielding

$$\sum_{j \in N} \varphi(\bar{c}_j) x_j \geq \varphi(\bar{c}_{\mathbf{0}}).$$