InFoMM – Optimisation
Lecture 5

Raphael Hauser

Oxford Mathematical Institute

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Example of Branch & Bound in Action

- *LP based branch-and-bound* is a divide-and-conquer strategy that can be applied to any IP or MIP problem.
- The problem is recursively broken down into easier problems of optimising the same objective function over smaller feasible sets, and the subproblems are arranged in a tree.
- To keep notation simple, we write $(\mathscr{F})$ for the problem with feasible set $\mathscr{F}$, and $(\mathscr{P})$ for the associated LP relaxation, where $\mathscr{P}$ is the formulation used for set $\mathscr{F}$.
- Each subproblem is solved approximately to obtain primal and dual bounds that are then used to formulate new problems and discard part of the solution domain.

Example

For illustration, we will solve the IP instance

$$
\begin{aligned}
(\mathscr{F}) \quad z = \max\, & 4x_1 - x_2 \\
\text{s.t. } & 7x_1 - 2x_2 \leq 14 \\
& x_2 \leq 3 \\
& 2x_1 - 2x_2 \leq 3 \\
& x \in \mathbb{Z}^2_+ .
\end{aligned}
$$

Step 1: Bounding $(\mathscr{F})$. To obtain a dual bound, we solve the LP relaxation

$$
\begin{aligned}
(\mathscr{P}) \quad z = \max\ & 4x_1 - x_2 \\
\text{s.t. } & 7x_1 - 2x_2 \leq 14 \\
& x_2 \leq 3 \\
& 2x_1 - 2x_2 \leq 3 \\
& x \geq 0
\end{aligned}
$$

of $(\mathscr{F})$. Introducing slack variables x_3, x_4, x_5 and applying the simplex algorithm, we obtain the optimal dictionary

$$
\begin{aligned}
x_1 &= \frac{20}{7} - \frac{1}{7}x_3 - \frac{2}{7}x_4 \\
x_2 &= 3 - 0 \cdot x_3 - x_4 \\
x_5 &= \frac{23}{7} + \frac{2}{7}x_3 - \frac{10}{7}x_4 \\
\bar{z} &= \frac{59}{7} - \frac{4}{7}x_3 - \frac{1}{7}x_4
\end{aligned}
$$

from which we read off the nonintegral solution $(\bar{x}_1, \bar{x}_2) = \left(\frac{20}{7}, 3\right)$ that corresponds to the dual bound $\bar{z} = \frac{59}{7}$.

To obtain a primal bound, we could try finding a feasible solution $\tilde{x}$ of (IP) via a heuristic and set $\underline{z} = 4\tilde{x}_1 - \tilde{x}_2$. We would also keep $\tilde{x}$ in memory as *incumbent*.

In this case, we did not produce a primal bound via a heuristic and can use $\underline{z} = -\infty$ as a valid lower bound.

Key Idea (Tightening Bounds)

So far we know that

$$\underline{z} \leq z \leq \overline{z}.$$

In subsequent iterations we will produce improved primal bounds $\underline{z} \leq \underline{z}_+$ and improved dual bounds $\overline{z}_+ \leq \overline{z}$ such that

$$\underline{z}_+ \leq z \leq \overline{z}_+$$

sandwiches the optimal objective value z in a narrower interval.

Each time the primal bound $\underline{z}$ increases, a new best feasible solution of $(\mathscr{F})$ has been found, and we update the incumbent.

Should we ever encounter a situation where $\underline{z} = \overline{z}$, the incumbent must be an optimal solution of $(\mathscr{F})$, and we can stop.

Often, we run the algorithm on a fixed time budget and stop early. The bounds then give an approximation guarantee, as the objective value $\underline{z}$ of the incumbent is at most $\overline{z} - \underline{z}$ away from optimal or, assuming positive objective values, the incumbent is within a factor of $\underline{z}/\overline{z}$ of optimal.

Step 2: Fractional branching. Since $\underline{z} < \overline{z}$, $(\mathscr{F})$ is not solved to optimality yet, so we need to branch. Since $\overline{x}_1$ is fractional, we distinguish the cases where $x_1 \leq \lfloor \overline{x}_1 \rfloor$ and $x_1 \geq \lceil \overline{x}_1 \rceil$. More generally, we can pick any index j such that $\overline{x}_j \notin \mathbb{Z}$ and set

$$\mathscr{F}_1 = \mathscr{F} \cap \{x : x_j \leq \lfloor \overline{x}_j \rfloor\},$$
$$\mathscr{F}_2 = \mathscr{F} \cap \{x : x_j \geq \lceil \overline{x}_j \rceil\}.$$

Of course, futher down the tree we can use the same branching rule to subdivide any $\mathscr{F}_j$ thus generated. In our case this leads to the two subproblems

$$(\mathscr{F}_1) \quad z = \max 4x_1 - x_2$$
$$\text{s.t. } 7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \leq 2$$
$$x \in \mathbb{Z}^2_+,$$

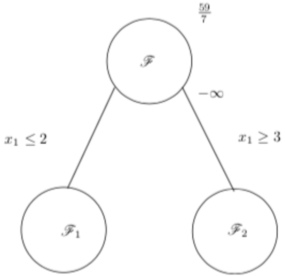
$$(\mathscr{F}_2) \quad z = \max 4x_1 - x_2$$
$$\text{s.t. } 7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \geq 3$$
$$x \in \mathbb{Z}^2_+.$$

Note that upon solving the LP relaxations of these two problems we will have

$$\max\{\bar{z}^{[1]}, \bar{z}^{[2]}\} < \bar{z},$$

since x_1 would have to be allowed to take the value $\frac{20}{7}$ for the value $\bar{z}$ to be attained. This will ensure that $\bar{z}$ decreases.

So far we obtain the partial enumeration tree below. Nodes $\mathscr{F}_1$ and $\mathscr{F}_2$ still need to be explored. We mark these nodes as *active*. The node $\mathscr{F}$ has been processes and is *inactive*.



Step 3: Choosing an an active node for processing. The list of active nodes is $\mathscr{F}_{\mathbf{1}}, \mathscr{F}_{\mathbf{2}}$. Later we will discuss breadth-first versus depth-first choices of the next active node to be processed. For now we arbitrarily select $\mathscr{F}_{\mathbf{1}}$.

Step 4: Bounding $(\mathscr{F}_{\mathbf{1}})$. Next we derive a bound $\bar{z}^{[\mathbf{1}]}$ by solving the LP relaxation

$$
\begin{aligned}
(\mathscr{P}_{\mathbf{1}}) \quad z = \max\, & 4x_{\mathbf{1}} - x_{\mathbf{2}} \\
\text{s.t.}\ & 7x_{\mathbf{1}} - 2x_{\mathbf{2}} \leq 14 \\
& x_{\mathbf{2}} \leq 3 \\
& 2x_{\mathbf{1}} - 2x_{\mathbf{2}} \leq 3 \\
& x_{\mathbf{1}} \leq 2 \\
& x \geq 0
\end{aligned}
$$

of problem $(\mathscr{F}_{\mathbf{1}}) \quad z^{[\mathbf{1}]} = \max\{c^{\mathbf{T}}x : x \in \mathscr{F}_{\mathbf{1}}\}$.

$(\mathscr{P}_{\mathbf{1}})$ differs from $(\mathscr{P})$ by ways of an additional constraint $x_{\mathbf{1}} \leq 2$. We will see that such problems can be solved via a warm start procedure that works out as follows:

- Instead of pivoting on the primal problem, keep solving the LP dual via the simplex algorithm.
- The optimal basic solution of the dual of $(\mathscr{P})$ is basic feasible (but not optimal) for the dual of $(\mathscr{P}_{\mathbf{1}})$. A new dual optimal solution is found in just a few pivot steps.

Using this approach, we find the optimal solution of $(\mathscr{P}_1)$,

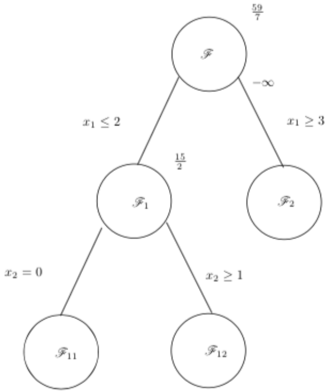
$$\begin{aligned}
x_1 &= 2 - x_6 \\
x_2 &= \frac{1}{2} + \frac{1}{2}x_5 + x_6 \\
x_3 &= 1 + x_5 + 5x_6 \\
x_4 &= \frac{5}{2} + \frac{1}{2}x_5 + 6x_6 \\
\bar{z}^{[1]} &= \frac{15}{2} - \frac{1}{2}x_5 - 3x_6.
\end{aligned}$$

Therefore, $\bar{z}^{[1]} = \frac{15}{2}$, $(\bar{x}_1^{[1]}, \bar{x}_2^{[1]}) = (2, \frac{1}{2})$.

Step 5: Branching. $(\bar{x}_1^{[1]}, \bar{x}_2^{[1]})$ is still not integral. Hence, $\mathscr{F}_1$ is not solved to optimality and cannot be pruned. Instead, we need to branch further.
Using the same approach as before, we introduce

$$\mathscr{F}_{11} = \mathscr{F}_1 \cap \{x : x_2 \le 0\} = \mathscr{F}_1 \cap \{x : x_2 = 0\} \quad (\text{since } x_2 \ge 0),$$
$$\mathscr{F}_{12} = \mathscr{F}_1 \cap \{x : x_2 \ge 1\}.$$

We now arrive at the partial enumeration tree below, and the new list of active nodes is $\mathscr{F}_{11}, \mathscr{F}_{12}, \mathscr{F}_2$.



Step 6: Choosing an active node. We arbitrarily choose $\mathscr{F}_2$ for processing.

Step 7: Bounding $(\mathscr{F}_2)$. We compute a bound $\bar{z}^{[2]}$ by solving the LP relaxation

$$
\begin{aligned}
(P_2) \quad z = \max\, & 4x_1 - x_2 \\
\text{s.t. } & 7x_1 - 2x_2 \leq 14 \\
& x_2 \leq 3 \\
& 2x_1 - 2x_2 \leq 3 \\
& x_1 \geq 3 \\
& x \geq 0
\end{aligned}
$$

of the problem

$$(\mathscr{F}_2) \quad z^{[1]} = \max\{c^{\mathrm{T}} x : \; x \in \mathscr{F}_2\}.$$

We express the new constraint $x_1 \geq 3$ in terms of the optimal nonbasic variables of (LP). We had found $x_1 = 20/7 - 1/7x_3 - 2/7x_4$. Thus, the new constraint is

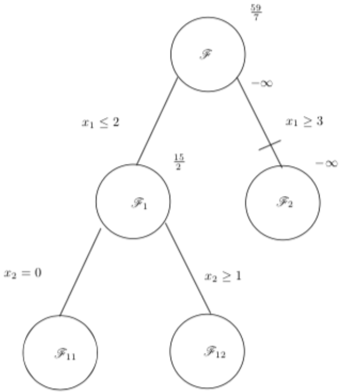
$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \leq -\frac{1}{7}.$$

After introducing a new slack variable x_6, we find that (LP_2) is equivalent to

$$\begin{aligned}
\bar{z} = \max \frac{59}{7} - \frac{4}{7}x_3 - \frac{1}{7}x_4 \\
\text{s.t. } x_1 &= \frac{20}{7} - \frac{1}{7}x_3 - \frac{2}{7}x_4 \\
x_2 &= 3 - 0 \cdot x_3 - x_4 \\
x_5 &= \frac{23}{7} + \frac{2}{7}x_3 - \frac{10}{7}x_4 \\
x_6 &= -\frac{1}{7} - \frac{1}{7}x_3 - \frac{2}{7}x_4 \\
x_1, \dots, x_6 &\geq 0.
\end{aligned}$$

But this LP is infeasible, because the last constraint contradicts $x_6 \geq 0$.

Hence, $\mathscr{F}_2$ can be *pruned by infeasibility*, and we arrive at the partial enumeration tree below.



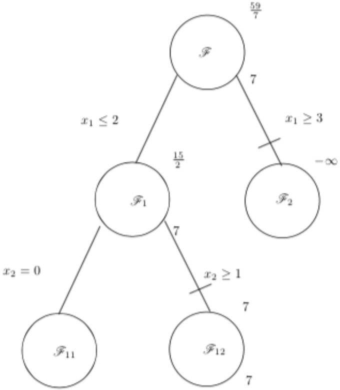
Key Idea (Pruning by bound)

Nodes whose bounds certify that they do not contain an optimal solution need not be further processed.

Step 8: Choosing an active node. From the list $\{\mathscr{F}_{\mathbf{11}}, \mathscr{F}_{\mathbf{12}}\}$ of active nodes we arbitrarily choose $\mathscr{F}_{\mathbf{12}}$.

Step 9: Bounding $(\mathscr{F}_{\mathbf{12}})$*.* Proceeding as above, we solve the LP relaxation of $(\mathscr{F}_{\mathbf{12}})$ and find the optimal solution $(\overline{x}_{\mathbf{1}}^{[\mathbf{12}]}, \overline{x}_{\mathbf{2}}^{[\mathbf{12}]}) = (2, 1)$ corresponding to the optimal value $\overline{z}^{[\mathbf{12}]} = 7$.

Since $\overline{x}^{[\mathbf{12}]}$ is integral, this is a feasible solution for $(\mathscr{F}_{\mathbf{1}})$ and provides a lower bound $\underline{z}^{[\mathbf{12}]} = 7$. In fact, $\mathscr{F}_{\mathbf{12}}$ can now be pruned *by optimality*. The partial enumeration tree is now as follows.



Key Idea (Pruning by optimality)

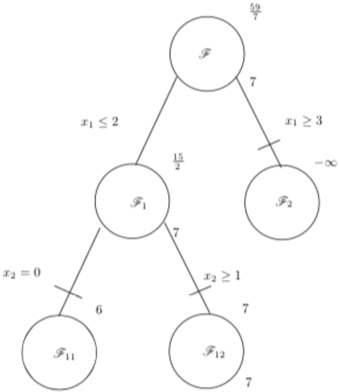
Nodes whose bounds certify that their subproblem has been solved to optimality need not be processed any further.

Step 10: Updating the incumbent. We store $(1,2)$ as the best integer solution found so far and update the lower bounds $\underline{z} \leftarrow \max(\underline{z}, 7) = 7$, $\underline{z}^{[\mathbf{1}]} \leftarrow \max(\underline{z}^{[\mathbf{1}]}, 7) = 7$.

Step 11: Choosing an active node. Only $\mathscr{F}_{\mathbf{11}}$ is active, so choose this node.

Step 12: Bounding. Proceeding as above, we solve the LP relaxation of $(\mathscr{F}_{\mathbf{11}})$ and find the optimal solution $(\bar{x}_{\mathbf{1}}^{[\mathbf{11}]}, \bar{x}_{\mathbf{2}}^{[\mathbf{11}]}) = \left(\frac{\mathbf{3}}{\mathbf{2}}, 0\right)$ with optimal value $\bar{z}^{[\mathbf{11}]} = 6$.

Since $\bar{z}^{[\mathbf{11}]} < \underline{z}$, we can prune $\mathscr{F}_{\mathbf{11}}$ *by bound* and arrive at the partial enumeration tree below.



Step 13: Termination. There are no further active nodes left, and the algorithm terminates, returning the optimal solution $z = 7$ and the maximiser $x = (2, 1)$ that achieves it.

General Branch & Bound Framework

Proposition (Divide and conquer)

Consider the problem

$$z = \max\{c^{\mathbf{T}} x : x \in \mathscr{F}\},$$

where $\mathscr{F}$ denotes the set of feasible solutions (as we saw, $\mathscr{F}$ is usually defined implicitly via constraints). If $\mathscr{F}$ can be decomposed into a union of simpler sets $\mathscr{F} = \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_k$ and if

$$z^{[j]} := \max\{c^{\mathbf{T}} x : x \in \mathscr{F}_j\} \quad (j = 1, \dots, k),$$

then

$$z = \max_j z^{[j]}.$$

Proof. $\mathscr{F}$ is a relaxation of $\mathscr{F}_j$, so that $z^{[j]} \leq z$ for all j, and hence, $\max_j z^{[j]} \leq z$. Let x^* be optimal for the master problem, i.e., $x^* \in \mathscr{F}$ such that $z = c^{\mathbf{T}} x^*$. Then $x^* \in \mathscr{F}_i$ for some i, so that $z = c^{\mathbf{T}} x^* \leq z^{[i]} \leq \max_j z^{[j]}$.

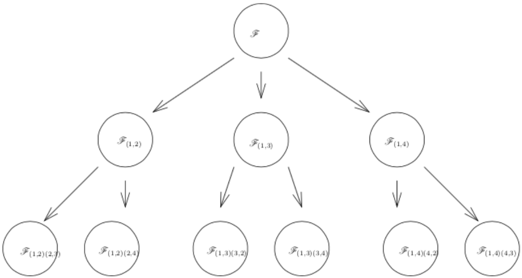
Example (Enumeration tree)

Let $\mathscr{F}$ be the set of feasible tours of the travelling salesman problem on a network of 4 cities. Let node 1 be the departure city.

$\mathscr{F}$ can be subdivided $\mathscr{F} = \mathscr{F}_{(\mathbf{1},\mathbf{2})} \cup \mathscr{F}_{(\mathbf{1},\mathbf{3})} \cup \mathscr{F}_{(\mathbf{1},\mathbf{4})}$ into the disjoint sets of tours that start with an arc $(1,2)$, $(1,3)$ or $(1,4)$ respectively.

Each of the sets $\mathscr{F}_{(\mathbf{1},\mathbf{2})}$, $\mathscr{F}_{(\mathbf{1},\mathbf{3})}$ and $\mathscr{F}_{(\mathbf{1},\mathbf{4})}$ can be further subdivided according to the choice of the second arc, $\mathscr{F}_{(\mathbf{1},\mathbf{2})} = \mathscr{F}_{(\mathbf{1},\mathbf{2})(\mathbf{2},\mathbf{3})} \cup \mathscr{F}_{(\mathbf{1},\mathbf{2})(\mathbf{2},\mathbf{4})}$ etc.

Finally, we see that each of these sets corresponds to a specific TSP tour and cannot be further subdivided. We have found an *enumeration tree* of the TSP tours.



Proposition (Bound propagation)

Consider the problem

$$z = \max\{c^{\mathrm{T}} x : x \in \mathscr{F}\},$$

and let $\mathscr{F} = \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_k$ *be a decomposition of its feasible domain into smaller sets. Let* $\underline{z}^{[j]} \leq z^{[j]} \leq \overline{z}^{[j]}$ *be lower and upper bounds on* $z^{[j]} = \max\{c^{\mathrm{T}} x : x \in \mathscr{F}_j\}$ *for all* j*. Then*

$$\underline{z} := \max_j \underline{z}^{[j]} \leq z \leq \max_j \overline{z}^{[j]} =: \overline{z}$$

gives an upper and lower bound on z*.*

Proof. Since $(\mathscr{F})$ is a relaxation of $(\mathscr{F}_j)$, we have $\underline{z}^{[j]} \leq z^{[j]} \leq z$ for all j, and hence,

$$\max_j \underline{z}^{[j]} \leq z.$$

On the other hand, by Proposition (Divide and conquer), we have

$$z = \max_j z^{[j]} \leq \max_j \overline{z}^{[j]}.$$

Proposition (Pruning by bound)

A branch $\mathscr{F}_j$ can be pruned when $\overline{z}^{[j]} \leq \underline{z}$.

Proof. By construction of $\underline{z}$, the incumbent x^* (our best solution for the root problem $(\mathscr{F})$ thus far encountered) satisfies

$$c^{\mathrm{T}} x^* = \underline{z} \geq \overline{z}^{[j]} \geq c^{\mathrm{T}} x \quad \forall x \in \mathscr{F}_j.$$

Therefore, the incumbent cannot be improved by searching over $\mathscr{F}_j$.

Proposition (Pruning by infeasibility)

If $\mathscr{F}_j = \emptyset$, then the corresponding branch can be pruned.

Proof. This is a special case of pruning by bound in which $\overline{z}^{[j]} = -\infty$.

Remark

It may not be obvious that $\mathscr{F}_j$ is empty, but this may be algorithmically detected, e.g., by LP relaxation.

Proposition (Pruning by optimality)

When $\underline{z}^{[j]} = \overline{z}^{[j]}$ for some j, then the branch corresponding to $\mathscr{F}_j$ needs no further consideration.

Proof. $z^{[j]} = \underline{z}^{[j]} = \overline{z}^{[j]}$ certfies that we have solved this branch to optimality. The optimal point of the branch has automatically become the incumbent if it is the best solution found so far for the root problem.

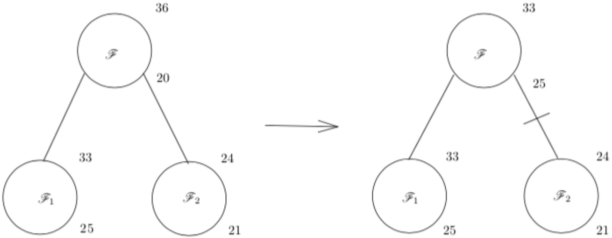


FIG. 3.2. *Pruning by bound.*

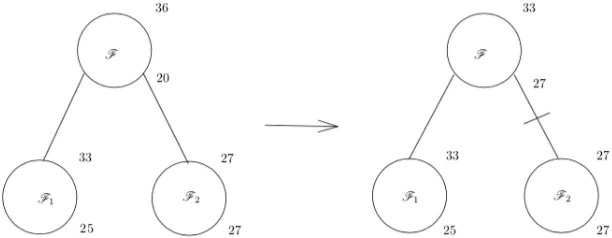


FIG. 3.3. *Pruning by optimality.*