Brady Metherall

11 November 2019

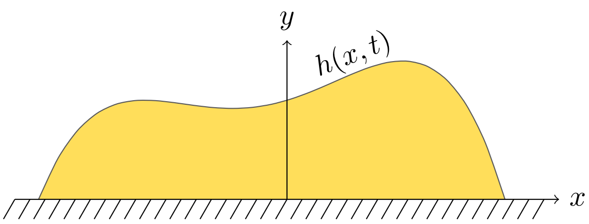


Figure 1: Beer spilled on a table.

1 Introduction

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g} \tag{1}$$

$\varepsilon := HL^{-1} \ll 1$

2 Flat Table

We shall begin by non-dimensionalizing (1) by scaling time by the characteristic length divided by the characteristic fluid speed, the pressure as hydrostatic, and by assuming the fluid motion is driven by the pressure gradient [1]. This yields

$$\begin{aligned} \mathrm{Re}'(u_t + u\,u_x + v\,u_y) &= -p_x + \varepsilon^2 u_{xx} + u_{yy}, \\ \mathrm{Re}'\varepsilon^2(v_t + u\,v_x + v\,v_y) &= \\ &\quad -p_y + \varepsilon^4 v_{xx} + \varepsilon^2 v_{yy} - 1, \end{aligned} \tag{2}$$

for the x and y components, respectively, and where Re' is the reduced Reynolds number. To leading order we obtain

$$\mathcal{O}(1): \quad p_x = u_{yy}, \quad \mathcal{O}(1): \quad p_y = -1. \tag{3}$$

After imposing the boundary conditions that $u = 0$ on $x = 0$, and $u_y = 0$ on $y = h$, where h is the height of the fluid, we find

$$u = -\frac{1}{2}(2hy - y^2)h_x. \tag{4}$$

We can now readily find the flux of the flow, and therefore, develop an equation of conservation. The mass flux is given as the flow of the fluid multiplied by its velocity—that is, the flux is $\int_0^h u\,\mathrm{d}y$. Conservation of mass now yields

$$h_t = \frac{1}{3}\nabla\cdot\left(h^3\nabla h\right). \tag{5}$$

$$\int_{\mathbb{R}} h\,\mathrm{d}x \tag{6}$$

$h(x,t) = t^\alpha f(\eta)$ where $\eta = xt^{-\beta}$
$\alpha = -1/5$ $\beta = 1/5$

$$\frac{-3}{5}(\eta f' + f) = (f^3 f)'$$

$$f = \left(\frac{9}{10}\right)^{1/3}\left(\eta_*^2 - \eta^2\right)^{1/3}$$

$$\begin{aligned}\eta_* &= \left(\frac{6075\Gamma^6\left(\frac{2}{3}\right)\Gamma^6\left(\frac{11}{6}\right)}{16\pi^9}\right)^{1/10} \\ &\approx 0.747412\end{aligned}$$

2.1 Polar

solve (5) and (6) in polar
$\alpha = -1/4$ $\beta = 1/8$

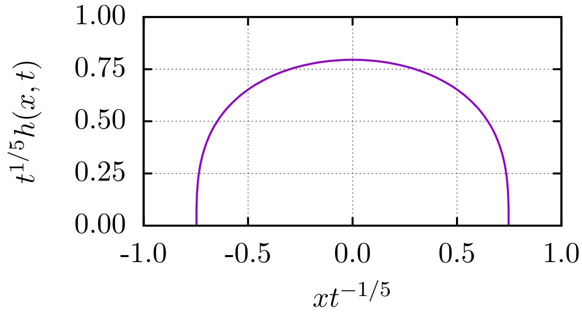
$$\frac{-3}{8}(2\eta f + \eta^2 f') = (\eta f^3 f')'$$

$$f = \left(\frac{9}{16}\right)^{1/3}\left(\eta_*^2 - \eta^2\right)^{1/3}$$

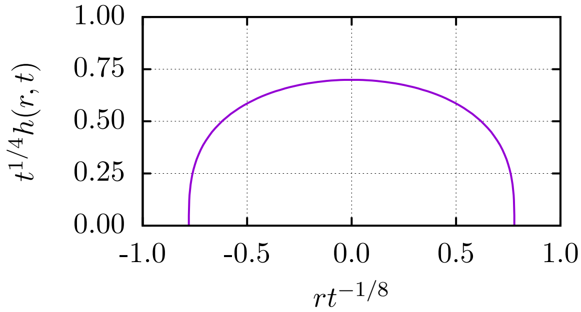
$$\begin{aligned}\eta_* &= \left(\frac{1024}{243\pi^3}\right)^{1/8} \\ &\approx 0.779212\end{aligned}$$

3 Angled Table

$$\begin{aligned}\mathrm{Re}'(u_t + u\,u_x + v\,u_y) &= \\ &\quad -\varepsilon p_x + \varepsilon^2 u_{xx} + u_{yy} + \sin\alpha,\end{aligned} \tag{7}$$



(a)



(b)

Figure 2

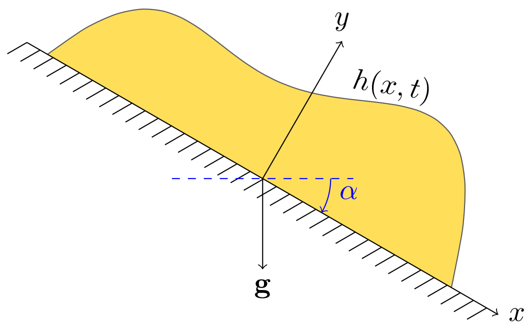


Figure 3: Beer spilled on a crooked table.

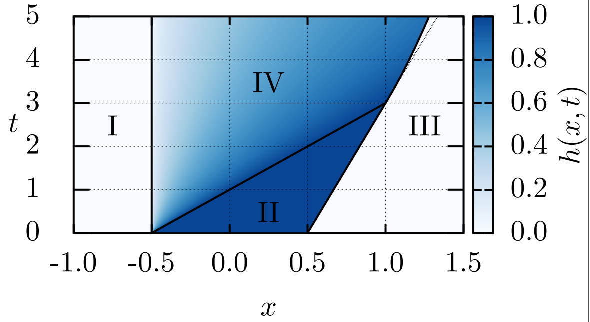


Figure 4: $\alpha = \pi/6$

$$\mathrm{Re}'\varepsilon(v_t + u\,v_x + v\,v_y) = \\ - p_y + \varepsilon^3 v_{xx} + \varepsilon v_{yy} - \cos\alpha \quad (8)$$

$$\mathcal{O}(1): \qquad u_{yy} = -\sin\alpha \qquad (9)$$

$$\mathcal{O}(1): \qquad p_y = -\cos\alpha \qquad (10)$$

$$u = \frac{1}{2}\sin\alpha\big(2hy - y^2\big) \qquad (11)$$

mass conservation becomes

$$h_t + \frac{1}{3}\sin\alpha(h^3)_x = 0 \qquad (12)$$

$$h_0(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (13)$$

[2, 3]

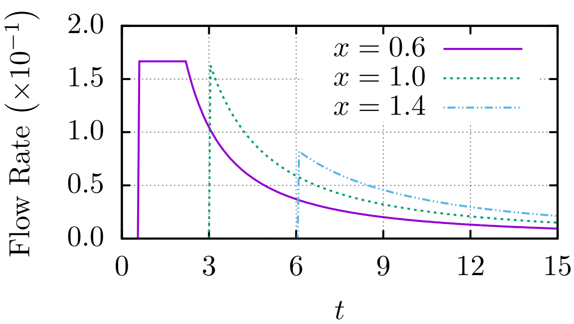


Figure 5: $\alpha = \pi/6$

4 Conclusion

References

[1] S. Howison, *Practical Applied Mathematics: Modelling, Analysis, Approximation.* Cambridge Texts in Applied Mathematics, Cambridge University Press, 2005.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, 8 ed., 2015.

[3] D. Zwillinger, *Handbook of Differential Equations.* Academic Press, 3 ed., 1998.