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To start off the discussion of the current modelling efforts for a tuneable laser, we begin with a review of the efforts to describe an 'average' model. The idea is to capture some of the physical the elements in the waveform (dispersion, modulation and gain/loss) described by an effective PDE, the solution of which gives the amplitude of the wave packet.

1 Average Model

The model for the amplitude in the 'average' model is presupposed to have the form

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \frac{\epsilon}{2}T^2 A + \frac{g}{2}A \tag{1}$$

where $A = A(T, z)$ is the complex amplitude of the pulse with $\beta_2 \in \mathbb{R}$ defining the dispersion, $\epsilon \in \mathbb{R}$, $\epsilon > 0$, determining the modulation and $g \in \mathbb{R}$, $g > 0$ giving the gain. Expression (1) is reminiscent of the nonlinear Schrödinger equation but linear in A so that the solution can take the form of a Gaussian wavepacket. Consequently, the ansatz for the amplitude is taken to be a function of the form

$$A(T, z) = \left(\frac{P_0}{1 - iC}\right)^{1/2} \exp\left(-\frac{\delta\Omega^2 T^2}{2(1 - iC)}\right) \mathrm{e}^{i\psi z} \tag{2}$$

so that the modulus, $|A|$, and the phase, $A/|A|$, are given by

$$|A| = \left(\frac{P_0}{\sqrt{1 + C^2}}\right)^{1/2} \exp\left(-\frac{\delta\Omega^2 T^2}{2(1 + C^2)}\right), \quad \frac{A}{|A|} = \left(\frac{1 + iC}{\sqrt{1 + C^2}}\right)^{1/2} \exp\left(-i\frac{\delta\Omega^2 T^2 C}{2(1 + C^2)}\right) \mathrm{e}^{i\psi z}.$$

The quantity $C \in \mathbb{R}$ is known as the chirp and contributes a constant phase of θ where $2\theta = \arctan C$, P_0 is the maximum value of $|A|^2$ at $C = 0$ (zero chirp), $\psi \in \mathbb{R}$ is the accumulated phase, and $\delta\Omega$ is the spectral half-width of $|A|^2$ since[1]

$$\hat{A}(\omega, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(t, z)\mathrm{e}^{-i\omega t}\,\mathrm{d}t = \left(\frac{P_0}{2\pi\delta\Omega^2}\right)^{1/2} \exp\left(-\frac{(1 - iC)\omega^2}{2\delta\Omega^2}\right) \mathrm{e}^{i\psi z},$$

and $|\hat{A}(\delta\Omega, z)|^2 = \mathrm{e}^{-1}|\hat{A}(0, z)|^2$. The corresponding half-width of the pulse duration comes from the expression for $|A|$ and gives

$$\delta T = \frac{\sqrt{1 + C^2}}{\delta\Omega}. \tag{3}$$

[1] If $f(t) = \mathrm{e}^{-\alpha t^2}$ then $\hat{f}(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t = \frac{1}{2\pi}(\frac{\pi}{\alpha})^{1/2}\mathrm{e}^{-\omega^2/4\alpha}$.

Applying the form (2) to expression (1) provides the algebraic condition

$$\begin{aligned}
i\psi &= -i\frac{\beta_2}{2}\delta\Omega^2(1-iC)^{-1}\left(-1+\delta\Omega^2(1-iC)^{-1}T^2\right)-\frac{\epsilon}{2}T^2+\frac{g}{2}\\
&= \frac{\beta_2\delta\Omega^2}{2(1+C^2)^2}(-C(1+C^2)+2\delta\Omega^2T^2C)-\frac{\epsilon}{2}T^2+\frac{g}{2}\\
&\quad + i\frac{\beta_2\delta\Omega^2}{2(1+C^2)^2}(1+C^2+\delta\Omega^2T^2(C^2-1)).
\end{aligned}$$

Treating this condition as a complex valued quadratic in T gives four conditions. Assuming that $\psi\in\mathbb{R}$, the real and imaginary components at $\mathcal{O}(1)$ and $\mathcal{O}(T^2)$ yield

$$\begin{aligned}
&\mathcal{O}(T^2)_{\mathrm{Im}}: 0 = C^2-1, && \mathcal{O}(1)_{\mathrm{Im}}: \psi = \frac{\beta_2\delta\Omega^2}{2(1+C^2)},\\
&\mathcal{O}(T^2)_{\mathrm{Re}}: \epsilon = \frac{2\beta_2\delta\Omega^4C}{(1+C^2)^2}, && \mathcal{O}(1)_{\mathrm{Re}}: g = \frac{\beta_2\delta\Omega^2C}{(1+C^2)}.
\end{aligned}$$

Starting with $\mathcal{O}(1)_{\mathrm{Re}}$ we note that $g>0$ implies that $\mathrm{sgn}(\beta_2C)=\mathrm{sgn}(\beta_2)\,\mathrm{sgn}(C)=1$. From $\mathcal{O}(T^2)_{\mathrm{Im}}$, $C=\pm1$ and therefore $C=\mathrm{sgn}(\beta_2)$, $\beta_2C=|\beta_2|$ and $\epsilon>0$, consistent with (1). The result is a two parameter family of solutions of the form (2) with

$$C=\mathrm{sgn}(\beta_2),\qquad \delta\Omega^2=\left(\frac{2\epsilon}{|\beta_2|}\right)^{1/2},\qquad \psi=\mathrm{sgn}(\beta_2)\left(\frac{\epsilon|\beta_2|}{8}\right)^{1/2},\qquad g=\left(\frac{\epsilon|\beta_2|}{2}\right)^{1/2}.$$

Moreover, we also see that the representation (2) as a classical solution of (1) imposes a subclass of solutions with $g=(\epsilon|\beta_2|/2)^{1/2}$. A useful pulse characterization is the half-width of the pulse duration, δT, which satisfies

$$\delta T^2=\left(\frac{2|\beta_2|}{\epsilon}\right)^{1/2}.$$

Since (1) is linear in A, any value of the peak power, P_0, is admissible. In practice, at high power levels, the gain drops as the fibre saturates. One can model this with a gain term that takes the form

$$g(P_0)=\frac{g_0}{1+\frac{\text{power in fibre}}{\text{saturation power}}}-\alpha \tag{4}$$

where g_0 is the low-power gain and α represents the net losses in the laser cavity. The power in the fibre depends on the frequency f and modulus of the pulse so that

$$f\int_{-\infty}^{\infty}|A(s)|^2\,\mathrm{d}s=\frac{\sqrt{\pi}f}{\delta\Omega}P_0=\Delta P_0,\qquad \Delta=\frac{\sqrt{\pi}f}{\delta\Omega}=\frac{\sqrt{\pi}f\delta T}{\sqrt{1+C^2}}=\sqrt{\pi}f\left(\frac{|\beta_2|}{2\epsilon}\right)^{1/4}$$

where Δ is the duty cycle of the pulse. Denoting the saturation power as P_{sat}, (4) can be inverted to give

$$P_0=\frac{P_{\mathrm{sat}}}{\sqrt{\pi}f}\left(\frac{2\epsilon}{|\beta_2|}\right)^{1/4}\left(g_0\left(\left(\frac{\epsilon|\beta_2|}{2}\right)^{1/2}+\alpha\right)^{-1}-1\right).$$

Figure **??** shows a select set of possible waveforms in $(\epsilon, |\beta_2|)$ parameter space. The representative hyperbola, $\epsilon|\beta_2| = 2g^2$ correspond to constant gain while the rays through the origin, $|\beta_2|/\epsilon = \delta T^4/4$, correspond to constant width. Curves of constant power level are also indicated. One of the main drawbacks of this model is the fixed chirp and its symmetric behaviour with respect to T. One way to alleviate some of these restrictions is to construct a discrete model whereby each of the modules in the tuneable laser are given by a transfer function that is motivated by the PDE (1). In particular, the dispersion and the modulation would correspond to transfer functions of the form

$$\hat{A}_{\text{out}}(\omega) = \hat{A}_{\text{in}}(\omega)\mathrm{e}^{i\beta_2\omega^2/2}, \qquad A_{\text{out}}(T) = A_{\text{in}}(T)\mathrm{e}^{-\epsilon T^2/2}, \tag{5}$$

respectively where the dispersion is naturally defined in the frequency domain. Between the modules, the pulse is assumed to propagate according to the representation (2) chosen to be consistent with the transfer functions. Details of this technique and its ability of predict a number of experimental effects can be found in [4].

2 A New Model

Rather than using a transfer function with a linear PDE, we instead return to generalized nonlinear Schrödinger equation [2, 5, 7, 9]

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} + i\gamma|A|^2 A + \frac{1}{2}g(A)A - \alpha A, \tag{6}$$

to represent the waveform. In this expression, β_3 is the third order dispersion coefficient, γ is the coefficient of nonlinearity, $g(A)$ is the gain, and α is the loss of the fibre. Using this expression as a starting point, the laser cavity is assumed to be composed of five independent processes—gain, nonlinearity, loss, dispersion, and modulation. Within each component of the laser cavity the other four process are assumed to be negligible, that is, each process is dominant only within one part of the laser, just as with the discrete model, but embracing any nonlinearities.

2.1 Gain

Considering the gain term as dominant as is expected with the the Er-doped gain fibre, equation (6) reduces to

$$\frac{\partial A}{\partial z} = \frac{1}{2}g(A)A, \qquad g(A) = \frac{g_0}{1 + E/E_{\text{sat}}}, \qquad E = \int_{-\infty}^{\infty} |A|^2 \,\mathrm{d}T, \tag{7}$$

where g_0 is a small signal gain, E is the energy of the pulse [3, 4, 7, 8, 9] and E_{sat} is the energy at which the gain begins to saturate. Multiplying (7) by $\bar{A}$, the complex conjugate of A, yields

$$2\bar{A}\frac{\partial A}{\partial z} = \frac{g_0|A|^2}{1 + E/E_{\text{sat}}}.$$

Adding this to its complex conjugate and integrating over T gives

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{g_0 E}{1 + E/E_{\mathrm{sat}}}. \tag{8}$$

For $E \ll E_{\mathrm{sat}}$ the energy grows exponentially, whereas for $E \gg E_{\mathrm{sat}}$ the gain has saturated and so the energy grows linearly. To obtain a closed form, integrate (8) over a gain fibre of length z and assume the energy increases from E to E_{out} so that

$$g_0 z = \log \frac{E_{\mathrm{out}}}{E} + \frac{E_{\mathrm{out}} - E}{E_{\mathrm{sat}}}$$

and by exponentiating, rearranging, and applying W_0, the positive branch of the Lambert W function,

$$W_0\left(\frac{E}{E_{\mathrm{sat}}}\mathrm{e}^{E/E_{\mathrm{sat}}}\mathrm{e}^{g_0 z}\right) = W_0\left(\frac{E_{\mathrm{out}}}{E_{\mathrm{sat}}}\mathrm{e}^{E_{\mathrm{out}}/E_{\mathrm{sat}}}\right) = \frac{E_{\mathrm{out}}}{E_{\mathrm{sat}}}.$$

This results in the closed form expression

$$E_{\mathrm{out}}(z) = E_{\mathrm{sat}} W_0\left(\frac{E}{E_{\mathrm{sat}}}\mathrm{e}^{E/E_{\mathrm{sat}}}\mathrm{e}^{g_0 z}\right) \tag{9}$$

with the desired property that $E_{\mathrm{out}}(0) = E$. Since $E \sim |A|^2$, the gain in terms of the amplitude is given by

$$G(A; E) = \left(\frac{E_{\mathrm{out}}(L_g)}{E}\right)^{1/2} A = \left(\frac{E_{\mathrm{sat}}}{E} W_0\left(\frac{E}{E_{\mathrm{sat}}}\mathrm{e}^{E/E_{\mathrm{sat}}}\mathrm{e}^{g_0 L_g}\right)\right)^{1/2} A, \tag{10}$$

where L_g is the length of the gain fibre.

2.2 Fibre Nonlinearity

The nonlinearity of the fibre depends on the parameter γ. In regions where this effect is dominant expression (6) gives

$$\frac{\partial A}{\partial z} - i\gamma |A|^2 A = 0, \tag{11}$$

so that $\frac{\partial}{\partial z}|A|^2 = 0$ suggesting that $A(T, z) = A_0(T)e^{i\varphi(T,z)}$. Substituting this representation into (11) and setting $\varphi(T, 0) = 0$ gives $\varphi(T, z) = \gamma|A|^2 z$. For a fibre of length L_f the effect of the nonlinearity is therefore

$$F(A) = Ae^{i\gamma|A|^2 L_f}. \tag{12}$$

2.3 Loss

Two sources of loss exist within the laser circuit: the loss due to the output coupler and the optical loss due to absorption and scattering. Combining these two effect gives a loss that takes the form

$$L(A) = Ce^{-\alpha L} A, \tag{13}$$

where C is the percentage lost due to the output coupler, and L is the length of the laser circuit.

2.4 Dispersion

Within the laser cavity, the dispersion is dominated by the chirped fibre Bragg grating (CFBG). In comparision, the dispersion due to the fibre is negligible.[2] The dispersive terms of (6) give

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} \tag{14}$$

and since dispersion acts in the frequency domain, it is convenient to use the Fourier transform of (14), giving the result that

$$\frac{\partial \mathcal{F}\left\{A\right\}}{\partial z} = i\frac{\omega^2}{2}\left(\beta_2 + \frac{\beta_3}{3}\omega\right)\mathcal{F}\left\{A\right\}.$$

The affect of dispersion is then

$$D(A) = \mathcal{F}^{-1}\left\{\mathrm{e}^{i\omega^2 L_D(\beta_2+\beta_3\omega/3)/2}\mathcal{F}\left\{A\right\}\right\}. \tag{15}$$

For a highly dispersive media the third order effects may need to be considered [2, 6]. However, for simplicity in the basic model, the third order effect will be neglected so we set $\beta_3 = 0$ for the subsequent analysis [2, 5].

2.5 Modulation

In the average model the amount of modulation is characterized by the parameter ϵ through the term $\frac{\epsilon}{2}T^2 A$. In the new model, the modulation is considered to be applied externally through its action on the spectrum and for simplicity the representation is taken as a Gaussian pulse

$$M(A) = \mathrm{e}^{-T^2/2T_M^2}A, \tag{16}$$

where T_M is a characteristic width of the modulation.[3]

2.6 Non-Dimensionalization

The structure of each process of the laser can be better understood by re-scaling the time, energy, and amplitude. Specifically, the time shall be scaled by the characteristic modulation time, the energy by the saturation energy, and the amplitude will be scaled so that it is consistent:

$$T = T_M\widetilde{T}, \qquad E = E_{\mathrm{sat}}\widetilde{E}, \qquad A = \left(\frac{E_{\mathrm{sat}}}{T_M}\right)^{1/2}\widetilde{A}.$$

[2] A 10 cm chirped grating can provide as much dispersion as 300 km of fibre [1].

[3] Taking the Fourier transform, $\mathcal{F}\left\{\mathrm{e}^{-T^2/2T_M^2}\right\} = \frac{1}{\sqrt{2\pi}}T_M\mathrm{e}^{-\omega^2T_M^2/2}$.

The new process maps, after dropping the tildes, become

$$G(A) = \left(E^{-1}W_0\left(aEe^E\right)\right)^{1/2} A, \qquad F(A) = A\mathrm{e}^{ib|A|^2}, \qquad L(A) = hA,$$
$$D(A) = \mathcal{F}^{-1}\left\{\mathrm{e}^{is^2\omega^2}\mathcal{F}\left\{A\right\}\right\}, \qquad M(A) = \mathrm{e}^{-T^2/2}A,$$

with four dimensionless parameter groups (see Table 1)

$$a = \mathrm{e}^{g_0 L_g} \sim 30, \qquad s = \sqrt{\frac{\beta_2 L_D}{2T_M^2}} \sim 0.1, \qquad b = \gamma L_F \frac{E_{\mathrm{sat}}}{T_M} \sim 0.1, \qquad h = C\mathrm{e}^{-\alpha L} \sim 0.25,$$

which control the behaviour of the laser.

2.7 Combining the Pieces

In this model the pulse is iteratively passed through each process, the order of which is now important. In this first realization, the pulse is first amplified by the gain fibre, then since the pulse's magnitude is greatest the nonlinearity needs to be considered. The pulse is then tapped off by the output coupler, and then passes through the grating and is modulated. The pulse after one complete circuit of the laser cavity is then passed back in to restart the process. Functionally this can be denoted as

$$\mathcal{L}(A) = M(D(L(F(G(A))))),$$

where $\mathcal{L}$ is one loop of the circuit. A solution to this model is one in which the envelope is unchanged after traversing every component in the cavity, that is, such that $|\mathcal{L}(A)| = |A|$.

3 Results

The model described is solved numerically and somewhat surprisingly, its convergence as a function of the characteristic parameter groups is highly complex. To gain some insight into this process we consider an linear analysis that ignores the nonlinear fibre effects.

3.1 Linear Solution

Since the pulse is modulated by a Gaussian, it is expected that the envelope will also converge to a Gaussian. Consider an initial pulse with envelope

$$|A_0| = \sqrt{P}\mathrm{e}^{-T^2/2\sigma^2},$$

after passing through the gain and loss pieces, and neglecting the fibre nonlinearity, the envelope will have the form

$$|A| = \sqrt{P}g(E)h\mathrm{e}^{-T^2/2\sigma^2}, \qquad g(E) = \left(\frac{W_0(aE\mathrm{e}^E)}{E}\right)^{1/2},$$

where $g(E)$ is the gain component. After the pulse travels through the grating, the envelope will maintain its Gaussian shape, however, it will have spread. This can be represented as

$$|A| = \sqrt{P} g(E) h \zeta(s,0) \mathrm{e}^{-T^2/2\tilde{\sigma}^2} \tag{17}$$

where $\zeta(s,b)$ is the decrease in the amplitude and $\widetilde{\sigma}^2$ is the new variance. Because the dispersive element conserves energy,

$$\int_{-\infty}^{\infty} P g(E)^2 h^2 \mathrm{e}^{-T^2/\sigma^2}\,\mathrm{d}T = \int_{-\infty}^{\infty} P g(E)^2 h^2 \zeta^2(s,0) \mathrm{e}^{-T^2/\tilde{\sigma}^2}\,\mathrm{d}T,$$

or more simply $\sigma = \zeta^2(s,0)\widetilde{\sigma}$. Therefore, the envelope as the pulse exits the grating given by (17) is

$$|A| = \sqrt{P} g(E) h \zeta(s,0) \exp\left(-\zeta^4(s,0)\frac{T^2}{2\sigma^2}\right).$$

Finally, the pulse is modulated and so that after one loop the total effect is given by

$$|A| = \sqrt{P} g(E) h \zeta(s,0) \exp\left(-\zeta^4(s,0)\frac{T^2}{2\sigma^2} - \frac{T^2}{2}\right). \tag{18}$$

In equilibrium the envelope of the pulse is unchanged after completing a full circuit, in other words,

$$\sqrt{P}\mathrm{e}^{-T^2/2\sigma^2} = \sqrt{P} g(E) h \zeta(s,0) \exp\left(-\zeta^4(s,0)\frac{T^2}{2\sigma^2} - \frac{T^2}{2}\right),$$

or more simply,

$$\left(\frac{W_0(aE\mathrm{e}^E)}{E}\right)^{1/2} h\zeta(s,0) = 1, \qquad -\frac{T^2}{2\sigma^2} = -\zeta^4(s,0)\frac{T^2}{2\sigma^2} - \frac{T^2}{2}. \tag{19}$$

The second condition simplifies to

$$\sigma = \sqrt{1 - \zeta^4(s,0)}$$

and the peak of the pulse can be found by integration to give

$$P = \frac{E}{\sqrt{\pi}\sigma}.$$

Furthermore, from (19) the equilibrium energy is found to be

$$E = \frac{\zeta^2(s,0)h^2}{1 - \zeta^2(s,0)h^2} \ln\left(a\zeta^2(s,0)h^2\right).$$

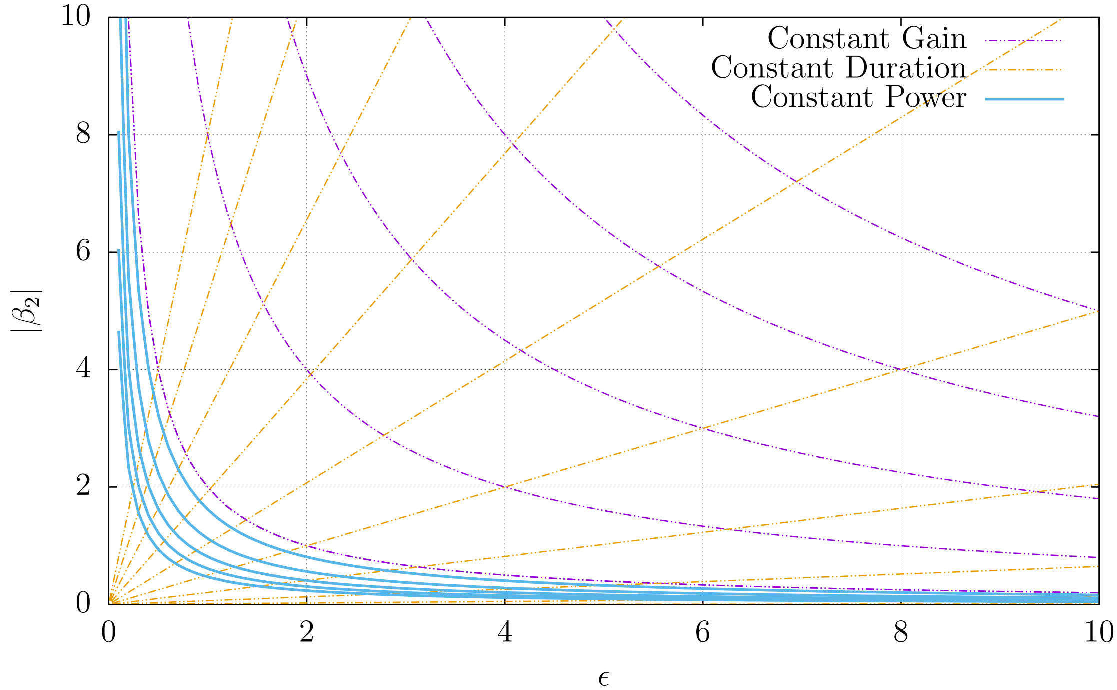

Figure 1:

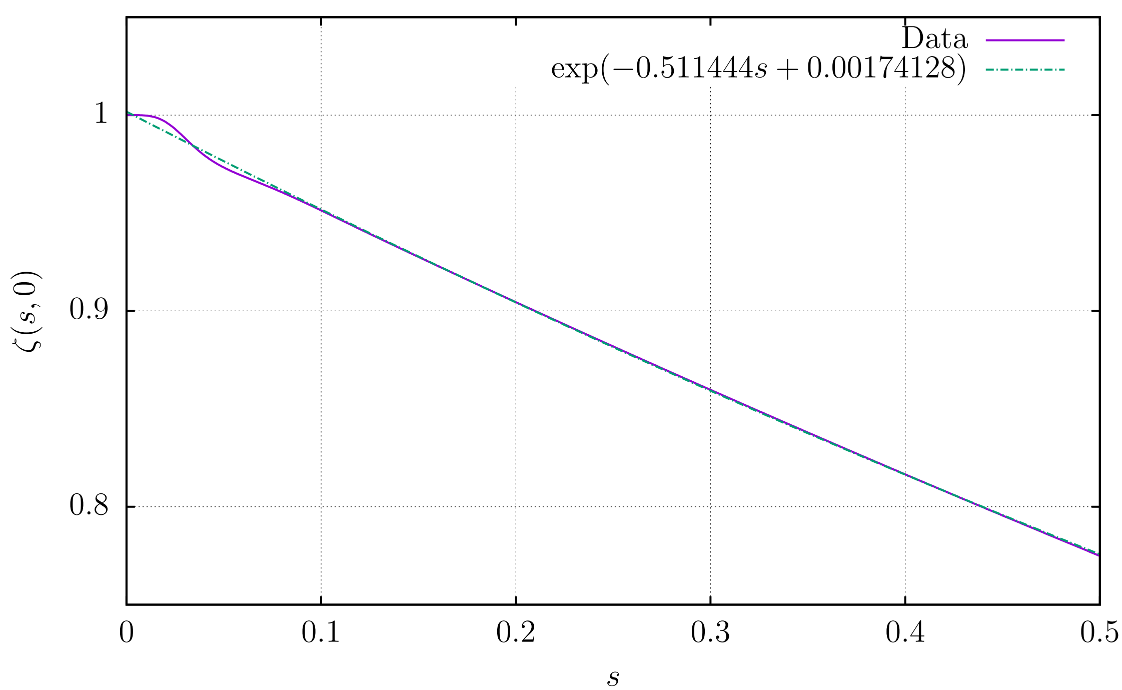

Figure 2:

Figure 3: Error is calculated by $\dfrac{\|A(\text{iteration } 75) - A(\text{iteration } 74)\|_2}{\|A(\text{iteration } 74)\|_2}$

Figure 4:

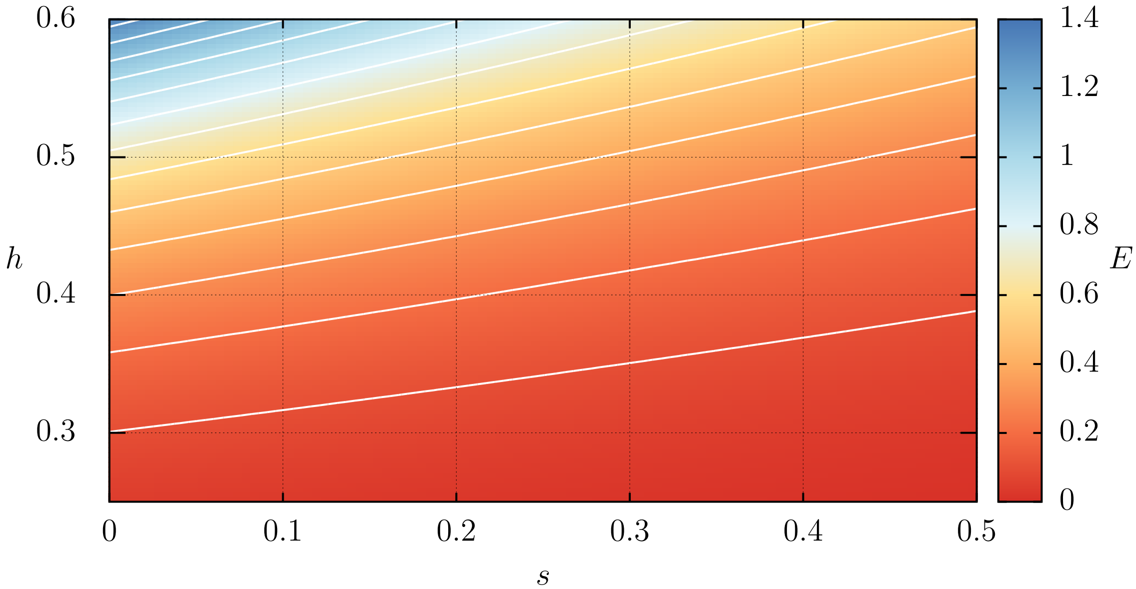

Figure 5: Equation (19) for $a = 30$

Parameter	Value
$\beta_2^g L_D$	10–2000ps^2
g_0	1–10m^{-1}
β_2^f	20–50ps^2/km
γ	0.001–0.01W^{-1}m^{-1} [2]
E_{sat}	10^4pJ
L_G	2–3m [4, 7, 9]

Table 1:

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