A Nonlinear Model for Dispersion-Tuned Actively Mode-Locked Lasers

by

Brady Metherall

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Abstract

A new nonlinear model is proposed for tuneable lasers. Using the generalized nonlinear Schrödinger equation as a starting point, expressions for the transformations undergone by the pulse are derived for each component in the cavity. These transformations are then composed to give the overall effect of one trip around the cavity. The linear version of this model is solved analytically, and the nonlinear version numerically. A consequence of this model being nonlinear is that it is able to exhibit wave breaking which prior models could not. We highlight the rich structure of the boundary of stability for a particular plane of the parameter space.

Acknowledgements

Thank some people that you like here.

Author's Declaration

I declare that the work in this thesis was carried out in accordance with the regulations of the University of Ontario Institute of Technology. The work is original except where indicated by special reference in the text and no part of the dissertation has been submitted for any other degree. Any views expressed in the dissertation are those of the author and in no way represent those of the University of Ontario Institute of Technology. This thesis has not been presented to any other university for examination either in Canada or elsewhere.

Brady Metherall
Friday 8$^{\text{th}}$ February, 2019

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Introduction

1.1 Tuneable Lasers

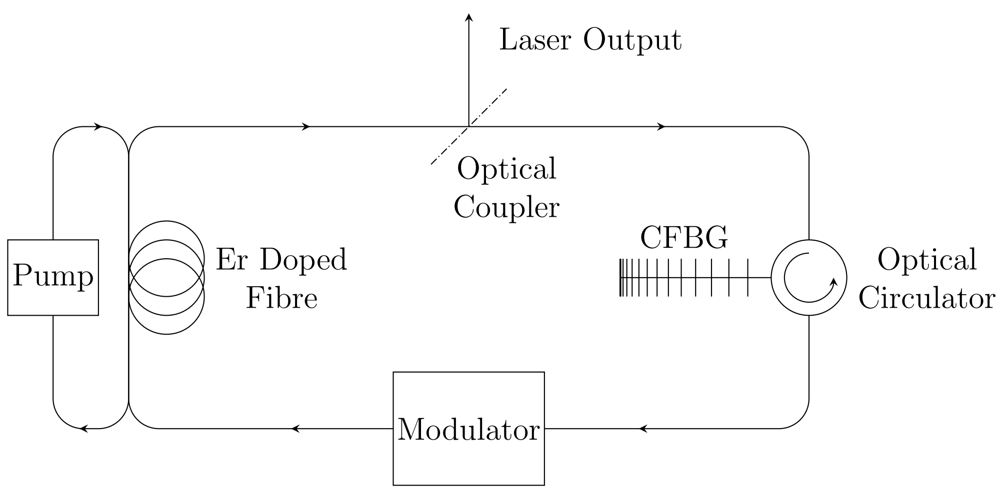


Figure 1.1: Laser cavity schematic.

1.1.1 Gain and Pump

1.1.2 Optical Coupler

1.1.3 Modulator

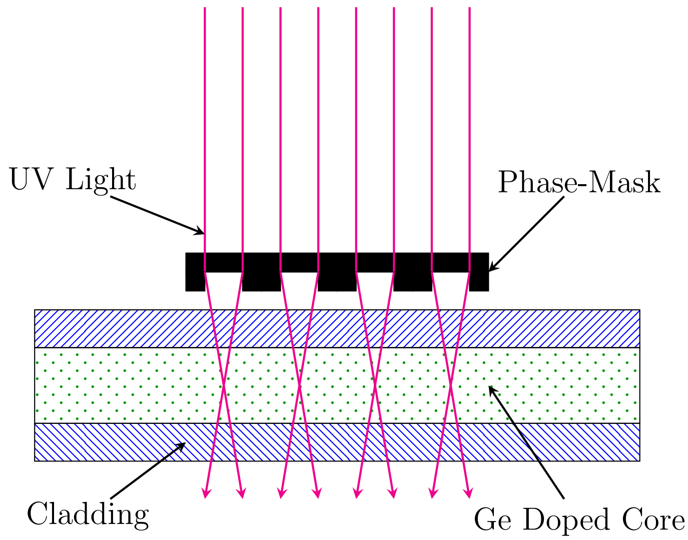
1.1.4 Fibre Bragg Grating

A fibre Bragg grating (FBG) is an optical fibre where the refractive index varies periodically along its length [7]. To achieve this, typically silica fibres are doped with Germanium, which when exposed to intense ultraviolet (UV) light alter the refractive index of the core [8]. The photosensitivity of the optical fibres can be increased by more than an order of magnitude by the Germanium doping [7].

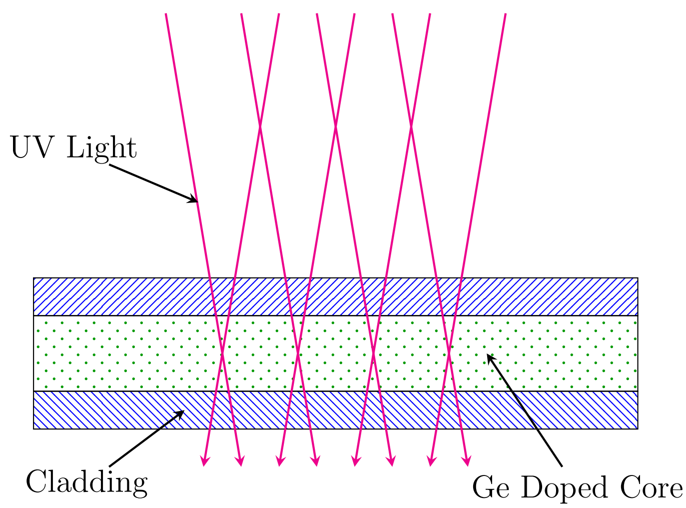
FBGs are manufactured using one of two methods—the phase-mask method [5, 6, 8], or the holographic side exposure method [5–8]—these are shown in Figure 1.2. Both methods cause the periodic nature of the refractive index through interference. In the phase-mask method (Figure 1.2a) a single beam of UV light passes through the phase-mask which acts as a series of lenses focusing the light at the core—this causes a sinusoidal interference pattern. Similarly, in the holographic side exposure method (Figure 1.2b), two beams of UV light are instead used to create the interference pattern.

The purpose of FBGs is that they act as reflective filters [5–8]. Due to the periodicity of the refractive index, light with the corresponding wavelength will be reflected, with all others passing through. This wavelength is defined by the Bragg condition [5–9]:

$$\lambda_B = 2\Lambda\bar{n}, \tag{1.1}$$

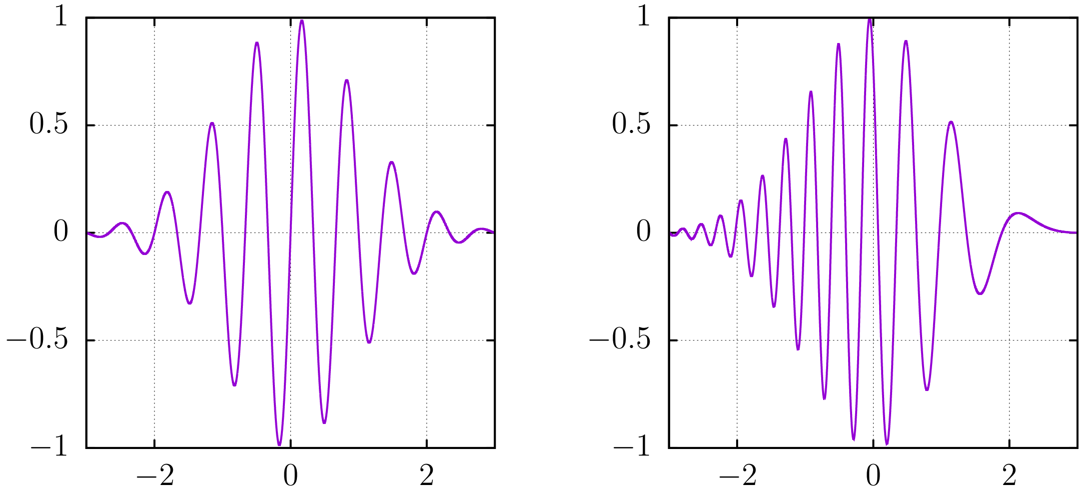


(a) **Phase-Mask:** The phase-mask focuses the ultraviolet light onto the centre of the core causing periodic constructive and destructive interference.



(b) **Holographic Side Exposure:** Two incident ultraviolet waves cause periodic constructive and destructive interference at the centre of the core.

Figure 1.2: Depictions of the two methods for manufacturing FBGs.



(a) Unchirped Gaussian pulse. (b) Linearly chirped Gaussian pulse.

Figure 1.3: In a unchirped pulse the frequency is constant, however, in a chirped pulse the frequency varies along the envelope.

where λ_B is the Bragg wavelength, Λ is the period of the grating, and $\bar{n}$ is the average index of refraction. This reflects wavelengths close to λ_B, known as the stop-band. In this way, FBGs as be used as filters.

Chirped Fibre Bragg Grating

Chirp is simply the term for a signal that has a non-constant frequency across it. Figure 1.3 shows examples of chirped and unchirped Gaussian pulses—the most common type of chirp is linear chirp, where the frequency varies linearly across the pulse. By using a chirped phase-mask a chirped fibre Bragg grating (CFBG) can be created. Since the period of the refractive index varies across the CFBG, so does when the Bragg condition, (1.1), is satisfied. This causes most wavelengths to be reflected by a CFBG, but with each wavelength penetrating to a different depth. A consequence of this is that a time delay is caused between wavelengths—this is depicted in Figure 1.4 with the upper portion showing the refractive index as a function of the depth. In this orientation, the red (dashed) wave is unable to penetrate as far as the blue

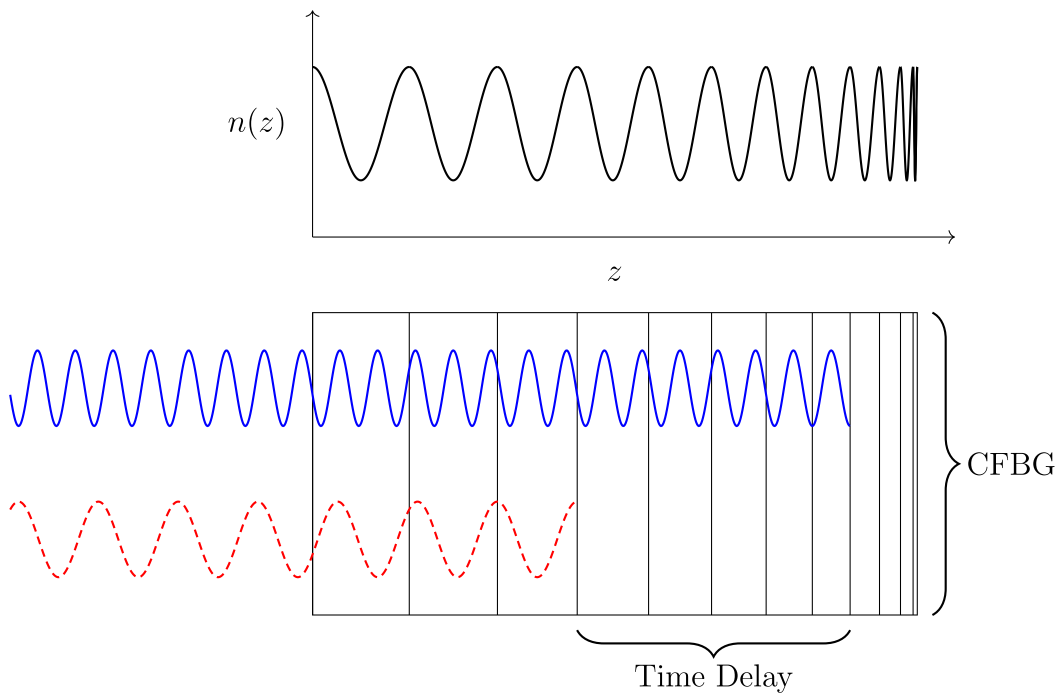


Figure 1.4: CFBG

(solid) wave since each wave is reflected where it matches the frequency of the refractive index.

The speed of light in an optical fibre is slightly dependent on the wavelength—this causes light with a longer wavelength to travel faster, and is known as chromatic dispersion. This is a large problem in fibre optic communications, the signal can spread and potentially becomes uninterpretable after large distances. However, chromatic dispersion can be counteracted using a CFBG [5,6,8] (in the opposite orientation of Figure 1.4). By forcing the longer wavelengths to travel farther the dispersion can be reversed, restoring the original signal[1]. However, in our case, we wish to accelerate the dispersion, simulating hundreds of metres of fibre, so the CGBG is used in the orientation shown in Figure 1.4.

[1] A 10 cm CFBG can compensate the dispersion of 300 km of fibre [5].

1.1.5 Optical Circulator

An optical circulator is an optical device that routes signals from port to port in a circular fashion [5, 6], the symbol for a four port optical circulator is shown in Figure 1.5. A signal entering from port 1 will be outputted from port 2; a signal entering from port 2 will exit from port 3; and so forth. Typically, optical circulators have three or four ports, with the first port being input only, and the final port being output only [6]. Optical circulators are most commonly used with devices that reflect signals instead of transmit them. For example, a signal may enter through port 1, exit through port 2, be reflected by an FBG, re-enter port 2, and finally exit through port 3.

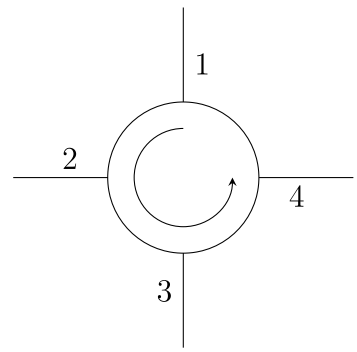


Figure 1.5: Symbol for a four port optical circulator.

1.2 Generalized Nonlinear Schödinger Equation

The standard equation for studying nonlinear optics is the generalized nonlinear Schrödinger equation (GNLSE) [7, 10–14],

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} + i\gamma|A|^2 A + \frac{1}{2}g(A)A - \alpha A. \tag{1.2}$$

Here, A is the complex pulse amplitude, β_2 and β_3 are the second-order dispersion or group delay dispersion and third-order dispersion, respectively. γ is the coefficient of nonlinearity, $g(A)$ is an amplifying term due to the gain, and α is the loss due to scattering and absorption.

This equation can be derived from the nonlinear wave equation, the derivation is presented in detail in [7, 10]. Within the derivation comoving coordinates are used so that the reference frame propagates with the group velocity. This is achieved with the substitution

$$T = t - \frac{z}{v_g}.$$

The GNLSE takes the same form as the Schrödinger equation with the inclusion of the cubic nonlinear term, hence its name. For this reason, it is sometimes referred to as the cubic nonlinear Schrödinger equation. For intensities approaching 1 GW/cm^2, the γ parameter must be replaced by $\gamma_0(1-b_s|A|^2)$, where b_s is a saturation parameter [10], this makes the addition of a quintic term to incorporate nonlinearities associated with such large powers. Furthermore, the β terms come from a Taylor expansion of the wavenumber, that is,

$$\begin{aligned} k(\omega) &= k_0 + \frac{\partial k}{\partial \omega}(\omega-\omega_0) + \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}(\omega-\omega_0)^2 + \frac{1}{6}\frac{\partial^3 k}{\partial \omega^3}(\omega-\omega_0)^3 + \dots, \\ &= \phi + \frac{1}{v_g}(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \frac{1}{6}\beta_3(\omega-\omega_0)^3 + \dots, \end{aligned}$$

where ϕ is the phase shift, and v_g is the group velocity. Typically, the third order effects must only be considered for ultrashort pulses—pulse widths less than ~ 5 ps—because of their large bandwidth [10].

1.3 Previous Modelling Efforts

Chapter 2

A New Model

2.1 Components

2.1.1 Gain

Within the Er-doped gain fibre, the gain term is dominant, and equation (1.2) reduces to

$$\frac{\partial A}{\partial z} = \frac{1}{2} g(A), \tag{2.1}$$

where $g(A)$ takes the form [1, 4, 9, 12–14]

$$g(A) = \frac{g_0}{1 + E/E_{\text{sat}}} A, \qquad E = \int_{-\infty}^{\infty} |A|^2 \, \mathrm{d}T, \tag{2.2}$$

where g_0 is a small signal gain, E is the energy of the pulse, and E_{sat} is the energy at which the gain begins to saturate. Multiplying (2.1) by $\bar{A}$, the complex conjugate of A, yields

$$\bar{A} \frac{\partial A}{\partial z} = \frac{1}{2} \frac{g_0 |A|^2}{1 + E/E_{\text{sat}}},$$

adding this to its complex conjugate and integrating over T gives

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{g_0 E}{1 + E/E_{\text{sat}}}. \tag{2.3}$$

For $E \ll E_{\text{sat}}$ the energy grows exponentially, whereas for $E \gg E_{\text{sat}}$ the gain has saturated and so the growth is linear. To obtain a closed form solution, (2.3) is integrated over a gain fibre of length z and assuming the energy increases from E to E_{out}, then

$$g_0 z = \log \frac{E_{\text{out}}}{E} + \frac{E_{\text{out}} - E}{E_{\text{sat}}},$$

and by exponentiating, rearranging, and applying W, the Lambert W function[1],

$$\begin{aligned} W\left(\frac{E}{E_{\text{sat}}} \mathrm{e}^{E/E_{\text{sat}}} \mathrm{e}^{g_0 z}\right) &= W\left(\frac{E_{\text{out}}}{E_{\text{sat}}} \mathrm{e}^{E_{\text{out}}/E_{\text{sat}}}\right) \\ &= \frac{E_{\text{out}}}{E_{\text{sat}}}, \end{aligned}$$

by (A.1). This results in the closed form expression

$$E_{\text{out}}(z) = E_{\text{sat}} W\left(\frac{E}{E_{\text{sat}}} \mathrm{e}^{E/E_{\text{sat}}} \mathrm{e}^{g_0 z}\right)$$

with the desired property that $E_{\text{out}}(0) = E$. Furthermore, since $E \sim |A|^2$, the gain in terms of the amplitude is given by

$$\begin{aligned} G(A; E) &= \left(\frac{E_{\text{out}}(L_g)}{E}\right)^{1/2} A \\ &= \left(\frac{E_{\text{sat}}}{E} W\left(\frac{E}{E_{\text{sat}}} \mathrm{e}^{E/E_{\text{sat}}} \mathrm{e}^{g_0 L_g}\right)\right)^{1/2} A, \end{aligned}$$

where L_g is the length of the gain fibre.

[1] see Appendix A

2.1.2 Fibre Nonlinearity

The nonlinearity of the fibre arises from the parameter γ; in regions where this affect is dominant expression (1.2) becomes

$$\frac{\partial A}{\partial z} - i\gamma |A|^2 A = 0. \tag{2.4}$$

This expression can be manipulated in a similar manner to the gain to show $\frac{\partial}{\partial z}|A|^2 = 0$, suggesting that $A(T, z) = A_0(T)e^{i\varphi(T,z)}$. Substituting this representation into (2.4) and setting $\varphi(T, 0) = 0$ gives $\varphi(T, z) = \gamma|A|^2 z$. For a fibre of length L_f the effect of the nonlinearity is thus

$$F(A) = Ae^{i\gamma|A|^2 L_f}.$$

2.1.3 Loss

Two sources of loss exist within the laser circuit: the loss due to the output coupler and the optical loss due to absorption and scattering. Combining these two effects give a loss that takes the form

$$L(A) = (1 - R)e^{-\alpha L}A,$$

where R is the reflectivity of the output coupler, and L is the total length of the laser circuit.

2.1.4 Dispersion

Within the laser cavity, the dispersion is dominated by the CFBG. In comparison, the dispersion due to the fibre is negligible. The dispersive terms of (1.2) give

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3}, \tag{2.5}$$

and since dispersion acts in the frequency domain it is convenient to take the Fourier transform of (2.5) [15], giving

$$\frac{\partial}{\partial z}\mathcal{F}\left\{A\right\} = i\frac{\omega^2}{2}\left(\beta_2 - \frac{\beta_3}{3}\omega\right)\mathcal{F}\left\{A\right\}.$$

The effect of dispersion is then

$$D(A) = \mathcal{F}^{-1}\left\{\mathrm{e}^{i\omega^2 L_D(\beta_2-\beta_3\omega/3)/2}\mathcal{F}\left\{A\right\}\right\}.$$

For a highly dispersive media the third order effects may need to be considered [10,16]. However, for simplicity and because of the nature of the grating, the third order effects will be neglected [7,10]—we set $\beta_3 = 0$ for the subsequent analysis.

2.1.5 Modulation

In the average model, the amount of modulation is characterized by the parameter ϵ through the term $\frac{\epsilon}{2}T^2 A$. In this new model, the modulation is considered to be applied externally through its action on the spectrum and for simplicity the representation is taken as the Gaussian

$$M(A) = \mathrm{e}^{-T^2/2T_M^2}A,$$

where T_M is a characteristic width of the modulation.

Parameter	Symbol	Value	Sources
Absorption of Fibre	α	0.01–0.3 m^{-1}	[13, 14, 17]
Fibre Dispersion	β_2^f	−50–50 $\mathrm{ps}^2/\mathrm{km}$	[4, 10, 12, 14, 16]
Fibre Nonlinearity	γ	0.001–0.01 $\mathrm{W}^{-1}\mathrm{m}^{-1}$	[10, 14]
Grating Dispersion	$\beta_2^g L_D$	10–2000 ps^2	[4, 5, 10, 18]
Length of Cavity	L	10–100 m	[12, 17]
Length of Fibre	L_f	0.15–1 m	[17]
Length of Gain Fibre	L_g	2–3 m	[4, 12–14]
Modulation Time	T_M	15–150 ps	[1, 4, 17]
Reflectivity of Optical Coupler	R	0.1–0.9	[12, 17]
Saturation Energy	E_{sat}	10^3–10^4 pJ	[14, 17]
Small Signal Gain	g_0	1–10 m^{-1}	[14, 17]

Table 2.1: Orders of magnitude of various parameters.

2.2 Non-Dimensionalization

The structure of each process of the laser can be better understood by re-scaling the time, energy, and amplitude. Specifically, the time shall be scaled by the characteristic modulation time which is related to the pulse duration, the energy by the saturation energy, and the amplitude will be scaled so that it is consistent:

$$T = T_M \widetilde{T}, \qquad E = E_{\text{sat}} \widetilde{E}, \qquad A = \left(\frac{E_{\text{sat}}}{T_M} \right)^{1/2} \widetilde{A}.$$

The new process maps, after dropping the tildes, become

$$G(A) = \left(E^{-1} W \left(aE\mathrm{e}^{E} \right) \right)^{1/2} A, \tag{2.6a}$$

$$F(A) = A\mathrm{e}^{ib|A|^2}, \tag{2.6b}$$

$$L(A) = hA, \tag{2.6c}$$

$$D(A) = \mathcal{F}^{-1} \left\{ \mathrm{e}^{is^2\omega^2} \mathcal{F}\left\{ A \right\} \right\}, \tag{2.6d}$$

$$M(A) = \mathrm{e}^{-T^2/2} A, \tag{2.6e}$$

with the four dimensionless parameters (see Table 2.1)

$$a = \mathrm{e}^{g_0 L_g} \sim 8 \times 10^3, \qquad s = \sqrt{\frac{\beta_2 L_D}{2T_M^2}} \sim 0.2,$$

$$b = \gamma L_f \frac{E_{\text{sat}}}{T_M} \sim 1, \qquad h = (1-R)\mathrm{e}^{-\alpha L} \sim 0.04,$$

which control the behaviour of the laser.

2.3 Combining the Effects

In this model the pulse is iteratively passed through each process, the order of which must now be considered. We are most interested in the output of the laser cavity, and so we shall start with the loss component. Next the pulse is passed though the CFBG, as well as the modulator. Finally, the pulse travels through the gain fibre to be amplified, and then we consider the effect of the nonlinearity since this is the region where the power is maximized. The pulse after one complete circuit of the laser cavity is then passed back in to restart the process. Functionally this can be denoted as

$$\mathcal{L}(A) = F(G(M(D(L(A))))),$$

where $\mathcal{L}$ is one loop of the laser. A solution to this model is one in which the envelope and chirp are unchanged after traversing every component in the cavity, that is, such that $\mathcal{L}(A) = A\mathrm{e}^{i\phi}$—for some $\phi \in \mathbb{R}$.

2.4 Solution to the Linear Model

In the case of $b = 0$—which shall be referred to as the linear model—a solution can be found analytically. It is expected the solution will take the form of a Gaussian.

There are a few reasons for this; the solution to the average model is a Gaussian, the equilibrium shape will be highly correlated to the shape of the modulation function ***, and since a Gaussian is a fixed point of the Fourier transform [15] dispersion will not alter the envelope.

Consider the initial pulse

$$A_0 = \sqrt{P} \exp\left(-(1+iC)\frac{T^2}{2\sigma^2}\right) \mathrm{e}^{i\phi_0},$$

where P is the peak power, C is the chirp, σ^2 is the variance, and ϕ_0 is the initial phase. After passing though the optical coupler the pulse will simply decay to $A_1 = hA_0$. The pulse then enters the CFBG, where it will maintain its Gaussian shape, however, it will spread [7, 9, 10]. This can be written as

$$A_2 = \sqrt{P} h\zeta \exp\left(-(1+i\widetilde{C})\frac{T^2}{2\widetilde{\sigma}^2}\right) \mathrm{e}^{i(\phi_0+\phi)},$$

where $\widetilde{\sigma}^2$ denotes the resulting variance, $\widetilde{C}$ denotes the resulting chirp, and ζ is the reduction of the amplitude caused by the spread. Next, the pulse is modulated:

$$A_3 = \sqrt{P} h\zeta \exp\left(-(1+i\widetilde{C})\frac{T^2}{2\widetilde{\sigma}^2} - \frac{T^2}{2}\right) \mathrm{e}^{i(\phi_0+\phi)}.$$

Finally, the pulse travels though the gain fibre where it is amplified to

$$A_4 = \sqrt{P} \left(\frac{W(aE\mathrm{e}^E)}{E}\right)^{1/2} h\zeta \exp\left(-(1+i\widetilde{C})\frac{T^2}{2\widetilde{\sigma}^2} - \frac{T^2}{2}\right) \mathrm{e}^{i(\phi_0+\phi)},$$

with E the energy of the pulse as it enters the gain fibre.

In equilibrium it must be that $A_0 = A_4\mathrm{e}^{-i\phi}$. More explicitly, this gives three conditions:

$$1 = \left(\frac{W(aE\mathrm{e}^{E})}{E}\right)^{1/2} h\zeta, \tag{2.7a}$$

$$\frac{1}{\sigma^2} = \frac{1}{\widetilde{\sigma}^2} + 1, \tag{2.7b}$$

$$\frac{C}{\sigma^2} = \frac{\widetilde{C}}{\widetilde{\sigma}^2}. \tag{2.7c}$$

2.4.1 Equilibrium Shape

Each of these processes has a relatively straight forward effect, with the exception of dispersion. After dispersion, the out-going variance, $\widetilde{\sigma}^2$, can be found by computing the effect of dispersion given by (2.6d). This yields the relations

$$\widetilde{\sigma}^2\sigma^2 = \left(\sigma^2 + 2Cs^2\right)^2 + 4s^4, \tag{2.8a}$$

$$\widetilde{C} = C + (1 + C^2)\frac{2s^2}{\sigma^2}, \tag{2.8b}$$

$$\phi = \frac{1}{2}\arctan\left(\frac{2s^2}{\sigma^2 + 2Cs^2}\right), \tag{2.8c}$$

$$\zeta = \left(\frac{\sigma}{\widetilde{\sigma}}\right)^{1/2} \tag{2.8d}$$

for the variance, chirp, phase shift, and spread, respectively, after the dispersive element, the full details can be found in Appendix B. Now, the out-going variance and chirp can be eliminated from this system of equations using (2.7b) and (2.7c):

$$\widetilde{\sigma}^2 = \frac{\sigma^2}{1 - \sigma^2},$$
$$\widetilde{C} = C\frac{1}{1 - \sigma^2}.$$

Combining this first expression with (2.8a) and expanding, we have that

$$\frac{\sigma^4}{1 - \sigma^2} = \sigma^4 + 4C^2s^4 + 4Cs^2\sigma^2 + 4s^4,$$

or written as a polynomial in σ,

$$0 = \sigma^6 + 4Cs^2\sigma^4 + \left(4s^4(C^2+1) - 4Cs^2\right)\sigma^2 - 4s^4(C^2+1).$$

The chirp can now be eliminated using (2.8b), the $1 + C^2$ can be reduced in order by noticing that

$$\begin{aligned} C\frac{1}{1-\sigma^2} &= C + (1+C^2)\frac{2s^2}{\sigma^2}, \\ 1 + C^2 &= \frac{\sigma^4}{2s^2(1-\sigma^2)}C. \end{aligned}$$

Furthermore, the chirp can be completely eliminated since

$$\begin{aligned} C &= \frac{\sigma^4}{2s^2(1-\sigma^2)} \pm \sqrt{\frac{\sigma^8}{16s^4(1-\sigma^2)^2} - 1}, \\ &= \frac{\sigma^4 \pm \sqrt{\sigma^8 - 16s^4(1-\sigma^2)^2}}{4s^2(1-\sigma^2)}. \end{aligned} \tag{2.9}$$

After simplifying algebraically, we arrive at

$$\begin{aligned} 0 &= \sigma^6 \pm \sqrt{\sigma^8 - 16s^4(1-\sigma^2)^2}(2-\sigma^2), \\ \frac{\sigma^6}{2-\sigma^2} &= \mp\sqrt{\sigma^8 - 16s^4(1-\sigma^2)^2}. \end{aligned}$$

As we shall see, σ is strictly less than 1 at equilibrium, and so, notice that the left hand side of this expression is strictly positive. Therefore, only the negative root of (2.9) will yield a solution. After squaring each side of this expression we obtain

$$\sigma^{12} = \sigma^8(2-\sigma^2)^2 - 16s^4(1-\sigma^2)^2(2-\sigma^2)^2,$$

which once fully simplified, yields the biquartic equation

$$\left(\sigma^2\right)^4 + 4s^4\left(\sigma^2\right)^3 - 20s^4\left(\sigma^2\right)^2 + 32s^4\left(\sigma^2\right) - 16s^4 = 0.$$

Since this is a quartic in σ^2 this can be solved analytically; the (positive, real) solution is

$$\sigma^2 = \sqrt{2}s\left(s^6 + 3s^2 + \sqrt{4+s^4}\left(1+s^4\right)\right)^{1/2} - s^4 - s^2\sqrt{4+s^4}. \tag{2.10}$$

Limit?***

2.4.2 Equilibrium Energy

From (2.7a) the equilibrium energy can be found, as well as the equilibrium peak power. This relation can be simplified by squaring both sides and rearranging to give

$$\frac{1}{h^2\zeta^2}E = W\left(aE\mathrm{e}^E\right).$$

Then, by taking the exponential of each side, and multiplying by this expression, we obtain[2]

$$\begin{aligned}\frac{1}{h^2\zeta^2}E\exp\left(\frac{1}{h^2\zeta^2}E\right) &= W\left(aE\mathrm{e}^E\right)\exp\left(W\left(aE\mathrm{e}^E\right)\right)\\ &= aE\mathrm{e}^E.\end{aligned}$$

[2]By (A.1).

Now, this can be written as

$$\begin{aligned} ah^2\zeta^2 &= \exp\left(\frac{1}{h^2\zeta^2}E - E\right), \\ \log\left(ah^2\zeta^2\right) &= E\left(\frac{1}{h^2\zeta^2} - 1\right). \end{aligned}$$

The energy of the pulse entering the gain fibre at equilibrium is thus

$$E = \frac{h^2\zeta^2}{1 - h^2\zeta^2}\log\left(ah^2\zeta^2\right).$$

This expression allows us to determine a restriction on the parameters for a solution to exist—in order for this energy to be positive, $ah^2\zeta^2 > 1$. The energy of the pulses as it enters the optical coupler can now be found, recall from (2.1) that the energy is defined as

$$E = \int_{-\infty}^{\infty} |A|^2 \, \mathrm{d}T;$$

the energy entering the optical coupler is then

$$\begin{aligned} E_* &= \int_{-\infty}^{\infty} |G(A)|^2 \, \mathrm{d}T, \\ &= \frac{W(aE\mathrm{e}^E)}{E}\int_{-\infty}^{\infty} |A|^2 \, \mathrm{d}T, \\ &= W(aE\mathrm{e}^E). \end{aligned} \tag{2.11}$$

We can now find the amplitude of the pulse as well as we have previously found the equilibrium shape. Again, from (2.1), it must be that $E_* = P\sigma\sqrt{\pi}$, or,

$$P = \frac{W(aE\mathrm{e}^E)}{\sigma\sqrt{\pi}}. \tag{2.12}$$

Chapter 3

Solution of the Nonlinear Model

3.1 Code

With the inclusion of the nonlinearity ($b > 0$) the model becomes too difficult to solve analytically, and we must resort to solving it numerically.

Appendix C

3.1.1 Validation

The code can be validated by comparing to the results of the linear model.

3.2 Nonlinear Model

The fibre adds a phase shift proportional to the power of the pulse, this in turn can inject higher frequency oscillations due to the dispersion which then reinforces the oscillations. These oscillations are then intensified with each trip around the cavity until the envelope of the pulse becomes mangled.

Figure **??** shows the envelope, Fourier transform, and chirp for a stable wave as well as a broken wave. In the case of the stable wave the envelope and the Fourier

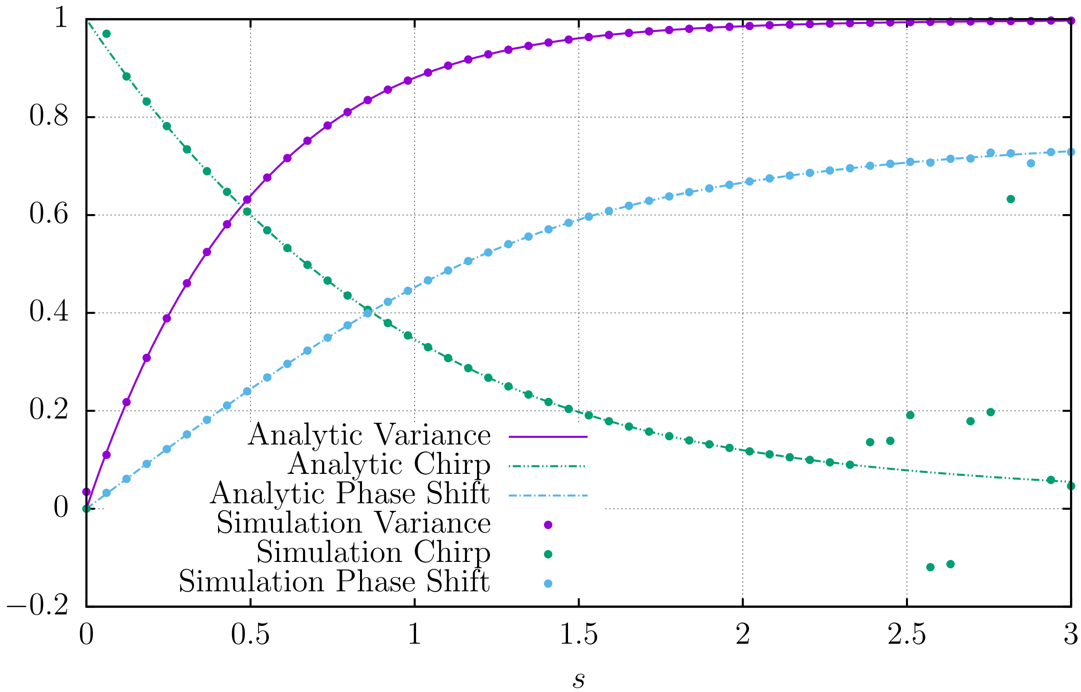


Figure 3.1: Simulation and analytic equilibrium variance, chirp, and phase shift as a function of s.

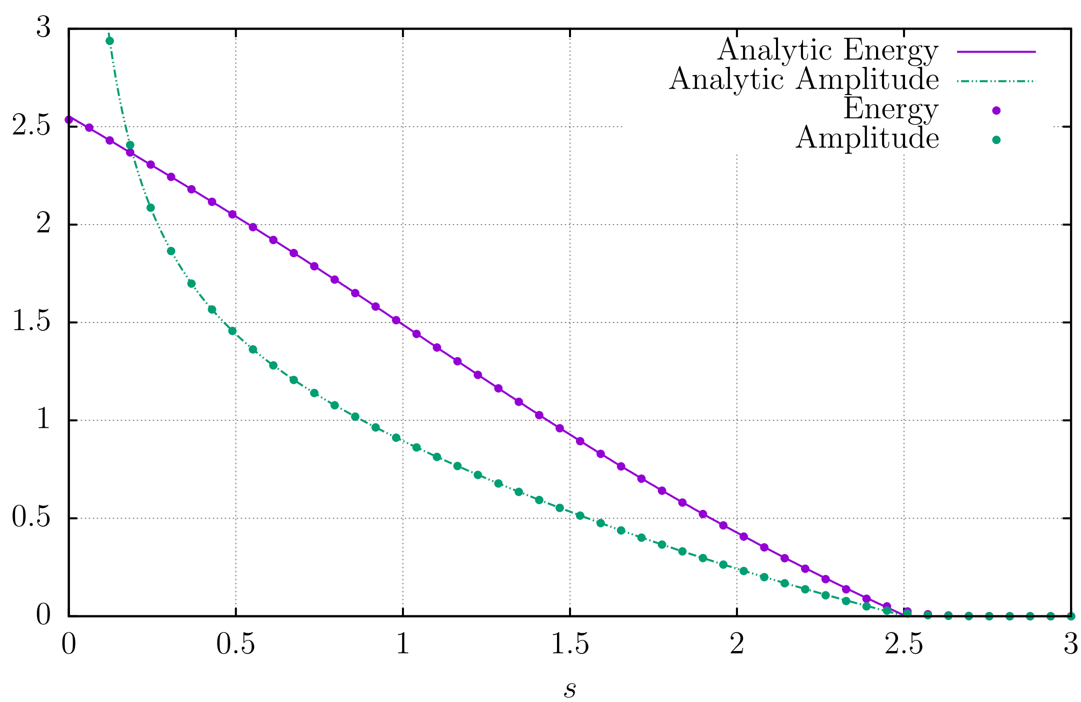


Figure 3.2: Equilibrium energy and peak power of the pulse as a function of s.

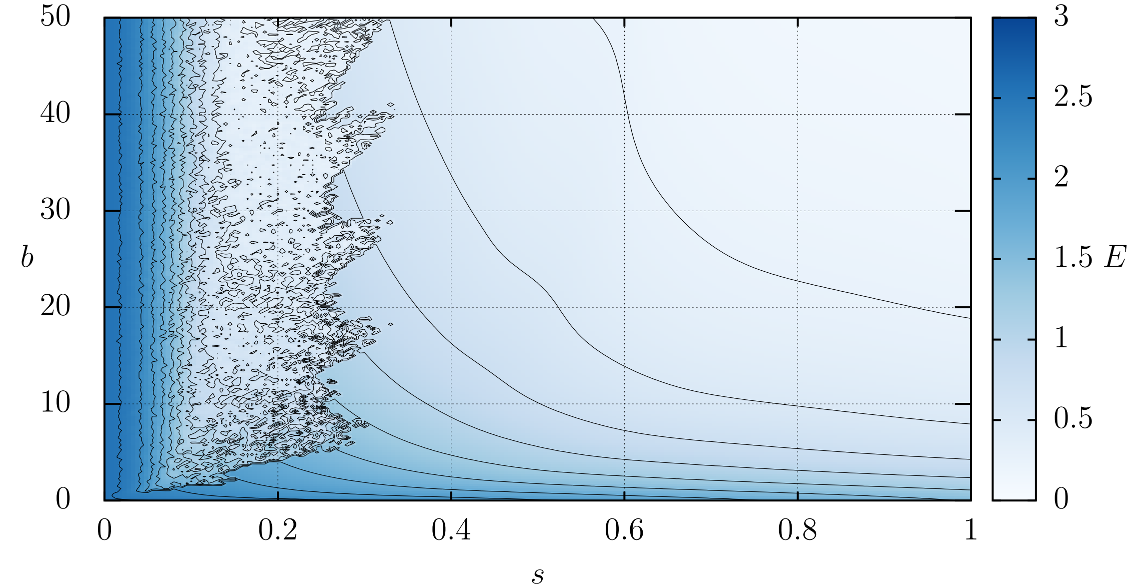


Figure 3.3

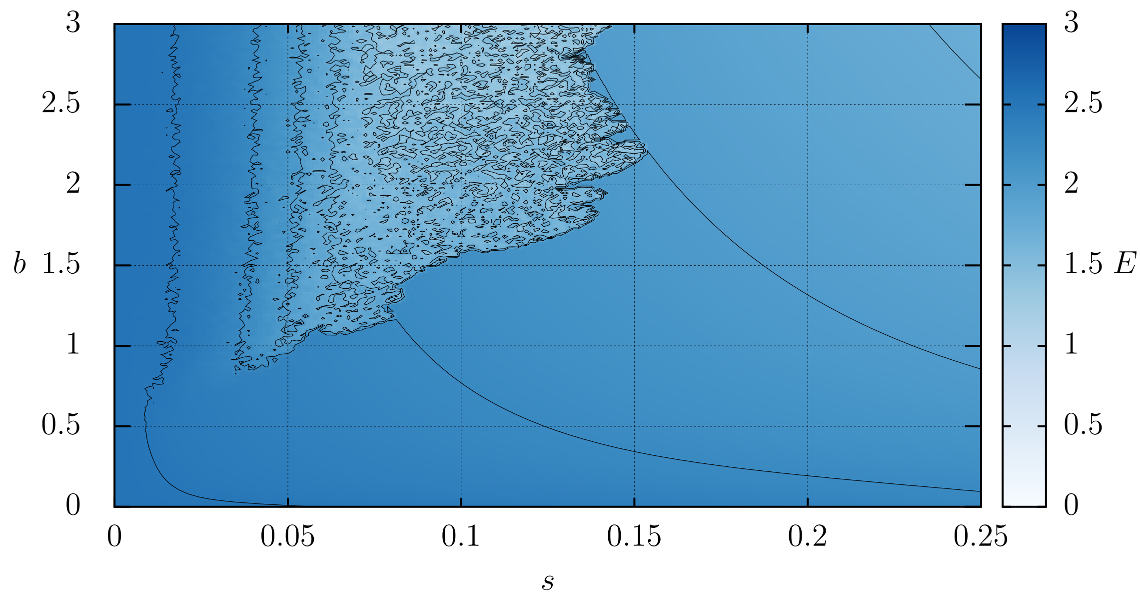


Figure 3.4

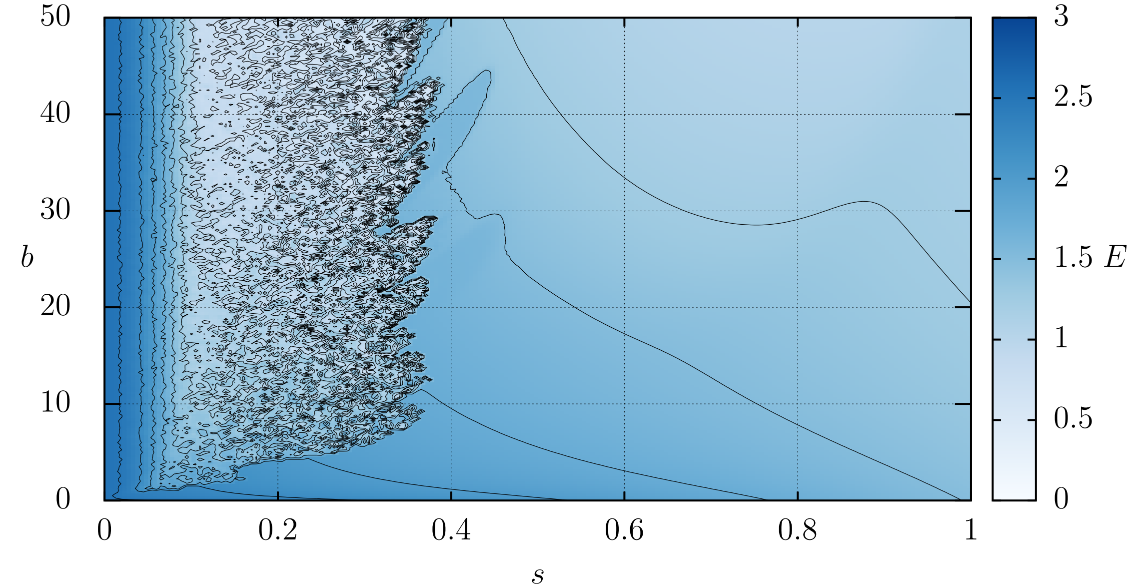


Figure 3.5

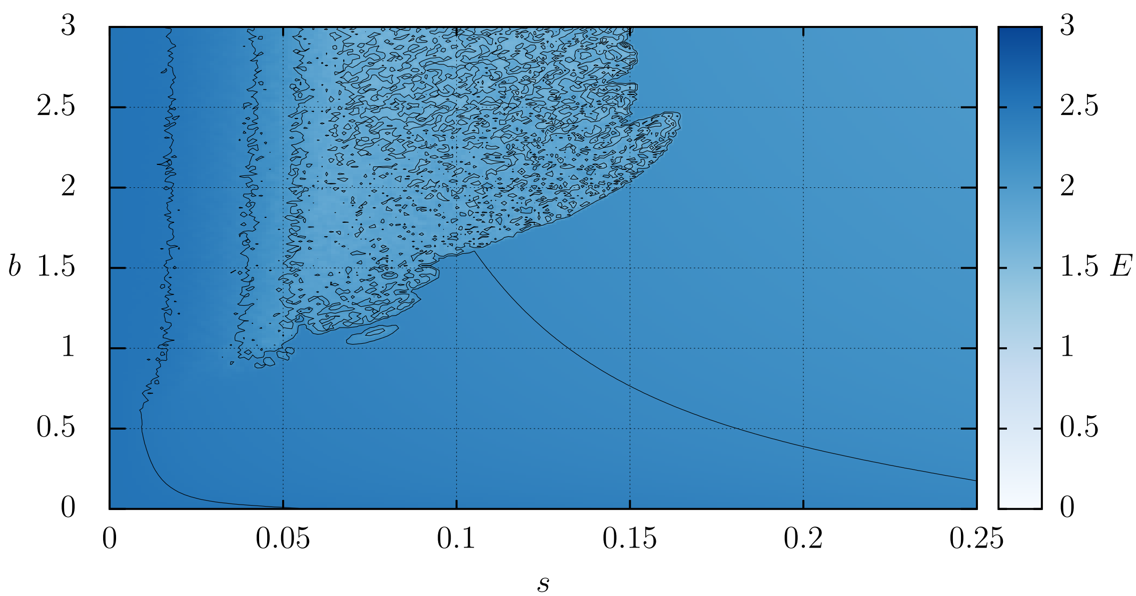


Figure 3.6

transform are Gaussian-esque, and the chirp is a very smooth function*** [19]. However, in the case of the broken wave the envelope and Fourier transform are oscillatory. Furthermore, the chirp is highly erratic.

$$\frac{\|A(\text{iteration } 25) - A(\text{iteration } 24)\|_2}{\|A(\text{iteration } 24)\|_2}.$$

Figure **??** shows the effect of switching these two components. Overall, the large scale structure is unchanged with the exception of $0.3 < s < 0.55$ and $20 < b$. In this region the waveform appears to have reached a period 2 equilibrium, that is, $\mathcal{L}(\mathcal{L}(A)) = A$.

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Appendix A

The Lambert W Function

The Lambert W function is defined to be the inverse of the function $f(x) = x\mathrm{e}^x$ and its graph is shown in Figure A.1. In other words, if $z = x\mathrm{e}^x$ then $x = W(z)$. Notice that by combining these relations we obtain the identities

$$z = W(z)\mathrm{e}^{W(z)}, \qquad\qquad x = W(x\mathrm{e}^x). \tag{A.1}$$

This function is called the Lambert W function because it is the logarithm of a special instance of Lambert's series—the letter W is used because of the work done by E. M. Wright [21].

Notice that the original function, $f(x) = x\mathrm{e}^x$, is *not* injective, and as a consequence, the W function is multi-valued on the interval $[-1/\mathrm{e}, 0)$. To alleviate this, occasionally the branch $W(x) \geq -1$ is denoted W_0 and is called the principal or upper branch, whereas the branch $W(x) < -1$ is denoted W_{-1} and is called the lower branch. However, in this work the W function will only take positive real values and so this distinction is not needed.

The Lambert W function has applications in various areas of math and physics [21]

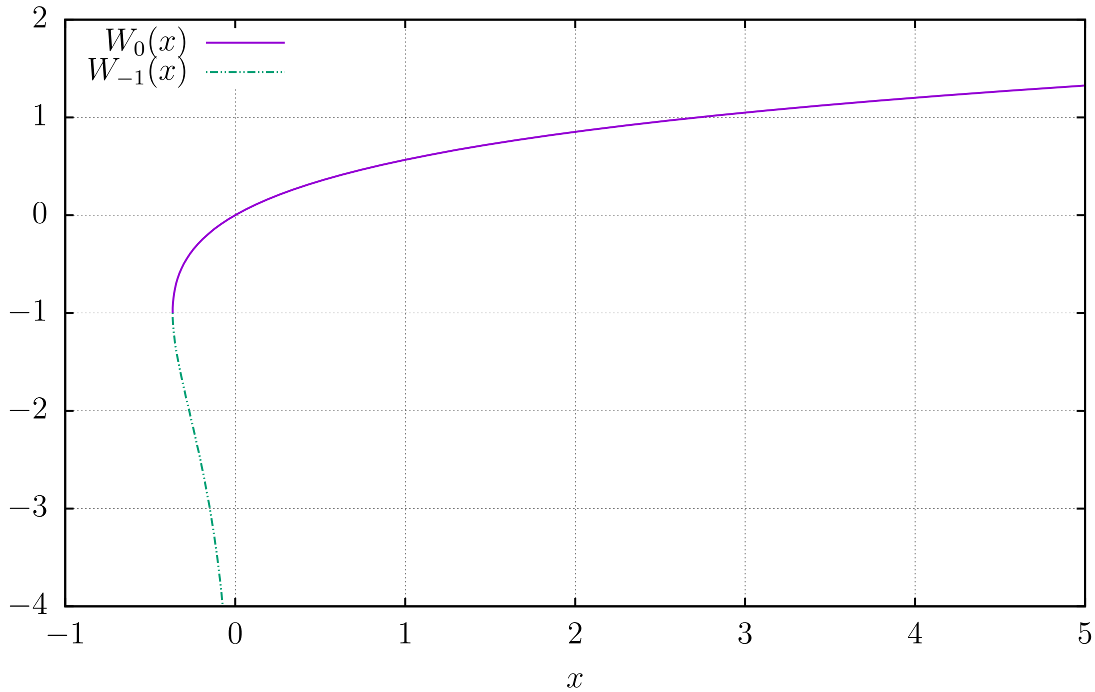

Figure A.1: The two branches of the Lambert W function.

including:

- Jet fuel problems
- Combustion problems
- Enzyme kinetics problems
- Linear constant coefficient differential delay equations
- Volterra equations.

Primarily, the W function arises when solving iterated exponentiation or certain algebraic equations. For example, consider the equation $z = x^x$. By taking the logarithm of each side we have

$$\begin{aligned}\log z &= x \log x, \\ &= \log x e^{\log x},\end{aligned}$$

which after applying the W function reduces to $W(\log z) = \log x$ by (A.1). Finally, x as a function of z can be written as $x = \exp(W(\log z))$.

Appendix B

Spread Due to Dispersion

The effect of dispersion can be computed analytically for several input pulses using (2.6d). In the case of a Gaussian pulse, we make use of the transforms [15]

$$\mathcal{F}\left\{\mathrm{e}^{-\eta T^2}\right\} = (2\eta)^{-1/2}\mathrm{e}^{-\omega^2/4\eta}, \qquad \mathcal{F}^{-1}\left\{\mathrm{e}^{-\eta\omega^2}\right\} = (2\eta)^{-1/2}\mathrm{e}^{-T^2/4\eta}.$$

From (2.6d) we have that

$$\begin{aligned} D\left(\mathrm{e}^{-\eta T^2}\right) &= \mathcal{F}^{-1}\left\{\mathrm{e}^{is^2\omega^2}\mathcal{F}\left\{\mathrm{e}^{-\eta T^2}\right\}\right\}, \\ &= (2\eta)^{-1/2}\mathcal{F}^{-1}\left\{\exp\left(-\omega^2\left(\frac{1}{4\eta} - is^2\right)\right)\right\}, \\ &= (1-4i\eta s^2)^{-1/2}\exp\left(-T^2\frac{a}{1-4is^2\eta}\right). \end{aligned}$$

For us, $\eta = \frac{1}{2}\frac{1+iC}{\sigma^2}$; making this substitution yields

$$D(A_1) = \left(1 + \frac{2Cs^2}{\sigma^2} - \frac{2s^2}{\sigma^2}i\right)^{-1/2}\exp\left(-T^2\frac{1+iC}{2\sigma^2 - 4is^2(1+iC)}\right).$$

This can be greatly simplified by first rationalizing the denominators to give

$$D(A_1) = \left(\frac{1 + \frac{2Cs^2}{\sigma^2} + \frac{2s^2}{\sigma^2} i}{\left(1 + \frac{2Cs^2}{\sigma^2}\right)^2 + \left(\frac{2s^2}{\sigma^2}\right)^2} \right)^{1/2} \exp\left(-T^2 \frac{(1 + iC)(2\sigma^2 + 4Cs^2 + 4is^2)}{(2\sigma^2 + 4Cs^2)^2 + 16s^4} \right),$$

and then by writing in polar / rectangular coordinates:

$$D(A_1) = \left(\left(1 + \frac{2Cs^2}{\sigma^2}\right)^2 + \left(\frac{2s^2}{\sigma^2}\right)^2 \right)^{-1/4} \exp\left(\frac{1}{2} i \arctan\left(\frac{\frac{2s^2}{\sigma^2}}{1 + \frac{2Cs^2}{\sigma^2}} \right) \right) \times \exp\left(-T^2 \frac{\sigma^2 \left[1 + i \left(C + (1 + C^2) \frac{2s^2}{\sigma^2} \right) \right]}{2\left[(\sigma^2 + 2Cs^2)^2 + 4s^4 \right]} \right).$$

Finally, this can be simplified further to

$$D(A_1) = \sigma \left(\left(\sigma^2 + 2Cs^2\right)^2 + 4s^4 \right)^{-1/4} \exp\left(\frac{1}{2} i \arctan\left(\frac{2s^2}{\sigma^2 + 2Cs^2} \right) \right) \times \exp\left(-T^2 \frac{\sigma^2 \left[1 + i \left(C + (1 + C^2) \frac{2s^2}{\sigma^2} \right) \right]}{2\left[(\sigma^2 + 2Cs^2)^2 + 4s^4 \right]} \right).$$

From this expression it is clear that at equilibrium

$$\begin{aligned} \widetilde{\sigma}^2 \sigma^2 &= \left(\sigma^2 + 2Cs^2\right)^2 + 4s^4, \\ \widetilde{C} &= C + (1 + C^2) \frac{2s^2}{\sigma^2}, \\ \phi &= \frac{1}{2} \arctan\left(\frac{2s^2}{\sigma^2 + 2Cs^2} \right), \\ \zeta &= \left(\frac{\sigma}{\widetilde{\sigma}} \right)^{1/2}. \end{aligned}$$

The expressions for the chirp and variance can be verified with the well known relations [7, 9, 10]

$$\left(\frac{T_1}{T_0} \right)^2 = \left(1 + \frac{C \beta_2 z}{T_0^2} \right)^2 + \left(\frac{\beta_2 z}{T_0^2} \right)^2, \qquad \widetilde{C} = C + (1 + C^2) \frac{\beta_2 z}{T_0^2}.$$

Where, within our non-dimensionalization, $T_0 = \sigma T_M$, $T_1 = \widetilde{\sigma} T_M$, $z = L_D$, and $\beta_2 z T_0^{-2} = 2s^2\sigma^{-2}$. In addition to this, the expression for ζ can be validated using conservation of energy. Within the CFBG energy is conserved, therefore,

$$\int_{-\infty}^{\infty} |A_1|^2 \, \mathrm{d}T = \int_{-\infty}^{\infty} |A_2|^2 \, \mathrm{d}T.$$

Which, after substituting the expressions from Section 2.4, reduces to

$$h^2 P \int_{-\infty}^{\infty} \mathrm{e}^{-T^2/\sigma^2} \, \mathrm{d}T = h^2 P \zeta \int_{-\infty}^{\infty} \mathrm{e}^{-T^2/\widetilde{\sigma}^2} \, \mathrm{d}T.$$

These expressions are easily integrated to show that $\sqrt{\pi\sigma^2} = \sqrt{\pi\widetilde{\sigma}^2}\zeta^2$, and finally that

$$\zeta = \left(\frac{\sigma}{\widetilde{\sigma}}\right)^{1/2}.$$

Appendix **C**

Code

```python
######################################################
#               Brady Metherall
#               MSc Thesis
######################################################
'''
This is the code used for my MSc thesis. This code
is separated into four parts each described below.
'''

################
#    Part I
################
'''
Function definitions and initialization
'''

import numpy as np
```

```
import scipy.special.lambertw as W
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit

def func(x, a, b):
    return a * np.exp(-x**2 / (2 * b**2))

def Energy(A, dx):
    return np.trapz(np.real(A * np.conj(A)), dx = dx)

# Functions for each component
def Gain(A, E, a = 8000):
    return np.real(np.sqrt(W(a * E * np.exp(E)) / E)) * A

def Loss(A, h = 0.04):
    return h * A

def Mod(A, T):
    return np.exp(-T**2 / 2) * A

def Fibre(A, b = 1.0):
    return np.exp(1j * b * np.abs(A)**2) * A

def Disp(A, T, s = 0.1):
    F = np.fft.fft(A)
    F = F * (np.abs(F) > 10**-4) # Numerical stability
    dw = np.pi / T[-1]
```

```
    w = np.fft.fftfreq(len(A)) * len(A) * dw
    return np.fft.ifft(F * np.exp(1j * w**2 * s**2)))

# 1 round trip
def Loop(A, T, dx, s, b, switch = False):
    A = Loss(A)
    if not switch:
        A = Disp(A, T, s)
        A = Mod(A, T)
    else:
        A = Mod(A, T)
        A = Disp(A, T, s)
    A = Gain(A, Energy(A, dx))
    A = Fibre(A, b)
    return A

N = 250 # Number of loops of the circuit
p = 2**12 # Number of points in the discretization
width = 64 # Size of window
E0 = 0.1 # Initial energy

# Initialization
T = np.linspace(-width, width, p, endpoint = False)
dx = T[1] - T[0]
A0 = 1 / np.cosh(2 * T) * np.exp(1j * np.pi / 4)
A0 = np.sqrt(E0 / Energy(A0, dx)) * A0 # Normalize
E = np.zeros(N)
```

```
data = np.zeros((2 * N, p))
A = A0

part = 4 # Select which part of the code to run

if part == 2:
    ################
    #   Part II
    ################
    '''
    On the fly animation of single realizations
    '''
    plt.ion()
    fig = plt.figure()
    ax = fig.add_subplot(111)
    line1, = ax.plot(T, np.real(A), 'r-', label = 'Real')
    line2, = ax.plot(T, np.imag(A), 'b-', label = 'Imaginary')
    line3, = ax.plot(T, np.abs(A), 'g-', label = 'Magnitude')

    fig.canvas.draw()
    fig.canvas.flush_events()

    plt.legend()
    plt.xlim(-2, 2)
    plt.ylim(-1, 1)

```

```
    # N round trips of the laser
    for i in range(N):
        # Animate the plot
        line1.set_ydata(np.real(A))
        line2.set_ydata(np.imag(A))
        line3.set_ydata(np.abs(A))
        fig.canvas.draw()
        fig.canvas.flush_events()
        #time.sleep(2)
        print i

        A = Loop(A, T, dx, 0.41, 2)
        E[i] = Energy(A, dx)
        #time.sleep(2)

    #np.savetxt('E.dat', E)

    #np.savetxt('Envelope.dat', np.vstack((T, np.abs(A))).T)

elif part == 3:
    ###############
    #   Part III
    ###############
    '''
    Run the simulation for an nxn grid in s-b space
    '''
    n = 201
```

```
    #z = np.zeros((n**2, 4))
    zoom = False

    if zoom:
        s = np.linspace(0, 0.25, num = n)
        #b = np.logspace(3.0, 5.0, num = n)
        b = np.linspace(0, 3, num = n)
    else:
        s = np.linspace(0, 1, num = n)
        #b = np.logspace(3.0, 6.0, num = n)
        b = np.linspace(0, 50, num = n)

    filename = '8000-04-01-DM.dat'
    open(filename, 'w').close()
    f = open(filename, 'ab')

    for k in range(n):
        print k
        z = np.zeros((n, 4))
        for j in range(n):
            A0 = 1 / np.cosh(2 * T) * np.exp(1j * np.pi / 4)
            A0 = np.sqrt(E0 / Energy(A0, dx)) * A0 #Normalize
            A = A0
            for i in range(25):
                old = np.abs(A)
                A = Loop(A, T, dx, s[j], b[k], switch = False
```

```
)
                new = np.abs(A)
            z[j] = s[j], b[k], np.sqrt(np.trapz((old - new)**2, dx = dx)) / np.sqrt(np.trapz(old**2, dx = dx)), Energy(A, dx)
        np.savetxt(f, z)
        f.write('\n')

    f.close()

elif part == 4:
    ################
    #   Part IV
    ################
    '''
    Compute features of linear model (E, P, sigma, C, phi)
    '''
    n = 50
    z = np.zeros((n, 6))
    s = np.linspace(0, 3, num = n)

    for j in range(n):
        A0 = 1 / np.cosh(2 * T) * np.exp(1j * np.pi / 4)
        A0 = np.sqrt(E0 / Energy(A0, dx)) * A0 # Normalize
        A = A0
        for i in range(25):
            old = np.angle(A[len(A)/2])
```

```
                A = Loop(A, T, dx, s[j], 0, switch = False)
                new = np.angle(A[len(A)/2])
                while old > new:
                    old -= 2 * np.pi
            r, sigma = curve_fit(func, T, np.abs(A))[0]
            z[j] = s[j], Energy(A, dx), np.abs(A)[len(A)/2]**2,
    sigma, -np.gradient(np.gradient(np.angle(A), dx), dx)[len(
    A)/2] * sigma**2, new - old

    np.savetxt('Linear.dat', z)

else:
    print 'Please enter a valid part number (2--4)'
```