A Nonlinear Model for Dispersion-Tuned Actively Mode-Locked Lasers

by

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Abstract

A new nonlinear model is proposed for tuneable lasers. Using the generalized nonlinear Schrödinger equation as a starting point, expressions for the transformations undergone by the pulse are derived for each component in the cavity. These transformations are then composed to give the overall effect of one trip around the cavity. The linear version of this model is solved analytically, and the nonlinear version numerically. A consequence of this model being nonlinear is that it is able to exhibit wave breaking which prior models could not. We highlight the rich structure of the boundary of stability for a particular plane of the parameter space.

Acknowledgements

Thank some people that you like here.

Author's Declaration

I declare that the work in this thesis was carried out in accordance with the regulations of the University of Ontario Institute of Technology. The work is original except where indicated by special reference in the text and no part of the dissertation has been submitted for any other degree. Any views expressed in the dissertation are those of the author and in no way represent those of the University of Ontario Institute of Technology. This thesis has not been presented to any other university for examination either in Canada or elsewhere.

Brady Metherall
Monday 21st January, 2019

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Introduction

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Classical lasers, such as a laser pointer or a Helium-Neon gas laser, are limited to a single wavelength since the light is generated by stimulated emission. Tuneable lasers on the other hand, have the ability to operate within a range of wavelengths [1–3]. As a result, tuneable lasers have applications in spectroscopy and high resolution imaging such as coherent anti-Stokes Raman spectroscopy and optical coherence tomography [1, 3, 4]. This article is concerned with dispersion-tuned actively mode-locked (DTAML) lasers. The laser cavity consists of four elements: the dispersive element, the modulator, the gain fibre, and the optical coupler. The gain fibre consists of either an Erbium or Ytterbium doped fibre, and dispersion is generated by the highly dispersive chirped fibre Bragg grating (CFBG).

To start off the discussion of the current modelling efforts for a tuneable laser, we begin with a review of the efforts to describe an 'average' model. The idea is to capture some of the physical elements in the waveform described by an effective PDE, the solution of which gives the amplitude of the wave packet.

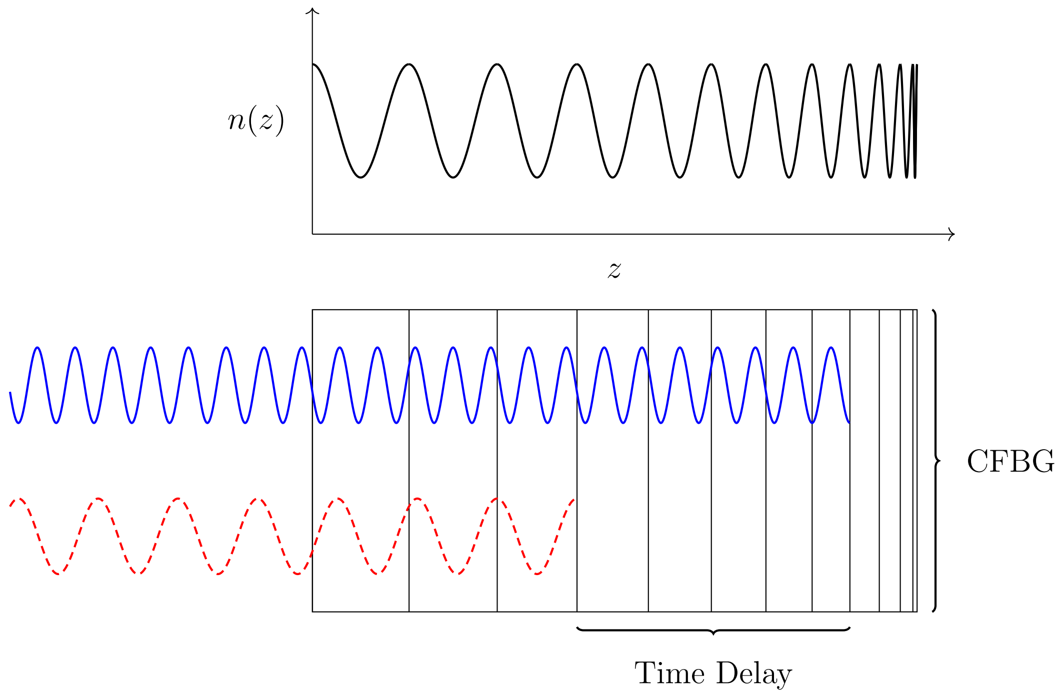

Figure 1.1: Chirped fibre Bragg grating schematic.

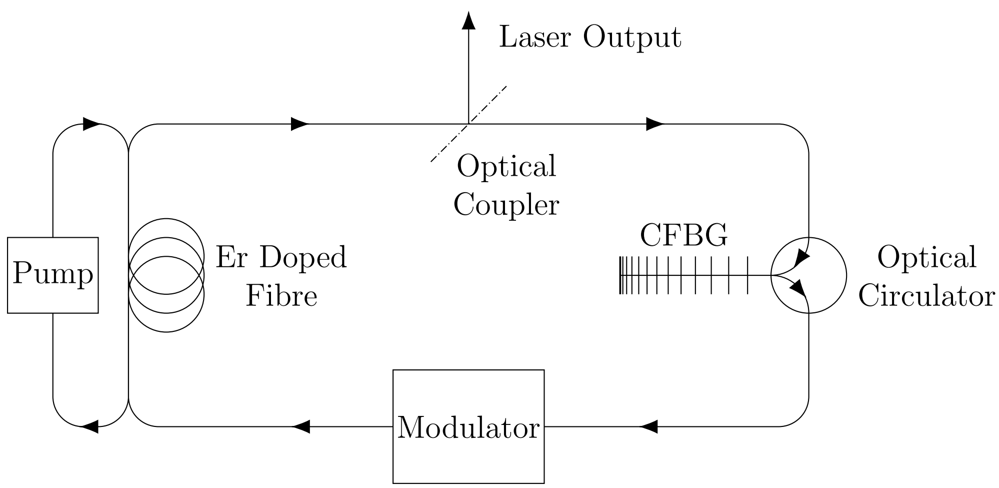

Figure 1.2: Laser cavity schematic.

Chapter 2

A New Model

Rather than using transfer functions with a linear PDE, we instead return to the generalized nonlinear Schrödinger equation [5–10]

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} + i\gamma|A|^2 A + \frac{1}{2}g(A)A - \alpha A, \tag{2.1}$$

to represent the waveform. In this expression, β_3 is the third order dispersion coefficient, γ is the coefficient of nonlinearity or self-phase modulation, $g(A)$ is the gain, and α is the loss of the fibre. Using this expression as a starting point, the laser cavity is assumed to be composed of five independent processes—gain, nonlinearity, loss, dispersion, and modulation. Within each component of the laser cavity the other four processes are assumed to be negligible, that is, each process is dominant only within one part of the laser, just as with the discrete model, but embracing any nonlinearities.

2.1 Components

2.1.1 Gain

Considering the gain term as dominant as is expected with the Er-doped gain fibre, equation (2.1) reduces to

$$\frac{\partial A}{\partial z} = \frac{1}{2}\frac{g_0}{1 + E/E_{\text{sat}}}A, \qquad\qquad E = \int_{-\infty}^{\infty} |A|^2 \, \mathrm{d}T, \tag{2.2}$$

where g_0 is a small signal gain, E is the energy of the pulse and E_{sat} is the energy at which the gain begins to saturate [1, 4, 7, 8, 10, 11]. Multiplying (2.2) by $\bar{A}$, the complex conjugate of A, yields

$$2\bar{A}\frac{\partial A}{\partial z} = \frac{g_0 |A|^2}{1 + E/E_{\text{sat}}}.$$

Adding this to its complex conjugate and integrating over T gives

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{g_0 E}{1 + E/E_{\text{sat}}}. \tag{2.3}$$

For $E \ll E_{\text{sat}}$ the energy grows exponentially, whereas for $E \gg E_{\text{sat}}$ the gain has saturated and so the energy grows linearly. To obtain a closed form solution, (2.3) is integrated over a gain fibre of length z and assume the energy increases from E to E_{out} so that

$$g_0 z = \log \frac{E_{\text{out}}}{E} + \frac{E_{\text{out}} - E}{E_{\text{sat}}}$$

and by exponentiating, rearranging, and applying W, the Lambert W function (See Appendix A),

$$W\left(\frac{E}{E_{\text{sat}}}\mathrm{e}^{E/E_{\text{sat}}}\mathrm{e}^{g_0 z}\right) = W\left(\frac{E_{\text{out}}}{E_{\text{sat}}}\mathrm{e}^{E_{\text{out}}/E_{\text{sat}}}\right) = \frac{E_{\text{out}}}{E_{\text{sat}}}.$$

This results in the closed form expression

$$E_{\mathrm{out}}(z) = E_{\mathrm{sat}} W \left(\frac{E}{E_{\mathrm{sat}}} \mathrm{e}^{E/E_{\mathrm{sat}}} \mathrm{e}^{g_0 z} \right) \tag{2.4}$$

with the desired property that $E_{\mathrm{out}}(0) = E$. Since $E \sim |A|^2$, the gain in terms of the amplitude is given by

$$G(A;E) = \left(\frac{E_{\mathrm{out}}(L_g)}{E} \right)^{1/2} A = \left(\frac{E_{\mathrm{sat}}}{E} W \left(\frac{E}{E_{\mathrm{sat}}} \mathrm{e}^{E/E_{\mathrm{sat}}} \mathrm{e}^{g_0 L_g} \right) \right)^{1/2} A, \tag{2.5}$$

where L_g is the length of the gain fibre.

2.1.2 Fibre Nonlinearity

The nonlinearity of the fibre depends on the parameter γ. In regions where this affect is dominant expression (2.1) becomes

$$\frac{\partial A}{\partial z} - i\gamma |A|^2 A = 0, \tag{2.6}$$

so that $\frac{\partial}{\partial z}|A|^2 = 0$ suggesting that $A(T,z) = A_0(T) e^{i\phi(T,z)}$. Substituting this representation into (2.6) and setting $\phi(T,0) = 0$ gives $\phi(T,z) = \gamma |A|^2 z$. For a fibre of length L_f the effect of the nonlinearity is therefore

$$F(A) = A e^{i\gamma |A|^2 L_f}. \tag{2.7}$$

2.1.3 Loss

Two sources of loss exist within the laser circuit: the loss due to the output coupler and the optical loss due to absorption and scattering. Combining these two effects give a loss that takes the form

$$L(A) = Re^{-\alpha L} A, \tag{2.8}$$

where R is the reflectivity of the output coupler, and L is the total length of the laser circuit.

2.1.4 Dispersion

Within the laser cavity, the dispersion is dominated by the chirped fibre Bragg grating (CFBG). In comparison, the dispersion due to the fibre is negligible[1]. The dispersive terms of (2.1) give

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} \tag{2.9}$$

and since dispersion acts in the frequency domain, it is convenient to use the Fourier transform of (2.9), giving the result that

$$\frac{\partial}{\partial z}\mathcal{F}\left\{A\right\} = i\frac{\omega^2}{2}\left(\beta_2 - \frac{\beta_3}{3}\omega\right)\mathcal{F}\left\{A\right\}.$$

The effect of dispersion is then

$$D(A) = \mathcal{F}^{-1}\left\{\mathrm{e}^{i\omega^2 L_D(\beta_2 - \beta_3\omega/3)/2}\mathcal{F}\left\{A\right\}\right\}. \tag{2.10}$$

For a highly dispersive media the third order effects may need to be considered [5,13]. However, for simplicity in the basic model and because of the nature of the grating, the third order effect will be neglected so we set $\beta_3 = 0$ for the subsequent analysis [5,6].

[1] A 10 cm chirped grating can provide as much dispersion as 300 km of fibre [12].

Parameter	Symbol	Value	Sources
Saturation Energy	E_{sat}	10^3–10^4 pJ	[8, 14]
Fibre Nonlinearity	γ	0.001–0.01 $\text{W}^{-1}\text{m}^{-1}$	[5, 8]
Small Signal Gain	g_0	1–10 m^{-1}	[8, 14]
Grating Dispersion	$\beta_2^g L_D$	10–2000 ps^2	[4, 5, 12, 15]
Fibre Dispersion	β_2^f	−50–50 ps^2/km	[4, 5, 8, 10, 13]
Modulation Time	T_M	15–150 ps	[1, 4, 14]
Length of Cavity	L	10–100 m	[10, 14]
Length of Gain Fibre	L_g	2–3 m	[4, 7, 8, 10]
Length of Fibre	L_f	0.15–1 m	[14]
Reflectivity of Optical Coupler	R	0.1–0.9	[10, 14]
Absorption of Fibre	α	0.01–0.3 m^{-1}	[7, 8, 14]

Table 2.1: Orders of magnitude of various parameters.

2.1.5 Modulation

In the average model, the amount of modulation is characterized by the parameter ϵ through the term $\frac{\epsilon}{2}T^2 A$. In the new model, the modulation is considered to be applied externally through its action on the spectrum and for simplicity the representation is taken as the Gaussian

$$M(A) = \mathrm{e}^{-T^2/2T_M^2} A, \tag{2.11}$$

where T_M is a characteristic width of the modulation.

2.2 Non-Dimensionalization

The structure of each process of the laser can be better understood by re-scaling the time, energy, and amplitude. Specifically, the time shall be scaled by the characteristic modulation time which is proportional to the pulse duration, the energy by the saturation energy, and the amplitude will be scaled so that it is consistent:

$$T = T_M \widetilde{T}, \qquad E = E_{\text{sat}} \widetilde{E}, \qquad A = \left(\frac{E_{\text{sat}}}{T_M}\right)^{1/2} \widetilde{A}.$$

The new process maps, after dropping the tildes, become

$$\begin{aligned} G(A) &= \left(E^{-1} W\left(aE\mathrm{e}^{E}\right)\right)^{1/2} A, \qquad F(A) = A\mathrm{e}^{ib|A|^2}, \qquad M(A) = \mathrm{e}^{-T^2/2} A, \\ D(A) &= \mathcal{F}^{-1}\left\{\mathrm{e}^{is^2\omega^2} \mathcal{F}\left\{A\right\}\right\}, \qquad L(A) = hA, \end{aligned}$$

with four dimensionless parameter groups (see Table 2.1)

$$a = \mathrm{e}^{g_0 L_g} \sim 8 \times 10^3, \quad s = \sqrt{\frac{\beta_2 L_D}{2T_M^2}} \sim 0.2, \quad b = \gamma L_f \frac{E_{\text{sat}}}{T_M} \sim 1, \quad h = (1-R)\mathrm{e}^{-\alpha L} \sim 0.04,$$

which control the behaviour of the laser.

2.3 Combining the Effects

In this model the pulse is iteratively passed through each process, the order of which is now important. We are most interested in the output of the laser cavity, and so we shall start with the loss component—what enters the optical coupler. Next the pulse is passed though the CFBG, as well as the modulator. Finally, the pulse travels through the gain fibre to be amplified, and then we consider the effect of the nonlinearity since this is the region where the power is maximized. The pulse after one complete circuit of the laser cavity is then passed back in to restart the process. Functionally this can be denoted as

$$\mathcal{L}(A) = F(G(M(D(L(A))))),$$

where $\mathcal{L}$ is one loop of the laser. A solution to this model is one in which the envelope and chirp are unchanged after traversing every component in the cavity, that is, such that $\mathcal{L}(A) = A$—potentially with a constant phase shift.

2.4 Solution to the Linear Model

Since the pulse is modulated by a Gaussian, it is expected that the equilibrium envelope will also be a Gaussian. Consider the initial pulse

$$A_0 = \sqrt{P}\exp\left(-(1+iC)\frac{T^2}{2\sigma^2}\right),$$

where P is the peak power. After passing though the optical coupler the pulse will simply decay, $A_1 = hA_0$. The pulse then enters the CFBG, where it will maintain its Gaussian shape, however, it will spread [5]. This can be written as

$$A_2 = \sqrt{P}h\left(\frac{\sigma}{\widetilde{\sigma}}\right)^{1/2}\exp\left(-(1+i\widetilde{C})\frac{T^2}{2\widetilde{\sigma}^2}\right),$$

where $\widetilde{\sigma}^2$ denotes the resulting variance, and $\widetilde{C}$ denotes the resulting chirp. Next, the pulse is modulated:

$$A_3 = \sqrt{P}h\left(\frac{\sigma}{\widetilde{\sigma}}\right)^{1/2}\exp\left(-(1+i\widetilde{C})\frac{T^2}{2\widetilde{\sigma}^2} - \frac{T^2}{2}\right).$$

Finally, the pulse travels though the gain fibre where it is amplified to

$$A_4 = \sqrt{P}\left(\frac{W(aE\mathrm{e}^E)}{E}\right)^{1/2} h\left(\frac{\sigma}{\widetilde{\sigma}}\right)^{1/2}\exp\left(-(1+i\widetilde{C})\frac{T^2}{2\widetilde{\sigma}^2} - \frac{T^2}{2}\right),$$

with E the energy of the pulse as it enters the gain fibre.

In equilibrium, it must be that $A_0 = A_4\mathrm{e}^{i\varphi}$, for some φ. More explicitly, this gives three conditions:

$$1 = \left(\frac{W(aE\mathrm{e}^{E})}{E}\right)^{1/2} h\left(\frac{\sigma}{\widetilde{\sigma}}\right)^{1/2}, \tag{2.12a}$$

$$\frac{1}{\sigma^2} = \frac{1}{\widetilde{\sigma}^2} + 1, \tag{2.12b}$$

$$\frac{C}{\sigma^2} = \frac{\widetilde{C}}{\widetilde{\sigma}^2}. \tag{2.12c}$$

2.4.1 Equilibrium Shape

After dispersion, the out-going variance $\widetilde{\sigma}$ can be found by computing the effect of dispersion given by ***. In the case of a Gaussian pulse [5] we have

$$\left(\frac{T_1}{T_0}\right)^2 = \left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2, \qquad \widetilde{C} = C + (1 + C^2)\frac{\beta_2 z}{T_0^2}$$

where $T_0 = \sigma T_M$, $T_1 = \widetilde{\sigma} T_M$, $z = L_D$, and $\beta_2 z T_0^{-2} = 2s^2\sigma^{-2}$ within our non-dimensionalization. Making these substitutions yields the relations

$$\widetilde{\sigma}^2\sigma^2 = \left(\sigma^2 + 2Cs^2\right)^2 + 4s^4, \tag{2.13a}$$

$$\widetilde{C} = C + (1 + C^2)\frac{2s^2}{\sigma^2}, \tag{2.13b}$$

for the variance and chirp after the dispersive element. Now, the out-going variance and chirp can be eliminated from this system of equations using (2.12b) and (2.12c):

$$\widetilde{\sigma}^2 = \frac{\sigma^2}{1 - \sigma^2},$$

$$\widetilde{C} = C\frac{1}{1 - \sigma^2}.$$

Combining this first expression with (2.13a) and expanding, we have that

$$\frac{\sigma^4}{1 - \sigma^2} = \sigma^4 + 4C^2 s^4 + 4Cs^2\sigma^2 + 4s^4,$$

or written as a polynomial in σ,

$$0 = \sigma^6 + 4Cs^2\sigma^4 + \left(4s^4(C^2+1) - 4Cs^2\right)\sigma^2 - 4s^4(C^2+1).$$

The chirp can now be eliminated using (2.13b), the $1 + C^2$ can be reduced in order by noticing that

$$\begin{aligned} C\frac{1}{1-\sigma^2} &= C + (1+C^2)\frac{2s^2}{\sigma^2} \\ 1 + C^2 &= \frac{\sigma^4}{2s^2(1-\sigma^2)}C. \end{aligned}$$

Furthermore, the chirp can be completely eliminated since

$$\begin{aligned} C &= \frac{\sigma^4}{2s^2(1-\sigma^2)} \pm \sqrt{\frac{\sigma^8}{16s^4(1-\sigma^2)^2} - 1}, \\ &= \frac{\sigma^4 \pm \sqrt{\sigma^8 - 16s^4(1-\sigma^2)^2}}{4s^2(1-\sigma^2)}. \end{aligned} \tag{2.14}$$

After simplifying algebraically, we arrive at

$$\begin{aligned} 0 &= \sigma^6 \pm \sqrt{\sigma^8 - 16s^4(1-\sigma^2)^2}(2-\sigma^2), \\ \frac{\sigma^6}{2-\sigma^2} &= \mp\sqrt{\sigma^8 - 16s^4(1-\sigma^2)^2}. \end{aligned}$$

Notice that the left hand side of this expression is strictly positive (σ is strictly less than 1 at equilibrium), therefore, only the negative root of (2.14) will yield a solution. After squaring each side of this expression we obtain

$$\sigma^{12} = \sigma^8(2-\sigma^2)^2 - 16s^4(1-\sigma^2)^2(2-\sigma^2)^2,$$

which once fully simplified, yields the biquartic equation

$$\left(\sigma^2\right)^4 + 4s^4\left(\sigma^2\right)^3 - 20s^4\left(\sigma^2\right)^2 + 32s^4\left(\sigma^2\right) - 16s^4 = 0.$$

Since this is a quartic in σ^2 this can be solved analytically; the (positive) solution is

$$\sigma^2 = \sqrt{2}s\left(s^6 + 3s^2 + \sqrt{4+s^4}(1+s^4)\right)^{1/2} - s^4 - s^2\sqrt{4+s^4}. \tag{2.15}$$

2.4.2 Equilibrium Energy

From (2.12a) the equilibrium energy can be found, as well as the equilibrium peak power. This relation can be simplified by squaring both sides and rearranging to give

$$\frac{1}{h^2\zeta}E = W\left(aE\mathrm{e}^E\right),$$

where $\zeta \equiv \dfrac{\sigma}{\widetilde{\sigma}} = \sqrt{1-\sigma^2}$. Then, by taking the exponential of each side, and multiplying by the original expression, we obtain

$$\begin{aligned}\frac{1}{h^2\zeta}E\exp\left(\frac{1}{h^2\zeta}E\right) &= W\left(aE\mathrm{e}^E\right)\exp\left(W\left(aE\mathrm{e}^E\right)\right)\\ &= aE\mathrm{e}^E,\end{aligned}$$

by (A.1). Now, this can be written as

$$\begin{aligned}ah^2\zeta &= \exp\left(\frac{1}{h^2\zeta}E - E\right),\\ \log\left(ah^2\zeta\right) &= E\left(\frac{1}{h^2\zeta} - 1\right).\end{aligned}$$

The energy of the pulse entering the gain fibre at equilibrium is thus

$$E = \frac{h^2\zeta}{1-h^2\zeta}\log\left(ah^2\zeta\right).$$

the pulse will converge if the argument of the log is greater than 1 The energy of the pulses as it enters the optical coupler can now be found, recall from (2.2) that the energy is defined as

$$E = \int_{-\infty}^{\infty} |A|^2 \,\mathrm{d}T;$$

the energy entering the optical coupler is then

$$\begin{aligned} E_* &= \int_{-\infty}^{\infty} |G(A)|^2 \,\mathrm{d}T, \\ &= \frac{W(aE\mathrm{e}^E)}{E}\int_{-\infty}^{\infty} |A|^2 \,\mathrm{d}T, \\ &= W(aE\mathrm{e}^E). \end{aligned} \tag{2.16}$$

Since we have previously found the equilibrium shape, we can now find the amplitude of the pulse as well. Again, from (2.2), it must be that $E_* = P\sigma\sqrt{\pi}$, or,

$$P = \frac{W(aE\mathrm{e}^E)}{\sigma\sqrt{\pi}}. \tag{2.17}$$

Chapter 3

Solution of the Nonlinear Model

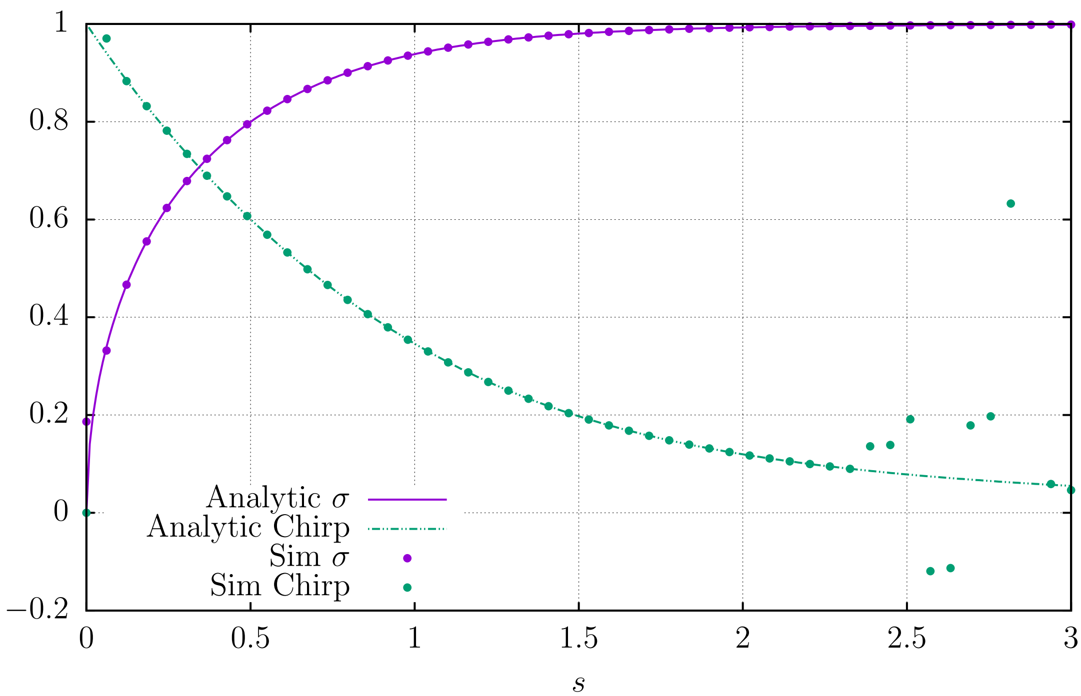

Figure 3.1: Simulation and analytic equilibrium standard deviation as a function of s.

In the linear model, the pulse converges to a Gaussian as long as $ah^2\zeta > 1$ so that the equilibrium energy from (**??**) is positive. However, this is not necessarily the case with the inclusion of the nonlinearity. The fibre adds a phase shift proportional to the power of the pulse, this in turn can inject higher frequency oscillations due to the

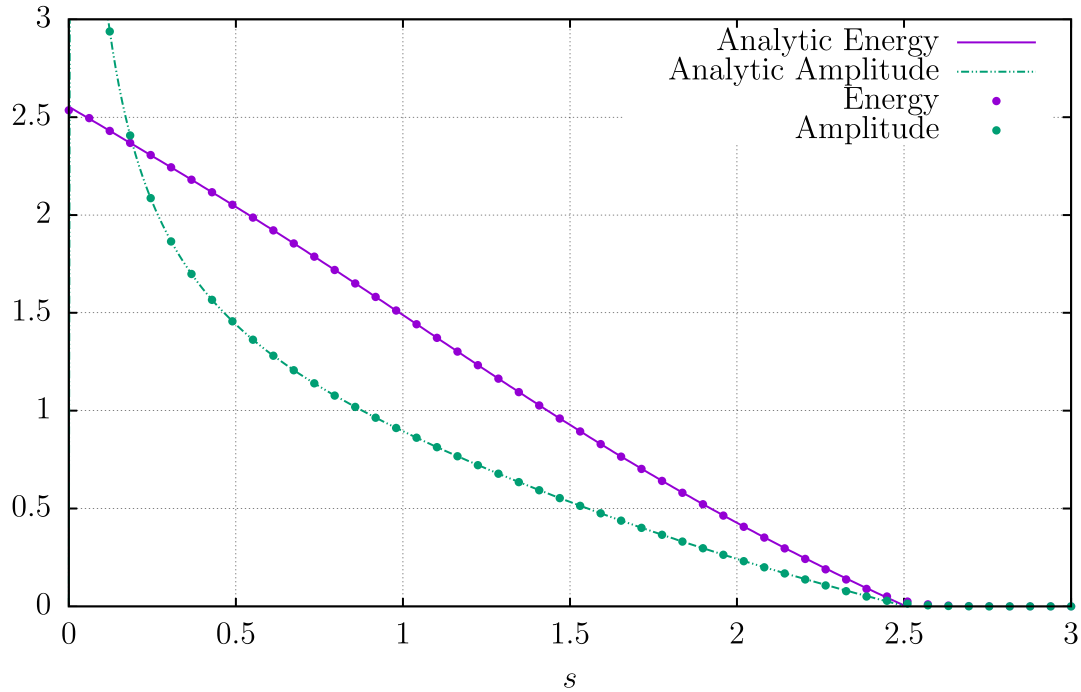

Figure 3.2: Equilibrium energy and peak power of the pulse as a function of s.

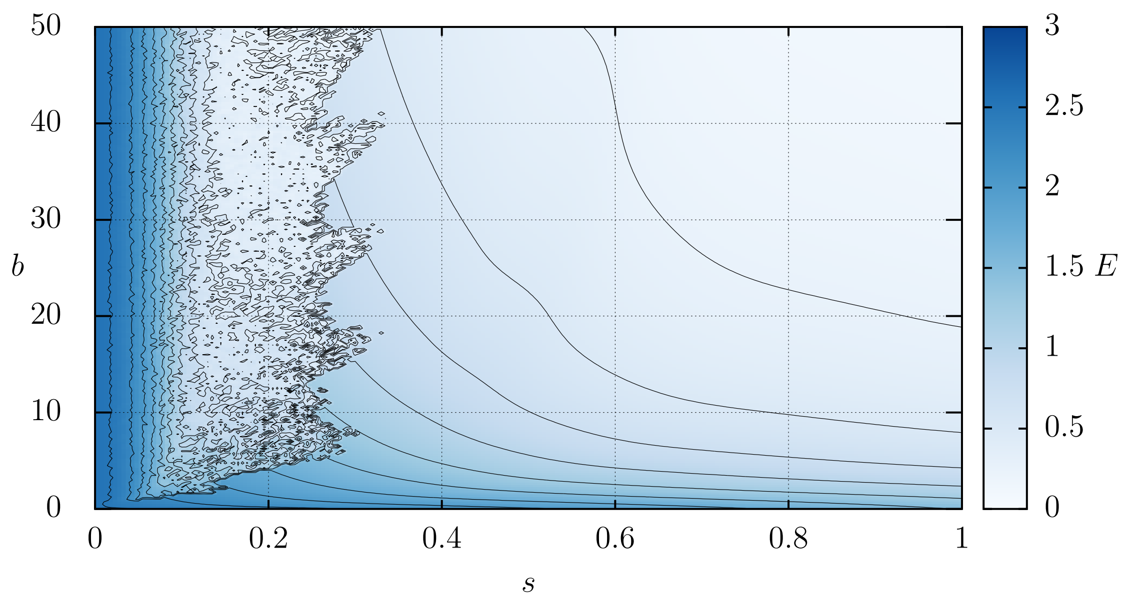

Figure 3.3

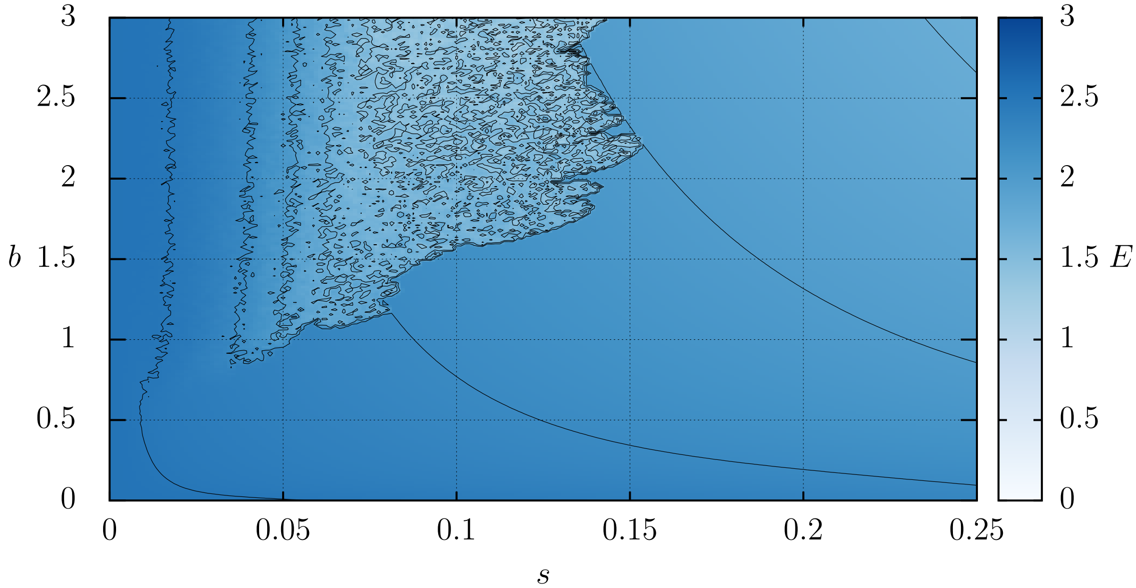

Figure 3.4

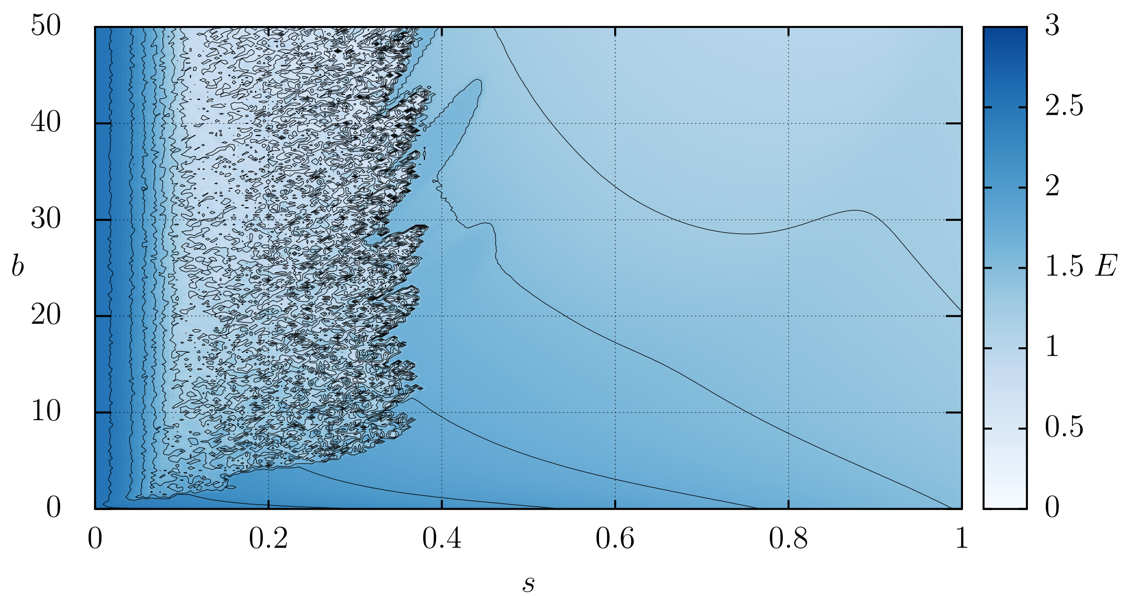

Figure 3.5

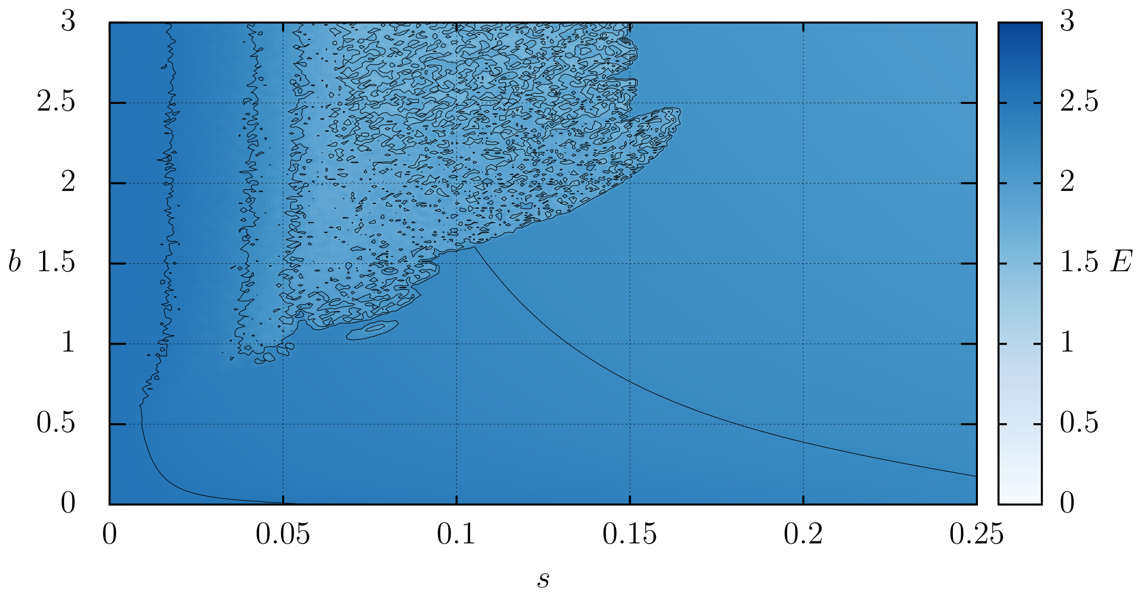

Figure 3.6

dispersion which then reinforces the oscillations. These oscillations are then intensified with each trip around the cavity until the envelope of the pulse becomes mangled.

Figure **??** shows the envelope, Fourier transform, and chirp for a stable wave as well as a broken wave. In the case of the stable wave the envelope and the Fourier transform are Gaussian-esque, and the chirp is a very smooth function in close agreement with [16]. However, in the case of the broken wave the envelope and Fourier transform are oscillatory. Furthermore, the chirp is highly erratic.

The breaking of the wave is not as predictable as one may expect—the structure of the boundary is quite rich. Note that in Figure **??** the wave breaks for the smaller of the two b values. This structure is highlighted in Figure **??** where the error is calculated by

$$\frac{\|A(\text{iteration } 40) - A(\text{iteration } 39)\|_2}{\|A(\text{iteration } 39)\|_2}.$$

Additionally, Figure **??** shows the energy of the pulse within the parameter space.

One interesting feature is that in the region $0.25 < s < 0.55$ the wave does not break regardless of b. This is because $b|A|^2$ is approximately constant for a particular s value, and so the phase shift added by the nonlinearity is constant, thus the wave is stable.

Since some of the individual mappings are nonlinear they do not commute—changing the order of the components should lead to different results. The nonlinearity should still follow the gain component since that is where it is most dominant. Furthermore, the loss is a linear mapping and so it will commute with both dispersion and modulation. Therefore, the only other unique case to consider is if dispersion follows modulation.

Figure **??** shows the effect of switching these two components. Overall, the large scale structure is unchanged with the exception of $0.3 < s < 0.55$ and $20 < b$. In this region the waveform appears to have reached a period 2 equilibrium, that is, $\mathcal{L}(\mathcal{L}(A)) = A$.

References

[1] C. S. Bohun, Y. Cher, L. J. Cummings, P. Howell, T. Mitre, L. Monasse, J. Mueller, and S. Rouillon, "Modelling and Specifying Dispersive Laser Cavities," vol. 2015, pp. 11–25, 2015.

[2] B. Burgoyne and A. Villeneuve, "Programmable lasers: design and applications," vol. 7580, pp. 7580 – 7580 – 15, 2010.

[3] S. Yamashita, Y. Nakazaki, R. Konishi, and O. Kusakari, "Wide and Fast Wavelength-Swept Fiber Laser Based on Dispersion Tuning for Dynamic Sensing," *Journal of Sensors*, vol. 2009, 2009.

[4] B. Burgoyne, A. Dupuis, and A. Villeneuve, "An Experimentally Validated Discrete Model for Dispersion-Tuned Actively Mode-Locked Lasers," *IEEE Journal of Selected Topics in Quantum Electronics*, vol. 20, pp. 390–398, Sept 2014.

[5] G. Agrawal, *Nonlinear Fiber Optics.* Academic Press, 5 ed., 2013.

[6] M. F. S. Ferreira, *Nonlinear Effects in Optical Fibers.* John Wiley & Sons, Inc., 2011.

[7] O. V. Shtyrina, A. V. Ivanenko, I. A. Yarutkina, A. V. Kemmer, A. S. Skidin, S. M. Kobtsev, and M. P. Fedoruk, "Experimental measurement and analytical estimation of the signal gain in an Er-doped fiber," *J. Opt. Soc. Am. B*, vol. 34, pp. 227–231, Feb 2017.

[8] I. Yarutkina, O. Shtyrina, M. Fedoruk, and S. Turitsyn, “Numerical modeling of fiber lasers with long and ultra-long ring cavity,” *Opt. Express*, vol. 21, pp. 12942–12950, May 2013.

[9] B. Burgoyne, N. Godbout, and S. Lacroix, “Nonlinear pulse propagation in optical fibers using second order moments,” *Opt. Express*, vol. 15, pp. 10075–10090, Aug 2007.

[10] J. Peng, H. Luo, and L. Zhan, “In-cavity soliton self-frequency shift ultrafast fiber lasers,” *Opt. Lett.*, vol. 43, pp. 5913–5916, Dec 2018.

[11] W. T. Silfvast, *Laser Fundamentals*. Cambridge University Press, 2 ed., 2004.

[12] G. Agrawal, *Fiber-Optic Communication Systems*. John Wiley & Sons, Inc., 3 ed., 2002.

[13] N. M. Litchinitser, B. J. Eggleton, and D. B. Patterson, “Fiber Bragg Gratings for Dispersion Compensation in Transmission: Theoretical Model and Design Criteria for Nearly Ideal Pulse Recompression,” *Journal of Lightwave Technology*, vol. 15, pp. 1303–1313, Aug 1997.

[14] B. Burgoyne. Private Communication, 2018.

[15] S. Li and K. T. Chan, “Electrical wavelength tunable and multiwavelength actively mode-locked fiber ring laser,” *Applied Physics Letters*, vol. 72, no. 16, pp. 1954–1956, 1998.

[16] Q. Chen, N. Lu, and F. Jiang, “Characterization of the dispersion of chirped fiber Bragg grating through Fourier transform spectrometry method,” vol. 6837, pp. 6837 – 6837 – 8, 2008.

[17] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," *Advances in Computational Mathematics*, vol. 5, Dec 1996.

[18] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. Academic Press, 7 ed., 2007.

Appendix A

The Lambert W Function

The Lambert W function is defined to be the inverse of the function $f(x) = x\mathrm{e}^x$ and its graph is shown in Figure A.1. In other words, if $z = x\mathrm{e}^x$ then $x = W(z)$. Notice that by combining these relations we obtain the identities

$$z = W(z)\mathrm{e}^{W(z)}, \qquad\qquad x = W(x\mathrm{e}^x). \tag{A.1}$$

This function is called the Lambert W function because it is the logarithm of a special instance of Lambert's series—the letter W is used because of the work done by E. M. Wright [17].

Notice that the original function, $f(x) = x\mathrm{e}^x$, is *not* injective, and as a consequence, the W function is multi-valued on the interval $[-1/\mathrm{e}, 0)$. To alleviate this, sometimes the branch $W(x) \geq -1$ is denoted W_0 and is called the upper or principal branch, whereas the branch $W(x) < -1$ is denoted W_{-1} and is called the lower branch. However, in this work the W function will only take positive real values and so this distinction is not needed.

The Lambert W function has applications in various areas of math and physics [17] including:

- Jet fuel problems

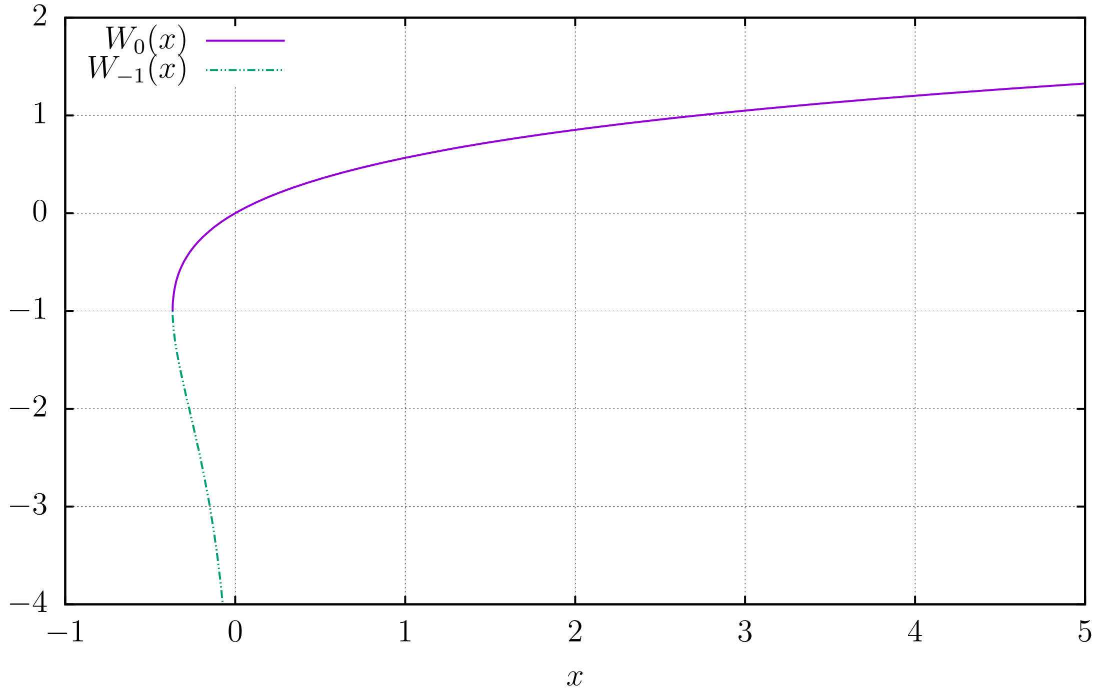

Figure A.1: The two branches of the Lambert W function.

- Combustion problems
- Enzyme kinetics problems
- Linear constant coefficient differential delay equations
- Volterra equations.

Primarily, the W function is used when solving iterated exponentiation, and solving certain algebraic equations. For example, consider the equation $z = x^x$. By taking the logarithm of each side we have

$$\begin{aligned} \log z &= x \log x, \\ &= \log x \mathrm{e}^{\log x}, \end{aligned}$$

which after applying the W function reduces to $W(\log z) = \log x$ by (A.1). Finally, the answer can be written as $x = \exp(W(\log z))$.

Appendix B

Spread Due to Dispersion

[18]

Appendix C

Code

```
import numpy as np
import scipy.special.lambertw as W
import matplotlib.pyplot as plt
import time

def Energy(A, dx):
    return np.trapz(np.real(A * np.conj(A)), dx = dx)

# Functions for each component
def Gain(A, E, a = 8000):
    return np.sqrt(W(a * E * np.exp(E)) / E) * A

def Loss(A, h = 0.04):
    return h * A

def Mod(A, T):
    return np.exp(-T**2 / 2) * A
```

```
def Fibre(A, b = 0.25):
    return np.exp(1j * b * np.real(np.abs(A)**2)) * A

def Disp(A, T, s = 0.2):
    F = np.fft.fft(A)
    F = (np.abs(F) > 10**-4) * F
    dw = np.pi / T[-1]
    w = np.fft.fftfreq(len(A)) * len(A) * dw
    return np.fft.ifft(F * np.exp(1j * (w**2 * s**2 - 0 * s**
    2 * w**3)))

# 1 round trip
def Loop(A, T, dx, s, b):
    A = Loss(A)
    A = Mod(A, T)
    A = Disp(A, T, s)

    A = Gain(A, Energy(A, dx))
    A = Fibre(A, b)
    return A

N = 500 # Number of loops of the circuit
p = 2**12 # Number of points in the discretization
width = 64 # Size of window
E0 = 1 # Initial energy

```

```
# Initialization
T = np.linspace(-width, width, p, endpoint = False)
dx = T[1] - T[0]
A0 = 1 / np.cosh(2 * T) * np.exp(1j * np.pi / 4)

#A0 = 1 / np.cosh(2 * T) * T**2 * np.exp(1j * np.pi / 4) * (1
    + np.sin(4 * T))
A0 = np.sqrt(E0 / Energy(A0, dx)) * A0 # Normalize
E = np.zeros(N)
data = np.zeros((2 * N, p))
A = A0

##########################################

dw = np.pi / T[-1]
w = np.fft.fftfreq(len(A)) * len(A) * dw

# Things for animated plot
plt.ion()
fig = plt.figure()
ax = fig.add_subplot(111)
line1, = ax.plot(T, np.real(A), 'r-', label = 'Real')
line2, = ax.plot(T, np.imag(A), 'b-', label = 'Imaginary')
line3, = ax.plot(T, np.angle(A), 'g-', label = 'Magnitude')

fig.canvas.draw()
fig.canvas.flush_events()
```

```

plt.legend()
#plt.xlim(-5000, 5000)
#plt.ylim(0, 0.02)

plt.xlim(-4, 4)
plt.ylim(-2, 2)
#plt.ylim(-0.1, 0.1)

dw = np.pi / T[-1]
w = np.fft.fftfreq(len(A)) * len(A) * dw

# N round trips of the laser
for i in range(N):
    # Animate the plot
    print i
    line1.set_ydata(np.real(A))
    line2.set_ydata(np.imag(A))
    line3.set_ydata(np.abs(A))
    fig.canvas.draw()
    fig.canvas.flush_events()
    #time.sleep(1)
    #print np.gradient(-np.gradient(np.angle(A), dx), dx)[len
    (A)/2]

    old = np.abs(A)
    A = Loop(A, T, dx, 0.2, 0)
```

```
    new = np.abs(A)

    #print np.sqrt(np.trapz((old - new)**2, dx = dx)) / np.sqrt(np.trapz(old**2, dx = dx))

print Energy(A, dx)
print np.abs(A)[len(A)/2]

    #np.savetxt('Envelope' + str(i) + '.dat', np.vstack((T, np.real(A), np.imag(A), np.abs(A), np.angle(A), w, np.abs(np.fft.fft(A)))).T)

#np.savetxt('E.dat', E)

#print np.sqrt(np.trapz((old - new)**2, dx = dx)) / np.sqrt(np.trapz(old**2, dx = dx))

#np.savetxt('Envelope.dat', np.vstack((T, np.real(A), np.imag(A), np.abs(A))).T)
#print np.abs(A[len(A)/2])
#print np.std(np.abs(A))

##################################################
'''
n = 41
#z = np.zeros((n**2, 4))
s = np.linspace(0, 1, num = n)
```

```
b = np.linspace(0, 60, num = n)
#count = 0

open('Test.dat', 'w').close()
f = open('Test.dat', 'ab')

for k in range(n):
    print k
    z = np.zeros((n, 4))
    for j in range(n):
        A0 = 1 / np.cosh(2 * T) * np.exp(1j * np.pi / 4)
        #A0 = np.exp(-T**2 / 2)
        A0 = np.sqrt(E0 / Energy(A0, dx)) * A0 # Normalize
        A = A0
        for i in range(25):
            old = np.abs(A)
            A = Loop(A, T, dx, s[j], b[k])
            new = np.abs(A)
        z[j] = s[j], b[k], np.sqrt(np.trapz((old - new)**2, dx = dx)) / np.sqrt(np.trapz(old**2, dx = dx)), Energy(A, dx)
        #count += 1
    np.savetxt(f, z)
    f.write('\n')

f.close()

```

```
#np.savetxt('Zeta.dat', np.vstack((s, z)).T)
#np.savetxt('a30.dat', z)
'''
##############################################3
'''
n = 51
z = np.zeros((n, 2))
s = np.linspace(0, 3.0, num = n)
count = 0

for j in range(n):
    print j
    A0 = 1 / np.cosh(2 * T) * np.exp(1j * np.pi / 4)
    #A0 = np.exp(-T**2 / 2)
    A0 = np.sqrt(E0 / Energy(A0, dx)) * A0 # Normalize
    A = A0
    for i in range(50):
        old = np.abs(A)
        A = Loop(A, T, dx, s[j], 0)
        new = np.abs(A)
    z[count] = s[j], Energy(A, dx)
    count += 1

np.savetxt('ETest.dat', z)

'''
```