1.4.5 Show that the divergence theorem is valid for a cube located at the center of a Cartesian coordinate system for a vector $\mathbf{A} = x\mathbf{u_x} + 2\mathbf{u_y}$. *Note: Diagram not shown.*

The divergence theorem states that the integral of a vector field over a closed surface is equal to the volume integral of the divergence of that field over the enclosed volume. That is,

$$\oint \mathbf{A} \bullet \mathbf{ds} = \int_{\Delta \mathrm{v}} (\nabla \bullet \mathbf{A}) \mathrm{dv}$$

We will compute both sides of this equation for the given vector field and surface and show that they are equal.

$$\begin{aligned}
\oint \mathbf{A} \bullet \mathbf{ds} &= \int_{x=-a}^{a} \int_{y=-a}^{a} (x\mathbf{u_x} + 2\mathbf{u_y}) \bullet (\mathrm{dydx}(\mathbf{u_z})) \\
&+ \int_{x=-a}^{a} \int_{y=-a}^{a} (x\mathbf{u_x} + 2\mathbf{u_y}) \bullet (\mathrm{dydx}(-\mathbf{u_z})) \\
&+ \int_{y=-a}^{a} \int_{z=-a}^{a} (x\mathbf{u_x} + 2\mathbf{u_y}) \bullet (\mathrm{dzdy}(\mathbf{u_x})) \\
&+ \int_{y=-a}^{a} \int_{z=-a}^{a} (x\mathbf{u_x} + 2\mathbf{u_y}) \bullet (\mathrm{dzdy}(-\mathbf{u_x})) \\
&+ \int_{x=-a}^{a} \int_{z=-a}^{a} (x\mathbf{u_x} + 2\mathbf{u_y}) \bullet (\mathrm{dzdx}(\mathbf{u_y})) \\
&+ \int_{x=-a}^{a} \int_{z=-a}^{a} (x\mathbf{u_x} + 2\mathbf{u_y}) \bullet (\mathrm{dzdx}(-\mathbf{u_x})) \\
&= \int_{y=-a}^{a} \int_{z=-a}^{a} x\mathrm{dzdy} - \int_{y=-a}^{a} \int_{z=-a}^{a} x\mathrm{dzdy} \\
&+ \int_{x=-a}^{a} \int_{z=-a}^{a} 2\mathrm{dzdx} - \int_{x=-a}^{a} \int_{z=-a}^{a} 2\mathrm{dzdx} \\
&= \int_{y=-a}^{a} \int_{z=-a}^{a} a\mathrm{dzdy} - \int_{y=-a}^{a} \int_{z=-a}^{a} -a\mathrm{dzdy} \\
&= 2a \int_{y=-a}^{a} \int_{z=-a}^{a} \mathrm{dzdy} \\
&= 2a \int_{y=-a}^{a} z|_{-a}^{a} \mathrm{dy} \\
&= 2a \int_{y=-a}^{a} 2a\mathrm{dy} \\
&= 4a^2 y|_{-a}^{a} \\
&= 8a^3
\end{aligned}$$

$$
\begin{aligned}
\int_{\Delta \mathrm{v}} (\nabla \bullet \mathbf{A}) \mathrm{dv} &= \int_{x=-a}^{a} \int_{y=-a}^{a} \int_{z=-a}^{a} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \mathrm{dzdydx} \\
&= \int_{x=-a}^{a} \int_{y=-a}^{a} \int_{z=-a}^{a} \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(0) \right) \mathrm{dzdydx} \\
&= \int_{x=-a}^{a} \int_{y=-a}^{a} \int_{z=-a}^{a} (1+0+0) \mathrm{dzdydx} \\
&= \int_{x=-a}^{a} \int_{y=-a}^{a} z|_{-a}^{a} \mathrm{dydx} \\
&= 2a \int_{x=-a}^{a} y|_{-a}^{a} \mathrm{dx} \\
&= 4a^2 x|_{-a}^{a} \\
&= 8a^3
\end{aligned}
$$

Both sides of the equation have the same value. Therefore, the divergence theorem is valid in this case.