1.3.2 Calculate the work required to move a mass m against a force field $\mathbf{F} = y\mathbf{u_x} + x\mathbf{u_y}$ along the path abc and along the path adc. Is this field conservative? *Note: Diagram not shown.*

By inspection of the diagram, the labeled points are $a = (0, 0)$, $b = (0, 5)$, $c = (5, 5)$, and $d = (0, 5)$. To find the work required to move along the two specified paths, we need to compute the path integrals of the force field.

$$\begin{aligned}
W_{abc} &= \int_a^b \mathbf{F} \bullet \mathbf{dl} + \int_b^c \mathbf{F} \bullet \mathbf{dl} \\
&= \int_a^b (y\mathbf{u_x} + x\mathbf{u_y}) \bullet (\mathrm{d}x\mathbf{u_x} + \mathrm{d}y\mathbf{u_y}) + \int_b^c (y\mathbf{u_x} + x\mathbf{u_y}) \bullet (\mathrm{d}x\mathbf{u_x} + \mathrm{d}y\mathbf{u_y}) \\
&= \int_a^b y\mathrm{d}x + \int_a^b x\mathrm{d}y + \int_b^c y\mathrm{d}x + \int_b^c x\mathrm{d}y \\
&= \int_{x=0}^0 y\mathrm{d}x + \int_{y=0}^5 x\mathrm{d}y + \int_{x=0}^5 y\mathrm{d}x + \int_{y=5}^5 x\mathrm{d}y \\
&= \int_{y=0}^5 0\mathrm{d}y + \int_{x=0}^5 5\mathrm{d}x \\
&= 5x\Big|_{x=0}^5 \\
&= 25
\end{aligned}$$

$$\begin{aligned}
W_{adc} &= \int_a^d \mathbf{F} \bullet \mathbf{dl} + \int_d^c \mathbf{F} \bullet \mathbf{dl} \\
&= \int_a^d (y\mathbf{u_x} + x\mathbf{u_y}) \bullet (\mathrm{d}x\mathbf{u_x} + \mathrm{d}y\mathbf{u_y}) + \int_d^c (y\mathbf{u_x} + x\mathbf{u_y}) \bullet (\mathrm{d}x\mathbf{u_x} + \mathrm{d}y\mathbf{u_y}) \\
&= \int_a^d y\mathrm{d}x + \int_a^d x\mathrm{d}y + \int_d^c y\mathrm{d}x + \int_d^c x\mathrm{d}y \\
&= \int_{x=0}^0 y\mathrm{d}x + \int_{y=0}^5 x\mathrm{d}y + \int_{x=0}^5 y\mathrm{d}x + \int_{y=5}^5 x\mathrm{d}y \\
&= \int_{y=0}^5 0\mathrm{d}y + \int_{x=0}^5 5\mathrm{d}x \\
&= 5x\Big|_{x=0}^5 \\
&= 25
\end{aligned}$$

The fact that both of the given paths have equal values for their path integrals is not sufficient to prove that $\mathbf{F}$ is conservative. It is possible that there are other paths between a and c that produce different values, and we cannot check all of them because there are infinitely many. However, it is sufficient to show that there is a scalar function f that satisfies $\mathbf{F} = \nabla f$, by the definition of a conservative field. That function is $f = xy$.

$$\begin{aligned}
\mathbf{F} &= \nabla f \\
y\mathbf{u_x} + x\mathbf{u_y} &= \frac{\partial f}{\partial x}\mathbf{u_x} + \frac{\partial f}{\partial y}\mathbf{u_y} \\
y\mathbf{u_x} + x\mathbf{u_y} &= \frac{\partial}{\partial x}(xy)\mathbf{u_x} + \frac{\partial}{\partial y}(xy)\mathbf{u_y} \\
y\mathbf{u_x} + x\mathbf{u_y} &= y\mathbf{u_x} + x\mathbf{u_y}
\end{aligned}$$

Therefore, $\mathbf{F}$ is conservative.