1.4.6 Show that the divergence theorem is valid for a sphere of radius a located at the center of a coordinate system for a vector $\mathbf{A} = r\mathbf{u_r}$.

The divergence theorem states that the integral of a vector field over a closed surface is equal to the volume integral of the divergence of that field over the enclosed volume. That is,

$$\oint \mathbf{A} \bullet \mathbf{ds} = \int_{\Delta \mathrm{v}} (\nabla \bullet \mathbf{A}) \mathrm{dv}$$

We will compute both sides of this equation for the given vector field and surface and show that they are equal.

$$\begin{aligned}
\oint \mathbf{A} \bullet \mathbf{ds} &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (r\mathbf{u_r}) \bullet (r^2 \sin\theta \mathrm{d}\phi \mathrm{d}\theta \mathbf{u_r}) \\
&= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^3 \sin\theta \mathrm{d}\phi \mathrm{d}\theta \\
&= \int_{\theta=0}^{\pi} r^3 \sin\theta \phi|_{\phi=0}^{2\pi} \mathrm{d}\theta \\
&= \int_{\theta=0}^{\pi} 2\pi r^3 \sin\theta \mathrm{d}\theta \\
&= -2\pi r^3 \cos\theta|_{\theta=0}^{\pi} \\
&= 4\pi r^3 \\
&= 4\pi a^3
\end{aligned}$$

$$
\begin{aligned}
\int_{\Delta \mathrm{v}} (\nabla \bullet \mathbf{A}) \mathrm{dv} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \frac{\partial}{\partial r} (r^3) \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} 3r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^3 \sin\theta |_{r=0}^{a} \mathrm{d}\theta\mathrm{d}\phi \\
&= \int_{\phi=0}^{2\pi} -r^3 \cos\theta |_{\theta=0}^{\pi} \mathrm{d}\phi \\
&= \int_{\phi=0}^{2\pi} 2r^3 \mathrm{d}\phi \\
&= 2r^3 \phi |_{\phi=0}^{2\pi} \\
&= 4\pi r^3 \\
&= 4\pi a^3
\end{aligned}
$$

Both sides of the equation have the same value. Therefore, the divergence theorem is valid in this case.