1.4.15 By direct differentiation, show that $\nabla^2(\frac{1}{r}) = 0$ at all points where $r \neq 0$ where $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$.

From **1.4.3**, we know that

$$\nabla\left(\frac{1}{r}\right) = -(r^2)^{-\frac{3}{2}}(x-x')\mathbf{u_x} - (r^2)^{-\frac{3}{2}}(y-y')\mathbf{u_y} - (r^2)^{-\frac{3}{2}}(z-z')\mathbf{u_z}$$

which we can use in the following way.

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \nabla \bullet \nabla\frac{1}{r} \\
&= \nabla \bullet \left[-(r^2)^{-\frac{3}{2}}(x-x')\mathbf{u_x} - (r^2)^{-\frac{3}{2}}(y-y')\mathbf{u_y} - (r^2)^{-\frac{3}{2}}(z-z')\mathbf{u_z}\right] \\
&= \frac{\partial}{\partial x}\left[-(r^2)^{-\frac{3}{2}}(x-x')\right] + \frac{\partial}{\partial y}\left[-(r^2)^{-\frac{3}{2}}(y-y')\right] + \frac{\partial}{\partial z}\left[-(r^2)^{-\frac{3}{2}}(z-z')\right] \\
&= \frac{3}{2}(r^2)^{-\frac{5}{2}}(2x-2x')(x-x') - (r^2)^{-\frac{3}{2}} \\
&\quad + \frac{3}{2}(r^2)^{-\frac{5}{2}}(2y-2y')(y-y') - (r^2)^{-\frac{3}{2}} \\
&\quad + \frac{3}{2}(r^2)^{-\frac{5}{2}}(2z-2z')(z-z') - (r^2)^{-\frac{3}{2}} \\
&= 3r^{-5}(x-x')^2 + 3r^{-5}(y-y')^2 + 3r^{-5}(z-z')^2 - 3r^{-3} \\
&= 3r^{-5}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right] - 3r^{-3} \\
&= 3r^{-5} \cdot r^2 - 3r^{-3} \\
&= 3r^{-3} - 3r^{-3} \\
&= 0
\end{aligned}$$

Therefore, the Laplacian of the inverse of r is equal to 0, except when $r \neq 0$, where it is undefined.