1.4.8 Evaluate the line integral of the vector function $\mathbf{A} = x\mathbf{u_x} + x^2\mathbf{u_y} + xyz\mathbf{u_z}$ around the square contour C. Integrate $\nabla \times \mathbf{A}$ over the surface bounded by C. Show that this example satisfies Stokes's theorem. *Note: Diagram not shown.*

Let the four corners of the square contour C be

$$a = (0,0), b = (1,0), c = (1,1), d = (0,1)$$

$$\begin{aligned}
\oint \mathbf{A} \bullet \mathbf{dl} &= \int_a^b (x\mathbf{u_x} + x^2\mathbf{u_y} + xyz\mathbf{u_z}) \bullet \mathrm{dx}\mathbf{u_x} \\
&+ \int_b^c (x\mathbf{u_x} + x^2\mathbf{u_y} + xyz\mathbf{u_z}) \bullet \mathrm{dy}\mathbf{u_y} \\
&+ \int_c^d (x\mathbf{u_x} + x^2\mathbf{u_y} + xyz\mathbf{u_z}) \bullet \mathrm{dx}\mathbf{u_x} \\
&+ \int_d^a (x\mathbf{u_x} + x^2\mathbf{u_y} + xyz\mathbf{u_z}) \bullet \mathrm{dy}\mathbf{u_y} \\
&= \int_{x=0}^1 x\mathrm{dx} + \int_{y=0}^1 x^2|_{x=1}\mathrm{dy} + \int_{x=1}^0 x\mathrm{dx} + \int_{y=1}^0 x^2|_{x=0}\mathrm{dy} \\
&= \frac{x^2}{2}\bigg|_0^1 + y\bigg|_0^1 - \frac{x^2}{2}\bigg|_0^1 - 0 \\
&= 1
\end{aligned}$$

Stokes's theorem states that the loop integral of a field is equal to the curl of the field on the surface enclosed by the loop. We can show that this theorem holds in this case by evaluating the appropriate surface integral and comparing the result to the loop integral above.

$$\begin{aligned}
\int_{\Delta\mathbf{s}} (\nabla \times \mathbf{A}) \bullet \mathbf{ds} &= \int_{\Delta\mathbf{s}} \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{u_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{u_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{u_z} \right] \bullet (\mathrm{dydx}\mathbf{u_z}) \\
&= \int_{x=0}^1 \int_{y=0}^1 \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(x) \right] \mathrm{dydx} \\
&= \int_{x=0}^1 \int_{y=0}^1 2x\mathrm{dydx} \\
&= \int_{x=0}^1 2xy|_{y=0}^1 \mathrm{dx} \\
&= \int_{x=0}^1 2x\mathrm{dx} \\
&= x^2|_{x=0}^1 \\
&= 1
\end{aligned}$$

The two sides of Stokes's theorem give the same value. Therefore, the theorem is valid in this case.