1.3.5 Calculate the closed-surface integral $\oint \mathbf{A} \bullet \mathbf{ds}$ if $\mathbf{A} = \mathbf{x}\mathbf{u_x} + \mathbf{y}\mathbf{u_y}$ and the surface is that of a cube. Apply the divergence theorem to solve the same integral. *Note: Diagram not shown.*

The cube has side length 2, and one of its corners is at the origin.

$$
\begin{aligned}
\oint \mathbf{A} \bullet \mathbf{ds} &= \int_{y=0}^{2} \int_{x=0}^{2} (x\mathbf{u_x} + y\mathbf{u_y}) \bullet (\mathrm{d}x\mathrm{d}y\mathbf{u_x}) \\
&\quad + \int_{y=0}^{2} \int_{x=0}^{2} (x\mathbf{u_x} + y\mathbf{u_y}) \bullet (\mathrm{d}x\mathrm{d}y(-\mathbf{u_x})) \\
&\quad + \int_{z=0}^{2} \int_{x=0}^{2} (x\mathbf{u_x} + y\mathbf{u_y}) \bullet (\mathrm{d}x\mathrm{d}z\mathbf{u_y}) \\
&\quad + \int_{z=0}^{2} \int_{x=0}^{2} (x\mathbf{u_x} + y\mathbf{u_y}) \bullet (\mathrm{d}x\mathrm{d}z(-\mathbf{u_y})) \\
&\quad + \int_{z=0}^{2} \int_{y=0}^{2} (x\mathbf{u_x} + y\mathbf{u_y}) \bullet (\mathrm{d}y\mathrm{d}z\mathbf{u_y}) \\
&\quad + \int_{z=0}^{2} \int_{y=0}^{2} (x\mathbf{u_x} + y\mathbf{u_y}) \bullet (\mathrm{d}y\mathrm{d}z(-\mathbf{u_y})) \\
&= \int_{z=0}^{2} \int_{x=0}^{2} 2\mathrm{d}x\mathrm{d}z + \int_{z=0}^{2} \int_{y=0}^{2} 2\mathrm{d}y\mathrm{d}z \\
&= \int_{z=0}^{2} 2x|_{x=0}^{2}\mathrm{d}z + \int_{z=0}^{2} 2y|_{y=0}^{2}\mathrm{d}z \\
&= \int_{z=0}^{2} 4\mathrm{d}z + \int_{z=0}^{2} 4\mathrm{d}z \\
&= 4z|_{z=0}^{2} + 4z|_{z=0}^{2} \\
&= 8 + 8 \\
&= 16
\end{aligned}
$$

We can apply the divergence theorem to solve the above integral in a different way.

$$
\begin{aligned}
\oint \mathbf{A} \bullet \mathbf{ds} &= \int_{\Delta v} (\nabla \bullet \mathbf{A}) \mathrm{d}v \\
\int_{\Delta v} (\nabla \bullet \mathbf{A}) \mathrm{d}v &= \int_{\Delta v} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) \mathrm{d}v \\
&= \int_{\Delta v} (1+1) \mathrm{d}v \\
&= \int_{z=0}^{2} \int_{y=0}^{2} \int_{x=0}^{2} 2 \mathrm{d}x \mathrm{d}y \mathrm{d}z \\
&= \int_{z=0}^{2} \int_{y=0}^{2} 2x|_{x=0}^{2} \mathrm{d}y \mathrm{d}z \\
&= \int_{z=0}^{2} \int_{y=0}^{2} 4 \mathrm{d}y \mathrm{d}z \\
&= \int_{z=0}^{2} 4y|_{y=0}^{2} \mathrm{d}z \\
&= \int_{z=0}^{2} 8 \mathrm{d}z \\
&= 8z|_{z=0}^{2} \\
&= 16
\end{aligned}
$$