1.4.2 Find the gradient of the function $H = x^2yz$ and also the directional derivative of H specified by the unit vector $\mathbf{u} = a(\mathbf{u_x} + \mathbf{u_y} + \mathbf{u_z})$ where a is a constant at the point (1,2,3). State the value for the constant a.

$$\begin{aligned}
\nabla H &= \frac{\partial H}{\partial x}\mathbf{u_x} + \frac{\partial H}{\partial y}\mathbf{u_y} + \frac{\partial H}{\partial z}\mathbf{u_z} \\
&= \frac{\partial}{\partial x}(x^2yz)\mathbf{u_x} + \frac{\partial}{\partial y}(x^2yz)\mathbf{u_y} + \frac{\partial}{\partial z}(x^2yz)\mathbf{u_z} \\
&= 2xyz\mathbf{u_x} + x^2z\mathbf{u_y} + x^2y\mathbf{u_z}
\end{aligned}$$

Let $p = \begin{pmatrix}1 & 2 & 3\end{pmatrix}$. The directional derivative of H at p along $\mathbf{u}$ is given by

$$\begin{aligned}
\nabla H(p) \bullet \mathbf{u} &= (2xyz\mathbf{u_x} + x^2z\mathbf{u_y} + x^2y\mathbf{u_z})|_p \bullet a(\mathbf{u_x} + \mathbf{u_y} + \mathbf{u_z}) \\
&= (2axyz + ax^2z + ax^2y)|_p \\
&= 2a \cdot 1 \cdot 2 \cdot 3 + a \cdot 1^2 \cdot 3 + a \cdot 1^2 \cdot 2 \\
&= 12a + 3a + 2a \\
&= 17a
\end{aligned}$$

The unit vector $\mathbf{u}$ has length 1 by definition, and we can use that to compute the value of a.

$$\begin{aligned}
||\mathbf{u}|| &= 1 \\
\sqrt{a^2 + a^2 + a^2} &= 1 \\
\sqrt{3a^2} &= 1 \\
3a^2 &= 1 \\
a^2 &= \frac{1}{3} \\
a &= \frac{\sqrt{3}}{3}
\end{aligned}$$

Therefore, $\nabla H(p) \bullet \mathbf{u} = \frac{17\sqrt{3}}{3} \approx 9.81$.