1.4.3 By direct differentiation show that

$$\nabla\left(\frac{1}{r}\right) = -\nabla'\left(\frac{1}{r}\right)$$

where $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ and ∇' denotes differentiation with respect to the variables x', y', and z'.

We will compute both sides of the equation and show that they are equal. For brevity, we will write $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$.

$$\begin{aligned}
\nabla\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x}(r^2)^{-\frac{1}{2}}\mathbf{u_x} + \frac{\partial}{\partial y}(r^2)^{-\frac{1}{2}}\mathbf{u_y} + \frac{\partial}{\partial z}(r^2)^{-\frac{1}{2}}\mathbf{u_z} \\
&= -\frac{1}{2}(r^2)^{-\frac{3}{2}}(2x-2x')\mathbf{u_x} \\
&\quad -\frac{1}{2}(r^2)^{-\frac{3}{2}}(2y-2y')\mathbf{u_y} \\
&\quad -\frac{1}{2}(r^2)^{-\frac{3}{2}}(2z-2z')\mathbf{u_z} \\
&= -(r^2)^{-\frac{3}{2}}(x-x')\mathbf{u_x} - (r^2)^{-\frac{3}{2}}(y-y')\mathbf{u_y} - (r^2)^{-\frac{3}{2}}(z-z')\mathbf{u_z} \\
&= -r^{-3}(x-x')\mathbf{u_x} - r^{-3}(y-y')\mathbf{u_y} - r^{-3}(z-z')\mathbf{u_z}
\end{aligned}$$

$$\begin{aligned}
-\nabla'\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x'}(r^2)^{-\frac{1}{2}}\mathbf{u_x} + \frac{\partial}{\partial y'}(r^2)^{-\frac{1}{2}}\mathbf{u_y} + \frac{\partial}{\partial z'}(r^2)^{-\frac{1}{2}}\mathbf{u_z} \\
&= \frac{1}{2}(r^2)^{-\frac{3}{2}}(2x'-2x)\mathbf{u_x} \\
&\quad +\frac{1}{2}(r^2)^{-\frac{3}{2}}(2y'-2y)\mathbf{u_y} \\
&\quad +\frac{1}{2}(r^2)^{-\frac{3}{2}}(2z'-2z)\mathbf{u_z} \\
&= -(r^2)^{-\frac{3}{2}}(x-x')\mathbf{u_x} - (r^2)^{-\frac{3}{2}}(y-y')\mathbf{u_y} - (r^2)^{-\frac{3}{2}}(z-z')\mathbf{u_z} \\
&= -r^{-3}(x-x')\mathbf{u_x} - r^{-3}(y-y')\mathbf{u_y} - r^{-3}(z-z')\mathbf{u_z}
\end{aligned}$$

Therefore, the given equation is true.