1.4.7 The water that flows in a channel with sides at $x = 0$ and $x = a$ has a velocity distribution $\mathbf{v}(x, z) = [(a/2)^2 - (x - a/2)^2]z^2\mathbf{u_y}$. The bottom of the river is at $z = 0$. A small paddle wheel with its axis parallel to the z axis is inserted into the channel and is free to rotate. Find the relative rates of rotation at the points

$$\left(x = \frac{a}{4}, z = 1\right), \left(x = \frac{a}{2}, z = 1\right), \text{and} \left(x = \frac{3a}{4}, z = 1\right)$$

Will the paddle wheel rotate if its axis is parallel to the x axis or the y axis? *Note: Diagram not shown.*

To find the relative rates of rotation at the given points, we must evaluate the curl of $\mathbf{v}$ in the $\mathbf{u_z}$ direction at those points. Let p_1, p_2, and p_3 be the three given points.

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{u_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{u_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{u_z} \\
&= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\left[\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2\right]z^2\right)\right]\mathbf{u_x} \\
&\quad + \left[\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(0)\right]\mathbf{u_y} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\left[\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2\right]z^2\right) - \frac{\partial}{\partial y}(0)\right]\mathbf{u_z} \\
&= -2z\left[\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2\right]\mathbf{u_x} + \left(z^2(a - 2x)\right)\mathbf{u_z}
\end{aligned}$$

$$\begin{aligned}
(\nabla \times \mathbf{v})|_{p_1} \bullet \mathbf{u_z} &= \left[-2 \cdot 1\left[\left(\frac{a}{2}\right)^2 - \left(\frac{a}{4} - \frac{a}{2}\right)^2\right]\mathbf{u_x} + \left[1^2\left(a - 2 \cdot \frac{a}{4}\right)\right]\mathbf{u_z}\right] \bullet \mathbf{u_z} \\
&= \frac{a}{2}
\end{aligned}$$

$$\begin{aligned}
(\nabla \times \mathbf{v})|_{p_2} \bullet \mathbf{u_z} &= \left[-2 \cdot 1\left[\left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - \frac{a}{2}\right)^2\right]\mathbf{u_x} + \left[1^2\left(a - 2 \cdot \frac{a}{2}\right)\right]\mathbf{u_z}\right] \bullet \mathbf{u_z} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(\nabla \times \mathbf{v})|_{p_3} \bullet \mathbf{u_z} &= \left[-2 \cdot 1\left[\left(\frac{a}{2}\right)^2 - \left(\frac{3a}{4} - \frac{a}{2}\right)^2\right]\mathbf{u_x} + \left[1^2\left(a - 2 \cdot \frac{3a}{4}\right)\right]\mathbf{u_z}\right] \bullet \mathbf{u_z} \\
&= -\frac{a}{2}
\end{aligned}$$

To determine whether or not the paddle wheel would rotate if oriented along the x or y axes, we must evaluate the curl of $\mathbf{v}$ in those directions.

$$
\begin{aligned}
(\nabla \times \mathbf{v}) \bullet \mathbf{u_x} &= \left[-2z\left[\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2\right]\mathbf{u_x} + \left(z^2(a-2x)\right)\mathbf{u_z}\right] \bullet \mathbf{u_x} \\
&= -2z\left[\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
(\nabla \times \mathbf{v}) \bullet \mathbf{u_y} &= \left[-2z\left[\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2\right]\mathbf{u_x} + \left(z^2(a-2x)\right)\mathbf{u_z}\right] \bullet \mathbf{u_y} \\
&= 0
\end{aligned}
$$

The paddle wheel rotates when oriented parallel to the x axis as long as $z \neq 0$ and $x \neq \frac{a}{2}$. It does not rotate when oriented parallel to the y axis.