Berkeley Engineering
MAS-E

Robust Optimization and Applications

Module 2: Robust Linear Programming

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Outline

E 238: Robust Optimization and Applications

Module 2: Robust Linear Programming

Part 1: Overview

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Objectives for This Module

- Learn how to apply robust optimization when the nominal problem is a linear program
- Understand the role of single inequality analysis in the development of the robust counterpart
- Explore a few practically relevant uncertainty models

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Part 2: Robust LP Framework

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Robust Optimization Recap

Curse of Uncertainty

"Nominal" optimization problem:

$$\min_x \; f_0(x) \; : \; f_i(x) \leq 0, \quad i = 1, \ldots, m.$$

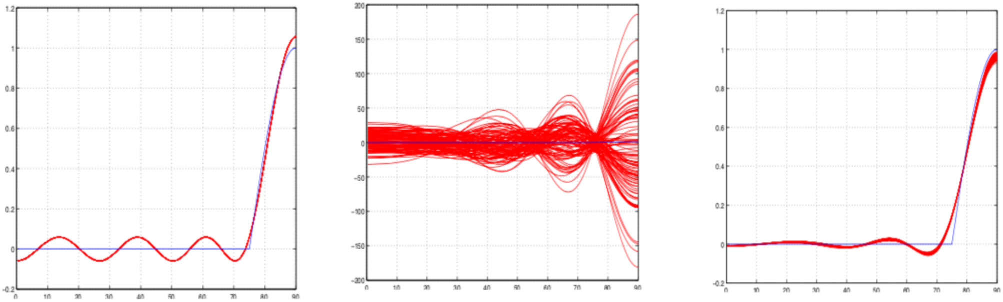
In practice, problem data is <u>uncertain</u>:

- <u>Estimation</u> errors affect problem parameters
- <u>Implementation</u> errors affect the decision taken

Uncertainties often lead to highly unstable solutions, or much degraded realized performance.

These problems are compounded in problems with multiple decision periods.

Example: Antenna Array Design



Antenna design: nominal, perturbed nominal, robust

The nominal solution, when implemented with a .01% relative error, gives a very bad result. A robust approach sacrifices a bit of performance but completely removes this high sensitivity issue.

Robust Counterpart

"Nominal" optimization problem:

$$\min_x f_0(x) \ : \ f_i(x) \leq 0, \quad i = 1, \ldots, m.$$

Robust counterpart:

$$\min_x \max_{u \in \mathcal{U}} f_0(x, u) \ : \ \forall\, u \in \mathcal{U}, \quad f_i(x, u) \leq 0, \quad i = 1, \ldots, m$$

- functions f_i now depend on a second variable u, the "uncertainty," which is constrained to lie in given set $\mathcal{U}$
- High complexity in general, except in some practically relevant cases

Robust LP Counterparts

Uncertainty in the Objective Function

Nominal problem:

$$\min_x \; c^T x \;:\; a_i^T x \leq b_i, \quad i = 1, \ldots, m.$$

Now assume that the cost vector c is only known to belong to a given set $\mathcal{U} \subseteq \mathbb{R}^n$.

The robust counterpart minimizes the worst-case cost:

$$\min_x \; \left(\max_{c \in \mathcal{U}} \; c^T x \right) \;:\; a_i^T x \leq b_i, \quad i = 1, \ldots, m.$$

Robust LP Counterparts

Uncertainty in Coefficients

Nominal problem:

$$\min_x \; c^T x \; : \; a_i^T x \leq b_i, \;\; i = 1, \ldots, m.$$

Now assume that a_i is only known to belong to a given set $\mathcal{U}_i \subseteq \mathbb{R}^n$.

The robust counterpart

$$\min_x \; c^T x \; : \; \forall a_i \in \mathcal{U}_i, \;\; a_i^T x \leq b_i, \;\; i = 1, \ldots, m$$

ensures that x remains always feasible despite the perturbations.

Support Function

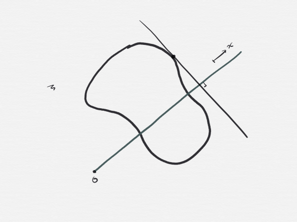
The robust constraint

$$\forall\, a \in \mathcal{U} \ : \ a^T x \leq b$$

can be expressed as

$$\phi_{\mathcal{U}}(x) \leq b,$$

where $\phi_{\mathcal{U}}$ is the support function of the set $\mathcal{U}$:



$$\phi_{\mathcal{U}}(x) := \max_{a \in \mathcal{U}} \ a^T x$$

Uncertainty in all Problem Data

Nominal problem:

$$\min_{x,t} \ t \ : \ c^T x + d \leq t, \ \ Ax \leq b.$$

Now assume that the problem data matrix

$$M := \begin{pmatrix} A & b \\ c^T & d \end{pmatrix}$$

is only known to belong to a given set $\mathcal{U} \subseteq \mathbb{R}^{(m+1)\times(n+1)}$.

Robust counterpart:

$$\min_{x,t} \ t \ : \ \forall M \in \mathcal{U}, \ \ M \begin{pmatrix} x \\ 1 \end{pmatrix} \leq \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

Formulation via Support Functions

Let $\mathcal{U}_i$, $i = 1, \ldots, m+1$ be the projection of set $\mathcal{U}$ on the subspace corresponding to the i-th row of M:

$$\mathcal{U}_i = \left\{ M^T e_i \ : \ M \in \mathcal{U} \right\},$$

where e_i is the i-th unit vector.
The constraint

$$\forall M \in \mathcal{U}, \quad M \begin{pmatrix} x \\ 1 \end{pmatrix} \leq \begin{pmatrix} 0 \\ t \end{pmatrix}$$

is equivalent to

$$\phi_{\mathcal{U}_i}(x) \leq v_i, \quad i = 1, \ldots, m+1$$

with $v = (0, t) \in \mathbb{R}^{n+1}$.

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Part 3: Single Inequality Analysis

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Single Inequality Analysis: Overview

Next we examine the robust counterpart to a single inequality constraint:

$$\forall\, a \in \mathcal{U}, \quad a^T x \leq b$$

where $\mathcal{U}$ takes the following forms:

- Scenario uncertainty: $\mathcal{U}$ is a finite set of "'"scenarios"
- $\mathcal{U}$ is a sphere, or more generally an ellipsoid
- $\mathcal{U}$ is a box
- Budgeted uncertainty: $\mathcal{U}$ is the intersection of a box and a "diamond" (l_1-norm ball)

The above constraint can be written

$$b \geq \phi_{\mathcal{U}}(x) := \max_{a \in \mathcal{U}} a^T x.$$

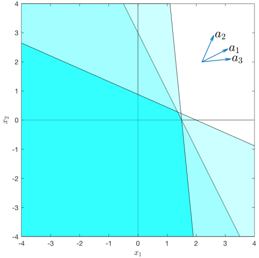
Scenario Uncertainty

Scenario uncertainty model:

$$\mathcal{U} = \left\{a^{(1)}, \ldots, a^{(K)}\right\},$$

with $a^{(k)} \in \mathbb{R}^n$ a "scenario", $k = 1, \ldots, K$. We have

$$\phi_{\mathcal{U}}(x) = \max_{a \in \mathcal{U}} a^T x = \max_{1 \le k \le K} (a^{(k)})^T x.$$



With three scenarios, the set

$$\left\{x \;:\; a^T x \le b \;:\; \forall\, a \in \mathcal{U}\right\}$$

is a polyhedron made up of three half-spaces.

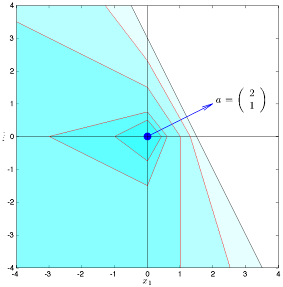
Box Uncertainty

Box uncertainty model:

$$\mathcal{U} = \{a : \|a - \hat{a}\|_\infty \le \rho\} = \{\hat{a} + \rho u : \|u\|_\infty \le 1\},$$

where $\hat{a}$, $\rho \ge 0$ are given. We have

$$\max_{a \in \mathcal{U}} a^T x = \hat{a}^T x + \rho \cdot \max_u \left\{ u^T x \ : \ \|u\|_\infty \le 1 \right\} = \hat{a}^T x + \rho \|x\|_1.$$



When $\mathcal{U}$ is a box, the set

$$\left\{ x \ : \ a^T x \le b \ : \ \forall\, a \in \mathcal{U} \right\}$$

is a polyhedron, with at most 2^n vertices.

Budgeted Uncertainty

Uncertainty model:

$$\mathcal{U} = \{\hat{a} + \rho u, \ \ \|u\|_1 \leq k, \ \ \|u\|_\infty \leq 1\},$$

with $k \in [1, n]$ an upper bound on the <u>number</u> of non-zero values in u. Then

$$\phi_{\mathcal{U}}(x) = \max_{a \in \mathcal{U}} \ a^T x = \hat{a}^T x + s_k(|x|),$$

where $s_k(z)$ is the sum of the top k entries in vector z.

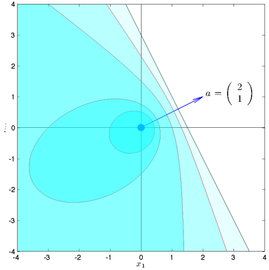
Spherical Uncertainty

Uncertainty model:

$$\mathcal{U} = \{a : \|a - \hat{a}\|_2 \le \rho\} = \{\hat{a} + \rho u : \|u\|_2 \le 1\}.$$

We have

$$\max_{a \in \mathcal{U}} a^T x = \hat{a}^T x + \rho \cdot \max_{u : \|u\|_2 \le 1} u^T x = \hat{a}^T x + \rho \|x\|_2.$$



When $\mathcal{U}$ is a sphere, the set

$$\{x : a^T x \le b : \forall a \in \mathcal{U}\}$$

is defined by a single second-order cone constraint.

Ellipsoidal Uncertainty

Ellipsoidal uncertainty model:

$$\mathcal{U} = \{a = \hat{a} + Ru \ : \ \|u\|_2 \leq 1\},$$

where $\hat{a}$ represents the nominal value of the coefficient vector, and matrix R determines the shape and size of the ellipse. Then

$$\phi_{\mathcal{U}}(x) = \max_{a \in \mathcal{U}} a^T x = \hat{a}^T x + \max_{u \,:\, \|u\|_2 \leq 1} (Ru)^T x = \hat{a}^T x + \|R^T x\|_2.$$

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Part 4: Robust LP Counterparts

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Robust Counterpart: Box Uncertainty

The robust counterpart:

$$\begin{aligned} &\min_x c^T x \\ &\text{s.t. } \forall\, a_i \in \mathcal{B}_i \;:\; a_i^T x \le b_i \qquad\qquad i = 1, \ldots, m, \end{aligned}$$

where $\mathcal{B}_i = \{\hat{a}_i + \rho_i u \;:\; \|u\|_\infty \le 1\}$, $i = 1, \ldots, m$, can be written

$$\begin{aligned} &\min_x c^T x \\ &\text{s.t. } \hat{a}_i^T x + \rho_i \|x\|_1 \le b_i \quad i = 1, \ldots, m. \end{aligned}$$

Robust Counterpart: Box Uncertainty

LP Representation

The robust counterpart:

$$\min_{x} c^T x$$
$$\text{s.t. } \hat{a}_i^T x + \rho_i \|x\|_1 \leq b_i \quad i = 1, \ldots, m$$

can be expressed in standard LP form as

$$\min_{x,v} c^T x$$
$$\text{s.t. } \hat{a}_i^T x + \rho_i \sum_{j=1}^{n} v_j \leq b_i, \quad i = 1, \ldots, m,$$
$$-v_j \leq x_j \leq v_j, \quad j = 1, \ldots, n.$$

Robust Counterpart: Budgeted Uncertainty

The robust counterpart

$$\begin{aligned} &\min_x \; c^T x \\ &\text{s.t. } \forall\, a_i \in \mathcal{B}_i \;:\; a_i^T x \leq b_i \quad i = 1, \ldots, m, \end{aligned}$$

where $\mathcal{B}_i = \{\hat{a}_i + \rho_i u \;:\; \|u\|_1 \leq k,\; \|u\|_\infty \leq 1\}$, $i = 1, \ldots, m$, can be written

$$\begin{aligned} &\min_x \; c^T x \\ &\text{s.t. } \hat{a}_i^T x + \rho_i s_k(|x|) \leq b_i \quad i = 1, \ldots, m. \end{aligned}$$

Robust LP With Budgeted Uncertainty

LP Representation

<u>Fact:</u> for any vector z

$$s_k(z) = \min_{\alpha} \; k\alpha + \sum_{i=1}^{n} \max(0, z_i - \alpha).$$

The robust counterpart writes as an LP:

$$\begin{aligned} \min_{x,v,\alpha} \; & c^T x \\ \text{s.t. } & \hat{a}_i^T x + \rho_i (k\alpha + v^T \mathbf{1}) \leq b_i, \quad i = 1, \ldots, m, \\ & v \geq 0, \quad v_j \geq x_j - \alpha, \quad v_j \geq -x_j - \alpha, \quad j = 1, \ldots, n. \end{aligned}$$

Ellipsoidal Uncertainty

The robust LP with ellipsoidal uncertainty:

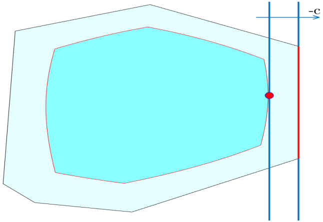
$$\min_x \ c^T x \ : \ \forall\, a_i \in \mathcal{E}_i \ : \ a_i^T x \le b_i i = 1, \ldots, m,$$

where $\mathcal{E}_i = \{\hat{a}_i + R_i u \ : \ \|u\|_2 \le 1\}$, $i = 1, \ldots, m$, is the SOCP

$$\min_x \ c^T x \ : \ \hat{a}_i^T x + \|R_i^T x\|_2 \le b_i \ \ i = 1, \ldots, m.$$

Ellipsoidal Uncertainty

Geometry



With spherical uncertainty, the robust counterpart's feasible set is inside the nominal feasible set, and has smooth boundaries, making the solution unique.

The nominal LP has many optimal points (red line), which means a solution might be very sensitive to data changes (such as if we change the direction of the objective slightly). The robust LP has a unique solution and is not very sensitive to changes in problem data.

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Part 5: Example: A Production Problem

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A Production Problem

A company produces two kinds of drugs, DrugI and DrugII, containing a specific active agent A, which is extracted from raw materials purchased on the market.

There are two kinds of raw materials, RawI and RawII, which can be used as sources of the active agent. The related production, cost and resource data are given next. The goal is to find the production plan which maximizes the profit of the company.

Production Problem Data

	DrugI	DrugII
Selling price, $ per 1000 packs	5,500	6,100
Content of agent A, g per 1000 packs	0.500	0.600
Manpower required, hours per 1000 packs	90.0	100.0
Equipment required, hours per 1000 packs	40.0	50.0

	Purchasing price ($ per kg)	Content of agent A (g per kg)
RawI	100.00	0.01
RawII	199.90	0.02

Budget ($)	Manpower (hours)	Equipment (hours)	Capacity of raw materials' storage (kg)
100,000	2,000	800	1,000

LP Model: Variables, Objective

- Variables: denote by x_{DrugI}, x_{DrugII} the amounts (in 1000 of packs) of Drug I and II produced, while x_{rmRawI}, $x_{rmRawII}$ denote the amounts (in kg) of raw materials to be purchased.
- Objective function: $f_0(x) = f_{\text{costs}}(x) - f_{\text{income}}(x)$, where

$$f_{\text{costs}}(x) = 100x_{\text{RawI}} + 199.90x_{\text{RawII}} + 700x_{\text{DrugI}} + 800x_{\text{DrugII}}$$

  represents the purchasing and operational costs, and

$$f_{\text{income}}(x) = 5,500x_{\text{DrugI}} + 6100x_{\text{DrugII}}$$

  contain the unit market prices as coefficients.

LP Model: Constraints

- Sign constraints: all variables are non-negative
- Storage: $x_{\mathrm{RawI}} + x_{\mathrm{RawII}} \leq 1000$
- Manpower: $90.0x_{\mathrm{DrugI}} + 100.0x_{\mathrm{DrugII}} \leq 2000$
- Equipment: $40.0x_{\mathrm{DrugI}} + 50.0x_{\mathrm{DrugII}} \leq 800$
- Budget: $100.0x_{\mathrm{RawI}} + 199.90x_{\mathrm{RawII}} + 700x_{\mathrm{DrugI}} + 800x_{\mathrm{DrugII}} \leq 100,000$
- Balance of active agent: says that the amount of raw material must be enough to produce the drugs

$$0.01x_{\mathrm{RawI}} + 0.02x_{\mathrm{RawII}} - 0.50x_{\mathrm{DrugI}} - 0.60x_{\mathrm{DrugII}} \geq 0$$

LP Model: More Constraints and Full Model

Putting this together we get the LP:

$$\min c^T x \ : \ Ax \leq b, \quad x \geq 0,$$

where $x = (x_{\text{RawI}}, x_{\text{RawII}}, x_{\text{DrugI}}, x_{\text{DrugII}})$, and

$$A = \begin{pmatrix} -0.01 & -0.02 & 0.500 & 0.600 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 90.0 & 100.0 \\ 0 & 0 & 40.0 & 50.0 \\ 100.0 & 199.9 & 700 & 800 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1000 \\ 2000 \\ 800 \\ 100000 \end{pmatrix},$$
$$c = \begin{pmatrix} 100 & 199.9 & -4800 & -5300 \end{pmatrix}.$$

<u>Nominal solution:</u> $x_{\text{RawI}} = 0$, $x_{\text{RawII}} = 438.79$, $x_{\text{DrugI}} = 17.55$, $x_{\text{DrugII}} = 0$, and profit $p^* = \$8819.66$.

Robust Counterpart: Uncertainty Model

The balance equation reads

$$0.01x_{\text{RawI}} + 0.02x_{\text{RawII}} - 0.05x_{\text{DrugI}} - 0.600x_{\text{DrugII}} = a^T x \geq 0,$$

where $a_1 = 0.01$, $a_2 = 0.02$ contain the content of agent A (per kg) in each raw material, and a_3, a_4 contain the content of agent A in each of the drugs.

Uncertainty model: amount of active agent in raw material is uncertain:

$$a_1 \in [0.00995, 0.01005], \quad a_2 \in [0.0196, 0.0204],$$

representing a 0.5% and 2% box uncertainty around the nominal values.

Behavior of Nominal and Robust Solutions

- If we disregard uncertainty in raw material's quality, and solve the nominal model, we obtain $x_{\text{RawI}} = 0$, $x_{\text{RawII}} = 438.79$, $x_{\text{DrugI}} = 17.552$, $x_{\text{DrugII}} = 0$, and profit $p^* = \$8819.66$
- If the parameter for RawII takes the worst-case value, the nominal solution is not feasible. Decreasing x_{RawII} to make the constraint feasible again, leads to a 21% reduction in profit
- Solving the robust counterpart instead, we get $x_{\text{RawI}} = 877.73$, $x_{\text{RawII}} = 0$, $x_{\text{DrugI}} = 17.47$, $x_{\text{DrugII}} = 0$, and profit $p^* = \$8294.56$. Profit is reduced by 5.95% only

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Part 6: Example: Robust Portfolio Optimization

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Returns and Portfolios

Consider n assets, denote by $p(i,t)$ the price of one share of asset i at time t. The return of asset i at time t is

$$r(i) := \frac{p(i,t) - p_i(t-1)}{p(i,t-1)}.$$

We may also use the log-return approximation:

$$r(i) \approx \log(1 + r(i)) = \log \frac{p(i,t)}{p(i,t-1)},$$

valid for small returns.

Now assume that we hold a portfolio, and denote by $x_i \geq 0$ the number of shares held in asset i. The portfolio return is

$$\sum_{i=1}^{n} r(i)x(i) = r^T x.$$

A Simple Nominal Model

Assuming the return vector r is given:

$$\max_{x \in \mathcal{X}} \; r^T x,$$

where $\mathcal{X}$ is a polytope of admissible portfolios, for example:

$$\mathcal{X} = \left\{ x \; : \; x \geq 0, \;\; \mathbf{1}^T x = 1 \right\}.$$

- The equality constraint fixes the budget (to a unit)
- The solution is trivial: simply let $x = e_i$ where i is any index such that $r_i = \max_j \; r_j$; it “puts all the eggs in the same basket”
- As such, it is highly sensitive to estimation errors ...

Uncertainty Models and Robust Counterparts

In practice, the return vector has to be estimated. We model estimation errors as $r \in \mathcal{U}$, where $\mathcal{U}$ is the uncertainty set. The following models are typically considered:

- Box uncertainty
- Ellipsoidal uncertainty
- Covariance-based models: Mahalanobis, diagonal covariance (see Module 7)

Robust counterpart:

$$\max_{x \in \mathcal{X}} \min_{r \in \mathcal{U}} r^T x.$$

Mahalanobis Uncertainty

$$\mathcal{U} = \left\{ r \ : \ (r - \hat{r})^T C^{-1} (r - \hat{r}) \leq \rho^2 \right\}$$

where

- $\hat{r}$ is the estimated mean
- C is the estimated covariance matrix

Motivated by the fact that for Gaussian returns, the set $\mathcal{U}$ is a sub-level set of the corresponding likelihood. (More on this is Module 7.)

Robust counterpart: penalizes the variance of the portfolio via the SOCP

$$\max_{x \in \mathcal{X}} \ \hat{r}^T x - \rho \sqrt{x^T C x}.$$

Uncertainty in the Mean Estimate

Model the estimation errors on the mean via the box model

$$\mathcal{U} = \{\hat{r} + \textbf{diag}(\sigma)u \ : \ \|u\|_\infty \le 1\},$$

where $\sigma > 0$ measures asset-wise estimation errors.

Robust counterpart: incorporates an additional penalty via the SOCP

$$\max_{x \in \mathcal{X}} \ \hat{r}^T x - \sigma^T |x| - \rho\sqrt{x^T C x}.$$

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Part 7: Summary

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Summary

- We can extend the robust LP approach to cases with uncertainties affecting all coefficients in the problem
- Without loss of generality we may assume that each constraint is independently affected by uncertainty
- Complexity of robust LP depends on our ability to compute the support function for each constraint efficiently
- For special uncertainty sets, the support function is easy to compute in closed-form, leading to tractable robust counterparts