

## Recitation #14: An overview of sequences and series - Full

### Warm up:

For each of the following sequences, list the first four terms (start each with  $n = 1$ ).

(a)  $a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right), a_1 = 1.$

**Solution:** **n=1:**  $a_1 = 1.$

**n=2:**  $a_2 = \frac{1}{2} \left( a_1 + \frac{2}{a_1} \right) = \frac{1}{2} \left( 1 + \frac{2}{1} \right) = \frac{3}{2}.$

**n=3:**  $a_3 = \frac{1}{2} \left( a_2 + \frac{2}{a_2} \right) = \frac{1}{2} \left( \frac{3}{2} + \frac{2}{\frac{3}{2}} \right) = \frac{17}{12}.$

**n=4:**  $a_4 = \frac{1}{2} \left( a_3 + \frac{2}{a_3} \right) = \frac{1}{2} \left( \frac{17}{12} + \frac{2}{\frac{17}{12}} \right) = \frac{577}{408}.$

(b)  $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)! \cdot 2n!},$  Recall that  $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n.$

**Solution:** **n=1:**  $a_1 = \frac{1}{2!2!} = \frac{1}{4}.$

**n=2:**  $a_2 = \frac{1 \cdot 3}{(2 \cdot 2)! \cdot 2 \cdot 2!} = \frac{3}{96} = \frac{1}{32}.$

**n=3:**  $a_3 = \frac{1 \cdot 3 \cdot 5}{6! \cdot 2 \cdot 3!}.$

**n=4:**  $a_4 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{8! \cdot 2 \cdot 4!}.$

**Instructor Notes:** For this handout, the problems are rather short. I suggest having all groups do each part.

## Group work:

**Problem 1** Give an explicit formula for each of the following sequences:

(a)  $\frac{2}{3}, \frac{-2}{7}, \frac{2}{11}, \frac{-2}{15}, \dots$

**Solution:**  $a_n = \frac{(-1)^{n+1} \cdot 2}{-1 + 4n}$ , starting at  $n = 1$ .

(b)  $-2, 6, -24, 120, -720, \dots$

**Solution:**  $a_n = (-1)^n (n+1)!$ , starting at  $n = 1$ .

(c)  $2, 8, 26, 80, 242, \dots$

**Solution:**  $a_n = 3^n - 1$ , starting at  $n = 1$ .

## Instructor Notes:

**Problem 2** For the sequence  $a_k = (2 - k)^k$

(a) calculate and list  $a_0, a_1, a_2, a_3$ , and  $a_4$ .

**Solution:**  $a_0 = (2 - 0)^0 = 1$ .  
 $a_1 = (2 - 1)^1 = 1$ .  
 $a_2 = (2 - 2)^2 = 0$ .  
 $a_3 = (2 - 3)^3 = -1$ .  
 $a_4 = (2 - 4)^4 = 16$ .

(b) Starting with  $k = 0$ , calculate and list  $S_0 = \sum_{k=0}^0 a_k$ ,  $S_1 = \sum_{k=0}^1 a_k$ ,  $S_2 = \sum_{k=0}^2 a_k$ ,  $S_3 = \sum_{k=0}^3 a_k$ , and  $S_4 = \sum_{k=0}^4 a_k$ . Write  $S_n$  in summation form and write  $S_\infty$  in summation form.

**Solution:**  $S_0 = \sum_{k=0}^0 a_k = a_0 = 1$ .

$S_1 = \sum_{k=0}^1 a_k = a_0 + a_1 = 1 + 1 = 2$ .

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$$S_2 = \sum_{k=0}^2 a_k = a_0 + a_1 + a_2 = 2 + 0 = 2.$$

$$S_3 = \sum_{k=0}^3 a_k = a_0 + a_1 + a_2 + a_3 = 2 + (-1) = 1.$$

$$S_4 = \sum_{k=0}^4 a_k = a_0 + a_1 + a_2 + a_3 + a_4 = 1 + 16 = 17.$$

$$S_n = \sum_{k=0}^n a_k.$$

$$S_\infty = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k.$$

**Instructor Notes:**

**Problem 3** For each of the following, write the series in the form  $\sum_{k=0}^{\infty} a_k$ .

(a)  $0.2 + 0.06 + 0.018 + 0.0054 + \dots$

**Solution:**  $\sum_{k=0}^{\infty} 0.2 \left( \frac{3}{10} \right)^k.$

(b)  $\frac{1}{2} + \frac{1}{3^2} - \frac{1}{3} - \frac{1}{4^2} + \frac{1}{4} + \frac{1}{5^2} - \frac{1}{5} - \frac{1}{6^2} \dots$

**Solution:**  $\sum_{k=1}^{\infty} \left[ (-1)^k \left( \frac{1}{k+2} + \frac{1}{(k+3)^2} \right) \right].$

**Instructor Notes:**

**Problem 4** Reindex the series

$$\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)}$$

in the form  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=-4}^{\infty} c_k$ .

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**Solution:** For the first series, we let  $i = k + 1$ . Then  $k = i - 1$  and, when  $k = 0$ ,  $i = 1$ . So we have that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)} &= \sum_{i=1}^{\infty} \frac{5}{(i-1+2)(i-1+1)} \\ &= \sum_{i=1}^{\infty} \frac{5}{i(i+1)} \\ &= \sum_{k=1}^{\infty} \frac{5}{k(k+1)}. \quad \text{Resubstituting } k=i.\end{aligned}$$

For the second series, we let  $i = k - 4$ . Then  $k = i + 4$  and, when  $k = 0$ ,  $i = -4$ . So we have that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)} &= \sum_{i=-4}^{\infty} \frac{5}{(i+4+2)(i+4+1)} \\ &= \sum_{i=-4}^{\infty} \frac{5}{(i+5)(i+6)} \\ &= \sum_{k=-4}^{\infty} \frac{5}{(k+5)(k+6)}. \quad \text{Resubstituting } k=i.\end{aligned}$$

Instructor Notes:

**Problem 5** If  $\sum_{k=0}^{\infty} a_k = 6$  and  $a_n = \frac{3}{2^n}$ , what is  $\sum_{k=4}^{\infty} a_k$ ?

**Solution:**

$$\begin{aligned}6 &= \sum_{k=0}^{\infty} a_k \\ &= a_0 + a_1 + a_2 + a_3 + \sum_{k=4}^{\infty} a_k \\ &= \frac{3}{1} + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \sum_{k=4}^{\infty} a_k \\ &= \frac{45}{8} + \sum_{k=4}^{\infty} a_k.\end{aligned}$$

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Thus,

$$\sum_{k=4}^{\infty} a_k = 6 - \frac{45}{8} = \frac{3}{8}.$$

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**Instructor Notes:**

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