Recitation #17: The Ratio, Root, Comparison, and Limit Comparison Tests

Group work:

Warm up:

For each of the following, answer **True** or **False**, and explain why.

(a) If $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n^2$ converges.

(b) If $a_n, b_n \geq 0$ and both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, then $\sum_{n=0}^{\infty} a_n b_n$ converges.

Solution: (a) **True**

Since $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. So, in particular, there exists an integer N such that $a_k < 1$ for all $k \geq N$. Then for all $k \geq N$, $a_k^2 < a_k$, and therefore we have that

$$\sum_{n=N}^{\infty} a_n^2 < \sum_{n=N}^{\infty} a_n.$$

Thus, by the Comparison Test, $\sum_{n=0}^{\infty} a_n^2$ is convergent.

(b) **True**

Just as in part (a) there exists an integer N such that $a_k < 1$ for all $k \geq N$. Then

$$\sum_{n=N}^{\infty} a_n b_n < \sum_{n=N}^{\infty} b_n$$

and thus, by the Comparison Test, $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

Group work:

Problem 1 (a) *Why can we not use the Comparison test with* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges?*

Learning outcomes:

(b) *Adjust* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges via the Comparison Test.*

(c) *Give a convergent series we can use in the Limit Comparison Test to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges.*

Solution: (a) *We cannot use the Comparison Test here because*

$$\frac{1}{k^2} < \frac{1}{k^2-5}$$

for all $k \geq 1$*. So we would just be showing the the series in question is greater than a series which converges, which does not give us any information.*

(b) *Notice that*

$$\frac{2}{k^2} > \frac{1}{k^2-5}$$

for all $k \geq 4$*. Since* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges,* $\sum_{k=1}^{\infty} \frac{2}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$*. Thus,* $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *converges.*

Therefore, the Comparison Test with $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *shows that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

(c) *For the Limit Comparison Test, we* **can** *use* $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$\begin{aligned} \lim_{k\to\infty} \frac{\frac{1}{k^2-5}}{\frac{1}{k^2}} &= \lim_{k\to\infty} \frac{k^2}{k^2-5} \\ &= 1. \end{aligned}$$

Thus, since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, by the Limit Comparison Test we know that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

Problem 2 *Determine if the following series converge or diverge.*

(a) $\sum_{n=1}^{\infty} \frac{(7n+1)^2 \cdot 2^n}{5^n}$

(b) $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$

(c) $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$

(d) $\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$

Solution: (a) ***Ratio Test***

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(7(n+1)+1)^2 \cdot 2^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(7n+1)^2 \cdot 2^n} \right]$$
$$= \lim_{n\to\infty} \frac{(7n+8)^2 \cdot 2}{5 \cdot (7n+1)^2}$$
$$= \frac{49 \cdot 2}{49 \cdot 5} = \frac{2}{5}.$$

Thus, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, *this series* $\boxed{\textit{converges}}$.

(b) ***Ratio Test***

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left[\frac{((k+1)!)^3}{(3(k+1))!} \cdot \frac{(3k)!}{(k!)^3} \right]$$
$$= \lim_{k\to\infty} \frac{(k+1)^3 (k!)^3}{(3k+3)(3k+2)(3k+1) \cdot (3k)!} \cdot \frac{(3k)!}{(k!)^3}$$
$$= \lim_{k\to\infty} \frac{(k+1)^3}{(3k+3)(3k+2)(3k+1)}$$
$$= \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{27}.$$

Thus, since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, *this series* $\boxed{\textit{converges}}$.

(c) *Use the* ***Limit Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n+1}{3n^3+1} \cdot \frac{n}{1}$$
$$= \frac{1}{3}.$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, by the Limit Comparison Test we know that* $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$ *diverges.*

(d) *Use the* ***Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Since we always have that $0 < \cos^2(n) < 1$, *we know that*

$$\frac{\cos^2(n)}{n^3+1} \le \frac{1}{n^3+1} < \frac{1}{n^3}.$$

Therefore, by the Comparison Test, we have that $\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$ *converges.*