Recitation #15: Sequences and Infinite Series

Warm up:

Find the limit of the following sequences as n tends to ∞.

(a) $a_n = \dfrac{n^{1000}}{2^n}$

(b) $b_n = \cos(n\pi)$

(c) $c_n = \cos(n!\pi)$

Group work:

Problem 1 *For each of the following sequences, find the limit as the number of terms approaches infinity.*

(a) $a_n = \left(\dfrac{n+1}{2n}\right)\left(\dfrac{n-2}{n}\right)^{\frac{n}{2}}$

(b) $a_n = \sqrt[n]{3^{2n+1}}$

(c) $a_n = \left(\sqrt{n^2+7} - n\right)$

(d) $a_n = \dfrac{(2n+3)!}{5n^3(2n)!}$

(e) $a_n = (2^n + 3^n)^{\frac{1}{n}}$

$$\textit{Hint: } a_n \geq (0+3^n)^{\frac{1}{n}} = 3 \textit{ and } a_n \leq (2\cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3$$

(f) $a_n = \dfrac{n^{365} + 5^n}{8^n + n^3}$

Problem 2 *Show that*

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

exists by proving that $a_n = \sqrt{n+1}-\sqrt{n}$ *is a bounded monotonic sequence. A hint is to show that* $f(x) = \sqrt{x+1}-\sqrt{x}$ *is a decreasing function by showing that* $f'(x) < 0$.

Problem 3 *Find the limit of the given sequence. Also, determine if it is a geometric sequence.*

(a) $a_n = \dfrac{n^2}{2^n}$

(b) $a_n = \dfrac{1}{3^n}$

(c) $a_n = \left(\dfrac{1}{n}\right)^4$

(d) $a_n = \dfrac{e^n + (-3)^n}{5^n}$

(e) $a_n = 3^{\frac{1}{n}}$

Problem 4 *Determine if the following series converge or diverge. If they converge, find the sum.*

(a) $e + 1 + e^{-1} + e^{-2} + e^{-3} + \dots$

(b) $\displaystyle\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k}$

(c) $\displaystyle\sum_{k=0}^{\infty} (\cos(1))^k$

(d) $\displaystyle\sum_{k=4}^{\infty} \frac{5\cdot 4^{k+3}}{7^{k-2}}$

(e) $\displaystyle\sum_{k=0}^{\infty} e^{5-2k}$

(f) $\displaystyle\sum_{k=0}^{\infty} \frac{e^k + (-7)^k}{5^k}$

(g) $\displaystyle\sum_{k=0}^{\infty} \left[\frac{5}{(k+1)(k+2)} + \left(-\frac{1}{2}\right)^k\right]$

(h) $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2}\right)$

Problem 5 *Convert the decimal* $2.456\overline{314}$ *to a fraction using geometric series.*

Problem 6 *Find all values of* x *for which the series*

$$f(x) = \sum_{k=0}^{\infty} \frac{(x+3)^k}{2^k}$$

converges.