Recitation #15: Sequences and Infinite Series - Solutions

Warm up:

Find the limit of the following sequences as n tends to ∞.

(a) $a_n = \dfrac{n^{1000}}{2^n}$

(b) $b_n = \cos(n\pi)$

(c) $c_n = \cos(n!\pi)$

Solution: (a) Note that $\lim\limits_{x\to\infty} \dfrac{x^a}{b^x} = 0$ for any constants a and $b > 1$. So $\lim\limits_{n\to\infty} a_n = 0$.

(b) If n is even, $b_n = \cos(n\pi) = 1$, but if n is odd, then $b_n = \cos(n\pi) = -1$. So $\lim\limits_{n\to\infty} b_n$ does not exist.

(c) If n is at least 2, then $n!$ is even. So $c_n = 1$ if n is at least 2. $\lim\limits_{n\to\infty} c_n = 1$.

Group work:

Problem 1 *For each of the following sequences, find the limit as the number of terms approaches infinity.*

(a) $a_n = \left(\dfrac{n+1}{2n}\right)\left(\dfrac{n-2}{n}\right)^{\frac{n}{2}}$

Solution: *Let* $f(x) = \left(\dfrac{x+1}{2x}\right)\left(\dfrac{x-2}{x}\right)^{\frac{x}{2}}$. *Then*

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} e^{\ln f(x)} \\ &= e^{\lim_{x\to\infty} \ln f(x)}.\end{aligned}$$

Learning outcomes:

So we need to compute the limit in the exponent. To this end

$$\begin{aligned}
\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty}\left[\ln\left(\frac{x+1}{2x}\right) + \ln\left(\frac{x-2}{x}\right)^{\frac{x}{2}}\right] \\
&= \lim_{x\to\infty} \ln\left(\frac{x+1}{2x}\right) + \lim_{x\to\infty}\left[\frac{x}{2}\ln\left(\frac{x-2}{x}\right)\right] \quad \textit{provided both limits exist} \\
&= \ln\left(\frac{1}{2}\right) + \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{2}{x}} \quad \textit{indeterminant of the form } \frac{0}{0} \\
&= \ln\left(\frac{1}{2}\right) + \lim_{x\to\infty}\frac{\frac{2x^{-2}}{1-\frac{2}{x}}}{-2x^{-2}} \quad \textit{L'Hospital's Rule} \\
&= \ln\left(\frac{1}{2}\right) + \lim_{x\to\infty}\frac{-1}{1-\frac{2}{x}} \\
&= \ln\left(\frac{1}{2}\right) - 1.
\end{aligned}$$

So

$$\lim_{x\to\infty} f(x) = e^{\ln\left(\frac{1}{2}\right)-1} = \frac{1}{2}e^{-1}$$

and therefore

$$\lim_{n\to\infty}\left(\frac{n+1}{2n}\right)\left(\frac{n-2}{n}\right)^{\frac{n}{2}} = \frac{1}{2}e^{-1}.$$

(b) $a_n = \sqrt[n]{3^{2n+1}}$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{3^{2n+1}} &= \lim_{n\to\infty}\left(3^{2n+1}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} 3^{2+\frac{1}{n}} \\
&= \lim_{n\to\infty} 3^2 \cdot 3^{\frac{1}{n}} \\
&= 9\lim_{n\to\infty} e^{\frac{1}{n}} \\
&= 9\cdot 3^0 \\
&= 9\cdot 1 = 9.
\end{aligned}$$

(c) $a_n = \left(\sqrt{n^2+7} - n\right)$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+7}-n\right) &= \lim_{n\to\infty}\left[\left(\sqrt{n^2+7}-n\right)\cdot\frac{\sqrt{n^2+7}+n}{\sqrt{n^2+7}+n}\right]\\
&= \lim_{n\to\infty}\frac{n^2+7-n^2}{n\sqrt{1+\frac{7}{n^2}}+n}\\
&= \lim_{n\to\infty}\frac{7}{n(\sqrt{1+\frac{7}{n^2}}+1)}\\
&= 0.
\end{aligned}$$

(d) $a_n = \dfrac{(2n+3)!}{5n^3(2n)!}$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\frac{(2n+3)!}{5n^3(2n)!} &= \lim_{n\to\infty}\frac{(2n+3)(2n+2)(2n+1)(2n)!}{5n^3(2n)!}\\
&= \lim_{n\to\infty}\frac{(2n+3)(2n+2)(2n+1)}{5n^3}\\
&= \frac{8}{5} \quad \textit{Compare the coefficients of the leading } n^3 \textit{terms}
\end{aligned}$$

(e) $a_n = (2^n+3^n)^{\frac{1}{n}}$

Hint: $a_n \geq (0+3^n)^{\frac{1}{n}} = 3$ *and* $a_n \leq (2\cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}}\cdot 3$

Solution: *From the hint*

$$3 = (0+3^n)^{\frac{1}{n}} \leq a_n \leq (2\cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}}\cdot 3.$$

So by the squeeze theorem, we have that

$$\begin{aligned}
&\lim_{n\to\infty} 3 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} 2^{\frac{1}{n}}\cdot 3\\
\Longrightarrow \qquad & 3 \leq \lim_{n\to\infty} a_n \leq 1\cdot 3 = 3.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} a_n = 3.$$

(f) $a_n = \dfrac{n^{365}+5^n}{8^n+n^3}$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\frac{n^{365}+5^n}{8^n+n^3} &= \lim_{n\to\infty}\frac{n^{365}+5^n}{8^n+n^3}\cdot\frac{\frac{1}{8^n}}{\frac{1}{8^n}}\\ &= \lim_{n\to\infty}\frac{\frac{n^{365}}{8^n}+\left(\frac{5}{8}\right)^n}{1+\frac{n^3}{8^n}}\\ &= \frac{0+0}{1+0}=0. \quad \text{due to growth rates, } \lim_{n\to\infty}\frac{n^k}{a^n}=0.\end{aligned}$$

Problem 2 *Show that*

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

exists by proving that $a_n=\sqrt{n+1}-\sqrt{n}$ *is a bounded monotonic sequence. A hint is to show that* $f(x)=\sqrt{x+1}-\sqrt{x}$ *is a decreasing function by showing that* $f'(x)<0$.

Solution: *Let* $f(x)=\sqrt{x+1}-\sqrt{x}$. *Then*

$$\begin{aligned}f'(x) &= \frac{1}{2\sqrt{x+1}}-\frac{1}{2\sqrt{x}}\\ &= \frac{\sqrt{x}-\sqrt{x+1}}{2\sqrt{x}\sqrt{x+1}}\\ &< 0\end{aligned}$$

since the denominator is clearly positive, and $\sqrt{x}<\sqrt{x+1}$. *Therefore* f *is decreasing, and so the original sequence is decreasing. Also notice that since*

$$\sqrt{x}<\sqrt{x+1}$$

we have that

$$0<\sqrt{x+1}-\sqrt{x}=f(x).$$

Thus the original sequence is bounded below by 0.

Therefore, since the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}$ *is bounded and monotone decreasing, the limit*

$$\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}$$

exists.

Problem 3 *Find the limit of the given sequence. Also, determine if it is a geometric sequence.*

(a) $a_n = \dfrac{n^2}{2^n}$

(b) $a_n = \dfrac{1}{3^n}$

(c) $a_n = \left(\dfrac{1}{n}\right)^4$

(d) $a_n = \dfrac{e^n + (-3)^n}{5^n}$

(e) $a_n = 3^{\frac{1}{n}}$

Solution: (a) $\lim_{n\to\infty} \dfrac{n^2}{2^n} = 0$ *growth rate*

(b) $\lim_{n\to\infty} \dfrac{1}{3^n} = \lim_{n\to\infty} \left(\dfrac{1}{3}\right)^n = 0$. *This is a geometric sequence with* $a = 1$ *and* $r = \dfrac{1}{3}$.

(c) $\lim_{n\to\infty} \left(\dfrac{1}{n}\right)^4 = 0$.

(d) $\lim_{n\to\infty} \dfrac{e^n + (-3)^n}{5^n} = \lim_{n\to\infty} \left[\left(\dfrac{e}{5}\right)^n + \left(\dfrac{-3}{5}\right)^n\right] = 0$.
This is the sum of two geometric sequences. For both, the initial term is $a = 1$. *For the first sequence the ratio is* $r_1 = \dfrac{e}{5}$, *and for the second the ratio is* $r_2 = \dfrac{-3}{5}$.

(e) $\lim_{n\to\infty} 3^{\frac{1}{n}} = 3^0 = 1$.

Problem 4 *Determine if the following series converge or diverge. If they converge, find the sum.*

(a) $e + 1 + e^{-1} + e^{-2} + e^{-3} + \dots$

Solution:

$$\begin{aligned}
e + 1 + e^{-1} + e^{-2} + e^{-3} + \dots &= e + \sum_{k=0}^{\infty} e^{-k} \\
&= e + \sum_{k=0}^{\infty} \left(e^{-1}\right)^k \qquad \text{geometric series, } r = e^{-1} < 1 \\
&= e + \frac{1}{1 - e^{-1}}.
\end{aligned}$$

Therefore, this series converges to $\left(e + \dfrac{1}{1 - e^{-1}}\right)$.

(b) $\displaystyle\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k}$

Solution: *Let us analyze the two different summands in this problem:*

(i) $\displaystyle\sum_{k=0}^{99} 2^k$

This is a finite sum from a geometric sequence, and so its sum is

$$\frac{a(1-r^n)}{1-r}.$$

Thus,

$$\sum_{k=0}^{99} 2^k = \frac{1(1-2^{100})}{1-2} = 2^{100}-1.$$

(ii) $\displaystyle\sum_{k=100}^{\infty} \frac{1}{2^k} = \sum_{k=100}^{\infty} \left(\frac{1}{2}\right)^k.$

This is a geometric series with $a = \dfrac{1}{2^{100}}$ *and* $r = \dfrac{1}{2}$. *So*

$$\sum_{k=100}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{100}}}{1-\frac{1}{2}} = \frac{1}{2^{99}}.$$

Therefore, combining parts (i) and (ii) we have that

$$\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k} = 2^{100} - 1 + \frac{1}{2^{99}}.$$

(c) $\displaystyle\sum_{k=0}^{\infty} (\cos(1))^k$

Solution: *This is a geometric series with* $a = 1$ *and* $r = \cos(1)$. *We know that* $-1 < \cos(1) < 1$, *and so* $|\cos(1)| < 1$. *Therefore, this geometric series converges and*

$$\sum_{k=0}^{\infty} (\cos(1))^k = \frac{1}{1-\cos(1)}.$$

(d) $\displaystyle\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}}$

Solution: *Let us first reindex this series. Let* $\ell = k - 4$. *Then* $k = \ell + 4$, *and when $k = 4$, $\ell = 0$. We then have that*

$$\begin{aligned}
\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}} &= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{\ell+4+3}}{7^{\ell+4-2}} \\
&= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{\ell+7}}{7^{\ell+2}} \\
&= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{7} \cdot 4^{\ell}}{7^{2} \cdot 7^{\ell}} \\
&= \frac{5 \cdot 4^7}{7^2} \sum_{\ell=0}^{\infty} \left(\frac{4}{7}\right)^{\ell} \quad \textit{assuming this series converges} \\
&= \frac{5 \cdot 4^7}{7^2} \cdot \frac{1}{1 - \frac{4}{7}} \quad \textit{geometric series with } a = 1, r = \frac{4}{7} \\
&= \frac{5 \cdot 4^7}{3 \cdot 7}.
\end{aligned}$$

Therefore, this series converges to $\dfrac{5 \cdot 4^7}{3 \cdot 7}$.

(e) $\displaystyle\sum_{k=0}^{\infty} e^{5-2k}$

Solution:

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{5-2k} &= \sum_{k=0}^{\infty} \left[e^5 \cdot \left(e^{-2}\right)^k\right] \\
&= e^5 \sum_{k=0}^{\infty} \left(e^{-2}\right)^k \quad \textit{assuming the series converges} \\
&= e^5 \cdot \frac{1}{1 - e^{-2}} \quad \textit{geometric series with } a = 1, r = e^{-2} < 1
\end{aligned}$$

Therefore, this series converges to $\dfrac{e^5}{1 - e^{-2}}$.

(f) $\displaystyle\sum_{k=0}^{\infty} \frac{e^k + (-7)^k}{5^k}$

Solution:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{e^k + (-7)^k}{5^k} &= \sum_{k=0}^{\infty} \left[\frac{e^k}{5^k} + \frac{(-7)^k}{5^k}\right] \\
&= \sum_{k=0}^{\infty} \left[\left(\frac{e}{5}\right)^k + \left(\frac{-7}{5}\right)^k\right].
\end{aligned}$$

If both of these series were convergent, then we would be able to split up the sum:

$$\sum_{k=0}^{\infty} \frac{e^k + (-7)^k}{5^k} \text{ “} = \text{” } \sum_{k=0}^{\infty} \left(\frac{e}{5}\right)^k + \sum_{k=0}^{\infty} \left(\frac{-7}{5}\right)^k.$$

The first series on the right hand side is a geometric series with $r = \frac{e}{5}$. *Since* $\left|\frac{e}{5}\right| < 1$, *this series converges. But the second series is a geometric series with* $r = \frac{-7}{5}$. *Since* $\left|\frac{-7}{5}\right| > 1$, *this series diverges.*

Therefore, the original series diverges.

(g) $\displaystyle\sum_{k=0}^{\infty} \left[\frac{5}{(k+1)(k+2)} + \left(-\frac{1}{2}\right)^k\right]$

Solution: *If both series converge, then we can break up the sum:*

$$\sum_{k=0}^{\infty} \left[\frac{5}{(k+1)(k+2)} + \left(-\frac{1}{2}\right)^k\right] = \sum_{k=0}^{\infty} \frac{5}{(k+1)(k+2)} + \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k.$$

Let us consider both series on the right hand side of this equation individually.

(i) $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$

This is a geometric series with $a = 1$ *and* $r = \frac{-1}{2}$. *Therefore, this series converges with*

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1 - \left(\frac{-1}{2}\right)} = \frac{2}{3}.$$

(ii) $\displaystyle\sum_{k=0}^{\infty} \frac{5}{(k+1)(k+2)}$

It may not be obvious yet, but this is a telescoping series. To see this, let us decompose $\frac{5}{(k+1)(k+2)}$ *as a partial fraction.*

$$\begin{aligned} \frac{5}{(k+1)(k+2)} &= \frac{A}{k+1} + \frac{B}{k+2} \\ \implies \quad 5 &= A(k+2) + B(k+1). \end{aligned}$$

We solve for A *and* B *by choosing “smart” values for* k*:*

$$\begin{aligned} &(k = -1) \quad \implies \quad A = 5 \\ &(k = -2) \quad \implies \quad -B = 5 \quad \implies \quad B = -5. \end{aligned}$$

So we see that

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+2}\right)=\sum_{k=0}^{\infty}\left[\frac{5}{k+1}-\frac{5}{k+2}\right].$$

Let

$$S_n=\sum_{k=0}^{n}\left[\frac{5}{k+1}-\frac{5}{k+2}\right].$$

Then we have that

$$\begin{aligned}
S_n&=\sum_{k=0}^{n}\left[\frac{5}{k+1}-\frac{5}{k+2}\right]\\
&=\left(\frac{5}{1}-\frac{5}{2}\right)+\left(\frac{5}{2}-\frac{5}{3}\right)+\left(\frac{5}{3}-\frac{5}{4}\right)+\ldots+\left(\frac{5}{n+1}-\frac{5}{n+2}\right)\\
&=\frac{5}{1}-\frac{5}{n+2}=5-\frac{5}{n+2}.
\end{aligned}$$

We then compute the sum by taking the limit of the sequence of partial sums:

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{5}{(k+1)(k+2)}&=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{5}{(k+1)(k+2)}\\
&=\lim_{n\to\infty}S_n\\
&=\lim_{n\to\infty}\left(5-\frac{5}{n+2}\right)\\
&=5.
\end{aligned}$$

Finally, we compute the sum of the original series as

$$\begin{aligned}
\sum_{k=0}^{\infty}\left[\frac{5}{(k+1)(k+2)}+\left(-\frac{1}{2}\right)^k\right]&=\sum_{k=0}^{\infty}\frac{5}{(k+1)(k+2)}+\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k\\
&=5+\frac{2}{3}=\frac{17}{3}.
\end{aligned}$$

(h) $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+2}\right)$

Solution: *This is a telescoping series. Let*

$$S_n=\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+2}\right).$$

Then,

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+2} \right) \\
&= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) \\
&+ \ldots + \left(\frac{1}{n-2} - \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) \\
&= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.
\end{aligned}$$

Note that the last equality above is because all of the non-red terms cancel (convince yourself of this). Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2} \right) &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} \left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right] \\
&= 1 + \frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

Problem 5 *Convert the decimal $2.456\overline{314}$ to a fraction using geometric series.*

Solution:

$$\begin{aligned}
2.456\overline{314} &= 2.456 + 0.000314 + 0.000000314 + \ldots \\
&= 2.456 + \frac{314}{1000^2} + \frac{314}{1000^3} + \ldots \\
&= 2.456 + \sum_{k=1}^{\infty} \left[\frac{314}{1000} \cdot \left(\frac{1}{1000} \right)^k \right] \\
&= \frac{2456}{1000} + \frac{\frac{314}{1000^2}}{1 - \frac{1}{1000}} \\
&= \frac{2456}{1000} + \frac{\frac{314}{1000^2}}{\frac{999}{1000}} \\
&= \frac{2456}{1000} + \frac{314}{999000} \\
&= \frac{2453544 + 314}{999000} \\
&= \frac{2453858}{999000} = \frac{1226929}{499500}.
\end{aligned}$$

Problem 6 *Find all values of x for which the series*

$$f(x) = \sum_{k=0}^{\infty} \frac{(x+3)^k}{2^k}$$

converges.

Solution: *First notice that*

$$f(x) = \sum_{k=0}^{\infty} \frac{(x+3)^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{x+3}{2}\right)^k$$

and so this is a geometric series with $a = 1$ and $r = \dfrac{x+3}{2}$. So this series converges when

$$\begin{aligned} & \left|\frac{x+3}{2}\right| < 1 \\ \iff \quad & -1 < \frac{x+3}{2} < 1 \\ \iff \quad & -2 < x+3 < 2 \\ \iff \quad & -5 < x < -1. \end{aligned}$$