

Recitation #19: Approximating functions with polynomials - Instructor Notes

Warm up:

For each of the following, write the given polynomial in summation notation starting with $k = 0$.

- (a) $\frac{3x}{2} - \frac{5x^2}{3} + \frac{7x^3}{4} - \frac{9x^4}{5} + \frac{11x^5}{6}$
- (b) $\frac{1}{2}x + \frac{1 \cdot 5}{4 \cdot 2!}x^3 + \frac{1 \cdot 5 \cdot 9}{8 \cdot 3!}x^5 - \frac{1 \cdot 5 \cdot 9 \cdot 13}{16 \cdot 4!}x^7$
- (c) $(x-1)^3 - \frac{(x-1)^4}{2!} + \frac{(x-1)^5}{4!} - \frac{(x-1)^6}{6!}$

Instructor Notes: Maybe give one problem per group. Allow 4 minutes for group work and 6 minutes for discussion. Make sure to note that we can “factor out” terms dealing only with x , but not with k (or n). Also, you might want to discuss starting with k equaling another number other than 0.

Group work:

Problem 1 Assuming that the function $f(x)$ is infinitely differentiable, and given that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + c_4(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

show that the coefficient c_4 of the $(x-a)^4$ term in the Taylor polynomial is $\frac{f^{(4)}(a)}{4!}$.

Instructor Notes: The lecture justifies the coefficients of $p_3(x)$. This problem should be done as a whole class discussion, perhaps for about 5 minutes.

Problem 2 Let $f(x) = \sin(2x)$. Find $p_3(x)$ about the point $a = \frac{\pi}{8}$.

Instructor Notes: Students often have difficulty “putting the pieces together”, particularly when $a \neq 0$. Watch for students not putting the polynomial as powers of $\left(x - \frac{\pi}{8}\right)$. They often will just use powers of x . They also will often not plug in $\frac{\pi}{8}$ into the derivative (leaving the derivative in terms of x). Another common error is forgetting the $k!$.

Problem 3 Let $f(x) = xe^{-x}$ on the interval $[-2, 8]$.

- (a) Write the Taylor polynomial $p_4(x)$ around $a = 3$.

$$\text{Fun facts: } f'(x) = -e^{-x}(x - 1)$$

$$f''(x) = e^{-x}(x - 2)$$

$$f^{(3)}(x) = -e^{-x}(x - 3)$$

$$f^{(4)}(x) = e^{-x}(x - 4)$$

- (b) Write $p_4(x)$ about $a = 3$ in summation notation. Also, write the remainder term $R_4(x)$.
- (c) Calculate $p_4(4.5)$ and, using $R_4(4.5)$, estimate how close $p_4(4.5)$ is to $f(4.5)$. Do the same for $p_4(1.5)$.
- (d) Use the remainder term $R_4(x)$ to estimate the maximum error for $p_4(x)$ on $[-2, 8]$.
- (e) How large must n be to assure that the n^{th} degree Taylor polynomial for $f(x) = xe^{-x}$ about $a = 3$ approximates $2e^{-2}$ within 10^{-5} ?

Instructor Notes: This is the longest of the problems. Part (a) is like the preceding problem, where they need to remember to plug 3 into the derivatives. For part (b), they also need to recognize that $R_4(x)$ deals with $f^{(5)}(x)$ and not $f^{(4)}(x)$. Writing $R_4(x) = \frac{f^{(5)}(c)}{5!}(x - 3)^5$ for some $c \in [-2, 8]$ will be sufficient at this stage (they will bound $f^{(5)}(c)$ in parts (c) and (d)).

Recitation #19: Approximating functions with polynomials - Instructor Notes

In part (c), the main issue is to bound $|f^{(5)}(x)| = |-e^{-5}(x-5)|$ on $[3, 4.5]$ and $[1.5, 3]$, and on $[2, 8]$ for part (d). This is a problem of finding the absolute max (and min) of a function on a closed interval. Students then need to find critical value(s) by finding when $f^{(6)}(x) = 0$ (or undefined) as well as consider endpoints $f^{(6)}(x) = e^{-x}(x-6)$, with critical $x = 6$. The last hurdle is to recognize we want to find the max of $|f^{(5)}(x)|$. This maximum occurs at $x = 3, x = 1.5$, and $x = -2$ respectively.
