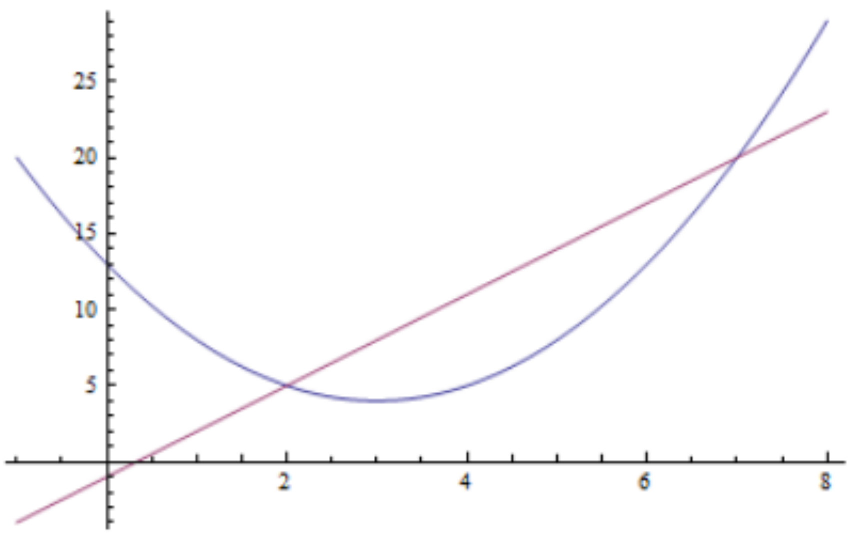
Recitation # 4: Volume by Shells & Length of Curves - Solutions

Group work:

Problem 1 *Set up an integral that will compute the volume of the solid generated by revolving the region bounded by the curves* $y = x^2 - 6x + 13$ *(i.e.* $x = 3 \pm \sqrt{y-4}$*) and* $y = 3x - 1$ *about:*



Use both the washer method as well as the shell method for each problem. Which method would you prefer for each problem? Why?

(a) *the* x*-axis*

Learning outcomes:

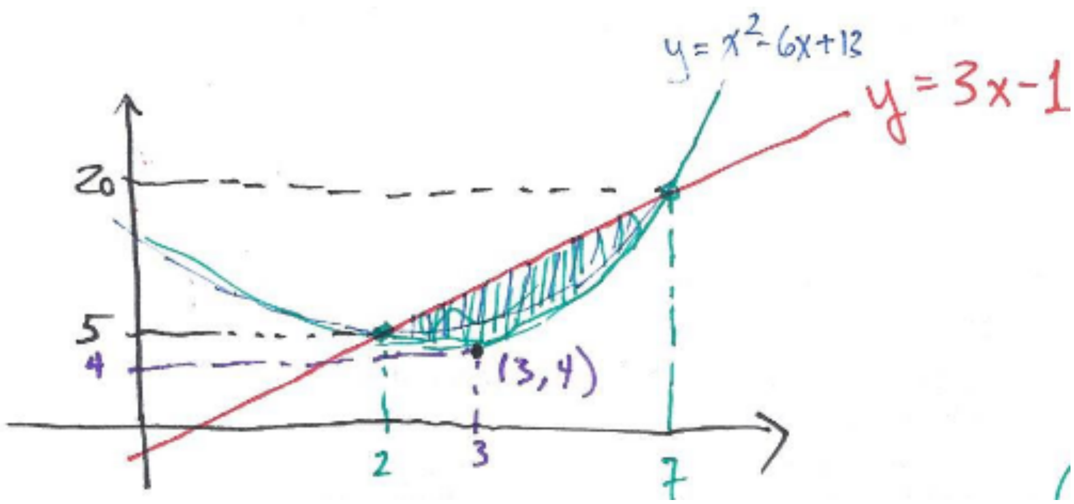
Solution: *First, we need to find the points where the curves intersect*

$$\begin{aligned} x^2 - 6x + 13 &= 3x - 1 \\ x^2 - 9x + 14 &= 0 \\ (x-2)(x-7) &= 0 \\ x &= 2, 7 \\ (2, 5), (7, 20). \end{aligned}$$

As we will see later, we also need to locate the vertex of the parabola $x^2 - 6x + 13$*. So we complete the square*

$$\begin{aligned} y &= x^2 - 6x + 13 \\ &= (x^2 - 6x + 9) + 13 - 9 \\ &= (x-3)^2 + 4. \end{aligned}$$

So the vertex of the parabola is $(3, 4)$*.*

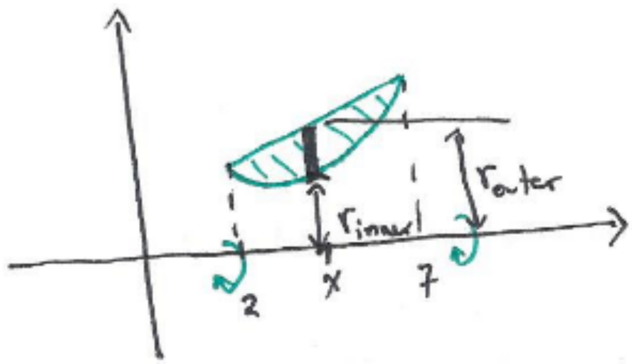


Washers: *For washers, the cross-sections must be* **perpendicular** *to the axis of rotation. So here we integrate along the x-axis. We have that*

$$\begin{aligned} r_{out} &= 3x - 1 \\ r_{in} &= x^2 - 6x + 13 \end{aligned}$$

and

$$\text{Volume of the region} = \pi \int_2^7 \left[(3x-1)^2 - (x^2 - 6x + 13)^2 \right] \, dx.$$



Shells: *For shells, the cross-sections must be* ***parallel*** *to the axis of rotation. So here we integrate along the y-axis, $5 \le y \le 20$. But we have a problem, namely the "bottom" of each cross-section changes at $y = 5$ due to the shape of the region (see the picture below). So we have two different cases:*

(1) *For $4 \le y \le 5$*

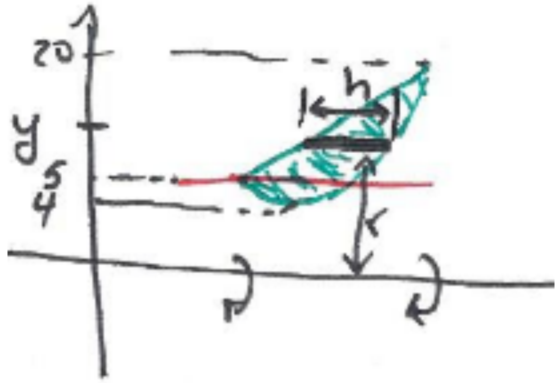
$$h = (3 + \sqrt{y-1}) - (3 - \sqrt{y-1}) = 2\sqrt{y-1}$$
$$r = y$$

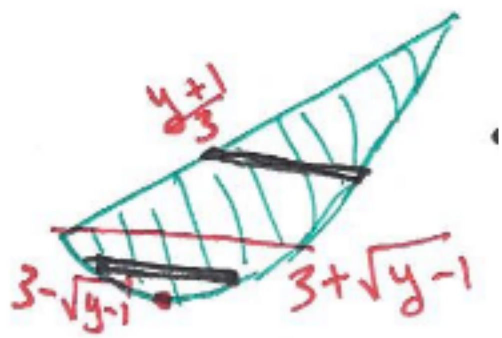
(2) *For $5 \le y \le 20$*

$$h = (3 + \sqrt{y-1}) - \left(\frac{1}{3}(y+1)\right)$$
$$r = y.$$

Thus

$$V = \int_4^{20} 2\pi r h \, dy = 2\pi \left[\int_4^5 y \cdot 2\sqrt{y-1}\, dy + \int_5^{20} y \left((3 + \sqrt{y+1}) - \frac{1}{3}(y+1)\right) dy\right]$$





It is pretty clear in this problem that the washer's method was easier than the shell's method.

(b) $y = -4$

Solution: **Washers:** *For washers, the cross-sections must be* ***perpendicular*** *to the axis of rotation. So here we integrate along the x-axis. We have that*

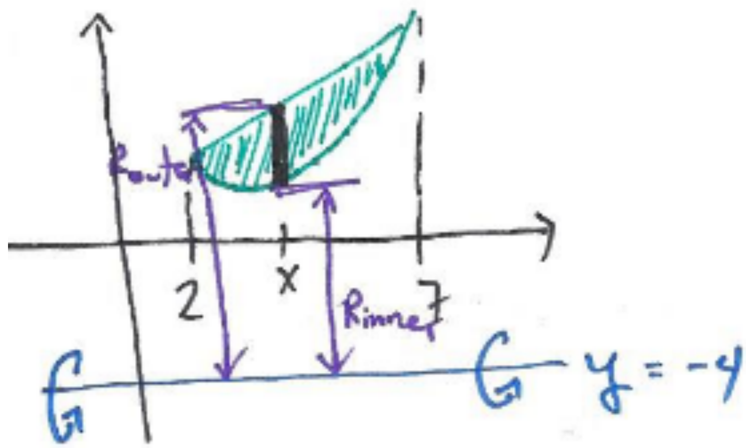
$$r_{out} = 4 + (3x - 1) = 3x + 3$$
$$r_{in} = 4 + (x^2 - 6x + 13) = x^2 - 6x + 17$$

and

$$V = \pi \int_2^7 \left[(3x+3)^2 - (x^2 - 6x + 17)^2\right]\, dx.$$

Shells: *For shells, the cross-sections must be* ***parallel*** *to the axis of rotation. So here we integrate along the y-axis. Just as before though, the equation determining the bottom of the shell changes at $y = 5$. The* *height of the shell over the two regions is the same as part (a), but now the radius of the shell is $4 + y$. So,*

$$V = 2\pi \left[\int_4^5 (4+y) \cdot 2\sqrt{y-1}\, dy + \int_5^{20} (4+y) \left((3 + \sqrt{y+1}) - \frac{1}{3}(y+1) \right) dy \right]$$



Again, it is pretty clear that the washer's method was easier than the shell's method for this problem.

(c) $y = 22$

Solution: **Washers:** *For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the x-axis. We have that*

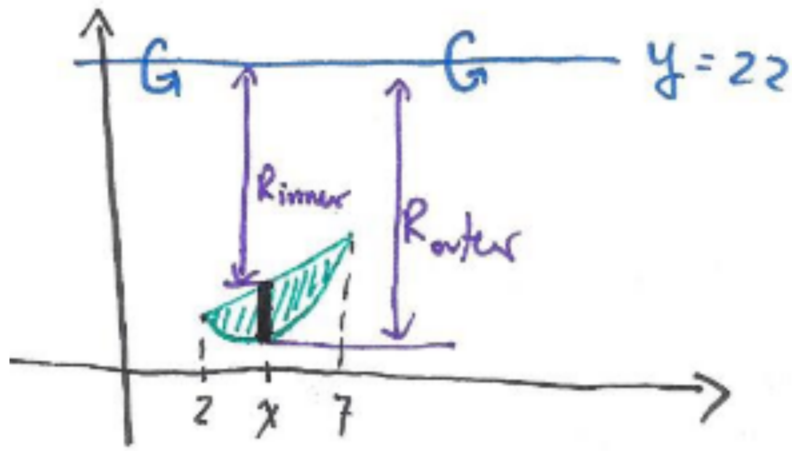
$$r_{out} = 22 - (x^2 - 6x + 13) = -x^2 + 6x + 9$$
$$r_{in} = 22 - (3x - 1) = -3x + 23$$

and so

$$V = \pi \int_2^7 \left[(-x^2 + 6x + 9)^2 - (-3x + 23)^2 \right] dx.$$

Shells: *For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the y-axis. Just as before though, the equation determining the bottom of the shell changes at $y = 5$. The height of the shell over the two regions is the same as part (a), but now the radius of each shell is $22 - y$. So,*

$$V = 2\pi \left[\int_4^5 (22-y) \cdot 2\sqrt{y-1}\, dy + \int_5^{20} (22-y) \left((3 + \sqrt{y+1}) - \frac{1}{3}(y+1) \right) dy \right]$$



Once again, the washer's method appears to be easier.

(d) *the y-axis*

Solution: **Washers:** *For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the y-axis (notice the change from the three preceding problems). Just as in the shells method for the previous three parts, we have to break up our region of integration at $y = 5$. We have the following two cases*

(1) for $4 \leq y \leq 5$

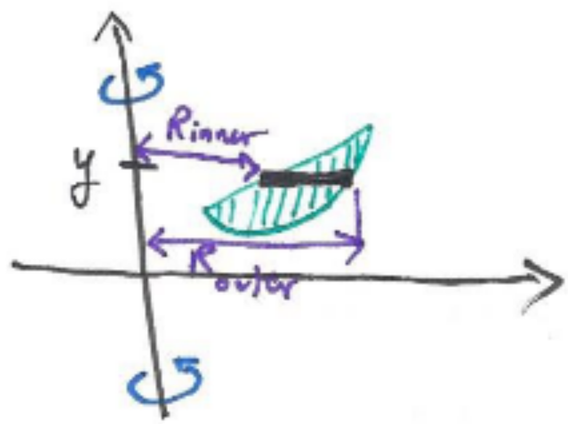
$$r_{out} = 3 + \sqrt{y-1}$$
$$r_{in} = 3 - \sqrt{y-1}$$

(2) for $5 \leq y \leq 20$

$$r_{out} = 3 + \sqrt{y-1}$$
$$r_{in} = \frac{1}{3}(y+1)$$

So,

$$V = \pi\left[\int_4^5 \left(\left(3+\sqrt{y-1}\right)^2 - \left(3-\sqrt{y-1}\right)^2\right)\,dy + \int_5^{20}\left(\left(3+\sqrt{y-1}\right)^2 - \frac{1}{9}(y+1)^2\right)\,dy\right].$$

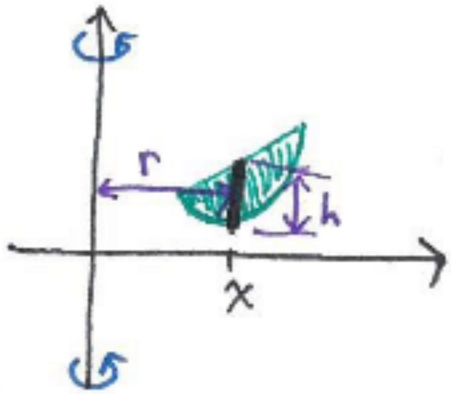


Shells: *For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the x-axis (again, notice the change from the three preceding problems). This time, we do not need to break up the region! Our shells will always have the parameters*

$$h = (3x - 1) - (x^2 - 6x + 13) = -x^2 + 9x - 14$$
$$r = x$$

So,

$$V = 2\pi \int_2^7 x(-x^2 + 9x - 14)\,dx.$$

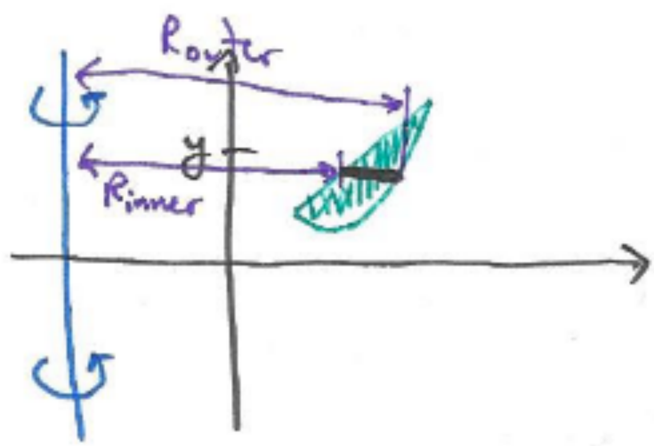


Now, notice that the shells method was a lot easier. The key for this region is that you want your cross sections to always be perpendicular to the x-axis. That way, the bounds for your parameters do not change over the entire region of integration.

(e) $x = -3$

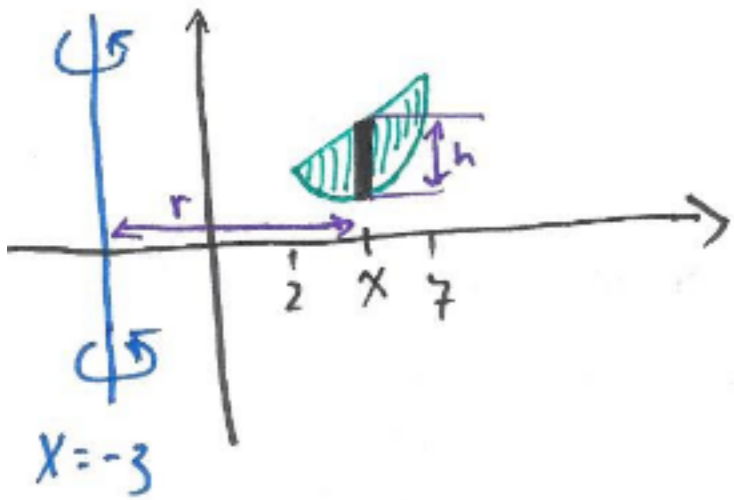
Solution: **Washers:** *For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the y-axis. Just as in part (d), we need to split the region up into two parts. But shifting the axis of rotation to $x = -3$ just adds three to all radii. So*

$$V = \pi \left[\int_4^5 \left(\left(6 + \sqrt{y-1}\right)^2 - \left(6 - \sqrt{y-1}\right)^2 \right) dy + \int_5^{20} \left(\left(6 + \sqrt{y-1}\right)^2 - \left(3 + \frac{1}{3}(y+1)\right)^2 \right) dy \right].$$



Shells: *For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the x-axis. Shifting the axis of rotation to $x = -3$ simply adds three to the radius of the shell, so*

$$V = 2\pi \int_2^7 (3 + x)(-x^2 + 9x - 14)\, dx.$$



The shells method was simpler.

(f) $x = 9$

Solution: **Washers:** *For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the y-axis. Just as in parts (d) and (e), we need to split the region up into two parts. Via the figure below, we compute*

(1) for $4 \leq y \leq 5$

$$r_{out} = 9 - \left(3 - \sqrt{y-1}\right) = 6 + \sqrt{y-1}$$
$$r_{in} = 9 - \left(3 + \sqrt{y-1}\right) = 6 - \sqrt{y-1}$$

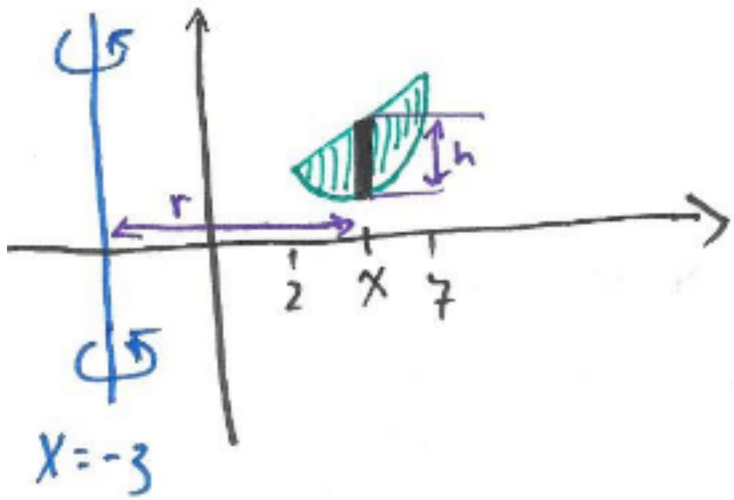
(2) for $5 \leq y \leq 20$

$$r_{out} = 9 - \left(\frac{1}{3}(y+1)\right)$$
$$r_{in} = 9 - \left(3 + \sqrt{y-1}\right) = 6 - \sqrt{y-1}$$

So,

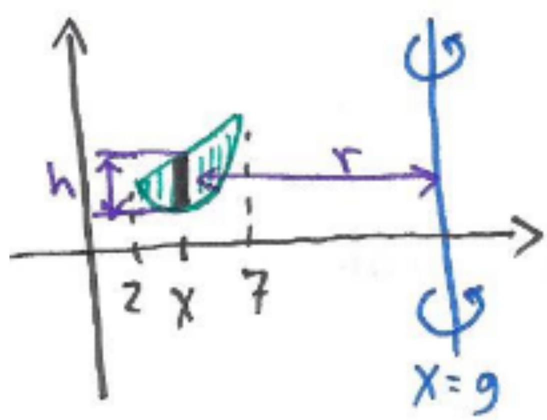
$$V = \pi \left[\int_4^5 \left(\left(6 + \sqrt{y-1}\right)^2 - \left(6 - \sqrt{y-1}\right)^2 \right) dy \right.$$

$$+\int_5^{20}\left(\left(9-\frac{1}{3}(y+1)\right)^2-\left(6-\sqrt{y-1}\right)^2\right)dy\Bigg].$$



Shells: *For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the x-axis. Shifting the axis of rotation to the right 9 units (from part (d)) does not change the height of the cylinder, and the radius changes to $9-x$. So*

$$V=2\pi\int_2^7(9-x)(-x^2+9x-14)\,dx.$$



Again, the shell method was much simpler.

Problem 2 **Set up** *an integral (or a sum of integrals) to find the perimeter of the region bounded by the curves* $y = 2x^2 - 5x + 13$ *and* $y = x^2 + 6x - 11$.

Solution: *Let* $f(x) = 2x^2 - 5x + 13$ *and* $g(x) = x^2 + 6x - 11$. *We first need to find the points where these two curves intersect. So we solve*

$$\begin{aligned} f(x) &= g(x) \\ 2x^2 - 5x + 13 &= x^2 + 6x - 11 \\ x^2 - 11x + 24 &= 0 \\ (x-3)(x-8) &= 0 \\ x &= 3, 8. \end{aligned}$$

Then the perimeter is $L_1 + L_2$ *where*

$$\begin{aligned} L_1 &= \int_3^8 \sqrt{1 + f'(x)^2}\, dx = \int_3^8 \sqrt{1 + (4x-5)^2}\, dx \\ L_2 &= \int_3^8 \sqrt{1 + g'(x)^2}\, dx = \int_3^8 \sqrt{1 + (2x+6)^2}\, dx. \end{aligned}$$

Problem 3 *Find the length of the following curves (length is in feet):*

(a) $y = \dfrac{1}{6}x^3 + \dfrac{1}{2x}$ *from* $\left(2, \dfrac{19}{12}\right)$ *to* $\left(3, \dfrac{14}{3}\right)$.

Solution:

$$\begin{aligned}
\text{Arc Length} &= \int_2^3 \sqrt{1 + y'(x)^2}\, dx \\
&= \int_2^3 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2}\, dx \\
&= \int_2^3 \sqrt{1 + \left(\frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}\right)}\, dx \\
&= \int_2^3 \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4}x^{-4}}\, dx \\
&= \int_2^3 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2}\, dx \\
&= \int_2^3 \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right) dx \\
&= \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_2^3 \\
&= \left(\frac{27}{6} - \frac{1}{6}\right) - \left(\frac{8}{6} - \frac{1}{4}\right) \\
&= 3 + \frac{1}{4} = \frac{13}{4}.
\end{aligned}$$

(b) $x = \frac{1}{9}e^{3y} + \frac{1}{4}e^{-3y}$ *from* $\left(\frac{13}{36}, 0\right)$ *to* $\left(\frac{265}{288}, \ln 2\right)$.

Solution:

$$
\begin{aligned}
\text{Arc Length} &= \int_0^{\ln 2} \sqrt{1 + x'(y)^2}\, dy \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{3}e^{3y} - \frac{3}{4}e^{-3y}\right)^2}\, dy \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{9}e^{6y} - \frac{1}{2} + \frac{9}{16}e^{-6y}\right)}\, dy \\
&= \int_0^{\ln 2} \sqrt{\frac{1}{9}e^{6y} + \frac{1}{2} + \frac{9}{16}e^{-6y}}\, dy \\
&= \int_0^{\ln 2} \sqrt{\left(\frac{1}{3}e^{3y} + \frac{3}{4}e^{-3y}\right)^2}\, dy \\
&= \int_0^{\ln 2} \left(\frac{1}{3}e^{3y} + \frac{3}{4}e^{-3y}\right) dy \\
&= \left[\frac{1}{9}e^{3y} - \frac{1}{4}e^{-3y}\right]_0^{\ln 2} \\
&\stackrel{*}{=} \left(\frac{8}{9} - \frac{1}{32}\right) - \left(\frac{1}{9} - \frac{1}{4}\right) \\
&= \frac{7}{9} + \frac{7}{32} = \frac{224 + 63}{288} = \frac{287}{288}.
\end{aligned}
$$

* *Note that*

$$e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$$

and

$$e^{-3\ln 2} = e^{\ln 2^{-3}} = 2^{-3} = \frac{1}{8}.$$