Recitation #19: Approximating functions with polynomials

Warm up:

For each of the following, write the given polynomial in summation notation starting with $k = 0$.

(a) $\dfrac{3x}{2} - \dfrac{5x^2}{3} + \dfrac{7x^3}{4} - \dfrac{9x^4}{5} + \dfrac{11x^5}{6}$

(b) $\dfrac{1}{2}x + \dfrac{1 \cdot 5}{4 \cdot 2!}x^3 + \dfrac{1 \cdot 5 \cdot 9}{8 \cdot 3!}x^5 - \dfrac{1 \cdot 5 \cdot 9 \cdot 13}{16 \cdot 4!}x^7$

(c) $(x-1)^3 - \dfrac{(x-1)^4}{2!} + \dfrac{(x-1)^5}{4!} - \dfrac{(x-1)^6}{6!}$

Group work:

Problem 1 *Assuming that the function $f(x)$ is infinitely differentiable, and given that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + c_4(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

show that the coefficient c_4 of the $(x-a)^4$ term in the Taylor polynomial is $\dfrac{f^{(4)}(a)}{4!}$.

Problem 2 *Let $f(x) = \sin(2x)$. Find $p_3(x)$ about the point $a = \dfrac{\pi}{8}$.*

Problem 3 *Let $f(x) = xe^{-x}$ on the interval $[-2, 8]$.*

(a) *Write the Taylor polynomial $p_4(x)$ around $a = 3$.*

$$\begin{aligned} \textit{Fun facts: } f'(x) &= -e^{-x}(x-1) \\ f''(x) &= e^{-x}(x-2) \\ f^{(3)}(x) &= -e^{-x}(x-3) \\ f^{(4)}(x) &= e^{-x}(x-4) \end{aligned}$$

(b) *Write* $p_4(x)$ *about* $a = 3$ *in summation notation. Also, write the remainder term* $R_4(x)$.

(c) *Calculate* $p_4(4.5)$ *and, using* $R_4(4.5)$, *estimate how close* $p_4(4.5)$ *is to* $f(4.5)$. *Do the same for* $p_4(1.5)$.

(d) *Use the remainder term* $R_4(x)$ *to estimate the maximum error for* $p_4(x)$ *on* $[-2, 6]$.

(e) *How large must* n *be to assure that the* n^{th} *degree Taylor polynomial for* $f(x) = xe^{-x}$ *about* $a = 3$ *approximates* $2e^{-2}$ *within* 10^{-5}*?*