Recitation #21: Taylor series

Warm up:

Find the Taylor series for:

(a) $27x^2 - 3x + 17$ centered at $a = 0$.

(b) $27x^2 - 3x + 17$ centered at $a = 1$.

Solution: (a) $27x^2 - 3x + 17$ is already a Taylor Series centered at 0 because it is already in the form $\sum_{k=0}^{\infty} c_k x^k$ with $c_0 = 17, c_1 = -3, c_2 = 27$, and the rest of the $c_k = 0$.

(b) Let $f(x) = 27x^2 - 3x + 17$. Then

$$\begin{aligned}
&f(1) = 27 - 3 + 17 = 41 \\
&f'(x) = 54x - 3 \quad \Longrightarrow \quad f'(1) = 54 - 3 = 51 \\
&f''(x) = 54 \quad \Longrightarrow \quad f''(1) = 54 \\
&f^{(3)}(x) = 0 \quad \Longrightarrow \quad f^{(3)}(1) = 0 \\
&\quad \vdots \qquad\qquad\qquad\qquad \vdots \\
&f^{(n)}(x) = 0 \quad \Longrightarrow \quad f^{(n)}(1) = 0.
\end{aligned}$$

So

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \boxed{41 + 51(x-1) + \frac{54}{2!}(x-1)^2}$$

Lastly, note that if you multiply this out then you will get back the original polynomial.

Group work:

Problem 1 *Find a Maclaurin series (and interval of convergence) for*

$$f(x) = x^3 \sin(x^5)$$

Solution: *We already know that*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Learning outcomes:

with interval of convergence $(-\infty, \infty)$. So we use this to compute

$$\begin{aligned} x^3 \sin(x^5) &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k (x^5)^{2k+1}}{(2k+1)!} \\ &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k x^{10k+5}}{(2k+1)!} \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{10k+8}}{(2k+1)!}} \end{aligned}$$

with interval of convergence $(-\infty, \infty)$.

Problem 2 *Find the first four non-zero terms of the Maclaurin Series for*

$$xe^{x^2} + \cos(x^3)$$

Solution: *We might be tempted to solve this problem using the definition of a Taylor Series since we only need the first four terms, but that is a bad idea because the derivatives will get very messy. Instead, we will use the known Maclaurin Series for e^x and $\cos(x)$.*

We already know that

$$e^x = \sum_{k=0}^{\infty} \frac{(x)^k}{k!}$$

with interval of convergence $(-\infty, \infty)$. So we use this to compute

$$\begin{aligned} xe^{x^2} &= x \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} \\ &= x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} \end{aligned}$$

with interval of convergence $(-\infty, \infty)$.

Similarly, we already know that

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

with interval of convergence $(-\infty, \infty)$. So we use this to compute

$$\begin{aligned} \cos(x^3) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x^3)^{2k}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k}}{(2k)!} \end{aligned}$$

with interval of convergence $(-\infty, \infty)$.

Therefore,

$$xe^{x^2} + \cos(x^3) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k}}{(2k)!} \qquad = \sum_{k=0}^{\infty} \left(\frac{x^{2k+1}}{k!} + \frac{(-1)^k x^{6k}}{(2k)!} \right)$$

Now, we just need to plug in numbers for k, starting with k=0, until we have four non-zero terms. Plugging in 0 through 3 gives:

$$\begin{aligned} &\left(\frac{x^{2(0)+1}}{(0)!} + \frac{(-1)^{(0)} x^{6(0)}}{(2(0))!} \right) \\ +&\left(\frac{x^{2(1)+1}}{(1)!} + \frac{(-1)^{(1)} x^{6(1)}}{(2(1))!} \right) \\ +&\left(\frac{x^{2(2)+1}}{(2)!} + \frac{(-1)^{(2)} x^{6(2)}}{(2(2))!} \right) \\ +&\left(\frac{x^{2(3)+1}}{(3)!} + \frac{(-1)^{(3)} x^{6(3)}}{(2(3))!} \right) \end{aligned}$$

Now, notice that you actually have more than four terms here because there are more than four powers of x. We will have to simplify thisa expression to get the first four terms.

$$x + 1 + x^3 + \frac{-x^6}{2} + \frac{x^5}{2} + \frac{x^{12}}{4} + \frac{x^7}{6} + \frac{-x^{18}}{12}$$

so the first four terms will be

$$\boxed{1 + x + x^3 + \frac{x^5}{2}}$$

Problem 3 *Find a function (closed expression) for the following series and the interval on which the function and the series are equal.*

$$x + x^4 + \frac{1}{2}x^7 + \frac{1}{6}x^{10} + \frac{1}{24}x^{13} + \ldots$$

Solution:

$$\begin{aligned} x + x^4 + \frac{1}{2}x^7 + \frac{1}{6}x^{10} + \frac{1}{24}x^{13} + \ldots &= x + x^4 + \frac{1}{2!}x^7 + \frac{1}{3!}x^{10} + \frac{1}{4!}x^{13} + \ldots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} x^{3k+1} \\ &= x \sum_{k=0}^{\infty} \frac{x^{3k}}{k!} \\ &= x \sum_{k=0}^{\infty} \frac{(x^3)^k}{k!} \\ &= \boxed{xe^{x^3}} \end{aligned}$$

which has interval of convergence $(-\infty, \infty)$.

Problem 4 *Compute the sum of the following series (Hint: You should use Taylor series.)*

(a) $1 - \ln 2 + \dfrac{(\ln 2)^2}{2!} - \dfrac{(\ln 2)^3}{3!} + \dots$

(b) $3 + \dfrac{9}{2!} + \dfrac{27}{3!} + \dfrac{81}{4!} + \dots$

Solution: (a)

$$\begin{aligned} 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots &= \sum_{k=0}^{\infty} \frac{(-\ln 2)^k}{k!} \\ &= e^{-\ln 2} = e^{\ln 2^{-1}} \\ &= 2^{-1} = \boxed{\frac{1}{2}}. \end{aligned}$$

(b)

$$\begin{aligned} 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots &= \sum_{k=1}^{\infty} \frac{3^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{3^k}{k!} - \frac{3^0}{0!} \\ &= \boxed{e^3 - 1}. \end{aligned}$$

Problem 5 *Find the Taylor Series for* $\sin(2x)$ *about* $a = \dfrac{\pi}{8}$.
Hint: Recall from a previous recitation that
$p_3(x) = \dfrac{\sqrt{2}}{2} + \sqrt{2}\left(x - \dfrac{\pi}{8}\right) - \sqrt{2}\left(x - \dfrac{\pi}{8}\right)^2 - \dfrac{2\sqrt{2}}{3}\left(x - \dfrac{\pi}{8}\right)^3$

Solution: *Let* $f(x) = \sin(2x)$*. Then*

$$\begin{aligned} &f\left(\frac{\pi}{8}\right) = \sin\left(\frac{2\pi}{8}\right) = \frac{\sqrt{2}}{2} \\ &f'(x) = 2\cos(2x) \quad \Longrightarrow \quad f'\left(\frac{\pi}{8}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \\ &f''(x) = -4\sin(2x) \quad \Longrightarrow \quad f''\left(\frac{\pi}{8}\right) = -4 \cdot \frac{\sqrt{2}}{2} = -2\sqrt{2} \\ &f^{(3)}(x) = -8\cos(2x) \quad \Longrightarrow \quad f^{(3)}\left(\frac{\pi}{8}\right) = -8 \cdot \frac{\sqrt{2}}{2} = -4\sqrt{2} \\ &f^{(4)}(x) = 16\cos(2x) \quad \Longrightarrow \quad f^{(4)}\left(\frac{\pi}{8}\right) = 16 \cdot \frac{\sqrt{2}}{2} = 8\sqrt{2}. \end{aligned}$$

Continuing this pattern, we see that

$$f^{(k)}\left(\frac{\pi}{8}\right) = (-1)^{\left\lceil \frac{k}{2} \right\rceil} 2^{k-1}\sqrt{2}$$

where $\left\lceil \frac{k}{2} \right\rceil$ *denotes the smallest integer greater than* $\frac{k}{2}$. *So, for example,* $\left\lceil \frac{1}{2} \right\rceil = 1$, $\left\lceil \frac{2}{2} \right\rceil = 1$, $\left\lceil \frac{3}{2} \right\rceil = 2$, *and so on.*

So from here we have that the Taylor series for $f(x)$ *is*

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^{\lceil \frac{k}{2} \rceil} 2^{k-1} \sqrt{2}}{k!} \left(x - \frac{\pi}{8}\right)^k}$$