# Section 7.5: Partial Fractions

## Warm up:

**Problem 1** *Give the general partial fraction decomposition for the following function. DO NOT SOLVE FOR THE CONSTANTS!*

$$f(x) = \frac{4x^3 - 7}{x^6 - x^2}$$

***Solution:*** *Factor the expression completely:*

$$\frac{4x^3 - 7}{x^6 - x^2} = \frac{4x^3 - 7}{x^2(x^4 - 1)} = \frac{4x^3 - 7}{x^2(x^2 - 1)(x^2 + 1)} = \frac{4x^3 - 7}{x^2(x + 1)(x - 1)(x^2 + 1)}$$

*Noting that the term $x^2$ is a* <u>repeated linear factor</u>, *we have:*

$$\boxed{f(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1} + \frac{Ex + F}{x^2 + 1}}$$

Remember that the $x^2$ term must be treated as a repeated linear factor, not as an irreducible quadratic! Do **NOT** write the expression in red below:

$$f(x) = \frac{Ax + B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1} + \frac{Ex + F}{x^2 + 1}$$

## Group work:

**Problem 2** *Without determining the coefficients, write the partial fraction decomposition of the following rational function:*

$$\frac{5x^{13} - 6x^{12} + 7x^3 - 5x - 18}{(2x - 3)(5x + 9)^3(x^2 + 9x + 19)(x^2 + 9x + 21)^2}$$

Learning outcomes:

***Solution:*** *The degree of the numerator is 13, whereas the degree of the denominator is 10. So if we perform long division, we will get a degree $13-10=3$ polynomial plus partial fractions for the remainder term:*

$$\frac{5x^{13}-6x^{12}+7x^3-5x-18}{(2x-3)(5x+9)^3(x^2+9x+19)(x^2+9x+21)^2} = Ax^3+Bx^2+Cx+D$$
$$+\frac{E}{2x-3}+\frac{F}{5x+9}+\frac{G}{(5x+9)^2}+\frac{H}{(5x+9)^3}+\frac{I}{x-i_1}+\frac{J}{x-i_2}$$
$$+\frac{Kx+L}{x^2+9x+21}+\frac{Mx+N}{(x^2+9x+21)^2}.$$

***Explanation of $i_1$ and $i_2$:*** *The quadratic $x^2+9x+19$ can be factored over the real numbers, since the discriminant $b^2-4ac=81-76>0$. The numbers $i_1$ and $i_2$ are the two real roots to this polynomial, ie*

$$i_1=\frac{-9+\sqrt{5}}{2} \qquad i_2=\frac{-9-\sqrt{5}}{2}.$$

*Note that the polynomial $x^2+9x+21$ is irreducible (over the real numbers) since its discriminant is less than 0.*

---

**Problem 3** *Evaluate:*

$$\int \frac{7x^3+18x+9}{x^4+9x^2}\,dx$$

*Hint: If $f(x)=7x^3+18x+9$, then $f(2)=101$, $f(1)=34$, and $f(-1)=-16$.*

***Solution:*** *First factor the denominator*

$$x^4+9x^2=x^2(x^2+9).$$

*The we can decompose the integrand as a partial fraction*

$$\frac{7x^3+18x+9}{x^2(x^2+9)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+9}$$

$$\begin{aligned}\implies \quad 7x^3+18x+9 &= Ax(x^2+9)+B(x^2+9)+(Cx+D)x^2\\ &= Ax^3+9Ax+Bx^2+9B+Cx^3+Dx^2\\ &= (A+C)x^3+(B+D)x^2+9Ax+9B.\end{aligned}$$

*By equating coefficients for powers of $x$ we have that*

$$\begin{array}{lllll} 9=9B & \implies & B=1 & & \\ 18=9A & \implies & A=2 & & \\ 0=B+D & \implies & 0=1+D & \implies & D=-1 \\ 7=A+C & \implies & 7=2+C & \implies & C=5. \end{array}$$

*Thus*

$$\begin{aligned}\int \frac{7x^3+18x+9}{x^4+9x^2}\,dx &= \int \left(\frac{2}{x}+\frac{1}{x^2}+\frac{5x-1}{x^2+9}\right)\,dx \\ &= 2\ln|x| - \frac{1}{x} + 5\int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx \\ &= 2\ln|x| - \frac{1}{x} + \frac{5}{2}\ln(x^2+9) - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.\end{aligned}$$

*Note that, in the previous step, we substituted* $u = x^2 + 9$ *for the first integral and* $u = \frac{x}{3}$ *in the second integral.*