Recitation #9: Trig Integrals and Trig Substitution

Group work:

Problem 1 *Evaluate the following integrals*

(a) $\displaystyle\int \tan^{23} x \sec^6 x \, dx$

Solution:

$$\begin{aligned}\int \tan^{23} x \sec^6 x \, dx &= \int \tan^{23} x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^{23} x \left(1+\tan^2 x\right)^2 \sec^2 x \, dx.\end{aligned}$$

We now substitute

$$u = \tan x \qquad \Longrightarrow \qquad du = \sec^2 x \, dx.$$

Then

$$\begin{aligned}\int \tan^{23} x \left(1+\tan^2 x\right)^2 \sec^2 x \, dx &= \int u^{23}(1+u^2)^2 \, du \\ &= \int u^{23}(1+2u^2+u^4) \, du \\ &= \int \left(u^{23}+2u^{25}+u^{27}\right) \, du \\ &= \frac{1}{24}u^{24} + \frac{1}{13}u^{26} + \frac{1}{28}u^{28} + C \\ &= \frac{1}{24}\tan^{24} x + \frac{1}{13}\tan^{26} x + \frac{1}{28}\tan^{28} x + C.\end{aligned}$$

(b) $\displaystyle\int \tan^2 x \sec x \, dx$ *Hint:* $\displaystyle\int \sec x \, dx = \ln|\sec x \tan x| + C$

Learning outcomes:

Solution:

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int \left(\sec^2 x - 1\right)\sec x\,dx\\ &= \int \sec^3 x\,dx - \int \sec x\,dx\\ &= \int \sec^3 x\,dx - \ln|\sec x \tan x| \qquad \textit{from the hint} \qquad (1)\end{aligned}$$

Now, in an attempt to evaluate $\int \sec^3 x\,dx$, *we use integration by parts with*

$$u = \sec x \qquad dv = \sec^2 x\,dx$$

$$du = \sec x \tan x\,dx \qquad v = \tan x.$$

So

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx. \qquad (2)$$

Combining equations (1) *and* (2) *yields*

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int \sec^3 x\,dx - \ln|\sec x \tan x|\\ \int \tan^2 x \sec x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx - \ln|\sec x \tan x|\\ 2\int \tan^2 x \sec x\,dx &= \sec x \tan x - \ln|\sec x \tan x| + C\\ \int \tan^2 x \sec x\,dx &= \frac{1}{2}\left(\sec x \tan x - \ln|\sec x \tan x|\right) + C.\end{aligned}$$

(c) $\int \tan^2 x \sin x\,dx$

Solution:

$$\begin{aligned}\int \tan^2 x \sin x\,dx &= \int \frac{\sin^2 x}{\cos^2 x}\sin x\,dx\\ &= \int \frac{1-\cos^2 x}{\cos^2 x}\sin x\,dx.\end{aligned}$$

Now we substitute

$$u = \cos x \qquad \Longrightarrow \qquad du = -\sin x\,dx, \quad -du = \sin x\,dx.$$

This gives us that

$$\begin{aligned}
\int \tan^2 x \sin x \, dx &= \int \frac{1-u^2}{u^2}(-1)\, du \\
&= \int \frac{u^2-1}{u^2}\, du \\
&= \int \left(1 - u^{-2}\right)\, du \\
&= u + \frac{1}{u} + C \\
&= \cos x + \sec x + C.
\end{aligned}$$

Instructor Notes: All three parts involve standard strategies learned in the online lessons. For each, split the problems between the groups. Then discuss each problem as a class, getting input from the group(s) that worked on that problem.

Problem 2 *Evaluate*

$$\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx.$$

Solution:

$$\begin{aligned}
\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx &= \int_{-\pi}^{0} \sqrt{\sin^2 x}\, dx \\
&= \int_{-\pi}^{0} |\sin x|\, dx.
\end{aligned}$$

Now, when $-\pi \le x \le 0$, $\sin x \le 0$. *Thus, on this region,* $|\sin x| = -\sin x$. *So we continue*

$$\begin{aligned}
\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx &= \int_{-\pi}^{0} -\sin x\, dx \\
&= \Big[\cos x\Big]_{-\pi}^{0} \\
&= \cos(0) - \cos(-\pi) = 1 - (-1) = 2.
\end{aligned}$$

Instructor Notes: You may want to do this problem as a whole class - perhaps play-acting by claiming that it is equal to $\int_{-\pi}^{0} \sin x \, dx$ rather than $\int_{-\pi}^{0} |\sin x| \, dx$

Problem 3 *Evaluate the following integrals*

(a)

$$\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3} \, dx.$$

Solution: *First notice that*

$$\begin{aligned}\sqrt{36x^2 - 25} &= 5\sqrt{\frac{36x^2}{25} - 1} \\ &= 5\sqrt{\left(\frac{6x}{5}\right)^2 - 1}.\end{aligned}$$

So we substitute

$$\frac{6x}{5} = \sec\theta \qquad \Longrightarrow \qquad x = \frac{5}{6}\sec\theta$$

which gives

$$dx = \frac{5}{6}\sec\theta\tan\theta \, d\theta.$$

Also, notice that

- *when* $x = -\frac{5}{3}$*:*

$$-\frac{5}{3} = \frac{5}{6}\sec\theta \qquad \Longrightarrow \qquad \sec\theta = 2 \qquad \Longrightarrow \qquad \theta = \frac{2\pi}{3}$$

- *and when* $x = -\frac{5}{6}$*:*

$$-\frac{5}{6} = \frac{5}{6}\sec\theta \qquad \Longrightarrow \qquad \sec\theta = -1 \qquad \Longrightarrow \qquad \theta = \pi.$$

Therefore

$$\begin{aligned}\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3} \, dx &= 5\int_{\frac{2\pi}{3}}^{\pi} \frac{\sqrt{\sec^2\theta - 1}}{\left(\frac{5}{6}\sec\theta\right)^3} \left(\frac{5}{6}\sec\theta\tan\theta\right) d\theta \\ &= 5 \cdot \left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan\theta|\tan\theta}{\sec^2\theta} \, d\theta.\end{aligned}$$

Now, notice that $\tan\theta < 0$ *whenever* $\dfrac{2\pi}{3} \le \theta \le \pi$. *So* $|\tan\theta| = -\tan\theta$. *We continue:*

$$\begin{aligned}
5\cdot\left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan\theta|\tan\theta}{\sec^2\theta}\,d\theta &= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\
&= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{1}\,d\theta \\
&= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \sin^2\theta\,d\theta \\
&= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}\left(1-\cos(2\theta)\right)\,d\theta \\
&= -\frac{18}{5}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{2\pi}{3}}^{\pi} \\
&= -\frac{18}{5}\left[(\pi - 0) - \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right)\right] \qquad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\
&= -\frac{18}{5}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right).
\end{aligned}$$

(b)

$$\int \frac{dx}{\left(x^2-6x+11\right)^2}.$$

Solution: *We begin by completing the square in the denominator*

$$x^2 - 6x - 11 = x^2 - 6x + 9 + 2 = (x-3)^2 + 2.$$

We then have that

$$\begin{aligned}
\int \frac{dx}{\left(x^2-6x+11\right)^2} &= \int \frac{1}{((x-3)^2+2)^2}\,dx \\
&= \frac{1}{4}\int \frac{1}{\left(\frac{(x-3)^2}{2}+1\right)^2}\,dx \\
&= \frac{1}{4}\int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2+1\right)^2}\,dx.
\end{aligned}$$

So we substitute

$$\frac{x-3}{\sqrt{2}} = \tan\theta \qquad \Longrightarrow \qquad x = \sqrt{2}\tan\theta + 3 \tag{3}$$

and then

$$dx = \sqrt{2}\sec^2\theta\,d\theta.$$

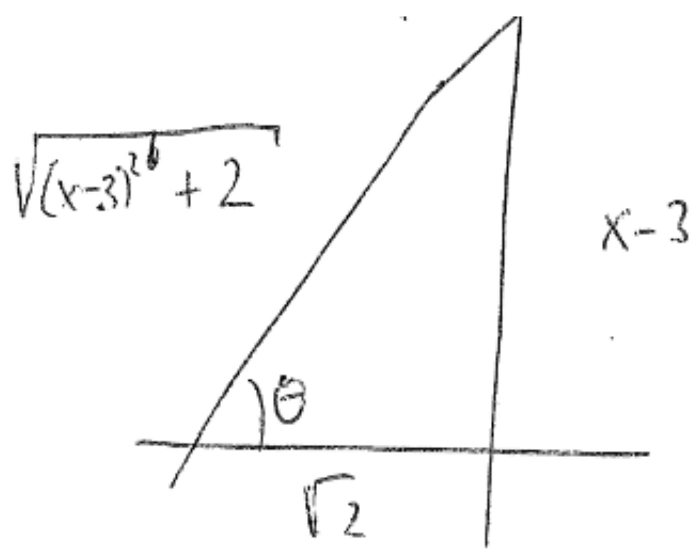
Continuing with the integral

$$\begin{aligned}
\frac{1}{4}\int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2+1\right)^2}\,dx &= \frac{1}{4}\int \frac{1}{\left(\tan^2\theta+1\right)^2}\sqrt{2}\sec^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \frac{1}{\sec^2\theta}\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \frac{1}{2}\left(1+\cos(2\theta)\right)\,d\theta \\
&= \frac{\sqrt{2}}{8}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C.
\end{aligned}$$

Now all that is left to do is to reverse-substitute for θ. First, from equation (3) we have that

$$\theta = \arctan\left(\frac{x-3}{\sqrt{2}}\right).$$

Now, we again use equation (3) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\cdot\frac{x-3}{\sqrt{(x-3)^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{(x-3)^2+2}}.$$

Thus

$$\int \frac{dx}{\left(x^2 - 6x + 11\right)^2} = \frac{\sqrt{2}}{8}\left(\arctan\left(\frac{x-3}{\sqrt{2}}\right) + \frac{\sqrt{2}(x-3)}{(x-3)^2+2}\right) + C.$$

Instructor Notes: Each of problems (a) through (c) involves one or more of the major points of trig substitution. Each of the three kinds of substitutions is represented, as well as working with absolute value issues in problem (a) (also could be brought up in problem (c)), completing the square, back substitution (c), and various trigonometric integrals. **Be adamant about substituting for** dx as well as the rest of the integrand. In (a), show the time-saving value of changing the limits in terms of θ.