Recitation 29: Lines and curves in space - Solutions

Warm up:

Find a vector-valued function for the line segment connecting the points $P = (-3, 7, 6)$ and $Q = (5, -4, 7)$ in such a way that the value at $t = 0$ is P and the value at $t = 1$ is Q. Also, find the point two-thirds of the way from P to Q.

Solution: The line segment $\vec{r}(t)$ from P to Q is

$$\begin{aligned}\vec{r}(t) &= (1-t)P + tQ \\ &= (1-t)\langle -3, 7, 6\rangle + t\langle 5, -4, 7\rangle \\ &= \boxed{\langle -3 + 8t, 7 - 11t, 6 + t\rangle \quad \text{for } 0 \le t \le 1}.\end{aligned}$$

The point two-thirds of the way from P to Q is

$$\begin{aligned}\vec{r}\left(\frac{2}{3}\right) &= \left\langle -3 + 8\left(\frac{2}{3}\right), 7 - 11\left(\frac{2}{3}\right), 6 + \frac{2}{3}\right\rangle \\ &= \boxed{\left\langle \frac{7}{3}, -\frac{1}{3}, \frac{20}{3}\right\rangle}\end{aligned}$$

Group work:

Problem 1 *Find a vector-valued function for the line through the point $(1, -2, 3)$ that is perpendicular to the lines*

$$\vec{r}_1(t) = \langle 7, 8, -2\rangle + t\langle 3, 5, 7\rangle \quad \textit{and} \quad \vec{r}_2(s) = \langle 4, -3, -7\rangle + s\langle 4, 9, -1\rangle$$

Solution: *Let $\vec{v}_1 = \langle 3, 5, 7\rangle$ and $\vec{v}_2 = \langle 4, 9, -1\rangle$. Then $\vec{v}_1$ is parallel to the line $\vec{r}_1$, and similarly for $\vec{v}_2$ and $\vec{r}_2$. So a vector perpendicular to both of the*

Learning outcomes:

lines $\vec{r}_1$ and $\vec{r}_2$ is

$$\begin{aligned}\vec{n} = \vec{v}_1 \times \vec{v}_2 &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & 7 \\ 4 & 9 & -1 \end{vmatrix} \\ &= (-5-63)\hat{\imath} - (-3-28)\hat{\jmath} + (27-20)\hat{k} \\ &= \langle -68, 31, 7 \rangle.\end{aligned}$$

So the equation of the line through $(1, -2, 3)$ and perpendicular to both $\vec{r}_1$ and $\vec{r}_2$ is

$$\begin{aligned}\vec{r}_3(t) &= \langle 1, -2, 3 \rangle + t\langle -68, 31, 7 \rangle \\ &= \boxed{\langle 1 - 68t, -2 + 31t, 3 + 7t \rangle \quad \text{for } -\infty < t < \infty}\end{aligned}$$

Problem 2 *Find the distance from the point $P(-1, 4, 3)$ to the line $\langle 8 + t, 3 - 3t, -26t \rangle$. Hint: The distance from the point to the line is the distance from the point P and the closest point on the line.*

Solution: *Let $P = (-1, 4, 3)$ and, for any time t, let $Q(t) = (8 + t, 3 - 3t, -26t)$. Then the distance from P to $Q(t)$ is given by*

$$\begin{aligned}D(t) &= \sqrt{(8 + t - (-1))^2 + (3 - 3t - 4)^2 + (-26t - 3)^2} \\ &= \sqrt{(9 + t)^2 + (-1 - 3t)^2 + (-3 - 26t)^2}.\end{aligned}$$

Instead of minimizing the distance $D(t)$, we will minimized the square of the distance $D^2(t)$, which leads to the same point. So

$$D^2(t) = (9 + t)^2 + (-1 - 3t)^2 + (-3 - 26t)^2.$$

To find the minimum of this function, we differentiate and find critical points

$$\begin{aligned}\frac{d}{dt}D^2(t) &= 2(9 + t) + 2(-1 - 3t)(-3) + 2(-3 - 26t)(-26) \\ &= (18 + 2t) + (6 + 18t) + (156 + 1352t) \\ &= 180 + 1372t := 0 \\ \implies \qquad t &= -\frac{180}{1372} = -\frac{45}{343}.\end{aligned}$$

Since there is exactly one critical point and since the second derivative is positive (it is the constant 1372), this value of t gives an absolute minimum. Therefore, the distance from P to the line is

$$\boxed{\sqrt{\left(9 - \frac{45}{343}\right)^2 + \left(-1 - 3\left(-\frac{45}{343}\right)\right)^2 + \left(-3 - 26\left(-\frac{45}{343}\right)\right)^2}}$$

Problem 3 *Show that the curve $\vec{r} = \langle t\cos t, t\sin t, t\rangle$ lies completely on the cone $z^2 = x^2 + y^2$.*

Solution: *We just need to check that the components of $\vec{r}$ satisfies the given equation. So we compute*

$$\begin{aligned} x^2 + y^2 &= (t\cos t)^2 + (t\sin t)^2 \\ &= t^2\cos^2 t + t^2\sin^2 t \\ &= t^2(\cos^2 t + \sin^2 t) \\ &= t^2 \\ &= z^2. \end{aligned}$$

Problem 4 *Match each of the following curves to the corresponding vector-valued function.*

(a) $\langle 3, t^2, 5\rangle$

(b) $\langle 3, t^2, t\rangle$

(c) $\langle 3, \sin t, \cos t\rangle$

(d) $\langle 3t, 5\sin t, 5\cos t\rangle$

(e) $\langle \sin t, \cos t, 2\cos t\rangle$

(f) $\langle 2\cos t, \sin t, \cos(3t)\rangle$

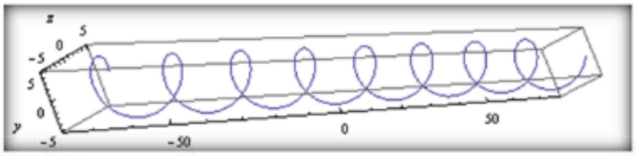

Figure 1

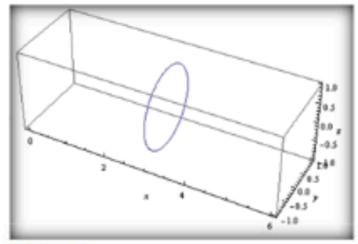

Figure 2

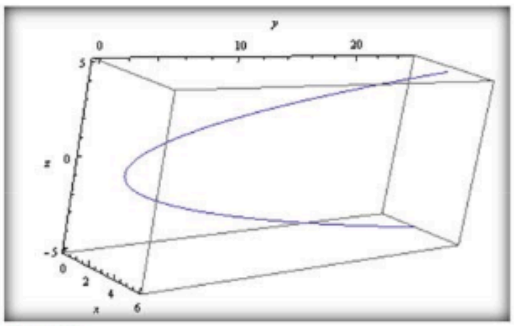

Figure 3

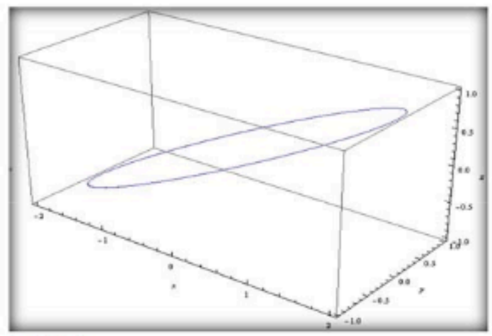

Figure 4

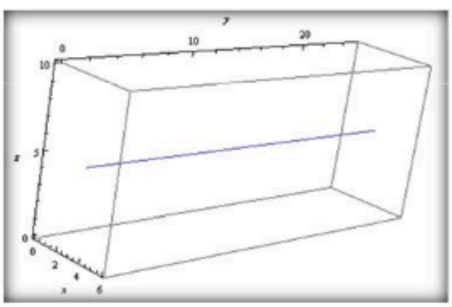

Figure 5

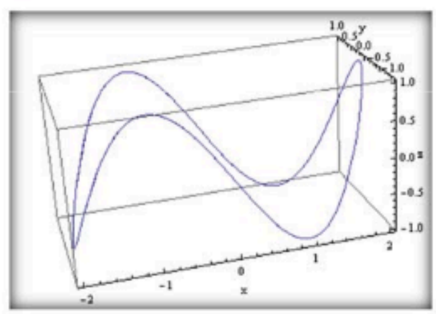

Figure 6

Solution: (a) **Figure 5**. *This is a line with both x and z held fixed.*

(b) **Figure 3**. *This is a parabola parallel to the yz-plane at $x = 3$.*

(c) **Figure 2**. *This is a circle of radius 1 in the plane $x = 3$.*

(d) **Figure 1**. *The x-component is linear, while the projection onto the yz-plane is a circle of radius 5. So this looks like a "spring".*

(e) **Figure 4**. *This is a circle with radius 1 when projected onto the xy-plane.*

(f) **Figure 6**. *This one is tricky. Maybe the best way to spot it is that it is an ellipse when projected onto the xy-plane, while the z-component varies between -1 and 1.*