3.10 Derivatives of Inverse Trig Functions (Solutions)

Warm up:

Explain what each of the following means:

(a) $\sin^{-1}(x)$

Solution: This denotes the inverse function to $sin(x)$, sometimes denoted by $\arcsin(x)$.

(b) $(\sin(x))^{-1}$

Solution: This means $\sin(x)$ raised to the -1 power, i.e. $\frac{1}{\sin(x)}$.

(c) $\sin\left(x^{-1}\right)$

Solution: This means $\sin\left(\frac{1}{x}\right)$.

(d) $f^{-1}(x)$

Solution: This denotes the inverse function of $f(x)$.

(e) $f(x^{-1})$

Solution: This means $f\left(\frac{1}{x}\right)$.

(f) $(f(x))^{-1}$

Solution: This means $f(x)$ raised to the -1 power, i.e. $\frac{1}{f(x)}$.

Group work:

Problem 1 *Find the derivatives of the following functions:*

(a) $f(x) = \sec^{-1}(\sqrt{x})$.

Solution: $f'(x) = \dfrac{1}{\sqrt{x}\sqrt{x-1}} \cdot \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x\sqrt{x-1}}$

(b) $g(x) = \ln(\sin^{-1}(x))$.

Solution: $g'(x) = \dfrac{1}{\sin^{-1}(x)} \cdot \dfrac{1}{\sqrt{1-x^2}}$.

(c) $h(x) = \dfrac{1}{\tan^{-1}(x^2+4)}$.

Solution: $h'(x) = -\left(\tan^{-1}(x^2+4)\right)^{-2} \cdot \dfrac{1}{1+(x^2+4)^2} \cdot (2x)$.

Problem 2 *Find the slope of the tangent line to the curve $y = f^{-1}(x)$ at $(4,7)$ if the slope of the tangent line to the curve $y = f(x)$ at $(7,4)$ is $\frac{2}{3}$.*

Solution: *Note that the statement "the slope of the tangent line to the curve $y = f(x)$ at $(7,4)$ is $\frac{2}{3}$" specifically means that $f'(7) = \frac{2}{3}$. The slope of the tangent line to the curve $y = f^{-1}(x)$ at $(4,7)$ is $(f^{-1})'(4)$, and so we use the formula for the derivative of the inverse function to compute:*

$(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{1}{\frac{2}{3}} = \dfrac{3}{2}$.

Problem 3 *Suppose that $f(x)$ is a differentiable function which is one-to-one. Given the table of values below, find the value of $(f^{-1})'(7)$.*

x	*1*	*7*	*11*
f(x)	*7*	*11*	*1*
f'(x)	*61*	*-17*	*71*

Solution: $(f^{-1})'(7) = \dfrac{1}{f'(f^{-1}(7))}$. *Since $f(1) = 7$, $f^{-1}(7) = 1$. Thus*

$$(f^{-1})'(7) = \frac{1}{f'(1)} = \frac{1}{61}.$$

Problem 4 *Find the derivative of f^{-1} at the following points without solving for f^{-1}.*

(a) $f(x) = x^2 + 1$ *(for $x \geq 0$) at the point* $(5, 2)$.

Solution: $(f^{-1})'(5) = \dfrac{1}{f'(2)}$. *Since* $f'(x) = 2x$, $f'(2) = 4$. *Thus,* $(f^{-1})'(5) = \dfrac{1}{4}$.

(b) $f(x) = x^2 - 2x - 3$ *(for $x \leq 1$) at the point* $(12, -3)$.

Solution: $(f^{-1})'(12) = \dfrac{1}{f'(-3)}$. *Since* $f'(x) = 2x - 2$, $f'(-3) = -6 - 2 = -8$. *Thus,* $(f^{-1})'(12) = -\dfrac{1}{8}$.