Recitation #9: Integration by parts and Trig Integrals - Solutions

Group work:

Problem 1 *Evaluate the following integrals*

(a) $\displaystyle\int_1^3 x^2 5^x \, dx$

Solution: *We proceed via integration by parts. Let*

$$u = x^2, \qquad dv = 5^x \, dx$$

so that

$$du = 2x \, dx, \qquad v = \frac{5^x}{\ln 5}.$$

Recall the formula for integration by parts is

$$\int_a^b u \, dv = \Big[uv \Big]_a^b - \int_a^b v \, du.$$

So we substitute

$$\begin{aligned} \int_1^3 x^2 5^x \, dx &= \left[\frac{x^2 5^x}{\ln(5)} \right]_1^3 - \int_1^3 2x \frac{5^x}{\ln(5)} \, dx \\ &= \frac{1}{\ln(5)} \left(9 \cdot 5^3 - 5 \right) - \frac{2}{\ln(5)} \int_1^3 x 5^x \, dx. \end{aligned}$$

For the remaining integral we again use integration by parts:

$$u = x, \qquad dv = 5^x$$

$$du = dx, \qquad v = \frac{5^x}{\ln(5)}.$$

Learning outcomes:

Thus,

$$\begin{aligned}
&\frac{1}{\ln(5)}\left(9\cdot 5^3 - 5\right) - \frac{2}{\ln(5)}\int_1^3 x5^x\,dx \\
&= \frac{5}{\ln(5)}(225-1) - \frac{2}{\ln(5)}\left(\left[\frac{x5^x}{\ln(5)}\right]_1^3 - \int_1^3 \frac{5^x}{\ln(5)}\,dx\right) \\
&= \frac{1120}{\ln(5)} - \frac{2}{\ln^2(5)}\left((3\cdot 5^3 - 5) - \left[\frac{5^x}{\ln(5)}\right]_1^3\right) \\
&= \frac{1}{\ln(5)}\left(1120 - \frac{740}{\ln(5)} + \frac{248}{\ln^2(5)}\right).
\end{aligned}$$

(b) $\displaystyle\int \sin(3x)e^{7x}\,dx$

Solution: *We begin by letting* $I = \displaystyle\int \sin(3x)e^{7x}\,dx$. *We then use integration by parts with*

$$u = e^{7x} \qquad dv = \sin(3x)\,dx$$

$$du = 7e^{7x}\,dx \qquad v = -\frac{1}{3}\cos(3x).$$

Then

$$\begin{aligned}
\int \sin(3x)e^{7x}\,dx = I &= -\frac{1}{3}e^{7x}\cos(3x) - \int -\frac{1}{3}(7e^{7x})\cos(3x)\,dx \\
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{3}\int e^{7x}\cos(3x)\,dx.
\end{aligned}$$

We then apply integration by parts again, this time with

$$u = e^{7x} \qquad dv = \cos(3x)\,dx$$

$$du = 7e^{7x}\,dx \qquad v = \frac{1}{3}\sin(3x).$$

This gives us

$$\begin{aligned}
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{3}\left[\frac{1}{3}e^{7x}\sin(3x) - \int \frac{1}{3}(7e^{7x})\sin(3x)\,dx\right] \\
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) - \frac{49}{9}\int e^{7x}\sin(3x)\,dx \\
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) - \frac{49}{9}I \\
\frac{58}{9}I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) \\
I &= \frac{9}{58}\left(-\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x)\right) + C.
\end{aligned}$$

(c) $\int x^{\frac{5}{3}} (\ln x)^2 \, dx$

Solution: *We begin with the substitution*

$$w = \ln x \qquad \Longrightarrow \qquad dw = \frac{1}{x}\, dx, \quad x = e^w.$$

Then

$$\begin{aligned}\int x^{\frac{5}{3}} (\ln x)^2 \, dx &= \int x^{\frac{8}{3}} (\ln x)^2 \cdot \frac{1}{x}\, dx \\ &= \int (e^w)^{\frac{8}{3}} w^2 \, dw \\ &= \int w^2 e^{\frac{8}{3}w} \, dw.\end{aligned}$$

We now use integration by parts, with

$$u = w^2 \qquad dv = e^{\frac{8}{3}w}\, dw$$

$$du = 2w\, dw \qquad v = \frac{3}{8} e^{\frac{8}{3}w}.$$

This gives us

$$\begin{aligned}\int x^{\frac{5}{3}} (\ln x)^2 \, dx &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \int \frac{3}{8}(2w) e^{\frac{8}{3}w}\, dw \\ &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \frac{3}{4}\int w e^{\frac{8}{3}w}\, dw.\end{aligned}$$

We apply integration by parts one last time with

$$u = w \qquad dv = e^{\frac{8}{3}w}\, dw$$

$$du = dw \qquad v = \frac{3}{8} e^{\frac{8}{3}w}$$

which yields

$$\begin{aligned}\int x^{\frac{5}{3}} (\ln x)^2 \, dx &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \frac{3}{4}\left(\frac{3}{8} w e^{\frac{8}{3}w} - \frac{3}{8}\int e^{\frac{8}{3}w}\, dw\right) \\ &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \frac{9}{32} w e^{\frac{8}{3}w} + \frac{27}{256} e^{\frac{8}{3}w} + C \\ &= \frac{3}{8} e^{\frac{8}{3}\ln x}\left((\ln x)^2 - \frac{3}{4}\ln x + \frac{9}{32}\right) + C \\ &= \frac{3}{8} x^{\frac{8}{3}}\left((\ln x)^2 - \frac{3}{4}\ln x + \frac{9}{32}\right) + C.\end{aligned}$$

Problem 2 *Evaluate the following integrals*

(a) $\displaystyle\int x^5 \cos\left(x^3\right)\, dx$

Solution: *We begin with the substitution*

$$w = x^3 \qquad \Longrightarrow \qquad dw = 3x^2\, dx, \quad \frac{1}{3}\, dw = x^2\, dx.$$

Then,

$$\begin{aligned}
\int x^5 \cos\left(x^3\right)\, dx &= \int x^3 \cos\left(x^3\right) \cdot x^2\, dx \\
&= \int w \cos(w) \cdot \frac{1}{3}\, dw \\
&= \frac{1}{3} \int w \cos(w)\, dw.
\end{aligned}$$

We then use integration by parts, with

$$u = w \qquad dv = \cos(w)\, dw$$

$$du = dw \qquad v = \sin(w)$$

which yields

$$\begin{aligned}
\int x^5 \cos\left(x^3\right)\, dx &= \frac{1}{3}\left(w \sin(w) - \int \sin(w)\, dw\right) \\
&= \frac{1}{3}\left(w \sin(w) + \cos(w)\right) + C \\
&= \frac{1}{3}\left(x^3 \sin\left(x^3\right) + \cos\left(x^3\right)\right) + C.
\end{aligned}$$

(b) $\displaystyle\int \cos\left(\sqrt{x}\right)\, dx$

Solution: *We begin with the subsitution*

$$w = \sqrt{x} \qquad \Longrightarrow \qquad dw = \frac{1}{2\sqrt{w}}\, dw, \quad 2\, dw = \frac{1}{\sqrt{x}}\, dw.$$

Then

$$\begin{aligned}
\int \cos\left(\sqrt{x}\right)\, dx &= \int \cos\left(\sqrt{x}\right) \cdot \frac{\sqrt{x}}{\sqrt{x}}\, dx \\
&= 2 \int w \cos(w)\, dw \\
&= 2(w \sin(w) + \cos(w)) + C \qquad \textit{From part (a)} \\
&= 2\left(\sqrt{x} \sin\left(\sqrt{x}\right) + \cos\left(\sqrt{x}\right)\right) + C.
\end{aligned}$$

(c) $\displaystyle\int x\cos x\sin x\,dx$

Solution: *First, recall that*

$$\sin(2x) = 2\sin x\cos x \qquad \Longrightarrow \qquad \sin x\cos x = \frac{1}{2}\sin(2x).$$

So we can rewrite the given integral as

$$\int x\cos x\sin x\,dx = \frac{1}{2}\int x\sin(2x)\,dx.$$

Now we use integration by parts with

$$u = x \qquad dv = \sin(2x)\,dx$$

$$du = dx \qquad v = -\frac{1}{2}\cos(2x).$$

This gives us that

$$\begin{aligned}\frac{1}{2}\int x\sin(2x)\,dx &= \frac{1}{2}\left(-\frac{1}{2}x\cos(2x) - \int -\frac{1}{2}\cos(2x)\,dx\right)\\ &= \frac{1}{4}\left(-x\cos(2x) + \frac{1}{2}\sin(2x)\right) + C.\end{aligned}$$

Problem 3 *Evaluate the following integrals*

(a) $\displaystyle\int \tan^{23}x\sec^6 x\,dx$

Solution:

$$\begin{aligned}\int \tan^{23}x\sec^6 x\,dx &= \int \tan^{23}x\sec^4 x\sec^2 x\,dx\\ &= \int \tan^{23}x\left(1+\tan^2 x\right)^2\sec^2 x\,dx.\end{aligned}$$

We now substitute

$$u = \tan x \qquad \Longrightarrow \qquad du = \sec^2 x\,dx.$$

Then

$$\begin{aligned}\int \tan^{23}x\left(1+\tan^2 x\right)^2\sec^2 x\,dx &= \int u^{23}(1+u^2)^2\,du\\ &= \int u^{23}(1+2u^2+u^4)\,du\\ &= \int \left(u^{23}+2u^{25}+u^{27}\right)\,du\\ &= \frac{1}{24}u^{24} + \frac{1}{13}u^{26} + \frac{1}{28}u^{28} + C\\ &= \frac{1}{24}\tan^{24}x + \frac{1}{13}\tan^{26}x + \frac{1}{28}\tan^{28}x + C.\end{aligned}$$

(b) $\displaystyle\int \tan^2 x \sec x \, dx$ *Hint:* $\displaystyle\int \sec x \, dx = \ln|\sec x \tan x| + C$

Solution:

$$\begin{aligned}
\int \tan^2 x \sec x \, dx &= \int \left(\sec^2 x - 1\right) \sec x \, dx \\
&= \int \sec^3 x \, dx - \int \sec x \, dx \\
&= \int \sec^3 x \, dx - \ln|\sec x \tan x| \qquad \textit{from the hint} \qquad (1)
\end{aligned}$$

Now, in an attempt to evaluate $\int \sec^3 x \, dx$*, we use integration by parts with*

$$u = \sec x \qquad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \qquad v = \tan x.$$

So

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx. \qquad (2)$$

Combining equations (1) *and* (2) *yields*

$$\begin{aligned}
\int \tan^2 x \sec x \, dx &= \int \sec^3 x \, dx - \ln|\sec x \tan x| \\
\int \tan^2 x \sec x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx - \ln|\sec x \tan x| \\
2\int \tan^2 x \sec x \, dx &= \sec x \tan x - \ln|\sec x \tan x| + C \\
\int \tan^2 x \sec x \, dx &= \frac{1}{2}\left(\sec x \tan x - \ln|\sec x \tan x|\right) + C.
\end{aligned}$$

(c) $\displaystyle\int \tan^2 x \sin x \, dx$

Solution:

$$\begin{aligned}
\int \tan^2 x \sin x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx \\
&= \int \frac{1 - \cos^2 x}{\cos^2 x} \sin x \, dx.
\end{aligned}$$

Now we substitute

$$u = \cos x \qquad \Longrightarrow \qquad du = -\sin x \, dx, \quad -du = \sin x \, dx.$$

This gives us that

$$\begin{aligned}\int \tan^2 x \sin x\, dx &= \int \frac{1-u^2}{u^2}(-1)\, du \\ &= \int \frac{u^2-1}{u^2}\, du \\ &= \int \left(1-u^{-2}\right)\, du \\ &= u + \frac{1}{u} + C \\ &= \cos x + \sec x + C.\end{aligned}$$

Problem 4 *Evaluate*

$$\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx.$$

Solution:

$$\begin{aligned}\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx &= \int_{-\pi}^{0} \sqrt{\sin^2 x}\, dx \\ &= \int_{-\pi}^{0} |\sin x|\, dx.\end{aligned}$$

Now, when $-\pi \leq x \leq 0$, $\sin x \leq 0$. *Thus, on this region,* $|\sin x| = -\sin x$. *So we continue*

$$\begin{aligned}\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx &= \int_{-\pi}^{0} -\sin x\, dx \\ &= \Big[\cos x\Big]_{-\pi}^{0} \\ &= \cos(0) - \cos(-\pi) = 1-(-1) = 2.\end{aligned}$$