Section 8.3: Separable Differential Equations

Warm up:

Which of the following differential equations are separable?

(a) $y' = \dfrac{ty}{t^2+1}$,

(b) $\dfrac{dy}{dx} = x^2 \sin(3y) - x^2$,

(c) $y' = t^2 - y$.

Solution: (a) Yes, it is separable. $y' = y \cdot \dfrac{t}{t^2+1}$.

(b) Yes, it is separable. $\dfrac{dy}{dx} = x^2 \left(\sin(3y) - 1\right)$.

(c) No, it is not separable. $t^2 - y$ can not be written in the form $F(t) \cdot G(y)$.

Group work:

Problem 1 *Find a specific solution to the differential equation* $\dfrac{dy}{dx} = x^{-2} \arctan(x)$ *if* $y(1) = 5$.

Solution: *First note that*

$$y = \int x^{-2} \arctan(x)\, dx.$$

To solve this integral, we use integration by parts with

$$u = \arctan(x) \qquad dv = x^{-2}\, dx$$

$$du = \frac{1}{1+x^2}\, dx \qquad v = -\frac{1}{x}.$$

Learning outcomes:

So

$$\int x^{-2} \arctan(x)\, dx = -\frac{1}{x} \arctan(x) + \int \frac{1}{x(1+x^2)}\, dx.$$

To complete this new integral, we use partial fractions.

$$\begin{aligned} \frac{1}{x(1+x^2)} &= \frac{A}{x} + \frac{Bx+C}{1+x^2} \\ \implies \quad 1 &= A(1+x^2) + (Bx+C)x \\ \implies \quad 1 &= (A+B)x^2 + Cx + A. \end{aligned}$$

Comparing coefficients, we see that $A = 1$*,* $C + 0$*, and* $B = -1$*.*

Thus

$$\begin{aligned} y &= \int x^{-2} \arctan(x)\, dx \\ &= -\frac{1}{x} \arctan(x) + \int \frac{1}{x(1+x^2)}\, dx \\ &= -\frac{1}{x} \arctan(x) + \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx \\ &= -\frac{1}{x} \arctan(x) + \ln|x| - \frac{1}{2} \ln(1+x^2) + C. \end{aligned}$$

To finish, we use the initial condition to solve for C*.*

$$\begin{aligned} 5 = y(1) &= -\frac{\pi}{4} + 0 - \frac{1}{2} \ln(2) + C \\ \implies \quad C &= 5 + \frac{\pi}{4} + \frac{1}{2} \ln(2). \end{aligned}$$

Therefore

$$y(t) = -\frac{1}{x} \arctan(x) + \ln|x| - \frac{1}{2} \ln(1+x^2) + 5 + \frac{\pi}{4} + \frac{1}{2} \ln(2).$$

Problem 2 *Find a specific solution to the initial value problem*

$$\frac{dy}{dx} = x^2 \sin(x), \qquad y(0) = 5.$$

Solution: *First, notice that*

$$y = \int x^2 \sin(x)\, dx.$$

To solve this integral, we use integration by parts twice.

$$u = x^2 \qquad dv = \sin(x)\, dx$$

$$du = 2x\,dx \qquad v = -\cos(x).$$

So

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + \int 2x\cos(x)\,dx.$$

Now we use

$$\begin{aligned} u &= 2x & dv &= \cos(x)\,dx \\ du &= 2\,dx & v &= \sin(x). \end{aligned}$$

Then

$$\begin{aligned} y &= \int x^2\sin(x)\,dx \\ &= -x^2\cos(x) + \int 2x\cos(x)\,dx \\ &= -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx \\ &= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C. \end{aligned}$$

Finally, to finish the problem, we solve for C.

$$\begin{aligned} & 5 = y(0) = 0 + 0 + 2 + C \\ \implies \quad & C = 3. \end{aligned}$$

Thus,

$$y(t) = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + 3.$$

Problem 3 *Solve the following differential equations assuming that $y(4) = 5$.*

(a) $y' = x + xy^2$

Solution:

$$\begin{aligned} & y' = x + xy^2 \\ \implies \quad & \frac{dy}{dx} = x(1+y^2) \\ \implies \quad & \frac{dy}{1+y^2} = x\,dx. \end{aligned}$$

*So this equation **is** separable. To solve, we integrate both sides of the equation:*

$$\begin{aligned} & \int \frac{1}{1+y^2}\,dy = \int x\,dx \\ \implies \quad & \arctan(y) = \frac{1}{2}x^2 + C \qquad (1) \\ \implies \quad & y = \tan\left(\frac{1}{2}x^2 + C\right). \end{aligned}$$

*To find C, we plug the initial condition $y(4) = 5$ into equation (**??**) and solve for C.*

$$\begin{aligned} & \arctan(5) = \frac{1}{2}(4)^2 + C = 8 + C \\ \implies \quad & C = \arctan(5) - 8. \end{aligned}$$

So

$$y = \tan\left(\frac{1}{2}x^2 + \arctan(5) - 8\right).$$

(b) $y' = e^{2x-y}$

Solution:

$$\begin{aligned} & y' = e^{2x-y} \\ \implies \quad & \frac{dy}{dx} = \frac{e^{2x}}{e^y} \\ \implies \quad & e^y\,dy = e^{2x}\,dx \end{aligned} \tag{2}$$

and so this ***is*** *a separable equation. To solve, we integrate both sides of equation (**??**).*

$$\begin{aligned} & \int e^y\,dy = \int e^{2x}\,dx \\ \implies \quad & e^y = \frac{1}{2}e^{2x} + C \qquad (3) \\ \implies \quad & y = \ln\left(\frac{1}{2}e^{2x} + C\right). \end{aligned}$$

*To find C, we plug into equation (**??**) and solve for C:*

$$\begin{aligned} & e^5 = \frac{1}{2}e^8 + C \\ \implies \quad & C = e^5 - \frac{1}{2}e^8. \end{aligned}$$

Therefore

$$y = \ln\left(\frac{1}{2}e^{2x} + e^5 - \frac{1}{2}e^8\right).$$