

Section 10.1: Approximating functions with polynomials

Warm up:

Problem 1 Write out the following polynomial without using sigma notation.

$$\sum_{n=0}^4 (n^2 + n + 1)x^n$$

Solution: $(0^2+0+1)+(1^2+1+1)x^1+(2^2+2+1)x^2+(3^2+3+1)x^3+(4^2+4+1)x^4$
 $= 1 + 3x + 7x^2 + 13x^3 + 21x^4$

Group work:

Problem 2 For each of the following, write the given polynomial in summation notation starting with $k = 0$.

(a) $\frac{3x}{2} - \frac{5x^2}{3} + \frac{7x^3}{4} - \frac{9x^4}{5} + \frac{11x^5}{6}$

(b) $\frac{1}{2}x + \frac{1 \cdot 5}{4 \cdot 2!}x^3 + \frac{1 \cdot 5 \cdot 9}{8 \cdot 3!}x^5 - \frac{1 \cdot 5 \cdot 9 \cdot 13}{16 \cdot 4!}x^7$

(c) $(x-1)^3 - \frac{(x-1)^4}{2!} + \frac{(x-1)^5}{4!} - \frac{(x-1)^6}{6!}$

Solution: (a) $\sum_{k=0}^4 (-1)^k (2k+3) \frac{x^{k+1}}{k+2}$.

(b) $\sum_{k=0}^3 \frac{1 \cdot 5 \cdot \dots \cdot (4k+1)}{2^{k+1}(k+1)!} x^{2k+1}$.

Learning outcomes:

Section 10.1: Approximating functions with polynomials

$$(c) \sum_{k=0}^3 \frac{(-1)^k}{(2k)!} (x-1)^{k+3}.$$

Problem 3 Assuming that the function $f(x)$ is infinitely differentiable, and given that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + c_4(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

show that the coefficient c_4 of the $(x-a)^4$ term in the Taylor polynomial is $\frac{f^{(4)}(a)}{4!}$.

Solution: Notice that we have the following:

$$f'(x) = f'(a) + f''(a)(x-a) + \frac{f^{(3)}(a)}{2}(x-a)^2 + 4c_4(x-a)^3 + \frac{f^{(5)}(a)}{4!}(x-a)^4$$

$$f''(x) = f''(a) + f^{(3)}(a)(x-a) + 4 \cdot 3c_4(x-a)^2 + \frac{f^{(5)}(a)}{3!}(x-a)^3$$

$$f^{(3)}(x) = f^{(3)}(a) + 4 \cdot 3 \cdot 2c_4(x-a) + \frac{f^{(5)}(a)}{2}(x-a)^2$$

$$f^{(4)}(x) = 4! \cdot c_4 + f^{(5)}(a)(x-a)$$

$$f^{(4)}(a) = 4! \cdot c_4 + 0$$

$$\implies \boxed{c_4 = \frac{f^{(4)}(a)}{4!}}.$$

Problem 4 Let $f(x) = \sin(2x)$. Find $p_3(x)$ about the point $a = \frac{\pi}{8}$.

Solution: First, note that around a

$$p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3.$$

So we compute

$$f'(x) = 2 \cos(2x) \implies f'\left(\frac{\pi}{8}\right) = \sqrt{2}$$

$$f''(x) = -4 \sin(2x) \implies f''\left(\frac{\pi}{8}\right) = -2\sqrt{2}$$

$$f^{(3)}(x) = -8 \cos(2x) \implies f^{(3)}\left(\frac{\pi}{8}\right) = -4\sqrt{2}.$$

Therefore

$$\boxed{p_3(x) = \frac{\sqrt{2}}{2} + \sqrt{2}\left(x - \frac{\pi}{8}\right) - \sqrt{2}\left(x - \frac{\pi}{8}\right)^2 - \frac{2\sqrt{2}}{3}\left(x - \frac{\pi}{8}\right)^3}.$$

Problem 5 Let $f(x) = xe^{-x}$ on the interval $[-2, 8]$.

- (a) Write the Taylor polynomial $p_4(x)$ around $a = 3$.

$$\text{Fun facts: } f'(x) = -e^{-x}(x - 1)$$

$$f''(x) = e^{-x}(x - 2)$$

$$f^{(3)}(x) = -e^{-x}(x - 3)$$

$$f^{(4)}(x) = e^{-x}(x - 4)$$

- (b) Write $p_4(x)$ about $a = 3$ in summation notation. Also, write the remainder term $R_4(x)$.

Extra Challenge Problems

- (c) Calculate $p_4(4.5)$ and, using $R_4(4.5)$, estimate how close $p_4(4.5)$ is to $f(4.5)$. Do the same for $p_4(1.5)$.
- (d) Use the remainder term $R_4(x)$ to estimate the maximum error for $p_4(x)$ on $[-2, 6]$.
- (e) How large must n be to assure that the n^{th} degree Taylor polynomial for $f(x) = xe^{-x}$ about $a = 3$ approximates $2e^{-2}$ within 10^{-5} ?

Solution: (a)

$$\begin{aligned} p_4(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \frac{f^{(4)}(a)}{4!}(x - a)^4 \\ &= 3e^{-3} - 2e^{-3}(x - 3) + \frac{e^{-3}}{2}(x - 3)^2 - \frac{e^{-3}}{4!}(x - 3)^4. \end{aligned}$$

- (b)

$$p_4(x) = \sum_{k=0}^4 \frac{(-1)^k e^{-3} (3 - k)}{k!} (x - 3)^k.$$

$$R_4(x) = f(x) - p_4(x) = \frac{f^{(5)}(c)}{5!} (x - 3)^5 = \frac{-e^{-c}(c - 5)}{5!} (x - 3)^5$$

for some c between x and 3 .

- (c)

$$p_4(4.5) = 3e^{-3} - 2e^{-3}(1.5) + \frac{e^{-3}}{2}(1.5)^2 - \frac{e^{-3}}{4!}(1.5)^4 \approx 0.0455$$

and

$$R_4(4.5) \leq \left| \max_{c \in [3, 4.5]} f^{(5)}(c) \right| \cdot \frac{(1.5)^5}{5!}.$$

Section 10.1: Approximating functions with polynomials

Now, $f^{(5)}(x) = -e^{-x}(x - 5)$. To see whether this is increasing or decreasing, we compute its derivative $f^{(6)}(x) = e^{-x}(x - 6) < 0$ on $[3, 4.5]$. Thus, $f^{(5)}(x)$ is decreasing on $[3, 4.5]$, and therefore its maximum occurs at $x = 3$. So we compute

$$\begin{aligned} R_4(4.5) &\leq \left| f^{(5)}(3) \right| \cdot \frac{(1.5)^5}{5!} \\ &= |(0.5)e^{-4.5}| \cdot \frac{(1.5)^5}{5!} \\ &\approx 0.00035. \end{aligned}$$

Now we do all of the same work for 1.5.

$$p_4(1.5) = 3e^{-3} - 2e^{-3}(-1.5) + \frac{e^{-3}}{2}(-1.5)^2 - \frac{e^{-3}}{4!}(-1.5)^4 \approx 0.344$$

$$\begin{aligned} R_4(4.5) &\leq \left| \max_{c \in [1.5, 3]} f^{(5)}(c) \right| \cdot \frac{|-1.5|^5}{5!} \\ &= |f^{(5)}(1.5)| \cdot \frac{(1.5)^5}{5!} \quad \text{f is similarly decreasing on } [1.5, 3] \\ &= |3.5e^{-1.5}| \cdot \frac{(1.5)^5}{5!} \\ &\approx 0.04942. \end{aligned}$$

(d)

$$R_4(x) \leq \left| \max_{c \in [-2, 6]} f^{(5)}(c) \right| \cdot \frac{|x - 3|^5}{5!}.$$

From part (c) we know that $f^{(6)}(x) = e^{-x}(x - 6)$. This function is non-positive on $[-2, 6]$, and the only zero is at $x = 6$. So $f^{(5)}(x)$ is decreasing on the entire interval and therefore attains its maximum value at $x = -2$. Thus

$$\begin{aligned} R_4(x) &\leq |f^{(5)}(-2)| \cdot \frac{|-5|^5}{5!} \\ &= 7e^2 \cdot \frac{5^5}{5!} \\ &= 1,346.96335 \end{aligned}$$

(e) First, note that $2e^{-2} = f(2)$. Then we need to find n so that

$$R_n \leq 10^{-5}.$$

Recall that

$$R_n(x) \leq \left| \max_{c \in [2, 3]} f^{(n+1)}(c) \right| \cdot \frac{|2 - 3|^{n+1}}{(n+1)!}.$$

Note that

$$f^{(n+1)}(x) = (-1)^{n+1}e^{-x}(x - (n+1)).$$

Section 10.1: Approximating functions with polynomials

- For n odd, $f^{(n+1)}(x)$ is decreasing and thus the max is at $c = 2$.
- For n even, $f^{(n+1)}(x)$ is increasing and thus the max is at $c = 3$.

So there are two cases:

Case 1: $n + 1$ is odd (and so n is even)

Then the maximum occurs at $c = 2$, and so

$$\begin{aligned} R_n(x) &\leq |(-1)^{n+1}e^{-2}(2 - (n + 1))| \cdot \frac{1}{(n + 1)!} \\ &= \left| \frac{(-1)^{n+1}(2 - n - 1)}{e^2(n + 1)!} \right|. \end{aligned}$$

Now, we want n so that

$$\left| \frac{(-1)^{n+1}(2 - n - 1)}{e^2(n + 1)!} \right| \leq 10^{-5}.$$

We solve for n (using a calculator) to see that $n \geq 7.4$, and so $R_n \leq 10^{-5}$ for $n = 8$.

Case 2: $n + 1$ is even (and so n is odd)

Now the maximum occurs at $c = 3$, and so

$$\begin{aligned} R_n(x) &\leq |(-1)^{n+1}e^{-3}(3 - (n + 1))| \cdot \frac{1}{(n + 1)!} \\ &= \left| \frac{(-1)^{n+1}(3 - n - 1)}{e^3(n + 1)!} \right|. \end{aligned}$$

Now, we want n so that

$$\left| \frac{(-1)^{n+1}(3 - n - 1)}{e^3(n + 1)!} \right| \leq 10^{-5}.$$

We solve for n (using a calculator) to see that $n \geq 6.8$, and so $R_n \leq 10^{-5}$ for $n = 7$.

Conclusion: The error R_n is smaller than 10^{-5} if n is at least 7.