Recitation # 5: Length of Curves & Surface Area

Group work:

Problem 1 *Find the length of the following curves (length is in feet):*

(a) $y = \frac{4}{3}x^{\frac{3}{2}}$ *from* $(0,0)$ *to* $\left(1, \frac{4}{3}\right)$.

Solution:

$$\begin{aligned}
\textit{Arc Length} &= \int_0^1 \sqrt{1 + y'(x)^2}\, dx \\
&= \int_0^1 \sqrt{1 + \left(2x^{\frac{1}{2}}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1 + 4x}\, dx
\end{aligned}$$

$$\begin{aligned}
u &= 1 + 4x \\
du &= 4dx \\
\frac{du}{4} &= dx
\end{aligned}$$

$$\begin{aligned}
u(0) &= 1 + 4(0) = 1 \\
u(1) &= 1 + 4(1) = 5
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\int_1^5 \sqrt{u}\, du \\
&= \frac{1}{4}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^5 \\
&= \frac{1}{4}\left(\frac{2}{3}(5)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right) = \frac{1}{4}\left(\frac{2}{3}(5)^{\frac{3}{2}} - \frac{2}{3}\right)
\end{aligned}$$

Learning outcomes:

(b) $x = \frac{1}{9}e^{3y} + \frac{1}{4}e^{-3y}$ *from* $\left(\frac{13}{36}, 0\right)$ *to* $\left(\frac{265}{288}, \ln 2\right)$.

Solution:

$$
\begin{aligned}
\text{Arc Length} &= \int_0^{\ln 2} \sqrt{1 + x'(y)^2}\, dy \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{3}e^{3y} - \frac{3}{4}e^{-3y}\right)^2}\, dy \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{9}e^{6y} - \frac{1}{2} + \frac{9}{16}e^{-6y}\right)}\, dy \\
&= \int_0^{\ln 2} \sqrt{\frac{1}{9}e^{6y} + \frac{1}{2} + \frac{9}{16}e^{-6y}}\, dy \\
&= \int_0^{\ln 2} \sqrt{\left(\frac{1}{3}e^{3y} + \frac{3}{4}e^{-3y}\right)^2}\, dy \\
&= \int_0^{\ln 2} \left(\frac{1}{3}e^{3y} + \frac{3}{4}e^{-3y}\right) dy \\
&= \left[\frac{1}{9}e^{3y} - \frac{1}{4}e^{-3y}\right]_0^{\ln 2} \\
&\overset{*}{=} \left(\frac{8}{9} - \frac{1}{32}\right) - \left(\frac{1}{9} - \frac{1}{4}\right) \\
&= \frac{7}{9} + \frac{7}{32} = \frac{224 + 63}{288} = \frac{287}{288}.
\end{aligned}
$$

* *Note that*

$$e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$$

and

$$e^{-3\ln 2} = e^{\ln 2^{-3}} = 2^{-3} = \frac{1}{8}.$$

Problem 2 *Find the surface area of the surface generated by revolving the curve given by*

(a) $x = 2y^3$ *from* $(0,0)$ *to* $(2,1)$ *about the* y*-axis.*

Solution: *The formula for the surface area is*

$$\text{Surface Area} = \int_0^1 2\pi f(y)\sqrt{1 + f'(y)^2}\, dy.$$

Since $x = f(y) = 2y^3$, *we know that* $f'(y) = 6y^2$. *Note that*

$$\begin{aligned}\sqrt{1+f'(y)^2}\,dy &= \sqrt{1+(6y^2)^2}\\ &= \sqrt{1+36y^4}\end{aligned}$$

and so

$$\begin{aligned}\textcolor{red}{\text{Surface Area}} &= \int_0^1 2\pi\left(2y^3\right)\left(\sqrt{1+36y^4}\right)\,dy\\ &= \int_0^1 4\pi y^3\sqrt{1+36y^4}\,dy\end{aligned}$$

$$\begin{aligned}u &= 1+36y^4\\ du &= 144y^3dy\\ \frac{du}{144} &= y^3dy\end{aligned}$$

$$\begin{aligned}u(0) &= 1+36(0)^4 = 1\\ u(1) &= 1+36(1)^4) = 37\end{aligned}$$

$$\begin{aligned}&= \frac{4\pi}{144}\int_1^{37}\sqrt{u}\,du\\ &= \frac{4\pi}{144}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{37}\\ &= \frac{4\pi}{144}\left[\left(\frac{2}{3}(37)^{\frac{3}{2}}\right)-\left(\frac{2}{3}(1)^{\frac{3}{2}}\right)\right]\\ &= \frac{(37)^{\frac{3}{2}}-1}{54}\pi\end{aligned}$$

(b) $y = \frac{1}{6}x^3 + \frac{1}{2x}$ *from* $\left(2, \frac{19}{12}\right)$ *to* $\left(3, \frac{14}{3}\right)$ *about the* x*-axis.*

Solution: *The formula for the surface area is*

$$\textcolor{red}{\text{Surface Area}} = \int_2^3 2\pi f(x)\sqrt{1+f'(x)^2}\,dx.$$

Since $y = f(x) = \frac{1}{6}x^3 + \frac{1}{2x}$, *we know that* $f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$. *Note* *that*

$$\begin{aligned}
\sqrt{1+f'(x)^2} &= \sqrt{1+\left(\frac{1}{2}x^2-\frac{1}{2}x^{-2}\right)^2} \\
&= \sqrt{1+\left(\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}x^{-4}\right)} \\
&= \sqrt{\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4}x^{-4}} \\
&= \sqrt{\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)^2} \\
&= \left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)
\end{aligned}$$

and so

$$\begin{aligned}
\text{Surface Area} &= \int_2^3 2\pi\left(\frac{1}{6}x^3+\frac{1}{2}x^{-1}\right)\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)\,dx \\
&= 2\pi\int_2^3\left(\frac{1}{12}x^5+\frac{1}{12}x+\frac{1}{4}x+\frac{1}{4}x^{-3}\right)\,dx \\
&= 2\pi\int_2^3\left(\frac{1}{12}x^5+\frac{1}{3}x+\frac{1}{4}x^{-3}\right)\,dx \\
&= 2\pi\left[\frac{1}{72}x^6+\frac{1}{6}x^2-\frac{1}{8}x^{-2}\right]_2^3 \\
&= 2\pi\left[\left(\frac{81}{8}+\frac{3}{2}-\frac{1}{72}\right)-\left(\frac{8}{9}+\frac{2}{3}-\frac{1}{32}\right)\right] \\
&= 2\pi\left(\frac{2916+432-4-256-192+9}{288}\right) \\
&= \frac{2905\pi}{144}.
\end{aligned}$$

Problem 3 Set up *an integral (or a sum of integrals) to find the perimeter of the region bounded by the curves $y = 2x^2 - 5x + 13$ and $y = x^2 + 6x - 11$.*

Solution: *Let $f(x) = 2x^2 - 5x + 13$ and $g(x) = x^2 + 6x - 11$. We first need to find the points where these two curves intersect. So we solve*

$$\begin{aligned}
f(x) &= g(x) \\
2x^2-5x+13 &= x^2+6x-11 \\
x^2-11x+24 &= 0 \\
(x-3)(x-8) &= 0 \\
x &= 3, 8.
\end{aligned}$$

Then the perimeter is $L_1 + L_2$ *where*

$$L_1 = \int_3^8 \sqrt{1 + f'(x)^2}\, dx = \int_3^8 \sqrt{1 + (4x-5)^2}\, dx$$
$$L_2 = \int_3^8 \sqrt{1 + g'(x)^2}\, dx = \int_3^8 \sqrt{1 + (2x+6)^2}\, dx.$$

Problem 4 *A steady wind blows a kite due west. The kite's height above the ground from horizontal position* $x = 0$ *ft. to* $x = 80$ *ft. is given by*

$$y = 150 - \frac{1}{40}(x-50)^2.$$

Set up the integral to find the distance traveled by the kite.

Solution:

$$\text{distance the kite traveled} = \int_0^{80} \sqrt{1 + y'(x)^2}\, dx$$

where $y(x) = 150 - \frac{1}{40}(x-50)^2$*. Then since* $y' = -\frac{1}{20}(x-50)$*, we have that*

$$\text{distance the kite traveled} = \int_0^{80} \sqrt{1 + \frac{(x-50)^2}{400}}\, dx.$$