Recitation # 7: Exponential Models and Integration By Parts

Group work:

Problem 1 *Vitameatavegamin is a strange substance that comes in two forms. V-I decays at a linear rate, while V-II decays at an exponential rate. Both have the property that 10 ounces will decrease to 7 ounces in 6 hours. For each of V-I and V-II, answer the following:*

(a) *If we started with 80 ounces, how much will there be 6 hours later?*

Solution: ***V-I:*** *Recall that in the linear decay model*

$$y(t) = -k \cdot t + y_0$$

where k denotes the rate of decay and y_0 is the initial amount. We are given that $y_0 = 80oz$. Clearly, we also have that

$$y'(t) = -k.$$

In the linear decay model, the rate of decay does not depend on the initial amount. So from the given information, we have that

$$-k = \frac{10oz - 7oz}{0hr - 6hr} = -\frac{1}{2}.$$

Thus, $y(t) = -\frac{1}{2}t + 80$, and therefore

$$y(6) = -\frac{1}{2}(6) + 80 = 77oz.$$

V-II: *Recall that in the exponential decay model*

$$y(t) = y_0 \cdot e^{-k \cdot t}$$

where again $y_0 = 80oz$ is the initial amount. Also notice that

$$\begin{aligned} y'(t) &= -ky_0e^{-kt} \\ &= -ky(t) \\ \implies \quad y'(0) &= -ky_0. \end{aligned}$$

It is given that it takes 6 hours for 10 ounces to decrease to 7 ounces. In other words, it takes 6 hours

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for 70% of the substance to remain. So we have that

$$\begin{aligned} y(6) &= \frac{7}{10} y_0 \\ \implies \quad y_0 e^{-k\cdot 6} &= \frac{7}{10} y_0 \\ \implies \quad e^{-6k} &= \frac{7}{10} \\ \implies \quad -6k &= \ln\left(\frac{7}{10}\right) = -\ln\left(\frac{10}{7}\right) \\ \implies \quad k &= \frac{1}{6}\ln\left(\frac{10}{7}\right). \end{aligned}$$

Thus,

$$\begin{aligned} y(6) &= 80e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot 6} \\ &= 80e^{-\ln\left(\frac{10}{7}\right)} \\ &= 80\cdot\frac{7}{10} = 56oz. \end{aligned}$$

(b) *How long will it take to decrease from 15 ounces to 7.5 ounces?*

Solution: ***V-I:*** *Recall from above that* $k = \frac{1}{2}$. *Then since* y_0 *is now 15, we have that*

$$y(t) = -\frac{1}{2}t + 15.$$

We want to find t *such that* $y(t) = 7.5$. *So we solve*

$$\begin{aligned} 7.5 &= -\frac{1}{2}t + 15 \\ -\frac{15}{2} &= -\frac{1}{2}t \\ t &= 15 \textit{ hours.} \end{aligned}$$

V-II: *Again, since* y_0 *is now 15, we know from above that*

$$y(t) = 15e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t}.$$

We want to find t *such that* $y(t) = 7.5 = \frac{15}{2}$. *So we solve*

$$\begin{aligned} \frac{15}{2} &= 15e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t} \\ \frac{1}{2} &= e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t} \\ \ln\left(\frac{1}{2}\right) &= -\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t \\ \ln\left(\frac{10}{7}\right)t &= -6\ln\left(\frac{1}{2}\right) = 6\ln 2 \\ t &= \frac{6\ln 2}{\ln\left(\frac{10}{7}\right)} \textit{ hours.} \end{aligned}$$

Problem 2 *Evaluate the following integrals*

(a) $\displaystyle\int_1^3 x^2 5^x \, dx$

Solution: *We proceed via integration by parts. Let*

$$u = x^2, \qquad dv = 5^x \, dx$$

so that

$$du = 2x \, dx, \qquad v = \frac{5^x}{\ln 5}.$$

Recall the formula for integration by parts is

$$\int_a^b u \, dv = \Big[uv\Big]_a^b - \int_a^b v \, du.$$

So we substitute

$$\begin{aligned}\int_1^3 x^2 5^x \, dx &= \left[\frac{x^2 5^x}{\ln(5)}\right]_1^3 - \int_1^3 2x \frac{5^x}{\ln(5)} \, dx \\ &= \frac{1}{\ln(5)} \left(9 \cdot 5^3 - 5\right) - \frac{2}{\ln(5)} \int_1^3 x 5^x \, dx.\end{aligned}$$

For the remaining integral we again use integration by parts:

$$u = x, \qquad dv = 5^x$$

$$du = dx, \qquad v = \frac{5^x}{\ln(5)}.$$

Thus,

$$\begin{aligned}&\frac{1}{\ln(5)} \left(9 \cdot 5^3 - 5\right) - \frac{2}{\ln(5)} \int_1^3 x 5^x \, dx \\ &= \frac{5}{\ln(5)}(225 - 1) - \frac{2}{\ln(5)} \left(\left[\frac{x 5^x}{\ln(5)}\right]_1^3 - \int_1^3 \frac{5^x}{\ln(5)} \, dx \right) \\ &= \frac{1120}{\ln(5)} - \frac{2}{\ln^2(5)} \left((3 \cdot 5^3 - 5) - \left[\frac{5^x}{\ln(5)}\right]_1^3 \right) \\ &= \frac{1}{\ln(5)} \left(1120 - \frac{740}{\ln(5)} + \frac{248}{\ln^2(5)} \right).\end{aligned}$$

(b) $\displaystyle\int \arcsin(x) \, dx$

Solution: *We do not know how to integrate* $\arcsin(x)$*, but we do know how to differentiate it, so we will use integration by parts.*

$$u = arcsin(x) \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1 - x^2}} \, dx \qquad v = x$$

This gives us

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \\ -\frac{1}{2}\,du &= \,dx \end{aligned}$$

$$\begin{aligned} \int \frac{x\,dx}{\sqrt{1-x^2}} &= \int \frac{-\,du}{2\sqrt{u}} \\ &= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du \\ &= -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + C \\ &= -\sqrt{1-x^2} + C \\ \int \arcsin(x)\,dx &= x\arcsin(x) - \int \frac{x\,dx}{\sqrt{1-x^2}} \\ &= x\arcsin(x) - \left(-\sqrt{1-x^2}\right) + C \\ &= x\arcsin(x) + \sqrt{1-x^2} + C \end{aligned}$$

(c) $\int x^{\frac{5}{3}}(\ln x)^2\,dx$

Solution: *We begin with the substitution*

$$w = \ln x \qquad \Longrightarrow \qquad dw = \frac{1}{x}\,dx, \quad x = e^w.$$

Then

$$\begin{aligned} \int x^{\frac{5}{3}}(\ln x)^2\,dx &= \int x^{\frac{8}{3}}(\ln x)^2 \cdot \frac{1}{x}\,dx \\ &= \int (e^w)^{\frac{8}{3}} w^2\,dw \\ &= \int w^2 e^{\frac{8}{3}w}\,dw. \end{aligned}$$

We now use integration by parts, with

$$\begin{aligned} u &= w^2 & dv &= e^{\frac{8}{3}w}\,dw \\ du &= 2w\,dw & v &= \frac{3}{8}e^{\frac{8}{3}w}. \end{aligned}$$

This gives us

$$\begin{aligned} \int x^{\frac{5}{3}}(\ln x)^2\,dx &= \frac{3}{8}w^2 e^{\frac{8}{3}w} - \int \frac{3}{8}(2w)e^{\frac{8}{3}w}\,dw \\ &= \frac{3}{8}w^2 e^{\frac{8}{3}w} - \frac{3}{4}\int w e^{\frac{8}{3}w}\,dw. \end{aligned}$$

We apply integration by parts one last time with

$$u = w \qquad dv = e^{\frac{8}{3}w}\,dw$$

$$du = \,dw \qquad v = \frac{3}{8}e^{\frac{8}{3}w}$$

which yields

$$\begin{aligned}
\int x^{\frac{5}{3}}(\ln x)^2\,dx &= \frac{3}{8}w^2e^{\frac{8}{3}w} - \frac{3}{4}\left(\frac{3}{8}we^{\frac{8}{3}w} - \frac{3}{8}\int e^{\frac{8}{3}w}\,dw\right)\\
&= \frac{3}{8}w^2e^{\frac{8}{3}w} - \frac{9}{32}we^{\frac{8}{3}w} + \frac{27}{256}e^{\frac{8}{3}w} + C\\
&= \frac{3}{8}e^{\frac{8}{3}\ln x}\left((\ln x)^2 - \frac{3}{4}\ln x + \frac{9}{32}\right) + C\\
&= \frac{3}{8}x^{\frac{8}{3}}\left((\ln x)^2 - \frac{3}{4}\ln x + \frac{9}{32}\right) + C.
\end{aligned}$$

Problem 3 *Evaluate the following integral*

$$\int \sin(3x)e^{7x}\,dx$$

Solution: *We begin by letting* $I = \int \sin(3x)e^{7x}\,dx$. *We then use integration by parts with*

$$u = e^{7x} \qquad dv = \sin(3x)\,dx$$

$$du = 7e^{7x}\,dx \qquad v = -\frac{1}{3}\cos(3x).$$

Then

$$\begin{aligned}
\int \sin(3x)e^{7x}\,dx = I &= -\frac{1}{3}e^{7x}\cos(3x) - \int -\frac{1}{3}(7e^{7x})\cos(3x)\,dx\\
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{3}\int e^{7x}\cos(3x)\,dx.
\end{aligned}$$

We then apply integration by parts again, this time with

$$u = e^{7x} \qquad dv = \cos(3x)\,dx$$

$$du = 7e^{7x}\,dx \qquad v = \frac{1}{3}\sin(3x).$$

This gives us

$$\begin{aligned}
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{3}\left[\frac{1}{3}e^{7x}\sin(3x) - \int \frac{1}{3}(7e^{7x})\sin(3x)\,dx\right]\\
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) - \frac{49}{9}\int e^{7x}\sin(3x)\,dx\\
I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) - \frac{49}{9}I\\
\frac{58}{9}I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x)\\
I &= \frac{9}{58}\left(-\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x)\right) + C.
\end{aligned}$$

Problem 4 *Evaluate the following integrals*

(a) $\int x^5 \cos\left(x^3\right)\, dx$

Solution: *We begin with the substitution*

$$w = x^3 \qquad \Longrightarrow \qquad dw = 3x^2\, dx, \quad \frac{1}{3}\, dw = x^2\, dx.$$

Then,

$$\begin{aligned}
\int x^5 \cos\left(x^3\right)\, dx &= \int x^3 \cos\left(x^3\right) \cdot x^2\, dx \\
&= \int w \cos(w) \cdot \frac{1}{3}\, dw \\
&= \frac{1}{3} \int w \cos(w)\, dw.
\end{aligned}$$

We then use integration by parts, with

$$u = w \qquad dv = \cos(w)\, dw$$

$$du = dw \qquad v = \sin(w)$$

which yields

$$\begin{aligned}
\int x^5 \cos\left(x^3\right)\, dx &= \frac{1}{3}\left(w \sin(w) - \int \sin(w)\, dw\right) \\
&= \frac{1}{3}\left(w \sin(w) + \cos(w)\right) + C \\
&= \frac{1}{3}\left(x^3 \sin\left(x^3\right) + \cos\left(x^3\right)\right) + C.
\end{aligned}$$

(b) $\int \cos\left(\sqrt{x}\right)\, dx$

Solution: *We begin with the subsitution*

$$w = \sqrt{x} \qquad \Longrightarrow \qquad dw = \frac{1}{2\sqrt{w}}\, dw, \quad 2\, dw = \frac{1}{\sqrt{x}}\, dw.$$

Then

$$\begin{aligned}
\int \cos\left(\sqrt{x}\right)\, dx &= \int \cos\left(\sqrt{x}\right) \cdot \frac{\sqrt{x}}{\sqrt{x}}\, dx \\
&= 2 \int w \cos(w)\, dw \\
&= 2(w \sin(w) + \cos(w)) + C \qquad \textit{From part (a)} \\
&= 2\left(\sqrt{x} \sin\left(\sqrt{x}\right) + \cos\left(\sqrt{x}\right)\right) + C.
\end{aligned}$$

(c) $\int x \cos x \sin x\, dx$

Solution: *First, recall that*

$$\sin(2x) = 2\sin x\cos x \qquad \Longrightarrow \qquad \sin x\cos x = \frac{1}{2}\sin(2x).$$

So we can rewrite the given integral as

$$\int x\cos x\sin x\,dx = \frac{1}{2}\int x\sin(2x)\,dx.$$

Now we use integration by parts with

$$u = x \qquad dv = \sin(2x)\,dx$$

$$du = dx \qquad v = -\frac{1}{2}\cos(2x).$$

This gives us that

$$\begin{aligned}\frac{1}{2}\int x\sin(2x)\,dx &= \frac{1}{2}\left(-\frac{1}{2}x\cos(2x) - \int -\frac{1}{2}\cos(2x)\,dx\right)\\ &= \frac{1}{4}\left(-x\cos(2x) + \frac{1}{2}\sin(2x)\right) + C.\end{aligned}$$