Section 10.3: Taylor series

Warm up:

Find the Taylor series for:

(a) $27x^2 - 3x + 17$ centered at $a = 0$.

(b) $27x^2 - 3x + 17$ centered at $a = 1$.

Solution: (a) $27x^2 - 3x + 17$ is already a Taylor Series centered at 0 because it is already in the form $\sum_{k=0}^{\infty} c_k x^k$ with $c_0 = 17, c_1 = -3, c_2 = 27$, and the rest of the $c_k = 0$.

(b) Let $f(x) = 27x^2 - 3x + 17$. Then

$$
\begin{aligned}
&f(1) = 27 - 3 + 17 = 41 \\
&f'(x) = 54x - 3 \quad \Longrightarrow \quad f'(1) = 54 - 3 = 51 \\
&f''(x) = 54 \quad \Longrightarrow \quad f''(1) = 54 \\
&f^{(3)}(x) = 0 \quad \Longrightarrow \quad f^{(3)}(1) = 0 \\
&\quad \vdots \qquad\qquad\qquad \vdots \\
&f^{(n)}(x) = 0 \quad \Longrightarrow \quad f^{(n)}(1) = 0.
\end{aligned}
$$

So

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \boxed{41 + 51(x-1) + \frac{54}{2!}(x-1)^2}$$

Lastly, note that if you multiply this out then you will get back the original polynomial.

Group work:

Problem 1 *Find a Maclaurin series (and interval of convergence) for*

$$f(x) = x^3 \sin(x^5)$$

Learning outcomes:

Solution: *We already know that*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

with interval of convergence $(-\infty, \infty)$*. So we use this to compute*

$$\begin{aligned} x^3 \sin(x^5) &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k (x^5)^{2k+1}}{(2k+1)!} \\ &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k x^{10k+5}}{(2k+1)!} \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{10k+8}}{(2k+1)!}} \end{aligned}$$

with interval of convergence $(-\infty, \infty)$*.*

Problem 2 *Find the first four non-zero terms of the Maclaurin Series for*

$$xe^{x^2} + \cos(x^3)$$

Solution: *We might be tempted to solve this problem using the definition of a Taylor Series since we only need the first four terms, but that is a bad idea because the derivatives will get very messy. Instead, we will use the known Maclaurin Series for* e^x *and* $\cos(x)$*.*

We already know that

$$e^x = \sum_{k=0}^{\infty} \frac{(x)^k}{k!}$$

with interval of convergence $(-\infty, \infty)$*. So we use this to compute*

$$\begin{aligned} xe^{x^2} &= x \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} \\ &= x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} \end{aligned}$$

with interval of convergence $(-\infty, \infty)$*.*

Similarly, we already know that

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

with interval of convergence $(-\infty, \infty)$. *So we use this to compute*

$$\begin{aligned}\cos(x^3) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x^3)^{2k}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k}}{(2k)!}\end{aligned}$$

with interval of convergence $(-\infty, \infty)$.

Therefore,

$$xe^{x^2} + \cos(x^3) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k}}{(2k)!} \quad = \sum_{k=0}^{\infty} \left(\frac{x^{2k+1}}{k!} + \frac{(-1)^k x^{6k}}{(2k)!} \right)$$

Now, we just need to plug in numbers for k, starting with k=0, until we have four non-zero terms. Plugging in 0 through 3 gives:

$$\begin{aligned} &\left(\frac{x^{2(0)+1}}{(0)!} + \frac{(-1)^{(0)} x^{6(0)}}{(2(0))!} \right) \\ +&\left(\frac{x^{2(1)+1}}{(1)!} + \frac{(-1)^{(1)} x^{6(1)}}{(2(1))!} \right) \\ +&\left(\frac{x^{2(2)+1}}{(2)!} + \frac{(-1)^{(2)} x^{6(2)}}{(2(2))!} \right) \\ +&\left(\frac{x^{2(3)+1}}{(3)!} + \frac{(-1)^{(3)} x^{6(3)}}{(2(3))!} \right) \end{aligned}$$

Now, notice that you actually have more than four terms here because there are more than four powers of x. *We will have to simplify thisa expression to get the first four terms.*

$x + 1 + x^3 + \frac{-x^6}{2} + \frac{x^5}{2} + \frac{x^{12}}{4} + \frac{x^7}{6} + \frac{-x^{18}}{12}$

so the first four terms will be

$\boxed{1 + x + x^3 + \frac{x^5}{2}}$

Problem 3 *Find a function (closed expression) for the following series and the interval on which the function and the series are equal.*

$$x + x^4 + \frac{1}{2}x^7 + \frac{1}{6}x^{10} + \frac{1}{24}x^{13} + \dots$$

Solution:

$$\begin{aligned} x + x^4 + \frac{1}{2}x^7 + \frac{1}{6}x^{10} + \frac{1}{24}x^{13} + \dots &= x + x^4 + \frac{1}{2!}x^7 + \frac{1}{3!}x^{10} + \frac{1}{4!}x^{13} + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}x^{3k+1} \\ &= x\sum_{k=0}^{\infty} \frac{x^{3k}}{k!} \\ &= x\sum_{k=0}^{\infty} \frac{(x^3)^k}{k!} \\ &= \boxed{xe^{x^3}} \end{aligned}$$

which has interval of convergence $(-\infty, \infty)$.

Problem 4 *Compute the sum of the following series (Hint: You should use Taylor series.)*

(a) $1 - \ln 2 + \dfrac{(\ln 2)^2}{2!} - \dfrac{(\ln 2)^3}{3!} + \dots$

(b) $3 + \dfrac{9}{2!} + \dfrac{27}{3!} + \dfrac{81}{4!} + \dots$

Solution: (a)

$$\begin{aligned} 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots &= \sum_{k=0}^{\infty} \frac{(-\ln 2)^k}{k!} \\ &= e^{-\ln 2} = e^{\ln 2^{-1}} \\ &= 2^{-1} = \boxed{\frac{1}{2}}. \end{aligned}$$

(b)

$$\begin{aligned} 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \ldots &= \sum_{k=1}^{\infty} \frac{3^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{3^k}{k!} - \frac{3^0}{0!} \\ &= \boxed{e^3 - 1}. \end{aligned}$$

Problem 5 *Find the Taylor Series for* $\sin(2x)$ *about* $a = \frac{\pi}{8}$.
Hint: Recall from a previous recitation that
$p_3(x) = \frac{\sqrt{2}}{2} + \sqrt{2}\left(x - \frac{\pi}{8}\right) - \sqrt{2}\left(x - \frac{\pi}{8}\right)^2 - \frac{2\sqrt{2}}{3}\left(x - \frac{\pi}{8}\right)^3$

Solution: *Let* $f(x) = \sin(2x)$. *Then*

$$f\left(\frac{\pi}{8}\right) = \sin\left(\frac{2\pi}{8}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = 2\cos(2x) \implies f'\left(\frac{\pi}{8}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$f''(x) = -4\sin(2x) \implies f''\left(\frac{\pi}{8}\right) = -4 \cdot \frac{\sqrt{2}}{2} = -2\sqrt{2}$$

$$f^{(3)}(x) = -8\cos(2x) \implies f^{(3)}\left(\frac{\pi}{8}\right) = -8 \cdot \frac{\sqrt{2}}{2} = -4\sqrt{2}$$

$$f^{(4)}(x) = 16\cos(2x) \implies f^{(4)}\left(\frac{\pi}{8}\right) = 16 \cdot \frac{\sqrt{2}}{2} = 8\sqrt{2}.$$

Continuing this pattern, we see that

$$f^{(k)}\left(\frac{\pi}{8}\right) = (-1)^{\left\lceil \frac{k}{2} \right\rceil} 2^{k-1}\sqrt{2}$$

where $\left\lceil \frac{k}{2} \right\rceil$ *denotes the smallest integer greater than* $\frac{k}{2}$. *So, for example,* $\left\lceil \frac{1}{2} \right\rceil = 1$, $\left\lceil \frac{2}{2} \right\rceil = 1$, $\left\lceil \frac{3}{2} \right\rceil = 2$, *and so on.*

So from here we have that the Taylor series for $f(x)$ *is*

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^{\left\lceil \frac{k}{2} \right\rceil} 2^{k-1}\sqrt{2}}{k!} \left(x - \frac{\pi}{8}\right)^k}$$