Recitation #19: Approximating functions with polynomials

Warm up:

Write out the following polynomial without using sigma notation.

$$\sum_{k=0}^{4}(n^2+n+1)x^n$$

Solution: $(0^2+0+1)+(1^2+1+1)x^1+(2^2+2+1)x^2+(3^2+3+1)x^3+(4^2+4+1)x^4$
$= 1+3x+7x^2+13x^2+21x^4$

Group work:

Problem 1 *For each of the following, write the given polynomial in summation notation starting with $k=0$.*

(a) $\dfrac{3x}{2}-\dfrac{5x^2}{3}+\dfrac{7x^3}{4}-\dfrac{9x^4}{5}+\dfrac{11x^5}{6}$

(b) $\dfrac{1}{2}x+\dfrac{1\cdot 5}{4\cdot 2!}x^3+\dfrac{1\cdot 5\cdot 9}{8\cdot 3!}x^5-\dfrac{1\cdot 5\cdot 9\cdot 13}{16\cdot 4!}x^7$

(c) $(x-1)^3-\dfrac{(x-1)^4}{2!}+\dfrac{(x-1)^5}{4!}-\dfrac{(x-1)^6}{6!}$

Solution: (a) $\displaystyle\sum_{k=0}^{4}(-1)^k(2k+3)\frac{x^{k+1}}{k+2}$.

(b) $\displaystyle\sum_{k=0}^{3}\frac{1\cdot 5\cdot\ldots\cdot(4k+1)}{2^{k+1}(k+1)!}x^{2k+1}$.

(c) $\displaystyle\sum_{k=0}^{3}\frac{(-1)^k}{(2k)!}(x-1)^{k+3}$.

Learning outcomes:

Problem 2 *Assuming that the function $f(x)$ is infinitely differentiable, and given that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + c_4(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

show that the coefficient c_4 of the $(x-a)^4$ term in the Taylor polynomial is $\dfrac{f^{(4)}(a)}{4!}$.

Solution: *Notice that we have the following:*

$$\begin{aligned}
f'(x) &= f'(a) + f''(a)(x-a) + \frac{f^{(3)}(a)}{2}(x-a)^2 + 4c_4(x-a)^3 + \frac{f^{(5)}(a)}{4!}(x-a)^4 \\
f''(x) &= f''(a) + f^{(3)}(a)(x-a) + 4\cdot 3c_4(x-a)^2 + \frac{f^{(5)}(a)}{3!}(x-a)^3 \\
f^{(3)}(x) &= f^{(3)}(a) + 4\cdot 3\cdot 2c_4(x-a) + \frac{f^{(5)}(a)}{2}(x-a)^2 \\
f^{(4)}(x) &= 4!\cdot c_4 + f^{(5)}(a)(x-a) \\
f^{(4)}(a) &= 4!\cdot c_4 + 0 \\
\implies & \quad \boxed{c_4 = \frac{f^{(4)}(a)}{4!}}.
\end{aligned}$$

Problem 3 *Let $f(x) = \sin(2x)$. Find $p_3(x)$ about the point $a = \dfrac{\pi}{8}$.*

Solution: *First, note that around a*

$$p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3.$$

So we compute

$$\begin{aligned}
f'(x) &= 2\cos(2x) &&\implies f'\left(\frac{\pi}{8}\right) = \sqrt{2} \\
f''(x) &= -4\sin(2x) &&\implies f''\left(\frac{\pi}{8}\right) = -2\sqrt{2} \\
f^{(3)}(x) &= -8\cos(2x) &&\implies f^{(3)}\left(\frac{\pi}{8}\right) = -4\sqrt{2}.
\end{aligned}$$

Therefore

$$\boxed{p_3(x) = \frac{\sqrt{2}}{2} + \sqrt{2}\left(x - \frac{\pi}{8}\right) - \sqrt{2}\left(x - \frac{\pi}{8}\right)^2 - \frac{2\sqrt{2}}{3}\left(x - \frac{\pi}{8}\right)^3}.$$

Problem 4 *Let $f(x) = xe^{-x}$ on the interval $[-2, 8]$.*

(a) *Write the Taylor polynomial $p_4(x)$ around $a = 3$.*

$$\begin{aligned}\text{Fun facts: } f'(x) &= -e^{-x}(x-1)\\ f''(x) &= e^{-x}(x-2)\\ f^{(3)}(x) &= -e^{-x}(x-3)\\ f^{(4)}(x) &= e^{-x}(x-4)\end{aligned}$$

(b) *Write $p_4(x)$ about $a = 3$ in summation notation. Also, write the remainder term $R_4(x)$.*

Extra Challenge Problems

(c) *Calculate $p_4(4.5)$ and, using $R_4(4.5)$, estimate how close $p_4(4.5)$ is to $f(4.5)$. Do the same for $p_4(1.5)$.*

(d) *Use the remainder term $R_4(x)$ to estimate the maximum error for $p_4(x)$ on $[-2, 6]$.*

(e) *How large must n be to assure that the n^{th} degree Taylor polynomial for $f(x) = xe^{-x}$ about $a = 3$ approximates $2e^{-2}$ within 10^{-5}?*

Solution: (a)

$$\begin{aligned}p_4(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}}{3!}(x-a)^3 + \frac{f^{(4)}}{4!}(x-a)^4\\ &= 3e^{-3} - 2e^{-3}(x-3) + \frac{e^{-3}}{2}(x-3)^2 - \frac{e^{-3}}{4!}(x-3)^4.\end{aligned}$$

(b)

$$p_4(x) = \sum_{k=0}^{4} \frac{(-1)^k e^{-3}(3-k)}{k!}(x-3)^k.$$

$$R_4(x) = f(x) - p_4(x) = \frac{f^{(5)}(c)}{5!}(x-3)^5 = \frac{-e^{-c}(c-5)}{5!}(x-3)^5$$

for some c between x and 3.

(c)

$$p_4(4.5) = 3e^{-3} - 2e^{-3}(1.5) + \frac{e^{-3}}{2}(1.5)^2 - \frac{e^{-3}}{4!}(1.5)^4 \approx 0.0455$$

and

$$R_4(4.5) \le \left| \max_{c\in[3,4.5]} f^{(5)}(c) \right| \cdot \frac{(1.5)^5}{5!}.$$

Now, $f^{(5)}(x) = -e^{-x}(x-5)$. To see whether this is increasing or decreasing, we compute its derivative $f^{(6)}(x) = e^{-x}(x-6) < 0$ on $[3, 4.5]$. Thus, $f^{(5)}(x)$ is decreasing on $[3, 4.5]$, and therefore its maximum occurs at $x = 3$. So we compute

$$\begin{aligned} R_4(4.5) \leq & \left| f^{(5)}(3) \right| \cdot \frac{(1.5)^5}{5!} \\ &= |(0.5)e^{-4.5}| \cdot \frac{(1.5)^5}{5!} \\ &\approx 0.00035. \end{aligned}$$

Now we do all of the same work for 1.5.

$$p_4(1.5) = 3e^{-3} - 2e^{-3}(-1.5) + \frac{e^{-3}}{2}(-1.5)^2 - \frac{e^{-3}}{4!}(-1.5)^4 \approx 0.344$$

$$\begin{aligned} R_4(4.5) \leq & \left| \max_{c \in [1.5,3]} f^{(5)}(c) \right| \cdot \frac{|-1.5|^5}{5!} \\ &= |f^{(5)}(1.5)| \cdot \frac{(1.5)^5}{5!} \qquad \textit{f is similarly decreasing on [1.5,3]} \\ &= |3.5e^{-1.5}| \cdot \frac{(1.5)^5}{5!} \\ &\approx 0.04942. \end{aligned}$$

(d)

$$R_4(x) \leq \left| \max_{c \in [-2,6]} f^{(5)}(c) \right| \cdot \frac{|x-3|^5}{5!}.$$

From part (c) we know that $f^{(6)}(x) = e^{-x}(x-6)$. This function is nonpositive on $[-2, 6]$, and the only zero is at $x = 6$. So $f^{(5)}(x)$ is decreasing on the entire interval and therefore attains its maximum value at $x = -2$. Thus

$$\begin{aligned} R_4(x) &\leq |f^{(5)}(-2)| \cdot \frac{|-5|^5}{5!} \\ &= 7e^2 \cdot \frac{5^5}{5!} \\ &= 1{,}346.96335 \end{aligned}$$

(e) *First, note that $2e^{-2} = f(2)$. Then we need to find n so that*

$$R_n \leq 10^{-5}.$$

Recall that

$$R_n(x) \leq \left| \max_{c \in [2,3]} f^{(n+1)}(c) \right| \cdot \frac{|2-3|^{n+1}}{(n+1)!}.$$

Note that

$$f^{(n+1)}(x) = (-1)^{n+1} e^{-x}(x - (n+1)).$$

- *For n odd, $f^{(n+1)}(x)$ is decreasing and thus the max is at $c = 2$.*
- *For n even, $f^{(n+1)}(x)$ is increasing and thus the max is at $c = 3$.*

So there are two cases:

Case 1: $n + 1$ is odd (and so n is even)
Then the maximum occurs at $c = 2$, and so

$$\begin{aligned} R_n(x) &\leq |(-1)^{n+1}e^{-2}(2-(n+1))| \cdot \frac{1}{(n+1)!} \\ &= \left|\frac{(-1)^{n+1}(2-n-1)}{e^2(n+1)!}\right|. \end{aligned}$$

Now, we want n so that

$$\left|\frac{(-1)^{n+1}(2-n-1)}{e^2(n+1)!}\right| \leq 10^{-5}.$$

We solve for n (using a calculator) to see that $n \geq 7.4$, and so $R_n \leq 10^{-5}$ for $n = 8$.

Case 2: $n + 1$ is even (and so n is odd)
Now the maximum occurs at $c = 3$, and so

$$\begin{aligned} R_n(x) &\leq |(-1)^{n+1}e^{-3}(3-(n+1))| \cdot \frac{1}{(n+1)!} \\ &= \left|\frac{(-1)^{n+1}(3-n-1)}{e^3(n+1)!}\right|. \end{aligned}$$

Now, we want n so that

$$\left|\frac{(-1)^{n+1}(3-n-1)}{e^3(n+1)!}\right| \leq 10^{-5}.$$

We solve for n (using a calculator) to see that $n \geq 6.8$, and so $R_n \leq 10^{-5}$ for $n = 7$.

Conclusion: The error R_n is smaller than 10^{-5} *if n is at least 7.*