Recitation # 11: Partial fractions and Improper Integrals - Solutions

Warm up:

True or False: It is possible for a region to be infinitely long but have a finite area.

Solution: True. Consider the region below the curve $y = \dfrac{1}{x^2}$, $x \geq 1$.

Group work:

Problem 1 *Without determining the coefficients, write the partial fraction decomposition of the following rational function:*

$$\frac{5x^{13} - 6x^{12} + 7x^3 - 5x - 18}{(2x-3)(5x+9)^3(x^2+9x+19)(x^2+9x+21)^2}$$

Solution: *The degree of the numerator is 13, whereas the degree of the denominator is 10. So if we perform long division, we will get a degree $13 - 10 = 3$ polynomial plus partial fractions for the remainder term:*

$$\begin{aligned}
&\frac{5x^{13} - 6x^{12} + 7x^3 - 5x - 18}{(2x-3)(5x+9)^3(x^2+9x+19)(x^2+9x+21)^2} = Ax^3 + Bx^2 + Cx + D \\
&+ \frac{E}{2x-3} + \frac{F}{5x+9} + \frac{G}{(5x+9)^2} + \frac{H}{(5x+9)^3} + \frac{I}{x-i_1} + \frac{J}{x-i_2} \\
&+ \frac{Kx+L}{x^2+9x+21} + \frac{Mx+N}{(x^2+9x+21)^2}.
\end{aligned}$$

Explanation of i_1 and i_2: *The quadratic $x^2 + 9x + 19$ can be factored over the real numbers, since the discriminant $b^2 - 4ac = 81 - 76 > 0$. The numbers i_1 and i_2 are the two real roots to this polynomial, ie*

$$i_1 = \frac{-9+\sqrt{5}}{2} \qquad i_2 = \frac{-9-\sqrt{5}}{2}.$$

Learning outcomes:

Note that the polynomial $x^2 + 9x + 21$ is irreducible (over the real numbers) since its discriminant is less than 0.

Problem 2 *Evaluate:*

$$\int \frac{7x^3 + 18x + 9}{x^4 + 9x^2}\,dx$$

Hint: If $f(x) = 7x^3 + 18x + 9$, then $f(2) = 101$, $f(1) = 34$, and $f(-1) = -16$.

Solution: *First factor the denominator*

$$x^4 + 9x^2 = x^2(x^2 + 9).$$

The we can decompose the integrand as a partial fraction

$$\frac{7x^3 + 18x + 9}{x^2(x^2 + 9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9}$$

$$\begin{aligned} \implies \quad 7x^3 + 18x + 9 &= Ax(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2 \\ &= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2 \\ &= (A + C)x^3 + (B + D)x^2 + 9Ax + 9B. \end{aligned}$$

By equating coefficients for powers of x we have that

$$\begin{aligned} &9 = 9B &&\implies && B = 1 \\ &18 = 9A &&\implies && A = 2 \\ &0 = B + D &&\implies && 0 = 1 + D &&\implies && D = -1 \\ &7 = A + C &&\implies && 7 = 2 + C &&\implies && C = 5. \end{aligned}$$

Thus

$$\begin{aligned} \int \frac{7x^3 + 18x + 9}{x^4 + 9x^2}\,dx &= \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{5x - 1}{x^2 + 9} \right) dx \\ &= 2\ln|x| - \frac{1}{x} + 5\int \frac{x}{x^2 + 9}\,dx - \int \frac{1}{x^2 + 9}\,dx \\ &= 2\ln|x| - \frac{1}{x} + \frac{5}{2}\ln(x^2 + 9) - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C. \end{aligned}$$

Note that, in the previous step, we substituted $u = x^2 + 9$ for the first integral and $u = \dfrac{x}{3}$ in the second integral.

Problem 3 *Review of limits:*

(a) $\displaystyle\lim_{x\to-\infty}\left(3x^{-6}+e^{5x}+\frac{\sin x}{x^2+3}\right)$

Solution: *Recall that the limit of a sum is the sum of the limits, provided that those limits exist.*

- $\displaystyle\lim_{x\to-\infty}3x^{-6}=\lim_{x\to-\infty}\frac{3}{x^6}=0.$
- $\displaystyle\lim_{x\to-\infty}e^{5x}=0.$
- $\displaystyle\lim_{x\to-\infty}\frac{\sin x}{x^2+3}=0$

  To rigorously prove this, you need to use the squeeze theorem.

Thus,

$$\lim_{x\to-\infty}\left(3x^{-6}+e^{5x}+\frac{\sin x}{x^2+3}\right)=0$$

(b) $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+4}}$

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+4}}&=\lim_{x\to\infty}\frac{x}{\sqrt{x^2}\cdot\sqrt{9+\frac{4}{x^2}}}\\&=\lim_{x\to\infty}\frac{x}{|x|\cdot\sqrt{9+\frac{4}{x^2}}}\\&=\lim_{x\to\infty}\frac{x}{x\cdot\sqrt{9+\frac{4}{x^2}}}\\&=\lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{4}{x^2}}}\\&=\frac{1}{\sqrt{9+0}}=\frac{1}{3}.\end{aligned}$$

(c) $\displaystyle\lim_{x\to-\infty}\arctan x$

Solution: $\displaystyle\lim_{x\to-\infty}\arctan x=-\frac{\pi}{2}.$

Problem 4 *In each of the following, determine if the given integral converges or diverges. If it converges, find the value.*

(a) $\displaystyle\int_{-1}^{\infty} \frac{3}{2x+1}\,dx$

Solution: *The function* $\frac{3}{2x+1}$ *has a vertical asymptote at* $x = -\frac{1}{2}$. *So we rewrite the original integral as*

$$\int_{-1}^{\infty} \frac{3}{2x+1}\,dx = \lim_{a\to -\frac{1}{2}^-} \int_{-1}^{a} \frac{3}{2x+1}\,dx + \lim_{b\to -\frac{1}{2}^+} \int_{b}^{0} \frac{3}{2x+1}\,dx + \lim_{c\to\infty} \int_{0}^{c} \frac{3}{2x+1}\,dx.$$

The latter integral does not exist. To see this, just note that

$$\begin{aligned}\lim_{c\to\infty} \int_0^c \frac{3}{2x+1}\,dx &= \lim_{c\to\infty} \left[\frac{3}{2}\ln|2x+1|\right]_0^c \\ &= \lim_{c\to\infty} \frac{3}{2}\ln|2c+1| = \infty.\end{aligned}$$

Therefore,

$$\int_{-1}^{\infty} \frac{3}{2x+1}\,dx \quad \textit{diverges.}$$

(b) $\displaystyle\int_{-\infty}^{\infty} xe^{-x}\,dx$

Solution:

$$\int_{-\infty}^{\infty} xe^{-x}\,dx = \lim_{a\to-\infty} \int_a^0 xe^{-x}\,dx + \lim_{b\to\infty} \int_0^b xe^{-x}\,dx.$$

The first of these integrals does not exist. To see this, we apply integration by parts

$$\begin{aligned}\lim_{a\to-\infty} \int_a^0 xe^{-x}\,dx &= \lim_{a\to-\infty} \left(\Big[-xe^{-x}\Big]_a^0 + \int_a^0 e^{-x}\,dx\right) \\ &= \lim_{a\to-\infty} \left(\big[0 + ae^{-a}\big] + \Big[-e^{-x}\Big]_a^0\right) \\ &= \lim_{a\to-\infty} \left(ae^{-a} + e^{-a} - 1\right) \\ &= \lim_{a\to-\infty} \left(e^{-a}(a+1) - 1\right) \\ &= -\infty.\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} xe^{-x}\,dx \quad \textit{diverges.}$$

(c) $\displaystyle\int_{6}^{\infty} \frac{2-4x}{2x^2-13x+20}\,dx$

Solution: *First notice that we can factor the denominator as*

$$2x^2 - 13x + 20 = (2x-5)(x-4).$$

So we find a partial fraction decomposition of the integrand

$$\begin{aligned} \frac{2-4x}{(2x-5)(x-4)} &= \frac{A}{2x-5} + \frac{B}{x-4} \\ \implies \qquad 2-4x &= A(x-4) + B(2x-5). \end{aligned}$$

We can solve for A and B via the following substitutions:

- $\left(\text{Let } x = \frac{5}{2}\right)$ $\quad 2 - 4\left(\frac{5}{2}\right) = A\left(\frac{5}{2} - 4\right)$
  $$-8 = -\frac{3}{2}A \qquad \implies \qquad A = \frac{16}{3}.$$
- $(\text{Let } x = 4)$ $\quad 2 - 16 = 3B \qquad \implies \qquad B = -\frac{14}{3}.$

Thus,

$$\begin{aligned} \int_6^\infty \frac{2-4x}{2x^2-13x+20}\,dx &= \lim_{a\to\infty} \int_6^a \left[\frac{\frac{16}{3}}{2x-5} - \frac{\frac{14}{3}}{x-4}\right] dx \\ &= \lim_{a\to\infty} \left[\frac{8}{3}\ln|2x-5| - \frac{14}{3}\ln|x-4|\right]_6^a \\ &= \lim_{a\to\infty} \left[\frac{8}{3}\left(\ln|2a-5| - \ln(7)\right) - \frac{14}{3}\left(\ln|a-4| - \ln(2)\right)\right] \\ &= \lim_{a\to\infty} \left[\frac{1}{3}\left(\ln(2a-5)^8 - \ln(a-4)^{14}\right) - \frac{8}{3}\ln(7) + \frac{14}{3}\ln(2)\right] \\ &= \lim_{a\to\infty} \left[\frac{1}{3}\ln\left(\frac{(2a-5)^8}{(a-4)^{14}}\right) - \frac{8}{3}\ln(7) + \frac{14}{3}\ln(2)\right] \\ &= -\infty. \end{aligned}$$

Thus,

$$\int_6^\infty \frac{2-4x}{2x^2-13x+20}\,dx \quad \textit{diverges.}$$

Problem 5 *Find the volume of the solid whose base is the region where $x \geq 1$, $y \geq 0$, and below the curve $y = \frac{1}{x^4}$, and whose cross sections perpendicular to the x-axis are squares.*

Solution: *The area of each cross-section is* $\left(\dfrac{1}{x^4}\right)^2 = \dfrac{1}{x^8}$. *Thus,*

$$\begin{aligned}
\textit{Volume} &= \int_1^{\infty} \frac{1}{x^8}\,dx \\
&= \lim_{a\to\infty}\left[-\frac{1}{7x^7}\right]_1^a \\
&= \lim_{a\to\infty}\left(-\frac{1}{7a^7}+\frac{1}{7}\right) \\
&= 0 + \frac{1}{7} = \frac{1}{7}.
\end{aligned}$$