

## Recitation # 12: Basic ideas of differential equations - Full

### Warm up:

Which of the following is a solution to the differential equation  $y'' + 9y = 0$ ?

- (a)  $y = e^{3t} + e^{-3t}$
- (b)  $y = C(t^2 + t)$
- (c)  $y = \sin(3t) + 6$
- (d)  $y = 5 \cos(3t) - 7 \sin(3t)$
- (e)  $y = A \cos(3t) + B \sin(3t)$  (where A and B are real numbers.)

**Solution:** (a)

$$y = e^{3t} + e^{-3t} \quad y' = 3e^{3t} - 3e^{-3t} \quad y'' = 9e^{3t} + 9e^{-3t}$$

So,

$$\begin{aligned} y'' + 9y &= (9e^{3t} + 9e^{-3t}) + 9 \cdot (e^{3t} + e^{-3t}) \\ &= 18e^{3t} + 18e^{-3t} \neq 0. \end{aligned}$$

Therefore, this is **not** a solution to  $y'' + 9y = 0$ .

(b)

$$y = C(t^2 + t) \quad y' = C(2t + 1) \quad y'' = 2C$$

So,

$$\begin{aligned} y'' + 9y &= 2C + 9C(t^2 + t) \\ &= 9Ct^2 + 9Ct + 2C \neq 0 \text{ if } C \neq 0. \end{aligned}$$

Therefore, this is **not** a solution to  $y'' + 9y = 0$  unless  $C = 0$ , in which case we get the *trivial* solution.

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Learning outcomes:

Recitation # 12: Basic ideas of differential equations - Full

(c)

$$y = \sin(3t) + 6 \quad y' = 3 \cos(3t) \quad y'' = -9 \sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -9 \sin(3t) + 9(\sin(3t) + 6) \\ &= 54 \neq 0. \end{aligned}$$

Therefore, this is **not** a solution to  $y'' + 9y = 0$ .

(d)

$$y = 5 \cos(3t) - 7 \sin(3t) \quad y' = -15 \sin(3t) - 21 \cos(3t) \quad y'' = -45 \cos(3t) + 63 \sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -45 \cos(3t) + 63 \sin(3t) + 9(5 \cos(3t) - 7 \sin(3t)) \\ &= -45 \cos(3t) + 63 \sin(3t) + 45 \cos(3t) - 63 \sin(3t) = 0. \end{aligned}$$

Therefore, this **is** a solution to  $y'' + 9y = 0$ .

(e)

$$y = A \cos(3t) + B \sin(3t) \quad y' = -3A \sin(3t) + 3B \cos(3t) \quad y'' = -9A \cos(3t) - 9B \sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -9A \cos(3t) - 9B \sin(3t) + 9(A \cos(3t) + B \sin(3t)) \\ &= 0. \end{aligned}$$

Therefore, this **is** a solution to  $y'' + 9y = 0$ .

**Instructor Notes:** This is a good exercise in evaluating differential equations with functions. Suggest that each member in the group try a different possible solution.

## Group work:

**Problem 1** Verify that, if  $y(0) = 0$ , that both  $f(x) = 1 - (x^2 + 1)^2$  **and**  $g(x) = 1 - (x^2 - 1)^2$  are solutions to the differential equation  $\frac{dy}{dx} = 4x\sqrt{1-y}$ .

Recitation # 12: Basic ideas of differential equations - Full

**Solution:**

$$\begin{aligned}f'(x) &= -2(x^2 + 1) \cdot 2x \\&= -4x\sqrt{(x^2 + 1)^2} \\&= -4x\sqrt{1 - 1 + (x^2 + 1)^2} \\&= -4x\sqrt{1 - [1 - (x^2 + 1)^2]} \\&= -4x\sqrt{1 - f(x)}.\end{aligned}$$

$$\begin{aligned}g'(x) &= -2(x^2 - 1) \cdot 2x \\&= -4x\sqrt{(x^2 - 1)^2} \\&= -4x\sqrt{1 - 1 + (x^2 - 1)^2} \\&= -4x\sqrt{1 - [1 - (x^2 - 1)^2]} \\&= -4x\sqrt{1 - g(x)}.\end{aligned}$$

**Instructor Notes:** Note that students will need to plug into both  $x$  and  $y$  in the differential equation. Also, make sure that students realize that an initial value problem can possibly have many solutions.

**Problem 2** Find a specific solution to the differential equation  $\frac{dy}{dx} = x^{-2} \arctan(x)$  if  $y(1) = 5$ .

**Solution:** First note that

$$y = \int x^{-2} \arctan(x) dx.$$

To solve this integral, we use integration by parts with

$$u = \arctan(x) \quad dv = x^{-2} dx$$

$$du = \frac{1}{1+x^2} dx \quad v = -\frac{1}{x}.$$

So

$$\int x^{-2} \arctan(x) dx = -\frac{1}{x} \arctan(x) + \int \frac{1}{x(1+x^2)} dx.$$

Recitation # 12: Basic ideas of differential equations - Full

To complete this new integral, we use partial fractions.

$$\begin{aligned}\frac{1}{x(1+x^2)} &= \frac{A}{x} + \frac{Bx+C}{1+x^2} \\ \Rightarrow 1 &= A(1+x^2) + (Bx+C)x \\ \Rightarrow 1 &= (A+B)x^2 + Cx + A.\end{aligned}$$

Comparing coefficients, we see that  $A = 1$ ,  $C = 0$ , and  $B = -1$ .

Thus

$$\begin{aligned}y &= \int x^{-2} \arctan(x) dx \\ &= -\frac{1}{x} \arctan(x) + \int \frac{1}{x(1+x^2)} dx \\ &= -\frac{1}{x} \arctan(x) + \int \left( \frac{1}{x} - \frac{x}{1+x^2} \right) dx \\ &= -\frac{1}{x} \arctan(x) + \ln|x| - \frac{1}{2} \ln(1+x^2) + C.\end{aligned}$$

To finish, we use the initial condition to solve for  $C$ .

$$\begin{aligned}5 = y(1) &= -\frac{\pi}{4} + 0 - \frac{1}{2} \ln(2) + C \\ \Rightarrow C &= 5 + \frac{\pi}{4} + \frac{1}{2} \ln(2).\end{aligned}$$

Therefore

$$y(t) = -\frac{1}{x} \arctan(x) + \ln|x| - \frac{1}{2} \ln(1+x^2) + 5 + \frac{\pi}{4} + \frac{1}{2} \ln(2).$$

**Instructor Notes:** This is a straight forward antiderivative problem. The real interest here is finding the antiderivative using integration by parts and then partial fractions.

**Problem 3** Find a specific solution to the initial value problem

$$\frac{dy}{dx} = x^2 \sin(x), \quad y(0) = 5.$$

**Solution:** First, notice that

$$y = \int x^2 \sin(x) dx.$$

Recitation # 12: Basic ideas of differential equations - Full

To solve this integral, we use integration by parts twice.

$$\begin{aligned}u &= x^2 & dv &= \sin(x) \, dx \\ du &= 2x \, dx & v &= -\cos(x).\end{aligned}$$

So

$$\int x^2 \sin(x) \, dx = -x^2 \cos(x) + \int 2x \cos(x) \, dx.$$

Now we use

$$\begin{aligned}u &= 2x & dv &= \cos(x) \, dx \\ du &= 2 \, dx & v &= \sin(x).\end{aligned}$$

Then

$$\begin{aligned}y &= \int x^2 \sin(x) \, dx \\ &= -x^2 \cos(x) + \int 2x \cos(x) \, dx \\ &= -x^2 \cos(x) + 2x \sin(x) - \int 2 \sin(x) \, dx \\ &= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C.\end{aligned}$$

Finally, to finish the problem, we solve for  $C$ .

$$\begin{aligned}5 &= y(0) = 0 + 0 + 2 + C \\ \implies C &= 3.\end{aligned}$$

Thus,

$$y(t) = -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + 3.$$

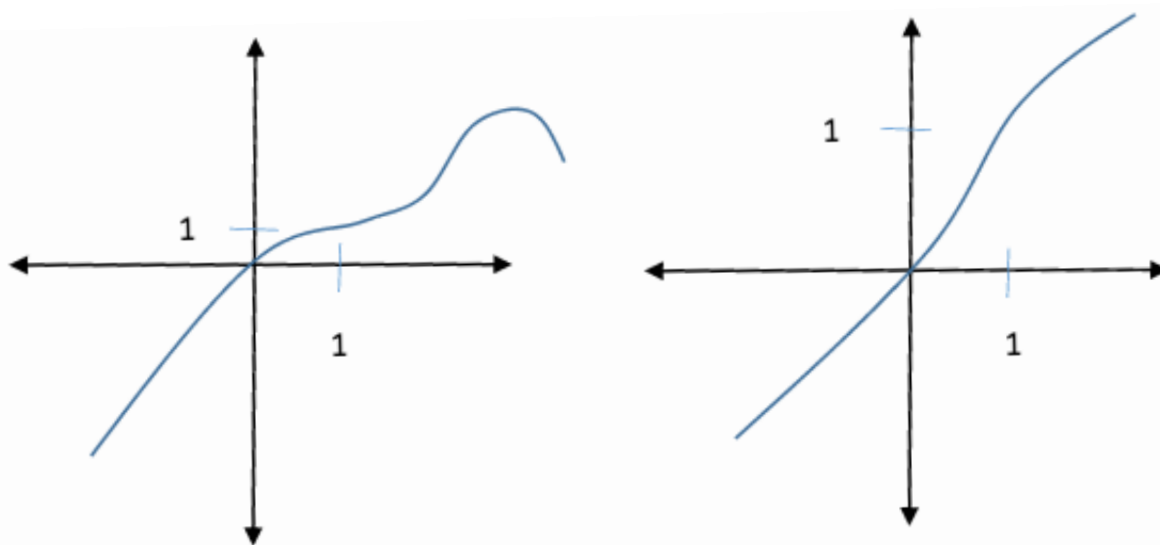
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**Instructor Notes:**

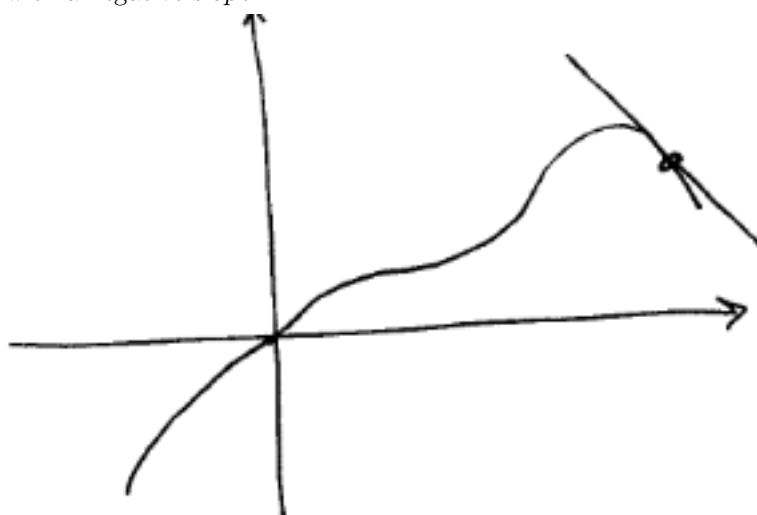
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**Problem 4** Explain why the functions with the given graphs cannot be solutions of the differential equation  $y' = e^x(y - 1)^2$ .

Recitation # 12: Basic ideas of differential equations - Full

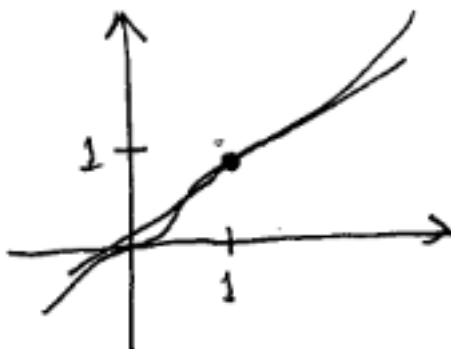


**Solution:** Since  $y' = e^x(y-1)^2$ , the derivative of  $y$  is always nonnegative. Thus, the first graph cannot satisfy this differential equation since it has a tangent line with a negative slope.



The second graph cannot satisfy the differential equation since the slope of the tangent line at  $x = 1$  is positive

Recitation # 12: Basic ideas of differential equations - Full



$$\text{but } \left[ \frac{dy}{dx} \right]_{x=0} = e^0(1-1) = 0.$$

**Instructor Notes:** This activity anticipates the idea of direction fields to be covered in Section 8.2. Students should eventually realize that differential equations are statements about the slope of the function at each point  $(x, y)$ . Issues such as where the slope equals 0 or where the slope is positive, negative, increasing, or decreasing should arise in the solution/discussion. The first graph has a negative slope at some places ( $e^x$  and  $(y-1)^2$  are always non-negative), and the second graph does not have a slope of 0 at the point  $(1, 1)$ .