

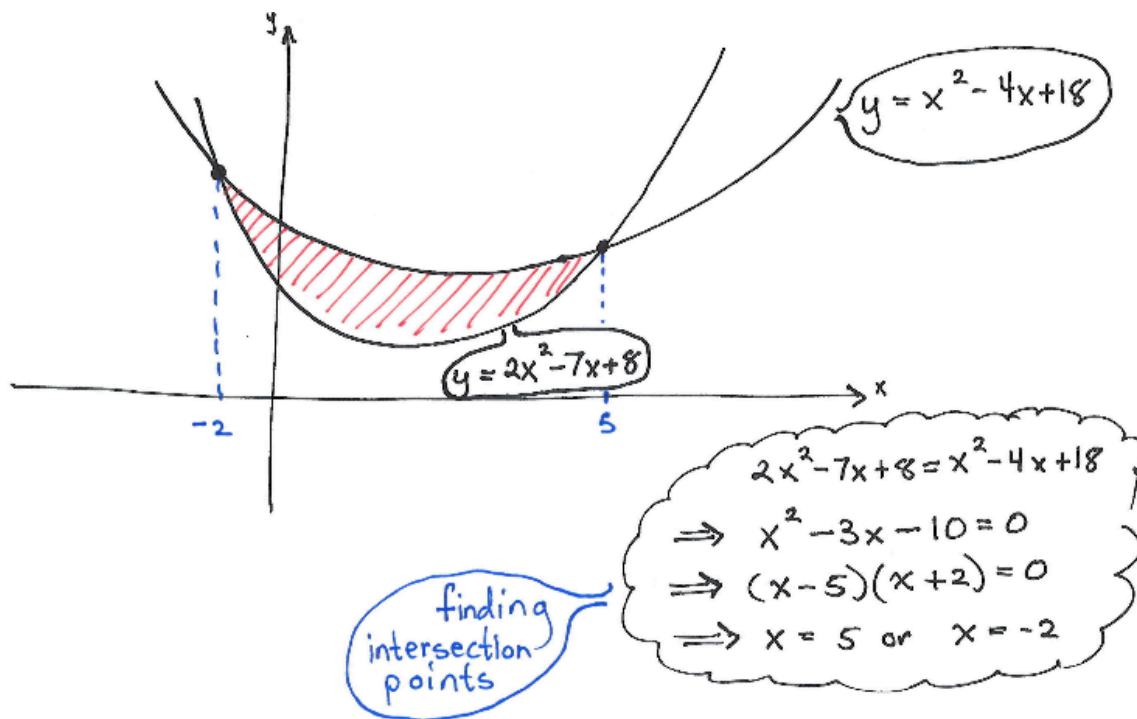
Recitation # 2 Regions Between Curves - Solutions

Group work:

Problem 1 Consider the region bounded by the curves $y = 2x^2 - 7x + 8$ and $y = x^2 - 4x + 18$.

(a) Draw a sketch of the graphs.

Solution:



(b) Find the area between these curves.

Learning outcomes:

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Solution: Let $y_1 = 2x^2 - 7x + 8$ and $y_2 = x^2 - 4x + 18$. By the solution to part (a), we know both that $y_1 - y_2 = x^2 - 3x - 10$ and that these two curves intersect at $x = -2, 5$. By checking the point $x = 0$ (or by looking at the graph from part (a)) we see that $y_2 \geq y_1$ on the interval $[-2, 5]$. So the area between the curves is:

$$\begin{aligned} \int_{-2}^5 (y_2 - y_1) dx &= \int_{-2}^5 (-x^2 + 3x + 10) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 10x \right]_{-2}^5 \\ &= \left(-\frac{125}{3} + \frac{75}{2} + 50 \right) - \left(\frac{8}{3} + 6 - 20 \right) \\ &= -\frac{133}{3} + \frac{75}{2} + 64 \\ &= \frac{-266 + 225 + 384}{6} = \frac{343}{6} \end{aligned}$$

- (c) Find the area of the region bounded by the curves $x = 2y^2 - 7y + 8$ and $x = y^2 - 4y + 18$.

Solution: This region is exactly the same as the **red region** from part (a), except it is rotated clockwise by 90° . Since the area of a region does not change under rotation, we have that the area of the new region is still $\frac{343}{6}$.

- (d) Find the area of the region bounded by the curves

(i) $y = 2x^2 - 7x$ and $y = x^2 - 4x + 10$.

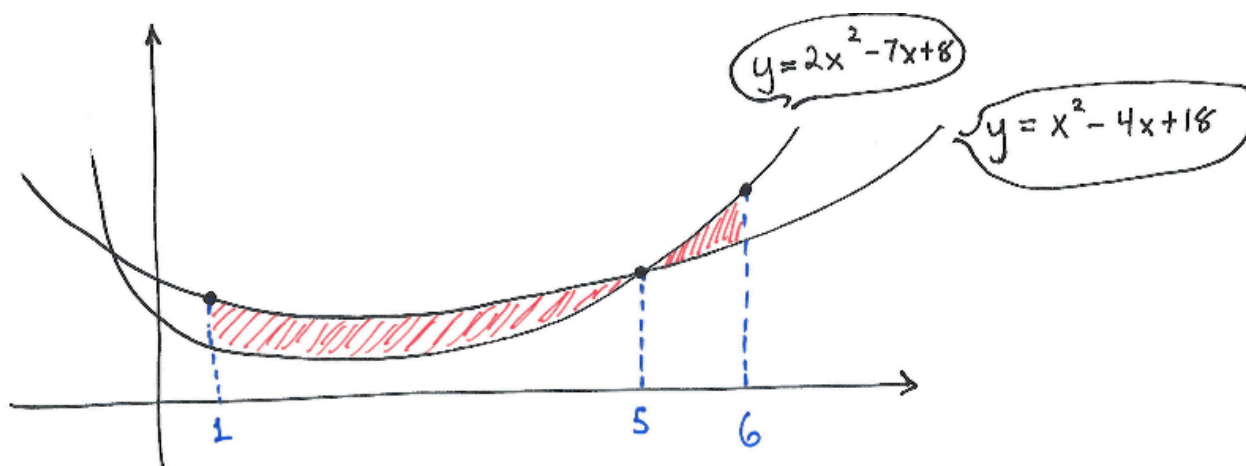
Solution: The region bounded by the curves $y = 2x^2 - 7x$ and $y = x^2 - 4x + 10$ is just the **red region** in part (a) translated downward by 8 units. Since the area of a region does not change under translation, we have that the area of the new region is still $\frac{343}{6}$.

(ii) $y = 2x^2 - 7x - 30$ and $y = x^2 - 4x - 20$.

Solution: This is the same as (i), except now the **red region** has been translated downward by 38 units. Therefore, the area of this region is still $\frac{343}{6}$.

- (e) Find the area of the region bounded by the curves $y = 2x^2 - 7x + 8$, $y = x^2 - 4x + 18$, $x = 1$, and $x = 6$.

Solution:



We already know that the graphs of the two functions intersect at the point $x = 5$. By checking the points $x = 1$ and $x = 6$ (or by looking at the graph above) we see that

$$y_2 \geq y_1 \quad \text{on} \quad [1, 5]$$

$$y_1 \geq y_2 \quad \text{on} \quad [5, 6]$$

Thus, the area between the curves is

$$\begin{aligned} & \int_1^5 (y_2 - y_1) dx + \int_5^6 (y_1 - y_2) dx \\ &= \int_1^5 (-x^2 + 3x + 10) dx + \int_5^6 (x^2 - 3x - 10) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 10x \right]_1^5 + \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 - 10x \right]_5^6 \\ &= \left[\left(-\frac{125}{3} + \frac{75}{2} + 50 \right) - \left(-\frac{1}{3} + \frac{3}{2} + 10 \right) \right] + \left[(72 - 54 - 60) - \left(\frac{125}{3} - \frac{75}{2} - 50 \right) \right] \\ &= -\frac{249}{3} + \frac{147}{2} + 48 = -83 + 48 + \frac{147}{2} = \frac{-70 + 147}{2} = \frac{77}{2} \end{aligned}$$

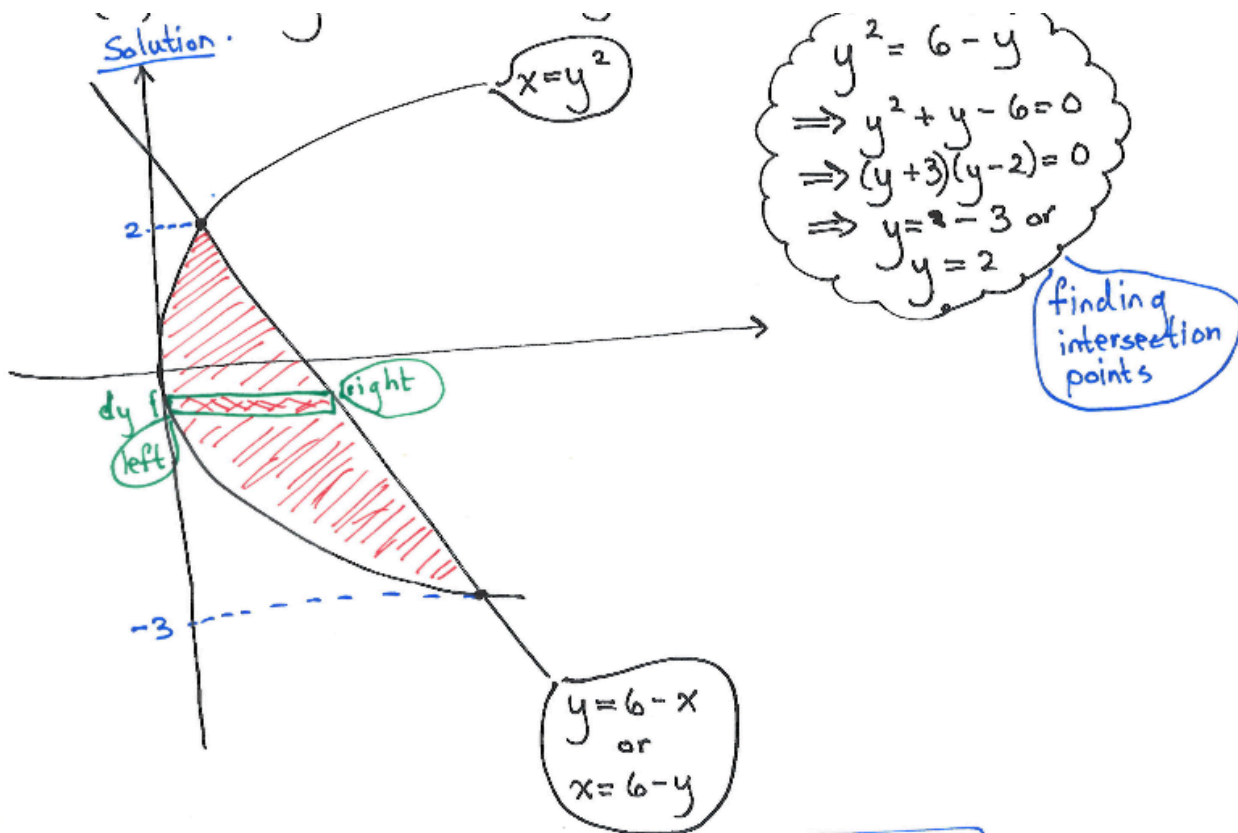
Instructor Notes: Have the students do (a) and (b), and then have a group present. Afterwards, discuss the variations (c) and (d) as a whole class.

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Problem 2 Set up a single integral that computes the area of the region bounded by the curves (and be sure to draw a sketch of the graphs).

(a) $x = y^2$ and $y = 6 - x$

Solution:



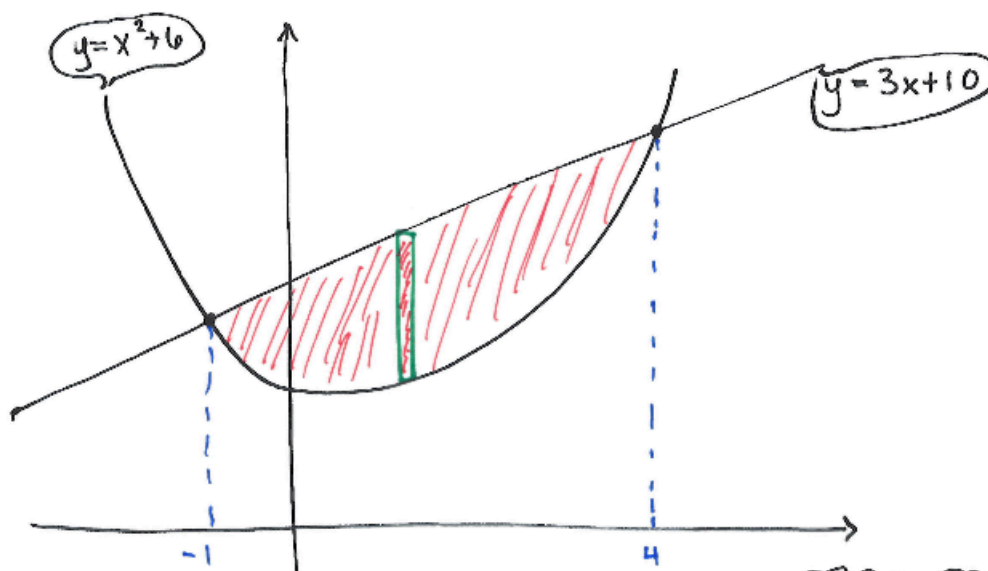
Thus,

$$\text{Area of region} = \int_{-3}^2 [(6 - y) - y^2] dy.$$

(b) $y = x^2 + 6$ and $y = 3x + 10$

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Solution:



To find the intersection points in the above picture, we solve

$$\begin{aligned} x^2 + 6 &= 3x + 10 \\ x^2 - 3x - 4 &= 0 \\ (x + 1)(x - 4) &= 0 \\ x &= -1, 4. \end{aligned}$$

So

$$\text{Area of region} = \int_{-1}^4 [(3x + 10) - (x^2 + 6)] dx.$$

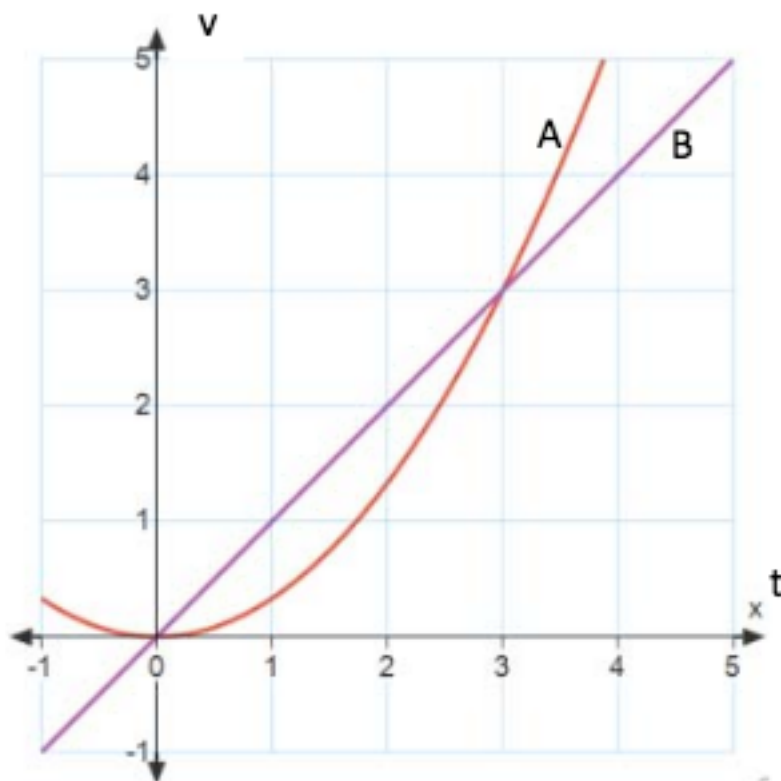
Instructor Notes: Split (a) and (b) among the groups. Note that (a) should be set up in terms of y while (b) should be set up in terms of x . Have groups present their solutions. Discuss with the students factors to consider when deciding whether to integrate in terms of x or y .

Problem 3 Two runners (A and B) run in a race in which the winner runs the farthest in 4 minutes. The runners' respective velocities are

$$v_A(t) = \frac{1}{3}t^2 \quad v_B(t) = t$$

The graphs of the runners' velocities is given below.

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- (a) Who is running the fastest 2 minutes into the race?

Solution: $v_A(2) = \frac{4}{3}$ and $v_B(2) = 2$. So B is running faster at the 2 minute mark of the race.

- (b) Who is winning the race 2 minutes into the race (and by how much)?

Solution: The distance that A covers in the first 2 minutes is

$$\int_0^2 v_A(t) dt = \int_0^2 \frac{1}{3}t^2 dt = \left[\frac{1}{9}t^3 \right]_0^2 = \frac{8}{9}.$$

The distance that B covers in the first 2 minutes is

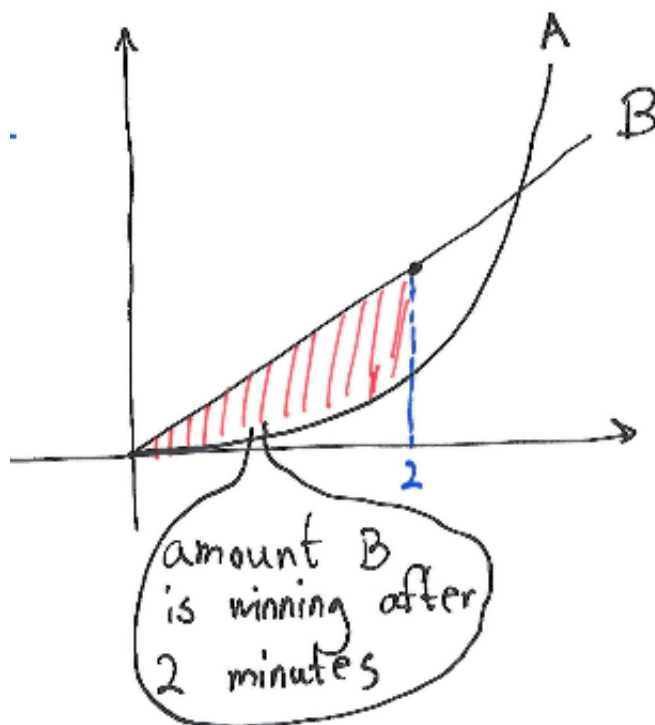
$$\int_0^2 v_B(t) dt = \int_0^2 t dt = \left[\frac{1}{2}t^2 \right]_0^2 = 2.$$

So B is winning after 2 minutes.

B is winning by $2 - \frac{8}{9} = \frac{10}{9}$. This could also be calculated by

$$\int_0^2 (v_B(t) - v_A(t)) dt.$$

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- (c) What special event occurs 3 minutes into the race?

Solution: Runner A matches runner B's velocity. ie, $v_A(3) = v_B(3)$.

- (d) Who wins the race (and by how much)?

Solution: The distance that A covers is

$$\int_0^4 v_A(t) dt = \int_0^4 \frac{1}{3}t^2 dt = \left[\frac{1}{9}t^3 \right]_0^4 = \frac{64}{9} = 7.\bar{1}.$$

The distance that B covers is

$$\int_0^4 v_B(t) dt = \int_0^4 t dt = \left[\frac{1}{2}t^2 \right]_0^4 = 8.$$

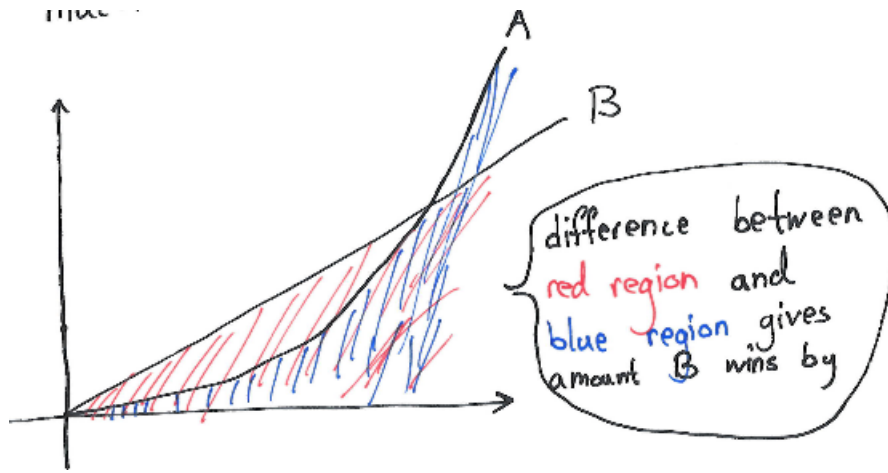
So runner B wins. The amount that B wins by is

$$8 - \frac{64}{9} = \frac{8}{9}.$$

This could have also been computed by

$$\int_0^4 (v_B(t) - v_A(t)) dt.$$

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Instructor Notes: Do this problem as a class discussion. The main point is to have students identify how each of the questions relates to the graph of the velocity functions.