Section 9.6: Alternating Series

Warm-Up

Problem 1 *Suppose* $\sum_{k=1}^{\infty} a_k$ *is an infinite series.*

A. *Explain what it means for a series* $\sum_{k=1}^{\infty} a_k$ *to converge absolutely.*

Solution: $\sum_{k=1}^{\infty} a_k$ *to converge absolutely if the associated series* $\sum_{k=1}^{\infty} |a_k|$ *converges.*

B. *Explain what it means for a series* $\sum_{k=1}^{\infty} a_k$ *to converge conditionally.*

Solution: $\sum_{k=1}^{\infty} a_k$ *to converge conditionally if the associated series* $\sum_{k=1}^{\infty} |a_k|$ *diverges but the original series* $\sum_{k=1}^{\infty} a_k$ *converges.*

Problem 2 *Understanding what an alternating series is.*

A. *Is the series* $\sum_{k=1}^{\infty} \sin(k)$ *alternating?*

Solution: *No; a series is alternating if and only if the terms* strictly *alternate in sign!*

B. *Suppose* $\{a_k\}$ *is a sequence. Is the series* $\sum_{k=1}^{\infty} (-1)^k a_k$ *alternating? If it is not, what assumption(s) would be needed on the terms in the sequence* $\{a_k\}$ *to ensure the series is alternating?*

Learning outcomes:

Solution: *No; there is no assumption here that $a_k > 0$ for all k, so we are not guaranteed that $(-1)^k a_k$ is strictly alternating in sign! If we assume that $a_k > 0$ for all k, then the series $\sum_{k=1}^{\infty}(-1)^k a_k$ would be alternating.*

Group Work

Problem 3 *Determine if the following series absolutely converge, conditionally converge, or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$

(e) $\displaystyle\sum_{n=4}^{\infty} \frac{(-2)^n}{n}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 + 2n + 1}{3n^4 + 1}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^2 e^{-n}\right]$

Solution: (a) *Since $\dfrac{1}{n+3}$ is positive and decreasing, the Alternating Series Test applies. Thus, this series converges. But we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges (Harmonic Series) and so $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ is conditionally convergent.*

(b) *Let* $a_n = \dfrac{(n+1)^n}{(2n)^n}$. *Applying the Root Test, we see that*

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \frac{n+1}{2n} \\ &= \frac{1}{2} < 1. \end{aligned}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^n}{(2n)^n}$ *converges, and therefore* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)^n}{(2n)^n}$ *is absolutely convergent.*

(c) *Let* $f(x) = x^2 e^{\frac{-x^3}{3}}$. *The function* $f(x)$ *is continuous, positive, and decreasing. So we apply the integral test*

$$\begin{aligned} \int_1^{\infty} x^2 e^{\frac{-x^3}{3}}\,dx &= \lim_{b\to\infty} \int_1^b x^2 e^{\frac{-x^3}{3}}\,dx \\ &= \lim_{b\to\infty} \int_{\frac{1}{3}}^{\frac{b^3}{3}} e^{-u}\,du \qquad u = \frac{x^3}{3},\ du = x^2\,dx \\ &= \lim_{b\to\infty} \left[-e^{\frac{-b^3}{3}} + e^{\frac{-1}{3}}\right] \\ &= 0 + e^{\frac{-1}{3}} = e^{\frac{-1}{3}}. \end{aligned}$$

So, by the Integral Test, $\displaystyle\sum_{n=1}^{\infty} n^2 e^{\frac{-n^3}{3}}$ *converges. Therefore,* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$ *is absolutely convergent.*

(d) *First, notice that*

$$\frac{5}{3^n} > \frac{5}{3^n + 3^{-n}}.$$

Then since $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n}$ *converges (geometric series with* $r = \frac{1}{3} < 1$*), we know that* $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n + 3^{-n}}$ *converges by the Comparison Test. Thus, the series* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$ *is absolutely convergent.*

(e) *Since the sequence* $\left\{\dfrac{(-2)^n}{n}\right\}$ *diverges, the series* $\displaystyle\sum_{n=4}^{\infty} \frac{(-2)^n}{n}$ *diverges by the Divergence Test.*

(f) *Notice that all the terms of this series are positive. Use the* ***Limit Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n+1}{3n^4+1} \cdot \frac{n^2}{1}$$
$$= \frac{1}{3}.$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges, by the Limit Comparison Test we know that* $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$ *converges and therefore is absolutely convergent.*

(g) *Notice that all the terms of this series are positive. Use the* ***Comparison Test*** *with* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$.

First, notice that for all $n \geq 3$,

$$\left(1 + \frac{1}{n}\right)^2 < 2.$$

Also, notice that $\sum_{n=1}^{\infty} e^{-n}$ *is a convergent geometric series. Therefore* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$ *converges, and so* $\sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^2 e^{-n}\right]$ *converges by the Comparison Test and therefore is absolutely convergent.*

Instructor Notes: These problems use a mix of tests. One of the main goals is for students to get practice determining which test to use.

(a) Limit Comparison Test with the harmonic series

(b) Root Test

(c) Integral Test

(d) Limit Comparison Test

(e) Divergence Test. Be sure to talk about "pulling out the -1" to get an alternating series in standard form. Talk about how the Alternating Series Test and the Divergence Test will take care of conditional convergence for most (but not all) alternating series that they will see.

Problem 4 (a) *Find an upper bound for how close* $\sum_{k=0}^{4} \frac{(-1)^k k}{4^k}$ *is to the value of* $\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k}$.

(b) *(Calculator Recommended) How many terms are needed to estimate* $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n!}$ *to within* 10^{-6}*?*

Solution: (a) *Recall from the lesson that the remainder* R_n *is given by*

$$R_n = |S - S_n| \leq a_{n+1}.$$

Also notice that

$$\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}$$

and

$$\sum_{k=0}^{4} \frac{(-1)^k k}{4^k} = \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}.$$

So

$$\begin{aligned}
&\left| \sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} - \sum_{k=0}^{4} \frac{(-1)^k k}{4^k} \right| \\
&= \left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}} - \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}} \right| \\
&= |S - S_5| \\
&= R_6 \leq a_6 = \boxed{\frac{5}{4^5}}
\end{aligned}$$

(b) *Since* $R_n \leq a_{n+1}$, *we need to find* n *so that*

$$\begin{aligned}
& a_{n+1} \leq 10^{-6} \\
\iff \quad & \frac{\ln(n+1)}{(n+1)!} \leq 10^{-6} \\
\iff \quad & \frac{(n+1)!}{\ln(n+1)} \geq 10^{6}
\end{aligned}$$

Using a calculator, we see that this inequality holds for $n \approx 8.8$. *So for* $\boxed{n \geq 9}$, $R_n < 10^{-6}$.

Instructor Notes: Split (a) and (b) among the groups.