Recitation # 12: Basic ideas of differential equations - Solutions

Warm up:

Which of the following is a solution to the differential equation $y'' + 9y = 0$?

(a) $y = e^{3t} + e^{-3t}$

(b) $y = C(t^2 + t)$

(c) $y = \sin(3t) + 6$

(d) $y = 5\cos(3t) - 7\sin(3t)$

(e) $y = A\cos(3t) + B\sin(3t)$ (where A and B are real numbers.)

Solution: (a)

$$y = e^{3t} + e^{-3t} \qquad y' = 3e^{3t} - 3e^{-3t} \qquad y'' = 9e^{3t} + 9e^{-3t}$$

So,

$$\begin{aligned} y'' + 9y &= (9e^{3t} + 9e^{-3t}) + 9 \cdot (e^{3t} + e^{-3t}) \\ &= 18e^{3t} + 18e^{-3t} \neq 0. \end{aligned}$$

Therefore, this is **not** a solution to $y'' + 9y = 0$.

(b)

$$y = C(t^2 + t) \qquad y' = C(2t + 1) \qquad y'' = 2C$$

So,

$$\begin{aligned} y'' + 9y &= 2C + 9C(t^2 + t) \\ &= 9Ct^2 + 9Ct + 2C \neq 0 \text{ if } C \neq 0. \end{aligned}$$

Therefore, this is **not** a solution to $y'' + 9y = 0$ unless $C = 0$, in which case we get the *trivial* solution.

Learning outcomes:

(c)

$$y = \sin(3t) + 6 \qquad y' = 3\cos(3t) \qquad y'' = -9\sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -9\sin(3t) + 9(\sin(3t) + 6) \\ &= 54 \neq 0. \end{aligned}$$

Therefore, this is **not** a solution to $y'' + 9y = 0$.

(d)

$$y = 5\cos(3t) - 7\sin(3t) \qquad y' = -15\sin(3t) - 21\cos(3t) \qquad y'' = -45\cos(3t) + 63\sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -45\cos(3t) + 63\sin(3t) + 9(5\cos(3t) - 7\sin(3t)) \\ &= -45\cos(3t) + 63\sin(3t) + 45\cos(3t) - 63\sin(3t) = 0. \end{aligned}$$

Therefore, this **is** a solution to $y'' + 9y = 0$.

(e)

$$y = A\cos(3t) + B\sin(3t) \qquad y' = -3A\sin(3t) + 3B\cos(3t) \qquad y'' = -9A\cos(3t) - 9B\sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -9A\cos(3t) - 9B\sin(3t) + 9(A\cos(3t) + B\sin(3t)) \\ &= 0. \end{aligned}$$

Therefore, this **is** a solution to $y'' + 9y = 0$.

Group work:

Problem 1 *Verify that, if $y(0) = 0$, that both $f(x) = 1 - (x^2+1)^2$* ***and*** *$g(x) = 1 - (x^2-1)^2$ are solutions to the differential equation $\frac{dy}{dx} = 4x\sqrt{1-y}$.*

Solution:

$$\begin{aligned} f'(x) &= -2(x^2+1) \cdot 2x \\ &= -4x\sqrt{(x^2+1)^2} \\ &= -4x\sqrt{1 - 1 + (x^2+1)^2} \\ &= -4x\sqrt{1 - [1 - (x^2+1)^2]} \\ &= -4x\sqrt{1 - f(x)}. \end{aligned}$$

$$
\begin{aligned}
g'(x) &= -2(x^2-1)\cdot 2x \\
&= -4x\sqrt{(x^2-1)^2} \\
&= -4x\sqrt{1-1+(x^2-1)^2} \\
&= -4x\sqrt{1-[1-(x^2-1)^2]} \\
&= -4x\sqrt{1-g(x)}.
\end{aligned}
$$

Problem 2 *Find a specific solution to the differential equation* $\frac{dy}{dx} = x^{-2}\arctan(x)$ *if* $y(1) = 5$.

Solution: *First note that*

$$y = \int x^{-2}\arctan(x)\,dx.$$

To solve this integral, we use integration by parts with

$$u = \arctan(x) \qquad dv = x^{-2}\,dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = -\frac{1}{x}.$$

So

$$\int x^{-2}\arctan(x)\,dx = -\frac{1}{x}\arctan(x) + \int \frac{1}{x(1+x^2)}\,dx.$$

To complete this new integral, we use partial fractions.

$$
\begin{aligned}
&\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \\
\implies\quad & 1 = A(1+x^2) + (Bx+C)x \\
\implies\quad & 1 = (A+B)x^2 + Cx + A.
\end{aligned}
$$

Comparing coefficients, we see that $A = 1$, $C + 0$, *and* $B = -1$.

Thus

$$
\begin{aligned}
y &= \int x^{-2}\arctan(x)\,dx \\
&= -\frac{1}{x}\arctan(x) + \int \frac{1}{x(1+x^2)}\,dx \\
&= -\frac{1}{x}\arctan(x) + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx \\
&= -\frac{1}{x}\arctan(x) + \ln|x| - \frac{1}{2}\ln(1+x^2) + C.
\end{aligned}
$$

To finish, we use the initial condition to solve for C.

$$\begin{aligned} 5 = y(1) &= -\frac{\pi}{4} + 0 - \frac{1}{2}\ln(2) + C \\ \implies \qquad C &= 5 + \frac{\pi}{4} + \frac{1}{2}\ln(2). \end{aligned}$$

Therefore

$$y(t) = -\frac{1}{x}\arctan(x) + \ln|x| - \frac{1}{2}\ln(1+x^2) + 5 + \frac{\pi}{4} + \frac{1}{2}\ln(2).$$

Problem 3 *Find a specific solution to the initial value problem*

$$\frac{dy}{dx} = x^2\sin(x), \qquad y(0) = 5.$$

Solution: *First, notice that*

$$y = \int x^2 \sin(x)\,dx.$$

To solve this integral, we use integration by parts twice.

$$u = x^2 \qquad dv = \sin(x)\,dx$$

$$du = 2x\,dx \qquad v = -\cos(x).$$

So

$$\int x^2\sin(x)\,dx = -x^2\cos(x) + \int 2x\cos(x)\,dx.$$

Now we use

$$u = 2x \qquad dv = \cos(x)\,dx$$

$$du = 2\,dx \qquad v = \sin(x).$$

Then

$$\begin{aligned} y &= \int x^2\sin(x)\,dx \\ &= -x^2\cos(x) + \int 2x\cos(x)\,dx \\ &= -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx \\ &= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C. \end{aligned}$$

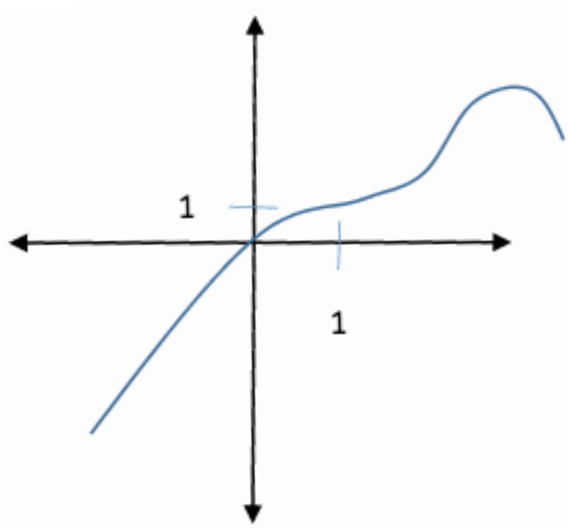
Finally, to finish the problem, we solve for C.

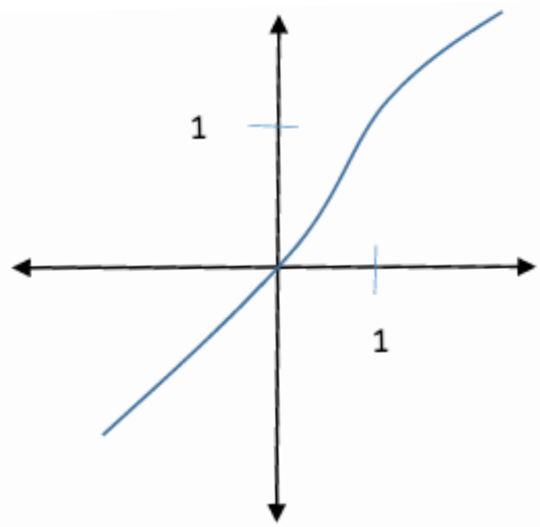
$$\begin{aligned} 5 = y(0) &= 0 + 0 + 2 + C \\ \implies \quad C &= 3. \end{aligned}$$

Thus,

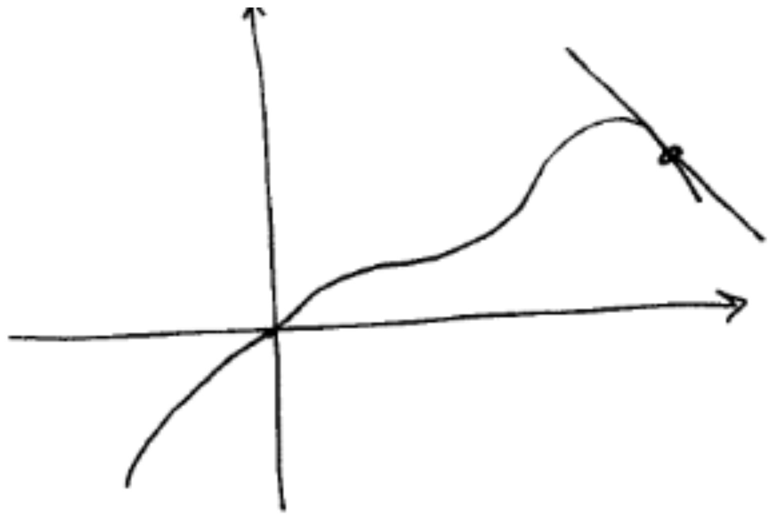
$$y(t) = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + 3.$$

Problem 4 *Explain why the functions with the given graphs cannot be solutions of the differential equation $y' = e^x(y-1)^2$.*

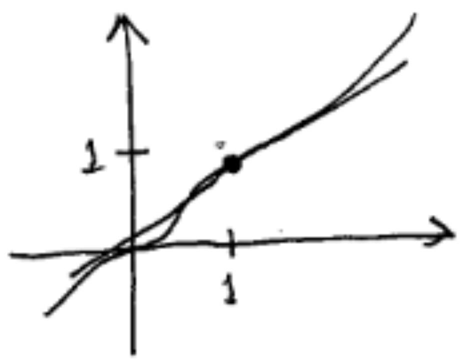




Solution: *Since $y' = e^x(y-1)^2$, the derivative of y is always nonnegative. Thus, the first graph cannot satisfy this differential equation since it has a tangent line with a negative slope.*



The second graph cannot satisfy the differential equation since the slope of the tangent line at $x = 1$ is positive



but $\left[\dfrac{dy}{dx}\right]_{x=0} = e^0(1-1) = 0.$