

Recitation #20: Properties of power series - Solutions

Warm up:

Suppose that $\sum_{k=0}^{\infty} c_k(x+5)^k$ converges when $x = -9$ and diverges when $x = -1$.

What can be said about the convergence and divergence of the following series?

(a) $\sum_{k=0}^{\infty} c_k$ (b) $\sum_{k=0}^{\infty} c_k(-5)^k$ (c) $\sum_{k=0}^{\infty} c_k(5)^k$

Solution: What is important to note first is that the interval of convergence for the series $\sum_{k=0}^{\infty} c_k(x+5)^k$ is $[-9, -1)$.

(a) Notice that

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} c_k(-4+5)^k.$$

Then since -4 is in the interval $[-9, -1)$, the series $\sum_{k=0}^{\infty} c_k$ converges.

(b) Since -10 is not in the interval $[-9, -1)$, the series

$$\sum_{k=0}^{\infty} c_k(-5)^k = \sum_{k=0}^{\infty} c_k(-10+5)^k$$

diverges.

(c) Since 0 is not in the interval $[-9, -1)$, the series

$$\sum_{k=0}^{\infty} c_k 5^k = \sum_{k=0}^{\infty} c_k(0+5)^k$$

diverges.

Group work:

Problem 1 If the series $\sum_{k=0}^{\infty} a_k(x-2)^k$ has an interval of convergence of $[-4, 8)$, determine the interval of convergence of the following series:

$$(a) \sum_{k=300}^{\infty} a_k(x-2)^k \quad (b) \sum_{k=0}^{\infty} a_k x^k \quad (c) \sum_{k=0}^{\infty} \left(a_k(x-2)^k + \left(\frac{1}{7} \right)^k x^k \right)$$

Solution: (a) This series has exactly the same interval of convergence, $[-4, 8)$, since a finite number of terms do not change whether or not a series converges.

(b) In the original interval of convergence $[-4, 8)$, the center is $x = 2$ and the radius of convergence is 6 (which includes the left endpoint, but not the right). So now, the center is $x = 0$. Taking an interval about 0 with radius 6, and adding in the left endpoint, gives the IOC of $[-6, 6]$.

(c) The interval of convergence of this series is the intersection of the interval of convergence for $\sum_{k=0}^{\infty} a_k(x-2)^k$ and $\sum_{k=0}^{\infty} \left(\frac{1}{7} \right)^k x^k$. To find the IOC for this second series, we use the Root Test.

$$\begin{aligned} \lim_{k \rightarrow \infty} \sqrt[k]{\left| \left(\frac{1}{7} \right)^k x^k \right|} &= \lim_{k \rightarrow \infty} \frac{1}{7} |x| \\ &= \frac{1}{7} |x|. \end{aligned}$$

So the series $\sum_{k=0}^{\infty} \left(\frac{1}{7} \right)^k x^k$ converges when

$$\begin{aligned} \frac{1}{7} |x| &< 1 \\ \implies |x| &< 7 \\ \implies -7 &< x < 7. \end{aligned}$$

We need to test the endpoints $x = \pm 7$, but using the divergence test we see that both

$$\sum_{k=0}^{\infty} (-1)^k \quad \text{and} \quad \sum_{k=0}^{\infty} 1^k$$

diverge. Therefore, the interval of convergence for $\sum_{k=0}^{\infty} \left(\frac{1}{7} \right)^k x^k$ is $(-7, 7)$.

Finally, the interval of convergence for the series

$$\sum_{k=0}^{\infty} \left(a_k (x-2)^k + \left(\frac{1}{7} \right)^k x^k \right)$$

is

$$[-4, 8) \cap (-7, 7) = \boxed{[-4, 7]}.$$

Problem 2 For each of the following, find the domain of $f(x)$ (i.e. find the interval of convergence).

(a) $f(x) = \sum_{k=1}^{\infty} \frac{(3x-2)^k}{k \cdot 3^k}$

(c) $f(x) = \sum_{k=2}^{\infty} \frac{x^{3k+2}}{(\ln k)^k}$

(b) $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}} x^k$

Solution: (a) We use the Ratio Test

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{(3x-2)^{k+1}}{(k+1)3^{k+1}} \cdot \frac{k \cdot 3^k}{(3x-2)^k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{k(3x-2)}{3(k+1)} \right| \\ &= \left| \frac{3x-2}{3} \right| = \frac{|3x-2|}{3}. \end{aligned}$$

We know that this series converges when

$$\begin{aligned} \frac{|3x-2|}{3} &< 1 \\ \implies |3x-2| &< 3 \\ \implies -3 &< 3x-2 < 3 \\ \implies -1 &< 3x < 5 \\ \implies -\frac{1}{3} &< x < \frac{5}{3}. \end{aligned}$$

We still need to check the endpoints. When $x = -\frac{1}{3}$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k3^k} \left(3 \left(-\frac{1}{3} \right) - 2 \right)^k &= \sum_{k=1}^{\infty} \frac{(-3)^k}{k3^k} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \end{aligned}$$

which converges (conditionally) by the alternating series test.

When $x = \frac{5}{3}$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k3^k} \left(3 \left(\frac{5}{3} \right) - 2 \right)^k &= \sum_{k=1}^{\infty} \frac{3^k}{k3^k} \\ &= \sum_{k=1}^{\infty} \frac{1}{k}\end{aligned}$$

which diverges since it is the Harmonic series.

Therefore, the interval of convergence is $\left[-\frac{1}{3}, \frac{5}{3} \right)$.

(b) We again apply the Ratio Test

$$\begin{aligned}\lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{k+1}}{\sqrt{(k+1)^2 + 1}} \cdot \frac{\sqrt{k^2 + 1}}{(-1)^k x^k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{x \sqrt{k^2 + 1}}{\sqrt{k^2 + 2k + 2}} \right| \\ &= |x|.\end{aligned}$$

So we know that this series converges when

$$|x| < 1 \quad \Longleftrightarrow \quad -1 < x < 1.$$

We still need to check the endpoints. When $x = -1$, the series

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2 + 1}}$$

diverges by the Limit Comparison Test (compare with $\sum_{k=1}^{\infty} \frac{1}{k}$).

When $x = 1$, the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2 + 1}}$$

converges conditionally by the alternating series test.

Therefore, the interval of convergence is $\left[-1, 1 \right]$.

(c) For this series we apply the Root Test

$$\begin{aligned}\lim_{k \rightarrow \infty} \sqrt[k]{\left| \frac{x^{3k+2}}{(\ln k)^k} \right|} &= \lim_{k \rightarrow \infty} \frac{|x|^3 \cdot |x|^{\frac{2}{k}}}{\ln k} \\ &= |x|^3 \cdot \lim_{k \rightarrow \infty} \frac{|x|^{\frac{2}{k}}}{\ln k} \\ &= |x|^3 \cdot 0 = 0.\end{aligned}$$

Therefore, the interval of convergence is $\left[-\infty, \infty \right)$.

Problem 3 In each of the following, give a power series (with an interval of convergence) for the given function. Assume that we know $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ on $(-1, 1)$.

(a) $f(x) = \frac{3}{5x-2}$

(b) $f(x) = \frac{3x^4}{5x^3-2}$

Solution: (a)

$$\begin{aligned} f(x) &= \frac{3}{5x-2} \\ &= \frac{3}{-2\left(1 - \left(\frac{5}{2}x\right)\right)} \\ &= -\frac{3}{2} \sum_{k=0}^{\infty} \left(\frac{5}{2}x\right)^k \\ &= \boxed{\sum_{k=0}^{\infty} \left(-\frac{3}{2}\right) \left(\frac{5}{2}x\right)^k}. \end{aligned}$$

Since this is a geometric series, it converges if and only if

$$\begin{aligned} &\left|\frac{5}{2}x\right| < 1 \\ \iff &|x| < \frac{2}{5} \\ \iff &-\frac{2}{5} < x < \frac{2}{5}. \end{aligned}$$

Therefore, the interval of convergence is $\boxed{\left(-\frac{2}{5}, \frac{2}{5}\right)}$

(b)

$$\begin{aligned} f(x) &= \frac{3x^4}{5x^3-2} \\ &= \frac{-3x^4}{2} \cdot \frac{1}{1 - \left(\frac{5}{2}x^3\right)} \\ &= -\frac{3}{2}x^4 \sum_{k=0}^{\infty} \left(\frac{5}{2}x^3\right)^k \\ &= \boxed{\sum_{k=0}^{\infty} \left(-\frac{3}{2}\right) \left(\frac{5}{2}\right)^k x^{3k+4}}. \end{aligned}$$

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This geometric series converges if and only if

$$\begin{aligned} \left| \frac{5}{2}x^3 \right| &< 1 \\ \iff |x^3| &< \frac{2}{5} \\ \iff |x| &< \sqrt[3]{\frac{2}{5}} \\ \iff -\sqrt[3]{\frac{2}{5}} &< x < \sqrt[3]{\frac{2}{5}}. \end{aligned}$$

Therefore, the interval of convergence is $\boxed{\left(-\sqrt[3]{\frac{2}{5}}, \sqrt[3]{\frac{2}{5}}\right)}$.

Problem 4 Consider $f(x) = \sum_{k=0}^{\infty} \frac{2^k x^k}{(k+1)^3}$.

- (a) Write out $p_3(x)$, the cubic polynomial which is the first three terms of this power series.
- (b) Find $p'_3(x)$ and $f'(x)$ and compare your answers.
- (c) Find $\int p_3(x) dx$ and $\int f(x) dx$ and compare your answers.

Solution: (a)

$$\begin{aligned} p_3(x) &= \frac{2^0 x^0}{(0+1)^3} + \frac{2^1 x^1}{(1+1)^3} + \frac{2^2 x^2}{(2+1)^3} + \frac{2^3 x^3}{(3+1)^3} \\ &= \boxed{1 + \frac{2x}{2^3} + \frac{4x^2}{3^3} + \frac{8x^3}{4^3}}. \end{aligned}$$

(b)

$$p'_3(x) = \boxed{0 + \frac{2}{2^3} + \frac{2 \cdot 4}{3^3}x + \frac{3 \cdot 8}{4^3}x^2}.$$

$$\begin{aligned} f'(x) &= \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^3} \cdot k \cdot x^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{k \cdot 2^k}{(k+1)^3} x^{k-1} \quad \text{Since the } k=0 \text{ term is } 0. \\ &= \boxed{\sum_{\ell=0}^{\infty} \frac{(\ell+1)2^{\ell+1}}{(\ell+2)^3} x^{\ell}} \quad \text{reindexing with } \ell = k-1 \end{aligned}$$

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Note that this sum agrees with $p'_3(x)$ when $\ell = 0, 1$, and 2 .

(c)

$$\int p_3(x) dx = \boxed{x + \frac{2x^2}{2^4} + \frac{4x^3}{3^4} + \frac{8x^4}{4^4} + C}.$$

$$\begin{aligned} \int f(x) dx &= \int \sum_{k=0}^{\infty} \frac{2^k x^k}{(k+1)^3} dx \\ &= \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^3} \cdot \left(\int x^k dx \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{2^k}{(k+1)^3} \cdot \frac{1}{k+1} \cdot x^{k+1} \right) + C \\ &= \sum_{\ell=1}^{\infty} \left(\frac{2^{\ell-1}}{\ell^4} x^{\ell} \right) + C \quad \text{reindexing with } \ell = k+1. \end{aligned}$$

Note again that this sum agrees with $\int p_3(x) dx$ when $\ell = 0, 1, 2$, and 3 (as it should).

Problem 5 Give a power series (with interval of convergence) for the given functions.

(a) $f(x) = \frac{1}{1+x^2}$

(b) $f(x) = \tan^{-1}(x)$

(c) $f(x) = \tan^{-1}(3x^2)$

Solution: (a)

$$\begin{aligned} f(x) &= \frac{1}{1+x^2} \\ &= \sum_{k=0}^{\infty} (-x^2)^k \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}} \end{aligned}$$

when

$$|-x^2| < 1 \quad \Longleftrightarrow \quad -1 < x < 1.$$

So the interval of convergence is $\boxed{(-1, 1)}$.

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(b) Notice that

$$f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

on the interval $(-1, 1)$. So

$$\begin{aligned} f(x) &= \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k} \right) dx \\ &= \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} x^{2k+1} \right) + C. \end{aligned}$$

We know that this series converges (at least) on $(-1, 1)$. We need to first find C , and then check the endpoints for convergence.

To find C , just notice that

$$0 = \arctan(0) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} 0^{2k+1} \right) + C = 0 + C.$$

and so $C = 0$.

Now to check the endpoints, we plug in -1 and 1 . When $x = -1$, we have the series

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} \quad x = -1$$

‘ and when $x = 1$ the series is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}. \quad x = 1$$

Both of these series converge conditionally by the Alternating Series Test. Thus,

$$f(x) = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}}$$

with interval of convergence $\boxed{[-1, 1]}$.

(c) By part (b), if $|3x^2| \leq 1$ then

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (3x^2)^{2k+1} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{2k+1} x^{4k+2}}.$$

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This series converges when

$$\begin{aligned} & |3x^2| \leq 1 \\ \Rightarrow & |x^2| \leq \frac{1}{3} \\ \Rightarrow & -\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}. \end{aligned}$$

So the interval of convergence is

$$\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right].$$
