

Recitation #9: Trig Integrals and Trig Substitution

Group work:

Problem 1 Evaluate the following integrals

(a) $\int \tan^{23} x \sec^6 x \, dx$

Solution:

$$\begin{aligned} \int \tan^{23} x \sec^6 x \, dx &= \int \tan^{23} x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^{23} x (1 + \tan^2 x)^2 \sec^2 x \, dx. \end{aligned}$$

We now substitute

$$u = \tan x \quad \implies \quad du = \sec^2 x \, dx.$$

Then

$$\begin{aligned} \int \tan^{23} x (1 + \tan^2 x)^2 \sec^2 x \, dx &= \int u^{23} (1 + u^2)^2 \, du \\ &= \int u^{23} (1 + 2u^2 + u^4) \, du \\ &= \int (u^{23} + 2u^{25} + u^{27}) \, du \\ &= \frac{1}{24} u^{24} + \frac{1}{13} u^{26} + \frac{1}{28} u^{28} + C \\ &= \frac{1}{24} \tan^{24} x + \frac{1}{13} \tan^{26} x + \frac{1}{28} \tan^{28} x + C. \end{aligned}$$

(b) $\int \tan^2 x \sec x \, dx$ *Hint:* $\int \sec x \, dx = \ln |\sec x \tan x| + C$

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Solution:

$$\begin{aligned}
 \int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx \\
 &= \int \sec^3 x \, dx - \int \sec x \, dx \\
 &= \int \sec^3 x \, dx - \ln |\sec x \tan x| \quad \text{from the hint} \quad (1)
 \end{aligned}$$

Now, in an attempt to evaluate $\int \sec^3 x \, dx$, we use integration by parts with

$$\begin{aligned}
 u &= \sec x & dv &= \sec^2 x \, dx \\
 du &= \sec x \tan x \, dx & v &= \tan x.
 \end{aligned}$$

So

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx. \quad (2)$$

Combining equations (1) and (2) yields

$$\begin{aligned}
 \int \tan^2 x \sec x \, dx &= \int \sec^3 x \, dx - \ln |\sec x \tan x| \\
 \int \tan^2 x \sec x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx - \ln |\sec x \tan x| \\
 2 \int \tan^2 x \sec x \, dx &= \sec x \tan x - \ln |\sec x \tan x| + C \\
 \int \tan^2 x \sec x \, dx &= \frac{1}{2} (\sec x \tan x - \ln |\sec x \tan x|) + C.
 \end{aligned}$$

(c) $\int \tan^2 x \sin x \, dx$

Solution:

$$\begin{aligned}
 \int \tan^2 x \sin x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx \\
 &= \int \frac{1 - \cos^2 x}{\cos^2 x} \sin x \, dx.
 \end{aligned}$$

Now we substitute

$$u = \cos x \quad \implies \quad du = -\sin x \, dx, \quad -du = \sin x \, dx.$$

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This gives us that

$$\begin{aligned}\int \tan^2 x \sin x \, dx &= \int \frac{1-u^2}{u^2}(-1) \, du \\ &= \int \frac{u^2-1}{u^2} \, du \\ &= \int (1-u^{-2}) \, du \\ &= u + \frac{1}{u} + C \\ &= \cos x + \sec x + C.\end{aligned}$$

Problem 2 Evaluate

$$\int_{-\pi}^0 \sqrt{1 - \cos^2 x} \, dx.$$

Solution:

$$\begin{aligned}\int_{-\pi}^0 \sqrt{1 - \cos^2 x} \, dx &= \int_{-\pi}^0 \sqrt{\sin^2 x} \, dx \\ &= \int_{-\pi}^0 |\sin x| \, dx.\end{aligned}$$

Now, when $-\pi \leq x \leq 0$, $\sin x \leq 0$. Thus, on this region, $|\sin x| = -\sin x$. So we continue

$$\begin{aligned}\int_{-\pi}^0 \sqrt{1 - \cos^2 x} \, dx &= \int_{-\pi}^0 -\sin x \, dx \\ &= \left[\cos x \right]_{-\pi}^0 \\ &= \cos(0) - \cos(-\pi) = 1 - (-1) = 2.\end{aligned}$$

Problem 3 Evaluate the following integrals

(a)

$$\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3} \, dx.$$

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Solution: First notice that

$$\begin{aligned}\sqrt{36x^2 - 25} &= 5\sqrt{\frac{36x^2}{25} - 1} \\ &= 5\sqrt{\left(\frac{6x}{5}\right)^2 - 1}.\end{aligned}$$

So we substitute

$$\frac{6x}{5} = \sec \theta \quad \implies \quad x = \frac{5}{6} \sec \theta$$

which gives

$$dx = \frac{5}{6} \sec \theta \tan \theta d\theta.$$

Also, notice that

- when $x = -\frac{5}{3}$:

$$-\frac{5}{3} = \frac{5}{6} \sec \theta \quad \implies \quad \sec \theta = 2 \quad \implies \quad \theta = \frac{2\pi}{3}$$

- and when $x = -\frac{5}{6}$:

$$-\frac{5}{6} = \frac{5}{6} \sec \theta \quad \implies \quad \sec \theta = -1 \quad \implies \quad \theta = \pi.$$

Therefore

$$\begin{aligned}\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3} dx &= 5 \int_{\frac{2\pi}{3}}^{\pi} \frac{\sqrt{\sec^2 \theta - 1}}{\left(\frac{5}{6} \sec \theta\right)^3} \left(\frac{5}{6} \sec \theta \tan \theta\right) d\theta \\ &= 5 \cdot \left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan \theta| \tan \theta}{\sec^2 \theta} d\theta.\end{aligned}$$

Now, notice that $\tan \theta < 0$ whenever $\frac{2\pi}{3} \leq \theta \leq \pi$. So $|\tan \theta| = -\tan \theta$.

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We continue:

$$\begin{aligned}
 5 \cdot \left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan \theta| \tan \theta}{\sec^2 \theta} d\theta &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \frac{\tan^2 \theta}{\sec^2 \theta} d\theta \\
 &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{\cos^2 \theta}{1} d\theta \\
 &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \sin^2 \theta d\theta \\
 &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2} (1 - \cos(2\theta)) d\theta \\
 &= -\frac{18}{5} \left[\theta - \frac{1}{2} \sin(2\theta) \right]_{\frac{2\pi}{3}}^{\pi} \\
 &= -\frac{18}{5} \left[(\pi - 0) - \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4} \right) \right] \quad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\
 &= -\frac{18}{5} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right).
 \end{aligned}$$

(b)

$$\int \frac{dx}{(x^2 - 6x + 11)^2}.$$

Solution: We begin by completing the square in the denominator

$$x^2 - 6x - 11 = x^2 - 6x + 9 + 2 = (x - 3)^2 + 2.$$

We then have that

$$\begin{aligned}
 \int \frac{dx}{(x^2 - 6x + 11)^2} &= \int \frac{1}{((x - 3)^2 + 2)^2} dx \\
 &= \frac{1}{4} \int \frac{1}{\left(\frac{(x-3)^2}{2} + 1\right)^2} dx \\
 &= \frac{1}{4} \int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2 + 1\right)^2} dx.
 \end{aligned}$$

So we substitute

$$\frac{x - 3}{\sqrt{2}} = \tan \theta \quad \implies \quad x = \sqrt{2} \tan \theta + 3 \quad (3)$$

and then

$$dx = \sqrt{2} \sec^2 \theta d\theta.$$

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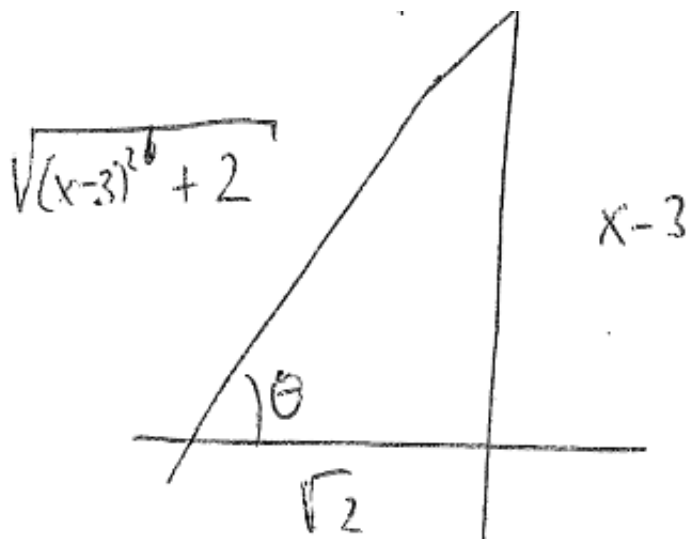
Continuing with the integral

$$\begin{aligned}\frac{1}{4} \int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2 + 1\right)^2} dx &= \frac{1}{4} \int \frac{1}{(\tan^2 \theta + 1)^2} \sqrt{2} \sec^2 \theta d\theta \\ &= \frac{\sqrt{2}}{4} \int \frac{1}{\sec^2 \theta} d\theta \\ &= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\ &= \frac{\sqrt{2}}{4} \int \frac{1}{2} (1 + \cos(2\theta)) d\theta \\ &= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C.\end{aligned}$$

Now all that is left to do is to reverse-substitute for θ . First, from equation (3) we have that

$$\theta = \arctan\left(\frac{x-3}{\sqrt{2}}\right).$$

Now, we again use equation (3) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta) = 2 \cdot \frac{x-3}{\sqrt{(x-3)^2 + 2}} \cdot \frac{\sqrt{2}}{\sqrt{(x-3)^2 + 2}}.$$

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Thus

$$\int \frac{dx}{(x^2 - 6x + 11)^2} = \frac{\sqrt{2}}{8} \left(\arctan \left(\frac{x-3}{\sqrt{2}} \right) + \frac{\sqrt{2}(x-3)}{(x-3)^2 + 2} \right) + C.$$
