Recitation # 4: Volume by Shells

Warm up:

Problem 1 *Determine whether you should integrate in terms of x or y in the given scenarios:*

(a) *You are revolving around the y-axis and using shells.*

Solution: *If you are revolving around the y-axis, then the axis is vertical. If you are using shells, the slices should be parallel to the axis so they should also be vertical. Therefore, you should integrate in terms of x.*

(b) *You are revolving around $y = 2$ and using washers.*

Solution: *If you are revolving around the line $y = 2$, then your axis is horizontal. If you are using washers, the slices should be perpendicular to the axis. Thus, the slices should be vertical. Thus, you should integrate in terms of x.*

Problem 2 *Determine whether you should use washers or shells:*

(a) *You are revolving around $x = 3$ and want to integrate in terms of y.*

Solution: *If you are revolving around $x = 3$, then the axis is vertical. If you want to integrate in terms of y, then you will have horizontal slices. Therefore, your slices will be perpendicular to the axis. Thus, you must use washers.*

(b) *You are revolving around the x-axis and want to integrate in terms of y.*

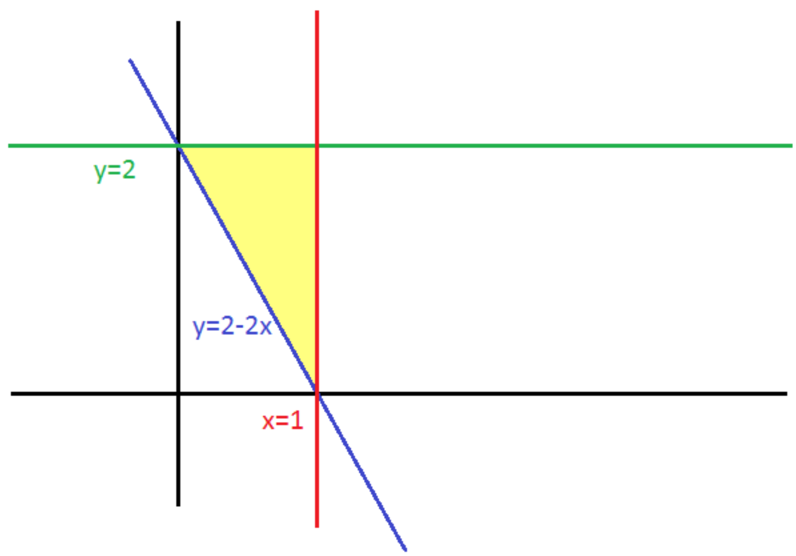
Solution: *If you are revolving around the x-axis, the axis is horizontal. If you want to integrate in terms of y, then you will have horizontal slices. Therefore, your slices will be parallel to the axis. Thus, you must use shells.*

Learning outcomes:

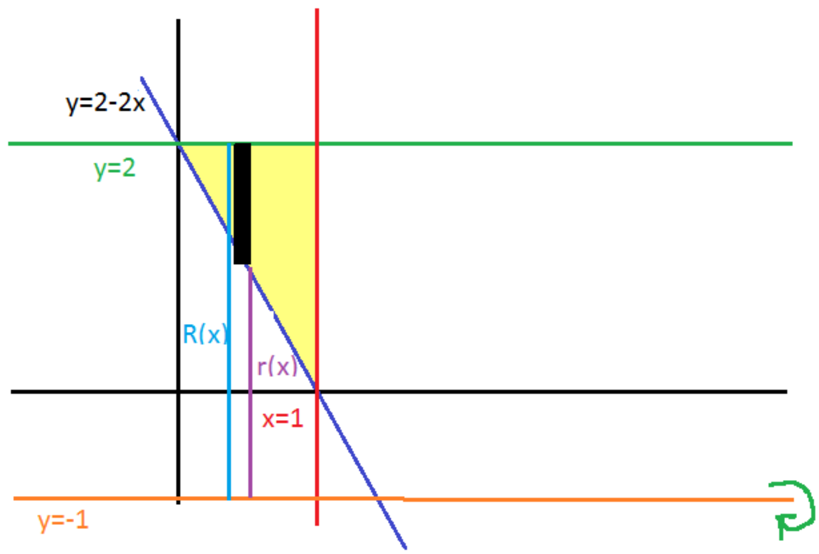
Group work:

Problem 3 *Consider the region bounded by $y = 2 - 2x$, $y = 2$, and $x = 1$. Use both the washer and shell method to find the volume of the solid formed by revolving this region around $y = -1$. Do your answers match?*

Solution:



<u>Washers</u>: *To use washers, we want slices perpendicular to the axis of revolution. The axis is $y = -1$ which is a horizontal line, so we want vertical slices. Therefore, we want to integrate in terms of x.*



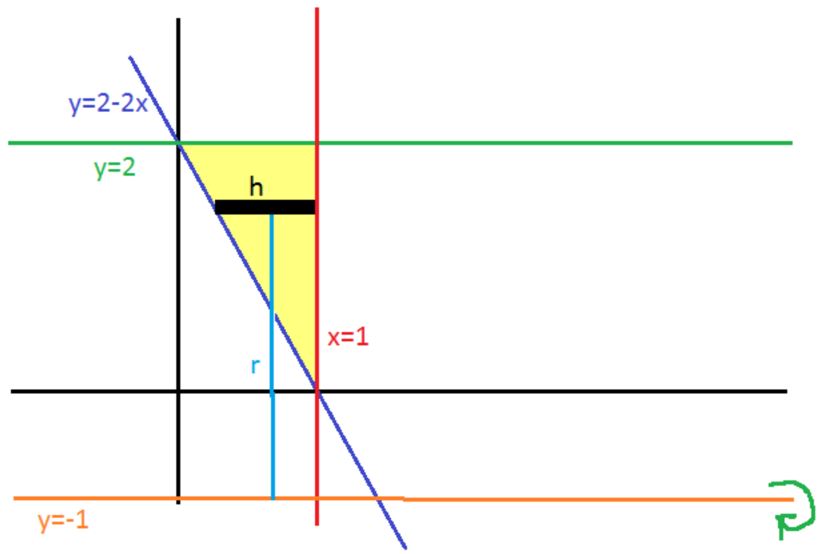
We can see from the figure that the x values range from 0 to 1. We have:

$$\begin{aligned} r_{out} &= R(x) = 2 - (-1) = 3 \\ r_{in} &= r(x) = (2 - 2x) - (-1) = 3 - 2x \end{aligned}$$

Thus,

$$\begin{aligned} V &= \pi \int_0^1 \left((3)^2 - (3 - 2x)^2 \right) \, dx \\ &= \pi \int_0^1 \left(9 - (9 - 12x + 4x^2) \right) \, dx \\ &= \pi \int_0^1 \left(-4x^2 + 12x) \right) \, dx \\ &= \pi \left[\frac{-4x^3}{3} + 6x^2 \right]_0^1 \\ &= \pi \left(\frac{-4(1)^3}{3} + 6(1)^2) \right) - \pi \left(\frac{-4(0)^3}{3} + 6(0)^2 \right) \\ &= \pi \frac{14}{3} \end{aligned}$$

Shells*: To use shells, we want slices parallel to the axis of revolution. The axis is $y = -1$ which is a horizontal line, so we want horizontal slices. Therefore,* *we want to integrate in terms of y. We need to solve $y = 2 - 2x$ for x. We get $x = 1 - \frac{y}{2}$.*



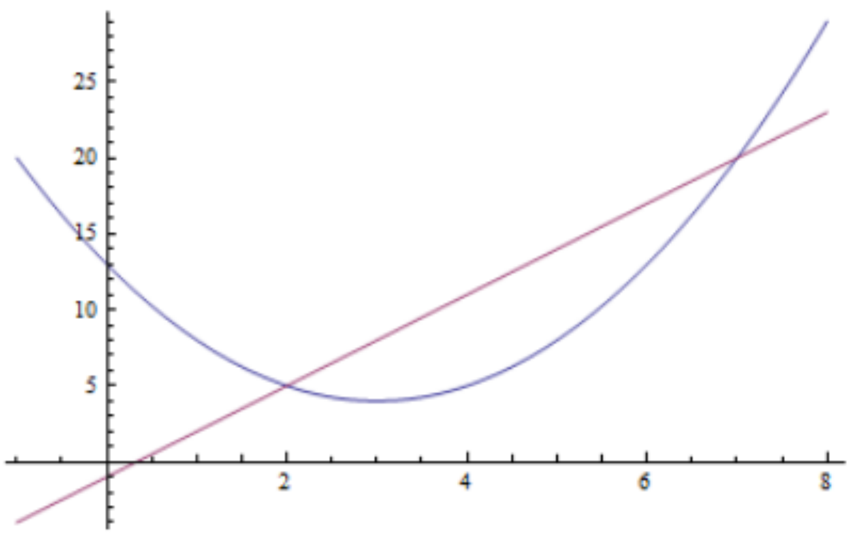
$$r = y - (-1) = y + 1$$
$$h = 1 - (1 - \frac{y}{2}) = 0.5y$$

We can see from the figure that y ranges from 0 to 2.

Thus,

$$\begin{aligned}
V &= 2\pi \int_0^2 (rh))\ dy \\
&= 2\pi \int_0^2 ((y+1)(0.5y))\ dy \\
&= \pi \int_0^2 \left(y^2 + y\right)\ dy \\
&= \pi \left[\frac{y^3}{3} + \frac{y^2}{2}\right]_0^1 \\
&= \pi \left(\frac{(2)^3}{3} + \frac{(2)^2}{2}\right) - \pi \left(\frac{(0)^3}{3} + \frac{(0)^2}{2}\right) \\
&= \pi \frac{14}{3}
\end{aligned}$$

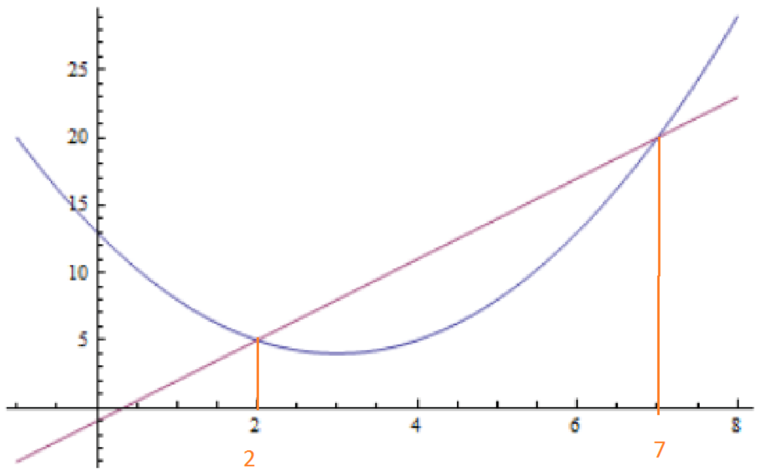
Problem 4 *Set up an integral that will compute the volume of the solid generated by revolving the region bounded by the curves* $y = x^2 - 6x + 13$ *(i.e.* $x = 3 \pm \sqrt{y-4}$*) and* $y = 3x - 1$ *about the given axes. Use the best/easiest method for each problem.*



(a) *the* x*-axis*

Solution: *First, we need to find the points where the curves intersect*

$$\begin{aligned} x^2 - 6x + 13 &= 3x - 1 \\ x^2 - 9x + 14 &= 0 \\ (x-2)(x-7) &= 0 \\ x &= 2, 7 \\ &(2,5), (7,20). \end{aligned}$$



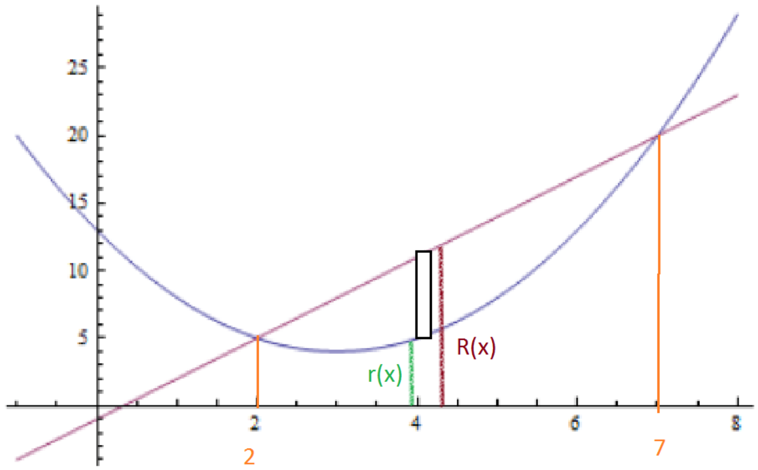
We can see from both the figure and the equations that we would prefer to integrate in terms of x. (If we were to integrate in terms of y, we would need two integrals.) Therefore, our slices will be vertical and perpendicular to the axis of rotation, so we will use washers.

Washers: *We have that*

$$\begin{aligned} r_{out} &= R(x) = 3x - 1 \\ r_{in} &= r(x) = x^2 - 6x + 13 \end{aligned}$$

and

$$\text{Volume of the region} = \pi \int_2^7 \left[(3x-1)^2 - (x^2 - 6x + 13)^2\right]\, dx.$$



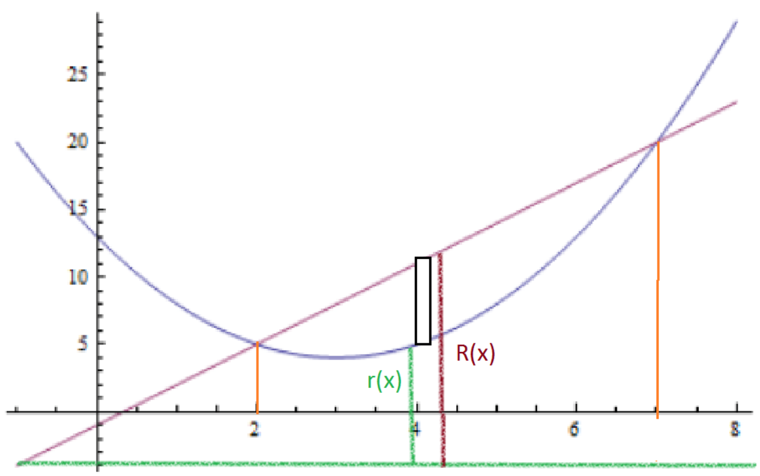
(b) $y = -4$

Solution: *Our axis is still horizontal so we will use washers again.*
Washers: *We have that*

$$r_{out} = (3x - 1) - (-4) = 3x + 3$$
$$r_{in} = (x^2 - 6x + 13) - (-4) = x^2 - 6x + 17$$

and

$$V = \pi \int_2^7 \left[(3x+3)^2 - (x^2 - 6x + 17)^2\right]\, dx.$$



(c) $y = 22$

Solution: **Washers:** *Again, since the axis is horizontal and we want to integrate with respect to x, we use washers. The only difference is the axis is above the curve this time. We have that*

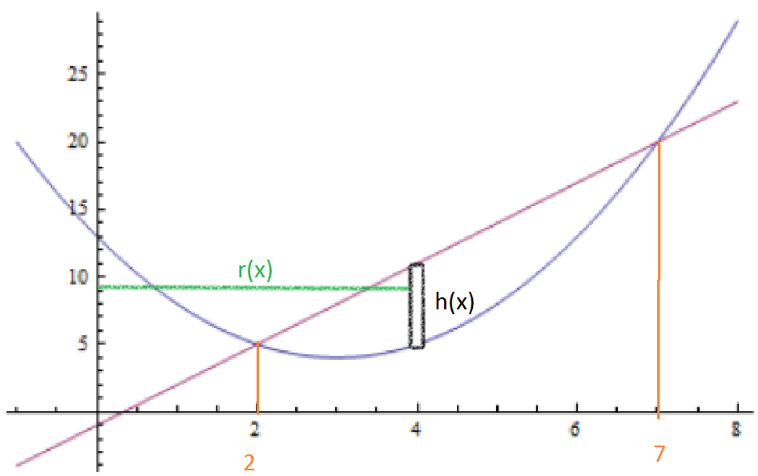
$$r_{out} = 22 - (x^2 - 6x + 13) = -x^2 + 6x + 9$$
$$r_{in} = 22 - (3x - 1) = -3x + 23$$

and so

$$V = \pi \int_2^7 \left[(-x^2 + 6x + 9)^2 - (-3x + 23)^2\right] dx.$$

(d) *the y-axis*

Solution: *Now we have a vertical axis. Since we want to integrate with respect to x and have parallel slices, we now need to use shells.*



Shells:

$$h = (3x - 1) - (x^2 - 6x + 13) = -x^2 + 9x - 14$$
$$r = x$$

So,

$$V = 2\pi \int_2^7 x(-x^2 + 9x - 14)\, dx.$$

(e) $x = -3$

Solution: *Again we have a vertical axis. Since we want to integrate with respect to x and have parallel slices, we use shells.*

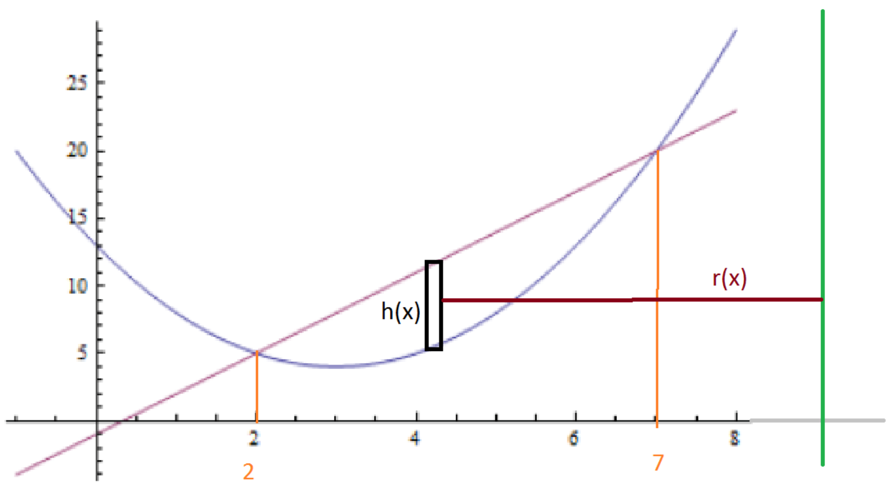
Shells:

Shifting the axis of rotation to $x = -3$ simply adds three to the radius of the shell, so

$$V = 2\pi \int_2^7 (3 + x)(-x^2 + 9x - 14)\, dx.$$

(f) $x = 9$

Solution: *Again we have a vertical axis. Since we want to integrate with respect to x and have parallel slices, we use shells.*



Shells: *Shifting the axis of rotation to the right 9 units (from part (d)) does not change the height of the cylinder, and the radius changes to $9-x$. So*

$$V = 2\pi \int_2^7 (9 - x)(-x^2 + 9x - 14)\, dx.$$