Recitation #13: Direction fields and Separable Differential Equations - Solutions

Warm up:

Which of the following differential equations are separable?

(a) $y' = \dfrac{ty}{t^2+1}$,

(b) $\dfrac{dy}{dx} = x^2 \sin(3y) - x^2$,

(c) $y' = t^2 - y$.

Solution: (a) Yes, it is separable. $y' = y \cdot \dfrac{t}{t^2+1}$.

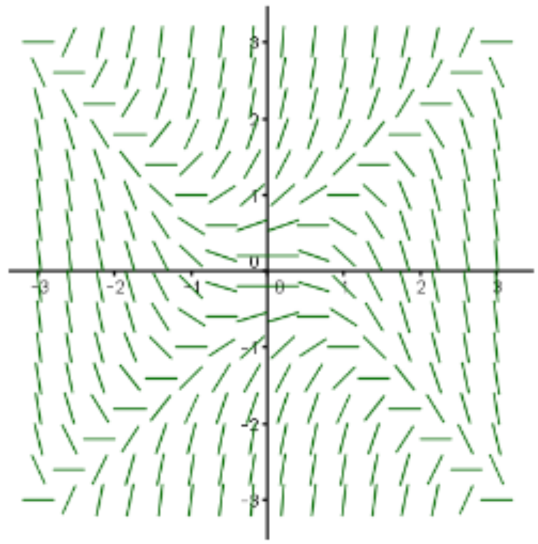
(b) Yes, it is separable. $\dfrac{dy}{dx} = x^2 (\sin(3y) - 1)$.

(c) No, it is not separable. $t^2 - y$ can not be written in the form $F(t) \cdot G(y)$.

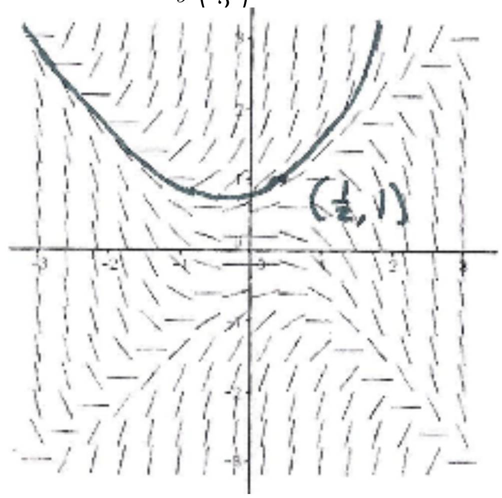
Group work:

Problem 1 (a) *The following is a direction field for the differential equation* $\dfrac{dy}{dx} = y^2 - x^2$.

Learning outcomes:



Sketch the solution such that $y\left(\frac{1}{2}\right) = 1$.



Solution:

(b) *Use Euler's Method to give a numerical estimate to the solution of the differential equation* $y' = y^2 - t^2$ *at* $y(2)$ *that goes through the point* $\left(\frac{1}{2}, 1\right)$. *Use* $\Delta t = 0.5$.

Solution:

T	y	$\frac{dy}{dt} = y^2 - t^2$	$y + \frac{dy}{dt} \cdot \Delta t$
0.5	1	0.75	1.375
1	1.375	0.890625	1.820313
1.5	1.820313	1.063538	2.352081
2	2.352081		

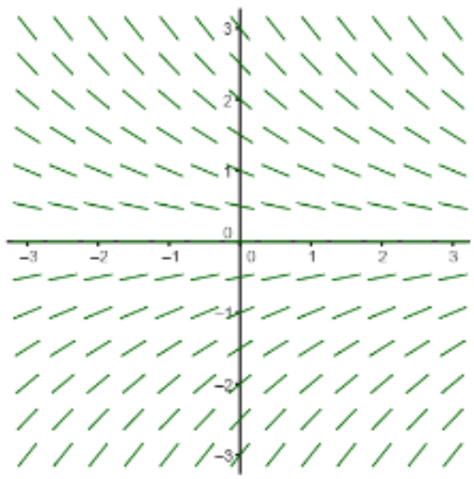
So, $y(2) \approx 2.352081$.

Problem 2 *Describe why the following direction field could be the direction field for the differential equation*

$$\frac{dy}{dt} = y\cos(t)$$

but ***not*** *for*

$$\frac{dy}{dt} = y\sin(t) \qquad \text{or} \qquad \frac{dy}{dt} = t\cos(y).$$



Solution: *Look along the line* $t = 0$ *(the* y*-axis).*

- *For* $\dfrac{dy}{dt} = y\sin(t)$*:*
  $\left[\dfrac{dy}{dt}\right]_{t=0} = y\sin(0) = 0$. *But this direction field does not have horizontal tangents at each point along* $t = 0$*. So it* ***cannot*** *be the direction field for* $\dfrac{dy}{dt} = y\sin(t)$.

- *For* $\dfrac{dy}{dt} = t\cos(y)$*:*
  $\left[\dfrac{dy}{dt}\right]_{t=0} = (0)\cos(y) = 0$. *But again this direction field does not have horizontal tangents at each point along* $t = 0$*. So it* ***cannot*** *be the direction field for* $\dfrac{dy}{dt} = t\cos(y)$.

- *For* $\dfrac{dy}{dt} y\cos(t)$*:*

  Along $t = 0$*, this equation is* $\dfrac{dy}{dt} = y\cos(0) = y$*. However, when* y *is positive, the slopes are negative. Also, when* y *is negative, the slopes are positive. So this is* **not** *the direction field for* $\dfrac{dy}{dt} = y\cos(t)$*, either.*

Problem 3 *Match each of the following differential equations with a corresponding direction field (if it is present):*

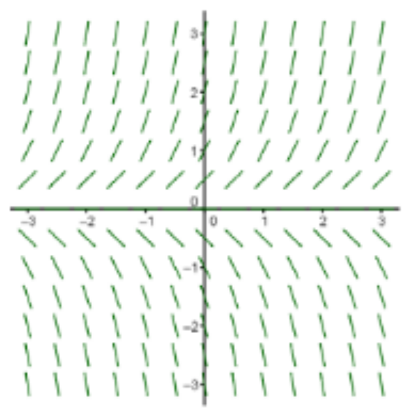
i. $y' = \dfrac{t}{2+y}$

ii. $y' = \cos(t+y)$

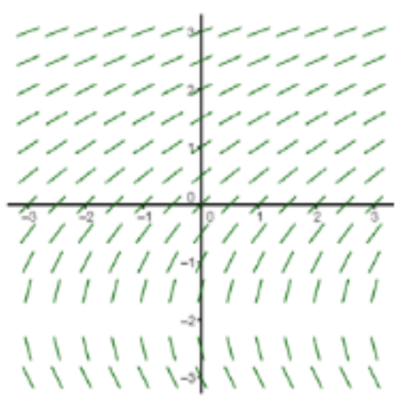
iii. $y' = 1 + y^2$

iv. $y' = ty$

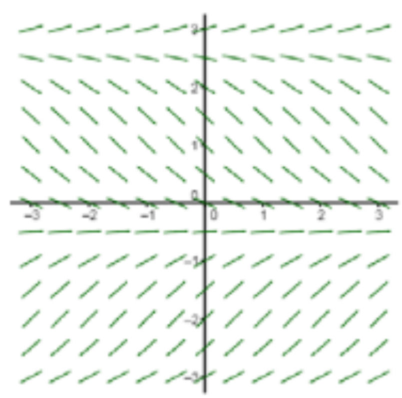
b.



b.



c.



d.

Solution: *Look along the line* $t = 0$ *(the* y*-axis).*

(i) $y' = \dfrac{t}{2+y}$ *and (iv)* $y' = ty$ *must both be identically* 0 *along the* y*-axis. However, none of the direction fields given have horizontal slopes along the* y*-axis. So none of them can be the direction field for (i) or (iv).*

(ii) At the origin, we have $y' = \cos(t+y) = \cos(0) = 1$*. So the direction field for (ii) must have slope* 1 *at the origin. This eliminates (b) and (d). Now look at the point* $\left(0, \dfrac{\pi}{2}\right)$*. We have* $\Big[y'\Big]_{(0,\frac{\pi}{2})} = \cos\left(\dfrac{\pi}{2}\right) = 0$*, so the direction field must be horizontal at that point. That means that it cannot be (a) or (c) either.*

(iii) Here $y' = 1 + y^2$*, so there is no* t *on the right hand side of the equation. Therefore,* y' *depends only on* y*. At* $y = 0$ *the slope is* 1*, then as* y *increases the slopes increase too. Similarly, as* y *gets more and more negative, the slope gets more and more positive. So it seems as if this direction field is (a).*

Problem 4 *Which of the following are separable differential equations? For those that are, solve them, assuming that* $y(4) = 5$*.*

(a) $y' = x^2 + y^2$

Solution: *This differential equation is* ***not*** *separable.*

(b) $y' = x + xy^2$

Solution:

$$\begin{aligned} & y' = x + xy^2 \\ \Longrightarrow \quad & \frac{dy}{dx} = x(1+y^2) \\ \Longrightarrow \quad & \frac{dy}{1+y^2} = x\,dx. \end{aligned}$$

So this equation ***is*** *separable. To solve, we integrate both sides of the equation:*

$$\begin{aligned} & \int \frac{1}{1+y^2}\,dy = \int x\,dx \\ \Longrightarrow \quad & \arctan(y) = \frac{1}{2}x^2 + C \qquad (1) \\ \Longrightarrow \quad & y = \tan\left(\frac{1}{2}x^2 + C\right). \end{aligned}$$

To find C, we plug the initial condition $y(4) = 5$ into equation (1) and solve for C.

$$\begin{aligned} \arctan(5) &= \frac{1}{2}(4)^2 + C = 8 + C \\ \implies \quad C &= \arctan(5) - 8. \end{aligned}$$

So

$$y = \tan\left(\frac{1}{2}x^2 + \arctan(5) - 8\right).$$

(c) $y' = e^{2x-y}$

Solution:

$$\begin{aligned} & y' = e^{2x-y} \\ \implies \quad & \frac{dy}{dx} = \frac{e^{2x}}{e^y} \\ \implies \quad & e^y\,dy = e^{2x}\,dx \end{aligned} \tag{2}$$

*and so this **is** a separable equation. To solve, we integrate both sides of equation (2).*

$$\int e^y\,dy = \int e^{2x}\,dx$$

$$\implies \quad e^y = \frac{1}{2}e^{2x} + C \tag{3}$$

$$\implies \quad y = \ln\left(\frac{1}{2}e^{2x} + C\right).$$

To find C, we plug into equation (3) and solve for C:

$$\begin{aligned} e^5 &= \frac{1}{2}e^8 + C \\ \implies \quad C &= e^5 - \frac{1}{2}e^8. \end{aligned}$$

Therefore

$$y = \ln\left(\frac{1}{2}e^{2x} + e^5 - \frac{1}{2}e^8\right).$$