Recitation #1 - Review of Integration

The following worksheet is designed to help review and/or sharpen your ability to differentiate and integrate functions encountered in a typical Calculus 1 course. These problems are all reasonable to expect to see on the quiz this coming Thursday.

Group work:

Problem 1 *Consider the function $f(x) = e^{2x}$. We know that $\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$ by the Chain Rule, and this lets us easily conclude that $\int e^{2x}\,dx = \frac{1}{2}e^{2x}$. This could of course be verified by u-substitution (if you know/remember this technique), but can also be understood the following way:*

The symbol $\int e^{2x}\,dx$ represents a function whose derivative is e^{2x}. Since taking a derivative of e^{2x} results in multiplying e^{2x} by 2, when we antidifferentiate e^{2x}, we must multiply by $\frac{1}{2}$.

You must be careful with this type of thought! Indeed, it works only when the argument of the function (in this case, the expression in the exponent) is LINEAR. "Linear in x" means the argument is of the form $ax + b$ in x!

(a) *Calculate $\frac{d}{dx}\left(e^{x^2}\right)$.*

Solution: $\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2}\left(2^x\right)' = e^{x^2}\left(2x\right)$

(b) *Suppose a student tries to apply the above logic to compute $\int e^{x^2}\,dx$. The student concludes that since $\frac{d}{dx}e^{x^2}\,dx = 2xe^{x^2}$, then:*

$$\int e^{x^2}\,dx = \frac{1}{2x}e^{x^2} \tag{1}$$

Since you know that $\int e^{x^2}\,dx$ is a function whose derivative is e^{x^2}, prove this student wrong by differentiating his/her answer (i.e. the RHS of Eqn 1).

Learning outcomes:

Solution: *First, the student forgot "+C" !*
Also, if the student is correct, then the derivative of his/her answer should be e^{x^2}*, but:*

$$\frac{d}{dx}\left(\frac{1}{2x}e^{x^2}\right) = \frac{\left(e^{x^2}\right)' \cdot 2x - e^{x^2}\,(2x)'}{(2x)^2} = \frac{4x^2e^{x^2} - 2e^{x^2}}{4x^2}$$

This is not e^{x^2}*!*

(c) *What insight does this reveal as to why this students' answer is wrong? Why can we think of antidifferentiating* e^{2x} *differently than antidifferentiating* e^{x^2}*?*

Solution: *Note that to differentiate* $\frac{1}{2x}e^{x^2}$*, we need the quotient rule!*
When we differentiate $\frac{1}{2}e^{x^2}$*, we don't need the quotient rule because* $\frac{1}{2}$ *is just a constant.*
Whenever the arguement is linear in x, the derivative of the argument will be a constant, so we won't need the quotient rule. Indeed, if $F'(x) = f(x)$*, then* $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$ *since*

$$\frac{d}{dx}\left(\frac{1}{a}F(ax+b)\right) = \frac{1}{a}F'(ax+b)\cdot(ax+b)' = \frac{1}{a}f(ax+b)\cdot a = f(ax+b)$$

Problem 2 *Categorize the following integrals into ones you know how to integrate after taking Calculus 1 and learning section 7.1 and ones you do not know how to integrate. If you have time, try to evaluate the integrals.*

a) $\int \left(3x^4 - \sqrt[3]{x^2} + \frac{2}{\sqrt[7]{x}}\right) dx$	*b)* $\int e^{x^2}\,dx$	*c)* $\int_0^{\pi/6} 4\sin(2x)\,dx$
d) $\int e^{-\frac{x}{3}}\,dx$	*e)* $\int \ln x\,dx$	*f)* $\int \frac{4x^3-3x}{2x^2}\,dx$
g) $\int \left(\sec(4x)\tan(4x) + 3\sec^2\frac{x}{5}\right) dx$	*h)* $\int_1^4 (\sqrt{x}-1)^2\,dx$	*i)* $\int_0^1 \sqrt{e^{3x}}\,dx$
j) $\int \cot^2(3x)\sec^2(3x)\,dx$	*k)* $\int \cos\sqrt{x}\,dx$	*l)* $\int \frac{2}{(3x)^2}$

Solution: *See attached solutions section II*

On your own:

Problem 3 *Differentiate the following functions.*

$a)\ y = (2x-7)^4$	$b)\ y = e^{\frac{x}{4}}$	$c)\ y = 7x^4 - 3\sqrt[5]{x} + \dfrac{2}{5x^2}$
$d)\ y = \ln(2x + \cos x)$	$e)\ y = 2xe^{-x}$	$f)\ y = \dfrac{\tan(3x)}{\sqrt{4-x}}$
$g)\ y = \csc\left(e^{4x}\right)$	$h)\ y = [\ln(4x^3 - 2x)]^3$	$i)\ y = e^{4\sqrt{x}}$
$j)\ y = 4e^{x \sin x}$	$k)\ y = 6x^9 - \dfrac{1}{8x^4} + \dfrac{2}{\sqrt[3]{2x-1}}$	$l)\ y = \dfrac{2}{(3x^2-1)^2}$

Solution: *See attached solutions section I*

Worksheet #1 Answers

Detailed Solutions follow this sheet!

I. a) $8(2x-7)^3$

b) $\frac{1}{4}e^{\frac{1}{4}x}$

c) $28x^3 - \frac{3}{5}x^{-\frac{4}{5}} - \frac{4}{5}x^{-3}$

d) $\dfrac{2-\sin x}{2x+\cos x}$

e) $2e^{-x} - 2xe^{-x}$

f) $\dfrac{3(4-x)^{1/2}\sec^2 3x + \frac{1}{2}(4-x)^{-1/2}\tan 3x}{4-x}$

g) $-4e^{4x}\csc(e^{4x})\cot(e^{4x})$

h) $3[\ln(4x^3-2x)]^2 \cdot \dfrac{12x^2-2}{4x^3-2x}$

i) $2x^{-1/2}e^{4x^{1/2}}$

j) $4e^{x\sin x}(\sin x + x\cos x)$

k) $54x^8 + \frac{1}{2}x^{-5} - \frac{4}{3}(2x-1)^{-4/3}$

l) $-24x(3x^2-1)^{-3}$

II. a) $\frac{3}{5}x^5 - \frac{3}{5}x^{5/3} - \frac{14}{6}x^{6/7} + C$

b) Cannot Integrate

c) 1

d) $-3e^{-\frac{1}{3}x} + C$

e) Cannot Integrate yet

f) $x^2 - \frac{3}{2}\ln|x| + C$

g) $\frac{1}{4}\sec 4x + 15\tan\frac{1}{5}x + C$

h) $\frac{5}{6}$

i) $\frac{2}{3}e^{3/2} - \frac{2}{3}$

j) $-\frac{1}{3}\cot 3x + C$.

k) Cannot integrate!

l) $-\frac{2}{9}x^{-1} + C$

Worksheet # 1 Solutions

I. a) $y = (2x-7)^4$

$y' = 4(2x-7)^3(2x-7)'$

$y' = 4(2x-7)^3 \cdot 2$

$\boxed{y' = 8(2x-7)^3}$ ← It's ok just to write this! The previous steps are meant to make sure everyone's ~~on~~ the same page.

b) $y = e^{\frac{x}{4}}$

$y = e^{\frac{1}{4}x}$ ← It's helpful to separate numbers and variables. Once again, you don't have to show this if you're comfortable with the algebra

$y' = e^{\frac{1}{4}x}\left(\frac{1}{4}x\right)'$

$\boxed{y' = \frac{1}{4}e^{\frac{1}{4}x}}$

c) $y = 7x^4 - 3\sqrt[5]{x} + \frac{2}{5x^2}$ ←

$y = 7x^4 - 3x^{\frac{1}{5}} + \frac{2}{5}\frac{1}{x^2}$ ←

In order to apply $\frac{d}{dx}(x^n) = nx^{n-1}$, we have to rewrite all of these terms as fractional / negative powers of x!

$y = 7x^4 - 3x^{\frac{1}{5}} + \frac{2}{5}x^{-2}$

$\boxed{y' = 28x^3 - \frac{3}{5}x^{-\frac{4}{5}} - \frac{4}{5}x^{-3}}$

d) $y = \ln(2x + \cos x)$

$y' = \frac{1}{2x+\cos x}(2x+\cos x)'$

$\boxed{y' = \frac{1}{2x+\cos x}(2-\sin x)}$

e) $y = 2xe^{-x}$

$y' = (2x)' e^{-x} + 2x (e^{-x})'$

$y' = 2e^{-x} + 2x\, e^{-x} (-x)'$

$\boxed{y' = 2e^{-x} - 2xe^{-x}}$

f) $y = \dfrac{\tan 3x}{\sqrt{4-x}}$

$y = \dfrac{\tan 3x}{(4-x)^{1/2}}$

$y' = \dfrac{(\tan 3x)' (4-x)^{1/2} - \tan 3x \left[(4-x)^{1/2}\right]'}{\left[(4-x)^{1/2}\right]^2}$

$y' = \dfrac{\sec^2 3x \cdot (3x)' (4-x)^{1/2} - \tan 3x \left[\frac{1}{2}(4-x)^{-1/2} \cdot (4-x)'\right]}{(4-x)}$

$= \boxed{\dfrac{3\sec^2 3x\, (4-x)^{1/2} + \frac{1}{2}\tan 3x\, (4-x)^{-1/2}}{4-x}}$

g) $y = \csc(e^{4x})$

$y' = -\csc e^{4x} \cot e^{4x} (e^{4x})'$

$\boxed{y' = -\csc e^{4x} \cot e^{4x} (e^{4x} \cdot 4)}$

h) $y = [\ln(4x^3 - 2x)]^3$

$y' = 3[\ln(4x^3-2x)]^2 \cdot [\ln(4x^3-2x)]'$

$y' = 3[\ln(4x^3-2x)]^2 \frac{1}{4x^3-2x} (4x^3-2x)'$

$$\boxed{y' = 3[\ln(4x^3-2x)]^2 \cdot \frac{1}{4x^3-2x} (12x^2-2)}$$

i) $y = e^{4\sqrt{x}}$

$y = e^{4x^{1/2}}$

$y' = e^{4x^{1/2}} (4x^{1/2})'$

$$\boxed{y' = e^{4x^{1/2}} \cdot 2x^{-1/2}}$$

j) $y = 4e^{x \sin x}$

$y' = 4e^{x\sin x} (\underbrace{x}\underbrace{\sin x}_{\text{Product Rule}})'$

$$\boxed{y' = 4e^{x\sin x}(\sin x + x\cos x)}$$

k) $y = 6x^9 - \frac{1}{8x^4} + \frac{2}{\sqrt[3]{2x-1}}$

$y = 6x^9 - \frac{1}{8}\frac{1}{x^4} + \frac{2}{(2x-1)^{1/3}}$

$y = 6x^9 - \frac{1}{8}x^{-4} + 2(2x-1)^{-1/3}$

$y' = 54x^8 + \frac{1}{2}x^{-5} - \frac{2}{3}(2x-1)^{-4/3} \cdot (2x-1)'$

$$\boxed{y' = 54x^8 + \frac{1}{2}x^{-5} - \frac{4}{3}(2x-1)^{-4/3}}$$

e) $y = \frac{2}{(3x^2-1)^2}$

$y = 2(3x^2-1)^{-2}$

$y' = -4(3x^2-1)^{-3}(3x^2-1)'$

$y' = -4(3x^2-1)^{-3}(6x)$

$\boxed{y' = -24x(3x^2-1)^{-3}}$

II. a) $\int \left(3x^4 - \sqrt[3]{x^2} + \frac{2}{\sqrt[7]{x}}\right) dx$

$= \int \left(3x^4 - x^{2/3} + 2 \cdot \frac{1}{x^{1/7}}\right) dx$

$= \int \left(3x^4 - x^{2/3} + 2x^{-1/7}\right) dx$

$= \boxed{\frac{3}{5}x^5 - \frac{3}{5}x^{5/3} - \frac{14}{6}x^{6/7} + C}$

b) Cannot be integrated! (In fact, one can prove there's no elementary derivative of e^{x^2}!).

* If you tried to integrate this, take the derivative of your answer. Is it equal to e^{x^2}?

c) $\int_0^{\pi/6} 4 \sin 2x \, dx$

$= -4 \cdot \frac{1}{2} \cos 2x \Big|_0^{\pi/6}$

$= -2 \cos \frac{\pi}{3} - (-2\cos 0)$

$= -2 \cdot \frac{1}{2} + 2$

$= \boxed{1}$

d) $\int e^{-\frac{x}{3}}\,dx$

$= \int e^{-\frac{1}{3}x}\,dx$

$= \boxed{-3e^{-\frac{1}{3}x} + C}$

e) $\boxed{\text{Cannot integrate yet!}}$ (We will learn how to do this later in the semester though!)

(For those curious: $\int \ln x\,dx = x\ln x - x + C$; you can check this by differentiating $x\ln x - x + C$!).

f) $\int \frac{4x^3 - 3x}{2x^2}\,dx$

DO NOT WRITE $\int \frac{4x^3-3x}{2x^2}\,dx = \frac{\int (4x^3-3x)\,dx}{\int 2x^2\,dx}$!

We need to write this as a sum of powers of x!

$$\int \frac{4x^3-3x}{2x^2}\,dx = \int \left(\frac{4x^3}{2x^2} - \frac{3x}{2x^2}\right)dx$$

$$= \int \left(\frac{4}{2}\cdot\frac{x^3}{x^2} - \frac{3}{2}\,\frac{x}{x^2}\right)dx$$

$$= \int \left(2x - \frac{3}{2}\,\frac{1}{x}\right)dx$$

$$= \boxed{x^2 - \frac{3}{2}\ln|x| + C}$$

g) $\int \left(\sec 4x \tan 4x + 3\sec^2 \frac{1}{5}x\right)dx$

$= \int \sec 4x \tan 4x\,dx + \int 3\sec^2 \frac{1}{5}x\,dx$

$$= \boxed{\frac{1}{4}\sec 4x + 3\cdot(5\tan\tfrac{1}{5}x) + C}$$

h) $\int_1^4 (\sqrt{x}-1)^2 \, dx$

$\downarrow$ FOIL

$$= \int_1^4 (x - 2\sqrt{x} + 1)\, dx$$

$$= \int_1^4 (x - 2x^{1/2} + 1)\, dx$$

$$= \left[\frac{1}{2}x^2 - \frac{4}{3}x^{3/2} + x\right|_1^4$$

$$= \left[\frac{1}{2}(4)^2 - \frac{4}{3}(4)^{3/2} + 4\right] - \left[\frac{1}{2}(1)^2 - \frac{4}{3}(1)^{3/2} + 1\right]$$

$$= \boxed{\frac{5}{6}}$$

i) $\int_0^1 \sqrt{e^{3x}}\, dx$

$$= \int_0^1 e^{\frac{3}{2}x}\, dx$$

$$= \left[\frac{2}{3}\cdot e^{\frac{3}{2}x}\right|_0^1$$

$$= \frac{2}{3}e^{\frac{3}{2}} - \frac{2}{3}e^0$$

$$= \boxed{\frac{2}{3}e^{3/2} - \frac{2}{3}}$$

j) $\int \cot^2 3x \sec^2 3x \, dx$

$$= \int \frac{\cos^2 3x}{\sin^2 3x} \sec^2 3x \, dx$$

$$= \int \frac{1}{\sin^2 3x}\, dx$$

(j) $= \int \csc^2 3x \, dx$

$= \boxed{-\frac{1}{3} \cot 3x + C}$

k) $\boxed{\text{Cannot integrate!}}$ (This can be shown to possess no elementary antiderivative!).

* If you tried to antidifferentiate this, differentiate your answer. Is it equal to $\cos\sqrt{x}$?

l) $\int \frac{2}{(3x)^2} \, dx$

$= \int \frac{2}{9x^2} \, dx$

$= \int \frac{2}{9} \frac{1}{x^2} \, dx$

$= \int \frac{2}{9} x^{-2} \, dx$

$= \boxed{-\frac{2}{9} x^{-1} + C}$