Sections 6.8 and 6.9: Exponential Models

Group work:

Problem 1 *Vitameatavegamin is a strange substance that comes in two forms. V-I decays at a linear rate, while V-II decays at an exponential rate. Both have the property that 10 ounces will decrease to 7 ounces in 6 hours. For each of V-I and V-II, answer the following:*

(a) *If we started with 80 ounces, how much will there be 6 hours later?*

Solution: ***V-I:*** *Recall that in the linear decay model*

$$y(t) = -k \cdot t + y_0$$

where k denotes the rate of decay and y_0 is the initial amount. We are given that $y_0 = 80oz$. Clearly, we also have that

$$y'(t) = -k.$$

In the linear decay model, the rate of decay does not depend on the initial amount. So from the given information, we have that

$$-k = \frac{10oz - 7oz}{0hr - 6hr} = -\frac{1}{2}.$$

Thus, $y(t) = -\frac{1}{2}t + 80$, and therefore

$$y(6) = -\frac{1}{2}(6) + 80 = 77oz.$$

V-II: *Recall that in the exponential decay model*

$$y(t) = y_0 \cdot e^{-k \cdot t}$$

where again $y_0 = 80oz$ is the initial amount. Also notice that

$$\begin{aligned} y'(t) &= -ky_0e^{-kt} \\ &= -ky(t) \\ \implies \quad y'(0) &= -ky_0. \end{aligned}$$

Learning outcomes:

It is given that it takes 6 hours for 10 ounces to decrease to 7 ounces. In other words, it takes 6 hours for 70% of the substance to remain. So we have that

$$\begin{aligned} y(6) &= \frac{7}{10}y_0 \\ \implies \qquad y_0e^{-k\cdot 6} &= \frac{7}{10}y_0 \\ \implies \qquad e^{-6k} &= \frac{7}{10} \\ \implies \qquad -6k &= \ln\left(\frac{7}{10}\right) = -\ln\left(\frac{10}{7}\right) \\ \implies \qquad k &= \frac{1}{6}\ln\left(\frac{10}{7}\right). \end{aligned}$$

Thus,

$$\begin{aligned} y(6) &= 80e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot 6} \\ &= 80e^{-\ln\left(\frac{10}{7}\right)} \\ &= 80\cdot\frac{7}{10} = 56oz. \end{aligned}$$

(b) *How long will it take to decrease from 15 ounces to 7.5 ounces?*

Solution: ***V-I:*** *Recall from above that* $k = \frac{1}{2}$. *Then since* y_0 *is now 15, we have that*

$$y(t) = -\frac{1}{2}t + 15.$$

We want to find t *such that* $y(t) = 7.5$. *So we solve*

$$\begin{aligned} 7.5 &= -\frac{1}{2}t + 15 \\ -\frac{15}{2} &= -\frac{1}{2}t \\ t &= 15 \textit{ hours.} \end{aligned}$$

V-II: *Again, since* y_0 *is now 15, we know from above that*

$$y(t) = 15e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t}.$$

We want to find t *such that* $y(t) = 7.5 = \dfrac{15}{2}$. *So we solve*

$$\begin{aligned}
\frac{15}{2} &= 15e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t} \\
\frac{1}{2} &= e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t} \\
\ln\left(\frac{1}{2}\right) &= -\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t \\
\ln\left(\frac{10}{7}\right)t &= -6\ln\left(\frac{1}{2}\right) = 6\ln 2 \\
t &= \frac{6\ln 2}{\ln\left(\frac{10}{7}\right)} \text{ hours.}
\end{aligned}$$