Recitation #26: Vectors in three dimensions and Dot Products - Full

Warm up:

If $\vec{u} = \hat{\imath} - 2\hat{\jmath}$ and $\vec{v} = 3\hat{\imath} + 4\hat{k}$, find $\vec{u} \cdot \vec{v}$.

Solution: Note that these vectors are in $\mathbb{R}^3$ and not $\mathbb{R}^2$.

$$\vec{u} \cdot \vec{v} = (1 \cdot 3) + (-2 \cdot 0) + (0 \cdot 4) = \boxed{3}.$$

Instructor Notes: Make sure that students realize that $\vec{u} = \langle 3, 4, 0\rangle$ and not $\langle 3, 4\rangle$.

Group work:

Problem 1 *Solve the following problems:*

(a) *Which of the points $(6, 2, 3)$, $(-5, -1, 4)$, and $(0, 3, 8)$ is closest to the xz-plane? Which point lies on the yz-plane?*

(b) *Write an equation of the circle of radius 2 centered at $(-3, 4, 1)$ that lies in a plane parallel to the xy-plane.*

(c) *Describe the sphere $x^2 + y^2 + z^2 + 6x - 14y - 2z = 5$ (ie, find its center and radius).*

(d) *Find a vector whose magnitude is 311 and is in the same direction as the vector $\langle 3, -6, 7\rangle$.*

Solution: (a) *The xz-plane has equation $y = 0$. The distance from a point (a, b, c) to $y = 0$ is just $|b|$. So*

$$\begin{aligned} &(6, 2, 3) \text{ has distance } 2 \\ &(-5, -1, 4) \text{ has distance } 1 \\ &(0, 3, 8) \text{ has distance } 3 \end{aligned}$$

Learning outcomes:

Therefore, the point $(-5,-1,4)$ *is closest to the* xz*-plane.*

The yz*-plane is* $x=0$*, and so the point* $(0,3,8)$ *is on the* yz*-plane.*

(b) *A plane parallel to the* xy*-plane has equation* $z=\#$*. We are looking for such a plane containing the point* $(-3,4,1)$*, and so the plane is* $z=1$*. Therefore, the equation is*

$$\boxed{(x+3)^2+(y-4)^2=4\ ,\ z=1.}$$

(c) *We complete the square with respect to all three variables.*

$$\begin{aligned} x^2+y^2+z^2+6x-14y-2z &= 5 \\ (x^2+6y+9)+(y^2-14y+49)+(z^2-2z+1) &= 5+9+49+1 \\ (x+3)^2+(y-7)^2+(z-1)^2 &= 64. \end{aligned}$$

So, the center of the sphere is $(-3,7,1)$ *and its radius is* 8*.*

(d) *Let* $\vec{v}=\langle 3,-6,7\rangle$*. Then*

$$\begin{aligned} |\vec{v}| &= \sqrt{3^2+(-6)^2+7^2} \\ &= \sqrt{9+36+49} \\ &= \sqrt{94}. \end{aligned}$$

So a unit vector in the same direction as $\vec{v}$ *is*

$$\frac{1}{\sqrt{94}}\langle 3,-6,7\rangle$$

and therefore a vector with magnitude 311 *in the same direction as* v *is*

$$\boxed{\frac{311}{\sqrt{94}}\langle 3,-6,7\rangle}$$

Instructor Notes:

Problem 2 *Find a vector (in the* xy*-plane) with length* 4 *that makes a* $\dfrac{\pi}{3}$ *radian angle with the vector* $\langle 3,4\rangle$*.*

Solution: *Let* $\vec{v}=\langle a,b\rangle$ *denote a vector that we are looking for, and let* $\vec{u}=\langle 3,4\rangle$*. First note that*

$$|\vec{u}|=\sqrt{9+16}=5.$$

So

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\left(\frac{\pi}{3}\right) = 5 \cdot 4 \cdot \frac{1}{2} = 10.$$

Then we have the following two equations:

$$10 = \vec{u} \cdot \vec{v} = 3a + 4b \tag{1}$$
$$16 = |\vec{v}|^2 = a^2 + b^2. \tag{2}$$

Solving equation (3) for a gives us

$$a = \frac{10 - 4b}{3}.$$

Plugging this into equation (4) yields

$$\begin{aligned}
\left(\frac{10 - 4b}{3}\right)^2 + b^2 &= 16 \\
(10 - 4b)^2 + 9b^2 &= 144 \\
16b^2 - 80b + 100 + 9b^2 &= 144 \\
25b^2 - 80b - 44 &= 0
\end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned}
b &= \frac{80 \pm \sqrt{(-80)^2 - 4(25)(-44)}}{2(25)} \\
&= \frac{80 \pm \sqrt{10800}}{50} \\
&= \frac{80 \pm 60\sqrt{3}}{50} \\
&= \frac{8 \pm 6\sqrt{3}}{5}.
\end{aligned}$$

We can choose either value for b. Choosing $b = \dfrac{8 + 6\sqrt{3}}{5}$ gives a value of $a = \dfrac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}$. Thus,

$$\vec{v} = \boxed{\left\langle \frac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}, \frac{8 + 6\sqrt{3}}{5} \right\rangle}$$

Instructor Notes: The students need to assimilate a lot of information in this problem. They need to "name" the unknown vector (say $\langle a, b\rangle$). Then, they need to realize that both $\langle 3,4\rangle \cdot \langle 3,4\rangle = 3a+4b$ and that $\left|\langle 3,4\rangle\right| \cdot \left|\langle a,b\rangle\right| \cos\left(\frac{\pi}{3}\right) = 5 \cdot 4 \cdot \frac{1}{2}$, giving $3a+4b=10$. Lastly, they also need to realize that $a^2+b^2=16$. A picture illustrating that there could be two such vectors would be helpful.

Problem 3 *Answer the following questions about* $proj_v u$.

(a) *Is* $proj_v u$ *a vector of the form* $c\vec{v}$ *or* $c\vec{u}$ *(where* c *is a real number)? ie, is* $proj_v u$ *parallel to* $\vec{u}$ *or* $\vec{v}$*?*

(b) *If* $\vec{u} = 5\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$ *and* $\vec{v} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$*, find* $proj_v u$.

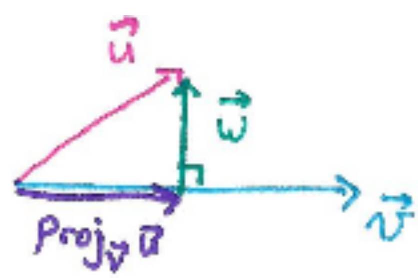
(c) *For* $\vec{u}$ *and* $\vec{v}$ *from part (b), write* $\vec{u}$ *as the sum of two perpendicular vectors, one of which is parallel to* $\vec{v}$.

Solution: (a) $\boxed{c\vec{v}}$

(b)

$$\begin{aligned} proj_v u &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\ &= \frac{10-24-12}{4+16+16}\langle 2,-4,4\rangle \\ &= \boxed{-\frac{13}{18}\langle 2,-4,4\rangle} \end{aligned}$$

(c) *A schematic picture of the situation is as follows:*



The vector which is parallel to $\vec{v}$ *is*

$$proj_v u = \boxed{\left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9} \right\rangle}$$

The vector which is orthogonal to $\vec{v}$ is

$$\vec{w} := \vec{u} - \text{proj}_v u = \langle 5, 6, -3 \rangle - \left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9} \right\rangle$$
$$= \boxed{\left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9} \right\rangle}$$

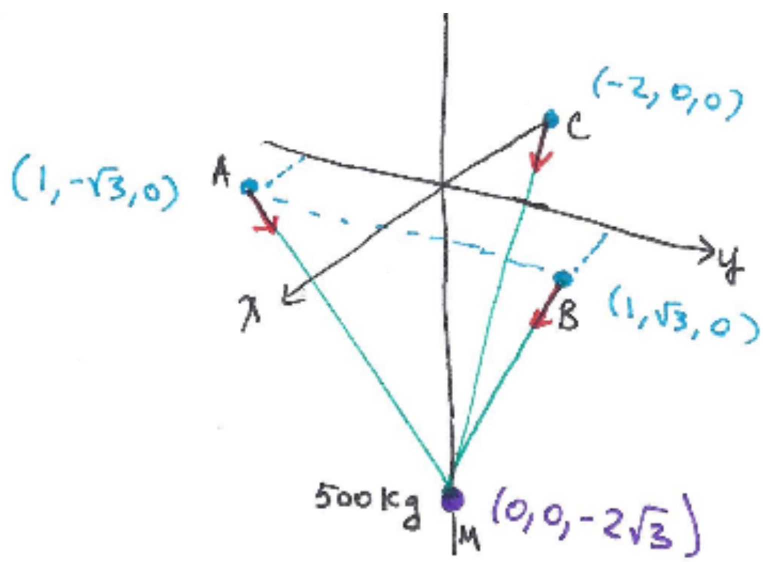
And, clearly, $\text{proj}_v u + \vec{w} = \text{proj}_v u + (\vec{u} - \text{proj}_v u) = \vec{u}$.

Instructor Notes: Working with projections.

Problem 4 *A* $500kg$ *lead hangs from three cables of equal length that are located at the points* $(-2, 0, 0)$, $(1, \sqrt{3}, 0)$, *and* $(1, -\sqrt{3}, 0)$. *The load is located at* $(0, 0, -2\sqrt{3})$. *Find the vectors describing the forces on the cables due to the load.*

Solution: *Let* $A = (1, -\sqrt{3}, 0)$, $B = (1, \sqrt{3}, 0)$, *and* $C = (-2, 0, 0)$, *and let* $M = (0, 0, -2\sqrt{3})$. *Let* $\vec{a}$, $\vec{b}$, *and* $\vec{c}$ *denote the vectors from* A, B, *and* C *to* M, *respectively. ie,*

$$\vec{a} = \langle 0 - 1, 0 - (-\sqrt{3}), -2\sqrt{3} - 0 \rangle = \langle -1, \sqrt{3}, -2\sqrt{3} \rangle$$
$$\vec{b} = \langle 0 - 1, 0 - \sqrt{3}, -2\sqrt{3} - 0 \rangle = \langle -1, -\sqrt{3}, -2\sqrt{3} \rangle$$
$$\vec{c} = \langle 0 - (-2), 0 - 0, -2\sqrt{3} - 0 \rangle = \langle 2, 0, -2\sqrt{3} \rangle.$$



Notice that

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 4$$

and so unit vectors in the directions of $\vec{a}$, $\vec{b}$, *and* $\vec{c}$ *are*

$$\begin{aligned}
\vec{u}_a &= \frac{1}{4}\langle -1, \sqrt{3}, -2\sqrt{3}\rangle \\
\vec{u}_b &= \frac{1}{4}\langle -1, -\sqrt{3}, -2\sqrt{3}\rangle \\
\vec{u}_c &= \frac{1}{4}\langle 2, 0, -2\sqrt{3}\rangle.
\end{aligned}$$

The force on M *due to gravity is*

$$\langle 0, 0, -500g\rangle$$

where g *is the gravitational constant. We need to find real numbers* x, y, *and* z *such that*

$$-\frac{1}{4}x - \frac{1}{4}y + \frac{1}{2}z = 0 \qquad \Longrightarrow \qquad x + y - 2z = 0 \tag{3}$$

$$\frac{\sqrt{3}}{4}x - \frac{\sqrt{3}}{4}y + 0z = 0 \qquad \Longrightarrow \qquad x - y = 0 \tag{4}$$

$$-\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}y - \frac{\sqrt{3}}{2}z = -500g \quad \Longrightarrow \quad x + y + z = \frac{1000g}{\sqrt{3}}. \tag{5}$$

By equation (4) *we have that* $x = y$. *Substituting this into equation* (3) *we also see that* $x = z$. *So* $x = y = z$. *We plug this into equation* (5) *to get that*

$$3x = \frac{1000g}{\sqrt{3}} \qquad \Longrightarrow \qquad x = \frac{1000g}{3\sqrt{3}}.$$

Thus,

- *The force along* AM *is* $\boxed{\frac{1000g}{3\sqrt{3}} \cdot \frac{1}{4}\langle -1, \sqrt{3}, -2\sqrt{3}\rangle}$.
- *The force along* BM *is* $\boxed{\frac{1000g}{3\sqrt{3}} \cdot \frac{1}{4}\langle -1, -\sqrt{3}, -2\sqrt{3}\rangle}$.
- *The force along* CM *is* $\boxed{\frac{1000g}{3\sqrt{3}} \cdot \frac{1}{4}\langle 2, 0, -2\sqrt{3}\rangle}$.

Instructor Notes: Students need to take into account that the mass, and not the force, is given. The students should also take advantage of the fact that the vectors are all of the same magnitude.

Problem 5 *Find the work done by a constant force of $10\hat{\imath}+18\hat{\jmath}-6\hat{k}$ that moves an object up a ramp from $(2,3,7)$ to $(4,9,15)$. Assume that distance is in feet and force in pounds. Also, find the angle between the force and the ramp.*

Solution: *First, let $\vec{F} = \langle 10, 18, -7\rangle$. Also, the vector from $(2,3,7)$ to $(4,9,15)$ is $\langle 2,6,8\rangle$. Let $\vec{d} = \langle 2,6,8\rangle$. Then the work done by the force is*

$$\vec{F}\cdot\vec{d} = 10\cdot 2 + 18\cdot 6 - 6\cdot 8 = \boxed{80\,ft\cdot lb}.$$

To calculate the angle, we compute

$$\cos\theta = \frac{\vec{F}\cdot\vec{d}}{|\vec{F}||\vec{d}|} = \frac{80}{\sqrt{460}\sqrt{104}}.$$

and so

$$\theta = \boxed{\cos^{-1}\left(\frac{80}{\sqrt{460}\sqrt{104}}\right) \approx 1.196 \textit{ radians}}$$

Instructor Notes: Simple work question.

Problem 6 *Suppose that the deli at the Tiny Sparrow grocery store sells roast beef for \$9 per pound, turkey for \$4 per pound, salami for \$5 per pound, and ham for \$7 per pound. For lunches this week, Sam the sandwhich maker buys 1.5 pounds of roast beef, 2 pounds of turkey, no salami, and half a pound of ham. How can you use a dot product to compute Sam's total bill from the deli?*

Solution: *The cost vector is*

$$\vec{c} = \langle 9,4,5,7\rangle.$$

The vector for Sam's order is

$$\vec{o} = \left\langle \frac{3}{2}, 2, 0, \frac{1}{2}\right\rangle.$$

Then Sam's bill is

$$\vec{c}\cdot\vec{o} = 9(1.5) + 4(2) + 5(0) + 7(0.5) = 13.5 + 8 + 0 + 3.5 = \boxed{25}.$$

Instructor Notes: