

## Recitation 27: Cross products - Solutions

### Warm up:

If  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  are vectors in 3-space  $\mathbb{R}^3$ , which of the following make sense?

- |  |  |  |
|--|--|--|
| (a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$  | (d) $(\vec{a} \cdot \vec{b}) + \vec{c}$  | (g) $\vec{a} \cdot (\vec{b} \times \vec{c})$ |
| (b) $(\vec{a} \cdot \vec{b})\vec{c}$         | (e) $(\vec{a} \times \vec{b}) + \vec{c}$ | (h) $\vec{a} \times (\vec{b} \cdot \vec{c})$ |
| (c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ | (f) $\vec{a} \cdot (\vec{b} + \vec{c})$  | (i) $(\vec{a} \times \vec{b})\vec{c}$        |

**Solution:** (a) Since  $\vec{a} \cdot \vec{b}$  is a scalar,  $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$  does **not** make sense.

(b) Now since  $\vec{a} \cdot \vec{b}$  is a scalar,  $(\vec{a} \cdot \vec{b})\vec{c}$  **does** make sense as regular scalar multiplication.

(c) Since  $\vec{a} \times \vec{b}$  is a vector,  $(\vec{a} \times \vec{b}) \cdot \vec{c}$  **does** make sense.

(d) This is of the form “scalar + vector”, which does **not** make sense.

(e) Since  $\vec{a} \times \vec{b}$  is a vector,  $(\vec{a} \times \vec{b}) + \vec{c}$  **does** make sense.

(f) This is of the form “vector · vector”, which **does** make sense.

(g) This is also of the form “vector · vector”, which **does** make sense.

(h) This is of the form “vector × scalar”, which does **not** make sense.

(i) Since  $\vec{a} \times \vec{b}$  is a vector, this does **not** make sense.

### Group work:

**Problem 1** Given three dimensional vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ , use dot product or cross product notation to describe the following vectors:

- (a) The vector projection of  $\vec{w}$  onto  $\vec{u}$ .

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Learning outcomes:

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- (b) A vector orthogonal to both  $\vec{u}$  and  $\vec{v}$ .
- (c) A vector with the length of  $\vec{v}$  and the direction of  $\vec{w}$ .
- (d) A vector orthogonal to  $\vec{u} \times \vec{v}$  and  $\vec{w}$ .

**Solution:** (a) This is the definition of vector projections.

$$\text{proj}_{\vec{u}} \vec{w} = \left( \frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$$

- (b) There are many such vectors, but one of them is

$$\vec{u} \times \vec{v}$$

- (c) Note that  $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$  so that  $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$ .

$$|\vec{v}| \left( \frac{\vec{w}}{|\vec{w}|} \right) = \frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\vec{w} \cdot \vec{w}}} \vec{w}$$

- (d)

$$(\vec{u} \times \vec{v}) \times \vec{w}$$

**Problem 2** Find a vector of length 7 that is perpendicular to both  $\langle 5, -1, 8 \rangle$  and  $\langle -2, 10, 5 \rangle$ .

**Solution:** Let  $\vec{u} = \langle 5, -1, 8 \rangle$  and  $\vec{v} = \langle -2, 10, 5 \rangle$ . Then a vector which is perpendicular to both  $\vec{u}$  and  $\vec{v}$  is  $\vec{w} := \vec{u} \times \vec{v}$ . So we calculate

$$\begin{aligned} \vec{w} = \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -1 & 8 \\ -2 & 10 & 5 \end{vmatrix} = (-5 - 80)\hat{i} - (25 + 16)\hat{j} + (50 - 2)\hat{k} \\ &= -85\hat{i} - 41\hat{j} + 48\hat{k} \end{aligned}$$

A unit vector in the same direction as  $\vec{w}$  is

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{(-85)^2 + (-41)^2 + 48^2}} \vec{w} = \frac{1}{\sqrt{11210}} \vec{w}.$$

Therefore, a vector with a magnitude of 7 in the same direction as  $\vec{w}$  is

$$\vec{t} = \frac{7}{|\vec{w}|} \vec{w} = \frac{7}{\sqrt{11210}} \langle -85, -41, 48 \rangle$$

**Problem 3** Find the area of the triangle in  $\mathbb{R}^3$  with vertices at  $P(2, -1, 0)$ ,  $Q(1, 1, 4)$  and  $R(2, -1, 6)$ .

**Solution:** The area of the triangle is  $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$ .

$$\vec{PR} = \langle 2, -1, 6 \rangle - \langle 2, -1, 0 \rangle = \langle 0, 0, 6 \rangle,$$

$$\vec{PQ} = \langle 1, 1, 4 \rangle - \langle 2, -1, 0 \rangle = \langle -1, 2, 4 \rangle.$$

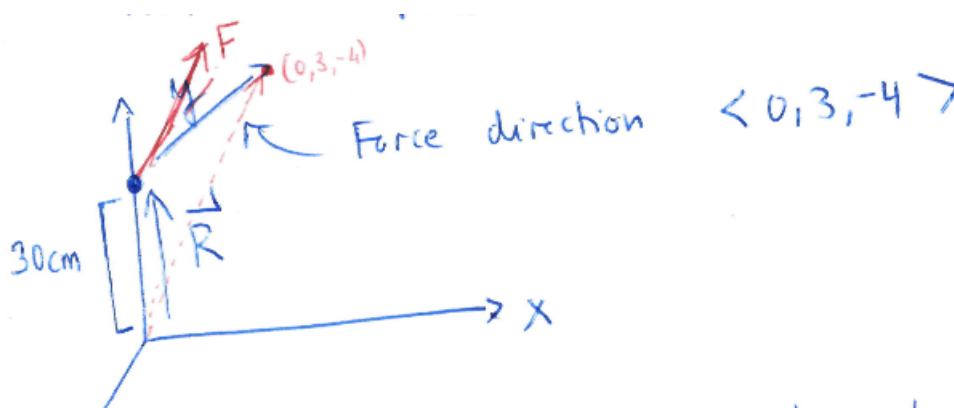
So

$$\begin{aligned} \vec{PQ} \times \vec{PR} &= (-\vec{i} + 2\vec{j} + 4\vec{k}) \times 6\vec{k} = -(\vec{i} \times \vec{k}) + 2(\vec{j} \times \vec{k}) + 24(\vec{k} \times \vec{k}) \\ &= -(-\vec{j}) + 2\vec{i} + 0 = \langle 2, 1, 0 \rangle. \end{aligned}$$

The area of the triangle is  $\frac{1}{2}\sqrt{2^2 + 1^2 + 0^2} = \frac{\sqrt{5}}{2}$ .

**Problem 4** A wrench that is 30cm long lies along the positive  $y$ -axis and grips a bolt at the origin. A force is applied in the direction  $\langle 0, 3, -4 \rangle$  at the end of the wrench. Find the magnitude of the force needed to supply 100J of torque to the bolt.

**Solution:** Below is a picture of the situation



Let

$$\vec{R} = \langle 0, 0.3, 0 \rangle$$

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denote the position vector of the end of the wrench. Since the force is in the direction  $\langle 0, 3, -4 \rangle$ , we know that the force vector satisfies

$$\vec{F} = c \langle 0, 3, -4 \rangle = \langle 0, 3c, -4c \rangle$$

for some constant  $c$ . Let  $\vec{t}$  denote the torque vector. Then  $\vec{t} = \vec{R} \times \vec{F}$  and so  $|\vec{t}| = |\vec{R} \times \vec{F}|$ . Thus,

$$\begin{aligned} 100 &= |\langle 0, 0.3, 0 \rangle \times \langle 0, 3c, -4c \rangle| \\ &= \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0.3 & 0 \\ 0 & 3c & -4c \end{vmatrix} \right| \\ &= |(-1.2c - 0)\hat{i} - (0 - 0)\hat{j} + (0 - 0)\hat{k}| \\ &= 1.2c = \frac{6}{5}c. \end{aligned}$$

Therefore

$$c = 100 \cdot \frac{5}{6} = \frac{500}{6} = \frac{250}{3}.$$

So, the magnitude of the force is

$$\begin{aligned} |\vec{F}| &= \frac{250}{3} \sqrt{0^2 + 3^2 + (-4)^2} \\ &= \frac{250}{3} \cdot 5 = \boxed{\frac{1250}{3} \text{ N}} \end{aligned}$$