Section 6.6 Surface Area

Group work:

Problem 1 *Find the surface area of the surface generated by revolving the curve given by*

(a) $x = 2y^3$ *from* $(0,0)$ *to* $(2,1)$ *about the* y*-axis.*

Solution: *The formula for the surface area is*

$$\text{Surface Area} = \int_0^1 2\pi f(y)\sqrt{1+f'(y)^2}\,dy.$$

Since $x = f(y) = 2y^3$*, we know that* $f'(y) = 6y^2$*. Note that*

$$\begin{aligned}\sqrt{1+f'(y)^2}\,dy &= \sqrt{1+(6y^2)^2} \\ &= \sqrt{1+36y^4}\end{aligned}$$

Learning outcomes:

and so

$$\begin{aligned}
\text{Surface Area} &= \int_0^1 2\pi \left(2y^3\right) \left(\sqrt{1+36y^4}\right) dy \\
&= \int_0^1 4\pi y^3 \sqrt{1+36y^4}\, dy
\end{aligned}$$

$$\begin{aligned}
u &= 1+36y^4 \\
du &= 144y^3 dy \\
\frac{du}{144} &= y^3 dy
\end{aligned}$$

$$\begin{aligned}
u(0) &= 1+36(0)^4 = 1 \\
u(1) &= 1+36(1)^4) = 37
\end{aligned}$$

$$\begin{aligned}
&= \frac{4\pi}{144} \int_1^{37} \sqrt{u}\, du \\
&= \frac{4\pi}{144} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^{37} \\
&= \frac{4\pi}{144} \left[\left(\frac{2}{3}(37)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right)\right] \\
&= \frac{(37)^{\frac{3}{2}} - 1}{54}\pi
\end{aligned}$$

(b) $y = \frac{1}{6}x^3 + \frac{1}{2x}$ *from* $\left(2, \frac{19}{12}\right)$ *to* $\left(3, \frac{14}{3}\right)$ *about the* x*-axis.*

Solution: *The formula for the surface area is*

$$\text{Surface Area} = \int_2^3 2\pi f(x)\sqrt{1+f'(x)^2}\, dx.$$

Since $y = f(x) = \frac{1}{6}x^3 + \frac{1}{2x}$, *we know that* $f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$. *Note* *that*

$$\begin{aligned}
\sqrt{1+f'(x)^2} &= \sqrt{1+\left(\frac{1}{2}x^2-\frac{1}{2}x^{-2}\right)^2} \\
&= \sqrt{1+\left(\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}x^{-4}\right)} \\
&= \sqrt{\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4}x^{-4}} \\
&= \sqrt{\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)^2} \\
&= \left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)
\end{aligned}$$

and so

$$\begin{aligned}
\textcolor{red}{\textit{Surface Area}} &= \int_2^3 2\pi \left(\frac{1}{6}x^3+\frac{1}{2}x^{-1}\right)\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)\,dx \\
&= 2\pi\int_2^3 \left(\frac{1}{12}x^5+\frac{1}{12}x+\frac{1}{4}x+\frac{1}{4}x^{-3}\right)\,dx \\
&= 2\pi\int_2^3 \left(\frac{1}{12}x^5+\frac{1}{3}x+\frac{1}{4}x^{-3}\right)\,dx \\
&= 2\pi\left[\frac{1}{72}x^6+\frac{1}{6}x^2-\frac{1}{8}x^{-2}\right]_2^3 \\
&= 2\pi\left[\left(\frac{81}{8}+\frac{3}{2}-\frac{1}{72}\right)-\left(\frac{8}{9}+\frac{2}{3}-\frac{1}{32}\right)\right] \\
&= 2\pi\left(\frac{2916+432-4-256-192+9}{288}\right) \\
&= \frac{2905\pi}{144}.
\end{aligned}$$