Recitation #18: Comparison Tests and Alternating Series - Solutions

Warm up:

For each of the following, answer **True** or **False**, and explain why.

(a) If $a_n \geq 0$ and $\displaystyle\sum_{n=0}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=0}^{\infty} a_n^2$ converges.

(b) If $a_n, b_n \geq 0$ and both $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ converge, then $\displaystyle\sum_{n=0}^{\infty} a_n b_n$ converges.

Solution: (a) **True**

Since $\displaystyle\sum_{n=0}^{\infty} a_n$ converges, $\displaystyle\lim_{n\to\infty} a_n = 0$. So, in particular, there exists an integer N such that $a_k < 1$ for all $k \geq N$. Then for all $k \geq N$, $a_k^2 < a_k$, and therefore we have that

$$\sum_{n=N}^{\infty} a_n^2 < \sum_{n=N}^{\infty} a_n.$$

Thus, by the Comparison Test, $\displaystyle\sum_{n=0}^{\infty} a_n^2$ is convergent.

(b) **True**

Just as in part (a) there exists an integer N such that $a_k < 1$ for all $k \geq N$. Then

$$\sum_{n=N}^{\infty} a_n b_n < \sum_{n=N}^{\infty} b_n$$

and thus, by the Comparison Test, $\displaystyle\sum_{n=0}^{\infty} a_n b_n$ is convergent.

Learning outcomes:

Group work:

Problem 1 (a) *Why can we not use the Comparison test with* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges?*

(b) *Adjust* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges via the Comparison Test.*

(c) *Give a convergent series we can use in the Limit Comparison Test to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges.*

Solution: (a) *We cannot use the Comparison Test here because*

$$\frac{1}{k^2} < \frac{1}{k^2-5}$$

for all $k \geq 1$. *So we would just be showing the the series in question is greater than a series which converges, which does not give us any information.*

(b) *Notice that*

$$\frac{2}{k^2} > \frac{1}{k^2-5}$$

for all $k \geq 4$. *Since* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges,* $\sum_{k=1}^{\infty} \frac{2}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$. *Thus,* $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *converges.*

Therefore, the Comparison Test with $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *shows that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

(c) *For the Limit Comparison Test, we* **can** *use* $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$\begin{aligned}\lim_{k\to\infty} \frac{\frac{1}{k^2-5}}{\frac{1}{k^2}} &= \lim_{k\to\infty} \frac{k^2}{k^2-5} \\ &= 1.\end{aligned}$$

Thus, since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, by the Limit Comparison Test we know that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

Problem 2 *Determine if the following series converge or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^2 e^{-n}\right]$

Solution: (a) *Use the* ***Limit Comparison Test*** *with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^3+1} \cdot \frac{n}{1} \\ &= \frac{1}{3}.\end{aligned}$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, by the Limit Comparison Test we know that* $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$ *diverges.*

(b) *Use the* ***Limit Comparison Test*** *with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^4+1} \cdot \frac{n^2}{1} \\ &= \frac{1}{3}.\end{aligned}$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges, by the Limit Comparison Test we know that* $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$ *converges.*

(c) *Use the* ***Comparison Test*** *with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Since we always have that $0 < \cos^2(n) < 1$, *we know that*

$$\frac{\cos^2(n)}{n^3+1} \leq \frac{1}{n^3+1} < \frac{1}{n^3}.$$

Therefore, by the Comparison Test, we have that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$ *converges.*

(d) *Use the* ***Comparison Test*** *with* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$.

First, notice that for all $n \geq 3$,

$$\left(1 + \frac{1}{n}\right)^2 < 2.$$

Also, notice that $\sum_{n=1}^{\infty} e^{-n}$ *is a convergent geometric series. Therefore* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$ *converges, and so* $\sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^2 e^{-n}\right]$ *converges by the Comparison Test.*

Problem 3 *Determine if the following series absolutely converge, conditionally converge, or diverge.*

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$

(e) $\sum_{n=4}^{\infty} \frac{(-2)^n}{n}$

Solution: (a) *Since* $\frac{1}{n+3}$ *is positive and decreasing, the Alternating Series Test applies. Thus, this series converges. But we know that* $\sum_{n=1}^{\infty} \frac{1}{n+3}$ *diverges (Harmonic Series) and so* $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ *is conditionally convergent.*

(b) *Let* $a_n = \frac{(n+1)^n}{(2n)^n}$. *Applying the Root Test, we see that*

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \frac{n+1}{2n} \\ &= \frac{1}{2} < 1. \end{aligned}$$

So $\sum_{n=1}^{\infty} \frac{(n+1)^n}{(2n)^n}$ *converges, and therefore* $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$ *is absolutely convergent.*

(c) *Let* $f(x) = x^2 e^{\frac{-x^3}{3}}$. *The function* $f(x)$ *is continuous, positive, and decreasing. So we apply the integral test*

$$\begin{aligned}
\int_1^\infty x^2 e^{\frac{-x^3}{3}}\,dx &= \lim_{b\to\infty}\int_1^b x^2 e^{\frac{-x^3}{3}}\,dx \\
&= \lim_{b\to\infty}\int_{\frac{1}{3}}^{\frac{b^3}{3}} e^{-u}\,du \qquad u=\frac{x^3}{3},\ du = x^2\,dx \\
&= \lim_{b\to\infty}\left[-e^{\frac{-b^3}{3}} + e^{\frac{-1}{3}}\right] \\
&= 0 + e^{\frac{-1}{3}} = e^{\frac{-1}{3}}.
\end{aligned}$$

So, by the Integral Test, $\sum_{n=1}^{\infty} n^2 e^{\frac{-n^3}{3}}$ *converges. Therefore,* $\sum_{n=1}^{\infty}(-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$ *is absolutely convergent.*

(d) *First, notice that*

$$\frac{5}{3^n} > \frac{5}{3^n + 3^{-n}}.$$

Then since $\sum_{n=0}^{\infty}\frac{5}{3^n}$ *converges (geometric series with* $r = \frac{1}{3} < 1$*), we know that* $\sum_{n=0}^{\infty}\frac{5}{3^n+3^{-n}}$ *converges by the Comparison Test. Thus, the series* $\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 5}{3^n+3^{-n}}$ *is absolutely convergent.*

(e) *Since the sequence* $\left\{\frac{(-2)^n}{n}\right\}$ *diverges, the series* $\sum_{n=4}^{\infty}\frac{(-2)^n}{n}$ *diverges by the Divergence Test.*

Problem 4 (a) *Find an upper bound for how close* $\sum_{k=0}^{4}\frac{(-1)^k k}{4^k}$ *is to the value of* $\sum_{k=0}^{\infty}\frac{(-1)^k k}{4^k}$.

(b) *How many terms are needed to estimate* $\sum_{n=1}^{\infty}\frac{(-1)^n \ln n}{n!}$ *to within* 10^{-6}*?*

Solution: (a) *Recall from the lesson that the remainder* R_n *is given by*

$$R_n = |S - S_n| \le a_{n+1}.$$

Also notice that

$$\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}$$

and

$$\sum_{k=0}^{4} \frac{(-1)^k k}{4^k} = \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}.$$

So

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} - \sum_{k=0}^{4} \frac{(-1)^k k}{4^k}\right| \\
&= \left|\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}} - \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}\right| \\
&= |S - S_5| \\
&= R_6 \leq a_6 = \boxed{\frac{5}{4^5}}
\end{aligned}$$

(b) *Since $R_n \leq a_{n+1}$, we need to find n so that*

$$\begin{aligned}
& a_{n+1} \leq 10^{-6} \\
\iff \quad & \frac{\ln(n+1)}{(n+1)!} \leq 10^{-6} \\
\iff \quad & \frac{(n+1)!}{\ln(n+1)} \geq 10^{6}
\end{aligned}$$

Using a calculator, we see that this inequality holds for $n \approx 8.8$. So for $\boxed{n \geq 9}$, $R_n < 10^{-6}$.