Section 9.5: The Divergence, Integral, Ratio and Root Tests - Solutions

Warm-Up

Problem 1 (a) *Why can we not use the Comparison test with* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges?*

(b) *Adjust* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges via the Comparison Test.*

(c) *Give a convergent series we can use in the Limit Comparison Test to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges.*

Solution: (a) *We cannot use the Comparison Test here because*

$$\frac{1}{k^2} < \frac{1}{k^2-5}$$

for all $k \geq 1$. *So we would just be showing the the series in question is greater than a series which converges, which does not give us any information.*

(b) *Notice that*

$$\frac{2}{k^2} > \frac{1}{k^2-5}$$

for all $k \geq 4$. *Since* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges,* $\sum_{k=1}^{\infty} \frac{2}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$. *Thus,* $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *converges.*

Therefore, the Comparison Test with $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *shows that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

Learning outcomes:

(c) *For the Limit Comparison Test, we* **can** use $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$\lim_{k\to\infty} \frac{\frac{1}{k^2-5}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-5} = 1.$$

Thus, since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, by the Limit Comparison Test we know that* $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

Problem 2 *For each of the following, answer* **True** *or* **False**, *and explain why.*

(a) *If* $\displaystyle\sum_{n=0}^{\infty} a_n$ *converges, then* $\displaystyle\sum_{n=0}^{\infty} (a_n + 0.001)$ *converges.*

(b) *Since* $\displaystyle\int_1^{\infty} x\sin(\pi x)\,dx$ *diverges then, by the Integral Test,* $\displaystyle\sum_{n=0}^{\infty} n\sin(\pi n)$ *diverges.*

(c) *Since* $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = 1$ *then, by the Integral Test,* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = 1$.

Solution: (a) **False**

Since $\displaystyle\sum_{n=0}^{\infty} a_n$ *converges, we know that* $\displaystyle\lim_{n\to\infty} a_n = 0$. *But then*

$$\lim_{n\to\infty} (a_n + 0.0001) = 0.0001 \neq 0$$

and so $\displaystyle\sum_{n=0}^{\infty} (a_n + 0.001)$ *diverges by the Divergence Test.*

(b) **False**

The Integral Test only holds for positive, decreasing functions. The function $f(x) = x\sin(\pi x)$ *is not always positive, nor is it always decreasing. So the Integral Test does not apply here.*

This problem is simpler than that though. Since $\sin(\pi n) = 0$ *for all integers* n, *we have that* $\displaystyle\sum_{n=0}^{\infty} n\sin(\pi n) = 0$.

(c) **False**

The Integral Test tells us that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, but it does* **not** *give us the sum (this sum is actually* $\dfrac{\pi^2}{6}$*).*

Group work:

Problem 3 *Assume* $\displaystyle\sum_{k=0}^{\infty} a_k = L$ *and* $b_k = 8$ *for all* k*.*

(a) *What is* $\displaystyle\lim_{k\to\infty}(a_k + b_k)$*?*

(b) *What is* $\displaystyle\lim_{k\to\infty}\sum_{n=0}^{k}(a_n + b_n)$*?*

(c) *What is* $\displaystyle\lim_{k\to\infty}\sum_{n=0}^{k}(a_{n+1} - a_n)$*?*

Solution: (a) *Since* $\displaystyle\sum_{k=0}^{\infty} a_k$ *converges, we know that* $\displaystyle\lim_{k\to\infty} a_k = 0$*. Therefore,*

$$\lim_{k\to\infty}(a_k + b_k) = 0 + 8 = \boxed{8}.$$

(b) *Since* $\displaystyle\lim_{n\to\infty}(a_n + b_n) = 8$*, the series* $\displaystyle\sum_{n=0}^{\infty}(a_n + b_n)$ *diverges by the Divergence Test. But* $\displaystyle\lim_{k\to\infty}\sum_{n=0}^{k}(a_n + b_n) = \sum_{n=0}^{\infty}(a_n + b_n)$*. Thus*

$$\lim_{k\to\infty}\sum_{n=0}^{k}(a_n + b_n) = \sum_{n=0}^{\infty}(a_n + b_n) = \boxed{\infty}.$$

(c) *Let* $\displaystyle S_k = \sum_{n=0}^{k}(a_{n+1} - a_n)$ *(and recall that* $\{S_k\}$ *is the sequence of partial* *sums). Then*

$$\begin{aligned}
S_k &= \sum_{n=0}^{k}(a_{n+1} - a_n) \\
&= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_k - a_{k-1}) + (a_{k+1} - a_k) \\
&= a_{k+1} - a_0.
\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\sum_{n=0}^{k}(a_{n+1} - a_n) = \lim_{k\to\infty} S_k = \lim_{k\to\infty} a_{k+1} - a_0 = \boxed{-a_0}.$$

Problem 4 *Determine if the following series converge or diverge.*

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(7n+1)^2\cdot 2^n}{5^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$, *where* $a_{n+1} = \dfrac{2n+5}{3n-1}\cdot a_n$ *and* $a_1 = 1$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2+2n+1}{3n^2+1}$

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

(e) $\displaystyle\sum_{k=1}^{\infty}\frac{(k!)^3}{(3k)!}$

Solution: (a) ***Ratio Test***

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\left[\frac{(7(n+1)+1)^2\cdot 2^{n+1}}{5^{n+1}}\cdot\frac{5^n}{(7n+1)^2\cdot 2^n}\right] \\
&= \lim_{n\to\infty}\frac{(7n+8)^2\cdot 2}{5\cdot(7n+1)^2} \\
&= \frac{49\cdot 2}{49\cdot 5} = \frac{2}{5}.
\end{aligned}$$

Thus, since $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$, *this series* $\boxed{\textit{converges}}$.

(b) ***Ratio Test***

Even though the terms in this series look a little weird, this is set up perfectly for the Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n+5}{3n-1} = \frac{2}{3}.$$

Thus, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, *this series* $\boxed{\textit{converges}}$.

(c) ***Divergence Test***

Notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{3n^2+1} = \frac{1}{3}.$$

Therefore, since $\lim_{n\to\infty} a_n \neq 0$, *by the Divergence Test this series* $\boxed{\textit{diverges}}$.

(d) ***Integral Test***

First, notice that $f(x) = \dfrac{1}{x(\ln x)^2}$ *is a decreasing and positive function on* $[2, \infty)$. *Then*

$$\begin{aligned}
\int_2^\infty f(x)\,dx &= \int_2^\infty \frac{1}{x(\ln x)^2}\,dx \\
&= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx \\
&= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-2}\,du \qquad {\color{red} u = \ln x,\ du = \frac{1}{x}\,dx} \\
&= \lim_{b\to\infty} \left[\frac{-1}{u}\right]_{\ln 2}^{\ln b} \\
&= \lim_{b\to\infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln 2}\right) \\
&= 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}.
\end{aligned}$$

Therefore, since the above integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ $\boxed{\textit{converges}}$ *by the Integral Test.*

(e) ***Ratio Test***

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \left[\frac{((k+1)!)^3}{(3(k+1))!} \cdot \frac{(3k)!}{(k!)^3} \right] \\
&= \lim_{k\to\infty} \frac{(k+1)^3(k!)^3}{(3k+3)(3k+2)(3k+1)\cdot(3k)!} \cdot \frac{(3k)!}{(k!)^3} \\
&= \lim_{k\to\infty} \frac{(k+1)^3}{(3k+3)(3k+2)(3k+1)} \\
&= \frac{1}{3\cdot 3\cdot 3} = \frac{1}{27}.
\end{aligned}$$

Thus, since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, *this series* $\boxed{\textit{converges}}$.

Problem 5 *Determine if the following series converge or diverge.*

(a) $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$

(b) $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$

(c) $\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$

(d) $\sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^2 e^{-n} \right]$

Solution: (a) *Use the* ***Limit Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^3+1} \cdot \frac{n}{1} \\
&= \frac{1}{3}.
\end{aligned}$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, by the Limit Comparison Test we know that* $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$ *diverges.*

(b) *Use the* ***Limit Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^4+1} \cdot \frac{n^2}{1} \\
&= \frac{1}{3}.
\end{aligned}$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges, by the Limit Comparison Test we know that* $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$ *converges.*

(c) *Use the* ***Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Since we always have that $0 < \cos^2(n) < 1$, *we know that*

$$\frac{\cos^2(n)}{n^3+1} \leq \frac{1}{n^3+1} < \frac{1}{n^3}.$$

Therefore, by the Comparison Test, we have that $\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$ *converges.*

(d) *Use the* ***Comparison Test*** *with* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$.

First, notice that for all $n \geq 3$,

$$\left(1 + \frac{1}{n}\right)^2 < 2.$$

Also, notice that $\sum_{n=1}^{\infty} e^{-n}$ *is a convergent geometric series. Therefore* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$ *converges, and so* $\sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^2 e^{-n}\right]$ *converges by the Comparison Test.*