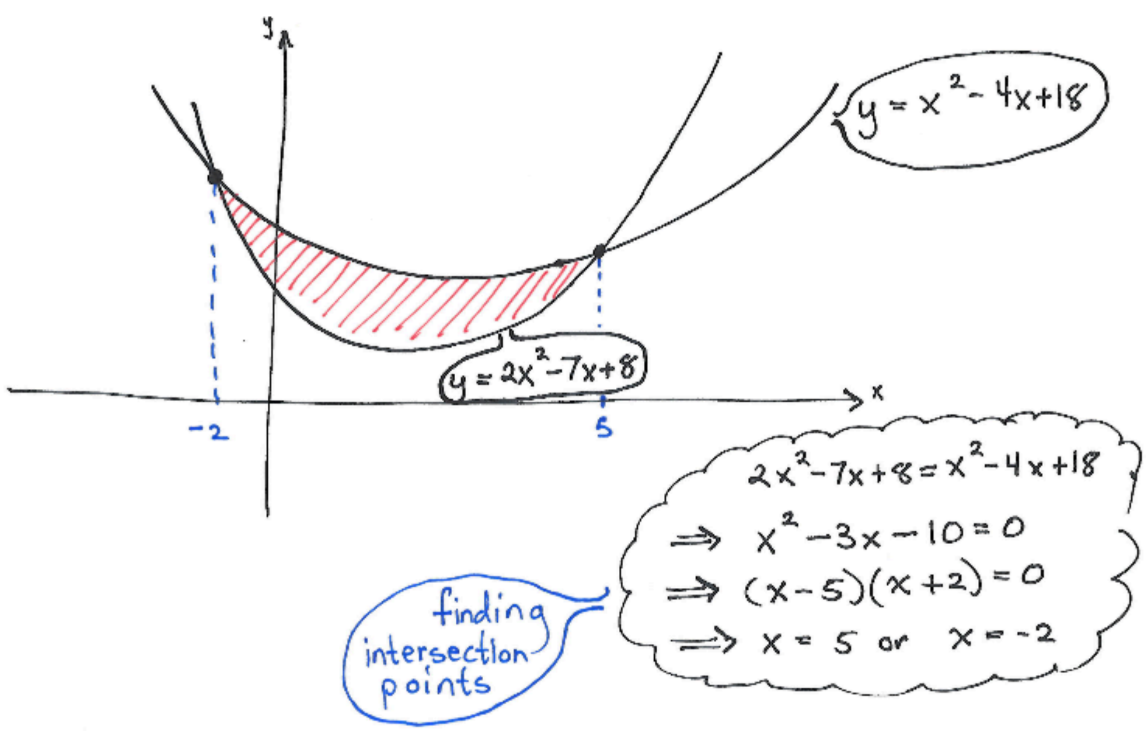
Recitation # 2 Regions Between Curves - Solutions

Group work:

Problem 1 *Consider the region bounded by the curves* $y = 2x^2 - 7x + 8$ *and* $y = x^2 - 4x + 18$.

(a) *Draw a sketch of the graphs.*

Solution:



(b) *Find the area between these curves.*

Learning outcomes:

Solution: *Let* $y_1 = 2x^2 - 7x + 8$ *and* $y_2 = x^2 - 4x + 18$. *By the solution to part (a), we know both that* $y_1 - y_2 = x^2 - 3x - 10$ *and that these two curves intersect at* $x = -2, 5$. *By checking the point* $x = 0$ *(or by looking at the graph from part (a)) we see that* $y_2 \geq y_1$ *on the interval* $[-2, 5]$. *So the area between the curves is:*

$$\begin{aligned}
\int_{-2}^{5} (y_2 - y_1)\, dx &= \int_{-2}^{5} (-x^2 + 3x + 10)\, dx \\
&= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 10x \right]_{-2}^{5} \\
&= \left(-\frac{125}{3} + \frac{75}{2} + 50 \right) - \left(\frac{8}{3} + 6 - 20 \right) \\
&= -\frac{133}{3} + \frac{75}{2} + 64 \\
&= \frac{-266 + 225 + 384}{6} = \frac{343}{6}
\end{aligned}$$

(c) *Find the area of the region bounded by the curves* $x = 2y^2 - 7y + 8$ *and* $x = y^2 - 4y + 18$.

Solution: *This region is exactly the same as the red region from part (a), except it is rotated clockwise by* $90°$. *Since the area of a region does not change under rotation, we have that the area of the new region is still* $\frac{343}{6}$.

(d) *Find the area of the region bounded by the curves*

(i) $y = 2x^2 - 7x$ *and* $y = x^2 - 4x + 10$.

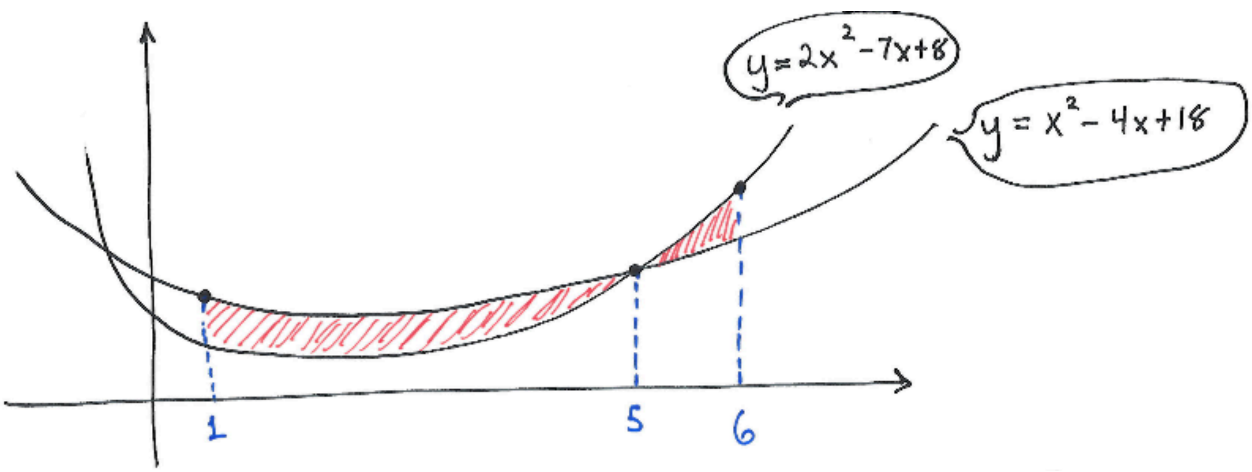
Solution: *The region bounded by the curves* $y = 2x^2 - 7x$ *and* $y = x^2 - 4x + 10$ *is just the red region in part (a) translated downward by 8 units. Since the area of a region does not change under translation, we have that the area of the new region is still* $\frac{343}{6}$.

(ii) $y = 2x^2 - 7x - 30$ *and* $y = x^2 - 4x - 20$.

Solution: *This is the same as (i), except now the red region has been translated downward by 38 units. Therefore, the area of this region is still* $\frac{343}{6}$.

(e) *Find the area of the region bounded by the curves* $y = 2x^2 - 7x + 8$, $y = x^2 - 4x + 18$, $x = 1$, *and* $x = 6$.

Solution:



We already know that the graphs of the two functions intersect at the point $x = 5$*. By checking the points* $x = 1$ *and* $x = 6$ *(or by looking at the graph above) we see that*

$$\begin{aligned} y_2 \geq y_1 \quad &\textit{on} \quad [1,5] \\ y_1 \geq y_2 \quad &\textit{on} \quad [5,6] \end{aligned}$$

Thus, the area between the curves is

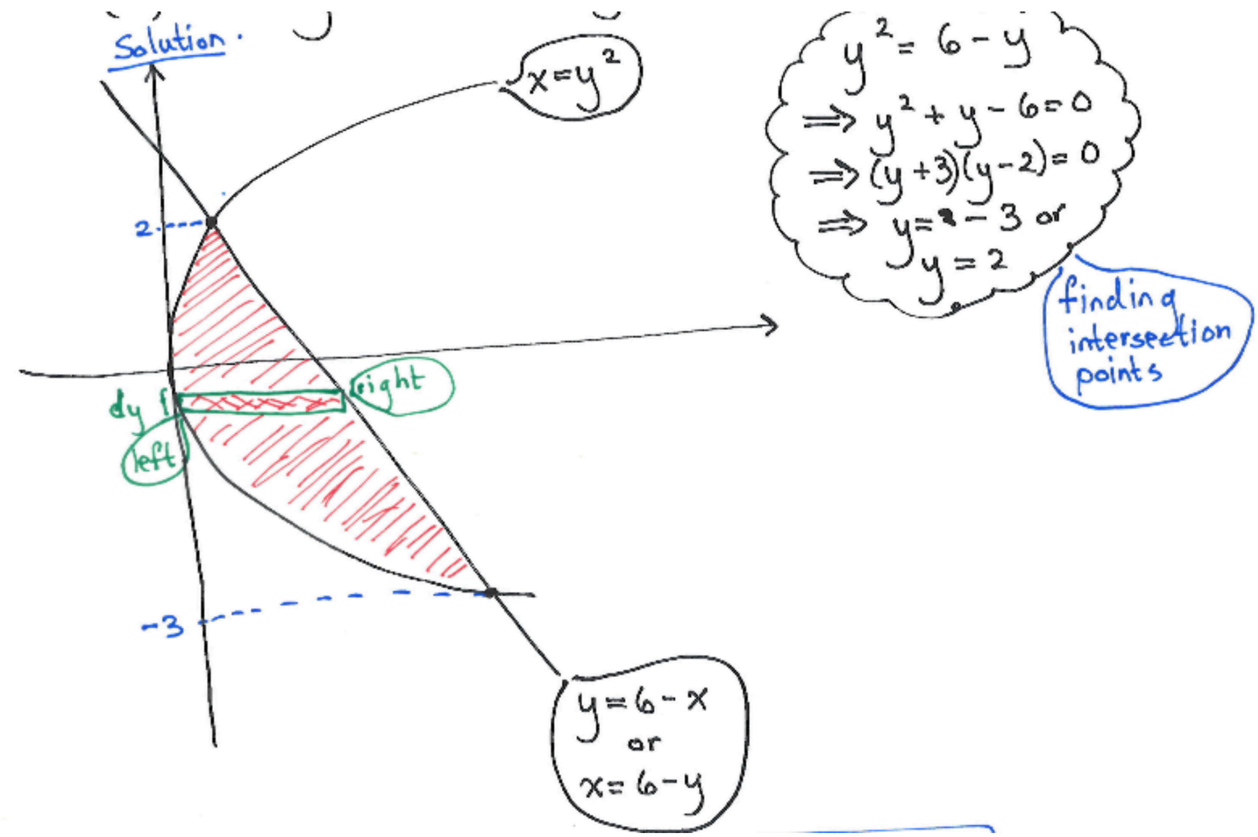
$$\begin{aligned}
&\int_1^5 (y_2 - y_1)\,dx + \int_5^6 (y_1 - y_2)\,dx \\
&= \int_1^5 (-x^2 + 3x + 10)\,dx + \int_5^6 (x^2 - 3x - 10)\,dx \\
&= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 10x \right]_1^5 + \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 - 10x \right]_5^6 \\
&= \left[\left(-\frac{125}{3} + \frac{75}{2} + 50 \right) - \left(-\frac{1}{3} + \frac{3}{2} + 10 \right) \right] + \left[(72 - 54 - 60) - \left(\frac{125}{3} - \frac{75}{2} - 50 \right) \right] \\
&= -\frac{249}{3} + \frac{147}{2} + 48 = -83 + 48 + \frac{147}{2} = \frac{-70 + 147}{2} = \frac{77}{2}
\end{aligned}$$

Instructor Notes: Have the students do (a) and (b), and then have a group present. Afterwards, discuss the variations (c) and (d) as a whole class.

Problem 2 *Set up a single integral that computes the area of the region bounded by the curves (and be sure to draw a sketch of the graphs).*

(a) $x = y^2$ and $y = 6 - x$

Solution:

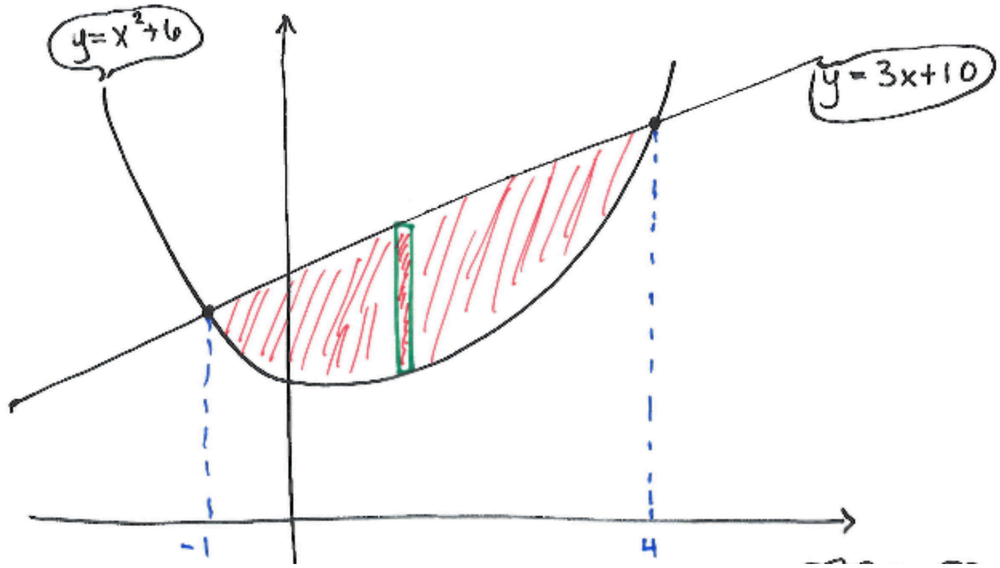


Thus,

$$\text{Area of region} = \int_{-3}^{2} [(6-y) - y^2]\, dy.$$

(b) $y = x^2 + 6$ and $y = 3x + 10$

Solution:



To find the intersection points in the above picture, we solve

$$\begin{aligned} x^2 + 6 &= 3x + 10 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0 \\ x &= -1, 4. \end{aligned}$$

So

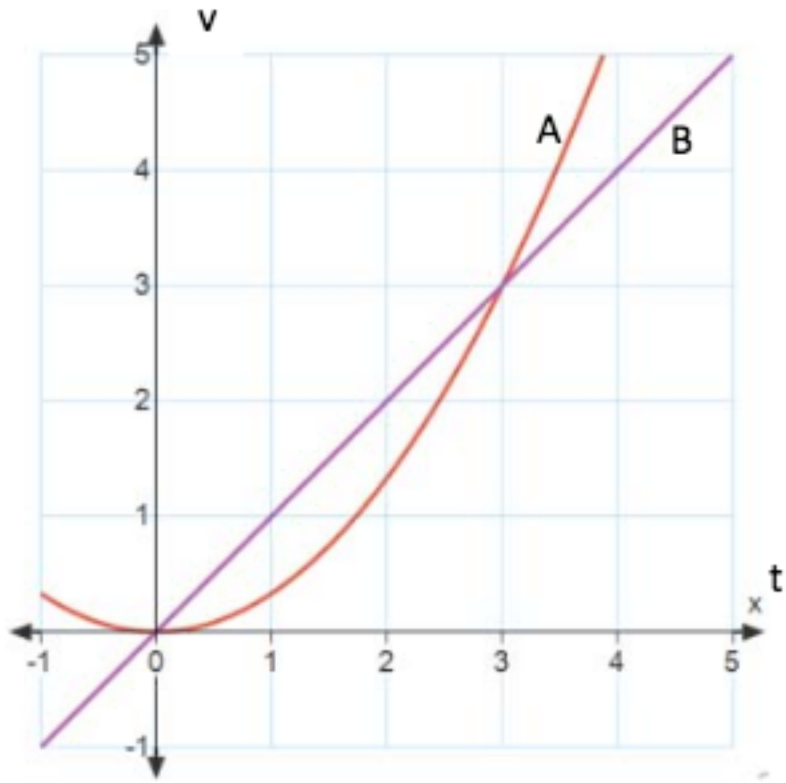
$$\text{Area of region} = \int_{-1}^{4} [(3x + 100) - (x^2 + 6)]\, dx.$$

Instructor Notes: Split (a) and (b) among the groups. Note that (a) should be set up in terms of y while (b) should be set up in terms of x. Have groups present their solutions. Discuss with the students factors to consider when deciding whether to integrate in terms of x or y.

Problem 3 *Two runners (A and B) run in a race in which the winner runs the farthest in 4 minutes. The runners' respective velocities are*

$$v_A(t) = \frac{1}{3}t^2 \qquad v_B(t) = t$$

The graphs of the runners' velocities is given below.



(a) *Who is running the fastest 2 minutes into the race?*

Solution: $v_A(2) = \dfrac{4}{3}$ *and* $v_B(2) = 2$. *So* B *is running faster at the* 2 *minute mark of the race.*

(b) *Who is winning the race 2 minutes into the race (and by how much)?*

Solution: *The distance that* A *covers in the first* 2 *minutes is*

$$\int_0^2 v_A(t)\,dt = \int_0^2 \frac{1}{3}t^2\,dt = \left[\frac{1}{9}t^3\right]_0^2 = \frac{8}{9}.$$

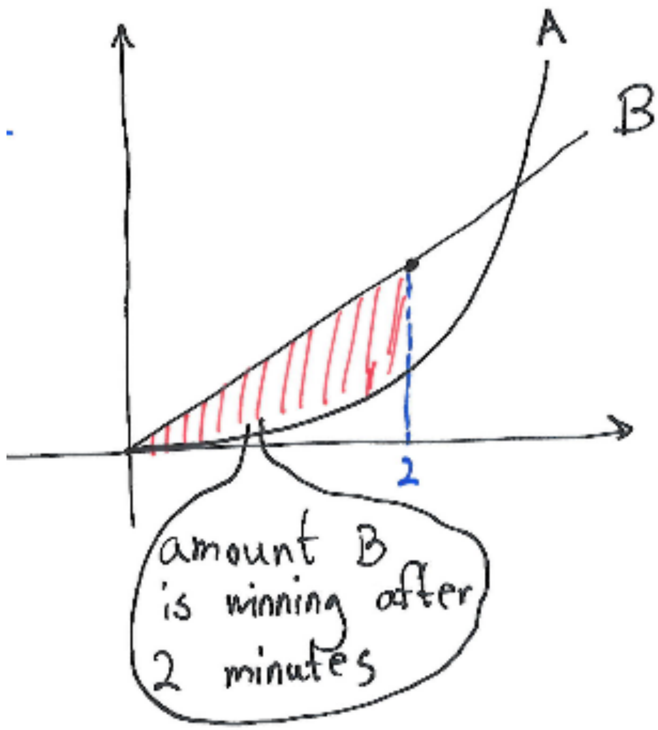
The distance that B *covers in the first* 2 *minutes is*

$$\int_0^2 v_B(t)\,dt = \int_0^2 t\,dt = \left[\frac{1}{2}t^2\right]_0^2 = 2.$$

So B *is winning after* 2 *minutes.*

B *is winning by* $2 - \dfrac{8}{9} = \dfrac{10}{9}$. *This could also be calculated by*

$$\int_0^2 (v_B(t) - v_A(t))\,dt.$$



(c) *What special event occurs 3 minutes into the race?*

Solution: *Runner A matches runner B's velocity. ie,* $v_A(3) = v_B(3)$.

(d) *Who wins the race (and by how much)?*

Solution: *The distance that A covers is*

$$\int_0^4 v_A(t)\,dt = \int_0^4 \frac{1}{3}t^2\,dt = \left[\frac{1}{9}t^3\right]_0^4 = \frac{64}{9} = 7.\overline{1}.$$

The distance that B covers is

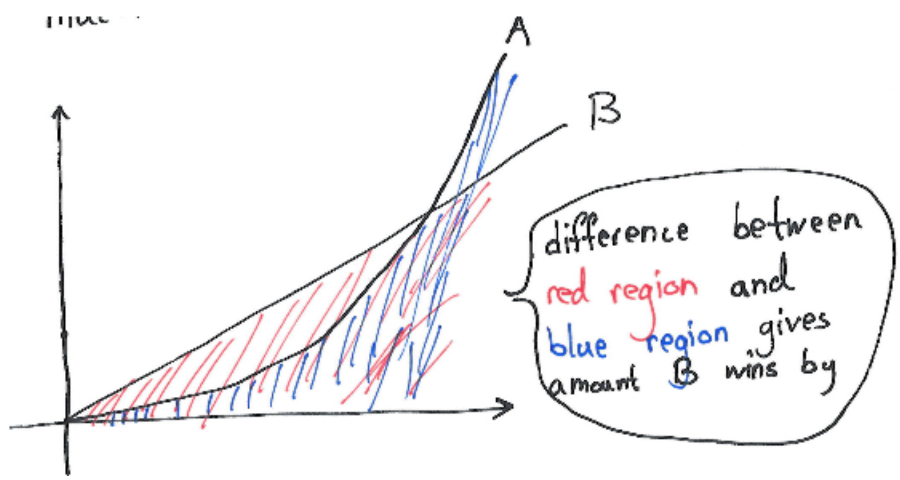
$$\int_0^4 v_B(t)\,dt = \int_0^4 t\,dt = \left[\frac{1}{2}t^2\right]_0^4 = 8.$$

So runner B wins. The amount that B wins by is

$$8 - \frac{64}{9} = \frac{8}{9}.$$

This could have also been computed by

$$\int_0^4 (v_B(t) - v_A(t))\,dt.$$



Instructor Notes: Do this problem as a class discussion. The main point is to have students identify how each of the questions relates to the graph of the velocity functions.