Section 7.3: Trig Integrals

Group work:

Problem 1 *Evaluate the following integrals*

(a) $\displaystyle\int \sin^3(4\theta)\cos^6(4\theta)\,\mathrm{d}\theta$

Solution: *If we try choosing* $u = \sin 4\theta$, *note that* $du = 4\cos 4\theta\, d\theta$, *which would leave us with a* $\cos^5 4\theta$ *term. This would* NOT *be easy to convert into powers of* $\sin 4\theta$*! Hence:*

$$\begin{aligned} u &= \cos 4\theta \\ du &= -4\sin 4\theta\, d\theta \\ d\theta &= -\frac{du}{4\sin 4\theta} \end{aligned}$$

Making the substitution into the integral gives:

$$\begin{aligned} \int \sin^3(4\theta)\cos^6 4\theta\, d\theta &= \int \sin^3 4\theta \;\cdot u^6 \left[-\frac{du}{4\sin 4\theta}\right] \\ &= -\frac{1}{4}\int \sin^2 4\theta \;\cdot u^6\, du \\ &= -\frac{1}{4}\int (1-\cos^2 4\theta)u^6\, du \\ &= -\frac{1}{4}\int (1-u^2)u^6\, du \\ &= -\frac{1}{4}\int (u^6-u^8)\, du \\ &= -\frac{1}{4}\left[\frac{1}{7}u^7 - \frac{1}{9}u^9\right] + C \\ &= \boxed{-\frac{1}{28}\cos^7 4\theta + \frac{1}{36}\cos^9 4\theta + C} \end{aligned}$$

(b) $\displaystyle\int \tan^{23} x \sec^6 x\, dx$

Learning outcomes:

Solution:

$$\begin{aligned}\int \tan^{23} x \sec^6 x \, dx &= \int \tan^{23} x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^{23} x \left(1 + \tan^2 x\right)^2 \sec^2 x \, dx.\end{aligned}$$

We now substitute

$$u = \tan x \qquad \Longrightarrow \qquad du = \sec^2 x \, dx.$$

Then

$$\begin{aligned}\int \tan^{23} x \left(1 + \tan^2 x\right)^2 \sec^2 x \, dx &= \int u^{23}(1+u^2)^2 \, du \\ &= \int u^{23}(1 + 2u^2 + u^4) \, du \\ &= \int \left(u^{23} + 2u^{25} + u^{27}\right) \, du \\ &= \frac{1}{24}u^{24} + \frac{1}{13}u^{26} + \frac{1}{28}u^{28} + C \\ &= \frac{1}{24}\tan^{24} x + \frac{1}{13}\tan^{26} x + \frac{1}{28}\tan^{28} x + C.\end{aligned}$$

(c) $\displaystyle\int \tan^2 x \sec x \, dx$ *Hint:* $\displaystyle\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Solution:

$$\begin{aligned}\int \tan^2 x \sec x \, dx &= \int \left(\sec^2 x - 1\right) \sec x \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \\ &= \int \sec^3 x \, dx - \ln|\sec x + \tan x| \qquad \textit{from the hint} \quad (1)\end{aligned}$$

Now, in an attempt to evaluate $\displaystyle\int \sec^3 x \, dx$, *we use integration by parts with*

$$\begin{aligned}u &= \sec x & dv &= \sec^2 x \, dx \\ du &= \sec x \tan x \, dx & v &= \tan x.\end{aligned}$$

So

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx. \qquad (2)$$

*Combining equations (**??**) and (**??**) yields*

$$\begin{aligned}
\int \tan^2 x \sec x \, dx &= \int \sec^3 x \, dx - \ln|\sec x + \tan x| \\
\int \tan^2 x \sec x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx - \ln|\sec x + \tan x| \\
2\int \tan^2 x \sec x \, dx &= \sec x \tan x - \ln|\sec x + \tan x| + C \\
\int \tan^2 x \sec x \, dx &= \frac{1}{2}\left(\sec x \tan x - \ln|\sec x + \tan x|\right) + C.
\end{aligned}$$

(d) $\int \tan^2 x \sin x \, dx$

Solution:

$$\begin{aligned}
\int \tan^2 x \sin x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx \\
&= \int \frac{1 - \cos^2 x}{\cos^2 x} \sin x \, dx.
\end{aligned}$$

Now we substitute

$$u = \cos x \qquad \Longrightarrow \qquad du = -\sin x \, dx, \quad -du = \sin x \, dx.$$

This gives us that

$$\begin{aligned}
\int \tan^2 x \sin x \, dx &= \int \frac{1 - u^2}{u^2} (-1) \, du \\
&= \int \frac{u^2 - 1}{u^2} \, du \\
&= \int \left(1 - u^{-2}\right) \, du \\
&= u + \frac{1}{u} + C \\
&= \cos x + \sec x + C.
\end{aligned}$$

Problem 2 *Evaluate*

$$\int_{-\pi}^{0} \sqrt{1 - \cos^2 x} \, dx.$$

Solution:

$$\begin{aligned}
\int_{-\pi}^{0} \sqrt{1 - \cos^2 x} \, dx &= \int_{-\pi}^{0} \sqrt{\sin^2 x} \, dx \\
&= \int_{-\pi}^{0} |\sin x| \, dx.
\end{aligned}$$

Now, when $-\pi \leq x \leq 0$, $\sin x \leq 0$*. Thus, on this region,* $|\sin x| = -\sin x$*. So we continue*

$$\begin{aligned}\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\,dx &= \int_{-\pi}^{0} -\sin x\,dx \\ &= \Big[\cos x\Big]_{-\pi}^{0} \\ &= \cos(0) - \cos(-\pi) = 1 - (-1) = 2.\end{aligned}$$