

Recitation #9: Integration by parts and Trig Integrals - Full

Group work:

Problem 1 Evaluate the following integrals

(a) $\int_1^3 x^2 5^x dx$

Solution: We proceed via integration by parts. Let

$$u = x^2, \quad dv = 5^x dx$$

so that

$$du = 2x dx, \quad v = \frac{5^x}{\ln 5}.$$

Recall the formula for integration by parts is

$$\int_a^b u dv = \left[uv \right]_a^b - \int_a^b v du.$$

So we substitute

$$\begin{aligned} \int_1^3 x^2 5^x dx &= \left[\frac{x^2 5^x}{\ln(5)} \right]_1^3 - \int_1^3 2x \frac{5^x}{\ln(5)} dx \\ &= \frac{1}{\ln(5)} (9 \cdot 5^3 - 5) - \frac{2}{\ln(5)} \int_1^3 x 5^x dx. \end{aligned}$$

For the remaining integral we again use integration by parts:

$$u = x, \quad dv = 5^x$$

$$du = dx, \quad v = \frac{5^x}{\ln(5)}.$$

Learning outcomes:

Recitation #9: Integration by parts and Trig Integrals - Full

Thus,

$$\begin{aligned}
 & \frac{1}{\ln(5)} (9 \cdot 5^3 - 5) - \frac{2}{\ln(5)} \int_1^3 x 5^x dx \\
 &= \frac{5}{\ln(5)} (225 - 1) - \frac{2}{\ln(5)} \left(\left[\frac{x 5^x}{\ln(5)} \right]_1^3 - \int_1^3 \frac{5^x}{\ln(5)} dx \right) \\
 &= \frac{1120}{\ln(5)} - \frac{2}{\ln^2(5)} \left((3 \cdot 5^3 - 5) - \left[\frac{5^x}{\ln(5)} \right]_1^3 \right) \\
 &= \frac{1}{\ln(5)} \left(1120 - \frac{740}{\ln(5)} + \frac{248}{\ln^2(5)} \right).
 \end{aligned}$$

(b) $\int \sin(3x)e^{7x} dx$

Solution: We begin by letting $I = \int \sin(3x)e^{7x} dx$. We then use integration by parts with

$$\begin{aligned}
 u &= e^{7x} & dv &= \sin(3x) dx \\
 du &= 7e^{7x} dx & v &= -\frac{1}{3} \cos(3x).
 \end{aligned}$$

Then

$$\begin{aligned}
 \int \sin(3x)e^{7x} dx &= I = -\frac{1}{3}e^{7x} \cos(3x) - \int -\frac{1}{3}(7e^{7x}) \cos(3x) dx \\
 I &= -\frac{1}{3}e^{7x} \cos(3x) + \frac{7}{3} \int e^{7x} \cos(3x) dx.
 \end{aligned}$$

We then apply integration by parts again, this time with

$$\begin{aligned}
 u &= e^{7x} & dv &= \cos(3x) dx \\
 du &= 7e^{7x} dx & v &= \frac{1}{3} \sin(3x).
 \end{aligned}$$

This gives us

$$\begin{aligned}
 I &= -\frac{1}{3}e^{7x} \cos(3x) + \frac{7}{3} \left[\frac{1}{3}e^{7x} \sin(3x) - \int \frac{1}{3}(7e^{7x}) \sin(3x) dx \right] \\
 I &= -\frac{1}{3}e^{7x} \cos(3x) + \frac{7}{9}e^{7x} \sin(3x) - \frac{49}{9} \int e^{7x} \sin(3x) dx \\
 I &= -\frac{1}{3}e^{7x} \cos(3x) + \frac{7}{9}e^{7x} \sin(3x) - \frac{49}{9}I \\
 \frac{58}{9}I &= -\frac{1}{3}e^{7x} \cos(3x) + \frac{7}{9}e^{7x} \sin(3x) \\
 I &= \frac{9}{58} \left(-\frac{1}{3}e^{7x} \cos(3x) + \frac{7}{9}e^{7x} \sin(3x) \right) + C.
 \end{aligned}$$

(c) $\int x^{\frac{5}{3}} (\ln x)^2 dx$

Solution: We begin with the substitution

$$w = \ln x \quad \implies \quad dw = \frac{1}{x} dx, \quad x = e^w.$$

Then

$$\begin{aligned} \int x^{\frac{5}{3}} (\ln x)^2 dx &= \int x^{\frac{8}{3}} (\ln x)^2 \cdot \frac{1}{x} dx \\ &= \int (e^w)^{\frac{8}{3}} w^2 dw \\ &= \int w^2 e^{\frac{8}{3}w} dw. \end{aligned}$$

We now use integration by parts, with

$$\begin{aligned} u &= w^2 & dv &= e^{\frac{8}{3}w} dw \\ du &= 2w dw & v &= \frac{3}{8} e^{\frac{8}{3}w}. \end{aligned}$$

This gives us

$$\begin{aligned} \int x^{\frac{5}{3}} (\ln x)^2 dx &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \int \frac{3}{8} (2w) e^{\frac{8}{3}w} dw \\ &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \frac{3}{4} \int w e^{\frac{8}{3}w} dw. \end{aligned}$$

We apply integration by parts one last time with

$$\begin{aligned} u &= w & dv &= e^{\frac{8}{3}w} dw \\ du &= dw & v &= \frac{3}{8} e^{\frac{8}{3}w} \end{aligned}$$

which yields

$$\begin{aligned} \int x^{\frac{5}{3}} (\ln x)^2 dx &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \frac{3}{4} \left(\frac{3}{8} w e^{\frac{8}{3}w} - \frac{3}{8} \int e^{\frac{8}{3}w} dw \right) \\ &= \frac{3}{8} w^2 e^{\frac{8}{3}w} - \frac{9}{32} w e^{\frac{8}{3}w} + \frac{27}{256} e^{\frac{8}{3}w} + C \\ &= \frac{3}{8} e^{\frac{8}{3} \ln x} \left((\ln x)^2 - \frac{3}{4} \ln x + \frac{9}{32} \right) + C \\ &= \frac{3}{8} x^{\frac{8}{3}} \left((\ln x)^2 - \frac{3}{4} \ln x + \frac{9}{32} \right) + C. \end{aligned}$$

Instructor Notes: All three parts involve standard strategies learned in the online lessons. For each, split the problems between the groups. Then discuss each problem as a class, getting input from the group(s) that worked on that problem.

Problem 2 Evaluate the following integrals

(a) $\int x^5 \cos(x^3) dx$

Solution: We begin with the substitution

$$w = x^3 \quad \implies \quad dw = 3x^2 dx, \quad \frac{1}{3} dw = x^2 dx.$$

Then,

$$\begin{aligned} \int x^5 \cos(x^3) dx &= \int x^3 \cos(x^3) \cdot x^2 dx \\ &= \int w \cos(w) \cdot \frac{1}{3} dw \\ &= \frac{1}{3} \int w \cos(w) dw. \end{aligned}$$

We then use integration by parts, with

$$\begin{aligned} u &= w & dv &= \cos(w) dw \\ du &= dw & v &= \sin(w) \end{aligned}$$

which yields

$$\begin{aligned} \int x^5 \cos(x^3) dx &= \frac{1}{3} \left(w \sin(w) - \int \sin(w) dw \right) \\ &= \frac{1}{3} (w \sin(w) + \cos(w)) + C \\ &= \frac{1}{3} (x^3 \sin(x^3) + \cos(x^3)) + C. \end{aligned}$$

(b) $\int \cos(\sqrt{x}) dx$

Solution: We begin with the substitution

$$w = \sqrt{x} \quad \implies \quad dw = \frac{1}{2\sqrt{w}} dw, \quad 2 dw = \frac{1}{\sqrt{x}} dx.$$

Recitation #9: Integration by parts and Trig Integrals - Full

Then

$$\begin{aligned}\int \cos(\sqrt{x}) \, dx &= \int \cos(\sqrt{x}) \cdot \frac{\sqrt{x}}{\sqrt{x}} \, dx \\&= 2 \int w \cos(w) \, dw \\&= 2(w \sin(w) + \cos(w)) + C \quad \text{From part (a)} \\&= 2(\sqrt{x} \sin(\sqrt{x}) + \cos(\sqrt{x})) + C.\end{aligned}$$

(c) $\int x \cos x \sin x \, dx$

Solution: First, recall that

$$\sin(2x) = 2 \sin x \cos x \quad \implies \quad \sin x \cos x = \frac{1}{2} \sin(2x).$$

So we can rewrite the given integral as

$$\int x \cos x \sin x \, dx = \frac{1}{2} \int x \sin(2x) \, dx.$$

Now we use integration by parts with

$$\begin{aligned}u &= x & dv &= \sin(2x) \, dx \\du &= dx & v &= -\frac{1}{2} \cos(2x).\end{aligned}$$

This gives us that

$$\begin{aligned}\frac{1}{2} \int x \sin(2x) \, dx &= \frac{1}{2} \left(-\frac{1}{2} x \cos(2x) - \int -\frac{1}{2} \cos(2x) \, dx \right) \\&= \frac{1}{4} \left(-x \cos(2x) + \frac{1}{2} \sin(2x) \right) + C.\end{aligned}$$

Instructor Notes: For this problem, you might just lead a whole class discussion - hopefully soliciting ideas from the students (on their own or with hints). You might want to note that some of the problems may be approached in different ways. For example, one could begin part (a) with either the substitution $u = x^3$ or by parts with $u = x^3$, $dv = x^2 \cos(x^3)$.

Problem 3 Evaluate the following integrals

Recitation #9: Integration by parts and Trig Integrals - Full

(a) $\int \tan^{23} x \sec^6 x \, dx$

Solution:

$$\begin{aligned} \int \tan^{23} x \sec^6 x \, dx &= \int \tan^{23} x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^{23} x (1 + \tan^2 x)^2 \sec^2 x \, dx. \end{aligned}$$

We now substitute

$$u = \tan x \quad \implies \quad du = \sec^2 x \, dx.$$

Then

$$\begin{aligned} \int \tan^{23} x (1 + \tan^2 x)^2 \sec^2 x \, dx &= \int u^{23} (1 + u^2)^2 \, du \\ &= \int u^{23} (1 + 2u^2 + u^4) \, du \\ &= \int (u^{23} + 2u^{25} + u^{27}) \, du \\ &= \frac{1}{24} u^{24} + \frac{1}{13} u^{26} + \frac{1}{28} u^{28} + C \\ &= \frac{1}{24} \tan^{24} x + \frac{1}{13} \tan^{26} x + \frac{1}{28} \tan^{28} x + C. \end{aligned}$$

(b) $\int \tan^2 x \sec x \, dx$ *Hint:* $\int \sec x \, dx = \ln |\sec x \tan x| + C$

Solution:

$$\begin{aligned} \int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \\ &= \int \sec^3 x \, dx - \ln |\sec x \tan x| \quad \text{from the hint} \quad (1) \end{aligned}$$

Now, in an attempt to evaluate $\int \sec^3 x \, dx$, we use integration by parts with

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x \, dx \\ du &= \sec x \tan x \, dx & v &= \tan x. \end{aligned}$$

So

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx. \quad (2)$$

Recitation #9: Integration by parts and Trig Integrals - Full

Combining equations (1) and (2) yields

$$\begin{aligned}\int \tan^2 x \sec x \, dx &= \int \sec^3 x \, dx - \ln |\sec x \tan x| \\ \int \tan^2 x \sec x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx - \ln |\sec x \tan x| \\ 2 \int \tan^2 x \sec x \, dx &= \sec x \tan x - \ln |\sec x \tan x| + C \\ \int \tan^2 x \sec x \, dx &= \frac{1}{2} (\sec x \tan x - \ln |\sec x \tan x|) + C.\end{aligned}$$

(c) $\int \tan^2 x \sin x \, dx$

Solution:

$$\begin{aligned}\int \tan^2 x \sin x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx \\ &= \int \frac{1 - \cos^2 x}{\cos^2 x} \sin x \, dx.\end{aligned}$$

Now we substitute

$$u = \cos x \quad \implies \quad du = -\sin x \, dx, \quad -du = \sin x \, dx.$$

This gives us that

$$\begin{aligned}\int \tan^2 x \sin x \, dx &= \int \frac{1 - u^2}{u^2} (-1) \, du \\ &= \int \frac{u^2 - 1}{u^2} \, du \\ &= \int (1 - u^{-2}) \, du \\ &= u + \frac{1}{u} + C \\ &= \cos x + \sec x + C.\end{aligned}$$

Instructor Notes: All three parts involve standard strategies learned in the online lessons. For each, split the problems between the groups. Then discuss each problem as a class, getting input from the group(s) that worked on that problem.

Recitation #9: Integration by parts and Trig Integrals - Full

Problem 4 Evaluate

$$\int_{-\pi}^0 \sqrt{1 - \cos^2 x} \, dx.$$

Solution:

$$\begin{aligned} \int_{-\pi}^0 \sqrt{1 - \cos^2 x} \, dx &= \int_{-\pi}^0 \sqrt{\sin^2 x} \, dx \\ &= \int_{-\pi}^0 |\sin x| \, dx. \end{aligned}$$

Now, when $-\pi \leq x \leq 0$, $\sin x \leq 0$. Thus, on this region, $|\sin x| = -\sin x$. So we continue

$$\begin{aligned} \int_{-\pi}^0 \sqrt{1 - \cos^2 x} \, dx &= \int_{-\pi}^0 -\sin x \, dx \\ &= \left[\cos x \right]_{-\pi}^0 \\ &= \cos(0) - \cos(-\pi) = 1 - (-1) = 2. \end{aligned}$$

Instructor Notes: You may want to do this problem as a whole class - perhaps play-acting by claiming that it is equal to $\int_{-\pi}^0 \sin x \, dx$ rather than

$$\int_{-\pi}^0 |\sin x| \, dx$$