Recitation 27: Cross products

Warm up:

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in 3-space $\mathbb{R}^3$, which of the following make sense?

(a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

(b) $(\vec{a} \cdot \vec{b})\vec{c}$

(c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$

(d) $(\vec{a} \cdot \vec{b}) + \vec{c}$

(e) $(\vec{a} \times \vec{b}) + \vec{c}$

(f) $\vec{a} \cdot (\vec{b} + \vec{c})$

(g) $\vec{a} \cdot (\vec{b} \times \vec{c})$

(h) $\vec{a} \times (\vec{b} \cdot \vec{c})$

(i) $(\vec{a} \times \vec{b})\vec{c}$

Solution: (a) Since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ does **not** make sense.

(b) Now since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b})\vec{c}$ **does** make sense as regular scalar multiplication.

(c) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ **does** make sense.

(d) This is of the form "scalar + vector", which does **not** make sense.

(e) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) + \vec{c}$ **does** make sense.

(f) This is of the form "vector $\cdot$ vector", which **does** make sense.

(g) This is also of the form "vector $\cdot$ vector", which **does** make sense.

(h) This is of the form "vector $\times$ scalar", which does **not** make sense.

(i) Since $\vec{a} \times \vec{b}$ is a vector, this does **not** make sense.

Instructor Notes: This problem can be split up among the groups if the instructor likes (with maybe 3 or so per group).

Learning outcomes:

Group work:

Problem 1 *Given three dimensional vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, use dot product or cross product notation to describe the following vectors:*

(a) *The vector projection of $\vec{w}$ onto $\vec{u}$.*

(b) *A vector orthogonal to both $\vec{u}$ and $\vec{v}$.*

(c) *A vector with the length of $\vec{v}$ and the direction of $\vec{w}$.*

(d) *A vector orthogonal to $\vec{u} \times \vec{v}$ and $\vec{w}$.*

Solution: (a) *This is the definition of vector projections.*

$$\text{proj}_u w = \boxed{\left(\frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\right) \vec{u}}$$

(b) *There are many such vectors, but one of them is*

$$\boxed{\vec{u} \times \vec{v}}$$

(c) *Note that $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ so that $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.*

$$\|\vec{v}\| \left(\frac{\vec{w}}{\|\vec{w}\|}\right) = \boxed{\frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\vec{w} \cdot \vec{w}}} \vec{w}}$$

(d)

$$\boxed{(\vec{u} \times \vec{v}) \times \vec{w}}$$

Instructor Notes: This problem and the Warm-up are meant to force the students to make sense of scalar vs. vector quantities, as well as what quantities the dot and cross products produce.

Problem 2 *Find a vector of length 7 that is perpendicular to both $\langle 5, -1, 8 \rangle$ and $\langle -2, 10, 5 \rangle$.*

Solution: *Let $\vec{u} = \langle 5, -1, 8\rangle$ and $\vec{v} = \langle -2, 10, 5\rangle$. Then a vector which is perpendicular to both $\vec{u}$ and $\vec{v}$ is $\vec{w} := \vec{u} \times \vec{v}$. So we calculate*

$$\begin{aligned}\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -1 & 8 \\ -2 & 10 & 5 \end{vmatrix} &= (-5 - 80)\hat{\imath} - (25 + 16)\hat{\jmath} + (50 - 2)\hat{k} \\ &= -85\hat{\imath} - 41\hat{\jmath} + 48\hat{k}\end{aligned}$$

A unit vector in the same direction as $\vec{w}$ is

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{(-85)^2 + (-41)^2 + 48^2}}\vec{w} = \frac{1}{\sqrt{11210}}\vec{w}.$$

Therefore, a vector with a magnitude of 7 in the same direction as $\vec{w}$ is

$$\vec{t} = \frac{7}{\|\vec{w}\|}\vec{w} = \boxed{\frac{7}{\sqrt{11210}}\langle -85, -41, 48\rangle}$$

Instructor Notes: Using cross product to find perpendicular vectors.

Problem 3 *Find the area of the triangle in $\mathbb{R}^3$ with vertices at $P(2, -1, 0)$, $Q(1, 1, 4)$ and $R(2, -1, 6)$.*

Solution: *The area of the triangle is $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$.*

$$\begin{aligned}\vec{PR} &= \langle 2, -1, 6\rangle - \langle 2, -1, 0\rangle = \langle 0, 0, 6\rangle, \\ \vec{PQ} &= \langle 1, 1, 4, \rangle - \langle 2, -1, 0\rangle = \langle -1, 2, 4\rangle.\end{aligned}$$

So

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= (-\vec{i} + 2\vec{j} + 4\vec{k}) \times 6\vec{k} = -(\vec{i} \times \vec{k}) + 2(\vec{j} \times \vec{k}) + 24(\vec{k} \times \vec{k}) \\ &= -(-\vec{j}) + 2\vec{i} + 0 = \langle 2, 1, 0\rangle.\end{aligned}$$

The area of the triangle is $\frac{1}{2}\sqrt{2^2 + 1^2 + 0^2} = \frac{\sqrt{5}}{2}$.

Instructor Notes: Students should know that we can find the areas of triangles and parallelograms in $\mathbb{R}^3$ by using the cross product.

Problem 4 *A wrench that is* 30*cm long lies along the positive* y*-axis and grips a bolt at the origin. A force is applied in the direction* $\langle 0, 3, -4\rangle$ *at the end of the wrench. Find the magnitude of the force needed to supply* $100J$ *of torque to the bolt.*

Solution: *Below is a picture of the situation*

Let

$$\vec{R} = \langle 0, 0.3, 0\rangle$$

denote the position vector of the end of the wrench. Since the force is in the direction $\langle 0, 3, -4\rangle$*, we know that the force vector satisfies*

$$\vec{F} = c\,\langle 0, 3, -4\rangle = \langle 0, 3c, -4c\rangle$$

for some constant c*. Let* $\vec{t}$ *denote the torque vector. Then* $\vec{t} = \vec{R} \times \vec{F}$ *and so* $\|\vec{t}\| = \|\vec{R} \times \vec{F}\|$*. Thus,*

$$\begin{aligned} 100 &= \|\langle 0, 0.3, 0\rangle \times \langle 0, 3c, -4c\rangle\| \\ &= \left\| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0.3 & 0 \\ 0 & 3c & -4c \end{vmatrix} \right\| \\ &= \|(-1.2c - 0)\hat{\imath} - (0 - 0)\hat{\jmath} + (0 - 0)\hat{k}\| \\ &= 1.2c = \frac{6}{5}c. \end{aligned}$$

Therefore

$$c = 100 \cdot \frac{5}{6} = \frac{500}{6} = \frac{250}{3}.$$

So, the magnitude of the force is

$$\begin{aligned} \|\vec{F}\| &= \frac{250}{3}\sqrt{0^2 + 3^2 + (-4)^2} \\ &= \frac{250}{3} \cdot 5 = \boxed{\frac{1250}{3}\,N} \end{aligned}$$

Instructor Notes: One goal in this problem is for students to make sense of the right-hand rule. The students need to know which direction of rotation tightens or loosens a bolt.