Section 7.2: Integration By Parts

Group work:

Problem 1 *Evaluate the following integrals*

(a) $\int_1^3 x^2 5^x \, dx$

Solution: *We proceed via integration by parts. Let*

$$u = x^2, \qquad dv = 5^x \, dx$$

so that

$$du = 2x \, dx, \qquad v = \frac{5^x}{\ln 5}.$$

Recall the formula for integration by parts is

$$\int_a^b u \, dv = \Big[uv \Big]_a^b - \int_a^b v \, du.$$

So we substitute

$$\begin{aligned} \int_1^3 x^2 5^x \, dx &= \left[\frac{x^2 5^x}{\ln(5)} \right]_1^3 - \int_1^3 2x \frac{5^x}{\ln(5)} \, dx \\ &= \frac{1}{\ln(5)} \left(9 \cdot 5^3 - 5 \right) - \frac{2}{\ln(5)} \int_1^3 x 5^x \, dx. \end{aligned}$$

For the remaining integral we again use integration by parts:

$$u = x, \qquad dv = 5^x$$

$$du = dx, \qquad v = \frac{5^x}{\ln(5)}.$$

Learning outcomes:

Thus,

$$\begin{aligned}
&\frac{1}{\ln(5)}\left(9\cdot 5^3-5\right)-\frac{2}{\ln(5)}\int_1^3 x5^x\,dx\\
&=\frac{5}{\ln(5)}(225-1)-\frac{2}{\ln(5)}\left(\left[\frac{x5^x}{\ln(5)}\right]_1^3-\int_1^3\frac{5^x}{\ln(5)}\,dx\right)\\
&=\frac{1120}{\ln(5)}-\frac{2}{\ln^2(5)}\left((3\cdot 5^3-5)-\left[\frac{5^x}{\ln(5)}\right]_1^3\right)\\
&=\frac{1}{\ln(5)}\left(1120-\frac{740}{\ln(5)}+\frac{248}{\ln^2(5)}\right).
\end{aligned}$$

(b) $\displaystyle\int \arcsin(x)\,dx$

Solution: *We do not know how to integrate* $\arcsin(x)$*, but we do know how to differentiate it, so we will use integration by parts.*

$$u = arcsin(x) \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$$

This gives us

$$\int \arcsin(x)\,dx = x\arcsin(x)-\int\frac{x\,dx}{\sqrt{1-x^2}}$$

$$\begin{aligned}
u &= 1-x^2\\
du &= -2x\,dx\\
-\frac{1}{2}\,du &= dx
\end{aligned}$$

$$\begin{aligned}
\int\frac{x\,dx}{\sqrt{1-x^2}} &= \int\frac{-\,du}{2\sqrt{u}}\\
&= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du\\
&= -\frac{1}{2}\cdot 2u^{\frac{1}{2}}+C\\
&= -\sqrt{1-x^2}+C\\
\int\arcsin(x)\,dx &= x\arcsin(x)-\int\frac{x\,dx}{\sqrt{1-x^2}}\\
&= x\arcsin(x)-\left(-\sqrt{1-x^2}\right)+C\\
&= x\arcsin(x)+\sqrt{1-x^2}+C
\end{aligned}$$

(c) $\displaystyle\int x^{\frac{5}{3}}(\ln x)^2\,dx$

Solution: *We begin with the substitution*

$$w = \ln x \qquad \Longrightarrow \qquad dw = \frac{1}{x}\,dx, \quad x = e^w.$$

Then

$$\begin{aligned}\int x^{\frac{5}{3}}(\ln x)^2\,dx &= \int x^{\frac{8}{3}}(\ln x)^2 \cdot \frac{1}{x}\,dx \\ &= \int (e^w)^{\frac{8}{3}} w^2\,dw \\ &= \int w^2 e^{\frac{8}{3}w}\,dw.\end{aligned}$$

We now use integration by parts, with

$$u = w^2 \qquad dv = e^{\frac{8}{3}w}\,dw$$

$$du = 2w\,dw \qquad v = \frac{3}{8}e^{\frac{8}{3}w}.$$

This gives us

$$\begin{aligned}\int x^{\frac{5}{3}}(\ln x)^2\,dx &= \frac{3}{8}w^2 e^{\frac{8}{3}w} - \int \frac{3}{8}(2w)e^{\frac{8}{3}w}\,dw \\ &= \frac{3}{8}w^2 e^{\frac{8}{3}w} - \frac{3}{4}\int we^{\frac{8}{3}w}\,dw.\end{aligned}$$

We apply integration by parts one last time with

$$u = w \qquad dv = e^{\frac{8}{3}w}\,dw$$

$$du = dw \qquad v = \frac{3}{8}e^{\frac{8}{3}w}$$

which yields

$$\begin{aligned}\int x^{\frac{5}{3}}(\ln x)^2\,dx &= \frac{3}{8}w^2 e^{\frac{8}{3}w} - \frac{3}{4}\left(\frac{3}{8}we^{\frac{8}{3}w} - \frac{3}{8}\int e^{\frac{8}{3}w}\,dw\right) \\ &= \frac{3}{8}w^2 e^{\frac{8}{3}w} - \frac{9}{32}we^{\frac{8}{3}w} + \frac{27}{256}e^{\frac{8}{3}w} + C \\ &= \frac{3}{8}e^{\frac{8}{3}\ln x}\left((\ln x)^2 - \frac{3}{4}\ln x + \frac{9}{32}\right) + C \\ &= \frac{3}{8}x^{\frac{8}{3}}\left((\ln x)^2 - \frac{3}{4}\ln x + \frac{9}{32}\right) + C.\end{aligned}$$

Problem 2 *Evaluate the following integral*

$$\int \sin(3x)e^{7x}\,dx$$

Solution: *We begin by letting* $I = \displaystyle\int \sin(3x)e^{7x}\,dx$. *We then use integration by parts with*

$$u = e^{7x} \qquad dv = \sin(3x)\,dx$$
$$du = 7e^{7x}\,dx \qquad v = -\frac{1}{3}\cos(3x).$$

Then

$$\begin{aligned}\int \sin(3x)e^{7x}\,dx = I &= -\frac{1}{3}e^{7x}\cos(3x) - \int -\frac{1}{3}(7e^{7x})\cos(3x)\,dx\\ I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{3}\int e^{7x}\cos(3x)\,dx.\end{aligned}$$

We then apply integration by parts again, this time with

$$u = e^{7x} \qquad dv = \cos(3x)\,dx$$
$$du = 7e^{7x}\,dx \qquad v = \frac{1}{3}\sin(3x).$$

This gives us

$$\begin{aligned}I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{3}\left[\frac{1}{3}e^{7x}\sin(3x) - \int \frac{1}{3}(7e^{7x})\sin(3x)\,dx\right]\\ I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) - \frac{49}{9}\int e^{7x}\sin(3x)\,dx\\ I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x) - \frac{49}{9}I\\ \frac{58}{9}I &= -\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x)\\ I &= \frac{9}{58}\left(-\frac{1}{3}e^{7x}\cos(3x) + \frac{7}{9}e^{7x}\sin(3x)\right) + C.\end{aligned}$$

Problem 3 *Evaluate the following integrals*

(a) $\displaystyle\int x^5\cos\left(x^3\right)\,dx$

Solution: *We begin with the substitution*

$$w = x^3 \qquad \Longrightarrow \qquad dw = 3x^2\,dx, \quad \frac{1}{3}\,dw = x^2\,dx.$$

Then,

$$\begin{aligned}\int x^5 \cos\left(x^3\right)\, dx &= \int x^3 \cos\left(x^3\right) \cdot x^2\, dx \\ &= \int w \cos(w) \cdot \frac{1}{3}\, dw \\ &= \frac{1}{3} \int w \cos(w)\, dw.\end{aligned}$$

We then use integration by parts, with

$$u = w \qquad dv = \cos(w)\, dw$$

$$du = dw \qquad v = \sin(w)$$

which yields

$$\begin{aligned}\int x^5 \cos\left(x^3\right)\, dx &= \frac{1}{3}\left(w \sin(w) - \int \sin(w)\, dw\right) \\ &= \frac{1}{3}\left(w \sin(w) + \cos(w)\right) + C \\ &= \frac{1}{3}\left(x^3 \sin\left(x^3\right) + \cos\left(x^3\right)\right) + C.\end{aligned}$$

(b) $\int \cos\left(\sqrt{x}\right)\, dx$

Solution: *We begin with the subsitution*

$$w = \sqrt{x} \qquad \Longrightarrow \qquad dw = \frac{1}{2\sqrt{w}}\, dw, \quad 2\, dw = \frac{1}{\sqrt{x}}\, dw.$$

Then

$$\begin{aligned}\int \cos\left(\sqrt{x}\right)\, dx &= \int \cos\left(\sqrt{x}\right) \cdot \frac{\sqrt{x}}{\sqrt{x}}\, dx \\ &= 2 \int w \cos(w)\, dw \\ &= 2(w \sin(w) + \cos(w)) + C \qquad \textit{From part (a)} \\ &= 2\left(\sqrt{x} \sin\left(\sqrt{x}\right) + \cos\left(\sqrt{x}\right)\right) + C.\end{aligned}$$

(c) $\int x \cos x \sin x\, dx$

Solution: *First, recall that*

$$\sin(2x) = 2 \sin x \cos x \qquad \Longrightarrow \qquad \sin x \cos x = \frac{1}{2} \sin(2x).$$

So we can rewrite the given integral as

$$\int x \cos x \sin x \, dx = \frac{1}{2} \int x \sin(2x) \, dx.$$

Now we use integration by parts with

$$u = x \qquad dv = \sin(2x) \, dx$$

$$du = dx \qquad v = -\frac{1}{2} \cos(2x).$$

This gives us that

$$\begin{aligned} \frac{1}{2} \int x \sin(2x) \, dx &= \frac{1}{2} \left(-\frac{1}{2} x \cos(2x) - \int -\frac{1}{2} \cos(2x) \, dx \right) \\ &= \frac{1}{4} \left(-x \cos(2x) + \frac{1}{2} \sin(2x) \right) + C. \end{aligned}$$