Recitation #19: Approximating functions with polynomials

Warm up:

Write out the following polynomial without using sigma notation.

$$\sum_{k=0}^{4}(n^2 + n + 1)x^n$$

Group work:

Problem 1 *For each of the following, write the given polynomial in summation notation starting with $k = 0$.*

(a) $\dfrac{3x}{2} - \dfrac{5x^2}{3} + \dfrac{7x^3}{4} - \dfrac{9x^4}{5} + \dfrac{11x^5}{6}$

(b) $\dfrac{1}{2}x + \dfrac{1 \cdot 5}{4 \cdot 2!}x^3 + \dfrac{1 \cdot 5 \cdot 9}{8 \cdot 3!}x^5 - \dfrac{1 \cdot 5 \cdot 9 \cdot 13}{16 \cdot 4!}x^7$

(c) $(x-1)^3 - \dfrac{(x-1)^4}{2!} + \dfrac{(x-1)^5}{4!} - \dfrac{(x-1)^6}{6!}$

Problem 2 *Assuming that the function $f(x)$ is infinitely differentiable, and given that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + c_4(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

show that the coefficient c_4 of the $(x-a)^4$ term in the Taylor polynomial is $\dfrac{f^{(4)}(a)}{4!}$.

Problem 3 *Let $f(x) = \sin(2x)$. Find $p_3(x)$ about the point $a = \dfrac{\pi}{8}$.*

Problem 4 *Let $f(x) = xe^{-x}$ on the interval $[-2, 8]$.*

(a) *Write the Taylor polynomial $p_4(x)$ around $a = 3$.*

$$\text{Fun facts: } f'(x) = -e^{-x}(x - 1)$$
$$f''(x) = e^{-x}(x - 2)$$
$$f^{(3)}(x) = -e^{-x}(x - 3)$$
$$f^{(4)}(x) = e^{-x}(x - 4)$$

(b) *Write $p_4(x)$ about $a = 3$ in summation notation. Also, write the remainder term $R_4(x)$.*

Extra Challenge Problems

(c) *Calculate $p_4(4.5)$ and, using $R_4(4.5)$, estimate how close $p_4(4.5)$ is to $f(4.5)$. Do the same for $p_4(1.5)$.*

(d) *Use the remainder term $R_4(x)$ to estimate the maximum error for $p_4(x)$ on $[-2, 6]$.*

(e) *How large must n be to assure that the nth degree Taylor polynomial for $f(x) = xe^{-x}$ about $a = 3$ approximates $2e^{-2}$ within 10^{-5}?*