Section 7.8: Improper Integrals

Warm up:

True or False: It is possible for a region to be infinitely long but have a finite area.

Solution: True. Consider the region below the curve $y = \dfrac{1}{x^2}$, $x \geq 1$.

Group work:

Problem 1 *Review of limits:*

(a) $\displaystyle\lim_{x\to-\infty}\left(3x^{-6} + e^{5x} + \frac{\sin x}{x^2+3}\right)$

Solution: *Recall that the limit of a sum is the sum of the limits, provided that those limits exist.*

- $\displaystyle\lim_{x\to-\infty} 3x^{-6} = \lim_{x\to-\infty}\frac{3}{x^6} = 0.$
- $\displaystyle\lim_{x\to-\infty} e^{5x} = 0.$
- $\displaystyle\lim_{x\to-\infty}\frac{\sin x}{x^2+3} = 0$

To rigorously prove this, you need to use the squeeze theorem.

Thus,

$$\lim_{x\to-\infty}\left(3x^{-6} + e^{5x} + \frac{\sin x}{x^2+3}\right) = 0$$

(b) $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+4}}$

Learning outcomes:

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{9x^2+4}} &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\cdot\sqrt{9+\frac{4}{x^2}}} \\
&= \lim_{x\to\infty} \frac{x}{|x|\cdot\sqrt{9+\frac{4}{x^2}}} \\
&= \lim_{x\to\infty} \frac{x}{x\cdot\sqrt{9+\frac{4}{x^2}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{4}{x^2}}} \\
&= \frac{1}{\sqrt{9+0}} = \frac{1}{3}.
\end{aligned}$$

(c) $\lim\limits_{x\to-\infty} \arctan x$

Solution: $\lim\limits_{x\to-\infty} \arctan x = -\dfrac{\pi}{2}.$

Problem 2 *Determine if the given integral converges or diverges. If it converges, find the value.*

$$\int_{-1}^{\infty} \frac{3}{2x+1}\,dx$$

Solution: *The function* $\dfrac{3}{2x+1}$ *has a vertical asymptote at* $x = -\dfrac{1}{2}$. *So we rewrite the original integral as*

$$\int_{-1}^{\infty} \frac{3}{2x+1}\,dx = \lim_{a\to-\frac{1}{2}^-} \int_{-1}^{a} \frac{3}{2x+1}\,dx + \lim_{b\to-\frac{1}{2}^+} \int_{b}^{0} \frac{3}{2x+1}\,dx + \lim_{c\to\infty} \int_{0}^{c} \frac{3}{2x+1}\,dx.$$

The latter integral does not exist. To see this, just note that

$$\begin{aligned}
\lim_{c\to\infty} \int_0^c \frac{3}{2x+1}\,dx &= \lim_{c\to\infty} \left[\frac{3}{2}\ln|2x+1|\right]_0^c \\
&= \lim_{c\to\infty} \frac{3}{2}\ln|2c+1| = \infty.
\end{aligned}$$

Therefore,

$$\int_{-1}^{\infty} \frac{3}{2x+1}\,dx \quad \textit{diverges.}$$

Problem 3 (a) *Show that*

$$\frac{9}{2x^2+3x} = \frac{3}{x} - \frac{6}{2x+3}$$

(b) *Determine if the integral*

$$\int_1^\infty \frac{9}{2x^2+3x}\,dx$$

converges or diverges. If it converges, give the value that it converges to.

Solution: *(a) Since $2x^2+3x = x(2x+3)$, we apply the method of partial fractions:*

$$\begin{aligned} \frac{9}{2x^2+3x} &= \frac{A}{x} + \frac{B}{2x+3} \\ \implies \quad 9 &= A(2x+3) + Bx. \end{aligned}$$

Letting $x = 0$ gives that

$$9 = 3A \quad \implies \quad A = 3.$$

Then letting $x = -\frac{3}{2}$, we see that

$$9 = -\frac{3}{2}B \quad \implies \quad B = -9 \cdot \frac{2}{3} = -6.$$

Therefore, plugging in our values for A and B gives us

$$\frac{9}{2x^2+3x} = \frac{3}{x} - \frac{6}{2x+3}.$$

(b) We have that

$$\begin{aligned} \int_1^\infty \frac{9}{2x^2+3x}\,dx &= \lim_{t\to\infty} \int_1^t \left(\frac{3}{x} - \frac{6}{2x+3}\right) dx \\ &= \lim_{t\to\infty} \left[3\ln|x| - \frac{6}{2}\ln|2x+3|\right]_1^t \qquad \textit{don't forget to divide the 6 by 2} \\ &= \lim_{t\to\infty} \left(3\ln|t| - 3\ln|2t+3| - 0 + 3\ln(5)\right) \qquad \ln(1) = 0 \\ &= \lim_{t\to\infty} \left(3\ln\left|\frac{5t}{2t+3}\right|\right) \qquad \textit{properties of logarithms} \\ &= \boxed{3\ln\left(\frac{5}{2}\right)} \qquad \textit{since } \ln(x) \textit{ is a continuous function} \end{aligned}$$

Problem 4 (a) *Show that*

$$\frac{6x-8}{x^3+4x} = \frac{2x+6}{x^2+4} - \frac{2}{x}$$

(b) *Determine if the integral*

$$\int_3^\infty \frac{6x-8}{x^3+4x}\,dx$$

converges or diverges. If it converges, give the value that it converges to.

Solution: *(a) Since $x^3+4x = x(x^2+4)$, we use partial fractions:*

$$\begin{aligned} \frac{6x-8}{x^3+4x} &= \frac{Ax+B}{x^2+4} + \frac{C}{x} \\ \implies \qquad 6x-8 &= (Ax+B)(x) + C(x^2+4). \end{aligned}$$

Letting $x=0$, we see that

$$-8 = 4C \quad \implies \quad C = -2.$$

To find A and B, let us plug in $C=-2$ and simplify:

$$\begin{aligned} 6x-8 &= Ax^2 + Bx - 2x^2 - 8 \\ &= (A-2)x^2 + Bx - 8. \end{aligned}$$

Aligning the respective coefficients, we see that

$$\begin{aligned} & A-2=0 \quad \text{and} \quad B=6 \\ \implies \; & A=2 \quad \textit{and} \quad B=6. \end{aligned}$$

Finally, plugging this into the original equation yields

$$\frac{6x-8}{x^3+4x} = \frac{2x+6}{x^2+4} - \frac{2}{x}.$$

(b) We have that

$$\begin{aligned} \int_3^\infty \frac{6x-8}{x^3+4x}\,dx &= \lim_{t\to\infty} \int_3^t \left(\frac{2x+6}{x^2+4} - \frac{2}{x}\right) dx \\ &= \lim_{t\to\infty} \left(\int_3^t \frac{2x}{x^2+4}\,dx + \int_3^t \frac{6}{x^2+4}\,dx - \int_3^t \frac{2}{x}\,dx \right). \end{aligned}$$

Let us evaluate each integral separately, combine them, and then take the limit.

(i)

$$\int_3^t \frac{2x}{x^2+4}\,dx = \int_{13}^{t^2+4} \frac{1}{w}\,dw \quad w = x^2+4,\ dw = 2x\,dx$$
$$= \ln(t^2+4) - \ln(13).$$

(ii)

$$\int_3^t \frac{6}{x^2+4}\,dx = \left[\frac{6}{2}\arctan\left(\frac{x}{2}\right)\right]_3^t$$
$$= 3\arctan\left(\frac{t}{2}\right) - 3\arctan\left(\frac{3}{2}\right).$$

(iii)

$$\int_3^t \frac{2}{x}\,dx = \Big[2\ln|x|\Big]_3^t$$
$$= 2\ln|t| - 2\ln(3).$$

We now combined these three expressions, and then compute the limit.

$$\int_3^\infty \frac{6x-8}{x^3+4x}\,dx$$
$$= \lim_{t\to\infty}\left[\big(\ln(t^2+4)-\ln(13)\big) + \left(3\arctan\left(\frac{t}{2}\right) - 3\arctan\left(\frac{3}{2}\right)\right) - \big(2\ln|t| - 2\ln(3)\big)\right]$$
$$= \lim_{t\to\infty}\left[\ln(t^2+4) - \ln t^2 + 3\arctan\left(\frac{t}{2}\right) - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)\right]$$
$$= \lim_{t\to\infty}\left[\ln\left(\frac{t^2+4}{t^2}\right) + 3\arctan\left(\frac{t}{2}\right) - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)\right]$$
$$= \ln(1) + 3\cdot\frac{\pi}{2} - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right) \quad \lim_{t\to\infty}\arctan(t) = \frac{\pi}{2}$$
$$= \boxed{\frac{3\pi}{2} - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)} \quad \textit{if you like, } -\ln 13 + \ln 9 = \ln\left(\frac{9}{13}\right)$$

Problem 5 *Given that* $\dfrac{37}{(2x-1)(x^2+9)} = \dfrac{4}{2x-1} - \dfrac{2x+1}{x^2+9}$, *evaluate:*

$$\int_3^\infty \frac{37}{(2x-1)(x^2+9)}\,dx$$

Solution:

$$\begin{aligned}\int_3^\infty \frac{37}{(2x-1)(x^2+9)}\,dx &= \lim_{b\to\infty}\int_3^b\left(\frac{4}{2x-1}-\frac{2x+1}{x^2+9}\right)dx\\ &= \lim_{b\to\infty}\int_3^b\left(\frac{4}{2x-1}-\frac{2x}{x^2+9}-\frac{1}{x^2+9}\right)dx \qquad (1)\end{aligned}$$

We compute the antiderivatives of each term separately:

- *For* $\int \frac{4}{2x-1}\,dx$*, let* $u = 2x-1 \to du = 2dx \to dx = \frac{du}{2}$.

  Thus, $\int \frac{4}{2x-1}\,dx = \int \frac{4}{u}\left(\frac{du}{2}\right) = 2\ln|u| + C = \underline{2\ \ln|2x+1| + C}$

- *For* $\int \frac{2x}{x^2+9}\,dx$*, let* $u = x^2+9 \to du = 2xdx \to dx = \frac{du}{2x}$.

  Thus, $\int \frac{2x}{x^2+9}\,dx = \int \frac{2x}{u}\left(\frac{du}{2x}\right) = \ln|u| + C = \underline{\ln(x^2+9) + C}$

- *For* $\int \frac{1}{x^2+9}\,dx$*, recall the important formula* $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$.

  Thus, $\int \frac{1}{x^2+9}\,dx = \underline{\frac{1}{3}\arctan\frac{x}{3} + C}$

*Substituting these results into (**??**) and dropping the constants since this is a definite integral gives:*

$$\int_3^\infty \frac{37}{(2x-1)(x^2+9)}\,dx = \lim_{b\to\infty}\left[2\ \ln|2x-1| - \ln(x^2+9) - \frac{1}{3}\arctan\frac{x}{3}\right|_3^b$$

To evaluate the resulting limit correctly, you MUST *combine the logarithms:*

$$\begin{aligned}
\int_3^\infty \frac{37}{(2x-1)(x^2+9)}\,dx &= \lim_{b\to\infty}\left[\ln(2x-1)^2 - \ln(x^2+9) - \frac{1}{3}\arctan\frac{x}{3}\right|_3^b \\
&= \lim_{b\to\infty}\left[\ln\left(\frac{4x^2-4x+1}{x^2+9}\right) - \frac{1}{3}\arctan\frac{x}{3}\right|_3^b \\
&= \lim_{b\to\infty}\left[\left\{\ln\left(\frac{4b^2-4b+1}{b^2+9}\right) - \frac{1}{3}\arctan\frac{b}{3}\right\} - \left\{\ln\left(\frac{25}{18}\right) - \frac{1}{3}\arctan 1\right\}\right] \\
&= \ln\left[\lim_{b\to\infty}\frac{4b^2-4b+1}{b^2+9}\right] - \frac{1}{3}\ln\left[\lim_{b\to\infty}\arctan\frac{b}{3}\right] - \ln\left(\frac{25}{18}\right) + \frac{1}{3}\cdot\frac{\pi}{4} \\
&= \ln 4 - \frac{1}{3}\cdot\frac{\pi}{2} - \ln\left(\frac{5}{18}\right) + \frac{\pi}{12} \\
&= \boxed{\ln\left(\frac{72}{25}\right) - \frac{\pi}{12}}
\end{aligned}$$