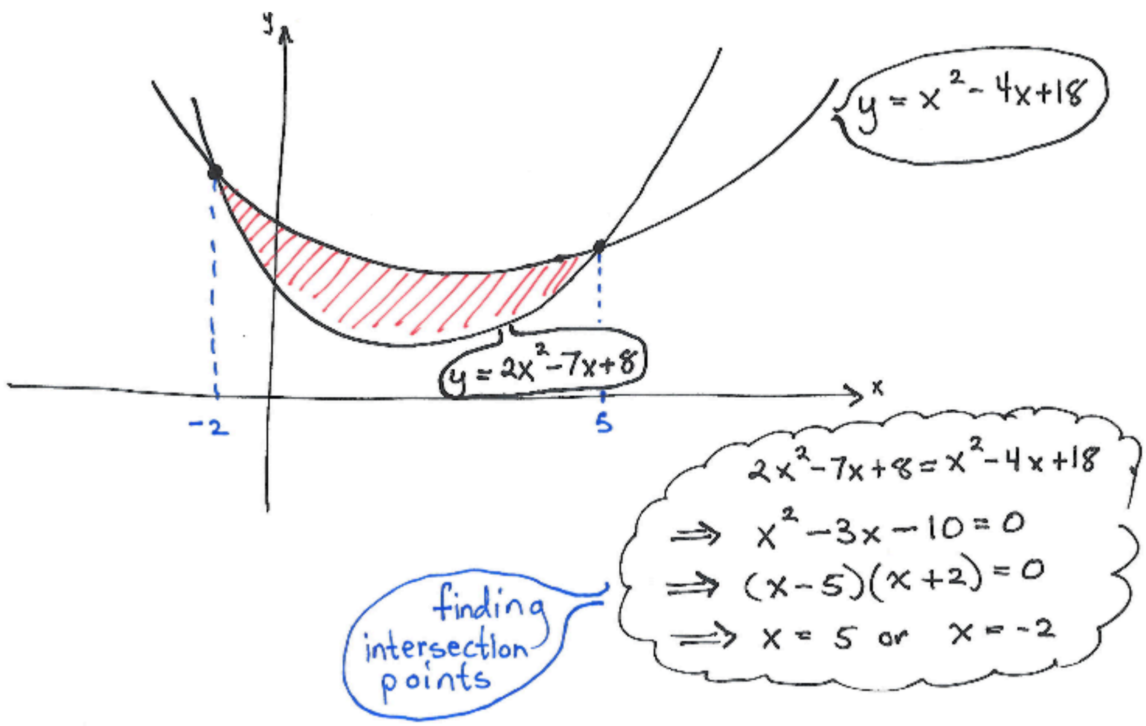
# Recitation # 2 Regions Between Curves - Solutions

## Group work:

**Problem 1** *Consider the region bounded by the curves $y = 2x^2 - 7x + 8$ and $y = x^2 - 4x + 18$.*

(a) *Draw a sketch of the graphs.*

***Solution:***




(b) *Find the area between these curves.*

Learning outcomes:

***Solution:*** *Let $y_1 = 2x^2 - 7x + 8$ and $y_2 = x^2 - 4x + 18$. By the solution to part (a), we know both that $y_1 - y_2 = x^2 - 3x - 10$ and that these two curves intersect at $x = -2, 5$. By checking the point $x = 0$ (or by looking at the graph from part (a)) we see that $y_2 \geq y_1$ on the interval $[-2, 5]$. So the area between the curves is:*

$$\begin{aligned}
\int_{-2}^{5} (y_2 - y_1)\,dx &= \int_{-2}^{5} (-x^2 + 3x + 10)\,dx \\
&= \left[ -\frac{1}{3}x^3 + \frac{3}{2}x^2 + 10x \right]_{-2}^{5} \\
&= \left( -\frac{125}{3} + \frac{75}{2} + 50 \right) - \left( \frac{8}{3} + 6 - 20 \right) \\
&= -\frac{133}{3} + \frac{75}{2} + 64 \\
&= \frac{-266 + 225 + 384}{6} = \frac{343}{6}
\end{aligned}$$

(c) *Find the area of the region bounded by the curves $x = 2y^2 - 7y + 8$ and $x = y^2 - 4y + 18$.*

***Solution:*** *This region is exactly the same as the red region from part (a), except it is rotated clockwise by $90^\circ$. Since the area of a region does not change under rotation, we have that the area of the new region is still $\frac{343}{6}$.*

(d) *Find the area of the region bounded by the curves*

*(i)* $y = 2x^2 - 7x$ and $y = x^2 - 4x + 10$.

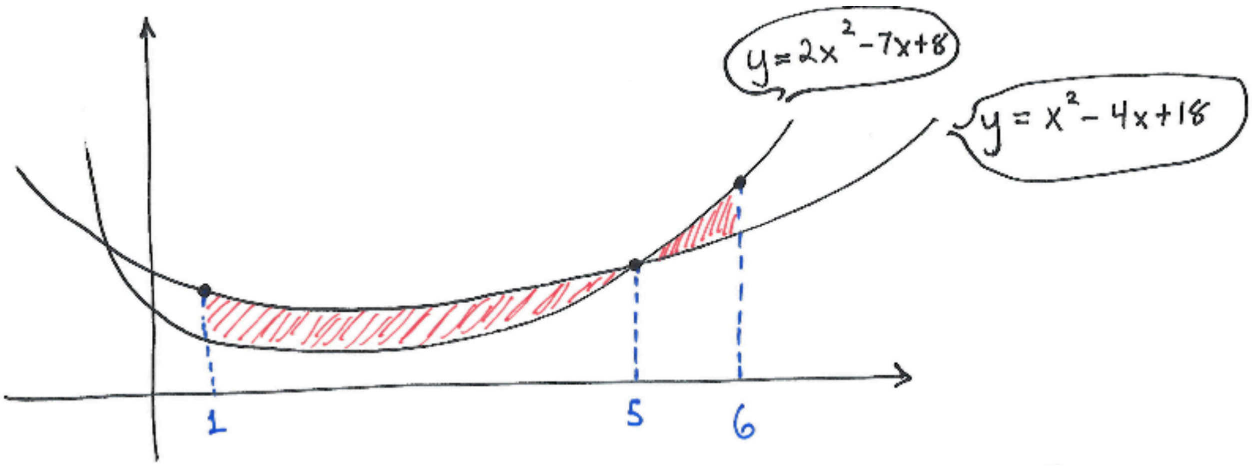
***Solution:*** *The region bounded by the curves $y = 2x^2 - 7x$ and $y = x^2 - 4x + 10$ is just the red region in part (a) translated downward by 8 units. Since the area of a region does not change under translation, we have that the area of the new region is still $\frac{343}{6}$.*

*(ii)* $y = 2x^2 - 7x - 30$ and $y = x^2 - 4x - 20$.

***Solution:*** *This is the same as (i), except now the red region has been translated downward by 38 units. Therefore, the area of this region is still $\frac{343}{6}$.*

(e) *Find the area of the region bounded by the curves $y = 2x^2 - 7x + 8$, $y = x^2 - 4x + 18$, $x = 1$, and $x = 6$.*

***Solution:***




*We already know that the graphs of the two functions intersect at the point $x = 5$. By checking the points $x = 1$ and $x = 6$ (or by looking at the graph above) we see that*

$$\begin{aligned} y_2 &\geq y_1 \quad \text{on} \quad [1,5] \\ y_1 &\geq y_2 \quad \text{on} \quad [5,6] \end{aligned}$$

*Thus, the area between the curves is*

$$\int_1^5 (y_2 - y_1)\,dx + \int_5^6 (y_1 - y_2)\,dx$$

$$= \int_1^5 (-x^2 + 3x + 10)\,dx + \int_5^6 (x^2 - 3x - 10)\,dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 10x\right]_1^5 + \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 - 10x\right]_5^6$$

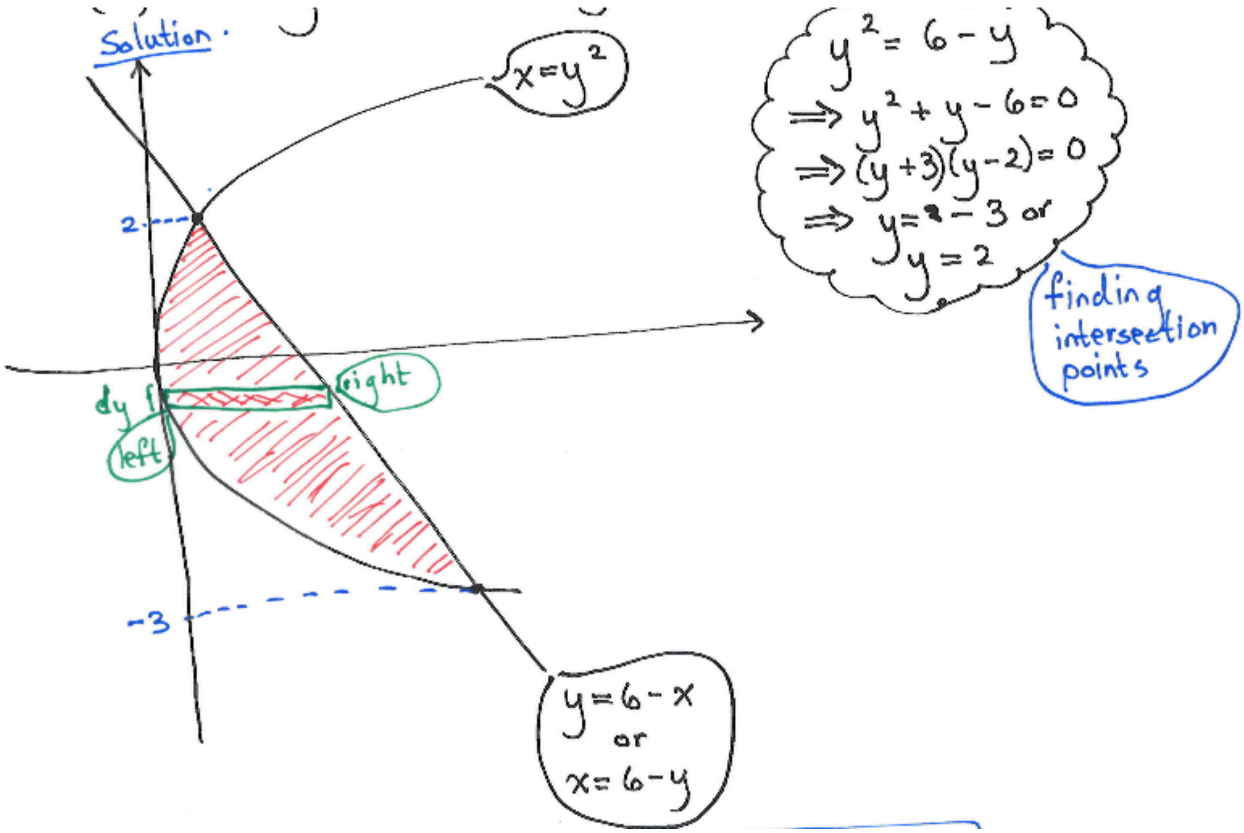
$$= \left[\left(-\frac{125}{3} + \frac{75}{2} + 50\right) - \left(-\frac{1}{3} + \frac{3}{2} + 10\right)\right] + \left[(72 - 54 - 60) - \left(\frac{125}{3} - \frac{75}{2} - 50\right)\right]$$

$$= -\frac{249}{3} + \frac{147}{2} + 48 = -83 + 48 + \frac{147}{2} = \frac{-70 + 147}{2} = \frac{77}{2}$$

**Problem 2** *Set up a single integral that computes the area of the region bounded by the curves (and be sure to draw a sketch of the graphs).*

(a) $x = y^2$ and $y = 6 - x$

***Solution:***

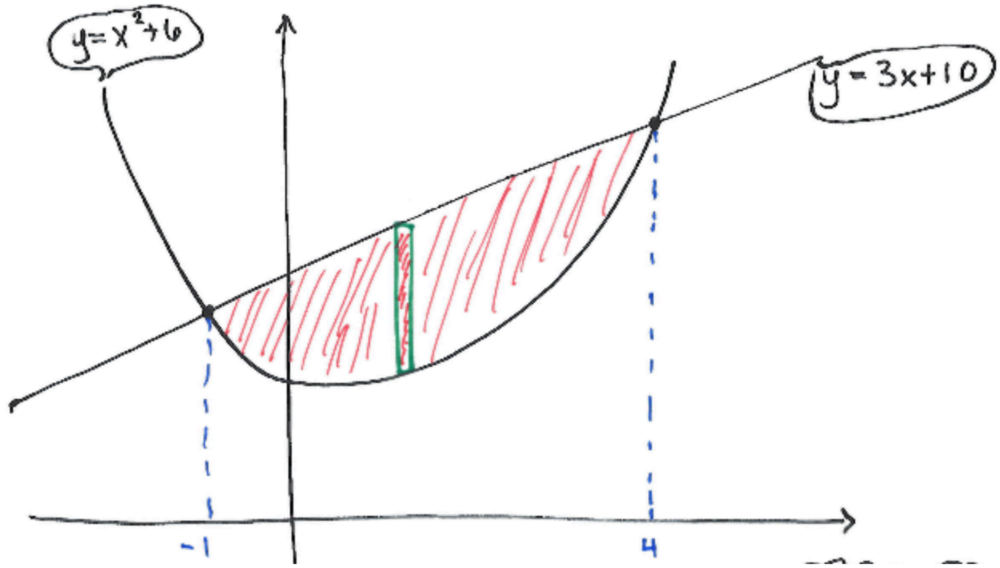



*Thus,*

$$\text{Area of region} = \int_{-3}^{2} [(6-y) - y^2]\, dy.$$

(b) $y = x^2 + 6$ and $y = 3x + 10$

***Solution:***




*To find the intersection points in the above picture, we solve*

$$\begin{aligned} x^2 + 6 &= 3x + 10 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0 \\ x &= -1, 4. \end{aligned}$$

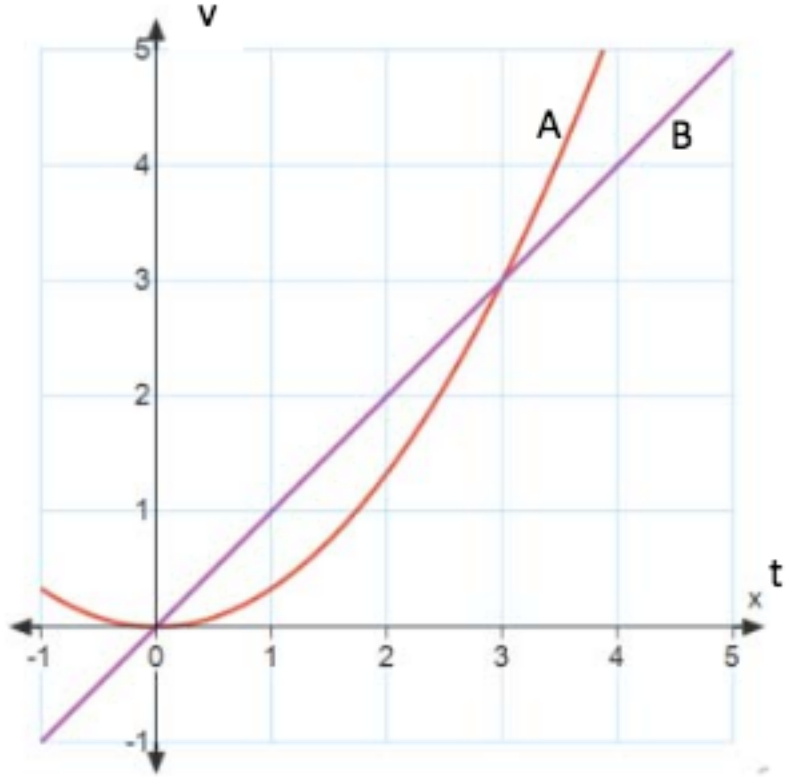
*So*

$$\text{Area of region} = \int_{-1}^{4} \left[(3x + 100) - (x^2 + 6)\right] dx.$$

**Problem 3** *Two runners (A and B) run in a race in which the winner runs the farthest in 4 minutes. The runners' respective velocities are*

$$v_A(t) = \frac{1}{3}t^2 \qquad v_B(t) = t$$

*The graphs of the runners' velocities is given below.*




(a) *Who is running the fastest 2 minutes into the race?*

***Solution:*** $v_A(2) = \frac{4}{3}$ *and* $v_B(2) = 2$. *So* $B$ *is running faster at the 2 minute mark of the race.*

(b) *Who is winning the race 2 minutes into the race (and by how much)?*

***Solution:*** *The distance that* $A$ *covers in the first 2 minutes is*

$$\int_0^2 v_A(t)\,dt = \int_0^2 \frac{1}{3}t^2\,dt = \left[\frac{1}{9}t^3\right]_0^2 = \frac{8}{9}.$$

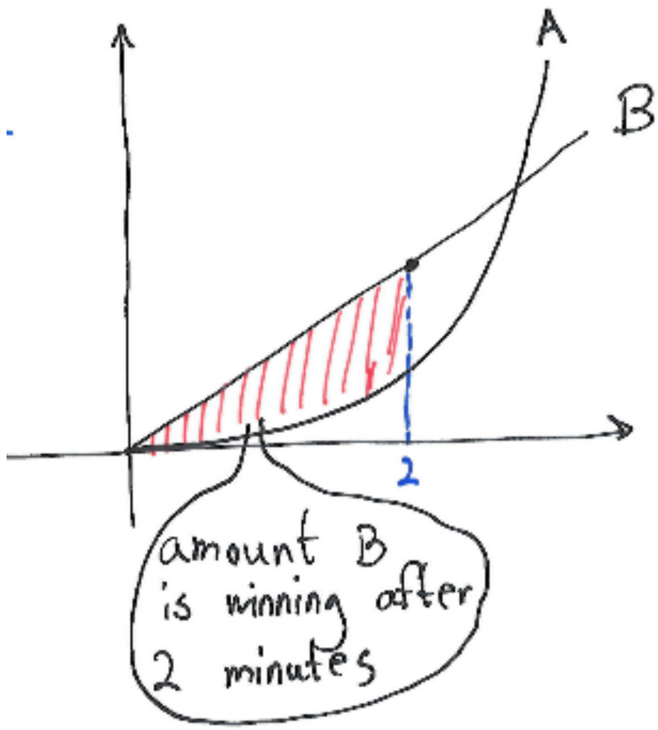
*The distance that* $B$ *covers in the first 2 minutes is*

$$\int_0^2 v_B(t)\,dt = \int_0^2 t\,dt = \left[\frac{1}{2}t^2\right]_0^2 = 2.$$

*So* $B$ *is winning after 2 minutes.*

$B$ *is winning by* $2 - \frac{8}{9} = \frac{10}{9}$. *This could also be calculated by*

$$\int_0^2 (v_B(t) - v_A(t))\,dt.$$




(c) *What special event occurs 3 minutes into the race?*

***Solution:*** *Runner A matches runner B's velocity. ie,* $v_A(3) = v_B(3)$.

(d) *Who wins the race (and by how much)?*

***Solution:*** *The distance that A covers is*

$$\int_0^4 v_A(t)\,dt = \int_0^4 \frac{1}{3}t^2\,dt = \left[\frac{1}{9}t^3\right]_0^4 = \frac{64}{9} = 7.\overline{1}.$$

*The distance that B covers is*

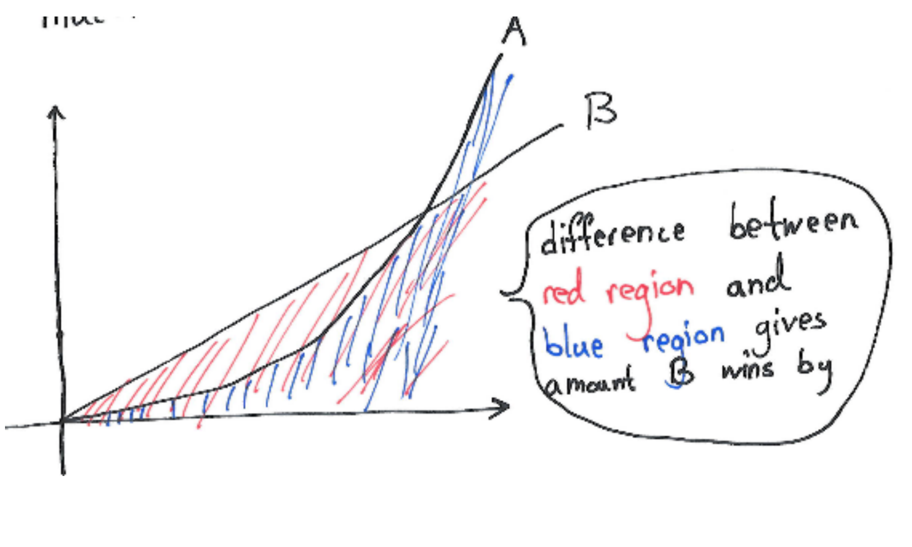
$$\int_0^4 v_B(t)\,dt = \int_0^4 t\,dt = \left[\frac{1}{2}t^2\right]_0^4 = 8.$$

*So runner B wins. The amount that B wins by is*

$$8 - \frac{64}{9} = \frac{8}{9}.$$

*This could have also been computed by*

$$\int_0^4 (v_B(t) - v_A(t))\,dt.$$


A
B
difference between
red region and
blue region gives
amount B wins by