Recitation # 7: Exponential models and approaches to integration - Solutions

Group work:

Problem 1 *Vitameatavegamin is a strange substance that comes in two forms. V-I decays at a linear rate, while V-II decays at an exponential rate. Both have the property that 10 ounces will decrease to 7 ounces in 6 hours. For each of V-I and V-II, answer the following:*

(a) *If we started with 80 ounces, how much will there be 6 hours later?*

Solution: ***V-I:*** *Recall that in the linear decay model*

$$y(t) = -k \cdot t + y_0$$

where k denotes the rate of decay and y_0 is the initial amount. We are given that $y_0 = 80oz$. Clearly, we also have that

$$y'(t) = -k.$$

In the linear decay model, the rate of decay does not depend on the initial amount. So from the given information, we have that

$$-k = \frac{10oz - 7oz}{0hr - 6hr} = -\frac{1}{2}.$$

Thus, $y(t) = -\frac{1}{2}t + 80$, and therefore

$$y(6) = -\frac{1}{2}(6) + 80 = 77oz.$$

V-II: *Recall that in the exponential decay model*

$$y(t) = y_0 \cdot e^{-k \cdot t}$$

where again $y_0 = 80oz$ is the initial amount. Also notice that

$$\begin{aligned} y'(t) &= -ky_0e^{-kt} \\ &= -ky(t) \\ \implies \quad y'(0) &= -ky_0. \end{aligned}$$

Learning outcomes:

It is given that it takes 6 hours for 10 ounces to decrease to 7 ounces. In other words, it takes 6 hours for 70% of the substance to remain. So we have that

$$\begin{aligned}
y(6) &= \frac{7}{10}y_0 \\
\implies \quad y_0 e^{-k\cdot 6} &= \frac{7}{10}y_0 \\
\implies \quad e^{-6k} &= \frac{7}{10} \\
\implies \quad -6k &= \ln\left(\frac{7}{10}\right) = -\ln\left(\frac{10}{7}\right) \\
\implies \quad k &= \frac{1}{6}\ln\left(\frac{10}{7}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
y(6) &= 80e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot 6} \\
&= 80e^{-\ln\left(\frac{10}{7}\right)} \\
&= 80\cdot\frac{7}{10} = 56oz.
\end{aligned}$$

(b) *How long will it take to decrease from 15 ounces to 7.5 ounces?*

Solution: ***V-I:*** *Recall from above that* $k = \frac{1}{2}$. *Then since* y_0 *is now* 15, *we have that*

$$y(t) = -\frac{1}{2}t + 15.$$

We want to find t *such that* $y(t) = 7.5$. *So we solve*

$$\begin{aligned}
7.5 &= -\frac{1}{2}t + 15 \\
-\frac{15}{2} &= -\frac{1}{2}t \\
t &= 15 \text{ hours.}
\end{aligned}$$

V-II: *Again, since* y_0 *is now* 15, *we know from above that*

$$y(t) = 15e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t}.$$

We want to find t *such that* $y(t) = 7.5 = \dfrac{15}{2}$. *So we solve*

$$\begin{aligned}
\frac{15}{2} &= 15e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t} \\
\frac{1}{2} &= e^{-\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t} \\
\ln\left(\frac{1}{2}\right) &= -\frac{1}{6}\ln\left(\frac{10}{7}\right)\cdot t \\
\ln\left(\frac{10}{7}\right) t &= -6\ln\left(\frac{1}{2}\right) = 6\ln 2 \\
t &= \frac{6\ln 2}{\ln\left(\frac{10}{7}\right)} \text{ hours.}
\end{aligned}$$

Problem 2 *Evaluate*

$$\int \frac{5x^3 - 6x + 2}{x-5}\, dx.$$

Solution: *When integrating a rational function (i.e., a fraction of polynomials) where the degree of the numerator is greater than or equal to the degree of the denominator, we need to use long division to simplify the expression.*

$$\begin{array}{r l}
 & 5x^2 + 25x + 119 \\
x-5 \,\big) & 5x^3 + 0x^2 - 6x + 2 \\
 & -(5x^3 - 25x^2) \\
\hline
 & 25x^2 - 6x \\
 & -(25x^2 - 125x) \\
\hline
 & 119x + 2 \\
 & -(119x - 595) \\
\hline
 & 597
\end{array}$$

recall polynomial long division

So

$$\frac{5x^3 - 6x + 2}{x-5} = 5x^2 + 25x + 119 + \frac{597}{x-5}$$

and therefore

$$\begin{aligned}\int \frac{5x^3-6x+2}{x-5}\,dx &= \int \left(5x^2+25x+119+\frac{597}{x-5}\right)dx \\ &= \frac{5}{3}x^3+\frac{25}{2}x^2+119x+597\int \frac{1}{x-5}\,dx.\end{aligned}$$

To evaluate

$$\int \frac{1}{x-5}\,dx$$

we technically should perform a substitution. But since $\frac{d}{dx}(x-5)=1$, *we can treat* $x-5$ *as a variable during integration. Thus*

$$\int \frac{1}{x-5}\,dx = \ln|x-5|$$

and therefore

$$\int \frac{5x^3-6x+2}{x-5}\,dx = \frac{5}{3}x^3+\frac{25}{2}x^2+119x+597\ln|x-5|+C.$$

Problem 3 *Evaluate*

$$\int \frac{5}{3^{2x}+3^{-2x}}\,dx.$$

Solution:

$$\begin{aligned}\int \frac{5}{3^{2x}+3^{-2x}}\,dx &= \int \frac{5}{3^{2x}+3^{-2x}}\cdot\frac{3^{2x}}{3^{2x}}\,dx \\ &= \int \frac{5\cdot 3^{2x}}{\left(3^{2x}\right)^2+1}\,dx.\end{aligned}$$

Now, let

$$u = 3^{2x}.$$

Then

$$du = 2\cdot 3^{2x}\cdot \ln(3)\,dx$$

and so

$$dx = \frac{1}{2\cdot 3^{2x}\cdot\ln(3)}\,du.$$

Thus

$$\begin{aligned}\int \frac{5}{3^{2x}+3^{-2x}}\,dx &= \int \frac{5}{u^2+1}\cdot\frac{1}{2\ln(3)}\,du \\ &= \frac{5}{2\ln(3)}\arctan(u)+C \\ &= \frac{5}{2\ln(3)}\arctan\left(3^{2x}\right)+C.\end{aligned}$$

Problem 4 *Evaluate the following integrals*

(a) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$

Solution: *Let* $u = 1+\sin x$*. Then* $du = \cos x\,dx$*, and so*

$$\begin{aligned}\int \frac{\cos x}{1+\sin x}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln(1+\sin(x)) + C.\end{aligned}$$

(b) $\displaystyle\int \frac{1}{\sin x - 1}\,dx$

Solution:

$$\begin{aligned}\int \frac{1}{\sin x-1}\,dx &= \int \frac{1}{\sin x - 1}\cdot\frac{\sin x+1}{\sin x+1}\,dx \qquad \textit{since } \frac{\sin x+1}{\sin x+1} = 1 \\ &= \int \frac{\sin x+1}{\sin^2 x - 1}\,dx \\ &= \int \frac{\sin x + 1}{-\cos^2 x}\,dx \qquad \textit{since } \cos^2 x + \sin^2 x = 1 \\ &= \int \frac{-\sin x}{\cos^2 x}\,dx - \int \frac{1}{\cos^2 x}\,dx.\end{aligned}$$

Now we split up and compute these two integrals separately.

$$\begin{aligned}\int \frac{-\sin x}{\cos x}\,dx &= \int \frac{1}{u^2}\,du \qquad \textit{where } u = \cos x,\ du = -\sin x\,dx \\ &= -\frac{1}{u} + C = \frac{-1}{\cos x} + C.\end{aligned}$$

Also,

$$\int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C.$$

So, we finally have that

$$\int \frac{1}{\sin x - 1}\,dx = \frac{-1}{\cos x} - \tan x + C.$$

Problem 5 *Evaluate the following integrals*

(a) $\displaystyle\int \frac{13}{\sqrt{12x - x^2 - 20}}\,dx$

Solution: *First notice that by completing the square, we have that*

$$\begin{aligned}
12x - x^2 - 20 &= -(x^2 - 12x) - 20 \\
&= -(x^2 - 12x + 36) - 20 + 36 \\
&= 16 - (x-6)^2 \\
&= 16\left(1 - \frac{(x-6)^2}{16}\right) \\
&= 16\left(1 - \left(\frac{x-6}{4}\right)^2\right).
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{13}{\sqrt{12x - x^2 - 20}}\,dx &= \int \frac{13}{\sqrt{16\left(1 - \left(\frac{x-6}{4}\right)^2\right)}}\,dx \\
&= \frac{13}{4}\int \frac{1}{\sqrt{1 - \left(\frac{x-6}{4}\right)^2}}\,dx.
\end{aligned}$$

Let

$$u = \frac{x-6}{4} = \frac{1}{4}(x-6).$$

Then

$$du = \frac{1}{4}\,dx \qquad \Longrightarrow \qquad 4\,du = dx.$$

Finally, we have that

$$\begin{aligned}
\int \frac{13}{\sqrt{12x - x^2 - 20}}\,dx &= 13\int \frac{1}{\sqrt{1-u^2}}\,du \\
&= 13\arcsin(u) + C \\
&= 13\arcsin\left(\frac{x-6}{4}\right) + C.
\end{aligned}$$

(b) $\displaystyle\int \frac{13x^3}{\sqrt{12x^6 - x^8 - 20x^4}}\,dx$

Solution:

$$\begin{aligned}
\int \frac{13x^3}{\sqrt{12x^6 - x^8 - 20x^4}}\,dx &= \int \frac{13x^3}{\sqrt{x^4}\cdot\sqrt{12x^2 - x^4 - 20}}\,dx \\
&= \int \frac{13x}{\sqrt{12x^2 - x^4 - 20}}\,dx \\
&= \frac{1}{2}\int \frac{13}{\sqrt{12u - u^2 - 20}}\,du && \textit{Let } u = x^2,\ du = 2x\,dx \\
&= \frac{1}{2}\left(13\arcsin\left(\frac{u-6}{4}\right)\right) + C && \textit{from part (a)} \\
&= \frac{13}{2}\arcsin\left(\frac{x^2-6}{4}\right) + C.
\end{aligned}$$

(c) $\displaystyle\int \frac{13e^{4x}}{\sqrt{12e^{6x} - e^{8x} - 20e^{4x}}}\,dx$

Solution:

$$\begin{aligned}
\int \frac{13e^{4x}}{\sqrt{12e^{6x} - e^{8x} - 20e^{4x}}}\,dx &= \int \frac{13e^{4x}}{\sqrt{e^{4x}}\cdot\sqrt{12e^{2x} - e^{4x} - 20}}\,dx \\
&= \int \frac{13e^{2x}}{\sqrt{12e^{2x} - e^{4x} - 20}}\,dx \\
&= \frac{1}{2}\int \frac{13}{\sqrt{12u - u^2 - 20}}\,du && \text{where } u = e^{2x},\ du = 2e^{2x}\,dx \\
&= \frac{1}{2}\left(13\arcsin\left(\frac{u-6}{4}\right)\right) + C && \textit{from part (a)} \\
&= \frac{13}{2}\arcsin\left(\frac{e^{2x}-6}{4}\right) + C.
\end{aligned}$$