Section 10.2: Properties of Power Series

Warm up:

Problem 1 *Suppose that* $\sum_{k=0}^{\infty} c_k(x+5)^k$ *converges when* $x = -9$ *and diverges when* $x = -1$. *What can be said about the convergence and divergence of the following series?*

(a) $\sum_{k=0}^{\infty} c_k$ (b) $\sum_{k=0}^{\infty} c_k(-5)^k$ (c) $\sum_{k=0}^{\infty} c_k(5)^k$

Solution: *What is important to note first is that the interval of convergence for the series* $\sum_{k=0}^{\infty} c_k(x+5)^k$ *is* $[-9, -1)$.

(a) *Notice that*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} c_k(-4+5)^k.$$

Then since -4 *is in the interval* $[-9, -1)$, *the series* $\sum_{k=0}^{\infty} c_k$ *converges.*

(b) *Since* -10 *is not in the interval* $[-9, -1)$, *the series*

$$\sum_{k=0}^{\infty} c_k(-5)^k = \sum_{k=0}^{\infty} c_k(-10+5)^k$$

diverges.

(c) *Since* 0 *is not in the interval* $[-9, -1)$, *the series*

$$\sum_{k=0}^{\infty} c_k 5^k = \sum_{k=0}^{\infty} c_k(0+5)^k$$

diverges.

Learning outcomes:

Group work:

Problem 2 *If the series* $\sum_{k=0}^{\infty} a_k(x-2)^k$ *has an interval of convergence of* $[-4, 8)$*, determine the interval of convergence of the following series:*

(a) $\sum_{k=300}^{\infty} a_k(x-2)^k$ (b) $\sum_{k=0}^{\infty} a_k x^k$ (c) $\sum_{k=0}^{\infty} \left(a_k(x-2)^k + \left(\frac{1}{7}\right)^k x^k \right)$

Solution: (a) *This series has exactly the same interval of convergence,* $\boxed{[-4, 8)}$*, since a finite number of terms do not change whether or not a series converges.*

(b) *In the original interval of convergence* $[-4, 8)$*, the center is* $x = 2$ *and the radius of convergence is* 6 *(which includes the left endpoint, but not the right). So now, the center is* $x = 0$*. Taking an interval about* 0 *with radius* 6*, and adding in the left endpoint, gives the IOC of* $\boxed{[-6, 6)}$.

(c) *The interval of convergence of this series is the intersection of the interval of convergence for* $\sum_{k=0}^{\infty} a_k(x-2)^k$ *and* $\sum_{k=0}^{\infty} \left(\frac{1}{7}\right)^k x^k$*. To find the IOC for this second series, we use the Root Test.*

$$\begin{aligned} \lim_{k\to\infty} \sqrt[k]{\left| \left(\frac{1}{7}\right)^k x^k \right|} &= \lim_{k\to\infty} \frac{1}{7}|x| \\ &= \frac{1}{7}|x|. \end{aligned}$$

So the series $\sum_{k=0}^{\infty} \left(\frac{1}{7}\right)^k x^k$ *converges when*

$$\begin{aligned} & \frac{1}{7}|x| < 1 \\ \implies \quad & |x| < 7 \\ \implies \quad & -7 < x < 7. \end{aligned}$$

We need to test the endpoints $x = \pm 7$*, but using the divergence test we see that both*

$$\sum_{k=0}^{\infty} (-1)^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} 1^k$$

diverge. Therefore, the interval of convergence for $\sum_{k=0}^{\infty} \left(\frac{1}{7}\right)^k x^k$ *is* $(-7, 7)$.

Finally, the interval of convergence for the series

$$\sum_{k=0}^{\infty}\left(a_k(x-2)^k+\left(\frac{1}{7}\right)^k x^k\right)$$

is

$$[-4,8)\cap(-7,7)=\boxed{[-4,7)}.$$

Problem 3 *For each of the following, find the domain of $f(x)$ (i.e. find the interval of convergence).*

(a) $f(x)=\displaystyle\sum_{k=1}^{\infty}\frac{(3x-2)^k}{k\cdot 3^k}$

(b) $f(x)=\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{k^2+1}}x^k$

(c) $f(x)=\displaystyle\sum_{k=2}^{\infty}\frac{x^{3k+2}}{(\ln k)^k}$

Solution: (a) *We use the Ratio Test*

$$\lim_{k\to\infty}\left|\frac{(3x-2)^{k+1}}{(k+1)3^{k+1}}\cdot\frac{k\cdot 3^k}{(3x-2)^k}\right|=\lim_{k\to\infty}\left|\frac{k(3x-2)}{3(k+1)}\right|$$
$$=\left|\frac{3x-2}{3}\right|=\frac{|3x-2|}{3}.$$

We know that this series converges when

$$\begin{aligned}&\frac{|3x-2|}{3}<1\\ \Longrightarrow\quad &|3x-2|<3\\ \Longrightarrow\quad &-3<3x-2<3\\ \Longrightarrow\quad &-1<3x<5\\ \Longrightarrow\quad &-\frac{1}{3}<x<\frac{5}{3}.\end{aligned}$$

We still need to check the endpoints. When $x=-\frac{1}{3}$ we have

$$\sum_{k=1}^{\infty}\frac{1}{k3^k}\left(3\left(-\frac{1}{3}\right)-2\right)^k=\sum_{k=1}^{\infty}\frac{(-3)^k}{k3^k}$$
$$=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

which converges (conditionally) by the alternating series test.

When $x = \dfrac{5}{3}$ *we have*

$$\sum_{k=1}^{\infty} \frac{1}{k3^k}\left(3\left(\frac{5}{3}\right)-2\right)^k = \sum_{k=1}^{\infty} \frac{3^k}{k3^k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges since it is the Harmonic series.

Therefore, the interval of convergence is $\boxed{\left[-\frac{1}{3}, \frac{5}{3}\right)}$.

(b) *We again apply the Ratio Test*

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{k+1}}{\sqrt{(k+1)^2+1}}\cdot\frac{\sqrt{k^2+1}}{(-1)^k x^k}\right| = \lim_{k\to\infty}\left|\frac{x\sqrt{k^2+1}}{\sqrt{k^2+2k+2}}\right| = |x|.$$

So we know that this series converges when

$$|x| < 1 \qquad \Longleftrightarrow \qquad -1 < x < 1.$$

We still need to check the endpoints. When $x = -1$*, the series*

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2+1}}$$

diverges by the Limit Comparison Test (compare with $\sum_{k=1}^{\infty} \frac{1}{k}$*).*

When $x = 1$*, the series*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}}$$

converges conditionally by the alternating series test.

Therefore, the interval of convergence is $\boxed{(-1, 1]}$.

(c) *For this series we apply the Root Test*

$$\lim_{k\to\infty}\sqrt[k]{\left|\frac{x^{3k+2}}{(\ln k)^k}\right|} = \lim_{k\to\infty}\frac{|x|^3\cdot|x|^{\frac{2}{k}}}{\ln k} = |x|^3\cdot\lim_{k\to\infty}\frac{|x|^{\frac{2}{k}}}{\ln k} = |x|^3\cdot 0 = 0.$$

Therefore, the interval of convergence is $\boxed{(-\infty, \infty)}$.

Problem 4 *In each of the following, give a power series (with an interval of convergence) for the given function. Assume that we know* $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ *on* $(-1, 1)$.

(a) $f(x) = \dfrac{3}{5x-2}$ (b) $f(x) = \dfrac{3x^4}{5x^3-2}$

Solution: (a)

$$\begin{aligned} f(x) &= \frac{3}{5x-2} \\ &= \frac{3}{-2\left(1-\left(\frac{5}{2}x\right)\right)} \\ &= -\frac{3}{2}\sum_{k=0}^{\infty}\left(\frac{5}{2}x\right)^k \\ &= \boxed{\sum_{k=0}^{\infty}\left(-\frac{3}{2}\right)\left(\frac{5}{2}x\right)^k}. \end{aligned}$$

Since this is a geometric series, it converges if and only if

$$\begin{aligned} & \left|\frac{5}{2}x\right| < 1 \\ \iff \quad & |x| < \frac{2}{5} \\ \iff \quad & -\frac{2}{5} < x < \frac{2}{5}. \end{aligned}$$

Therefore, the interval of convergence is $\boxed{\left(-\frac{2}{5}, \frac{2}{5}\right)}$

(b)

$$\begin{aligned} f(x) &= \frac{3x^4}{5x^3-2} \\ &= \frac{-3x^4}{2}\cdot\frac{1}{1-\left(\frac{5}{2}x^3\right)} \\ &= -\frac{3}{2}x^4\sum_{k=0}^{\infty}\left(\frac{5}{2}x^3\right)^k \\ &= \boxed{\sum_{k=0}^{\infty}\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right)^k x^{3x+4}}. \end{aligned}$$

This geometric series converges if and only if

$$
\begin{aligned}
& \left|\frac{5}{2}x^3\right| < 1 \\
\iff \quad & |x^3| < \frac{2}{5} \\
\iff \quad & |x| < \sqrt[3]{\frac{2}{5}} \\
\iff \quad & -\sqrt[3]{\frac{2}{5}} < x < \sqrt[3]{\frac{2}{5}}.
\end{aligned}
$$

Therefore, the interval of convergence is $\boxed{\left(-\sqrt[3]{\frac{2}{5}}, \sqrt[3]{\frac{2}{5}}\right)}$.

Problem 5 *Consider* $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{2^k x^k}{(k+1)^3}$.

(a) *Write out* $p_3(x)$*, the cubic polynomial which is the first three terms of this power series.*

(b) *Find* $p_3'(x)$ *and* $f'(x)$ *and compare your answers.*

(c) *Find* $\displaystyle\int p_3(x)\,dx$ *and* $\displaystyle\int f(x)\,dx$ *and compare your answers.*

Solution: (a)

$$
\begin{aligned}
p_3(x) &= \frac{2^0x^0}{(0+1)^3} + \frac{2^1x^1}{(1+1)^3} + \frac{2^2x^2}{(2+1)^3} + \frac{2^3x^3}{(3+1)^3} \\
&= \boxed{1 + \frac{2x}{2^3} + \frac{4x^2}{3^3} + \frac{8x^3}{4^3}}.
\end{aligned}
$$

(b)

$$
p_3'(x) = \boxed{0 + \frac{2}{2^3} + \frac{2\cdot 4}{3^3}x + \frac{3\cdot 8}{4^3}x^2}.
$$

$$
\begin{aligned}
f'(x) &= \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^3}\cdot k\cdot x^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{k\cdot 2^k}{(k+1)^3}x^{k-1} && \text{Since the k=0 term is 0.} \\
&= \boxed{\sum_{\ell=0}^{\infty} \frac{(\ell+1)2^{\ell+1}}{(\ell+2)^3}x^{\ell}} && \text{reindexing with } \ell = k-1
\end{aligned}
$$

Note that this sum agrees with $p_3'(x)$ when $\ell = 0$, 1, and 2.

(c)

$$\int p_3(x)\,dx = \boxed{x + \frac{2x^2}{2^4} + \frac{4x^3}{3^4} + \frac{8x^4}{4^4} + C}.$$

$$\begin{aligned}\int f(x)\,dx &= \int \sum_{k=0}^{\infty} \frac{2^k x^k}{(k+1)^3}\,dx \\ &= \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^3} \cdot \left(\int x^k\,dx\right) \\ &= \sum_{k=0}^{\infty} \left(\frac{2^k}{(k+1)^3} \cdot \frac{1}{k+1} \cdot x^{k+1}\right) + C \\ &= \sum_{\ell=1}^{\infty} \left(\frac{2^{\ell-1}}{\ell^4} x^\ell\right) + C \quad \textit{reindexing with } \ell = k+1.\end{aligned}$$

Note again that this sum agrees with $\int p_3(x)\,dx$ when $\ell = 0$, 1, 2, and 3 (as it should).

Problem 6 *Give a power series (with interval of convergence) for the given functions.*

(a) $f(x) = \dfrac{1}{1+x^2}$ (b) $f(x) = \tan^{-1}(x)$ (c) $f(x) = \tan^{-1}(3x^2)$

Solution: (a)

$$\begin{aligned}f(x) &= \frac{1}{1+x^2} \\ &= \sum_{k=0}^{\infty} (-x^2)^k \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}}\end{aligned}$$

when

$$|-x^2| < 1 \qquad \Longleftrightarrow \qquad -1 < x < 1.$$

So the interval of convergence is $\boxed{(-1,1)}$.

(b) *Notice that*

$$f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

on the interval $(-1,1)$. *So*

$$\begin{aligned} f(x) &= \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k} \right) dx \\ &= \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} x^{2k+1} \right) + C. \end{aligned}$$

We know that this series converges (at least) on $(-1,1)$. *We need to first find* C*, and then check the endpoints for convergence.*

To find C*, just notice that*

$$0 = \arctan(0) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} 0^{2k+1} \right) + C = 0 + C.$$

and so $C = 0$.

Now to check the endpoints, we plug in -1 *and* 1. *When* $x = -1$*, we have the series*

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} \qquad x = -1$$

‘ *and when* $x = 1$ *the series is*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}. \qquad x = 1$$

Both of these series converge conditionally by the Alternating Series Test. Thus,

$$f(x) = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}}$$

with interval of convergence $\boxed{[-1,1]}$.

(c) *By part (b), if* $|3x^2| \leq 1$ *then*

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (3x^2)^{2k+1} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{2k+1} x^{4k+2}}.$$

This series converges when

$$
\begin{aligned}
& |3x^2| \le 1 \\
\implies \quad & |x^2| \le \frac{1}{3} \\
\implies \quad & -\frac{1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}.
\end{aligned}
$$

So the interval of convergence is

$$\boxed{\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]}.$$