Recitation # 24: Calculus in polar coordinates - Solutions

Warm up:

(a) True or False: The slope of the tangent line to the curve $r = f(\theta)$ at the point (r_0, θ_0) is given by $f'(\theta_0)$.

(b) True or False: The area enclosed by the curve $r = 2\cos(\theta)$ is

$$\int_0^{2\pi} \frac{1}{2}(2\cos(\theta))^2 d\theta = \int_0^{2\pi} 1 - \cos(2\theta)d\theta = 2\pi.$$

Solution: (a) **False.** The slope of the tangent line to the curve $r = f(\theta)$ at (r_0, θ_0) is given by

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0} \neq f'(\theta_0).$$

(b) **False.** The curve $r = \cos(\theta)$ is a circle of radius 1 centered at (1,0). The curve traces out the circle twice for θ in $[0, 2\pi]$. So the area enclosed is just $\int_0^\pi \frac{1}{2}(2\cos(\theta))^2 d\theta = \pi$.

Group work:

Problem 1 *Find the equation of the tangent line to $r = 2 - \sin\theta$ at $\theta = \dfrac{\pi}{3}$. Also, determine for what values of θ the tangent lines to the curve are vertical or horizontal.*

Solution: *We use the formula from the warm-up to find $\dfrac{dy}{dx}$ when $f(\theta) = 2 - \sin\theta$:*

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{(-\cos\theta)(\sin\theta) + (2 - \sin\theta)(\cos\theta)}{(-\cos\theta)(\cos\theta) - (2 - \sin\theta)(\sin\theta)}.$$

Learning outcomes:

So

$$\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = \frac{(-\cos\frac{\pi}{3})(\sin\frac{\pi}{3}) + (2-\sin\frac{\pi}{3})(\cos\frac{\pi}{3})}{(-\cos\frac{\pi}{3})(\cos\frac{\pi}{3}) - (2-\sin\frac{\pi}{3})(\sin\frac{\pi}{3})}$$

$$= \frac{\left(-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) + \left(2-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\cdot\frac{1}{2}\right) - \left(2-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{-\sqrt{3}+4-\sqrt{3}}{-1-4\sqrt{3}+3}$$

$$= \frac{4-2\sqrt{3}}{2-4\sqrt{3}}$$

$$= \frac{2-\sqrt{3}}{1-2\sqrt{3}}.$$

Also, when $\theta = \frac{\pi}{3}$, $r = 2 - \frac{\sqrt{3}}{2} = \frac{4-\sqrt{3}}{2}$. *Therefore*

$$x = r\cos\theta = \frac{4-\sqrt{3}}{2}\cdot\frac{1}{2} = 1 - \frac{\sqrt{3}}{4}$$

$$y = r\sin\theta = \frac{4-\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{3}{4}.$$

Thus, the equation of the tangent line when $\theta = \frac{\pi}{3}$ *is*

$$\boxed{y - \left(\sqrt{3}-\frac{3}{4}\right) = \frac{2-\sqrt{3}}{1-2\sqrt{3}}\left(x - \left(1-\frac{\sqrt{3}}{4}\right)\right)}$$

To find all vertical and horizontal tangent lines, we need to find where the numerator and denominator of $\frac{dy}{dx}$ *are equal to* 0.

Numerator:

$$(-\cos\theta)(\sin\theta) + (2-\sin\theta)(\cos\theta) = 0$$

$$-\sin\theta\cos\theta + 2\cos\theta - \sin\theta\cos\theta = 0$$

$$2\cos\theta - 2\sin\theta\cos\theta = 0$$

$$2\cos\theta(1-\sin\theta) = 0$$

$$\cos\theta = 0 \quad\text{or}\quad \sin\theta = 1$$

$$\theta = \frac{\pi}{2} + k\pi \quad\text{or}\quad \theta = \frac{\pi}{2} + 2k\pi \quad\text{for } k \text{ an integer}$$

$$\theta = \frac{\pi}{2} + k\pi \quad\text{for } k \text{ an integer}$$

Denominator:

$$(-\cos\theta)(\cos\theta) - (2 - \sin\theta)(\sin\theta) = 0$$
$$-\cos^2\theta - 2\sin\theta + \sin^2\theta = 0$$
$$-1 + \sin^2\theta - 2\sin\theta + \sin^2\theta = 0$$
$$2\sin^2\theta - 2\sin\theta - 1 = 0$$

$$\sin\theta = \frac{2 \pm \sqrt{4 + 8}}{4} \quad \textcolor{red}{\textit{throw away the "+"}}$$

$$\sin\theta = \frac{1 - \sqrt{3}}{2} \quad \textcolor{red}{\textit{since it is out of the range of } \sin}$$

$$\theta = \sin^{-1}\left(\frac{1 - \sqrt{3}}{2}\right) + 2k\pi \quad \textit{for k an integer}$$

Note that $\sin\theta = \dfrac{1 - \sqrt{3}}{2}$ *twice during a period. The other one occurs at*
$\pi - \sin^{-1}\left(\dfrac{1 - \sqrt{3}}{2}\right).$

Then, since these two collections of angles are disjoint, the horizontal tangent lines occur when

$$\boxed{\theta = \frac{\pi}{2} + k\pi} \quad \textit{for k an integer}$$

and the vertical tangent lines occur when

$$\boxed{\theta = \sin^{-1}\left(\frac{1 - \sqrt{3}}{2}\right) + 2k\pi, \theta = \pi - \sin^{-1}\left(\frac{1 - \sqrt{3}}{2}\right) + 2k\pi} \quad \textit{for k an integer}$$

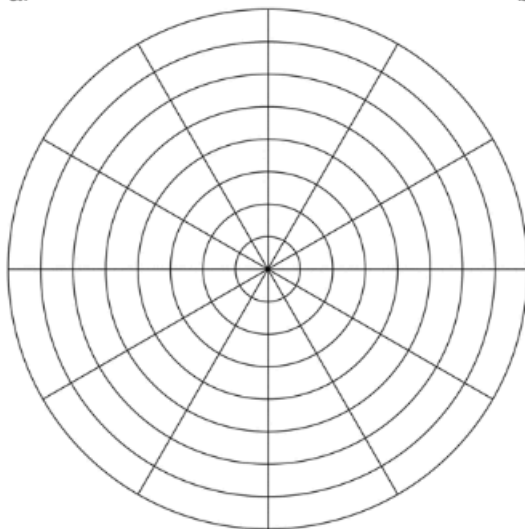**Problem 2** *Graph each region and then SET UP an integral for the area of the region:*

(a) *Outside the small loop and inside the large loop of $r = 3 - 6\sin\theta$.*

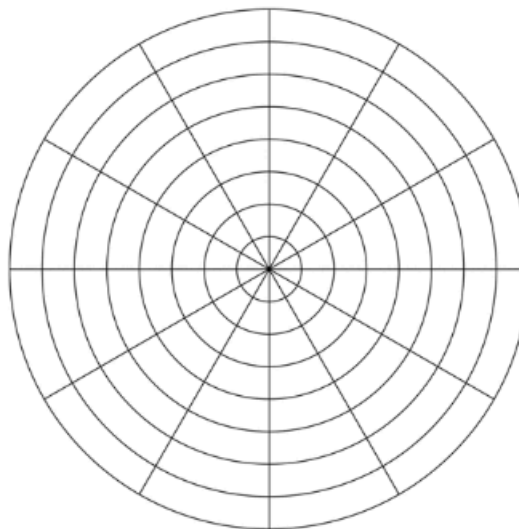(b) *Inside both of the curves $r = 4\cos\theta$ and $r = 1 - \cos\theta$.*

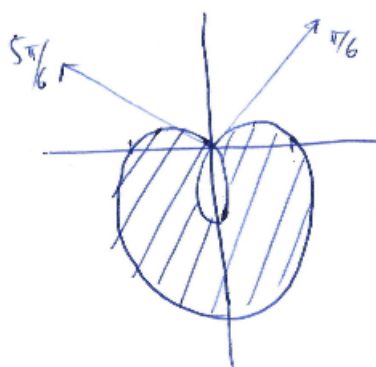*Note that you do not need to evaluate these integrals.*

a.

b.

Solution: (a) *The graph of the region is given below.*

To find the area inside the smaller loop, we need to first find the values of
θ *for which* $r = 0$. *So we compute*

$$3 - 6\sin\theta = 0$$
$$\implies \quad \sin\theta = \frac{1}{2}$$
$$\implies \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Hence, the area of the smaller loop is

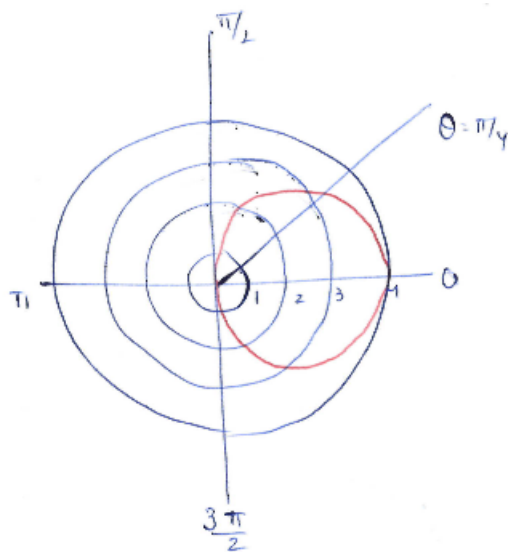$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 6\sin\theta)^2 \, d\theta.$$

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To find the area of the outer loop, we just integrate over the other values of θ:

$$\frac{1}{2} \int_0^{\frac{\pi}{6}} (3 - 6 \sin \theta)^2 \, d\theta + \frac{1}{2} \int_{\frac{5\pi}{6}}^{2\pi} (3 - 6 \sin \theta)^2 \, d\theta.$$

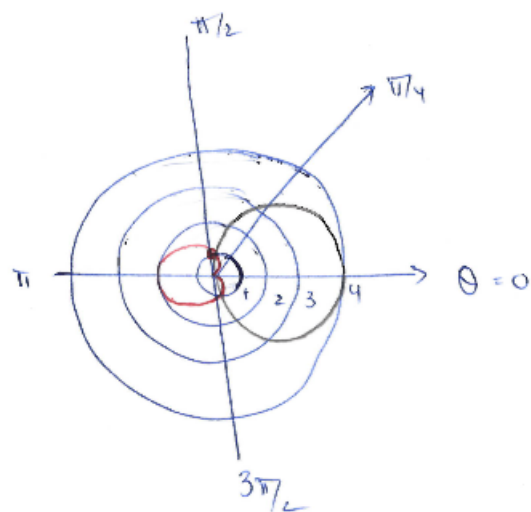*Therefore, the area of the region inside the outer loop and outside the inner loop is*

$$\frac{1}{2} \left[\int_0^{\frac{\pi}{6}} (3 - 6 \sin \theta)^2 \, d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 6 \sin \theta)^2 \, d\theta + \int_{\frac{5\pi}{6}}^{2\pi} (3 - 6 \sin \theta)^2 \, d\theta \right]$$
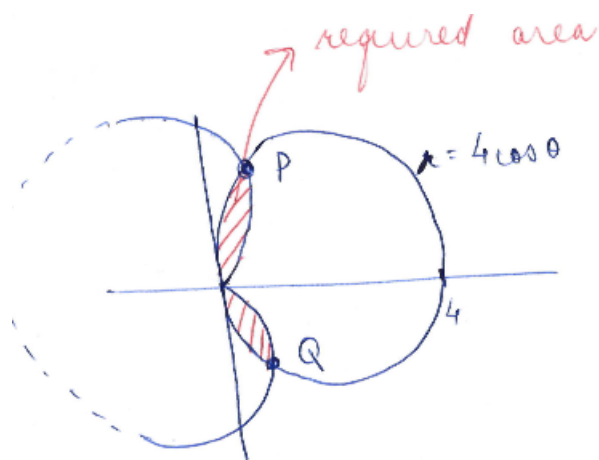
(b) *We first graph $r = 4 \cos \theta$.*

Now we graph $r = 1 - \cos \theta$

Here are the two graphs in the same picture

To find the points where the two curves intersect, we solve

$$4\cos\theta = 1 - \cos\theta$$
$$5\cos\theta = 1$$
$$\cos\theta = \frac{1}{5}$$
$$\theta = \pm\cos^{-1}\left(\frac{1}{5}\right) := \theta_0.$$

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Using the symmetry of the graph we find the area of one leaf and then double it. Therefore, the area between the curves is

$$2\left[\frac{1}{2}\int_0^{\theta_0}(1-\cos\theta)^2\,d\theta+\frac{1}{2}\int_{\theta_0}^{\frac{\pi}{2}}(4\cos\theta)^2\,d\theta\right]$$

In the following picture, the left-hand integral is (1) and the right-hand integral is (2).