Recitation #13: Direction fields and Separable Differential Equations - Full

Warm up:

Which of the following differential equations are separable?

(a) $y' = \dfrac{ty}{t^2+1}$,

(b) $\dfrac{dy}{dx} = x^2 \sin(3y) - x^2$,

(c) $y' = t^2 - y$.

Solution: (a) Yes, it is separable. $y' = y \cdot \dfrac{t}{t^2+1}$.

(b) Yes, it is separable. $\dfrac{dy}{dx} = x^2(\sin(3y) - 1)$.

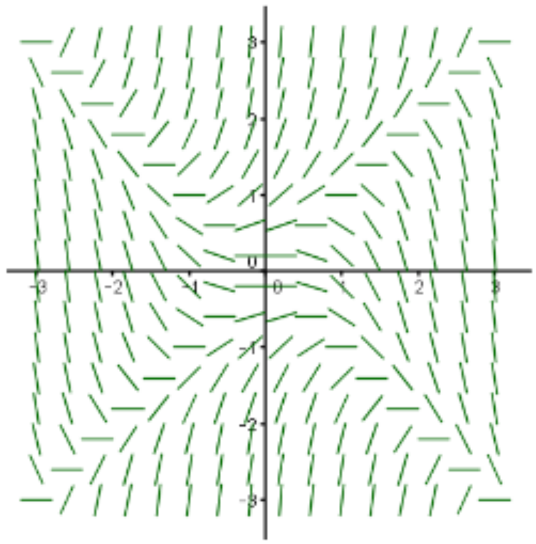
(c) No, it is not separable. $t^2 - y$ can not be written in the form $F(t) \cdot G(y)$.

Instructor Notes:

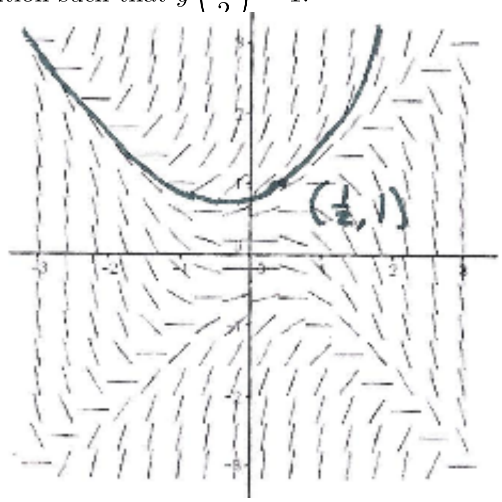
Group work:

Problem 1 (a) *The following is a direction field for the differential equation* $\dfrac{dy}{dx} = y^2 - x^2$.

Learning outcomes:



Sketch the solution such that $y\left(\frac{1}{2}\right) = 1$.



Solution:

(b) *Use Euler's Method to give a numerical estimate to the solution of the differential equation* $y' = y^2 - t^2$ *at* $y(2)$ *that goes through the point* $\left(\frac{1}{2}, 1\right)$. *Use* $\Delta t = 0.5$.

Solution:

T	y	$\frac{dy}{dt} = y^2 - t^2$	$y + \frac{dy}{dt} \cdot \Delta t$
0.5	1	0.75	1.375
1	1.375	0.890625	1.820313
1.5	1.820313	1.063538	2.352081
2	2.352081		

So, $y(2) \approx 2.352081$.

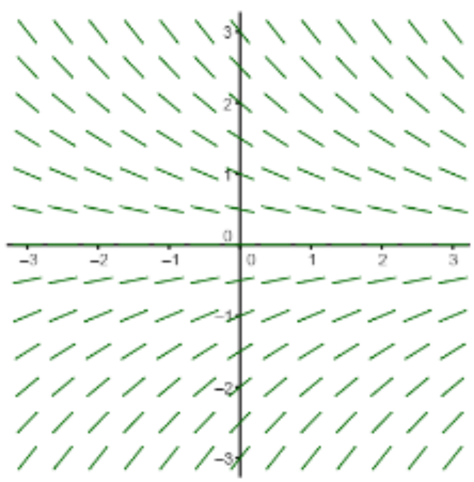
Instructor Notes: The major point here is that (a) and (b) are the same problem, presented with two different representations.

Problem 2 *Describe why the following direction field could be the direction field for the differential equation*

$$\frac{dy}{dt} = y\cos(t)$$

but ***not*** *for*

$$\frac{dy}{dt} = y\sin(t) \qquad \text{or} \qquad \frac{dy}{dt} = t\cos(y).$$



Solution: *Look along the line* $t = 0$ *(the* y*-axis).*

- *For* $\frac{dy}{dt} = y\sin(t)$*:*

  $\left[\frac{dy}{dt}\right]_{t=0} = y\sin(0) = 0$. *But this direction field does not have horizontal tangents at each point along* $t = 0$. *So it* ***cannot*** *be the direction field for* $\frac{dy}{dt} = y\sin(t)$.

- *For* $\frac{dy}{dt} = t\cos(y)$*:*
  $\left[\frac{dy}{dt}\right]_{t=0} = (0)\cos(y) = 0$. *But again this direction field does not have horizontal tangents at each point along* $t = 0$. *So it* ***cannot*** *be the direction field for* $\frac{dy}{dt} = t\cos(y)$.

- *For* $\frac{dy}{dt} y\cos(t)$*:*

  Along $t = 0$, *this equation is* $\frac{dy}{dt} = y\cos(0) = y$. *However, when* y *is positive, the slopes are negative. Also, when* y *is negative, the slopes are positive. So this is* ***not*** *the direction field for* $\frac{dy}{dt} = y\cos(t)$, *either.*

Instructor Notes: Students should examine when t varies across quadrants, combined with the sign of y at points (t, y) (with $y' = t\cos y$, y is going through "quadrants"). Checking where $y' = 0$ can reveal why the other two differential equations are not satisfied. This could all just be a whole class discussion with the instructor bringing up strategies.

Problem 3 *Match each of the following differential equations with a corresponding direction field (if it is present):*

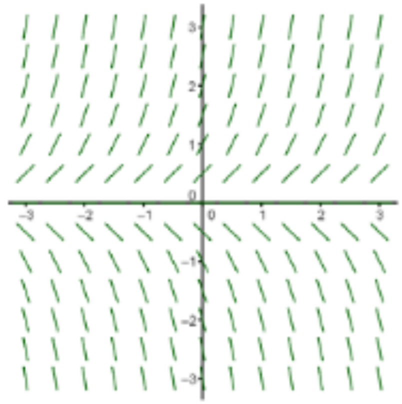
i. $y' = \frac{t}{2+y}$

ii. $y' = \cos(t+y)$

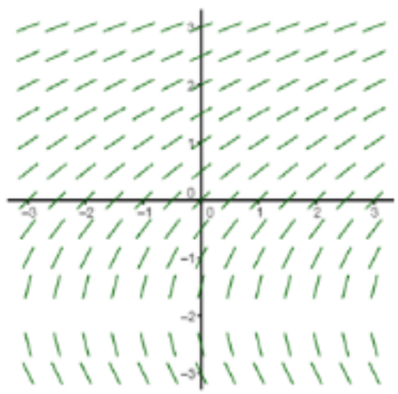
iii. $y' = 1 + y^2$

iv. $y' = ty$

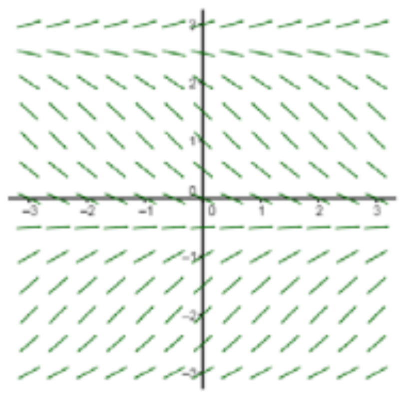
b.



b.



c.



d.

Solution: *Look along the line $t = 0$ (the y-axis).*

(i) $y' = \dfrac{t}{2+y}$ *and (iv)* $y' = ty$ *must both be identically* 0 *along the y-axis. However, none of the direction fields given have horizontal slopes along the y-axis. So none of them can be the direction field for (i) or (iv).*

(ii) At the origin, we have $y' = \cos(t+y) = \cos(0) = 1$*. So the direction field for (ii) must have slope 1 at the origin. This eliminates (b) and (d). Now look at the point* $\left(0, \dfrac{\pi}{2}\right)$*. We have* $\Big[y'\Big]_{(0,\frac{\pi}{2})} = \cos\left(\dfrac{\pi}{2}\right) = 0$*, so the direction field must be horizontal at that point. That means that it cannot be (a) or (c) either.*

(iii) Here $y' = 1 + y^2$*, so there is no t on the right hand side of the equation. Therefore, y' depends only on y. At $y = 0$ the slope is 1, then as y increases the slopes increase too. Similarly, as y gets more and more negative, the slope gets more and more positive. So it seems as if this direction field is (a).*

Instructor Notes: Several strategies exist. Make sure to ask what special quality direction fields of autonomous differential equations have. Depending on time, this also could all be done as a whole class.

Problem 4 *Which of the following are separable differential equations? For those that are, solve them, assuming that $y(4) = 5$.*

(a) $y' = x^2 + y^2$

Solution: *This differential equation is* ***not*** *separable.*

(b) $y' = x + xy^2$

Solution:

$$\begin{aligned} & y' = x + xy^2 \\ \implies \quad & \frac{dy}{dx} = x(1 + y^2) \\ \implies \quad & \frac{dy}{1 + y^2} = x\,dx. \end{aligned}$$

So this equation ***is*** *separable. To solve, we integrate both sides of the equation:*

$$\begin{aligned} & \int \frac{1}{1+y^2}\,dy = \int x\,dx \\ \implies \quad & \arctan(y) = \frac{1}{2}x^2 + C \qquad (1) \\ \implies \quad & y = \tan\left(\frac{1}{2}x^2 + C\right). \end{aligned}$$

To find C, we plug the initial condition $y(4) = 5$ into equation (1) and solve for C.

$$\begin{aligned} & \arctan(5) = \frac{1}{2}(4)^2 + C = 8 + C \\ \implies \quad & C = \arctan(5) - 8. \end{aligned}$$

So

$$y = \tan\left(\frac{1}{2}x^2 + \arctan(5) - 8\right).$$

(c) $y' = e^{2x-y}$

Solution:

$$\begin{aligned}
& y' = e^{2x-y} \\
\implies \quad & \frac{dy}{dx} = \frac{e^{2x}}{e^y} \\
\implies \quad & e^y\,dy = e^{2x}\,dx \qquad (2)
\end{aligned}$$

and so this **is** *a separable equation. To solve, we integrate both sides of equation (2).*

$$\begin{aligned}
& \int e^y\,dy = \int e^{2x}\,dx \\
\implies \quad & e^y = \frac{1}{2}e^{2x} + C \qquad (3) \\
\implies \quad & y = \ln\left(\frac{1}{2}e^{2x} + C\right).
\end{aligned}$$

To find C, we plug into equation (3) and solve for C:

$$\begin{aligned}
& e^5 = \frac{1}{2}e^8 + C \\
\implies \quad & C = e^5 - \frac{1}{2}e^8.
\end{aligned}$$

Therefore

$$y = \ln\left(\frac{1}{2}e^{2x} + e^5 - \frac{1}{2}e^8\right).$$

Instructor Notes: Part (b) is the only non-separable equation. Students may need help recognizing that they can divide by the entire right side of both sides. Also, some results may only define y implicitly.