Recitation # 11: Improper Integrals and Differential Equations

Warm up:

True or False: It is possible for a region to be infinitely long but have a finite area.

Solution: True. Consider the region below the curve $y = \dfrac{1}{x^2}$, $x \geq 1$.

Group work:

Problem 1 *Review of limits:*

(a) $\displaystyle\lim_{x\to-\infty}\left(3x^{-6} + e^{5x} + \frac{\sin x}{x^2+3}\right)$

Solution: *Recall that the limit of a sum is the sum of the limits, provided that those limits exist.*

- $\displaystyle\lim_{x\to-\infty} 3x^{-6} = \lim_{x\to-\infty}\frac{3}{x^6} = 0.$
- $\displaystyle\lim_{x\to-\infty} e^{5x} = 0.$
- $\displaystyle\lim_{x\to-\infty}\frac{\sin x}{x^2+3} = 0$

  To rigorously prove this, you need to use the squeeze theorem.

Thus,

$$\lim_{x\to-\infty}\left(3x^{-6} + e^{5x} + \frac{\sin x}{x^2+3}\right) = 0$$

(b) $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+4}}$

Learning outcomes:

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+4}} &= \lim_{x\to\infty}\frac{x}{\sqrt{x^2}\cdot\sqrt{9+\frac{4}{x^2}}}\\ &= \lim_{x\to\infty}\frac{x}{|x|\cdot\sqrt{9+\frac{4}{x^2}}}\\ &= \lim_{x\to\infty}\frac{x}{x\cdot\sqrt{9+\frac{4}{x^2}}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{4}{x^2}}}\\ &= \frac{1}{\sqrt{9+0}} = \frac{1}{3}.\end{aligned}$$

(c) $\lim\limits_{x\to-\infty}\arctan x$

Solution: $\lim\limits_{x\to-\infty}\arctan x = -\dfrac{\pi}{2}.$

Problem 2 *Determine if the given integral converges or diverges. If it converges, find the value.*

$$\int_{-1}^{\infty}\frac{3}{2x+1}\,dx$$

Solution: *The function* $\dfrac{3}{2x+1}$ *has a vertical asymptote at* $x=-\dfrac{1}{2}$*. So we rewrite the original integral as*

$$\int_{-1}^{\infty}\frac{3}{2x+1}\,dx = \lim_{a\to-\frac{1}{2}^-}\int_{-1}^{a}\frac{3}{2x+1}\,dx + \lim_{b\to-\frac{1}{2}^+}\int_{b}^{0}\frac{3}{2x+1}\,dx + \lim_{c\to\infty}\int_{0}^{c}\frac{3}{2x+1}\,dx.$$

The latter integral does not exist. To see this, just note that

$$\begin{aligned}\lim_{c\to\infty}\int_0^c\frac{3}{2x+1}\,dx &= \lim_{c\to\infty}\left[\frac{3}{2}\ln|2x+1|\right]_0^c\\ &= \lim_{c\to\infty}\frac{3}{2}\ln|2c+1| = \infty.\end{aligned}$$

Therefore,

$$\int_{-1}^{\infty}\frac{3}{2x+1}\,dx \text{ diverges.}$$

Problem 3 (a) *Show that*

$$\frac{9}{2x^2+3x} = \frac{3}{x} - \frac{6}{2x+3}$$

(b) *Determine if the integral*

$$\int_1^\infty \frac{9}{2x^2+3x}\,dx$$

converges or diverges. If it converges, give the value that it converges to.

Solution: (a) *Since* $2x^2+3x = x(2x+3)$, *we apply the method of partial fractions:*

$$\begin{aligned} & \frac{9}{2x^2+3x} = \frac{A}{x} + \frac{B}{2x+3} \\ \Longrightarrow \quad & 9 = A(2x+3) + Bx. \end{aligned}$$

Letting $x=0$ *gives that*

$$9 = 3A \quad \Longrightarrow \quad A = 3.$$

Then letting $x = -\frac{3}{2}$, *we see that*

$$9 = -\frac{3}{2}B \quad \Longrightarrow \quad B = -9\cdot\frac{2}{3} = -6.$$

Therefore, plugging in our values for A *and* B *gives us*

$$\frac{9}{2x^2+3x} = \frac{3}{x} - \frac{6}{2x+3}.$$

(b) *We have that*

$$\begin{aligned} \int_1^\infty \frac{9}{2x^2+3x}\,dx &= \lim_{t\to\infty}\int_1^t \left(\frac{3}{x} - \frac{6}{2x+3}\right) dx \\ &= \lim_{t\to\infty}\left[3\ln|x| - \frac{6}{2}\ln|2x+3|\right]_1^t \quad \textit{don't forget to divide the 6 by 2} \\ &= \lim_{t\to\infty}\left(3\ln|t| - 3\ln|2t+3| - 0 + 3\ln(5)\right) \quad \ln(1)=0 \\ &= \lim_{t\to\infty}\left(3\ln\left|\frac{5t}{2t+3}\right|\right) \quad \textit{properties of logarithms} \\ &= \boxed{3\ln\left(\frac{5}{2}\right)} \quad \textit{since } \ln(x) \textit{ is a continuous function} \end{aligned}$$

Problem 4 (a) *Show that*

$$\frac{6x-8}{x^3+4x} = \frac{2x+6}{x^2+4} - \frac{2}{x}$$

(b) *Determine if the integral*

$$\int_3^\infty \frac{6x-8}{x^3+4x}\,dx$$

converges or diverges. If it converges, give the value that it converges to.

Solution: *(a) Since $x^3+4x = x(x^2+4)$, we use partial fractions:*

$$\begin{aligned} \frac{6x-8}{x^3+4x} &= \frac{Ax+B}{x^2+4} + \frac{C}{x} \\ \implies \quad 6x-8 &= (Ax+B)(x) + C(x^2+4). \end{aligned}$$

Letting $x=0$, we see that

$$-8 = 4C \quad \implies \quad C = -2.$$

To find A and B, let us plug in $C=-2$ and simplify:

$$\begin{aligned} 6x-8 &= Ax^2 + Bx - 2x^2 - 8 \\ &= (A-2)x^2 + Bx - 8. \end{aligned}$$

Aligning the respective coefficients, we see that

$$\begin{aligned} & A-2 = 0 \quad \text{and} \quad B = 6 \\ \implies \quad & A = 2 \quad \textit{and} \quad B = 6. \end{aligned}$$

Finally, plugging this into the original equation yields

$$\frac{6x-8}{x^3+4x} = \frac{2x+6}{x^2+4} - \frac{2}{x}.$$

(b) We have that

$$\begin{aligned} \int_3^\infty \frac{6x-8}{x^3+4x}\,dx &= \lim_{t\to\infty} \int_3^t \left(\frac{2x+6}{x^2+4} - \frac{2}{x}\right) dx \\ &= \lim_{t\to\infty} \left(\int_3^t \frac{2x}{x^2+4}\,dx + \int_3^t \frac{6}{x^2+4}\,dx - \int_3^t \frac{2}{x}\,dx\right). \end{aligned}$$

Let us evaluate each integral separately, combine them, and then take the limit.

(i)

$$\int_3^t \frac{2x}{x^2+4}\,dx = \int_{13}^{t^2+4} \frac{1}{w}\,dw \quad w = x^2+4,\ dw = 2x\,dx$$
$$= \ln(t^2+4) - \ln(13).$$

(ii)

$$\int_3^t \frac{6}{x^2+4}\,dx = \left[\frac{6}{2}\arctan\left(\frac{x}{2}\right)\right]_3^t$$
$$= 3\arctan\left(\frac{t}{2}\right) - 3\arctan\left(\frac{3}{2}\right).$$

(iii)

$$\int_3^t \frac{2}{x}\,dx = \Big[2\ln|x|\Big]_3^t$$
$$= 2\ln|t| - 2\ln(3).$$

We now combined these three expressions, and then compute the limit.

$$\int_3^\infty \frac{6x-8}{x^3+4x}\,dx$$
$$= \lim_{t\to\infty}\left[\left(\ln(t^2+4)-\ln(13)\right) + \left(3\arctan\left(\frac{t}{2}\right) - 3\arctan\left(\frac{3}{2}\right)\right) - \left(2\ln|t| - 2\ln(3)\right)\right]$$
$$= \lim_{t\to\infty}\left[\ln(t^2+4) - \ln t^2 + 3\arctan\left(\frac{t}{2}\right) - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)\right]$$
$$= \lim_{t\to\infty}\left[\ln\left(\frac{t^2+4}{t^2}\right) + 3\arctan\left(\frac{t}{2}\right) - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)\right]$$
$$= \ln(1) + 3\cdot\frac{\pi}{2} - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right) \quad \lim_{t\to\infty}\arctan(t) = \frac{\pi}{2}$$
$$= \boxed{\frac{3\pi}{2} - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)} \quad \textit{if you like, } -\ln 13 + \ln 9 = \ln\left(\frac{9}{13}\right)$$

Problem 5 *Which of the following is a solution to the differential equation $y'' + 9y = 0$?*

(a) $y = e^{3t} + e^{-3t}$

(b) $y = C(t^2 + t)$

(c) $y = \sin(3t) + 6$

(d) $y = 5\cos(3t) - 7\sin(3t)$

(e) $y = A\cos(3t) + B\sin(3t)$ *(where A and B are real numbers.)*

Solution: (a)

$$y = e^{3t} + e^{-3t} \qquad y' = 3e^{3t} - 3e^{-3t} \qquad y'' = 9e^{3t} + 9e^{-3t}$$

So,

$$\begin{aligned} y'' + 9y &= (9e^{3t} + 9e^{-3t}) + 9 \cdot (e^{3t} + e^{-3t}) \\ &= 18e^{3t} + 18e^{-3t} \neq 0. \end{aligned}$$

Therefore, this is **not** *a solution to* $y'' + 9y = 0$.

(b)

$$y = C(t^2 + t) \qquad y' = C(2t + 1) \qquad y'' = 2C$$

So,

$$\begin{aligned} y'' + 9y &= 2C + 9C(t^2 + t) \\ &= 9Ct^2 + 9Ct + 2C \neq 0 \text{ if } C \neq 0. \end{aligned}$$

Therefore, this is **not** *a solution to* $y'' + 9y = 0$ *unless* $C = 0$, *in which case we get the trivial solution.*

(c)

$$y = \sin(3t) + 6 \qquad y' = 3\cos(3t) \qquad y'' = -9\sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -9\sin(3t) + 9(\sin(3t) + 6) \\ &= 54 \neq 0. \end{aligned}$$

Therefore, this is **not** *a solution to* $y'' + 9y = 0$.

(d)

$$y = 5\cos(3t) - 7\sin(3t) \qquad y' = -15\sin(3t) - 21\cos(3t) \qquad y'' = -45\cos(3t) + 63\sin(3t)$$

So,

$$\begin{aligned} y'' + 9y &= -45\cos(3t) + 63\sin(3t) + 9(5\cos(3t) - 7\sin(3t)) \\ &= -45\cos(3t) + 63\sin(3t) + 45\cos(3t) - 63\sin(3t) = 0. \end{aligned}$$

Therefore, this **is** *a solution to* $y'' + 9y = 0$.

(e)

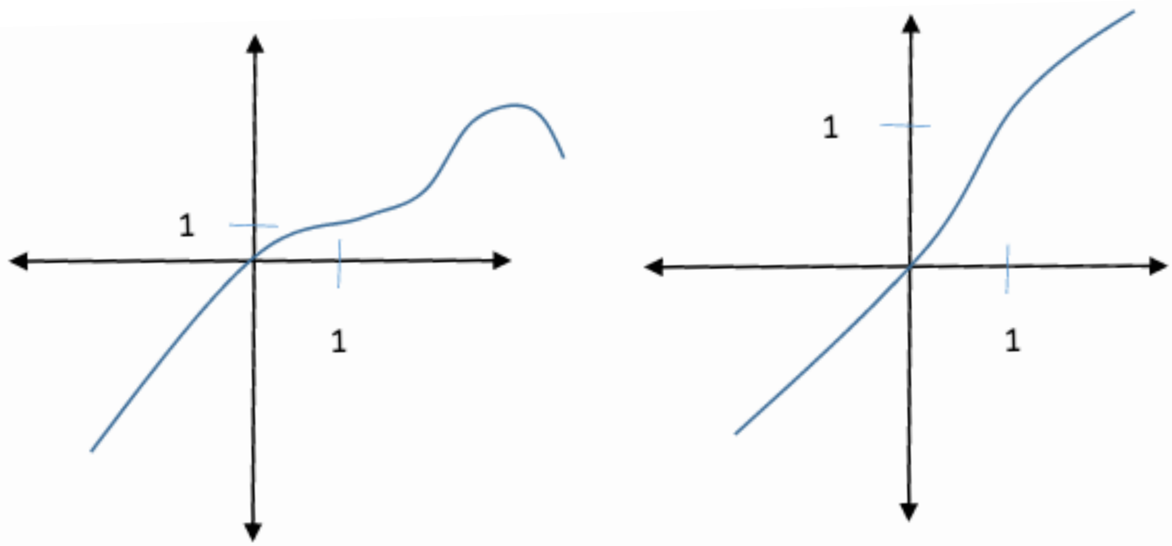
$$y = A\cos(3t)+B\sin(3t) \qquad y' = -3A\sin(3t)+3B\cos(3t) \qquad y'' = -9A\cos(3t)-9B\sin(3t)$$

So,

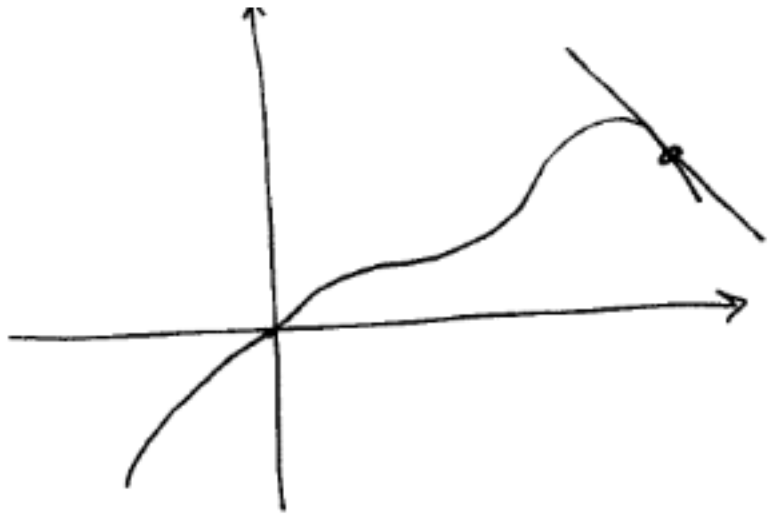
$$\begin{aligned} y'' + 9y &= -9A\cos(3t) - 9B\sin(3t) + 9(A\cos(3t) + B\sin(3t)) \\ &= 0. \end{aligned}$$

Therefore, this ***is*** *a solution to $y'' + 9y = 0$.*

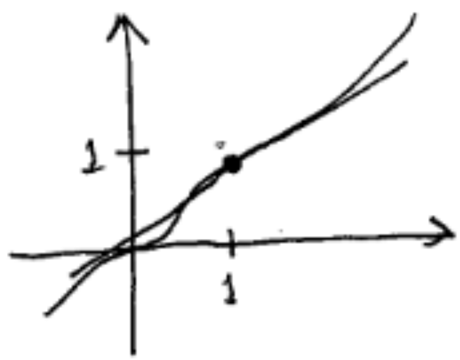
Problem 6 *Explain why the functions with the given graphs cannot be solutions of the differential equation $y' = e^x(y-1)^2$.*



Solution: *Since $y' = e^x(y-1)^2$, the derivative of y is always nonnegative. Thus, the first graph cannot satisfy this differential equation since it has a tangent line with a negative slope.*



The second graph cannot satisfy the differential equation since the slope of the tangent line at $x = 1$ is positive



but $\left[\dfrac{dy}{dx}\right]_{x=0} = e^0(1-1) = 0.$