Recitation #26: Vectors in three dimensions and Dot Products - Solutions

Warm up:

If $\vec{u} = \hat{\imath} - 2\hat{\jmath}$ and $\vec{v} = 3\hat{\imath} + 4\hat{k}$, find $\vec{u} \cdot \vec{v}$.

Solution: Note that these vectors are in $\mathbb{R}^3$ and not $\mathbb{R}^2$.

$$\vec{u} \cdot \vec{v} = (1 \cdot 3) + (-2 \cdot 0) + (0 \cdot 4) = \boxed{3}.$$

Group work:

Problem 1 *Solve the following problems:*

(a) *Which of the points $(6, 2, 3)$, $(-5, -1, 4)$, and $(0, 3, 8)$ is closest to the xz-plane? Which point lies on the yz-plane?*

(b) *Write an equation of the circle of radius 2 centered at $(-3, 4, 1)$ that lies in a plane parallel to the xy-plane.*

(c) *Describe the sphere $x^2 + y^2 + z^2 + 6x - 14y - 2z = 5$ (ie, find its center and radius).*

(d) *Find a vector whose magnitude is 311 and is in the same direction as the vector $\langle 3, -6, 7 \rangle$.*

Solution: (a) *The xz-plane has equation $y = 0$. The distance from a point (a, b, c) to $y = 0$ is just $|b|$. So*

$$\begin{aligned} &(6, 2, 3) \text{ has distance } 2 \\ &(-5, -1, 4) \text{ has distance } 1 \\ &(0, 3, 8) \text{ has distance } 3 \end{aligned}$$

Therefore, the point $(-5, -1, 4)$ is closest to the xz-plane.

The yz-plane is $x = 0$, and so the point $(0, 3, 8)$ is on the yz-plane.

Learning outcomes:

(b) *A plane parallel to the* xy*-plane has equation* $z = \#$. *We are looking for such a plane containing the point* $(-3, 4, 1)$, *and so the plane is* $z = 1$. *Therefore, the equation is*

$$\boxed{(x+3)^2 + (y-4)^2 = 4\ ,\ z = 1.}$$

(c) *We complete the square with respect to all three variables.*

$$\begin{aligned} x^2 + y^2 + z^2 + 6x - 14y - 2z &= 5 \\ (x^2 + 6y + 9) + (y^2 - 14y + 49) + (z^2 - 2z + 1) &= 5 + 9 + 49 + 1 \\ (x+3)^2 + (y-7)^2 + (z-1)^2 &= 64. \end{aligned}$$

So, the center of the sphere is $(-3, 7, 1)$ *and its radius is* 8.

(d) *Let* $\vec{v} = \langle 3, -6, 7 \rangle$. *Then*

$$\begin{aligned} |\vec{v}| &= \sqrt{3^2 + (-6)^2 + 7^2} \\ &= \sqrt{9 + 36 + 49} \\ &= \sqrt{94}. \end{aligned}$$

So a unit vector in the same direction as $\vec{v}$ *is*

$$\frac{1}{\sqrt{94}}\langle 3, -6, 7 \rangle$$

and therefore a vector with magnitude 311 *in the same direction as* v *is*

$$\boxed{\frac{311}{\sqrt{94}}\langle 3, -6, 7 \rangle}$$

Problem 2 *Find a vector (in the* xy*-plane) with length* 4 *that makes a* $\frac{\pi}{3}$ *radian angle with the vector* $\langle 3, 4 \rangle$.

Solution: *Let* $\vec{v} = \langle a, b \rangle$ *denote a vector that we are looking for, and let* $\vec{u} = \langle 3, 4 \rangle$. *First note that*

$$|\vec{u}| = \sqrt{9 + 16} = 5.$$

So

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\left(\frac{\pi}{3}\right) = 5 \cdot 4 \cdot \frac{1}{2} = 10.$$

Then we have the following two equations:

$$10 = \vec{u} \cdot \vec{v} = 3a + 4b \tag{1}$$

$$16 = |\vec{v}|^2 = a^2 + b^2. \tag{2}$$

Solving equation (3) for a gives us

$$a = \frac{10 - 4b}{3}.$$

Plugging this into equation (4) yields

$$\begin{aligned}
\left(\frac{10-4b}{3}\right)^2 + b^2 &= 16 \\
(10-4b)^2 + 9b^2 &= 144 \\
16b^2 - 80b + 100 + 9b^2 &= 144 \\
25b^2 - 80b - 44 &= 0
\end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned}
b &= \frac{80 \pm \sqrt{(-80)^2 - 4(25)(-44)}}{2(25)} \\
&= \frac{80 \pm \sqrt{10800}}{50} \\
&= \frac{80 \pm 60\sqrt{3}}{50} \\
&= \frac{8 \pm 6\sqrt{3}}{5}.
\end{aligned}$$

We can choose either value for b. Choosing $b = \dfrac{8+6\sqrt{3}}{5}$ gives a value of $a = \dfrac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}$. Thus,

$$\vec{v} = \boxed{\left\langle \frac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}, \frac{8+6\sqrt{3}}{5} \right\rangle}$$

Problem 3 *Answer the following questions about $\text{proj}_v u$.*

(a) *Is $\text{proj}_v u$ a vector of the form $c\vec{v}$ or $c\vec{u}$ (where c is a real number)? ie, is $\text{proj}_v u$ parallel to $\vec{u}$ or $\vec{v}$?*

(b) *If $\vec{u} = 5\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$ and $\vec{v} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$, find $\text{proj}_v u$.*

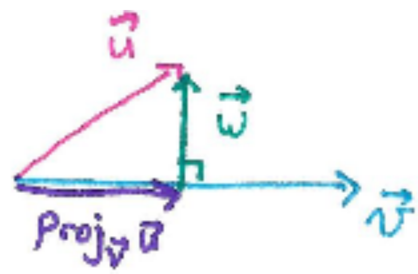
(c) *For $\vec{u}$ and $\vec{v}$ from part (b), write $\vec{u}$ as the sum of two perpendicular vectors, one of which is parallel to $\vec{v}$.*

Solution: (a) $\boxed{c\vec{v}}$

(b)

$$\begin{aligned} \text{proj}_v u &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\ &= \frac{10-24-12}{4+16+16}\langle 2,-4,4\rangle \\ &= \boxed{-\frac{13}{18}\langle 2,-4,4\rangle} \end{aligned}$$

(c) *A schematic picture of the situation is as follows:*



The vector which is parallel to $\vec{v}$ is

$$\text{proj}_v u = \boxed{\left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9}\right\rangle}$$

The vector which is orthogonal to $\vec{v}$ is

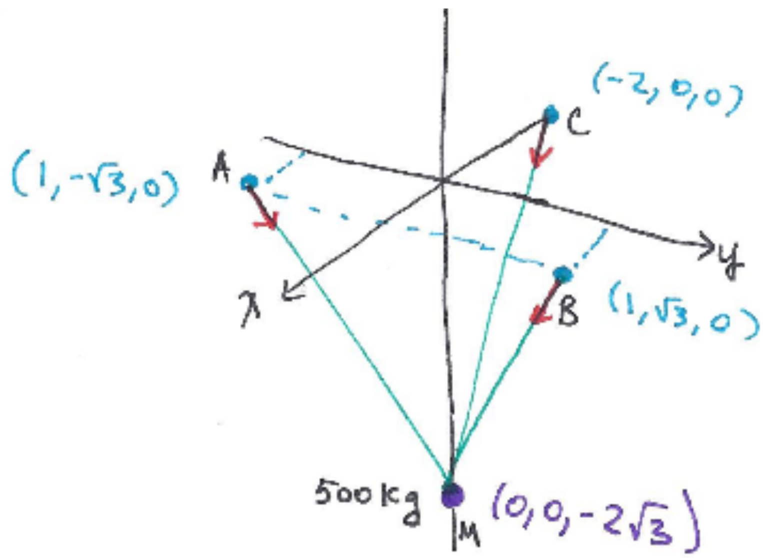
$$\begin{aligned} \vec{w} := \vec{u} - \text{proj}_v u = \langle 5,6,-3\rangle - \left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9}\right\rangle \\ = \boxed{\left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9}\right\rangle} \end{aligned}$$

And, clearly, $\text{proj}_v u + \vec{w} = \text{proj}_v u + (\vec{u} - \text{proj}_v u) = \vec{u}$.

Problem 4 *A 500kg lead hangs from three cables of equal length that are located at the points* $(-2,0,0)$, $(1,\sqrt{3},0)$, *and* $(1,-\sqrt{3},0)$. *The load is located at* $(0,0,-2\sqrt{3})$. *Find the vectors describing the forces on the cables due to the load.*

Solution: *Let $A = (1, -\sqrt{3}, 0)$, $B = (1, \sqrt{3}, 0)$, and $C = (-2, 0, 0)$, and let $M = (0, 0, -2\sqrt{3})$. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ denote the vectors from A, B, and C to M, respectively. ie,*

$$\begin{aligned}\vec{a} &= \langle 0-1, 0-(-\sqrt{3}), -2\sqrt{3}-0\rangle = \langle -1, \sqrt{3}, -2\sqrt{3}\rangle\\ \vec{b} &= \langle 0-1, 0-\sqrt{3}, -2\sqrt{3}-0\rangle = \langle -1, -\sqrt{3}, -2\sqrt{3}\rangle\\ \vec{c} &= \langle 0-(-2), 0-0, -2\sqrt{3}-0\rangle = \langle 2, 0, -2\sqrt{3}\rangle.\end{aligned}$$



Notice that

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 4$$

and so unit vectors in the directions of $\vec{a}$, $\vec{b}$, and $\vec{c}$ are

$$\begin{aligned}\vec{u}_a &= \frac{1}{4}\langle -1, \sqrt{3}, -2\sqrt{3}\rangle\\ \vec{u}_b &= \frac{1}{4}\langle -1, -\sqrt{3}, -2\sqrt{3}\rangle\\ \vec{u}_c &= \frac{1}{4}\langle 2, 0, -2\sqrt{3}\rangle.\end{aligned}$$

The force on M due to gravity is

$$\langle 0, 0, -500g\rangle$$

where g is the gravitational constant. We need to find real numbers x, y, and z

such that

$$-\frac{1}{4}x-\frac{1}{4}y+\frac{1}{2}z=0 \quad\Longrightarrow\quad x+y-2z=0 \tag{3}$$

$$\frac{\sqrt{3}}{4}x-\frac{\sqrt{3}}{4}y+0z=0 \quad\Longrightarrow\quad x-y=0 \tag{4}$$

$$-\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}y-\frac{\sqrt{3}}{2}z=-500g \quad\Longrightarrow\quad x+y+z=\frac{1000g}{\sqrt{3}}. \tag{5}$$

By equation (4) we have that $x=y$. Substituting this into equation (3) we also see that $x=z$. So $x=y=z$. We plug this into equation (5) to get that

$$3x=\frac{1000g}{\sqrt{3}} \qquad\Longrightarrow\qquad x=\frac{1000g}{3\sqrt{3}}.$$

Thus,

- *The force along AM is* $\boxed{\frac{1000g}{3\sqrt{3}}\cdot\frac{1}{4}\langle -1,\sqrt{3},-2\sqrt{3}\rangle}$.
- *The force along BM is* $\boxed{\frac{1000g}{3\sqrt{3}}\cdot\frac{1}{4}\langle -1,-\sqrt{3},-2\sqrt{3}\rangle}$.
- *The force along CM is* $\boxed{\frac{1000g}{3\sqrt{3}}\cdot\frac{1}{4}\langle 2,0,-2\sqrt{3}\rangle}$.

Problem 5 *Find the work done by a constant force of $10\hat{\imath}+18\hat{\jmath}-6\hat{k}$ that moves an object up a ramp from $(2,3,7)$ to $(4,9,15)$. Assume that distance is in feet and force in pounds. Also, find the angle between the force and the ramp.*

Solution: *First, let $\vec{F}=\langle 10,18,-7\rangle$. Also, the vector from $(2,3,7)$ to $(4,9,15)$ is $\langle 2,6,8\rangle$. Let $\vec{d}=\langle 2,6,8\rangle$. Then the work done by the force is*

$$\vec{F}\cdot\vec{d}=10\cdot 2+18\cdot 6-6\cdot 8=\boxed{80\,ft\cdot lb}.$$

To calculate the angle, we compute

$$\cos\theta=\frac{\vec{F}\cdot\vec{d}}{|\vec{F}||\vec{d}|}=\frac{80}{\sqrt{460}\sqrt{104}}.$$

and so

$$\theta=\boxed{\cos^{-1}\left(\frac{80}{\sqrt{460}\sqrt{104}}\right)\approx 1.196\ radians}$$

Problem 6 *Suppose that the deli at the Tiny Sparrow grocery store sells roast beef for \$9 per pound, turkey for \$4 per pound, salami for \$5 per pound, and ham for \$7 per pound. For lunches this week, Sam the sandwhich maker buys 1.5 pounds of roast beef, 2 pounds of turkey, no salami, and half a pound of ham. How can you use a dot product to compute Sam's total bill from the deli?*

Solution: *The cost vector is*

$$\vec{c} = \langle 9, 4, 5, 7 \rangle.$$

The vector for Sam's order is

$$\vec{o} = \left\langle \frac{3}{2}, 2, 0, \frac{1}{2} \right\rangle.$$

Then Sam's bill is

$$\vec{c} \cdot \vec{o} = 9(1.5) + 4(2) + 5(0) + 7(0.5) = 13.5 + 8 + 0 + 3.5 = \boxed{25}.$$