Recitation #19: Approximating functions with polynomials - Full

Warm up:

For each of the following, write the given polynomial in summation notation starting with $k = 0$.

(a) $\dfrac{3x}{2} - \dfrac{5x^2}{3} + \dfrac{7x^3}{4} - \dfrac{9x^4}{5} + \dfrac{11x^5}{6}$

(b) $\dfrac{1}{2}x + \dfrac{1 \cdot 5}{4 \cdot 2!}x^3 + \dfrac{1 \cdot 5 \cdot 9}{8 \cdot 3!}x^5 - \dfrac{1 \cdot 5 \cdot 9 \cdot 13}{16 \cdot 4!}x^7$

(c) $(x-1)^3 - \dfrac{(x-1)^4}{2!} + \dfrac{(x-1)^5}{4!} - \dfrac{(x-1)^6}{6!}$

Solution: (a) $\displaystyle\sum_{k=0}^{4} (-1)^k (2k+3)\frac{x^{k+1}}{k+2}$.

(b) $\displaystyle\sum_{k=0}^{3} \frac{1 \cdot 5 \cdot \ldots \cdot (4k+1)}{2^{k+1}(k+1)!}x^{2k+1}$.

(c) $\displaystyle\sum_{k=0}^{3} \frac{(-1)^k}{(2k)!}(x-1)^{k+3}$.

Instructor Notes: Maybe give one problem per group. Allow 4 minutes for group work and 6 minutes for discussion. Make sure to note that we can "factor out" terms dealing only with x, but not with k (or n). Also, you might want to discuss starting with k equaling another number other than 0.

Learning outcomes:

Group work:

Problem 1 *Assuming that the function $f(x)$ is infinitely differentiable, and given that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + c_4(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

show that the coefficient c_4 of the $(x-a)^4$ term in the Taylor polynomial is $\dfrac{f^{(4)}(a)}{4!}$.

Solution: *Notice that we have the following:*

$$f'(x) = f'(a) + f''(a)(x-a) + \frac{f^{(3)}(a)}{2}(x-a)^2 + 4c_4(x-a)^3 + \frac{f^{(5)}(a)}{4!}(x-a)^4$$

$$f''(x) = f''(a) + f^{(3)}(a)(x-a) + 4 \cdot 3c_4(x-a)^2 + \frac{f^{(5)}(a)}{3!}(x-a)^3$$

$$f^{(3)}(x) = f^{(3)}(a) + 4 \cdot 3 \cdot 2c_4(x-a) + \frac{f^{(5)}(a)}{2}(x-a)^2$$

$$f^{(4)}(x) = 4! \cdot c_4 + f^{(5)}(a)(x-a)$$

$$f^{(4)}(a) = 4! \cdot c_4 + 0$$

$$\implies \quad \boxed{c_4 = \frac{f^{(4)}(a)}{4!}}.$$

Instructor Notes: The lecture justifies the coefficients of $p_3(x)$. This problem should be done as a whole class discussion, perhaps for about 5 minutes.

Problem 2 *Let $f(x) = \sin(2x)$. Find $p_3(x)$ about the point $a = \dfrac{\pi}{8}$.*

Solution: *First, note that around a*

$$p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3.$$

So we compute

$$f'(x) = 2\cos(2x) \implies f'\left(\frac{\pi}{8}\right) = \sqrt{2}$$

$$f''(x) = -4\sin(2x) \implies f''\left(\frac{\pi}{8}\right) = -2\sqrt{2}$$

$$f^{(3)}(x) = -8\cos(2x) \implies f^{(3)}\left(\frac{\pi}{8}\right) = -4\sqrt{2}.$$

Therefore

$$\boxed{p_3(x) = \frac{\sqrt{2}}{2} + \sqrt{2}\left(x - \frac{\pi}{8}\right) - \sqrt{2}\left(x - \frac{\pi}{8}\right)^2 - \frac{2\sqrt{2}}{3}\left(x - \frac{\pi}{8}\right)^3}.$$

Instructor Notes: Students often have difficulty "putting the pieces together", particularly when $a \neq 0$. Watch for students not putting the polynomial as powers of $\left(x - \frac{\pi}{8}\right)$. They often will just use powers of x. They also will often not plug in $\frac{\pi}{8}$ into the derivative (leaving the derivative in terms of x). Another common error is forgetting the $k!$.

Problem 3 *Let $f(x) = xe^{-x}$ on the interval $[-2, 8]$.*

(a) *Write the Taylor polynomial $p_4(x)$ around $a = 3$.*

$$\begin{aligned} \textit{Fun facts: } f'(x) &= -e^{-x}(x-1) \\ f''(x) &= e^{-x}(x-2) \\ f^{(3)}(x) &= -e^{-x}(x-3) \\ f^{(4)}(x) &= e^{-x}(x-4) \end{aligned}$$

(b) *Write $p_4(x)$ about $a = 3$ in summation notation. Also, write the remainder term $R_4(x)$.*

(c) *Calculate $p_4(4.5)$ and, using $R_4(4.5)$, estimate how close $p_4(4.5)$ is to $f(4.5)$. Do the same for $p_4(1.5)$.*

(d) *Use the remainder term $R_4(x)$ to estimate the maximum error for $p_4(x)$ on $[-2, 8]$.*

(e) *How large must n be to assure that the n^{th} degree Taylor polynomial for $f(x) = xe^{-x}$ about $a = 3$ approximates $2e^{-2}$ within 10^{-5}?*

Solution: (a)

$$\begin{aligned} p_4(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}}{3!}(x-a)^3 + \frac{f^{(4)}}{4!}(x-a)^4 \\ &= 3e^{-3} - 2e^{-3}(x-3) + \frac{e^{-3}}{2}(x-3)^2 - \frac{e^{-3}}{4!}(x-3)^4. \end{aligned}$$

(b)

$$p_4(x) = \sum_{k=0}^{4} \frac{(-1)^k e^{-3}(3-k)}{k!}(x-3)^k.$$

$$R_4(x) = f(x) - p_4(x) = \frac{f^{(5)}(c)}{5!}(x-3)^5 = \frac{-e^{-c}(c-5)}{5!}(x-3)^5$$

for some c between x and 3.

(c)

(d)

(e)

Instructor Notes: This is the longest of the problems. Part (a) is like the preceeding problem, where they need to remember to plug 3 into the derivatives. For part (b), they also need to recognize that $R_4(x)$ deals with $f^{(5)}(x)$ and not $f^{(4)}(x)$. Writing $R_4(x) = \dfrac{f^{(5)}(c)}{5!}(x-3)^5$ for some $c \in [-2, 8]$ will be sufficient at this stage (they will bound $f^{(5)}(c)$ in parts (c) and (d)).

In part (c), the main issue is to bound $|f^{(5)}(x)| = |-e^{-5}(x-5)|$ on $[3, 4.5]$ and $[1.5, 3]$, and on $[2, 8]$ for part (d). This is a problem of finding the absolute max (and min) of a function on a closed interval. Students then need to find critical value(s) by finding when $f^{(6)}(x) = 0$ (or undefined) as well as consider endpoints $f^{(6)}(x) = e^{-x}(x-6)$, with critical $x = 6$. The last hurdle is to recognize we want to find the max of $|f^{(5)}(x)|$. This maximum occurs at $x = 3, x = 1.5$, and $x = -2$ respectively.