Problem 1 (a) *Show that*

$$\frac{9}{2x^2+3x} = \frac{3}{x} - \frac{6}{2x+3}$$

(b) *Determine if the integral*

$$\int_1^\infty \frac{9}{2x^2+3x}\,dx$$

converges or diverges. If it converges, give the value that it converges to.

Solution: *(a) Since* $2x^2 + 3x = x(2x+3)$*, we apply the method of partial fractions:*

$$\begin{aligned} \frac{9}{2x^2+3x} &= \frac{A}{x} + \frac{B}{2x+3} \\ \implies \quad 9 &= A(2x+3) + Bx. \end{aligned}$$

Letting $x = 0$ *gives that*

$$9 = 3A \quad \implies \quad A = 3.$$

Then letting $x = -\frac{3}{2}$*, we see that*

$$9 = -\frac{3}{2}B \quad \implies \quad B = -9 \cdot \frac{2}{3} = -6.$$

Therefore, plugging in our values for A *and* B *gives us*

$$\frac{9}{2x^2+3x} = \frac{3}{x} - \frac{6}{2x+3}.$$

(b) We have that

$$\begin{aligned} \int_1^\infty \frac{9}{2x^2+3x}\,dx &= \lim_{t\to\infty} \int_1^t \left(\frac{3}{x} - \frac{6}{2x+3}\right) dx \\ &= \lim_{t\to\infty} \left[3\ln|x| - \frac{6}{2}\ln|2x+3|\right]_1^t \quad \text{don't forget to divide the 6 by 2} \\ &= \lim_{t\to\infty} \left(3\ln|t| - 3\ln|2t+3| - 0 + 3\ln(5)\right) \quad \ln(1) = 0 \\ &= \lim_{t\to\infty} \left(3\ln\left|\frac{5t}{2t+3}\right|\right) \quad \text{properties of logarithms} \\ &= \boxed{3\ln\left(\frac{5}{2}\right)} \quad \text{since ln(x) is a continuous function} \end{aligned}$$

Learning outcomes:

Problem 2 (a) *Show that*

$$\frac{6x-8}{x^3+4x} = \frac{2x+6}{x^2+4} - \frac{2}{x}$$

(b) *Determine if the integral*

$$\int_3^\infty \frac{6x-8}{x^3+4x}\,dx$$

converges or diverges. If it converges, give the value that it converges to.

Solution: (a) *Since* $x^3+4x = x(x^2+4)$, *we use partial fractions:*

$$\begin{aligned} \frac{6x-8}{x^3+4x} &= \frac{Ax+B}{x^2+4} + \frac{C}{x} \\ \implies \quad 6x-8 &= (Ax+B)(x) + C(x^2+4). \end{aligned}$$

Letting $x=0$, *we see that*

$$-8 = 4C \quad \implies \quad C = -2.$$

To find A *and* B, *let us plug in* $C=-2$ *and simplify:*

$$\begin{aligned} 6x-8 &= Ax^2 + Bx - 2x^2 - 8 \\ &= (A-2)x^2 + Bx - 8. \end{aligned}$$

Aligning the respective coefficients, we see that

$$\begin{aligned} & A-2=0 \quad \text{and} \quad B=6 \\ \implies \; & A=2 \quad \text{and} \quad B=6. \end{aligned}$$

Finally, plugging this into the original equation yields

$$\frac{6x-8}{x^3+4x} = \frac{2x+6}{x^2+4} - \frac{2}{x}.$$

(b) *We have that*

$$\begin{aligned} \int_3^\infty \frac{6x-8}{x^3+4x}\,dx &= \lim_{t\to\infty} \int_3^t \left(\frac{2x+6}{x^2+4} - \frac{2}{x} \right) dx \\ &= \lim_{t\to\infty} \left(\int_3^t \frac{2x}{x^2+4}\,dx + \int_3^t \frac{6}{x^2+4}\,dx - \int_3^t \frac{2}{x}\,dx \right). \end{aligned}$$

Let us evaluate each integral separately, combine them, and then take the limit.

(i)

$$\int_3^t \frac{2x}{x^2+4}\,dx = \int_{13}^{t^2+4} \frac{1}{w}\,dw \quad w = x^2+4,\ dw = 2x\,dx$$
$$= \ln(t^2+4) - \ln(13).$$

(ii)

$$\int_3^t \frac{6}{x^2+4}\,dx = \left[\frac{6}{2}\arctan\left(\frac{x}{2}\right)\right]_3^t$$
$$= 3\arctan\left(\frac{t}{2}\right) - 3\arctan\left(\frac{3}{2}\right).$$

(iii)

$$\int_3^t \frac{2}{x}\,dx = \Big[2\ln|x|\Big]_3^t$$
$$= 2\ln|t| - 2\ln(3).$$

We now combined these three expressions, and then compute the limit.

$$\int_3^\infty \frac{6x-8}{x^3+4x}\,dx$$
$$= \lim_{t\to\infty}\left[\big(\ln(t^2+4)-\ln(13)\big) + \left(3\arctan\left(\frac{t}{2}\right) - 3\arctan\left(\frac{3}{2}\right)\right) - \big(2\ln|t| - 2\ln(3)\big)\right]$$
$$= \lim_{t\to\infty}\left[\ln(t^2+4) - \ln t^2 + 3\arctan\left(\frac{t}{2}\right) - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)\right]$$
$$= \lim_{t\to\infty}\left[\ln\left(\frac{t^2+4}{t^2}\right) + 3\arctan\left(\frac{t}{2}\right) - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)\right]$$
$$= \ln(1) + 3\cdot\frac{\pi}{2} - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right) \quad \lim_{t\to\infty}\arctan(t) = \frac{\pi}{2}$$
$$= \boxed{\frac{3\pi}{2} - \ln 13 + \ln 9 - 3\arctan\left(\frac{3}{2}\right)} \quad \textit{if you like, } -\ln 13 + \ln 9 = \ln\left(\frac{9}{13}\right)$$