Section 12.4: Cross Products

Warm up:

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in 3-space $\mathbb{R}^3$, which of the following make sense?

(a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

(b) $(\vec{a} \cdot \vec{b})\vec{c}$

(c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$

(d) $(\vec{a} \cdot \vec{b}) + \vec{c}$

(e) $(\vec{a} \times \vec{b}) + \vec{c}$

(f) $\vec{a} \cdot (\vec{b} + \vec{c})$

(g) $\vec{a} \cdot (\vec{b} \times \vec{c})$

(h) $\vec{a} \times (\vec{b} \cdot \vec{c})$

(i) $(\vec{a} \times \vec{b})\vec{c}$

Solution: (a) Since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ does **not** make sense.

(b) Now since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b})\vec{c}$ **does** make sense as regular scalar multiplication.

(c) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ **does** make sense.

(d) This is of the form "scalar + vector", which does **not** make sense.

(e) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) + \vec{c}$ **does** make sense.

(f) This is of the form "vector · vector", which **does** make sense.

(g) This is also of the form "vector · vector", which **does** make sense.

(h) This is of the form "vector × scalar", which does **not** make sense.

(i) Since $\vec{a} \times \vec{b}$ is a vector, this does **not** make sense.

Group work:

Problem 1 *Given three dimensional vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, use dot product or cross product notation to describe the following vectors:*

(a) *The vector projection of $\vec{w}$ onto $\vec{u}$.*

(b) *A vector orthogonal to both $\vec{u}$ and $\vec{v}$.*

Learning outcomes:

(c) *A vector with the length of $\vec{v}$ and the direction of $\vec{w}$.*

(d) *A vector orthogonal to $\vec{u} \times \vec{v}$ and $\vec{w}$.*

Solution: (a) *This is the definition of vector projections.*

$$\mathrm{proj}_u w = \boxed{\left(\frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\right) \vec{u}}$$

(b) *There are many such vectors, but one of them is*

$$\boxed{\vec{u} \times \vec{v}}$$

(c) *Note that $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ so that $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.*

$$|\vec{v}| \left(\frac{\vec{w}}{|\vec{w}|}\right) = \boxed{\frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\vec{w} \cdot \vec{w}}} \vec{w}}$$

(d)

$$\boxed{(\vec{u} \times \vec{v}) \times \vec{w}}$$

Problem 2 *Let $\vec{u} = \langle 5, -1, 8 \rangle$ and $\vec{v} = \langle -2, 10, 5 \rangle$.*

(a) *Find a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.*

(b) *Verify that your answer is perpendicular to both $\vec{u}$ and $\vec{v}$*

(c) *Find a vector of length 7 perpendicular to both $\vec{u}$ and $\vec{c}$.*

Solution: (a) *Let $\vec{u} = \langle 5, -1, 8 \rangle$ and $\vec{v} = \langle -2, 10, 5 \rangle$. Then a vector which is perpendicular to both $\vec{u}$ and $\vec{v}$ is $\vec{w} := \vec{u} \times \vec{v}$. So we calculate*

$$\begin{aligned} \vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -1 & 8 \\ -2 & 10 & 5 \end{vmatrix} &= (-5 - 80)\hat{\imath} - (25 + 16)\hat{\jmath} + (50 - 2)\hat{k} \\ &= -85\hat{\imath} - 41\hat{\jmath} + 48\hat{k} \end{aligned}$$

(b) *To verify perpendicularity, we take the dot product.*

$$\vec{u} \cdot \vec{w} = \langle 5, -1, 8 \rangle \cdot \langle -85, -41, 48 \rangle = 5(-85) - 1(-41) + 8(48) = -425 + 41 + 384 = 0$$

$$\vec{v} \cdot \vec{w} = \langle -2, 10, 5 \rangle \cdot \langle -85, -41, 48 \rangle = -2(-85) + 10(-41) + 5(48) = 170 - 410 + 240 = 0$$

(c) *A unit vector in the same direction as $\vec{w}$ is*

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{(-85)^2 + (-41)^2 + 48^2}}\vec{w} = \frac{1}{\sqrt{11210}}\vec{w}.$$

Therefore, a vector with a magnitude of 7 in the same direction as $\vec{w}$ is

$$\vec{t} = \frac{7}{|\vec{w}|}\vec{w} = \boxed{\frac{7}{\sqrt{11210}}\langle -85, -41, 48\rangle}$$

Problem 3 *Find the area of the triangle in $\mathbb{R}^3$ with vertices at $P(2,-1,0)$, $Q(1,1,4)$ and $R(2,-1,6)$.*

Solution: *The area of the triangle is $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$.*

$$\begin{aligned} \vec{PR} &= \langle 2,-1,6\rangle - \langle 2,-1,0\rangle = \langle 0,0,6\rangle, \\ \vec{PQ} &= \langle 1,1,4,\rangle - \langle 2,-1,0\rangle = \langle -1,2,4\rangle. \end{aligned}$$

So

$$\begin{aligned} \vec{PQ} \times \vec{PR} &= (-\vec{i} + 2\vec{j} + 4\vec{k}) \times 6\vec{k} = -(\vec{i} \times \vec{k}) + 2(\vec{j} \times \vec{k}) + 24(\vec{k} \times \vec{k}) \\ &= -(-\vec{j}) + 2\vec{i} + 0 = \langle 2,1,0\rangle. \end{aligned}$$

The area of the triangle is $\frac{1}{2}\sqrt{2^2 + 1^2 + 0^2} = \frac{\sqrt{5}}{2}$.