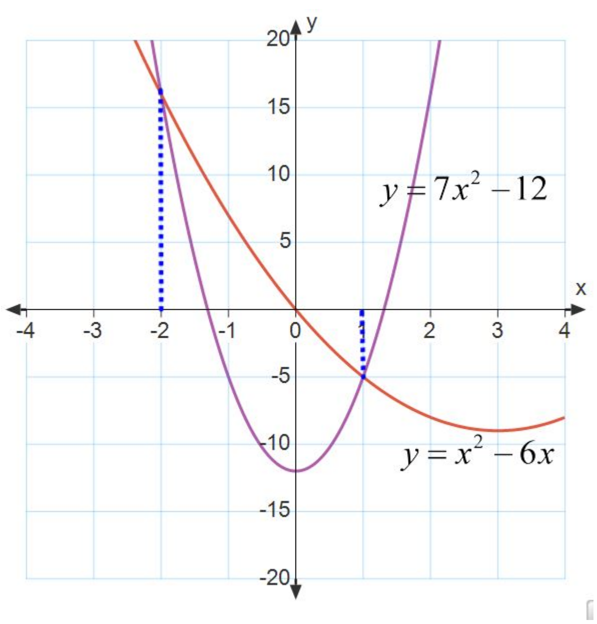
Section 6.2: Regions Between Curves

Group work:

Problem 1 *Consider the region bounded by the curves $y = 7x^2 - 12$ and $y = x^2 - 6x$.*

(a) *Draw a sketch of the graphs.*

Solution: *Set the curves equal to each other to find the intersection points:*
$7x^2 - 12 = x^2 - 6x$
$6x^2 + 6x - 12 = 0$
$6(x^2 + x - 2) = 0$
$6(x - 1)(x + 2) = 0$
$x = 1 \text{ or } x = -2$



Learning outcomes:

(b) *Find the area between these curves.*

Solution: *By the solution to part (a), we know that these two curves intersect at* $x = -2, 1$. *By checking the point* $x = 0$ *(or by looking at the graph from part (a)) we see that* $y = 7x^2 - 12 \leq y = x^2 - 6x$ *on the interval* $[-2, 1]$. *Said another way, y=*$7x^2 - 12$ *is on top and* $y = x^2 - 6x$ *is on bottom on the interval [-2,1], and we will use "top - bottom" to find the area between the curves. So the area between the curves is:*

$$\begin{aligned}
\int_{-2}^{1} (x^2 - 6x) - (7x^2 - 12)\, dx &= \int_{-2}^{1} (-6x^2 - 6x + 12)\, dx \\
&= \Big[-2x^3 - 3x^2 + 12x \Big]_{-2}^{1} \\
&= (-2(1) - 3(1) + 12(1)) - \big(-2(-2)^3 - 3(-2)^2 + 12(-2)\big) \\
&= 7 - (-20) = 27
\end{aligned}$$

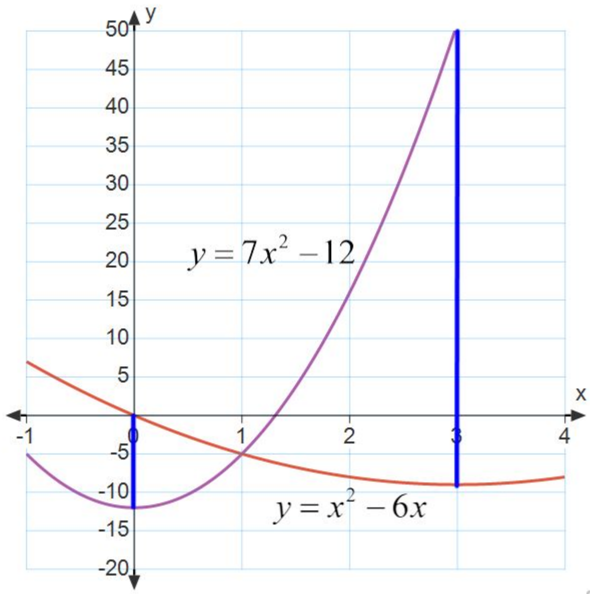
(c) *Find the area of the region bounded by the curves* $x = 7y^2 - 12$ *and* $x = y^2 - 6y$.

Solution: *This region is exactly the same as the region from part (a), except it is rotated clockwise by* 90°. *Since the area of a region does not change under rotation, we have that the area of the new region is still 27.*

(d) *Find the area of the region bounded by the curves* $y = 7x^2$ *and* $y = x^2 - 6x + 12$.

Solution: *The region bounded by the curves* $y = 7x^2$ *and* $y = x^2 - 6x + 12$ *is just the region in part (a) translated upward by 12 units. Since the area of a region does not change under translation, we have that the area of the new region is still 27.*

(e) *Find the area of the region bounded by the curves* $y = 7x^2 - 12$, $y = x^2 - 6x$, $x = 0$, *and* $x = 3$.



Solution: *We already know that the graphs of the two functions intersect at the point $x = 1$. By checking the points $x = 0$ and $x = 3$ (or by looking at the graph above) we see that*

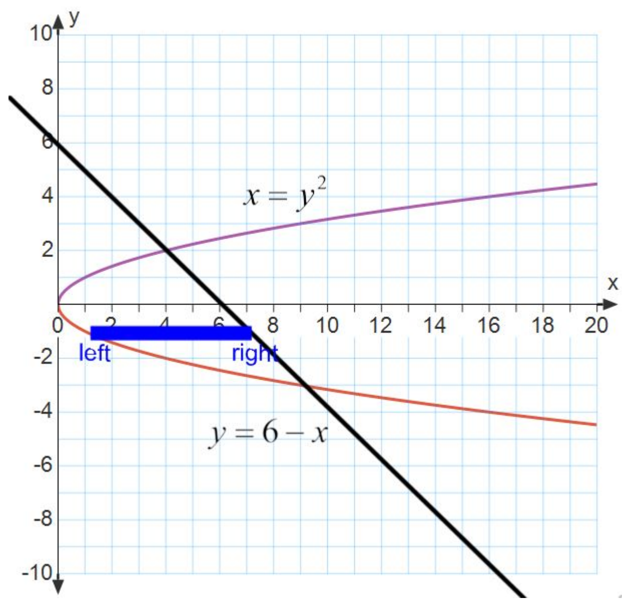
$$\begin{aligned} y = x^2 - 6x \geq y = 7x^2 - 12 \quad &\textit{on} \quad [0, 1] \\ y = 7x^2 - 12 \geq y = x^2 - 6x \quad &\textit{on} \quad [1, 3] \end{aligned}$$

Thus, the area between the curves is

$$\begin{aligned} &\int_0^1 (x^2 - 6x) - (7x^2 - 1)\, dx + \int_1^3 (7x^2 - 12) - (x^2 - 6x)\, dx \\ &= \int_0^1 (-6^2 - 6x + 12)\, dx + \int_1^3 (6x^2 + 6x - 12)\, dx \\ &= \Big[-2x^3 - 3x^2 + 12x\Big]_0^1 + \Big[2x^3 + 3x^2 - 12x\Big]_1^3 \\ &= [(-2 - 3 + 12) - 0] + [(2(27) + 3(9) - 12(3)) - (2 + 3 - 12)] \\ &= 7 + 54 + 27 - 36 + 7 = 59 \end{aligned}$$

Problem 2 *Set up two different integrals that compute the area of the region bounded by the curves $x = y^2$ and $y = 6 - x$ (and be sure to draw a sketch of the graphs).*

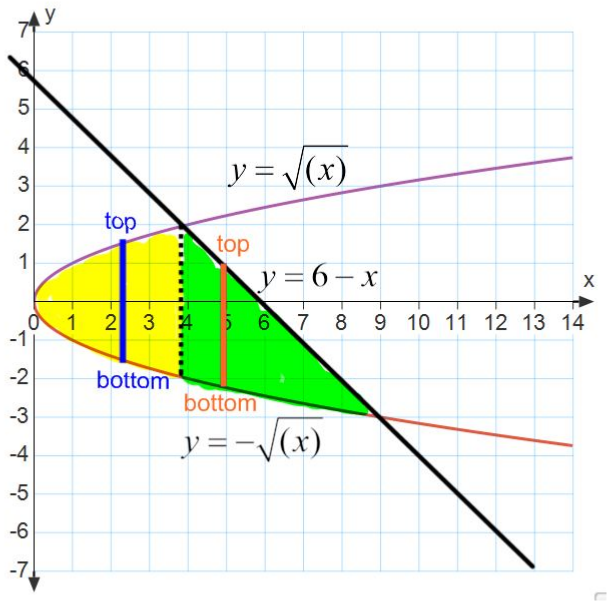
Solution: ***In terms of y:***
First we find the intersection points:
$y^2 = 6 - y$
$y^2 + y - 6 = 0$
$(y + 3)(y - 2) = 0$
$y = -3$ *or* $y = 2$



Thus,

$$\text{Area of region} = \int_{-3}^{2} [(6 - y) - y^2]\, dy.$$

In terms of x:
We need to rewrite $x = y^2$ as the two functions $y = \sqrt{x}$ and $y = -\sqrt{x}$. Then, we need two integrals because the top function changes where $y = \sqrt{x}$ intersects $y = 6 - x$. We also need to find our intersection points in terms of x. Since we already know the y-values of the points, we can plug the y-values into either function to get the x-values:
$x = 6 - y$
$x(2) = 6 - 2 = 4$
$x(-3) = 6 - (-3) = 9$



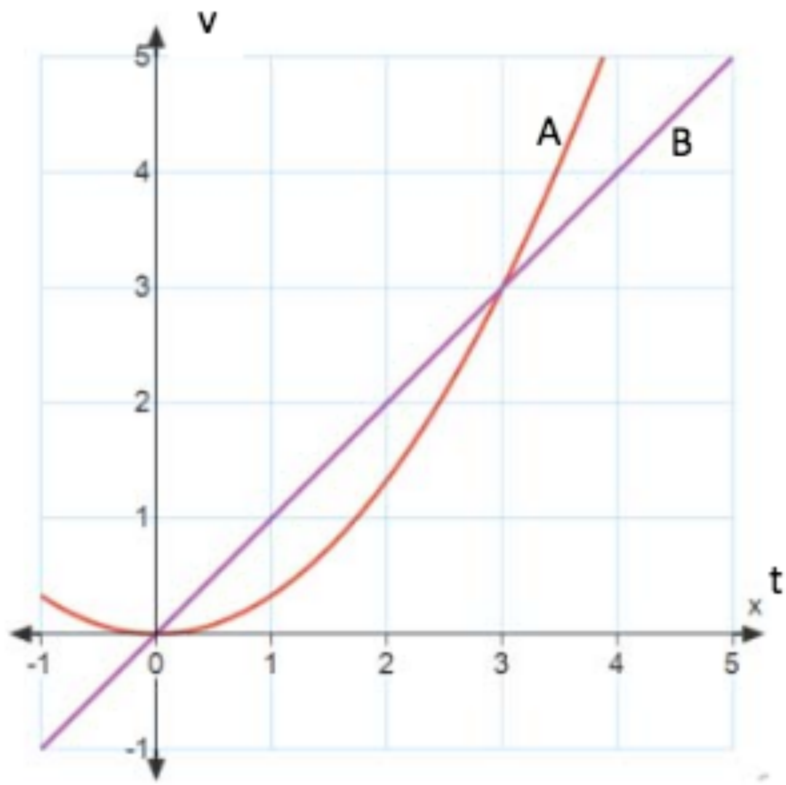
Thus,

$$\textcolor{red}{\text{Area of region}} = \int_0^4 (\sqrt{x} - (-\sqrt{x}))\,dx + \int_4^9 ((6 - x) - (-\sqrt{x}))\,dx.$$

Problem 3 *Two runners (A and B) run in a race in which the winner runs the greater distance after 4 minutes. The runners' respective velocities are*

$$v_A(t) = \frac{1}{3}t^2 \qquad v_B(t) = t$$

The graphs of the runners' velocities is given below.



(a) *Who is running faster 2 minutes into the race?*

Solution: $v_A(2) = \frac{4}{3}$ and $v_B(2) = 2$. *So B is running faster at the 2 minute mark of the race.*

(b) *Who is winning the race 2 minutes into the race (and by how much)?*

Solution: *The distance that A covers in the first 2 minutes is*

$$\int_0^2 v_A(t)\,dt = \int_0^2 \frac{1}{3}t^2\,dt = \left[\frac{1}{9}t^3\right]_0^2 = \frac{8}{9}.$$

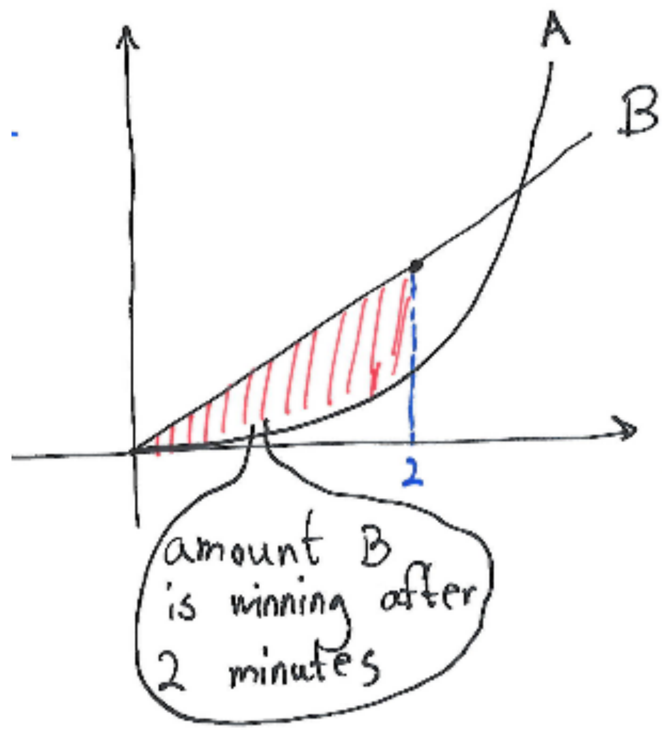
The distance that B covers in the first 2 minutes is

$$\int_0^2 v_B(t)\,dt = \int_0^2 t\,dt = \left[\frac{1}{2}t^2\right]_0^2 = 2.$$

So B is winning after 2 minutes.

B is winning by $2 - \frac{8}{9} = \frac{10}{9}$. This could also be calculated by

$$\int_0^2 (v_B(t) - v_A(t))\,dt.$$



(c) *What special event occurs 3 minutes into the race?*

Solution: *Runner A matches runner B's velocity. ie, $v_A(3) = v_B(3)$.*

(d) *Who wins the race (and by how much)?*

Solution: *The distance that A covers is*

$$\int_0^4 v_A(t)\,dt = \int_0^4 \frac{1}{3}t^2\,dt = \left[\frac{1}{9}t^3\right]_0^4 = \frac{64}{9} = 7.\overline{1}.$$

The distance that B covers is

$$\int_0^4 v_B(t)\,dt = \int_0^4 t\,dt = \left[\frac{1}{2}t^2\right]_0^4 = 8.$$

So runner B wins. The amount that B wins by is

$$8 - \frac{64}{9} = \frac{8}{9}.$$

This could have also been computed by

$$\int_0^4 (v_B(t) - v_A(t))\,dt.$$