Recitation # 5: Length of Curves & Surface Area - Solutions

Group work:

Problem 1 *Find the length of the following curves (length is in feet):*

(a) $y = \frac{1}{6}x^3 + \frac{1}{2x}$ *from* $\left(2, \frac{19}{12}\right)$ *to* $\left(3, \frac{14}{3}\right)$.

Solution:

$$\begin{aligned}
\text{Arc Length} &= \int_2^3 \sqrt{1 + y'(x)^2}\, dx \\
&= \int_2^3 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2}\, dx \\
&= \int_2^3 \sqrt{1 + \left(\frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}\right)}\, dx \\
&= \int_2^3 \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4}x^{-4}}\, dx \\
&= \int_2^3 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2}\, dx \\
&= \int_2^3 \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)\, dx \\
&= \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_2^3 \\
&= \left(\frac{27}{6} - \frac{1}{6}\right) - \left(\frac{8}{6} - \frac{1}{4}\right) \\
&= 3 + \frac{1}{4} = \frac{13}{4}.
\end{aligned}$$

(b) $x = \frac{1}{9}e^{3y} + \frac{1}{4}e^{-3y}$ *from* $\left(\frac{13}{36}, 0\right)$ *to* $\left(\frac{265}{288}, \ln 2\right)$.

Learning outcomes:

Solution:

$$
\begin{aligned}
\text{Arc Length} &= \int_0^{\ln 2} \sqrt{1 + x'(y)^2}\, dy \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{3}e^{3y} - \frac{3}{4}e^{-3y}\right)^2}\, dy \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{9}e^{6y} - \frac{1}{2} + \frac{9}{16}e^{-6y}\right)}\, dy \\
&= \int_0^{\ln 2} \sqrt{\frac{1}{9}e^{6y} + \frac{1}{2} + \frac{9}{16}e^{-6y}}\, dy \\
&= \int_0^{\ln 2} \sqrt{\left(\frac{1}{3}e^{3y} + \frac{3}{4}e^{-3y}\right)^2}\, dy \\
&= \int_0^{\ln 2} \left(\frac{1}{3}e^{3y} + \frac{3}{4}e^{-3y}\right) dy \\
&= \left[\frac{1}{9}e^{3y} - \frac{1}{4}e^{-3y}\right]_0^{\ln 2} \\
&\overset{*}{=} \left(\frac{8}{9} - \frac{1}{32}\right) - \left(\frac{1}{9} - \frac{1}{4}\right) \\
&= \frac{7}{9} + \frac{7}{32} = \frac{224 + 63}{288} = \frac{287}{288}.
\end{aligned}
$$

* *Note that*

$$e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$$

and

$$e^{-3\ln 2} = e^{\ln 2^{-3}} = 2^{-3} = \frac{1}{8}.$$

Problem 2 *Find the surface area of the surface generated by revolving the curve given by*

(a) $y = \frac{1}{6}x^3 + \frac{1}{2x}$ *from* $\left(2, \frac{19}{12}\right)$ *to* $\left(3, \frac{14}{3}\right)$ *about the x-axis.*

Solution: *The formula for the surface area is*

$$\text{Surface Area} = \int_2^3 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

Since $y = f(x) = \frac{1}{6}x^3 + \frac{1}{2x}$, *we know that* $f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$. *Note* *that*

$$\begin{aligned}\sqrt{1+f'(x)^2} &= \sqrt{1+\left(\frac{1}{2}x^2-\frac{1}{2}x^{-2}\right)^2}\\ &= \sqrt{1+\left(\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}x^{-4}\right)}\\ &= \sqrt{\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4}x^{-4}}\\ &= \sqrt{\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)^2}\\ &= \left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)\end{aligned}$$

and so

$$\begin{aligned}\text{Surface Area} &= \int_2^3 2\pi\left(\frac{1}{6}x^3+\frac{1}{2}x^{-1}\right)\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)\,dx\\ &= 2\pi\int_2^3\left(\frac{1}{12}x^5+\frac{1}{12}x+\frac{1}{4}x+\frac{1}{4}x^{-3}\right)\,dx\\ &= 2\pi\int_2^3\left(\frac{1}{12}x^5+\frac{1}{3}x+\frac{1}{4}x^{-3}\right)\,dx\\ &= 2\pi\left[\frac{1}{72}x^6+\frac{1}{6}x^2-\frac{1}{8}x^{-2}\right]_2^3\\ &= 2\pi\left[\left(\frac{81}{8}+\frac{3}{2}-\frac{1}{72}\right)-\left(\frac{8}{9}+\frac{2}{3}-\frac{1}{32}\right)\right]\\ &= 2\pi\left(\frac{2916+432-4-256-192+9}{288}\right)\\ &= \frac{2905\pi}{144}.\end{aligned}$$

(b) $x=\frac{1}{9}e^{3y}+\frac{1}{4}e^{-3y}$ *from* $\left(\frac{13}{36},0\right)$ *to* $\left(\frac{265}{288},\ln 2\right)$ *about the y-axis.*

Solution: *The formula for the surface area is*

$$\text{Surface Area} = \int_0^{\ln 2} 2\pi f(y)\sqrt{1+f'(y)^2}\,dy.$$

Since $x=f(y)=\frac{1}{9}e^{3y}+\frac{1}{4}e^{-3y}$*, we know that* $f'(y)=\frac{1}{3}e^{3y}-\frac{3}{4}e^{-3y}$.

Note that

$$\begin{aligned}
\sqrt{1+f'(y)^2}\,dy &= \sqrt{1+\left(\frac{1}{3}e^{3y}-\frac{3}{4}e^{-3y}\right)^2} \\
&= \sqrt{1+\left(\frac{1}{9}e^{6y}-\frac{1}{2}+\frac{9}{16}e^{-6y}\right)} \\
&= \sqrt{\frac{1}{9}e^{6y}+\frac{1}{2}+\frac{9}{16}e^{-6y}} \\
&= \sqrt{\left(\frac{1}{3}e^{3y}+\frac{3}{4}e^{-3y}\right)^2} \\
&= \frac{1}{3}e^{3y}+\frac{3}{4}e^{-3y}
\end{aligned}$$

and so

$$\begin{aligned}
\textit{Surface Area} &= \int_0^{\ln 2} 2\pi\left(\frac{1}{9}e^{3y}+\frac{1}{4}e^{-3y}\right)\left(\frac{1}{3}e^{3y}+\frac{3}{4}e^{-3y}\right)dy \\
&= 2\pi\int_0^{\ln 2}\left(\frac{1}{27}e^{6y}+\frac{1}{6}+\frac{3}{16}e^{-6y}\right)dy \\
&= 2\pi\left[\frac{1}{162}e^{6y}+\frac{1}{6}y-\frac{1}{32}e^{-6y}\right]_0^{\ln 2} \\
&= 2\pi\left[\left(\frac{32}{81}+\frac{\ln 2}{6}-\frac{1}{2048}\right)-\left(\frac{1}{162}+0-\frac{1}{32}\right)\right] \\
&= \frac{\pi}{3}\left(\frac{9655}{3072}+\ln 2\right).
\end{aligned}$$