

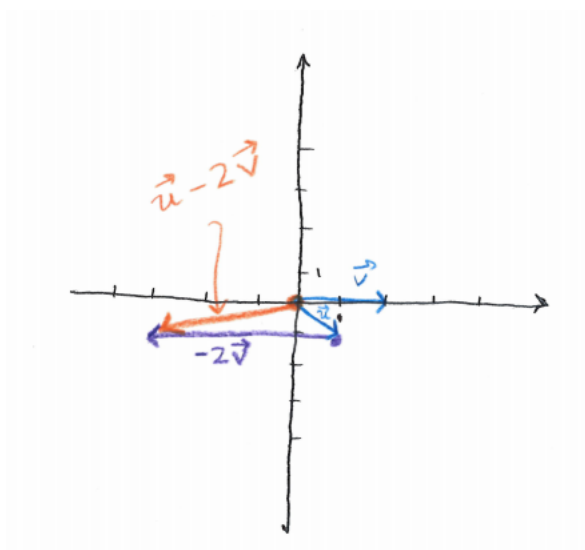
Recitation #25: Vectors in the plane - Full

Warm up:

- (a) What is the difference between the notations \hat{i} , \hat{j} , \hat{u} and \mathbf{i} , \mathbf{j} , \mathbf{u} ?
- (b) Sketch the vectors $\mathbf{u} = \langle 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0 \rangle$. Now using your sketch of these vectors, sketch $\mathbf{u} - 2\mathbf{v}$.

Solution: (a) There is no difference. The “hat” versions are typically written by hand, while the “bold” versions are typically in print.

(b) To add vectors, we put the tail of the second vector on the head of the first.



Instructor Notes: Reminds students that there are different equivalent vector notations. Also, students should understand vector addition geometrically.

Group work:

Problem 1 Suppose that $\mathbf{u} = \langle 5, -1 \rangle$ and $\mathbf{v} = \langle 2, 3 \rangle$. Find the following quantities:

- (a) $-\mathbf{v}$
- (b) $3\mathbf{u} - 4\mathbf{v}$
- (c) $|\mathbf{u}|$
- (d) $|\mathbf{u} - 2\mathbf{v}|$

Solution:

- (a) $-\mathbf{v} = \langle -2, -3 \rangle$
- (b) $3\mathbf{u} - 4\mathbf{v} = \langle 15, -4 \rangle - \langle 8, 12 \rangle = \langle 7, -16 \rangle$.
- (c) $|\mathbf{u}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$.
- (d) $|\mathbf{u} - 2\mathbf{v}| = |\langle 1, -7 \rangle| = \sqrt{1^2 + (-7)^2} = \sqrt{50} = 5\sqrt{2}$.

Instructor Notes: Simple vector operations

Problem 2 Suppose that $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$. Find the following:

- (a) A unit vector in the same direction of \mathbf{u} .
- (b) All unit vectors parallel to \mathbf{u} . (How does differ from part (a)?)
- (c) Two vector parallel to \mathbf{u} with length 10.

Solution:

- (a) $|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = 5$. A unit vector in the same direction is $\frac{\mathbf{u}}{|\mathbf{u}|} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$.

- (b) Parallel unit vectors are $\pm \frac{\mathbf{u}}{|\mathbf{u}|}$, which are $\langle \frac{3}{5}, \frac{-4}{5} \rangle$ and $\langle \frac{-3}{5}, \frac{4}{5} \rangle$. Note that parallel vectors include vectors in the opposite direction.
- (c) Since \mathbf{u} has length 5, two parallel vectors of length 10 are $\pm 2\mathbf{u}$, which are $\langle 6, -8 \rangle$ and $\langle -6, 8 \rangle$.

Instructor Notes: This question gauges whether they know the difference between vectors in the same direction and parallel vectors.

Problem 3 Assume that $\vec{u} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$ and $\vec{v} = \frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}$.

- (a) Show that \vec{u} and \vec{v} are unit vectors.
- (b) Write \hat{i} as $a_1\vec{u} + b_1\vec{v}$ for some real numbers a_1 and b_1 .
- (c) Write \hat{j} as $a_2\vec{u} + b_2\vec{v}$ for some real numbers a_2 and b_2 .

Solution: (a) First, to show that both vectors are unit vectors, we compute their magnitudes that they are equal to one.

$$|\vec{u}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$|\vec{v}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.$$

(b)

$$\begin{aligned}\hat{i} &= a_1\vec{u} + b_1\vec{v} \\ &= a_1\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) + b_1\left(\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}\right) \\ &= \left(\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}b_1\right)\hat{i} + \left(\frac{\sqrt{3}}{2}a_1 - \frac{1}{2}b_1\right)\hat{j}.\end{aligned}\tag{1}$$

Therefore, we must have that

$$\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}b_1 = 1 \quad \text{and} \quad \frac{\sqrt{3}}{2}a_1 - \frac{1}{2}b_1 = 0.$$

Solving the right-hand equation for b_1 we have that

$$b_1 = \sqrt{3}a_1.$$

Plugging this into the left-hand equation gives

$$\begin{aligned}1 &= \frac{1}{2}a_1 + \frac{\sqrt{3}}{2} \cdot \sqrt{3}a_1 \\ \implies 1 &= \frac{1}{2}a_1 + \frac{3}{2}a_1 = 2a_1 \\ \implies a_1 &= \frac{1}{2}.\end{aligned}$$

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So $b_1 = \frac{\sqrt{3}}{2}$, and therefore

$$\hat{i} = \frac{1}{2}\vec{u} + \frac{\sqrt{3}}{2}\vec{v}.$$

(c) In the same manner as in equation (1) we have that

$$\hat{j} = \left(\frac{1}{2}a_2 + \frac{\sqrt{3}}{2}b_2\right)\hat{i} + \left(\frac{\sqrt{3}}{2}a_2 - \frac{1}{2}b_2\right)\hat{j}$$

and so

$$\frac{1}{2}a_2 + \frac{\sqrt{3}}{2}b_2 = 0 \quad \text{and} \quad \frac{\sqrt{3}}{2}a_2 - \frac{1}{2}b_2 = 1.$$

Now, solving the left-hand equation for a_2 gives

$$a_2 = -\sqrt{3}b_2.$$

Then plugging into the right-hand equation yields

$$\begin{aligned} 1 &= \frac{\sqrt{3}}{2} \cdot (-\sqrt{3}b_2) - \frac{1}{2}b_2 \\ \implies 1 &= -\frac{3}{2}b_2 - \frac{1}{2}b_2 = -2b_2 \\ \implies b_2 &= -\frac{1}{2}. \end{aligned}$$

So $a_2 = \frac{\sqrt{3}}{2}$, and therefore

$$\hat{j} = \frac{\sqrt{3}}{2}\vec{u} - \frac{1}{2}\vec{v}$$

Instructor Notes: Students tend to have issues writing a vector as a linear combination of other vectors.
