Recitation #15: Infinite Series, Divergence and Integral Tests

Problem 1 *Suppose* $\{a_n\}_{n\geq 1}$ *is a sequence and* $\sum_{n=1}^{\infty} a_n$ *converges to* $L > 0$. *Let* $s_n = \sum_{k=1}^{n} a_k$. *Circle all of the statements that* MUST *be true.*

A. $\lim_{n\to\infty} a_n = L$ B. $\lim_{n\to\infty} a_n = 0$ C. $\lim_{n\to\infty} s_n = 0$

D. $\lim_{n\to\infty} s_n = L$ E. $\sum_{n=1}^{\infty} s_n$ *MUST diverge.* F. $\sum_{n=1}^{\infty}(a_n + 1) = L + 1$

G. *The divergence test tells us* $\sum_{n=1}^{\infty} a_n$ *converges to* L.

Solution: A. ***False***

Since $\{a_n\}_{n\geq 1} = L$, $\{a_n\}_{n\geq 1}$ *is a* convergent *series, so* $\lim_{n\to\infty} a_n = 0$. *Since* $L > 0$, *there is no way that* $\lim_{n\to\infty} a_n = L$.

B. ***True***

If $\lim_{n\to\infty} a_n \neq 0$, *the divergence test implies* $\sum_{n=1}^{\infty} a_n$ *diverges! Anytime a series* $\sum_{n=1}^{\infty} a_n$ *converges, it MUST be true that* $\lim_{n\to\infty} a_n = 0$.

C. ***False***

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{n=1}^{\infty} a_n = L > 0$$

D. ***True***

Some essential facts are:

– $\sum_{n=1}^{\infty} a_n$ *converges iff* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ *exists*

– *When* $\lim_{n\to\infty} s_n$ *does exist,* $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.

– *The series* $\sum_{n=1}^{\infty} a_n$ *likewise diverges iff the* $\lim_{n\to\infty} s_n$ *does not exist.*

Here, we are given $\sum_{n=1}^{\infty} a_n$ *converges to* $L > 0$, *which tells us immediately that* $\lim_{n\to\infty} s_n = L$.

E. ***True***

Since $\lim_{n\to\infty} s_n = L \neq 0$, *the divergence test tells us immediately that* $\sum_{n=1}^{\infty} s_n$ *MUST diverge.*

Learning outcomes:

F. **False**

Since $\sum_{n=1}^{\infty} a_n$ *converges,* $\lim_{n\to\infty} a_n = 0$*. Thus,* $\lim_{n\to\infty}(a_n+1) = 1$*, and the divergence test immediately tells us that* $\sum_{n=1}^{\infty}(a_n+1)$ *MUST diverge!*

G. **False**

The divergence test NEVER *can be used to conclude that a series converges!*

Problem 2 *For each of the following, answer* **True** *or* **False***, and explain why.*

(a) *If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\sum_{n=0}^{\infty}(a_n+0.001)$ *converges.*

(b) *Since* $\int_1^{\infty} x\sin(\pi x)\,dx$ *diverges then, by the Integral Test,* $\sum_{n=0}^{\infty} n\sin(\pi n)$ *diverges.*

(c) *Since* $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$ *then, by the Integral Test,* $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1$.

Solution: (a) **False**

Since $\sum_{n=0}^{\infty} a_n$ *converges, we know that* $\lim_{n\to\infty} a_n = 0$*. But then*

$$\lim_{n\to\infty}(a_n+0.0001) = 0.0001 \neq 0$$

and so $\sum_{n=0}^{\infty}(a_n+0.001)$ *diverges by the Divergence Test.*

(b) **False**

The Integral Test only holds for positive, decreasing functions. The function $f(x) = x\sin(\pi x)$ *is not always positive, nor is it always decreasing. So the Integral Test does not apply here.*

This problem is simpler than that though. Since $\sin(\pi n) = 0$ *for all integers* n*, we have that* $\sum_{n=0}^{\infty} n\sin(\pi n) = 0$.

(c) **False**

The Integral Test tells us that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, but it does* **not** *give us the sum (this sum is actually* $\frac{\pi^2}{6}$*).*

Problem 3 *Assume* $\sum_{k=0}^{\infty} a_k = L$ *and* $b_k = 8$ *for all* k.

(a) *What is* $\lim_{k\to\infty}(a_k + b_k)$*?*

(b) *What is* $\lim_{k\to\infty}\sum_{n=0}^{k}(a_n + b_n)$*?*

(c) *What is* $\lim_{k\to\infty}\sum_{n=0}^{k}(a_{n+1} - a_n)$*?*

Solution: (a) *Since* $\sum_{k=0}^{\infty} a_k$ *converges, we know that* $\lim_{k\to\infty} a_k = 0$*. Therefore,*

$$\lim_{k\to\infty}(a_k + b_k) = 0 + 8 = \boxed{8}.$$

(b) *Since* $\lim_{n\to\infty}(a_n+b_n) = 8$*, the series* $\sum_{n=0}^{\infty}(a_n+b_n)$ *diverges by the Divergence Test. But* $\lim_{k\to\infty}\sum_{n=0}^{k}(a_n+b_n) = \sum_{n=0}^{\infty}(a_n + b_n)$*. Thus*

$$\lim_{k\to\infty}\sum_{n=0}^{k}(a_n + b_n) = \sum_{n=0}^{\infty}(a_n + b_n) = \boxed{\infty}.$$

(c) *Let* $S_k = \sum_{n=0}^{k}(a_{n+1} - a_n)$ *(and recall that* $\{S_k\}$ *is the sequence of partial sums). Then*

$$\begin{aligned} S_k &= \sum_{n=0}^{k}(a_{n+1} - a_n) \\ &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_k - a_{k-1}) + (a_{k+1} - a_k) \\ &= a_{k+1} - a_0. \end{aligned}$$

Thus,

$$\lim_{k\to\infty}\sum_{n=0}^{k}(a_{n+1} - a_n) = \lim_{k\to\infty} S_k = \lim_{k\to\infty} a_{k+1} - a_0 = \boxed{-a_0}.$$

Problem 4 *Determine if the following series converge or diverge. If they converge, find the sum.*

(a) $\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k}$

Solution: *Let us analyze the two different summands in this problem:*

(i) $\sum_{k=0}^{99} 2^k$

This is a finite sum from a geometric sequence, and so its sum is

$$\frac{a(1-r^n)}{1-r}.$$

Thus,

$$\sum_{k=0}^{99} 2^k = \frac{1(1-2^{100})}{1-2} = 2^{100} - 1.$$

(ii) $\displaystyle\sum_{k=100}^{\infty} \frac{1}{2^k} = \sum_{k=100}^{\infty} \left(\frac{1}{2}\right)^k.$

This is a geometric series with $a = \dfrac{1}{2^{100}}$ *and* $r = \dfrac{1}{2}$. *So*

$$\sum_{k=100}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{100}}}{1-\frac{1}{2}} = \frac{1}{2^{99}}.$$

Therefore, combining parts (i) and (ii) we have that

$$\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k} = 2^{100} - 1 + \frac{1}{2^{99}}.$$

(b) $\displaystyle\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}}$

Solution: *Let us first reindex this series. Let* $\ell = k - 4$. *Then* $k = \ell + 4$, *and when* $k = 4$, $\ell = 0$. *We then have that*

$$\begin{aligned}
\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}} &= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{\ell+4+3}}{7^{\ell+4-2}} \\
&= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{\ell+7}}{7^{\ell+2}} \\
&= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{7} \cdot 4^{\ell}}{7^2 \cdot 7^{\ell}} \\
&= \frac{5 \cdot 4^7}{7^2} \sum_{\ell=0}^{\infty} \left(\frac{4}{7}\right)^{\ell} \qquad \textit{assuming this series converges} \\
&= \frac{5 \cdot 4^7}{7^2} \cdot \frac{1}{1-\frac{4}{7}} \qquad \textit{geometric series with } a = 1, r = \frac{4}{7} \\
&= \frac{5 \cdot 4^7}{3 \cdot 7}.
\end{aligned}$$

Therefore, this series converges to $\dfrac{5 \cdot 4^7}{3 \cdot 7}$.

(c) $\displaystyle\sum_{k=0}^{\infty} e^{5-2k}$

Solution:

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{5-2k} &= \sum_{k=0}^{\infty} \left[e^5 \cdot \left(e^{-2}\right)^k\right] \\
&= e^5 \sum_{k=0}^{\infty} \left(e^{-2}\right)^k \qquad \textit{assuming the series converges} \\
&= e^5 \cdot \frac{1}{1-e^{-2}} \qquad \textit{geometric series with } a = 1, r = e^{-2} < 1
\end{aligned}$$

Therefore, this series converges to $\dfrac{e^5}{1-e^{-2}}$.

(d) $\displaystyle\sum_{i=1}^{\infty}\left(\frac{2}{i^2+2i}\right)$ *Hint:* $\dfrac{2}{i^2+2i}=\dfrac{1}{i}-\dfrac{1}{i+2}$ *by partial fractions*

Solution: *This is a telescoping series. Let*

$$S_n=\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+2}\right).$$

Then,

$$\begin{aligned}S_n&=\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+2}\right)\\&=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\\&+\ldots+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)\\&=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}.\end{aligned}$$

Note that the last equality above is because all of the non-red terms cancel (convince yourself of this). Then

$$\begin{aligned}\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+2}\right)&=\lim_{n\to\infty}S_n\\&=\lim_{n\to\infty}\left[1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]\\&=1+\frac{1}{2}=\frac{3}{2}.\end{aligned}$$

Problem 5 *Determine if the following series converge or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2+2n+1}{3n^2+1}$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

Solution: (a) ***Divergence Test***

Notice that

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^2+2n+1}{3n^2+1}=\frac{1}{3}.$$

Therefore, since $\lim_{n\to\infty}a_n\neq 0$, *by the Divergence Test this series* $\boxed{\textit{diverges}}$.

(b) ***Integral Test***

First, notice that $f(x) = \dfrac{1}{x(\ln x)^2}$ *is a decreasing and positive function on* $[2, \infty)$. *Then*

$$\begin{aligned}
\int_2^\infty f(x)\,dx &= \int_2^\infty \frac{1}{x(\ln x)^2}\,dx \\
&= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx \\
&= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-2}\,du \quad {\color{red} u = \ln x,\ du = \frac{1}{x}\,dx} \\
&= \lim_{b\to\infty} \left[\frac{-1}{u}\right]_{\ln 2}^{\ln b} \\
&= \lim_{b\to\infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln 2}\right) \\
&= 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}.
\end{aligned}$$

Therefore, since the above integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ $\boxed{\textit{converges}}$ *by the Integral Test.*

Problem 6 *For a sequence* $\{a_n\}_{n\geq 1}$ *let* $s_n = \displaystyle\sum_{k=1}^{n} a_k$ *denote its sequence of partial sums. Now, suppose that* $\{a_n\}_{n\geq 1}$ *is a sequence such that* $s_n = \dfrac{4n^2+9}{1-2n}$.

(a) *Find* $a_1 + a_2 + a_3$.

(b) *Find* $a_8 + a_9 + a_{10}$.

(c) *Determine whether* $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.*

(d) *Determine whether* $\displaystyle\sum_{k=1}^{\infty} s_k$ *converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.*

Solution: (a) *Note by definition that* $a_1 + a_2 + a_3 = s_3$. *Using the formula given for* s_n *with* $n = 3$ *gives:*

$$a_1 + a_2 + a_3 = \frac{4(3)^2+9}{1-2(3)} = \boxed{-9}.$$

(b) *Note that by definition:*

$$\begin{aligned}
s_{10} &= a_1 + \cdots + a_7 + a_8 + a_9 + a_{10} \\
s_7 &= a_1 + \cdots + a_7
\end{aligned}$$

so $a_8 + a_9 + a_{10} = s_{10} - s_7$. *Using the formula for* s_n, *we have:*

$$s_10 = \frac{4(10)^2+9}{1-2(10)} = -\frac{409}{19}, \qquad s_7 = \frac{4(7)^2+9}{1-2(7)} = -\frac{205}{13}$$

Thus, $\boxed{a_8 + a_9 + a_{10} = -\frac{409}{19} + \frac{205}{13}}$.

(c) *To determine this, we note that:*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{4n^2+9}{1-2n} = -\infty.$$

Since $\lim_{n\to\infty} s_n$ *does not exist,* $\boxed{\sum_{k=1}^{\infty} a_k \text{ diverges by the Divergence Test}}$.

(d) *We showed that* $\lim_{n\to\infty} s_n = -\infty$*, so* $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by the Divergence Test}}$.