

## Recitation # 24: Calculus in polar coordinates

### Warm up:

- (a) True or False: The slope of the tangent line to the curve  $r = f(\theta)$  at the point  $(r_0, \theta_0)$  is given by  $f'(\theta_0)$ .
- (b) True or False: The area enclosed by the curve  $r = 2 \cos(\theta)$  is

$$\int_0^{2\pi} \frac{1}{2} (2 \cos(\theta))^2 d\theta = \int_0^{2\pi} 1 - \cos(2\theta) d\theta = 2\pi.$$

**Solution:** (a) **False.** The slope of the tangent line to the curve  $r = f(\theta)$  at  $(r_0, \theta_0)$  is given by

$$\frac{dy}{dx} = \frac{f'(\theta_0) \sin \theta_0 + f(\theta_0) \cos \theta_0}{f'(\theta_0) \cos \theta_0 - f(\theta_0) \sin \theta_0} \neq f'(\theta_0).$$

- (b) **False.** The curve  $r = \cos(\theta)$  is a circle of radius 1 centered at  $(1, 0)$ . The curve traces out the circle **twice** for  $\theta$  in  $[0, 2\pi]$ . So the area enclosed is just  $\int_0^\pi \frac{1}{2} (2 \cos(\theta))^2 d\theta = \pi$ .

**Instructor Notes:** The point of *b* is for the students to realize that you need to think about the curve before blindly using a formula.

---

### Group work:

**Problem 1** Find the equation of the tangent line to  $r = 2 - \sin \theta$  at  $\theta = \frac{\pi}{3}$ . Also, determine for what values of  $\theta$  the tangent lines to the curve are vertical or horizontal.

---

Learning outcomes:

Recitation # 24: Calculus in polar coordinates

**Solution:** We use the formula from the warm-up to find  $\frac{dy}{dx}$  when  $f(\theta) = 2 - \sin \theta$ :

$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} \\ &= \frac{(-\cos \theta)(\sin \theta) + (2 - \sin \theta)(\cos \theta)}{(-\cos \theta)(\cos \theta) - (2 - \sin \theta)(\sin \theta)}.\end{aligned}$$

So

$$\begin{aligned}\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} &= \frac{(-\cos \frac{\pi}{3})(\sin \frac{\pi}{3}) + (2 - \sin \frac{\pi}{3})(\cos \frac{\pi}{3})}{(-\cos \frac{\pi}{3})(\cos \frac{\pi}{3}) - (2 - \sin \frac{\pi}{3})(\sin \frac{\pi}{3})} \\ &= \frac{\left(-\frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) + \left(2 - \frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right)}{\left(-\frac{1}{2} \cdot \frac{1}{2}\right) - \left(2 - \frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right)} \\ &= \frac{-\sqrt{3} + 4 - \sqrt{3}}{-1 - 4\sqrt{3} + 3} \\ &= \frac{4 - 2\sqrt{3}}{2 - 4\sqrt{3}} \\ &= \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}.\end{aligned}$$

Also, when  $\theta = \frac{\pi}{3}$ ,  $r = 2 - \frac{\sqrt{3}}{2} = \frac{4 - \sqrt{3}}{2}$ . Therefore

$$\begin{aligned}x &= r \cos \theta = \frac{4 - \sqrt{3}}{2} \cdot \frac{1}{2} = 1 - \frac{\sqrt{3}}{4} \\ y &= r \sin \theta = \frac{4 - \sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{3}{4}.\end{aligned}$$

Thus, the equation of the tangent line when  $\theta = \frac{\pi}{3}$  is

$$\boxed{y - \left(\sqrt{3} - \frac{3}{4}\right) = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}} \left(x - \left(1 - \frac{\sqrt{3}}{4}\right)\right)}$$

To find all vertical and horizontal tangent lines, we need to find where the numerator and denominator of  $\frac{dy}{dx}$  are equal to 0.

Recitation # 24: Calculus in polar coordinates

**Numerator:**

$$\begin{aligned}
 (-\cos \theta)(\sin \theta) + (2 - \sin \theta)(\cos \theta) &= 0 \\
 -\sin \theta \cos \theta + 2 \cos \theta - \sin \theta \cos \theta &= 0 \\
 2 \cos \theta - 2 \sin \theta \cos \theta &= 0 \\
 2 \cos \theta(1 - \sin \theta) &= 0 \\
 \cos \theta = 0 \quad \text{or} \quad \sin \theta = 1 \\
 \theta = \frac{\pi}{2} + k\pi \quad \text{or} \quad \theta = \frac{\pi}{2} + 2k\pi \quad \text{for } k \text{ an integer} \\
 \theta = \frac{\pi}{2} + k\pi \quad \text{for } k \text{ an integer}
 \end{aligned}$$

**Denominator:**

$$\begin{aligned}
 (-\cos \theta)(\cos \theta) - (2 - \sin \theta)(\sin \theta) &= 0 \\
 -\cos^2 \theta - 2 \sin \theta + \sin^2 \theta &= 0 \\
 -1 + \sin^2 \theta - 2 \sin \theta + \sin^2 \theta &= 0 \\
 2 \sin^2 \theta - 2 \sin \theta - 1 &= 0 \\
 \sin \theta = \frac{2 \pm \sqrt{4 + 8}}{4} \quad \text{throw away the “+”} \\
 \sin \theta = \frac{1 - \sqrt{3}}{2} \quad \text{since it is out of the range of sin} \\
 \theta = \sin^{-1} \left( \frac{1 - \sqrt{3}}{2} \right) + 2k\pi \quad \text{for } k \text{ an integer}
 \end{aligned}$$

Note that  $\sin \theta = \frac{1 - \sqrt{3}}{2}$  twice during a period. The other one occurs at  $\pi - \sin^{-1} \left( \frac{1 - \sqrt{3}}{2} \right)$ .

Then, since these two collections of angles are disjoint, the horizontal tangent lines occur when

$$\boxed{\theta = \frac{\pi}{2} + k\pi} \quad \text{for } k \text{ an integer}$$

and the vertical tangent lines occur when

$$\boxed{\theta = \sin^{-1} \left( \frac{1 - \sqrt{3}}{2} \right) + 2k\pi, \theta = \pi - \sin^{-1} \left( \frac{1 - \sqrt{3}}{2} \right) + 2k\pi} \quad \text{for } k \text{ an integer}$$

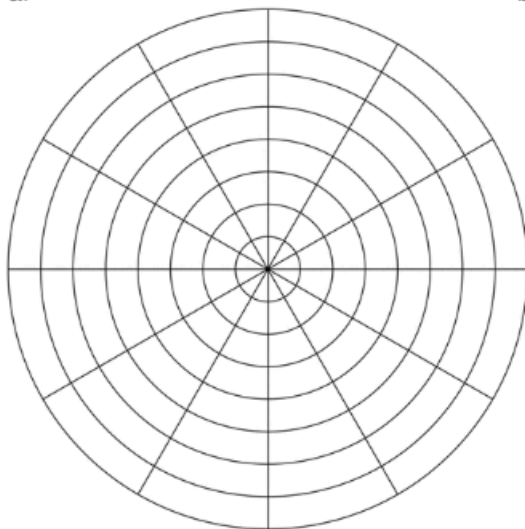
**Instructor Notes:** They probably haven't seen how to find the other angle for which  $\sin \theta = \frac{1 - \sqrt{3}}{2}$  since Pre-Calculus.

**Problem 2** Graph each region and then SET UP an integral for the area of the region:

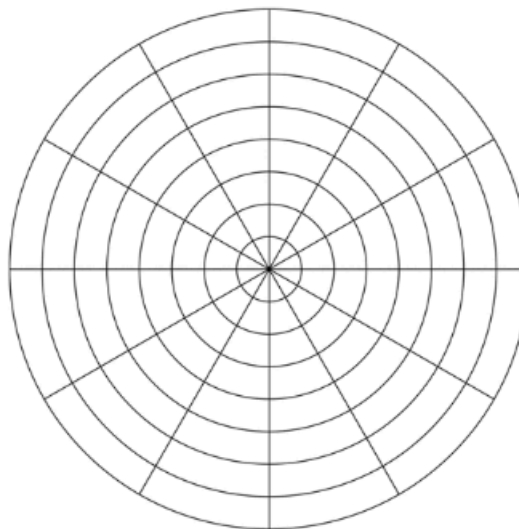
- (a) Outside the small loop and inside the large loop of  $r = 3 - 6 \sin \theta$ .
- (b) Inside both of the curves  $r = 4 \cos \theta$  and  $r = 1 - \cos \theta$ .

Note that you do not need to evaluate these integrals.

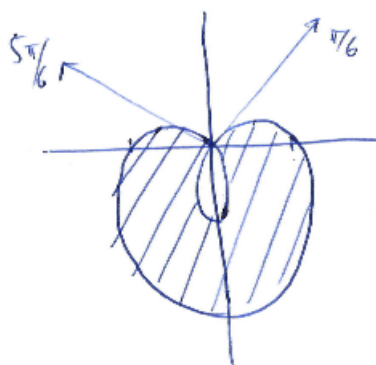
a.



b.



**Solution:** (a) The graph of the region is given below.



Recitation # 24: Calculus in polar coordinates

To find the area inside the smaller loop, we need to first find the values of  $\theta$  for which  $r = 0$ . So we compute

$$\begin{aligned} 3 - 6 \sin \theta &= 0 \\ \implies \sin \theta &= \frac{1}{2} \\ \implies \theta &= \frac{\pi}{6}, \frac{5\pi}{6}. \end{aligned}$$

Hence, the area of the smaller loop is

$$\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 6 \sin \theta)^2 d\theta.$$

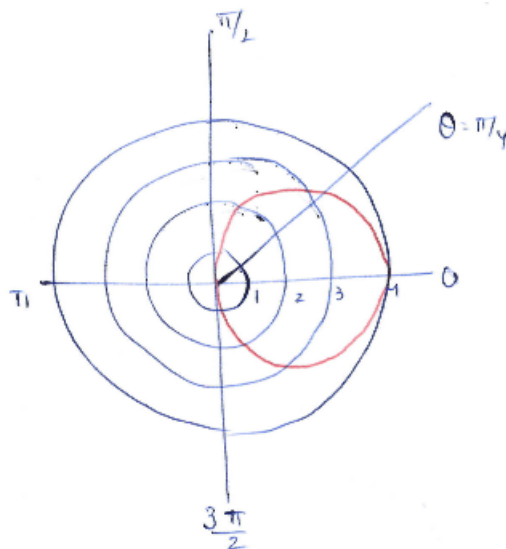
To find the area of the outer loop, we just integrate over the other values of  $\theta$ :

$$\frac{1}{2} \int_0^{\frac{\pi}{6}} (3 - 6 \sin \theta)^2 d\theta + \frac{1}{2} \int_{\frac{5\pi}{6}}^{2\pi} (3 - 6 \sin \theta)^2 d\theta.$$

Therefore, the area of the region inside the outer loop and outside the inner loop is

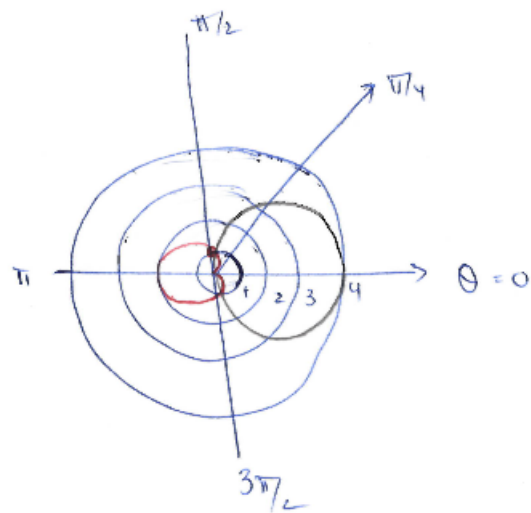
$$\frac{1}{2} \left[ \int_0^{\frac{\pi}{6}} (3 - 6 \sin \theta)^2 d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 6 \sin \theta)^2 d\theta + \int_{\frac{5\pi}{6}}^{2\pi} (3 - 6 \sin \theta)^2 d\theta \right]$$

(b) We first graph  $r = 4 \cos \theta$ .

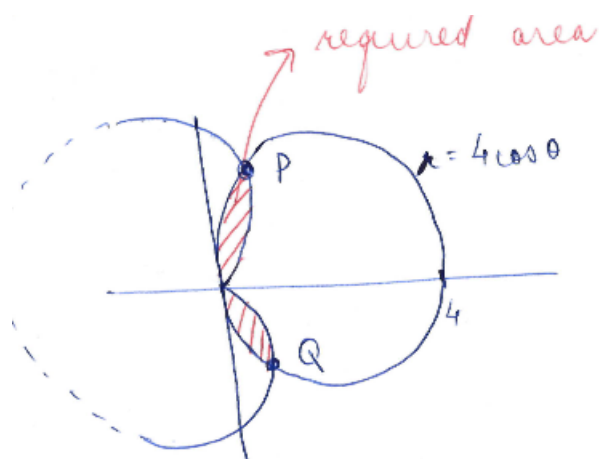


Recitation # 24: Calculus in polar coordinates

Now we graph  $r = 1 - \cos \theta$



Here are the two graphs in the same picture



To find the points where the two curves intersect, we solve

$$4 \cos \theta = 1 - \cos \theta$$

$$5 \cos \theta = 1$$

$$\cos \theta = \frac{1}{5}$$

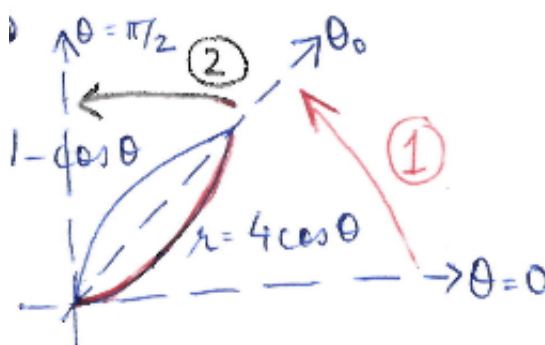
$$\theta = \pm \cos^{-1} \left( \frac{1}{5} \right) := \theta_0.$$

Recitation # 24: Calculus in polar coordinates

Using the symmetry of the graph we find the area of one leaf and then double it. Therefore, the area between the curves is

$$2 \left[ \frac{1}{2} \int_0^{\theta_0} (1 - \cos \theta)^2 d\theta + \frac{1}{2} \int_{\theta_0}^{\frac{\pi}{2}} (4 \cos \theta)^2 d\theta \right]$$

In the following picture, the left-hand integral is (1) and the right-hand integral is (2).



**Instructor Notes:** If you have time at the end, you can go over how you would evaluate these integrals.