Recitation 23: Parametric equations & Polar coordinates

Warm up:

Describe the motion given by $x = 8$, $y = 7\sin(t)$ for all t.

Solution: The parametric curve keeps oscillating up and down the vertical line segment between the points $(8, -7)$ and $(8, 7)$. This is because $x = 8$ is fixed, and $y = 7\sin(t)$ oscillates between -7 and 7 as t varies.

Group work:

Problem 1 *Try to figure out the shape of the following curve and then eliminate the parameter and check your intuition.*

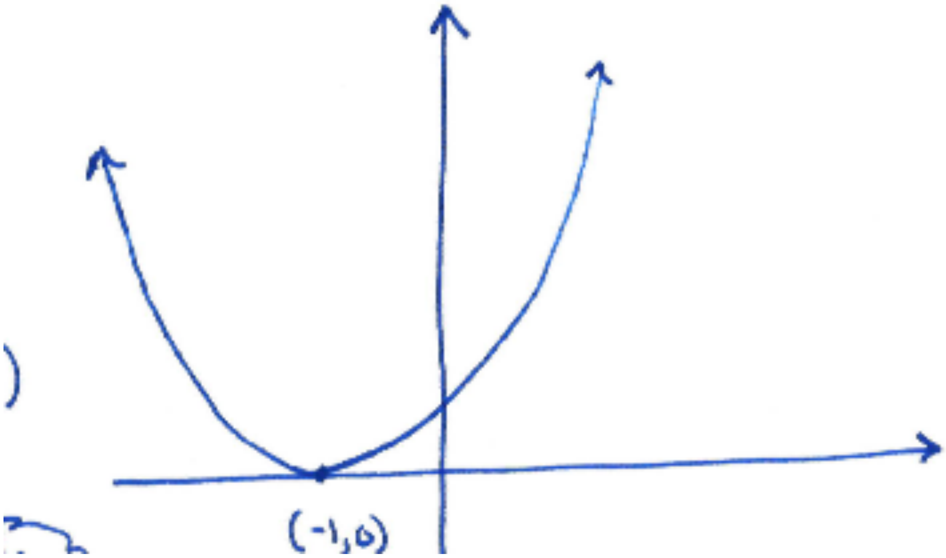
$$x = \ln t - 1 \qquad y = (\ln t)^2$$

Solution: *First, note that these two functions are both only defined when $0 < t < \infty$. Also, we see that $\ln t = x + 1$, and so*

$$y = (\ln t)^2 = (x + 1)^2.$$

So this graph is a parabola that opens up and has vertex $(-1, 0)$.

Learning outcomes:



Problem 2 *Find parametric equations for the path of a particle moving around the circle*

$$(x-3)^2 + (y+7)^2 = 4$$

(a) *one time around clockwise starting at* $(5,-7)$.

(b) *three times around counterclockwise starting at* $(5,-7)$.

(c) *halfway around clockwise starting at* $(1,-7)$.

Solution: *First, notice that this is the equation of the circle with radius* 2 *centered at* $(3,-7)$.

(a) *The point* $(5,-7)$ *is the "right-most" point on the circle. In order to parameterize the circle one time around going counter-clockwise we use the parameterization*

$$x = 3 + 2\cos t \qquad y = -7 + 2\sin t \qquad 0 \le t < 2\pi.$$

In order to traverse the circle one time clockwise, we just negate the t *in the above parameterization. So we get*

$$\boxed{x = 3 + 2\cos(-t) \qquad y = -7 + 2\sin(-t) \qquad 0 \le t < 2\pi}.$$

Remark: Since cos *is an even function and* sin *is an odd function, this solution is equivalent to*

$$x = 3 + 2\cos t \qquad y = -7 - 2\sin t \qquad 0 \le t < 2\pi$$

(b) *To traverse the circle three times counter-clockwise, we just triple the domain of t from the parameterization above. So we have that*

$$\boxed{x = 3 + 2\cos t \qquad y = -7 + 2\sin t \qquad 0 \le t < 6\pi}.$$

(c) *Note that this problem starts at $(1, -7)$, not $(5, -7)$. So this parameterization begins at the "left-most" point of the circle. Therefore, this is just the "second half" of the answer to part (a). So a parameterization for this problem is*

$$\boxed{x = 3 + 2\cos(-t) \qquad y = -7 + 2\sin(-t) \qquad \pi \le t < 2\pi}.$$

Problem 3 *Find the intersection point(s) of the lines*

$$x = -6 + 9t, \qquad y = 3 - 2t \tag{1}$$

and

$$x = 3 + t, \qquad y = -4 - 2t. \tag{2}$$

Do they intersect at the same time?

Solution: *Line 1 is the line with slope $-\frac{2}{9}$ passing through $(-6, 3)$. So it has equation*

$$\begin{aligned} & y - 3 = -\frac{2}{9}(x - (-6)) \\ \implies \quad & y = -\frac{2}{9}x - \frac{4}{3} + 3 = -\frac{2}{9}x + \frac{5}{3}. \end{aligned}$$

Line 2 is the line with slope -2 passing through $(3, -4)$. So it has equation

$$\begin{aligned} & y + 4 = -2(x - 3) \\ \implies \quad & y = -2x + 2. \end{aligned}$$

Since these lines have different slopes, they intersect in a single point. To find this point, we set the equations equal to each other:

$$\begin{aligned} & -\frac{2}{9}x + \frac{5}{3} = -2x + 2 \\ \implies \quad & \frac{16}{9}x = \frac{1}{3} \\ \implies \quad & x = \frac{3}{16} \\ \implies \quad & y = -2\left(\frac{3}{16}\right) + 2 = \frac{13}{8}. \end{aligned}$$

Therefore, the intersection point is

$$\boxed{\left(\frac{3}{16}, \frac{13}{8}\right)}.$$

To see if they intersect in the same point, let us first find the t value which makes the x-coordinate of line 1 equal to $\frac{3}{16}$.

$$\begin{aligned} &\frac{3}{16} = -6 + 9t \\ \implies \qquad &t = \frac{1}{9}\left(\frac{3}{16} + 6\right) = \frac{11}{16}. \end{aligned}$$

So now, we plug this into the equation for $x(t)$ for line 2 and see if we get $\frac{3}{16}$.

$$\begin{aligned} x\left(\frac{11}{16}\right) &= 3 + \frac{11}{16} \\ &= \frac{59}{16} \neq \frac{3}{16}. \end{aligned}$$

Therefore, these lines do **not** *intersect at the same time.*

Problem 4 *Consider the curve defined by the parameterization $x = t^2$, $y = t^3 - 3t$. Show that this curve has two tangent lines at $(3, 0)$, and find the equations of the tangent lines there.*

Solution: *First, recall that*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Then, since $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3$, we have that

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}.$$

Now,

$$\begin{aligned} &x(t) = 3 \\ \iff \qquad &t^2 = 3 \\ \iff \qquad &t = \pm\sqrt{3}. \end{aligned}$$

Also, notice that both $y(\sqrt{3}) = 0 = y(-\sqrt{3})$. So the given parametric curve intersects the point $(3,0)$ at two times, when $t = \pm\sqrt{3}$. At these two times, the tangent lines have slopes

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{(3)(3)-3}{2\sqrt{3}} = \sqrt{3} \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{(3)(3)-3}{-2\sqrt{3}} = -\sqrt{3}.$$

So the equations of the tangent lines are

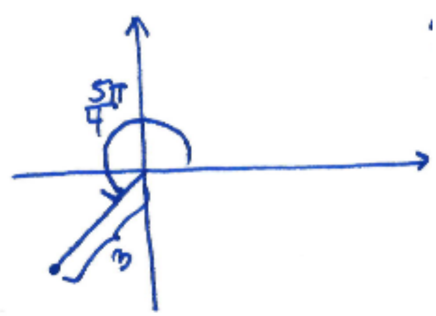
$$\boxed{y = \sqrt{3}(x-3)}$$
$$\boxed{y = -\sqrt{3}(x-3)}.$$

Problem 5 *Plot the following (polar) points in the xy-plane and then rewrite them as rectangular coordinates.*

(a) $\left(3, \frac{5\pi}{4}\right)$ (b) $\left(3, -\frac{5\pi}{4}\right)$ (c) $\left(-3, \frac{5\pi}{4}\right)$ (d) $\left(-3, -\frac{5\pi}{4}\right)$

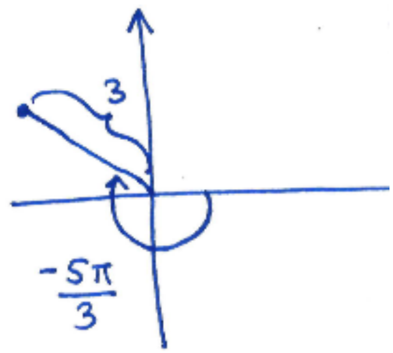
Solution: (a)

$$x = 3\cos\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$
$$y = 3\sin\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$



(b)

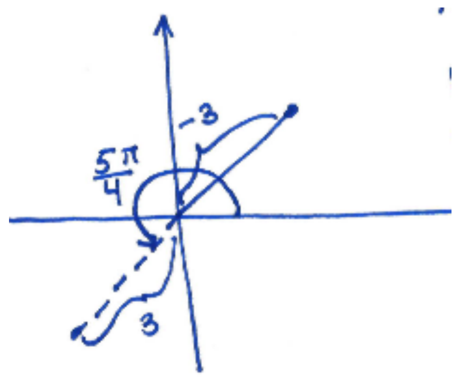
$$x = 3\cos\left(-\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$
$$y = 3\sin\left(-\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$



(c)

$$x = -3\cos\left(\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

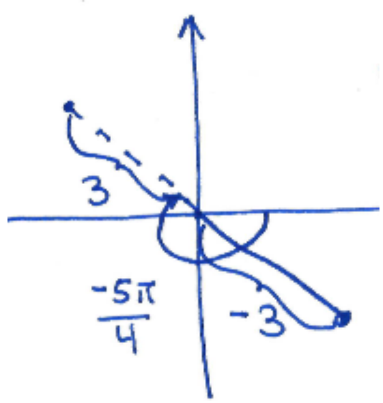
$$y = -3\sin\left(\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$



(d)

$$x = -3\cos\left(-\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

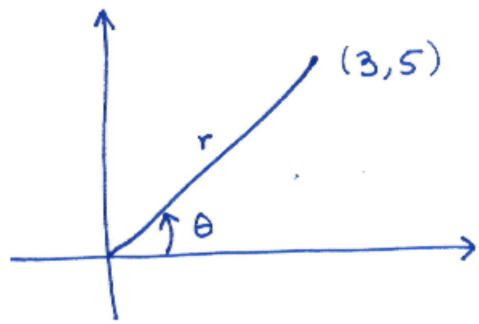
$$y = -3\sin\left(-\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$



Problem 6 *Rewrite the rectangular point $(3, 5)$ in polar coordinates in three different ways.*

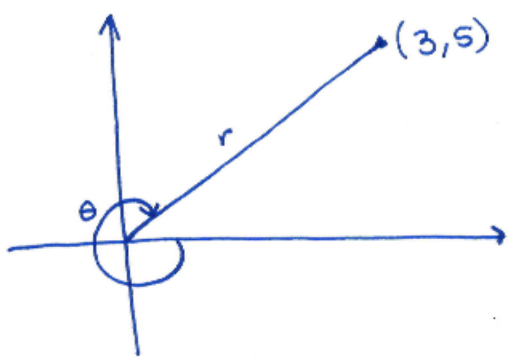
Solution: *i.*

$$r = \sqrt{3^2 + 5^2} = \sqrt{34} \qquad \theta = \arctan\left(\frac{5}{3}\right)$$



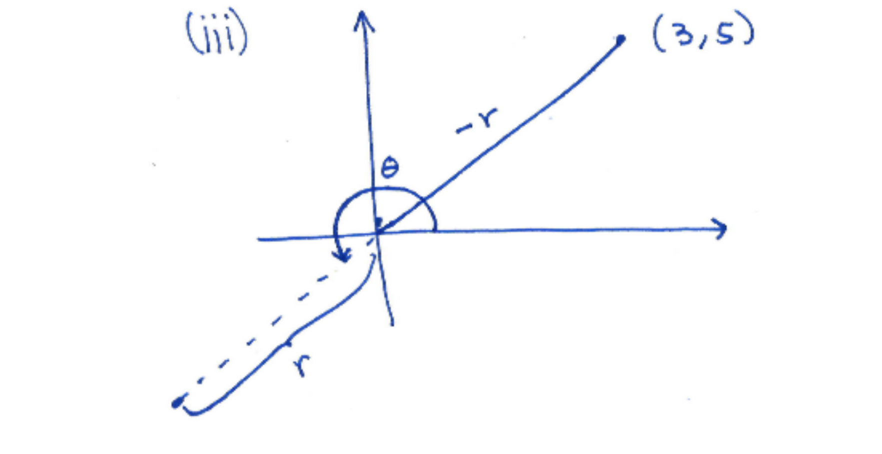
ii.

$$r = \sqrt{34} \qquad \theta = \arctan\left(\frac{5}{3}\right) - 2\pi$$

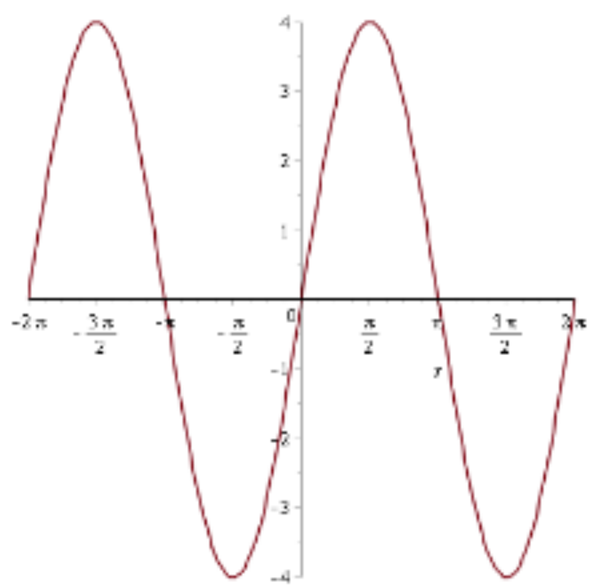


iii.

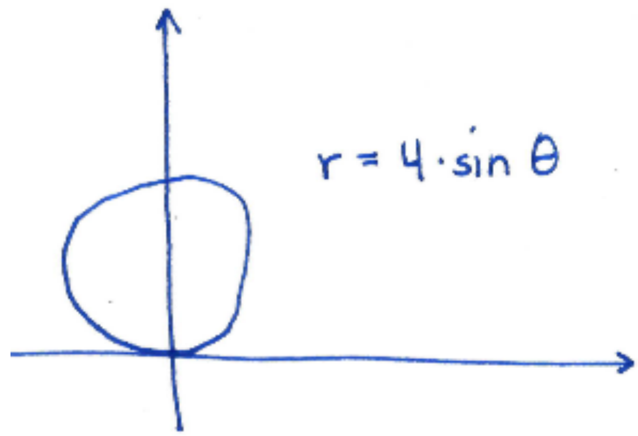
$$r = -\sqrt{34} \qquad \theta = \arctan\left(\frac{5}{3}\right) + \pi$$



Problem 7 *The graph of the curve $r = 4\sin\theta$ is a circle. Use the graph below to sketch this circle. Can you verify this algebraically? What is the period of the polar curve? Is $0 \le \theta \le 2\pi$ necessary to complete the graph?*



Solution: *Graphing this equation in the picture below, we see that this is a circle with radius 2 and center $(0, 2)$.*



To verify this algebraically,

$$\begin{aligned} 4 = 4\sin\theta \quad &\Longrightarrow \quad r^2 = 4r\sin\theta \\ &\Longrightarrow \quad x^2 + y^2 = 4y \\ &\Longrightarrow \quad x^2 + y^2 - 4y = 0 \\ &\Longrightarrow \quad x^2 + y^2 - 4y + 4 = 4 \\ &\Longrightarrow \quad x^2 + (y-2)^2 = 2^2. \end{aligned}$$

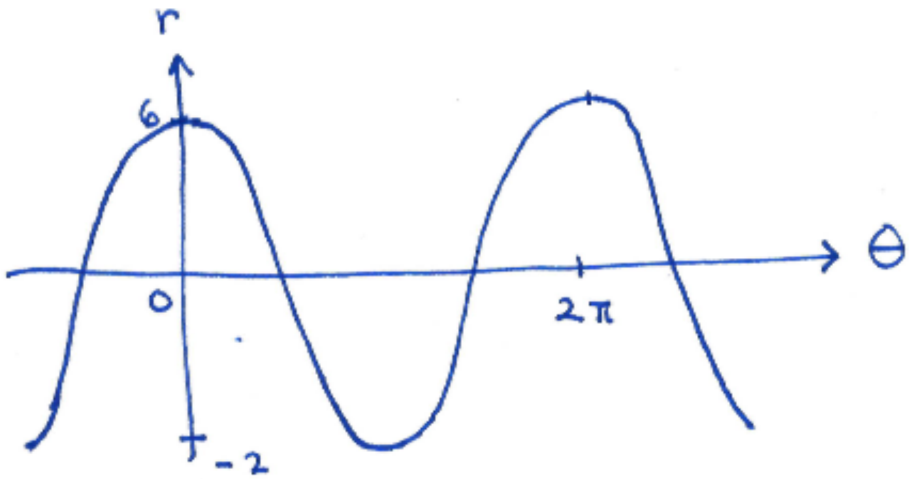
To find the period of the polar curve, we convert both x and y into parametric equations with parameter θ.

$$\begin{aligned} x &= r\cos\theta = 4\sin\theta\cos\theta = 2\sin(2\theta) \\ y &= r\sin\theta = 4\sin^2\theta = 4\cdot\frac{1}{2}(1-\cos(2\theta)) = 2(1-\cos(2\theta)). \end{aligned}$$

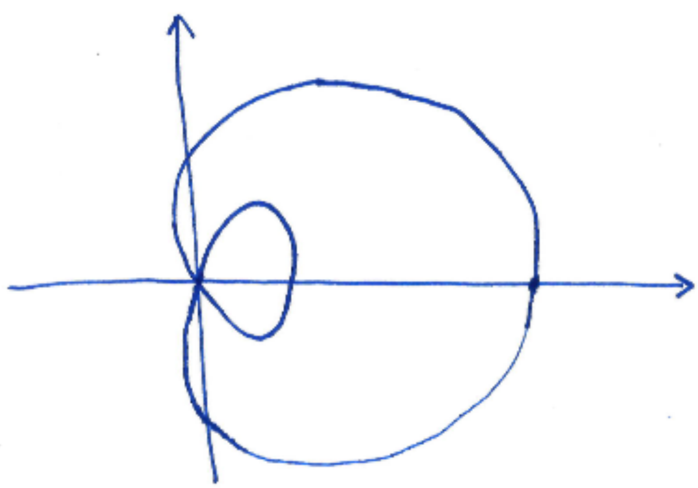
The period of both $\sin(2\theta)$ and $\cos(2\theta)$ is π. So the entire graph of the equation $r = 4\sin\theta$ is traversed over the region $0 \le \theta < \pi$.

Problem 8 *Graph $r = 2 + 4\cos\theta$ using the "Cartesian-to-Polar" method.*

Solution: *First, we graph $r = 2 + 4\cos\theta$ as if r and θ were Cartesian coordinates.*



We then use this to draw the following graph in the xy-plane



If it helps, here are the parametric equations for this graph

$$x = r\cos\theta = (2 + 4\cos\theta)\cos\theta = 2\cos\theta + 4\cos^2\theta = 2(\cos\theta + 1 + \cos(2\theta))$$
$$y = 4\sin\theta = (2 + 4\cos\theta)\sin\theta = 2\sin\theta + 2\sin(2\theta).$$