# Section 11.3: Calculus in Polar Coordinates

## Warm up:

(a) True or False: The slope of the tangent line to the curve $r = f(\theta)$ at the point $(r_0, \theta_0)$ is given by $f'(\theta_0)$.

(b) True or False: The area enclosed by the curve $r = 2\cos(\theta)$ is

$$\int_0^{2\pi} \frac{1}{2}(2\cos(\theta))^2 d\theta = \int_0^{2\pi} 1 - \cos(2\theta) d\theta = 2\pi.$$

**Solution:** (a) **False.** The slope of the tangent line to the curve $r = f(\theta)$ at $(r_0, \theta_0)$ is given by

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0} \neq f'(\theta_0).$$

(b) **False.** The curve $r = \cos(\theta)$ is a circle of radius 1 centered at (1,0). The curve traces out the circle twice for $\theta$ in $[0, 2\pi]$. So the area enclosed is just $\int_0^{\pi} \frac{1}{2}(2\cos(\theta))^2 d\theta = \pi$.

## Group work:

**Problem 1** *Find the equation of the tangent line to $r = 2 - \sin\theta$ at $\theta = \frac{\pi}{3}$. Also, determine for what values of $\theta$ the tangent lines to the curve are vertical or horizontal. Find the equations of the horizontal and vertical tangent lines.*

***Solution:*** *We use the formula from the warm-up to find $\frac{dy}{dx}$ when $f(\theta) = 2 - \sin\theta$:*

$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \\ &= \frac{(-\cos\theta)(\sin\theta) + (2 - \sin\theta)(\cos\theta)}{(-\cos\theta)(\cos\theta) - (2 - \sin\theta)(\sin\theta)}.\end{aligned}$$

Learning outcomes:

*So*

$$\begin{aligned}
\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} &= \frac{(-\cos\frac{\pi}{3})(\sin\frac{\pi}{3}) + (2-\sin\frac{\pi}{3})(\cos\frac{\pi}{3})}{(-\cos\frac{\pi}{3})(\cos\frac{\pi}{3}) - (2-\sin\frac{\pi}{3})(\sin\frac{\pi}{3})} \\
&= \frac{\left(-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) + \left(2-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\cdot\frac{1}{2}\right) - \left(2-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} \\
&= \frac{-\sqrt{3}+4-\sqrt{3}}{-1-4\sqrt{3}+3} \\
&= \frac{4-2\sqrt{3}}{2-4\sqrt{3}} \\
&= \frac{2-\sqrt{3}}{1-2\sqrt{3}}.
\end{aligned}$$

*Also, when* $\theta = \frac{\pi}{3}$, $r = 2 - \frac{\sqrt{3}}{2} = \frac{4-\sqrt{3}}{2}$. *Therefore*

$$\begin{aligned}
x = r\cos\theta &= \frac{4-\sqrt{3}}{2}\cdot\frac{1}{2} = 1 - \frac{\sqrt{3}}{4} \\
y = r\sin\theta &= \frac{4-\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{3}{4}.
\end{aligned}$$

*Thus, the equation of the tangent line when* $\theta = \frac{\pi}{3}$ *is*

$$\boxed{y - \left(\sqrt{3} - \frac{3}{4}\right) = \frac{2-\sqrt{3}}{1-2\sqrt{3}}\left(x - \left(1 - \frac{\sqrt{3}}{4}\right)\right)}$$

*To find all vertical and horizontal tangent lines, we need to find where the numerator and denominator of* $\frac{dy}{dx}$ *are equal to* 0.

**Numerator:**

$$\begin{aligned}
(-\cos\theta)(\sin\theta) + (2-\sin\theta)(\cos\theta) &= 0 \\
-\sin\theta\cos\theta + 2\cos\theta - \sin\theta\cos\theta &= 0 \\
2\cos\theta - 2\sin\theta\cos\theta &= 0 \\
2\cos\theta(1-\sin\theta) &= 0 \\
\cos\theta = 0 \quad \textit{or} \quad \sin\theta &= 1 \\
\theta = \frac{\pi}{2} + k\pi \quad \textit{or} \quad \theta &= \frac{\pi}{2} + 2k\pi \quad \textit{for } k \textit{ an integer} \\
\theta &= \frac{\pi}{2} + k\pi \quad \textit{for } k \textit{ an integer}
\end{aligned}$$

**Denominator:**

$$\begin{aligned}
(-\cos\theta)(\cos\theta) - (2-\sin\theta)(\sin\theta) &= 0 \\
-\cos^2\theta - 2\sin\theta + \sin^2\theta &= 0 \\
-1 + \sin^2\theta - 2\sin\theta + \sin^2\theta &= 0 \\
2\sin^2\theta - 2\sin\theta - 1 &= 0 \\
\sin\theta &= \frac{2 \pm \sqrt{4+8}}{4} \quad \text{throw away the “+”} \\
\sin\theta &= \frac{1-\sqrt{3}}{2} \quad \text{since it is out of the range of sin} \\
\theta &= \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) + 2k\pi \quad \text{for } k \text{ an integer}
\end{aligned}$$

*Note that* $\sin\theta = \dfrac{1-\sqrt{3}}{2}$ *twice during a period. The other one occurs at* $\pi - \sin^{-1}\left(\dfrac{1-\sqrt{3}}{2}\right)$.

*Then, since these two collections of angles are disjoint, the horizontal tangent lines occur when*

$$\boxed{\theta = \frac{\pi}{2} + k\pi} \quad \text{for } k \text{ an integer}$$

*and the vertical tangent lines occur when*

$$\boxed{\theta = \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) + 2k\pi, \theta = \pi - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) + 2k\pi} \quad \text{for } k \text{ an integer}$$

*Now we can find the equations for the horizontal and vertical tangent lines by recalling that* $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$.

***Equation of Horizontal tangent lines:***
$f(\theta) = 2 - \sin(\theta)$ *has a period of* $2\pi$ *so we will have horizontal tangent lines at* $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$.

*At* $\theta = \dfrac{\pi}{2}, r = f\left(\dfrac{\pi}{2}\right) = 2 - \sin\left(\dfrac{\pi}{2}\right) = 2 - 1 = 1$. *Therefore, one horizontal tangent line is* $y = r\sin(\theta) = 1 \cdot \sin\left(\dfrac{\pi}{2}\right) = 1$. $\boxed{y = 1}$

*At* $\theta = \dfrac{3\pi}{2}, r = f\left(\dfrac{3\pi}{2}\right) = 2 - \sin\left(\dfrac{3\pi}{2}\right) = 2 - (-1) = 3$. *Therefore, one horizontal tangent line is* $y = r\sin(\theta) = 3 \cdot \sin\left(\dfrac{3\pi}{2}\right) = -3$. $\boxed{y = -3}$

***Equation of Vertical tangent lines:***

*We start with* $\sin\theta = \dfrac{1-\sqrt{3}}{2}$. *We know* $f(\theta) = 2 - \sin(\theta) = 2 - \dfrac{1-\sqrt{3}}{2}$.
*Therefore, we have* $x = r\cos(\theta) = \left(2 - \dfrac{1-\sqrt{3}}{2}\right)\cos(\theta)$.

*We can find* $\cos(\theta)$ *using a right triangle:*
$\sin(\theta) = \dfrac{1-\sqrt{3}}{2}$ *but this value is negative so you would have to consider this to have a value of* $\theta$ *in the fourth quadrant. We know* $\cos(x) = \cos(-x)$ *so we will consider the corresponding triangle in the first quadrant with angle* $\sin(-\theta) = -\sin(\theta) = \dfrac{\sqrt{3}-1}{2}$.

$\sin\theta = \dfrac{\sqrt{3}-1}{2} = \dfrac{\textit{opposite}}{\textit{hypotenuse}}$ *so consider a triangle with opposite* $= \sqrt{3}-1$ *and hypotenuse = 2. Then, the adjacent* $= \sqrt{4-(\sqrt{3}-1)^2}$. *Thus,* $\cos(\theta) = \dfrac{\textit{adjacent}}{\textit{hypotenuse}} = \dfrac{\sqrt{4-(\sqrt{3}-1)^2}}{2}$.

*Therefore,* $x = r\cos(\theta) = \left(2 - \dfrac{1-\sqrt{3}}{2}\right)\cos(\theta) = \left(2 - \dfrac{1-\sqrt{3}}{2}\right)\dfrac{\sqrt{4-(\sqrt{3}-1)^2}}{2}$.

$$\boxed{x = \left(2 - \frac{1-\sqrt{3}}{2}\right)\frac{\sqrt{4-(\sqrt{3}-1)^2}}{2}}$$

*Because this equation is symmetric across the* $y$*-axis (Note:* $f(\pi-\theta) = 2 - \sin(\pi-\theta) = 2 - \sin(\theta)$*), the equation of the other vertical tangent line is*

$$\boxed{x = -\left(2 - \frac{1-\sqrt{3}}{2}\right)\frac{\sqrt{4-(\sqrt{3}-1)^2}}{2}}.$$

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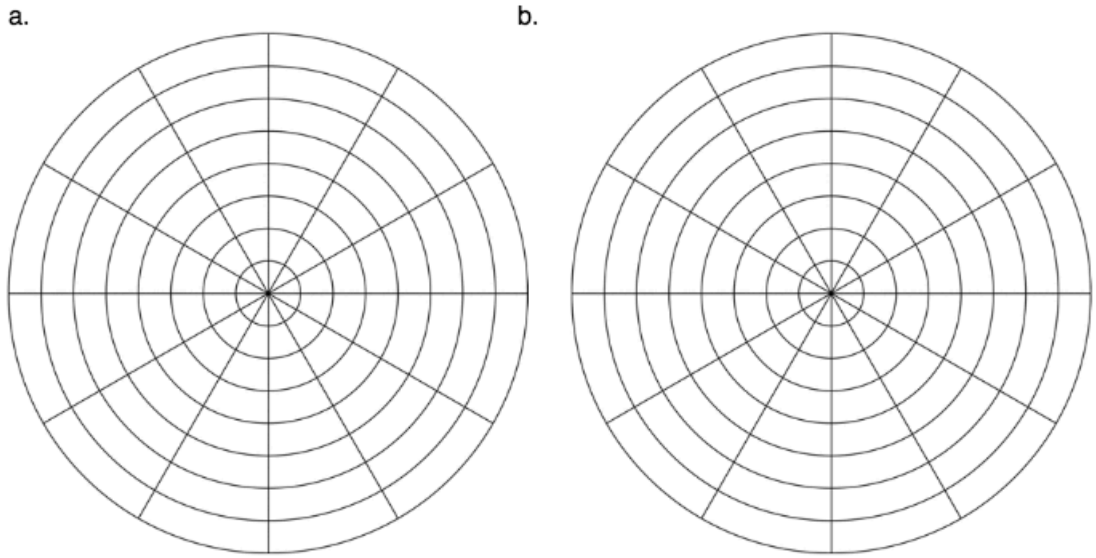
**Problem 2** *Graph each region and then SET UP an integral for the area of the region:*

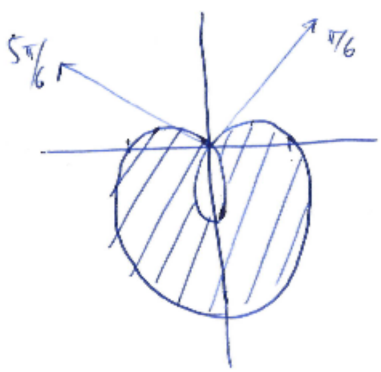
(a) *Outside the small loop and inside the large loop of* $r = 3 - 6\sin\theta$.

(b) *Inside both of the curves* $r = 4\cos\theta$ *and* $r = 1 - \cos\theta$.

*Note that you do not need to evaluate these integrals.*



**Solution:** (a) *The graph of the region is given below.*




*To find the area inside the smaller loop, we need to first find the values of $\theta$ for which $r = 0$. So we compute*

$$\begin{aligned} & 3 - 6\sin\theta = 0 \\ \implies \quad & \sin\theta = \frac{1}{2} \\ \implies \quad & \theta = \frac{\pi}{6}, \frac{5\pi}{6}. \end{aligned}$$

*Hence, the area of the smaller loop is*

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 6\sin\theta)^2 \, d\theta.$$

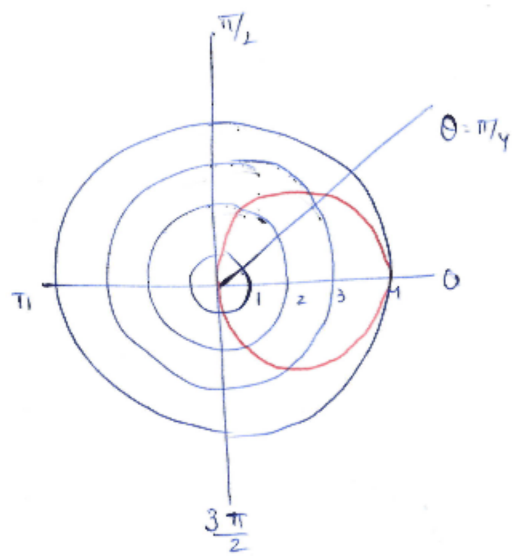
*To find the area of the outer loop, we just integrate over the other values of $\theta$:*

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} (3-6\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\frac{5\pi}{6}}^{2\pi} (3-6\sin\theta)^2\,d\theta.$$

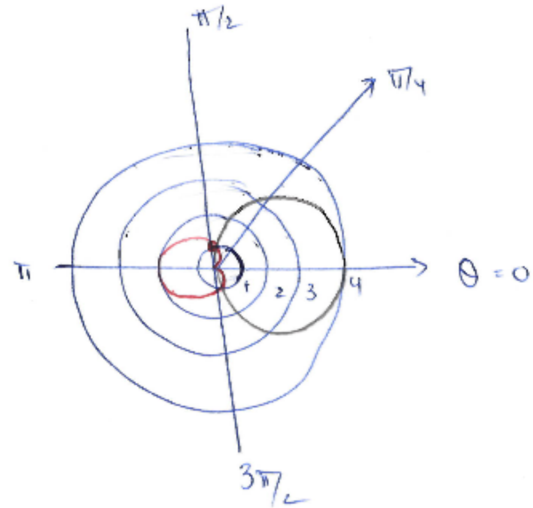
*Therefore, the area of the region inside the outer loop and outside the inner loop is*

$$\boxed{\frac{1}{2}\left[\int_0^{\frac{\pi}{6}} (3-6\sin\theta)^2\,d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3-6\sin\theta)^2\,d\theta + \int_{\frac{5\pi}{6}}^{2\pi} (3-6\sin\theta)^2\,d\theta\right]}$$

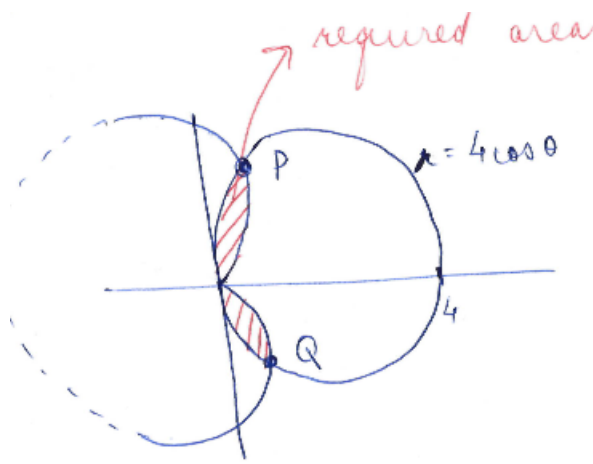
(b) *We first graph $r = 4\cos\theta$.*




*Now we graph $r = 1-\cos\theta$*




*Here are the two graphs in the same picture*




*To find the points where the two curves intersect, we solve*

$$
\begin{aligned}
4\cos\theta &= 1-\cos\theta \\
5\cos\theta &= 1 \\
\cos\theta &= \frac{1}{5} \\
\theta &= \pm\cos^{-1}\left(\frac{1}{5}\right) := \theta_0.
\end{aligned}
$$

*Using the symmetry of the graph we find the area of one leaf and then double it. Therefore, the area between the curves is*

$$\boxed{2\left[\frac{1}{2}\int_0^{\theta_0}(1-\cos\theta)^2\,d\theta+\frac{1}{2}\int_{\theta_0}^{\frac{\pi}{2}}(4\cos\theta)^2\,d\theta\right]}$$

*In the following picture, the left-hand integral is (1) and the right-hand integral is (2).*