

Recitation #13: An overview of sequences and series

Warm up:

For each of the following sequences, list the first four terms (start each with $n = 1$).

(a) $a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right), a_1 = 1.$

Solution: **n=1:** $a_1 = 1.$

n=2: $a_2 = \frac{1}{2} \left(a_1 + \frac{2}{a_1} \right) = \frac{1}{2} \left(1 + \frac{2}{1} \right) = \frac{3}{2}.$

n=3: $a_3 = \frac{1}{2} \left(a_2 + \frac{2}{a_2} \right) = \frac{1}{2} \left(\frac{3}{2} + \frac{2}{\frac{3}{2}} \right) = \frac{17}{12}.$

n=4: $a_4 = \frac{1}{2} \left(a_3 + \frac{2}{a_3} \right) = \frac{1}{2} \left(\frac{17}{12} + \frac{2}{\frac{17}{12}} \right) = \frac{577}{408}.$

(b) $a_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2n)! \cdot 2n!},$ Recall that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \dots (n-1) \cdot n.$

Solution: **n=1:** $a_1 = \frac{1}{2!2!} = \frac{1}{4}.$

n=2: $a_2 = \frac{1 \cdot 3}{(2 \cdot 2)! \cdot 2 \cdot 2!} = \frac{3}{96} = \frac{1}{32}.$

n=3: $a_3 = \frac{1 \cdot 3 \cdot 5}{6! \cdot 2 \cdot 3!}.$

n=4: $a_4 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{8! \cdot 2 \cdot 4!}.$

Group work:

Problem 1 Give an explicit formula for each of the following sequences:

(a) $\frac{2}{3}, \frac{-2}{7}, \frac{2}{11}, \frac{-2}{15}, \dots$

Solution: $a_n = \frac{(-1)^{n+1} \cdot 2}{-1 + 4n},$ starting at $n = 1.$

(b) $-2, 6, -24, 120, -720, \dots$

Solution: $a_n = (-1)^n (n+1)!,$ starting at $n = 1.$

Learning outcomes:

Problem 2 For the sequence $a_k = (2 - k)^k$

(a) calculate and list a_0, a_1, a_2, a_3 , and a_4 .

Solution: $a_0 = (2 - 0)^0 = 1.$

$$a_1 = (2 - 1)^1 = 1.$$

$$a_2 = (2 - 2)^2 = 0.$$

$$a_3 = (2 - 3)^3 = -1.$$

$$a_4 = (2 - 4)^4 = 16.$$

(b) Starting with $k = 0$, calculate and list $S_0 = \sum_{k=0}^0 a_k$, $S_1 = \sum_{k=0}^1 a_k$, $S_2 = \sum_{k=0}^2 a_k$, $S_3 = \sum_{k=0}^3 a_k$, and

$S_4 = \sum_{k=0}^4 a_k$. Write S_n in summation form and write S_∞ in summation form.

Solution: $S_0 = \sum_{k=0}^0 a_k = a_0 = 1.$

$$S_1 = \sum_{k=0}^1 a_k = a_0 + a_1 = 1 + 1 = 2.$$

$$S_2 = \sum_{k=0}^2 a_k = a_0 + a_1 + a_2 = 2 + 0 = 2.$$

$$S_3 = \sum_{k=0}^3 a_k = a_0 + a_1 + a_2 + a_3 = 2 + (-1) = 1.$$

$$S_4 = \sum_{k=0}^4 a_k = a_0 + a_1 + a_2 + a_3 + a_4 = 1 + 16 = 17.$$

$$S_n = \sum_{k=0}^n a_k.$$

$$S_\infty = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k.$$

Problem 3 Reindex the series

$$\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)}$$

in the form $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=-4}^{\infty} c_k$.

Solution: For the first series, we let $i = k + 1$. Then $k = i - 1$ and, when $k = 0$, $i = 1$. So we have that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)} &= \sum_{i=1}^{\infty} \frac{5}{(i-1+2)(i-1+1)} \\ &= \sum_{i=1}^{\infty} \frac{5}{i(i+1)} \\ &= \sum_{k=1}^{\infty} \frac{5}{k(k+1)}. \quad \text{Resubstituting } k=i.\end{aligned}$$

For the second series, we let $i = k - 4$. Then $k = i + 4$ and, when $k = 0$, $i = -4$. So we have that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)} &= \sum_{i=-4}^{\infty} \frac{5}{(i+4+2)(i+4+1)} \\ &= \sum_{i=-4}^{\infty} \frac{5}{(i+5)(i+6)} \\ &= \sum_{k=-4}^{\infty} \frac{5}{(k+5)(k+6)}. \quad \text{Resubstituting } k=i.\end{aligned}$$

Problem 4 If $\sum_{k=0}^{\infty} a_k = 6$ and $a_n = \frac{3}{2^n}$, what is $\sum_{k=4}^{\infty} a_k$?

Solution:

$$\begin{aligned}6 &= \sum_{k=0}^{\infty} a_k \\ &= a_0 + a_1 + a_2 + a_3 + \sum_{k=4}^{\infty} a_k \\ &= \frac{3}{1} + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \sum_{k=4}^{\infty} a_k \\ &= \frac{45}{8} + \sum_{k=4}^{\infty} a_k.\end{aligned}$$

Thus,

$$\sum_{k=4}^{\infty} a_k = 6 - \frac{45}{8} = \frac{3}{8}.$$