

Recitation #22: Working with Taylor series

You should memorize the Maclaurin series for $\cos(x)$, $\sin(x)$, e^x , and $\frac{1}{1-x}$. If you need the Maclaurin series for $\ln(1+x)$, $\arctan(x)$, or the binomial series these will be given to you.

$$\begin{aligned}\ln(1+x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} & -1 < x \leq 1 \\ \arctan(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} & -1 \leq x \leq 1\end{aligned}$$

Warm up:

True or False: To approximate $\frac{\pi}{3}$, one could substitute $x = \sqrt{3}$ into the Maclaurin series for $\tan^{-1} x$?

Solution: **False.** The power series representation

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

only converges on $[-1, 1]$. Since $\sqrt{3}$ is outside of the IOC, one cannot substitute $x = \sqrt{3}$ into this series to approximate $\frac{\pi}{3}$.

Group work:

Problem 1 Use power series to evaluate the limit

$$\lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{1-\cos x}$$

Solution: For $-1 < x \leq 1$, we know that

$$\ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (x^2)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k}}{k}.$$

For $-\infty < x < \infty$ we also know that

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Learning outcomes:

Since 0 is within both of these intervals, we can substitute these formulas into the limit:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{1-\cos x} &= \lim_{x \rightarrow 0} \frac{\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k}}{k}}{1 - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots}{\frac{x^2}{2!} - \frac{x^4}{4!} + \dots} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots\right)}{x^2 \left(\frac{1}{2!} - \frac{x^2}{4!} + \dots\right)} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots}{\frac{1}{2!} - \frac{x^2}{4!} + \dots} \\
 &= \frac{1}{\frac{1}{2}} = \boxed{2}.
 \end{aligned}$$

Problem 2 Given that

$$f(t) = \int_0^t x^2 \tan^{-1}(x^4) dx$$

approximate $f\left(\frac{1}{3}\right)$ with the first four non-zero terms of a power series. Estimate how close this approximation is.

Solution:

$$\begin{aligned}
 f\left(\frac{1}{3}\right) &= \int_0^{\frac{1}{3}} x^2 \arctan(x^4) dx \\
 &= \int_0^{\frac{1}{3}} \left(x^2 \sum_{k=0}^{\infty} \frac{(-1)^k (x^4)^{2k+1}}{2k+1} \right) dx \\
 &\approx \int_0^{\frac{1}{3}} x^2 \left(x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} \right) dx \quad \text{Using 4 terms of the power series} \\
 &= \int_0^{\frac{1}{3}} \left(x^6 - \frac{x^{14}}{3} + \frac{x^{22}}{5} - \frac{x^{30}}{7} \right) dx \\
 &= \left[\frac{x^7}{7} - \frac{x^{15}}{45} + \frac{x^{23}}{115} - \frac{x^{31}}{217} \right]_0^{\frac{1}{3}} \\
 &= \boxed{\frac{1}{3^7 \cdot 7} - \frac{1}{3^{15} \cdot 45} + \frac{1}{3^{23} \cdot 115} - \frac{1}{3^{31} \cdot 217}}.
 \end{aligned}$$

The error of the series is less than the first truncated term, which here is the fifth term

$$x^2 \cdot \frac{(x^4)^9}{9} = \frac{x^{38}}{9}.$$

So we integrate

$$\int_0^{\frac{1}{3}} \frac{x^{38}}{9} dx = \left[\frac{x^{39}}{351} \right]_0^{\frac{1}{3}} = \frac{1}{3^{39} \cdot 351}.$$

Therefore, an upper bound for the error is $\boxed{\frac{1}{3^{39} \cdot 351}}$

Problem 3 Use power series to determine a (series) solution to the initial value problem

$$y'' - xy' + y = 0 \quad y(0) = 1 \quad y'(0) = 0$$

Solution: Assume that $y(x)$ is a power series solution centered at 0. ie,

$$y = \sum_{k=0}^{\infty} c_k x^k \quad \text{where} \quad c_k = \frac{y^{(k)}(0)}{k!}.$$

We are given that $y(0) = 1$ and $y'(0) = 0$. Using these, we can find c_0 and c_1 :

$$c_0 = \frac{y^{(0)}(0)}{0!} = \frac{1}{1} = 1$$

$$c_1 = \frac{y'(0)}{1!} = \frac{0}{1} = 0.$$

To find c_k , we need to first find $y^{(k)}(0)$. Using the differential equation, we can directly find $y''(0)$.

$$y''(0) - 0 \cdot y'(0) + y(0) = 0$$

$$y''(0) - 0 + 1 = 0$$

$$y''(0) = -1.$$

So

$$c_2 = \frac{y''(0)}{2!} = \frac{-1}{2}.$$

To find $y^{(3)}(0)$, we need to differentiate the differential equation (using the product rule):

$$\frac{d}{dx} (y'' - xy' + y) = \frac{d}{dx} (0)$$

$$y^{(3)} - (y' + xy'') + y' = 0$$

$$y^{(3)} - xy'' = 0$$

$$y^{(3)}(0) - 0 \cdot y''(0) = 0$$

$$y^{(3)}(0) = 0.$$

In exactly the same manner, we can compute

$$y^{(4)}(0) = -1$$

$$y^{(5)}(0) = 0$$

$$y^{(6)}(0) = -3$$

$$y^{(7)}(0) = 0$$

$$y^{(8)}(0) = -5(3)$$

$$y^{(9)}(0) = 0$$

$$y^{(10)}(0) = -7(5)(3)$$

$$y^{(11)}(0) = 0$$

$$y^{(12)}(0) = -9(7)(5)(3)$$

In particular, notice that $y^{(k)}(0) = 0$ when k is odd. So in finding c_k , we split into the cases when k is odd and when k is even.

$$c_{2k+1} = 0$$

$$c_{2k} = \frac{-3 \cdot 5 \cdot 7 \cdot \dots \cdot (2k-3)}{(2k)!} = \frac{-1}{2^k k! (2k-1)}.$$

Thus, the power series solution to the original initial value problem is

$$\sum_{k=0}^{\infty} \left(\frac{-1}{2^k k! (2k-1)} \right) x^{2k}$$

Problem 4 Identify the function represented by the power series

$$\sum_{k=0}^{\infty} \frac{k(k-1)x^k}{7^k}$$

Solution:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k(k-1)x^k}{7^k} &= x^2 \sum_{k=0}^{\infty} \frac{k(k-1)x^{k-2}}{7^k} \\ &= x^2 \frac{d^2}{dx^2} \left(\sum_{k=2}^{\infty} \frac{x^k}{7^k} \right) \\ &= x^2 \frac{d^2}{dx^2} \left(\sum_{k=2}^{\infty} \left(\frac{x}{7} \right)^k \right) \\ &= x^2 \frac{d^2}{dx^2} \left(\frac{\frac{x^2}{49}}{1 - \frac{x}{7}} \right) \quad \text{geometric series, valid for } \left| \frac{x}{7} \right| < 1 \\ &= x^2 \frac{d^2}{dx^2} \left(\frac{x^2}{7(7-x)} \right) \\ &= x^2 \frac{d}{dx} \left(\frac{2x(49-7x) - (-7)(x^2)}{49(7-x)^2} \right) \\ &= x^2 \frac{d}{dx} \left(\frac{98x - 7x^2}{49(7-x)^2} \right) \\ &= x^2 \left(\frac{(98-14x)(49)(7-x)^2 - (-98)(7-x)(98x-7x^2)}{49^2(7-x)^4} \right) \\ &= x^2 \left(\frac{(98-14x)(7-x) + 2(98x-7x^2)}{49(7-x)^3} \right) \\ &= x^2 \left(\frac{(14-2x)(7-x) + 2x(14-x)}{7(7-x)^3} \right) \\ &= x^2 \left(\frac{98-14x-14x+2x^2+28x-2x^2}{7(7-x)^3} \right) \\ &= \boxed{\frac{14x}{(7-x)^3}} \end{aligned}$$

when

$$\left| \frac{x}{7} \right| < 1 \quad \Longleftrightarrow \quad |x| < 7.$$
