Recitation #9: Trig Integrals and Trig Substitution

Group work:

Problem 1 *Evaluate the following integrals*

(a) $\int \tan^{23} x \sec^6 x \, dx$

Solution:

$$\begin{aligned} \int \tan^{23} x \sec^6 x \, dx &= \int \tan^{23} x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^{23} x \left(1 + \tan^2 x\right)^2 \sec^2 x \, dx. \end{aligned}$$

We now substitute

$$u = \tan x \qquad \Longrightarrow \qquad du = \sec^2 x \, dx.$$

Then

$$\begin{aligned} \int \tan^{23} x \left(1 + \tan^2 x\right)^2 \sec^2 x \, dx &= \int u^{23} (1 + u^2)^2 \, du \\ &= \int u^{23} (1 + 2u^2 + u^4) \, du \\ &= \int \left(u^{23} + 2u^{25} + u^{27}\right) \, du \\ &= \frac{1}{24} u^{24} + \frac{1}{13} u^{26} + \frac{1}{28} u^{28} + C \\ &= \frac{1}{24} \tan^{24} x + \frac{1}{13} \tan^{26} x + \frac{1}{28} \tan^{28} x + C. \end{aligned}$$

(b) $\int \tan^2 x \sec x \, dx$ *Hint:* $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Learning outcomes:

Solution:

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int \left(\sec^2 x - 1\right)\sec x\,dx \\ &= \int \sec^3 x\,dx - \int \sec x\,dx \\ &= \int \sec^3 x\,dx - \ln|\sec x + \tan x| \qquad \textit{from the hint} \quad (1)\end{aligned}$$

Now, in an attempt to evaluate $\int \sec^3 x\,dx$*, we use integration by parts with*

$$u = \sec x \qquad dv = \sec^2 x\,dx$$

$$du = \sec x \tan x\,dx \qquad v = \tan x.$$

So

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx. \tag{2}$$

Combining equations (1) *and* (2) *yields*

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int \sec^3 x\,dx - \ln|\sec x + \tan x| \\ \int \tan^2 x \sec x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx - \ln|\sec x + \tan x| \\ 2\int \tan^2 x \sec x\,dx &= \sec x \tan x - \ln|\sec x + \tan x| + C \\ \int \tan^2 x \sec x\,dx &= \frac{1}{2}\left(\sec x \tan x - \ln|\sec x + \tan x|\right) + C.\end{aligned}$$

(c) $\int \tan^2 x \sin x\,dx$

Solution:

$$\begin{aligned}\int \tan^2 x \sin x\,dx &= \int \frac{\sin^2 x}{\cos^2 x}\sin x\,dx \\ &= \int \frac{1-\cos^2 x}{\cos^2 x}\sin x\,dx.\end{aligned}$$

Now we substitute

$$u = \cos x \qquad \Longrightarrow \qquad du = -\sin x\,dx, \quad -du = \sin x\,dx.$$

This gives us that

$$\begin{aligned}\int \tan^2 x \sin x \, dx &= \int \frac{1-u^2}{u^2}(-1)\, du \\ &= \int \frac{u^2-1}{u^2}\, du \\ &= \int \left(1-u^{-2}\right)\, du \\ &= u + \frac{1}{u} + C \\ &= \cos x + \sec x + C.\end{aligned}$$

Problem 2 *Evaluate*

$$\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx.$$

Solution:

$$\begin{aligned}\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx &= \int_{-\pi}^{0} \sqrt{\sin^2 x}\, dx \\ &= \int_{-\pi}^{0} |\sin x|\, dx.\end{aligned}$$

Now, when $-\pi \le x \le 0$, $\sin x \le 0$. *Thus, on this region,* $|\sin x| = -\sin x$. *So we continue*

$$\begin{aligned}\int_{-\pi}^{0} \sqrt{1-\cos^2 x}\, dx &= \int_{-\pi}^{0} -\sin x\, dx \\ &= \Big[\cos x\Big]_{-\pi}^{0} \\ &= \cos(0) - \cos(-\pi) = 1 - (-1) = 2.\end{aligned}$$

Problem 3 *Evaluate the following integrals*

(a)

$$\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2-25}}{x^3}\, dx.$$

Solution: *First notice that*

$$\begin{aligned}\sqrt{36x^2 - 25} &= 5\sqrt{\frac{36x^2}{25} - 1} \\ &= 5\sqrt{\left(\frac{6x}{5}\right)^2 - 1}.\end{aligned}$$

So we substitute

$$\frac{6x}{5} = \sec\theta \qquad \Longrightarrow \qquad x = \frac{5}{6}\sec\theta$$

which gives

$$dx = \frac{5}{6}\sec\theta\tan\theta\, d\theta.$$

Also, notice that

- *when* $x = -\frac{5}{3}$*:*

$$-\frac{5}{3} = \frac{5}{6}\sec\theta \qquad \Longrightarrow \qquad \sec\theta = 2 \qquad \Longrightarrow \qquad \theta = \frac{2\pi}{3}$$

- *and when* $x = -\frac{5}{6}$*:*

$$-\frac{5}{6} = \frac{5}{6}\sec\theta \qquad \Longrightarrow \qquad \sec\theta = -1 \qquad \Longrightarrow \qquad \theta = \pi.$$

Therefore

$$\begin{aligned}\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3}\, dx &= 5\int_{\frac{2\pi}{3}}^{\pi} \frac{\sqrt{\sec^2\theta - 1}}{\left(\frac{5}{6}\sec\theta\right)^3}\left(\frac{5}{6}\sec\theta\tan\theta\right) d\theta \\ &= 5\cdot\left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan\theta|\tan\theta}{\sec^2\theta}\, d\theta.\end{aligned}$$

Now, notice that $\tan\theta < 0$ *whenever* $\frac{2\pi}{3} \leq \theta \leq \pi$*. So* $|\tan\theta| = -\tan\theta$*.*

We continue:

$$
\begin{aligned}
5 \cdot \left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan\theta|\tan\theta}{\sec^2\theta}\,d\theta &= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\
&= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{1}\,d\theta \\
&= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \sin^2\theta\,d\theta \\
&= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}\left(1-\cos(2\theta)\right)\,d\theta \\
&= -\frac{18}{5}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{2\pi}{3}}^{\pi} \\
&= -\frac{18}{5}\left[(\pi - 0) - \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right)\right] \qquad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\
&= -\frac{18}{5}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right).
\end{aligned}
$$

(b)

$$\int \frac{dx}{\left(x^2-6x+11\right)^2}.$$

Solution: *We begin by completing the square in the denominator*

$$x^2 - 6x - 11 = x^2 - 6x + 9 + 2 = (x-3)^2 + 2.$$

We then have that

$$
\begin{aligned}
\int \frac{dx}{\left(x^2-6x+11\right)^2} &= \int \frac{1}{((x-3)^2+2)^2}\,dx \\
&= \frac{1}{4}\int \frac{1}{\left(\frac{(x-3)^2}{2}+1\right)^2}\,dx \\
&= \frac{1}{4}\int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2+1\right)^2}\,dx.
\end{aligned}
$$

So we substitute

$$\frac{x-3}{\sqrt{2}} = \tan\theta \qquad \Longrightarrow \qquad x = \sqrt{2}\tan\theta + 3 \tag{3}$$

and then

$$dx = \sqrt{2}\sec^2\theta\,d\theta.$$

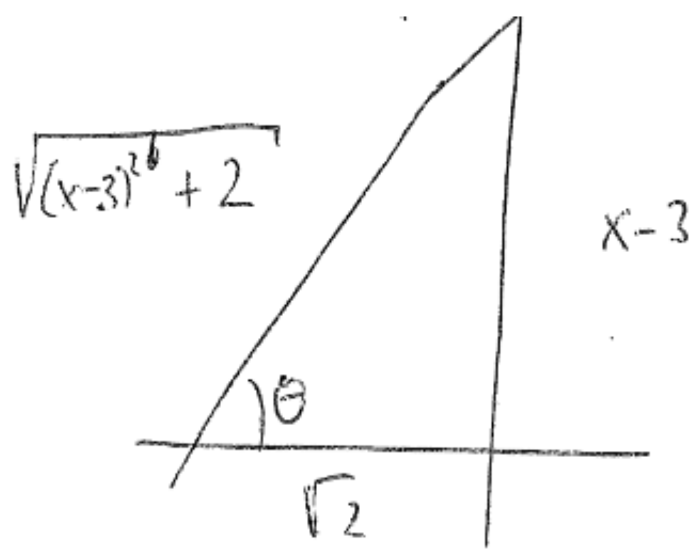
Continuing with the integral

$$
\begin{aligned}
\frac{1}{4}\int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2+1\right)^2}\,dx &= \frac{1}{4}\int \frac{1}{\left(\tan^2\theta+1\right)^2}\sqrt{2}\sec^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \frac{1}{\sec^2\theta}\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \frac{1}{2}\left(1+\cos(2\theta)\right)\,d\theta \\
&= \frac{\sqrt{2}}{8}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C.
\end{aligned}
$$

Now all that is left to do is to reverse-substitute for θ. First, from equation (3) we have that

$$\theta = \arctan\left(\frac{x-3}{\sqrt{2}}\right).$$

Now, we again use equation (3) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\cdot\frac{x-3}{\sqrt{(x-3)^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{(x-3)^2+2}}.$$

Thus

$$\int \frac{dx}{\left(x^2 - 6x + 11\right)^2} = \frac{\sqrt{2}}{8}\left(\arctan\left(\frac{x-3}{\sqrt{2}}\right) + \frac{\sqrt{2}(x-3)}{(x-3)^2+2}\right) + C.$$