

## Recitation #16: The Divergence, Integral, Ratio and Root Tests - Full

### Group work:

**Problem 1** For each of the following, answer **True** or **False**, and explain why.

- (a) If  $\sum_{n=0}^{\infty} a_n$  converges, then  $\sum_{n=0}^{\infty} (a_n + 0.001)$  converges.
- (b) Since  $\int_1^{\infty} x \sin(\pi x) dx$  diverges then, by the Integral Test,  $\sum_{n=0}^{\infty} n \sin(\pi n)$  diverges.
- (c) Since  $\int_1^{\infty} \frac{1}{x^2} dx = 1$  then, by the Integral Test,  $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1$ .

**Solution:** (a) **False**

Since  $\sum_{n=0}^{\infty} a_n$  converges, we know that  $\lim_{n \rightarrow \infty} a_n = 0$ . But then

$$\lim_{n \rightarrow \infty} (a_n + 0.0001) = 0.0001 \neq 0$$

and so  $\sum_{n=0}^{\infty} (a_n + 0.001)$  diverges by the Divergence Test.

(b) **False**

The Integral Test only holds for positive, decreasing functions. The function  $f(x) = x \sin(\pi x)$  is not always positive, nor is it always decreasing. So the Integral Test does not apply here.

This problem is simpler than that though. Since  $\sin(\pi n) = 0$  for all integers  $n$ , we have that  $\sum_{n=0}^{\infty} n \sin(\pi n) = 0$ .

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(c) **False**

The Integral Test tells us that  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  converges, but it does **not** give us the sum (this sum is actually  $\frac{\pi^2}{6}$ ).

**Instructor Notes:** For part (b) we need  $f(x)$  to be a decreasing function for the Integral Test to (necessarily) hold. All groups should do all of the parts.

**Problem 2** Assume  $\sum_{k=0}^{\infty} a_k = L$  and  $b_k = 8$  for all  $k$ .

- (a) What is  $\lim_{k \rightarrow \infty} (a_k + b_k)$ ?
- (b) What is  $\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_n + b_n)$ ?
- (c) What is  $\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_{n+1} - a_n)$ ?

**Solution:** (a) Since  $\sum_{k=0}^{\infty} a_k$  converges, we know that  $\lim_{k \rightarrow \infty} a_k = 0$ . Therefore,

$$\lim_{k \rightarrow \infty} (a_k + b_k) = 0 + 8 = \boxed{8}.$$

(b) Since  $\lim_{n \rightarrow \infty} (a_n + b_n) = 8$ , the series  $\sum_{n=0}^{\infty} (a_n + b_n)$  diverges by the Divergence

Test. But  $\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_n + b_n) = \sum_{n=0}^{\infty} (a_n + b_n)$ . Thus

$$\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_n + b_n) = \sum_{n=0}^{\infty} (a_n + b_n) = \boxed{\infty}.$$

(c) Let  $S_k = \sum_{n=0}^k (a_{n+1} - a_n)$  (and recall that  $\{S_k\}$  is the sequence of partial

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sums). Then

$$\begin{aligned} S_k &= \sum_{n=0}^k (a_{n+1} - a_n) \\ &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \dots + (a_k - a_{k-1}) + (a_{k+1} - a_k) \\ &= a_{k+1} - a_0. \end{aligned}$$

Thus,

$$\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_{n+1} - a_n) = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} a_{k+1} - a_0 = \boxed{-a_0}.$$

**Instructor Notes:** This question was adapted from midterm #2 in Spring 2013. Students had difficulty distinguishing between a question dealing with sequences vs. a question dealing with series.

**Problem 3** Determine if the following series converge or diverge.

- (a)  $\sum_{n=1}^{\infty} \frac{(7n+1)^2 \cdot 2^n}{5^n}$
- (b)  $\sum_{n=1}^{\infty} a_n$ , where  $a_{n+1} = \frac{2n+5}{3n-1} \cdot a_n$  and  $a_1 = 1$ .
- (c)  $\sum_{n=0}^{\infty} \frac{n^2 + 2n + 1}{3n^2 + 1}$
- (d)  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
- (e)  $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$

**Solution:** (a) **Ratio Test**

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \left[ \frac{(7(n+1)+1)^2 \cdot 2^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(7n+1)^2 \cdot 2^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{(7n+8)^2 \cdot 2}{5 \cdot (7n+1)^2} \\ &= \frac{49 \cdot 2}{49 \cdot 5} = \frac{2}{5}. \end{aligned}$$

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Thus, since  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ , this series converges.

(b) **Ratio Test**

Even though the terms in this series look a little weird, this is set up perfectly for the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{2n+5}{3n-1} = \frac{2}{3}.$$

Thus, since  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ , this series converges.

(c) **Divergence Test**

Notice that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{3n^2 + 1} = \frac{1}{3}.$$

Therefore, since  $\lim_{n \rightarrow \infty} a_n \neq 0$ , by the Divergence Test this series diverges.

(d) **Integral Test**

First, notice that  $f(x) = \frac{1}{x(\ln x)^2}$  is a decreasing and positive function on  $[2, \infty)$ . Then

$$\begin{aligned} \int_2^\infty f(x) dx &= \int_2^\infty \frac{1}{x(\ln x)^2} dx \\ &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx \\ &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-2} du \quad u = \ln x, du = \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{-1}{u} \right]_{\ln 2}^{\ln b} \\ &= \lim_{b \rightarrow \infty} \left( \frac{-1}{\ln b} + \frac{1}{\ln 2} \right) \\ &= 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}. \end{aligned}$$

Therefore, since the above integral converges, the series  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  converges by the Integral Test.

(e) **Ratio Test**

$$\begin{aligned}\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= \lim_{k \rightarrow \infty} \left[ \frac{((k+1)!)^3}{(3(k+1))!} \cdot \frac{(3k)!}{(k!)^3} \right] \\ &= \lim_{k \rightarrow \infty} \frac{(k+1)^3 (k!)^3}{(3k+3)(3k+2)(3k+1) \cdot (3k)!} \cdot \frac{(3k)!}{(k!)^3} \\ &= \lim_{k \rightarrow \infty} \frac{(k+1)^3}{(3k+3)(3k+2)(3k+1)} \\ &= \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{27}.\end{aligned}$$

Thus, since  $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1$ , this series converges.

**Instructor Notes:** Let the students experiment with what tests to use. Perhaps give two problems per group.

**Problem 4** How many terms are needed to estimate  $\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$  to within  $10^{-4}$ ?

What is the estimate for the sum of the series?

**Solution:** Let  $f(x) = \frac{1}{x^2 + 1}$ . Note that  $f(x)$  is continuous, decreasing, and positive for  $x \geq 1$ , and  $a_k = f(k)$ .

If  $S = \sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$  is the actual value of the sum, and  $S_n = \sum_{k=1}^n \frac{1}{k^2 + 1}$  is the  $n^{\text{th}}$  partial sum, then the remainder  $R_n = S - S_n$  can be bounded above by

$$R_n < \int_n^{\infty} f(x) dx.$$

So we need to find  $n$  so that

$$\int_n^{\infty} \frac{1}{x^2 + 1} dx < 10^{-4}. \quad (1)$$

We compute

$$\begin{aligned}\int_n^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \int_n^b \frac{1}{x^2 + 1} dx \\ &= \lim_{b \rightarrow \infty} \left[ \arctan(x) \right]_n^b \\ &= \lim_{b \rightarrow \infty} (\arctan(b) - \arctan(n)) \\ &= \frac{\pi}{2} - \arctan(n).\end{aligned}$$

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Plugging into equation (1) we see that we want

$$\begin{aligned}\frac{\pi}{2} - \arctan(n) &< 10^{-4} \\ \implies \arctan(n) &> \frac{\pi}{2} - 10^{-4} \\ \implies n &> \tan\left(\frac{\pi}{2} - 10^{-4}\right) \approx 9999.999967 \approx 10,000.\end{aligned}$$

Therefore, 10,000 terms will estimate  $\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$  to within  $10^{-4}$ . Using a computer, we can also compute

$$\sum_{k=1}^{10,000} \frac{1}{k^2 + 1} \approx 1.07657.$$

**Instructor Notes:** This should be a straightforward calculation, but calculators may be needed.