Recitation #10: Trigonometric substitutions - Solutions

Group work:

Problem 1 *Evaluate the following integrals*

(a)

$$\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3}\, dx.$$

Solution: *First notice that*

$$\begin{aligned}\sqrt{36x^2 - 25} &= 5\sqrt{\frac{36x^2}{25} - 1} \\ &= 5\sqrt{\left(\frac{6x}{5}\right)^2 - 1}.\end{aligned}$$

So we substitute

$$\frac{6x}{5} = \sec\theta \qquad \Longrightarrow \qquad x = \frac{5}{6}\sec\theta$$

which gives

$$dx = \frac{5}{6}\sec\theta\tan\theta\, d\theta.$$

Also, notice that

- *when* $x = -\frac{5}{3}$*:*

$$-\frac{5}{3} = \frac{5}{6}\sec\theta \qquad \Longrightarrow \qquad \sec\theta = 2 \qquad \Longrightarrow \qquad \theta = \frac{2\pi}{3}$$

- *and when* $x = -\frac{5}{6}$*:*

$$-\frac{5}{6} = \frac{5}{6}\sec\theta \qquad \Longrightarrow \qquad \sec\theta = -1 \qquad \Longrightarrow \qquad \theta = \pi.$$

Learning outcomes:

Therefore

$$\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2-25}}{x^3}\,dx = 5\int_{\frac{2\pi}{3}}^{\pi} \frac{\sqrt{\sec^2\theta-1}}{\left(\frac{5}{6}\sec\theta\right)^3}\left(\frac{5}{6}\sec\theta\tan\theta\right)\,d\theta$$
$$= 5\cdot\left(\frac{6}{5}\right)^2\int_{\frac{2\pi}{3}}^{\pi}\frac{|\tan\theta|\tan\theta}{\sec^2\theta}\,d\theta.$$

Now, notice that $\tan\theta < 0$ *whenever* $\frac{2\pi}{3} \le \theta \le \pi$. *So* $|\tan\theta| = -\tan\theta$. *We continue:*

$$5\cdot\left(\frac{6}{5}\right)^2\int_{\frac{2\pi}{3}}^{\pi}\frac{|\tan\theta|\tan\theta}{\sec^2\theta}\,d\theta = -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi}\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$
$$= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi}\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{1}\,d\theta$$
$$= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi}\sin^2\theta\,d\theta$$
$$= -\frac{36}{5}\int_{\frac{2\pi}{3}}^{\pi}\frac{1}{2}\left(1-\cos(2\theta)\right)\,d\theta$$
$$= -\frac{18}{5}\left[\theta-\frac{1}{2}\sin(2\theta)\right]_{\frac{2\pi}{3}}^{\pi}$$
$$= -\frac{18}{5}\left[(\pi-0)-\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{4}\right)\right] \qquad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$= -\frac{18}{5}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right).$$

(b)

$$\int\frac{dx}{\left(x^2-6x+11\right)^2}.$$

Solution: *We begin by completing the square in the denominator*

$$x^2-6x-11 = x^2-6x+9+2 = (x-3)^2+2.$$

We then have that

$$\int\frac{dx}{\left(x^2-6x+11\right)^2} = \int\frac{1}{((x-3)^2+2)^2}\,dx$$
$$= \frac{1}{4}\int\frac{1}{\left(\frac{(x-3)^2}{2}+1\right)^2}\,dx$$
$$= \frac{1}{4}\int\frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2+1\right)^2}\,dx.$$

So we substitute

$$\frac{x-3}{\sqrt{2}} = \tan\theta \qquad \Longrightarrow \qquad x = \sqrt{2}\tan\theta + 3 \tag{1}$$

and then

$$dx = \sqrt{2}\sec^2\theta\, d\theta.$$

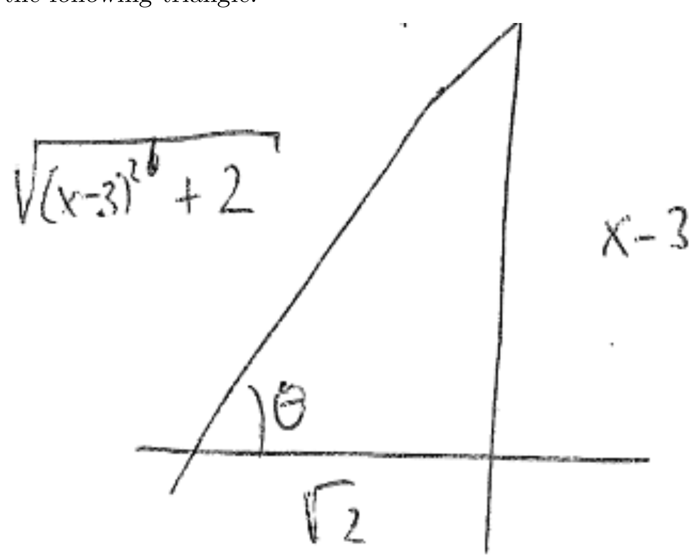
Continuing with the integral

$$\begin{aligned}
\frac{1}{4}\int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2+1\right)^2}\,dx &= \frac{1}{4}\int \frac{1}{\left(\tan^2\theta+1\right)^2}\sqrt{2}\sec^2\theta\, d\theta \\
&= \frac{\sqrt{2}}{4}\int \frac{1}{\sec^2\theta}\,d\theta \\
&= \frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta \\
&= \frac{\sqrt{2}}{4}\int \frac{1}{2}\left(1+\cos(2\theta)\right)\, d\theta \\
&= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C.
\end{aligned}$$

Now all that is left to do is to reverse-substitute for θ. First, from equation (1) we have that

$$\theta = \arctan\left(\frac{x-3}{\sqrt{2}}\right).$$

Now, we again use equation (1) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot \frac{x-3}{\sqrt{(x-3)^2+2}} \cdot \frac{\sqrt{2}}{\sqrt{(x-3)^2+2}}.$$

Thus

$$\int \frac{dx}{\left(x^2-6x+11\right)^2} = \frac{\sqrt{2}}{8}\left(\arctan\left(\frac{x-3}{\sqrt{2}}\right) + \frac{\sqrt{2}(x-3)}{(x-3)^2+2}\right) + C.$$

(c)

$$\int \frac{x^2}{\sqrt{4x-x^2}}\,dx.$$

Solution: *Again, we begin by completing the square in the denominator, and then factoring*

$$\begin{aligned} 4x - x^2 &= -(x^2-4x) \\ &= -(x^2-4x+4)+4 \\ &= -(x-2)^2+4 \\ &= 4\left(-\frac{(x-2)^2}{4}+1\right) \\ &= 4\left(1-\left(\frac{x-2}{2}\right)^2\right). \end{aligned}$$

So

$$\begin{aligned} \int \frac{x^2}{\sqrt{4x-x^2}}\,dx &= \int \frac{x^2}{\sqrt{4\left(1-\left(\frac{x-2}{2}\right)^2\right)}}\,dx \\ &= \frac{1}{2}\int \frac{x^2}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}}\,dx. \end{aligned}$$

We make the substitution

$$\frac{x-2}{2} = \sin\theta \qquad \Longrightarrow \qquad x = 2\sin\theta + 2 \tag{2}$$

which gives

$$dx = 2\cos\theta\,d\theta.$$

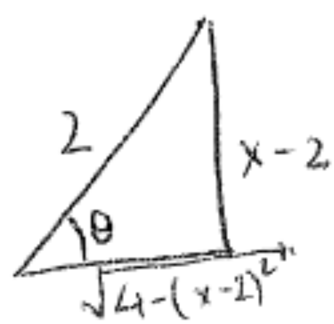
Continuing with the integral, we have that

$$\begin{aligned}
\frac{1}{2}\int \frac{x^2}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}}\,dx &= \frac{1}{2}\int \frac{(2\sin\theta+2)^2}{\sqrt{1-\sin^2\theta}}\cdot 2\cos\theta\,d\theta \\
&= \int (2\sin\theta+2)^2\,d\theta \\
&= \int (4\sin^2\theta+8\sin\theta+4)\,d\theta \\
&= \int (2(1-\cos(2\theta))+8\sin\theta+4)\,d\theta \\
&= \int (6+8\sin\theta-2\cos(2\theta))\,d\theta \\
&= 6\theta-8\cos\theta-\sin(2\theta)+C.
\end{aligned}$$

Now all that is left to do is to reverse-substitute for θ. First, from equation (2) we have that

$$\theta = \arcsin\left(\frac{x-2}{2}\right).$$

Now, we again use equation (2) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\begin{aligned}
\cos\theta &= \frac{\sqrt{4-(x-2)^2}}{2} \\
\sin(2\theta) &= 2\sin\theta\cos\theta = 2\cdot\frac{x-2}{2}\cdot\frac{\sqrt{4-(x-2)^2}}{2}.
\end{aligned}$$

Thus

$$\int \frac{x^2}{\sqrt{4x-x^2}}\,dx = 6\arcsin\left(\frac{x-2}{2}\right) - 4\sqrt{4-(x-2)^2} - \frac{(x-2)\sqrt{4-(x-2)^2}}{2}.$$

(d)

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx.$$

Solution: *First, notice that*

$$\sqrt{e^{2x}+9} = \sqrt{9\left(\frac{e^{2x}}{9}+1\right)} = 3\sqrt{\left(\frac{e^x}{3}\right)^2+1}.$$

So

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx = \frac{1}{3}\int \frac{e^x}{\sqrt{\left(\frac{e^x}{3}\right)^2+1}}\,dx.$$

We make the substitution

$$\frac{e^x}{3} = \tan\theta \qquad \Longrightarrow \qquad 3\tan\theta = e^x \tag{3}$$

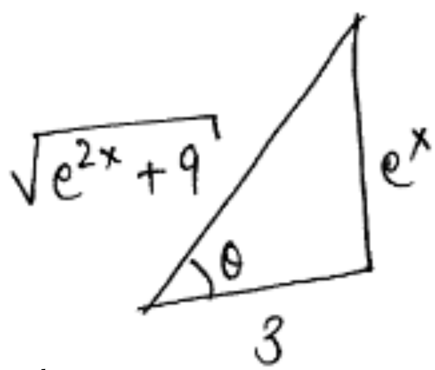
which gives

$$e^x\,dx = 3\sec^2\theta\,d\theta.$$

Continuing with the integral, we have that

$$\begin{aligned}\frac{1}{3}\int \frac{e^x}{\sqrt{\left(\frac{e^x}{3}\right)^2+1}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{\tan^2\theta+1}}\cdot 3\sec^2\theta\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

Now all that is left to do is to reverse-substitute for θ. We use equation (3) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\begin{aligned}\sec\theta &= \frac{\sqrt{e^{2x}+9}}{3} \\ \tan\theta &= \frac{e^x}{3}.\end{aligned}$$

Thus

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx = \ln\left(\frac{\sqrt{e^{2x}+9}+e^x}{3}\right) + C.$$

(e)

$$\int \frac{dx}{x^{\frac{1}{2}} - 9x^{\frac{3}{2}}}.$$

Solution:

$$\begin{aligned}
\int \frac{dx}{x^{\frac{1}{2}} - 9x^{\frac{3}{2}}} &= \int \frac{1}{x^{\frac{1}{2}}(1-9x)}\, dx \\
&= \int \frac{1}{x^{\frac{1}{2}}\left(1 - \left(3x^{\frac{1}{2}}\right)^2\right)}\, dx \\
&= \frac{2}{3}\int \frac{1}{1-u^2}\, dx \qquad \text{where } u = 3x^{\frac{1}{2}} \\
&= \frac{2}{3}\int \frac{1}{1-\sin^2\theta}\cos\theta\, d\theta \qquad \text{where } u = \sin\theta \\
&= \frac{2}{3}\int \frac{\cos\theta}{\cos^2\theta}\, d\theta \\
&= \frac{2}{3}\int \sec\theta\, d\theta \\
&= \frac{2}{3}\ln|\sec\theta + \tan\theta| + C \\
&= \frac{2}{3}\ln\left|\frac{1}{\sqrt{1-u^2}} + \frac{u}{\sqrt{1-u^2}}\right| + C \\
&= \frac{2}{3}\ln\left(\frac{1+3\sqrt{x}}{\sqrt{1-9x}}\right) + C
\end{aligned}$$