Recitation #25: Vectors in the plane - Solutions

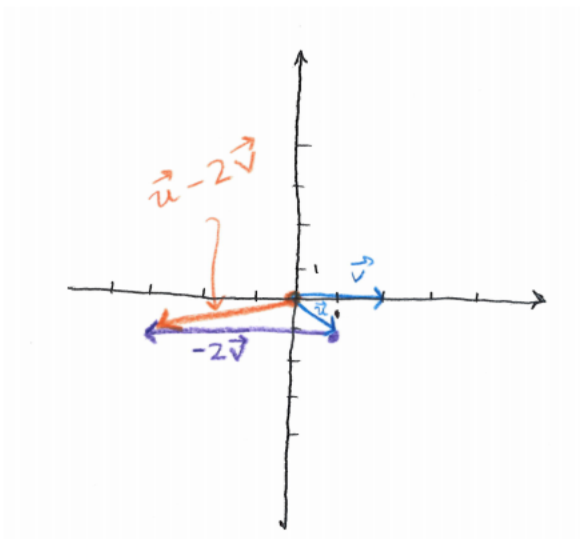
Warm up:

(a) What is the difference between the notations $\hat{\imath}$, $\hat{\jmath}$, $\hat{u}$ and $\mathbf{i}, \mathbf{j}, \mathbf{u}$?

(b) Sketch the vectors $\mathbf{u} = \langle 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0 \rangle$. Now using your sketch of these vectors, sketch $\mathbf{u} - 2\mathbf{v}$.

Solution: (a) There is no difference. The "hat" versions are typically written by hand, while the "bold" versions are typically in print.

(b) To add vectors, we put the tail of the second vector on the head of the first.



Learning outcomes:

Group work:

Problem 1 *Suppose that* $\mathbf{u} = \langle 5, -1 \rangle$ *and* $\mathbf{v} = \langle 2, 3 \rangle$. *Find the following quantities:*

(a) $-\mathbf{v}$

(b) $3\mathbf{u} - 4\mathbf{v}$

(c) $|\mathbf{u}|$

(d) $|\mathbf{u} - 2\mathbf{v}|$

Solution: (a) $-\mathbf{v} = \langle -2, -3 \rangle$

(b) $3\mathbf{u} - 4\mathbf{v} = \langle 15, -4 \rangle - \langle 8, 12 \rangle = \langle 7, -16 \rangle$.

(c) $|\mathbf{u}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$.

(d) $|\mathbf{u} - 2\mathbf{v}| = |\langle 1, -7 \rangle| = \sqrt{1^2 + (-7)^2} = \sqrt{50} = 5\sqrt{2}$.

Problem 2 *Suppose that* $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$. *Find the following:*

(a) *A unit vector in the same direction of* $\mathbf{u}$.

(b) *All unit vectors parallel to* $\mathbf{u}$. *(How does differ from part (a)?)*

(c) *Two vector parallel to* $\mathbf{u}$ *with length 10.*

Solution: (a) $|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = 5$. *A unit vector in the same direction is* $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \langle \dfrac{3}{5}, \dfrac{-4}{5} \rangle$.

(b) *Parallel unit vectors are* $\pm\dfrac{\mathbf{u}}{|\mathbf{u}|}$, *which are* $\langle \dfrac{3}{5}, \dfrac{-4}{5} \rangle$ *and* $\langle \dfrac{-3}{5}, \dfrac{4}{5} \rangle$. *Note that parallel vectors include vectors in the opposite direction.*

(c) *Since* $\mathbf{u}$ *has length 5, two parallel vectors of length 10 are* $\pm 2\mathbf{u}$, *which are* $\langle 6, -8 \rangle$ *and* $\langle -6, 8 \rangle$.

Problem 3 *Assume that* $\vec{u} = \dfrac{1}{2}\hat{\imath} + \dfrac{\sqrt{3}}{2}\hat{\jmath}$ *and* $\vec{v} = \dfrac{\sqrt{3}}{2}\hat{\imath} - \dfrac{1}{2}\hat{\jmath}$.

(a) *Show that* $\vec{u}$ *and* $\vec{v}$ *are unit vectors.*

(b) *Write* $\hat{\imath}$ *as* $a_1\vec{u} + b_1\vec{v}$ *for some real numbers* a_1 *and* b_1.

(c) *Write* $\hat{\jmath}$ *as* $a_2\vec{u} + b_2\vec{v}$ *for some real numbers* a_2 *and* b_2.

Solution: (a) *First, to show that both vectors are unit vectors, we compute their magnitudes that they are equal to one.*

$$|\vec{u}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$
$$|\vec{v}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.$$

(b)

$$\begin{aligned}\hat{\imath} &= a_1\vec{u} + b_1\vec{v}\\ &= a_1\left(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}\right) + b_1\left(\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}\right)\\ &= \left(\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}b_1\right)\hat{\imath} + \left(\frac{\sqrt{3}}{2}a_1 - \frac{1}{2}b_1\right)\hat{\jmath}. \qquad (1)\end{aligned}$$

Therefore, we must have that

$$\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}b_1 = 1 \qquad \text{and} \qquad \frac{\sqrt{3}}{2}a_1 - \frac{1}{2}b_1 = 0.$$

Solving the right-hand equation for b_1 *we have that*

$$b_1 = \sqrt{3}a_1.$$

Plugging this into the left-hand equation gives

$$\begin{aligned} & 1 = \frac{1}{2}a_1 + \frac{\sqrt{3}}{2}\cdot\sqrt{3}a_1\\ \Longrightarrow\quad & 1 = \frac{1}{2}a_1 + \frac{3}{2}a_1 = 2a_1\\ \Longrightarrow\quad & a_1 = \frac{1}{2}.\end{aligned}$$

So $b_1 = \frac{\sqrt{3}}{2}$, *and therefore*

$$\boxed{\hat{\imath} = \frac{1}{2}\vec{u} + \frac{\sqrt{3}}{2}\vec{v}}.$$

(c) *In the same manner as in equation (1) we have that*

$$\hat{\jmath} = \left(\frac{1}{2}a_2 + \frac{\sqrt{3}}{2}b_2\right)\hat{\imath} + \left(\frac{\sqrt{3}}{2}a_2 - \frac{1}{2}b_2\right)\hat{\jmath}$$

and so

$$\frac{1}{2}a_2 + \frac{\sqrt{3}}{2}b_2 = 0 \qquad \text{and} \qquad \frac{\sqrt{3}}{2}a_2 - \frac{1}{2}b_2 = 1.$$

Now, solving the left-hand equation for a_2 gives

$$a_2 = -\sqrt{3}b_2.$$

Then plugging into the right-hand equation yields

$$\begin{aligned} & 1 = \frac{\sqrt{3}}{2} \cdot (-\sqrt{3}b_2) - \frac{1}{2}b_2 \\ \Longrightarrow \quad & 1 = -\frac{3}{2}b_2 - \frac{1}{2}b_2 = -2b_2 \\ \Longrightarrow \quad & b_2 = -\frac{1}{2}. \end{aligned}$$

So $a_2 = \dfrac{\sqrt{3}}{2}$, and therefore

$$\boxed{\hat{\jmath} = \frac{\sqrt{3}}{2}\vec{u} - \frac{1}{2}\vec{v}}$$