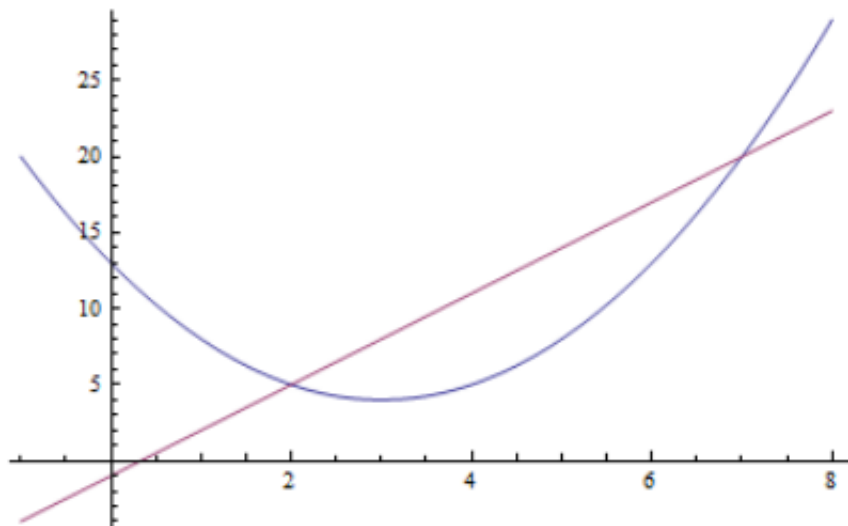


Recitation # 4: Volume by Shells & Length of Curves - Solutions

Group work:

Problem 1 Set up an integral that will compute the volume of the solid generated by revolving the region bounded by the curves $y = x^2 - 6x + 13$ (i.e. $x = 3 \pm \sqrt{y - 4}$) and $y = 3x - 1$ about:



Use both the washer method as well as the shell method for each problem. Which method would you prefer for each problem? Why?

(a) the x -axis

Learning outcomes:

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Solution: First, we need to find the points where the curves intersect

$$x^2 - 6x + 13 = 3x - 1$$

$$x^2 - 9x + 14 = 0$$

$$(x - 2)(x - 7) = 0$$

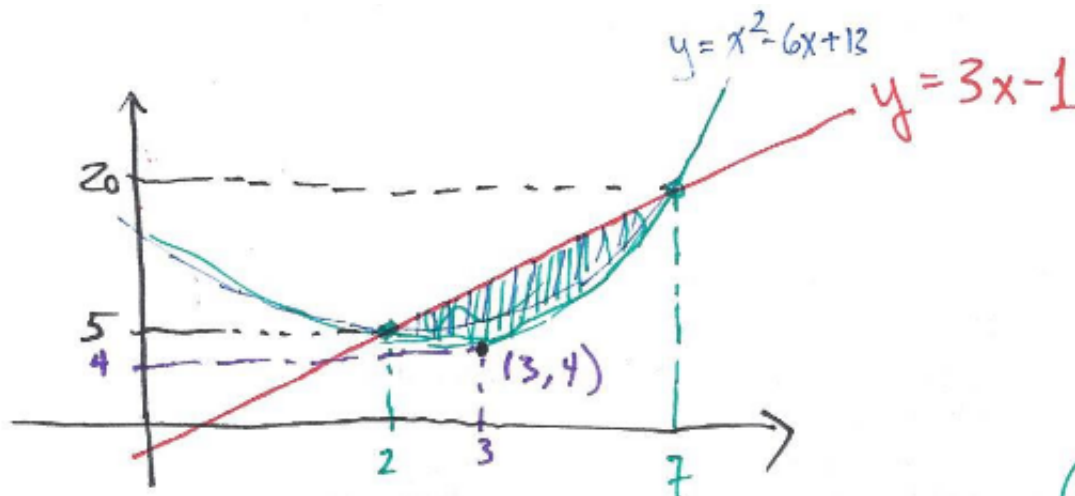
$$x = 2, 7$$

$$(2, 5), (7, 20).$$

As we will see later, we also need to locate the vertex of the parabola $x^2 - 6x + 13$. So we complete the square

$$\begin{aligned} y &= x^2 - 6x + 13 \\ &= (x^2 - 6x + 9) + 13 - 9 \\ &= (x - 3)^2 + 4. \end{aligned}$$

So the vertex of the parabola is $(3, 4)$.



Washers: For washers, the cross-sections must be **perpendicular** to the axis of rotation. So here we integrate along the x -axis. We have that

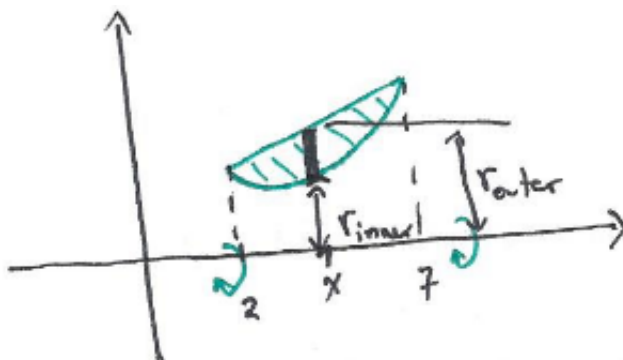
$$r_{out} = 3x - 1$$

$$r_{in} = x^2 - 6x + 13$$

and

$$\text{Volume of the region} = \pi \int_2^7 [(3x - 1)^2 - (x^2 - 6x + 13)^2] dx.$$

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Shells: For shells, the cross-sections must be **parallel** to the axis of rotation. So here we integrate along the y -axis, $5 \leq y \leq 20$. But we have a problem, namely the “bottom” of each cross-section changes at $y = 5$ due to the shape of the region (see the picture below). So we have two different cases:

(1) For $4 \leq y \leq 5$

$$h = (3 + \sqrt{y-1}) - (3 - \sqrt{y-1}) = 2\sqrt{y-1}$$

$$r = y$$

(2) For $5 \leq y \leq 20$

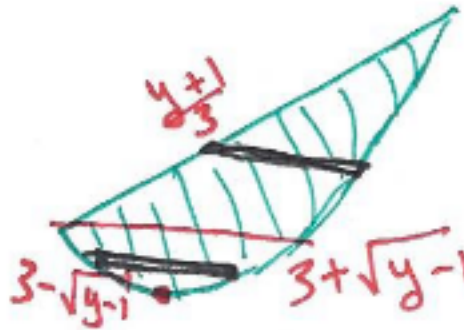
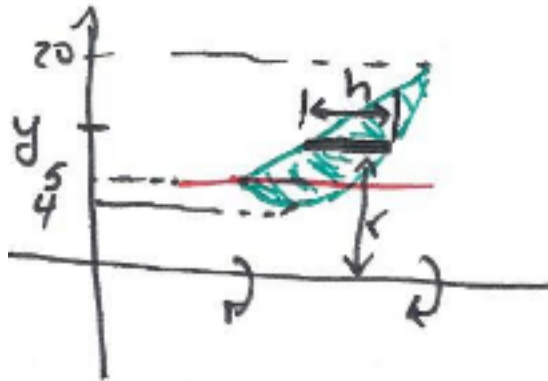
$$h = (3 + \sqrt{y-1}) - \left(\frac{1}{3}(y+1)\right)$$

$$r = y.$$

Thus

$$V = \int_4^{20} 2\pi r h \, dy = 2\pi \left[\int_4^5 y \cdot 2\sqrt{y-1} \, dy + \int_5^{20} y \left((3 + \sqrt{y-1}) - \frac{1}{3}(y+1) \right) \, dy \right]$$

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It is pretty clear in this problem that the washer's method was easier than the shell's method.

(b) $y = -4$

Solution: Washers: For washers, the cross-sections must be **perpendicular** to the axis of rotation. So here we integrate along the x -axis. We have that

$$r_{out} = 4 + (3x - 1) = 3x + 3$$

$$r_{in} = 4 + (x^2 - 6x + 13) = x^2 - 6x + 17$$

and

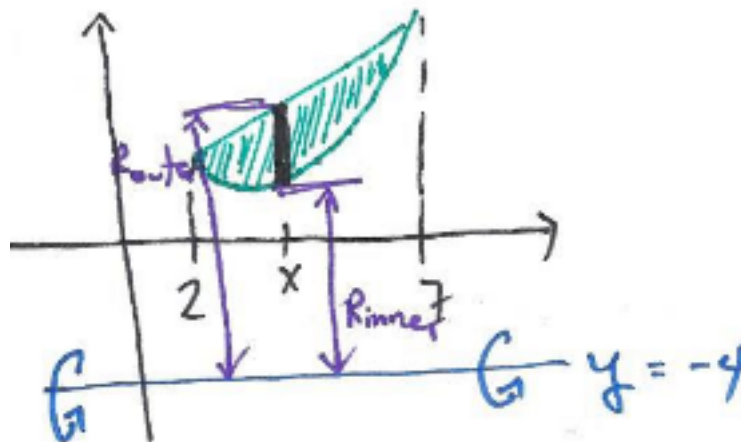
$$V = \pi \int_2^7 [(3x + 3)^2 - (x^2 - 6x + 17)^2] dx.$$

Shells: For shells, the cross-sections must be **parallel** to the axis of rotation. So here we integrate along the y -axis. Just as before though, the equation determining the bottom of the shell changes at $y = 5$. The

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height of the shell over the two regions is the same as part (a), but now the radius of the shell is $4 + y$. So,

$$V = 2\pi \left[\int_4^5 (4 + y) \cdot 2\sqrt{y-1} \, dy + \int_5^{20} (4 + y) \left((3 + \sqrt{y+1}) - \frac{1}{3}(y+1) \right) dy \right]$$



Again, it is pretty clear that the washer's method was easier than the shell's method for this problem.

(c) $y = 22$

Solution: Washers: For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the x -axis. We have that

$$\begin{aligned} r_{out} &= 22 - (x^2 - 6x + 13) = -x^2 + 6x + 9 \\ r_{in} &= 22 - (3x - 1) = -3x + 23 \end{aligned}$$

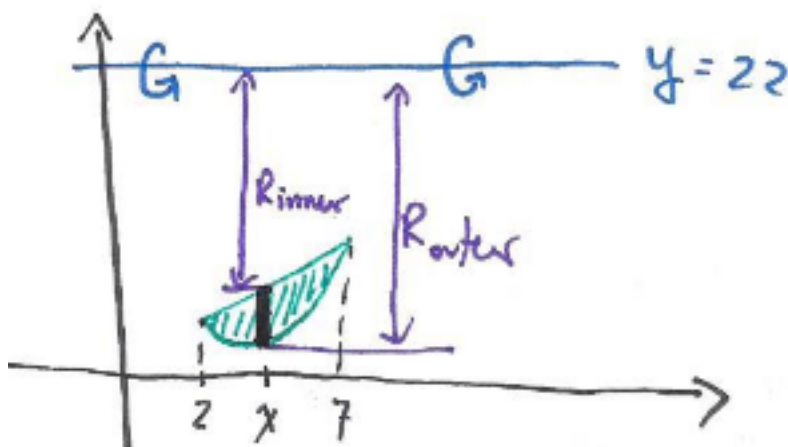
and so

$$V = \pi \int_2^7 [(-x^2 + 6x + 9)^2 - (-3x + 23)^2] \, dx.$$

Shells: For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the y -axis. Just as before though, the equation determining the bottom of the shell changes at $y = 5$. The height of the shell over the two regions is the same as part (a), but now the radius of each shell is $22 - y$. So,

$$V = 2\pi \left[\int_4^5 (22 - y) \cdot 2\sqrt{y-1} \, dy + \int_5^{20} (22 - y) \left((3 + \sqrt{y+1}) - \frac{1}{3}(y+1) \right) dy \right]$$

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Once again, the washer's method appears to be easier.

(d) the y -axis

Solution: Washers: For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the y -axis (notice the change from the three preceding problems). Just as in the shells method for the previous three parts, we have to break up our region of integration at $y = 5$. We have the following two cases

(1) for $4 \leq y \leq 5$

$$r_{out} = 3 + \sqrt{y-1}$$

$$r_{in} = 3 - \sqrt{y-1}$$

(2) for $5 \leq y \leq 20$

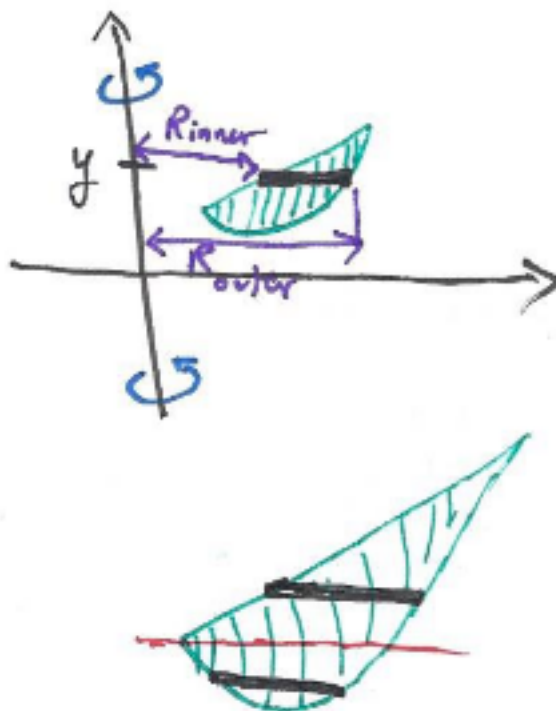
$$r_{out} = 3 + \sqrt{y-1}$$

$$r_{in} = \frac{1}{3}(y+1)$$

So,

$$V = \pi \left[\int_4^5 \left((3 + \sqrt{y-1})^2 - (3 - \sqrt{y-1})^2 \right) dy + \int_5^{20} \left((3 + \sqrt{y-1})^2 - \frac{1}{9}(y+1)^2 \right) dy \right].$$

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Shells: For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the x -axis (again, notice the change from the three preceding problems). This time, we do not need to break up the region! Our shells will always have the parameters

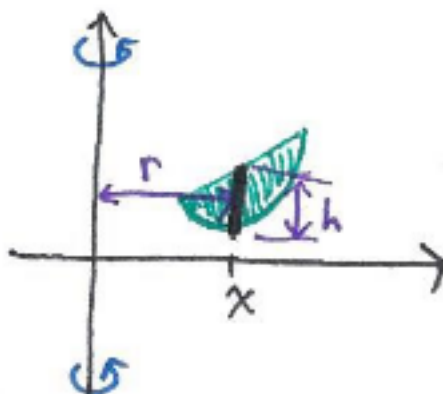
$$h = (3x - 1) - (x^2 - 6x + 13) = -x^2 + 9x - 14$$

$$r = x$$

So,

$$V = 2\pi \int_2^7 x(-x^2 + 9x - 14) dx.$$

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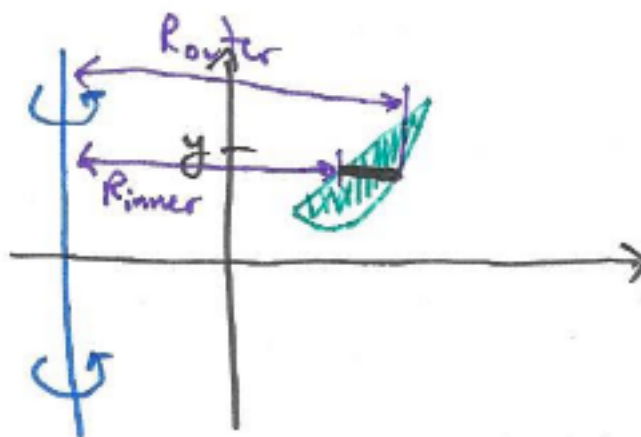


Now, notice that the shells method was a lot easier. The key for this region is that you want your cross sections to always be perpendicular to the x -axis. That way, the bounds for your parameters do not change over the entire region of integration.

(e) $x = -3$

Solution: Washers: For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the y -axis. Just as in part (d), we need to split the region up into two parts. But shifting the axis of rotation to $x = -3$ just adds three to all radii. So

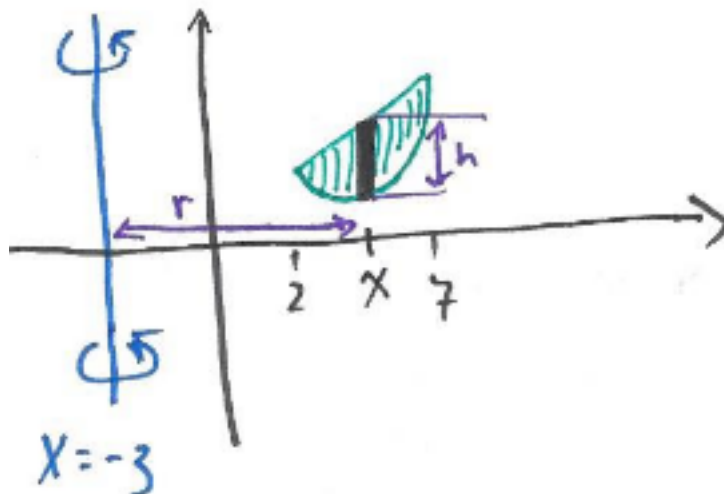
$$V = \pi \left[\int_4^5 \left((6 + \sqrt{y-1})^2 - (6 - \sqrt{y-1})^2 \right) dy + \int_5^{20} \left((6 + \sqrt{y-1})^2 - \left(3 + \frac{1}{3}(y+1) \right)^2 \right) dy \right].$$



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Shells: For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the x -axis. Shifting the axis of rotation to $x = -3$ simply adds three to the radius of the shell, so

$$V = 2\pi \int_2^7 (3+x)(-x^2 + 9x - 14) dx.$$



The shells method was simpler.

(f) $x = 9$

Solution: Washers: For washers, the cross-sections must be perpendicular to the axis of rotation. So here we integrate along the y -axis. Just as in parts (d) and (e), we need to split the region up into two parts. Via the figure below, we compute

(1) for $4 \leq y \leq 5$

$$r_{out} = 9 - (3 - \sqrt{y-1}) = 6 + \sqrt{y-1}$$

$$r_{in} = 9 - (3 + \sqrt{y-1}) = 6 - \sqrt{y-1}$$

(2) for $5 \leq y \leq 20$

$$r_{out} = 9 - \left(\frac{1}{3}(y+1)\right)$$

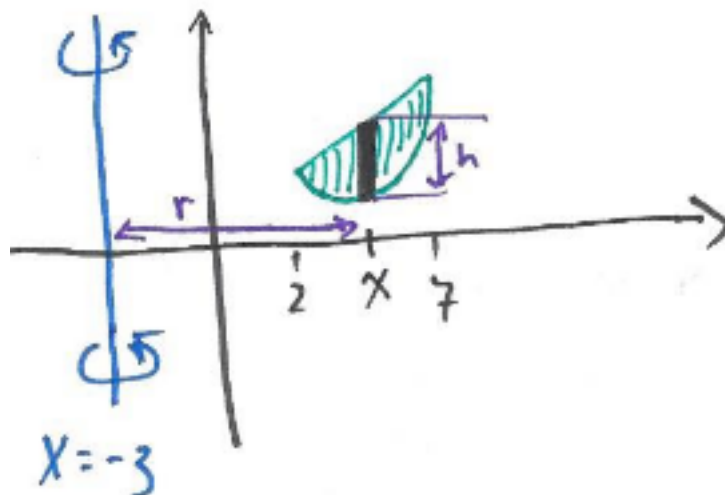
$$r_{in} = 9 - (3 + \sqrt{y-1}) = 6 - \sqrt{y-1}$$

So,

$$V = \pi \left[\int_4^5 \left((6 + \sqrt{y-1})^2 - (6 - \sqrt{y-1})^2 \right) dy \right]$$

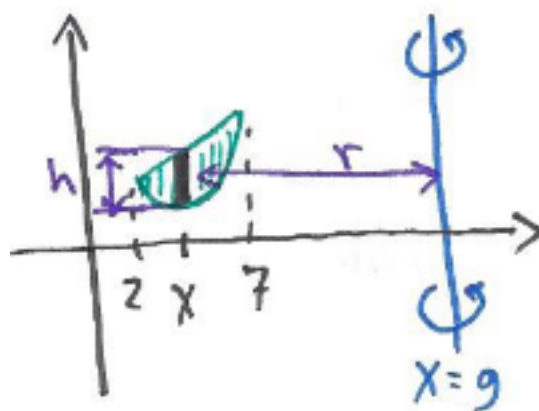
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$$+ \int_5^{20} \left(\left(9 - \frac{1}{3}(y+1) \right)^2 - \left(6 - \sqrt{y-1} \right)^2 \right) dy \Bigg].$$



Shells: For shells, the cross-sections must be parallel to the axis of rotation. So here we integrate along the x -axis. Shifting the axis of rotation to the right 9 units (from part (d)) does not change the height of the cylinder, and the radius changes to $9 - x$. So

$$V = 2\pi \int_2^7 (9 - x)(-x^2 + 9x - 14) dx.$$



Again, the shell method was much simpler.

Problem 2 Set up an integral (or a sum of integrals) to find the perimeter of the region bounded by the curves $y = 2x^2 - 5x + 13$ and $y = x^2 + 6x - 11$.

Solution: Let $f(x) = 2x^2 - 5x + 13$ and $g(x) = x^2 + 6x - 11$. We first need to find the points where these two curves intersect. So we solve

$$\begin{aligned} f(x) &= g(x) \\ 2x^2 - 5x + 13 &= x^2 + 6x - 11 \\ x^2 - 11x + 24 &= 0 \\ (x - 3)(x - 8) &= 0 \\ x &= 3, 8. \end{aligned}$$

Then the perimeter is $L_1 + L_2$ where

$$\begin{aligned} L_1 &= \int_3^8 \sqrt{1 + f'(x)^2} dx = \int_3^8 \sqrt{1 + (4x - 5)^2} dx \\ L_2 &= \int_3^8 \sqrt{1 + g'(x)^2} dx = \int_3^8 \sqrt{1 + (2x + 6)^2} dx. \end{aligned}$$

Problem 3 A steady wind blows a kite due west. The kite's height above the ground from horizontal position $x = 0$ ft. to $x = 80$ ft. is given by

$$y = 150 - \frac{1}{40}(x - 50)^2.$$

Set up the integral to find the distance traveled by the kite.

Solution:

$$\text{distance the kite traveled} = \int_0^{80} \sqrt{1 + y'(x)^2} dx$$

where $y(x) = 150 - \frac{1}{40}(x - 50)^2$. Then since $y' = -\frac{1}{20}(x - 50)$, we have that

$$\text{distance the kite traveled} = \int_0^{80} \sqrt{1 + \frac{(x - 50)^2}{400}} dx.$$