Section 9.3: Infinite Series

Warm-Up:

Problem 1 *Given a sequence $\{a_k\}_{k\geq 1}$, explain how the sequence $\{s_k\}_{k\geq 1}$ of partial sums can be used to determine if the series $\sum_{k=1}^{\infty} a_k$ converges or diverges.*

Hint: *Recall that by definition $s_n = \sum_{k=1}^{n} a_k$*

Solution: Note that $s_n = a_1 + a_2 + a_3 + \ldots + a_n$. Thus, s_n is a sequence whose n^{th} term is the result of *adding* the first n terms in the sequence $\{a_k\}$. If we continue adding the terms in the sequence $\{a_k\}$ indefinitely, we will arrive at the infinite series $\sum_{k=1}^{n} a_k$. This result is obtained by taking the limit of the sequence $\{s_n\}$! Formally, we can write:

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$$

Thus, we say $\sum_{k=1}^{\infty} a_k$ converges if and only if $\lim_{n\to\infty} s_n$ exists and $\sum_{k=1}^{\infty} a_k$ diverges if and only if $\lim_{n\to\infty} s_n$ does not exist.

Group work:

Problem 2 *Determine if the following series converge or diverge. If they converge, find the sum.*

(a) $e + 1 + e^{-1} + e^{-2} + e^{-3} + \ldots$

Learning outcomes:

Solution:

$$\begin{aligned} e + 1 + e^{-1} + e^{-2} + e^{-3} + \ldots &= e + \sum_{k=0}^{\infty} e^{-k} \\ &= e + \sum_{k=0}^{\infty} \left(e^{-1}\right)^k \qquad \text{geometric series, } r = e^{-1} < 1 \\ &= e + \frac{1}{1-e^{-1}}. \end{aligned}$$

Therefore, this series converges to $\left(e + \dfrac{1}{1-e^{-1}}\right)$.

(b) $\displaystyle\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k}$

Solution: *Let us analyze the two different summands in this problem:*

(i) $\displaystyle\sum_{k=0}^{99} 2^k$

This is a finite sum from a geometric sequence, and so its sum is

$$\frac{a(1-r^n)}{1-r}.$$

Thus,

$$\sum_{k=0}^{99} 2^k = \frac{1(1-2^{100})}{1-2} = 2^{100} - 1.$$

(ii) $\displaystyle\sum_{k=100}^{\infty} \frac{1}{2^k} = \sum_{k=100}^{\infty} \left(\frac{1}{2}\right)^k.$

This is a geometric series with $a = \dfrac{1}{2^{100}}$ and $r = \dfrac{1}{2}$. *So*

$$\sum_{k=100}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{100}}}{1-\frac{1}{2}} = \frac{1}{2^{99}}.$$

Therefore, combining parts (i) and (ii) we have that

$$\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k} = 2^{100} - 1 + \frac{1}{2^{99}}.$$

(c) $\displaystyle\sum_{k=0}^{\infty} (\cos(1))^k$

Solution: *This is a geometric series with $a = 1$ and $r = \cos(1)$. We know that $-1 < \cos(1) < 1$, and so $|\cos(1)| < 1$. Therefore, this geometric series converges and*

$$\sum_{k=0}^{\infty} (\cos(1))^k = \frac{1}{1-\cos(1)}.$$

(d) $\displaystyle\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}}$

Solution: *Let us first reindex this series. Let $\ell = k-4$. Then $k = \ell+4$, and when $k = 4$, $\ell = 0$. We then have that*

$$\begin{aligned}
\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}} &= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{\ell+4+3}}{7^{\ell+4-2}} \\
&= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{\ell+7}}{7^{\ell+2}} \\
&= \sum_{\ell=0}^{\infty} \frac{5 \cdot 4^{7} \cdot 4^{\ell}}{7^{2} \cdot 7^{\ell}} \\
&= \frac{5 \cdot 4^7}{7^2} \sum_{\ell=0}^{\infty} \left(\frac{4}{7}\right)^{\ell} \qquad \textit{assuming this series converges} \\
&= \frac{5 \cdot 4^7}{7^2} \cdot \frac{1}{1-\frac{4}{7}} \qquad \textit{geometric series with } a = 1, r = \frac{4}{7} \\
&= \frac{5 \cdot 4^7}{3 \cdot 7}.
\end{aligned}$$

Therefore, this series converges to $\dfrac{5 \cdot 4^7}{3 \cdot 7}$.

(e) $\displaystyle\sum_{k=0}^{\infty} e^{5-2k}$

Solution:

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{5-2k} &= \sum_{k=0}^{\infty} \left[e^5 \cdot \left(e^{-2}\right)^k\right] \\
&= e^5 \sum_{k=0}^{\infty} \left(e^{-2}\right)^k \qquad \textit{assuming the series converges} \\
&= e^5 \cdot \frac{1}{1-e^{-2}} \qquad \textit{geometric series with } a = 1, r = e^{-2} < 1
\end{aligned}$$

Therefore, this series converges to $\dfrac{e^5}{1-e^{-2}}$.

(f) $\displaystyle\sum_{i=1}^{\infty}\left(\frac{2}{i^2+2i}\right)$ *Hint:* $\dfrac{2}{i^2+2i}=\dfrac{1}{i}-\dfrac{1}{i+2}$ *by partial fractions*

Solution: *This is a telescoping series. Let*

$$S_n=\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+2}\right).$$

Then,

$$\begin{aligned}
S_n &= \sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+2}\right)\\
&= \left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\\
&\quad+\ldots+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)\\
&= 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}.
\end{aligned}$$

Note that the last equality above is because all of the non-red terms cancel (convince yourself of this). Then

$$\begin{aligned}
\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+2}\right) &= \lim_{n\to\infty} S_n\\
&= \lim_{n\to\infty}\left[1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]\\
&= 1+\frac{1}{2}=\frac{3}{2}.
\end{aligned}$$