Section 10.4: Working with Taylor Series

You should memorize the Maclaurin series for $\cos(x), \sin(x), e^x$, and $\dfrac{1}{1-x}$. If you need the Maclaurin series for $\ln(1+x)$, $\arctan(x)$, or the binomial series these will be given to you.

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \qquad -1 < x \le 1$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad -1 \le x \le 1$$

Warm up:

True or False: To approximate $\dfrac{\pi}{3}$, one could substitute $x = \sqrt{3}$ into the Maclaurin series for $\tan^{-1} x$?

Solution: **False.** The power series representation

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

only converges on $[-1, 1]$. Since $\sqrt{3}$ is outside of the IOC, one cannot substitute $x = \sqrt{3}$ into this series to approximate $\dfrac{\pi}{3}$.

Group work:

Problem 1 *Use power series to evaluate the limit*

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{1 - \cos x}$$

Solution: *We essentially want to use the Taylor series for these functions centered at $x = 0$ to compute the limit. To do so, we will only need to write*

Learning outcomes:

out a few terms and perform some basic algebra. To justify why we can use the Taylor series expansion, note for $-1 < x \leq 1$, we know that

$$\ln(1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x^2)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{k}.$$

For $-\infty < x < \infty$ we also know that

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Since 0 is within both of these intervals, we can substitute these formulas into the limit:

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} &= \lim_{x\to 0} \frac{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{k}}{1-\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}} \\
&= \lim_{x\to 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right)} \quad \textcolor{red}{\textit{Write out the first few terms in each Taylor series}} \\
&= \lim_{x\to 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots}{\frac{x^2}{2!}-\frac{x^4}{4!}+\dots} \\
&= \lim_{x\to 0} \frac{x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots\right)}{x^2\left(\frac{1}{2!}-\frac{x^2}{4!}+\dots\right)} \\
&= \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots}{\frac{1}{2!}-\frac{x^2}{4!}+\dots}
\end{aligned}$$

$$= \frac{1}{\frac{1}{2}} = \boxed{2}.$$

Problem 2 *Given that*

$$f(t) = \int_0^t x^2 \tan^{-1}(x^4)\, dx$$

approximate $f\left(\frac{1}{3}\right)$ with the first four non-zero terms of a power series. Estimate how close this approximation is.

Solution:

$$
\begin{aligned}
f\left(\frac{1}{3}\right) &= \int_0^{\frac{1}{3}} x^2 \arctan(x^4)\,dx \\
&= \int_0^{\frac{1}{3}} \left(x^2 \sum_{k=0}^{\infty} \frac{(-1)^k (x^4)^{2k+1}}{2k+1}\right) dx \\
&\approx \int_0^{\frac{1}{3}} x^2 \left(x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7}\right) dx \quad \textit{Using 4 terms of the power series} \\
&= \int_0^{\frac{1}{3}} \left(x^6 - \frac{x^{14}}{3} + \frac{x^{22}}{5} - \frac{x^{30}}{7}\right) dx \\
&= \left[\frac{x^7}{7} - \frac{x^{15}}{45} + \frac{x^{23}}{115} - \frac{x^{31}}{217}\right]_0^{\frac{1}{3}} \\
&= \boxed{\frac{1}{3^7 \cdot 7} - \frac{1}{3^{15} \cdot 45} + \frac{1}{3^{23} \cdot 115} - \frac{1}{3^{31} \cdot 217}}.
\end{aligned}
$$

The error of the series is less than the first truncated term, which here is the fifth term

$$x^2 \cdot \frac{(x^4)^9}{9} = \frac{x^{38}}{9}.$$

So we integrate

$$\int_0^{\frac{1}{3}} \frac{x^{38}}{9}\,dx = \left[\frac{x^{39}}{351}\right]_0^{\frac{1}{3}} = \frac{1}{3^{39} \cdot 351}.$$

Therefore, an upper bound for the error is $\boxed{\dfrac{1}{3^{39} \cdot 351}}$

Problem 3 *Use power series to determine a (series) solution to the initial value problem*

$$y'' - xy' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

Solution: *Assume that $y(x)$ is a power series solution centered at 0. ie,*

$$y = \sum_{k=0}^{\infty} c_k x^k \qquad \text{where} \qquad c_k = \frac{y^{(k)}(0)}{k!}.$$

We are given that $y(0) = 1$ and $y'(0) = 0$. Using these, we can find c_0 and c_1:

$$
\begin{aligned}
c_0 &= \frac{y^{(0)}(0)}{0!} = \frac{1}{1} = 1 \\
c_1 &= \frac{y'(0)}{1!} = \frac{0}{1} = 0.
\end{aligned}
$$

To find c_k, we need to first find $y^{(k)}(0)$. Using the differential equation, we can directly find $y''(0)$.

$$\begin{aligned} y''(0) - 0 \cdot y'(0) + y(0) &= 0 \\ y''(0) - 0 + 1 &= 0 \\ y''(0) &= -1. \end{aligned}$$

So

$$c_2 = \frac{y''(2)}{2!} = \frac{-1}{2}.$$

To find $y^{(3)}(0)$, we need to differentiate the differential equation (using the product rule):

$$\begin{aligned} \frac{d}{dx}\left(y'' - xy' + y\right) &= \frac{d}{dx}(0) \\ y^{(3)} - (y' + xy'') + y' &= 0 \\ y^{(3)} - xy'' &= 0 \\ y^{(3)}(0) - 0 \cdot y''(0) &= 0 \\ y^{(3)}(0) &= 0. \end{aligned}$$

In exactly the same manner, we can compute

$$\begin{aligned} &y^{(4)}(0) = -1 \\ &y^{(5)}(0) = 0 \\ &y^{(6)}(0) = -3 \\ &y^{(7)}(0) = 0 \\ &y^{(8)}(0) = -5(3) \\ &y^{(9)}(0) = 0 \\ &y^{(10)}(0) = -7(5)(3) \\ &y^{(11)}(0) = 0 \\ &y^{(12)}(0) = -9(7)(5)(3) \end{aligned}$$

In particular, notice that $y^{(k)}(0) = 0$ when k is odd. So in finding c_k, we split into the cases when k is odd and when k is even.

$$\begin{aligned} &c_{2k+1} = 0 \\ &c_{2k} = \frac{-3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2k-3)}{(2k)!} = \frac{-1}{2^k k!(2k-1)}. \end{aligned}$$

Thus, the power series solution to the original initial value problem is

$$\boxed{\sum_{k=0}^{\infty} \left(\frac{-1}{2^k k!(2k-1)}\right) x^{2k}}$$

Problem 4 *Identify the function represented by the power series*

$$\sum_{k=0}^{\infty} \frac{k(k-1)x^k}{7^k}$$

Solution:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{k(k-1)x^k}{7^k} &= x^2 \sum_{k=0}^{\infty} \frac{k(k-1)x^{k-2}}{7^k} \\
&= x^2 \frac{d^2}{dx^2}\left(\sum_{k=2}^{\infty} \frac{x^k}{7^k}\right) \\
&= x^2 \frac{d^2}{dx^2}\left(\sum_{k=2}^{\infty} \left(\frac{x}{7}\right)^k\right) \\
&= x^2 \frac{d^2}{dx^2}\left(\frac{\frac{x^2}{49}}{1-\frac{x}{7}}\right) \qquad \textit{geometric series, valid for } \left|\frac{x}{7}\right| < 1 \\
&= x^2 \frac{d^2}{dx^2}\left(\frac{x^2}{7(7-x)}\right) \\
&= x^2 \frac{d}{dx}\left(\frac{2x(49-7x)-(-7)(x^2)}{49(7-x)^2}\right) \\
&= x^2 \frac{d}{dx}\left(\frac{98x-7x^2}{49(7-x)^2}\right) \\
&= x^2 \left(\frac{(98-14x)(49)(7-x)^2-(-98)(7-x)(98x-7x^2)}{49^2(7-x)^4}\right) \\
&= x^2 \left(\frac{(98-14x)(7-x)+2(98x-7x^2)}{49(7-x)^3}\right) \\
&= x^2 \left(\frac{(14-2x)(7-x)+2x(14-x)}{7(7-x)^3}\right) \\
&= x^2 \left(\frac{98-14x-14x+2x^2+28x-2x^2}{7(7-x)^3}\right) \\
&= \boxed{\frac{14x}{(7-x)^3}}
\end{aligned}$$

when

$$\left|\frac{x}{7}\right| < 1 \qquad \Longleftrightarrow \qquad |x| < 7.$$