Recitation #15: Sequences and Infinite Series - Instructor Notes

Warm up:

Find the limit of the following sequences as n tends to ∞.

(a) $a_n = \dfrac{n^{1000}}{2^n}$

(b) $b_n = \cos(n\pi)$

(c) $c_n = \cos(n!\pi)$

Instructor Notes:

Group work:

Problem 1 *For each of the following sequences, find the limit as the number of terms approaches infinity.*

(a) $a_n = \left(\dfrac{n+1}{2n}\right)\left(\dfrac{n-2}{n}\right)^{\frac{n}{2}}$

(b) $a_n = \sqrt[n]{3^{2n+1}}$

(c) $a_n = \left(\sqrt{n^2+7} - n\right)$

(d) $a_n = \dfrac{(2n+3)!}{5n^3(2n)!}$

(e) $a_n = (2^n + 3^n)^{\frac{1}{n}}$

$$\textit{Hint: } a_n \geq (0+3^n)^{\frac{1}{n}} = 3 \textit{ and } a_n \leq (2\cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3$$

(f) $a_n = \dfrac{n^{365} + 5^n}{8^n + n^3}$

Instructor Notes: For this problem, give each group two of the problems to report on. Give about 8 minutes (or less?) for the group work and about $10 - 15$ minutes for discussion. On (a), L'Hospital's Rule is involved. On (e), the squeeze theorem should be involved. On (f), the students may use "known" facts of the relative growth of polynomial vs. exponential terms.

Problem 2 *Show that*

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

exists by proving that $a_n = \sqrt{n+1}-\sqrt{n}$ *is a bounded monotonic sequence. A hint is to show that* $f(x) = \sqrt{x+1}-\sqrt{x}$ *is a decreasing function by showing that* $f'(x) < 0$.

Instructor Notes: Perhaps do as a whole class discussion. Emphasize careful writing of reasoning.

Problem 3 *Find the limit of the given sequence. Also, determine if it is a geometric sequence.*

(a) $a_n = \dfrac{n^2}{2^n}$

(b) $a_n = \dfrac{1}{3^n}$

(c) $a_n = \left(\dfrac{1}{n}\right)^4$

(d) $a_n = \dfrac{e^n + (-3)^n}{5^n}$

(e) $a_n = 3^{\frac{1}{n}}$

Instructor Notes: These limits should be relatively easy to analyze. The students need to identify the "r" if it is a geometric sequence (and note that the exponent n is the variable). On (d) and (f), they should argue that they are looking at the sum of two geometric sequences. Maybe give one per group with about 8 minutes for discussion.

Problem 4 *Determine if the following series converge or diverge. If they converge, find the sum.*

(a) $e + 1 + e^{-1} + e^{-2} + e^{-3} + \dots$

(b) $\displaystyle\sum_{k=0}^{99} 2^k + \sum_{k=100}^{\infty} \frac{1}{2^k}$

(c) $\displaystyle\sum_{k=0}^{\infty} (\cos(1))^k$

(d) $\displaystyle\sum_{k=4}^{\infty} \frac{5 \cdot 4^{k+3}}{7^{k-2}}$

(e) $\displaystyle\sum_{k=0}^{\infty} e^{5-2k}$

(f) $\displaystyle\sum_{k=0}^{\infty} \frac{e^k + (-7)^k}{5^k}$

(g) $\displaystyle\sum_{k=0}^{\infty} \left[\frac{5}{(k+1)(k+2)} + \left(-\frac{1}{2}\right)^k\right]$

(h) $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2}\right)$

Instructor Notes: Assign two per group. Most of these involve geometric series and one (part (b)) involves a finite geometric sum, whose "trick" is presented in the lecture.

Students must identify the "r" and pay attention to both the indices and the exponents (for example, $7^{k+3} = 7^3 \cdot 7^k$). Encourage multiple methods (there are 3 methods presented in the lecture) and make sure students clearly explain their reasoning on paper.

Problem 5 *Convert the decimal* $2.456\overline{314}$ *to a fraction using geometric series.*

Instructor Notes: This is a common type of problem in this section. Students have (most likely) never seen a problem with a non-repeating part to the decimal. This should probably be done as a whole class.

Problem 6 *Find all values of x for which the series*

$$f(x) = \sum_{k=0}^{\infty} \frac{(x+3)^k}{2^k}$$

converges.

Instructor Notes: The purpose of this problem is to give the students a preview of the idea of an interval of convergence (to be covered in Chapter 10). Students need to be careful with absolute values and behavior at endpoints. This could be skipped (or presented as a "take home and think about it" question).