Recitation #20: Properties of power series

Warm up:

Suppose that $\displaystyle\sum_{k=0}^{\infty} c_k(x+5)^k$ converges when $x = -9$ and diverges when $x = -1$.
What can be said about the convergence and divergence of the following series?

(a) $\displaystyle\sum_{k=0}^{\infty} c_k$

(b) $\displaystyle\sum_{k=0}^{\infty} c_k(-5)^k$

(c) $\displaystyle\sum_{k=0}^{\infty} c_k(5)^k$

Solution: What is important to note first is that the interval of convergence for the series $\displaystyle\sum_{k=0}^{\infty} c_k(x+5)^k$ is $[-9, -1)$.

(a) Notice that
$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} c_k(-4+5)^k.$$

Then since -4 is in the interval $[-9, -1)$, the series $\displaystyle\sum_{k=0}^{\infty} c_k$ converges.

(b) Since -10 is not in the interval $[-9, -1)$, the series
$$\sum_{k=0}^{\infty} c_k(-5)^k = \sum_{k=0}^{\infty} c_k(-10+5)^k$$
diverges.

(c) Since 0 is not in the interval $[-9, -1)$, the series
$$\sum_{k=0}^{\infty} c_k 5^k = \sum_{k=0}^{\infty} c_k(0+5)^k$$
diverges.

Learning outcomes:

Group work:

Problem 1 *If the series* $\sum\limits_{k=0}^{\infty} a_k (x-2)^k$ *has an interval of convergence of* $[-4, 8)$,
determine the interval of convergence of the following series:

(a) $\sum\limits_{k=300}^{\infty} a_k (x-2)^k$ 　　　 (b) $\sum\limits_{k=0}^{\infty} a_k x^k$ 　　　 (c) $\sum\limits_{k=0}^{\infty} \left(a_k (x-2)^k + \left(\frac{1}{7}\right)^k x^k \right)$

Solution: 　　(a) *This series has exactly the same interval of convergence,* $\boxed{[-4, 8)}$, *since a finite number of terms do not change whether or not a series converges.*

(b) *In the original interval of convergence* $[-4, 8)$, *the center is* $x = 2$ *and the radius of convergence is 6 (which includes the left endpoint, but not the right). So now, the center is* $x = 0$. *Taking an interval about 0 with radius 6, and adding in the left endpoint, gives the IOC of* $\boxed{[-6, 6)}$.

(c) *The interval of convergence of this series is the intersection of the interval of convergence for* $\sum\limits_{k=0}^{\infty} a_k (x-2)^k$ *and* $\sum\limits_{k=0}^{\infty} \left(\frac{1}{7}\right)^k x^k$. *To find the IOC for this second series, we use the Root Test.*

$$\lim_{k \to \infty} \sqrt[k]{\left| \left(\frac{1}{7}\right)^k x^k \right|} = \lim_{k \to \infty} \frac{1}{7} |x|$$
$$= \frac{1}{7} |x|.$$

So the series $\sum\limits_{k=0}^{\infty} \left(\frac{1}{7}\right)^k x^k$ *converges when*

$$\frac{1}{7} |x| < 1$$
$$\implies \quad |x| < 7$$
$$\implies \quad -7 < x < 7.$$

We need to test the endpoints $x = \pm 7$, *but using the divergence test we see that both*

$$\sum_{k=0}^{\infty} (-1)^k \quad \text{and} \quad \sum_{k=0}^{\infty} 1^k$$

diverge. Therefore, the interval of convergence for $\sum\limits_{k=0}^{\infty} \left(\frac{1}{7}\right)^k x^k$ *is* $(-7, 7)$.

Finally, the interval of convergence for the series

$$\sum_{k=0}^{\infty} \left(a_k(x-2)^k + \left(\frac{1}{7}\right)^k x^k \right)$$

is

$$[-4, 8) \cap (-7, 7) = \boxed{[-4, 7)}.$$

Problem 2 *For each of the following, find the domain of $f(x)$ (i.e. find the interval of convergence).*

(a) $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(3x-2)^k}{k \cdot 3^k}$

(c) $f(x) = \displaystyle\sum_{k=2}^{\infty} \frac{x^{3k+2}}{(\ln k)^k}$

(b) $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}} x^k$

Solution: (a) *We use the Ratio Test*

$$\lim_{k\to\infty} \left| \frac{(3x-2)^{k+1}}{(k+1)3^{k+1}} \cdot \frac{k \cdot 3^k}{(3x-2)^k} \right| = \lim_{k\to\infty} \left| \frac{k(3x-2)}{3(k+1)} \right|$$

$$= \left| \frac{3x-2}{3} \right| = \frac{|3x-2|}{3}.$$

We know that this series converges when

$$\frac{|3x-2|}{3} < 1$$
$$\implies \quad |3x-2| < 3$$
$$\implies \quad -3 < 3x-2 < 3$$
$$\implies \quad -1 < 3x < 5$$
$$\implies \quad -\frac{1}{3} < x < \frac{5}{3}.$$

We still need to check the endpoints. When $x = -\dfrac{1}{3}$ we have

$$\sum_{k=1}^{\infty} \frac{1}{k3^k} \left(3\left(-\frac{1}{3}\right) - 2 \right)^k = \sum_{k=1}^{\infty} \frac{(-3)^k}{k3^k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

which converges (conditionally) by the alternating series test.

When $x = \dfrac{5}{3}$ we have

$$\sum_{k=1}^{\infty} \frac{1}{k3^k} \left(3\left(\frac{5}{3} \right) - 2 \right)^k = \sum_{k=1}^{\infty} \frac{3^k}{k3^k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges since it is the Harmonic series.

Therefore, the interval of convergence is $\boxed{\left[-\dfrac{1}{3}, \dfrac{5}{3} \right)}$.

(b) *We again apply the Ratio Test*

$$\lim_{k \to \infty} \left| \frac{(-1)^{k+1} x^{k+1}}{\sqrt{(k+1)^2 + 1}} \cdot \frac{\sqrt{k^2 + 1}}{(-1)^k x^k} \right| = \lim_{k \to \infty} \left| \frac{x\sqrt{k^2 + 1}}{\sqrt{k^2 + 2k + 2}} \right|$$

$$= |x|.$$

So we know that this series converges when

$$|x| < 1 \qquad \Longleftrightarrow \qquad -1 < x < 1.$$

We still need to check the endpoints. When $x = -1$, the series

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2 + 1}}$$

diverges by the Limit Comparison Test (compare with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$).

When $x = 1$, the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2 + 1}}$$

converges conditionally by the alternating series test.

Therefore, the interval of convergence is $\boxed{(-1, 1]}$.

(c) *For this series we apply the Root Test*

$$\lim_{k \to \infty} \sqrt[k]{\left| \frac{x^{3k+2}}{(\ln k)^k} \right|} = \lim_{k \to \infty} \frac{|x|^3 \cdot |x|^{\frac{2}{k}}}{\ln k}$$

$$= |x|^3 \cdot \lim_{k \to \infty} \frac{|x|^{\frac{2}{k}}}{\ln k}$$

$$= |x|^3 \cdot 0 = 0.$$

Therefore, the interval of convergence is $\boxed{(-\infty, \infty)}$.

Problem 3 *In each of the following, give a power series (with an interval of convergence) for the given function. Assume that we know* $\sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$ *on* $(-1, 1)$.

(a) $f(x) = \dfrac{3}{5x-2}$

(b) $f(x) = \dfrac{3x^4}{5x^3-2}$

Solution: (a)

$$
\begin{aligned}
f(x) &= \frac{3}{5x-2} \\
&= \frac{3}{-2\left(1-\left(\frac{5}{2}x\right)\right)} \\
&= -\frac{3}{2}\sum_{k=0}^{\infty}\left(\frac{5}{2}x\right)^k \\
&= \boxed{\sum_{k=0}^{\infty}\left(-\frac{3}{2}\right)\left(\frac{5}{2}x\right)^k}.
\end{aligned}
$$

Since this is a geometric series, it converges if and only if

$$
\left|\frac{5}{2}x\right| < 1
$$
$$
\Longleftrightarrow \quad |x| < \frac{2}{5}
$$
$$
\Longleftrightarrow \quad -\frac{2}{5} < x < \frac{2}{5}.
$$

Therefore, the interval of convergence is $\boxed{\left(-\dfrac{2}{5}, \dfrac{2}{5}\right)}$

(b)

$$
\begin{aligned}
f(x) &= \frac{3x^4}{5x^3-2} \\
&= \frac{-3x^4}{2}\cdot\frac{1}{1-\left(\frac{5}{2}x^3\right)} \\
&= -\frac{3}{2}x^4\sum_{k=0}^{\infty}\left(\frac{5}{2}x^3\right)^k \\
&= \boxed{\sum_{k=0}^{\infty}\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right)^k x^{3x+4}}.
\end{aligned}
$$

This geometric series converges if and only if

$$\left|\frac{5}{2}x^3\right| < 1$$

$$\Longleftrightarrow \quad |x^3| < \frac{2}{5}$$

$$\Longleftrightarrow \quad |x| < \sqrt[3]{\frac{2}{5}}$$

$$\Longleftrightarrow \quad -\sqrt[3]{\frac{2}{5}} < x < \sqrt[3]{\frac{2}{5}}.$$

Therefore, the interval of convergence is $\boxed{\left(-\sqrt[3]{\frac{2}{5}}, \sqrt[3]{\frac{2}{5}}\right)}$.

Problem 4 *Consider* $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{2^k x^k}{(k+1)^3}.$

(a) *Write out $p_3(x)$, the cubic polynomial which is the first three terms of this power series.*

(b) *Find $p_3'(x)$ and $f'(x)$ and compare your answers.*

(c) *Find $\displaystyle\int p_3(x)\,dx$ and $\displaystyle\int f(x)\,dx$ and compare your answers.*

Solution: (a)

$$p_3(x) = \frac{2^0 x^0}{(0+1)^3} + \frac{2^1 x^1}{(1+1)^3} + \frac{2^2 x^2}{(2+1)^3} + \frac{2^3 x^3}{(3+1)^3}$$

$$= \boxed{1 + \frac{2x}{2^3} + \frac{4x^2}{3^3} + \frac{8x^3}{4^3}}.$$

(b)

$$p_3'(x) = \boxed{0 + \frac{2}{2^3} + \frac{2\cdot 4}{3^3}x + \frac{3\cdot 8}{4^3}x^2}.$$

$$f'(x) = \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^3} \cdot k \cdot x^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{k\cdot 2^k}{(k+1)^3} x^{k-1} \quad \textit{Since the k=0 term is 0.}$$

$$= \boxed{\sum_{\ell=0}^{\infty} \frac{(\ell+1)2^{\ell+1}}{(\ell+2)^3} x^\ell} \quad \textit{reindexing with $\ell = k-1$}$$

Note that this sum agrees with $p_3'(x)$ when $\ell = 0$, 1, and 2.

(c)

$$\int p_3(x)\, dx = \boxed{x + \frac{2x^2}{2^4} + \frac{4x^3}{3^4} + \frac{8x^4}{4^4} + C}.$$

$$
\begin{aligned}
\int f(x)\, dx &= \int \sum_{k=0}^{\infty} \frac{2^k x^k}{(k+1)^3}\, dx \\
&= \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^3} \cdot \left(\int x^k\, dx \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{2^k}{(k+1)^3} \cdot \frac{1}{k+1} \cdot x^{k+1} \right) + C \\
&= \sum_{\ell=1}^{\infty} \left(\frac{2^{\ell-1}}{\ell^4} x^{\ell} \right) + C \quad \textcolor{red}{\text{reindexing with } \ell = k+1.}
\end{aligned}
$$

Note again that this sum agrees with $\int p_3(x)\, dx$ when $\ell = 0$, 1, 2, and 3 (as it should).

Problem 5 *Give a power series (with interval of convergence) for the given functions.*

(a) $f(x) = \dfrac{1}{1+x^2}$ (b) $f(x) = \tan^{-1}(x)$ (c) $f(x) = \tan^{-1}(3x^2)$

Solution: (a)

$$
\begin{aligned}
f(x) &= \frac{1}{1+x^2} \\
&= \sum_{k=0}^{\infty} (-x^2)^k \\
&= \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}}
\end{aligned}
$$

when

$$|-x^2| < 1 \qquad \Longleftrightarrow \qquad -1 < x < 1.$$

So the interval of convergence is $\boxed{(-1, 1)}$.

7

(b) *Notice that*

$$f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

on the interval $(-1, 1)$. *So*

$$f(x) = \int\left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right)dx$$
$$= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2k+1}x^{2k+1}\right) + C.$$

We know that this series converges (at least) on $(-1, 1)$. *We need to first find* C, *and then check the endpoints for convergence.*

To find C, *just notice that*

$$0 = \arctan(0) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2k+1}0^{2k+1}\right) + C = 0 + C.$$

and so $C = 0$.

Now to check the endpoints, we plug in -1 *and* 1. *When* $x = -1$, *we have the series*

$$\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1} \qquad \textcolor{red}{x = -1}$$

' and when $x = 1$ *the series is*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}. \qquad \textcolor{red}{x = 1}$$

Both of these series converge conditionally by the Alternating Series Test. Thus,

$$f(x) = \boxed{\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}}$$

with interval of convergence $\boxed{[-1, 1]}$.

(c) *By part (b), if* $|3x^2| \leq 1$ *then*

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}(3x^2)^{2k+1} = \boxed{\sum_{k=0}^{\infty}\frac{(-1)^k 3^{2k+1}}{2k+1}x^{4k+2}}.$$

8

This series converges when

$$|3x^2| \leq 1$$
$$\implies \quad |x^2| \leq \frac{1}{3}$$
$$\implies \quad -\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}.$$

So the interval of convergence is

$$\boxed{\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]}.$$