Section 9.2: Sequences

Warm up:

Find the limit of the following sequences as n tends to ∞.

(a) $a_n = \dfrac{n^{1000}}{2^n}$

(b) $b_n = \cos(n\pi)$

(c) $c_n = \cos(n!\pi)$

Solution: (a) Note that $\lim_{x\to\infty} \dfrac{x^a}{b^x} = 0$ for any constants a and $b > 1$. So $\lim_{n\to\infty} a_n = 0$.

(b) If n is even, $b_n = \cos(n\pi) = 1$, but if n is odd, then $b_n = \cos(n\pi) = -1$. So $\lim_{n\to\infty} b_n$ does not exist.

(c) If n is at least 2, then $n!$ is even. So $c_n = 1$ if n is at least 2. $\lim_{n\to\infty} c_n = 1$.

Group work:

Problem 1 *Find the limit of the given sequence. Also, determine if it is a geometric sequence.*

(a) $a_n = \dfrac{n^2}{2^n}$

(b) $a_n = \dfrac{1}{3^n}$

(c) $a_n = \left(\dfrac{1}{n}\right)^4$

(d) $a_n = \dfrac{e^n + (-3)^n}{5^n}$

(e) $a_n = 3^{\frac{1}{n}}$

Solution: (a) $\lim_{n\to\infty} \dfrac{n^2}{2^n} = 0$ *growth rate*

(b) $\lim_{n\to\infty} \dfrac{1}{3^n} = \lim_{n\to\infty} \left(\dfrac{1}{3}\right)^n = 0$. *This is a geometric sequence with* $a = 1$ *and* $r = \dfrac{1}{3}$.

Learning outcomes:

(c) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^4 = 0.$

(d) $\lim_{n\to\infty} \frac{e^n + (-3)^n}{5^n} = \lim_{n\to\infty} \left[\left(\frac{e}{5}\right)^n + \left(\frac{-3}{5}\right)^n\right] = 0.$

This is the sum of two geometric sequences. For both, the initial term is $a = 1$. *For the first sequence the ratio is* $r_1 = \frac{e}{5}$, *and for the second the ratio is* $r_2 = \frac{-3}{5}$.

(e) $\lim_{n\to\infty} 3^{\frac{1}{n}} = 3^0 = 1.$

Problem 2 *Show that*

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

exists by proving that $a_n = \sqrt{n+1} - \sqrt{n}$ *is a bounded monotonic sequence. A hint is to show that* $f(x) = \sqrt{x+1} - \sqrt{x}$ *is a decreasing function by showing that* $f'(x) < 0$.

Solution: *Let* $f(x) = \sqrt{x+1} - \sqrt{x}$. *Then*

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} \\ &= \frac{\sqrt{x} - \sqrt{x+1}}{2\sqrt{x}\sqrt{x+1}} \\ &< 0 \end{aligned}$$

since the denominator is clearly positive, and $\sqrt{x} < \sqrt{x+1}$. *Therefore* f *is decreasing, and so the original sequence is decreasing. Also notice that since*

$$\sqrt{x} < \sqrt{x+1}$$

we have that

$$0 < \sqrt{x+1} - \sqrt{x} = f(x).$$

Thus the original sequence is bounded below by 0.

Therefore, since the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ *is bounded and monotone decreasing, the limit*

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

exists.

Problem 3 *For each of the following sequences, find the limit as the number of terms approaches infinity.*

(a) $a_n = \left(\frac{n+1}{2n}\right)\left(\frac{n-2}{n}\right)^{\frac{n}{2}}$

Solution: *Let* $f(x) = \left(\frac{x+1}{2x}\right)\left(\frac{x-2}{x}\right)^{\frac{x}{2}}$. *Then*

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} e^{\ln f(x)} \\ &= e^{\lim_{x\to\infty} \ln f(x)}.\end{aligned}$$

So we need to compute the limit in the exponent. To this end

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty}\left[\ln\left(\frac{x+1}{2x}\right) + \ln\left(\frac{x-2}{x}\right)^{\frac{x}{2}}\right] \\ &= \lim_{x\to\infty} \ln\left(\frac{x+1}{2x}\right) + \lim_{x\to\infty}\left[\frac{x}{2}\ln\left(\frac{x-2}{x}\right)\right] \quad \textit{provided both limits exist} \\ &= \ln\left(\frac{1}{2}\right) + \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{2}{x}} \quad \textit{indeterminant of the form } \frac{0}{0} \\ &= \ln\left(\frac{1}{2}\right) + \lim_{x\to\infty}\frac{\frac{2x^{-2}}{1-\frac{2}{x}}}{-2x^{-2}} \quad \textit{L'Hospital's Rule} \\ &= \ln\left(\frac{1}{2}\right) + \lim_{x\to\infty}\frac{-1}{1-\frac{2}{x}} \\ &= \ln\left(\frac{1}{2}\right) - 1.\end{aligned}$$

So

$$\lim_{x\to\infty} f(x) = e^{\ln\left(\frac{1}{2}\right)-1} = \frac{1}{2}e^{-1}$$

and therefore

$$\lim_{n\to\infty}\left(\frac{n+1}{2n}\right)\left(\frac{n-2}{n}\right)^{\frac{n}{2}} = \frac{1}{2}e^{-1}.$$

(b) $a_n = \sqrt[n]{3^{2n+1}}$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{3^{2n+1}} &= \lim_{n\to\infty} \left(3^{2n+1}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} 3^{2+\frac{1}{n}} \\
&= \lim_{n\to\infty} 3^2 \cdot 3^{\frac{1}{n}} \\
&= 9 \lim_{n\to\infty} e^{\frac{1}{n}} \\
&= 9 \cdot 3^0 \\
&= 9 \cdot 1 = 9.
\end{aligned}$$

(c) $a_n = \left(\sqrt{n^2+7} - n\right)$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left(\sqrt{n^2+7} - n\right) &= \lim_{n\to\infty} \left[\left(\sqrt{n^2+7} - n\right) \cdot \frac{\sqrt{n^2+7}+n}{\sqrt{n^2+7}+n}\right] \\
&= \lim_{n\to\infty} \frac{n^2+7-n^2}{n\sqrt{1+\frac{7}{n^2}}+n} \\
&= \lim_{n\to\infty} \frac{7}{n(\sqrt{1+\frac{7}{n^2}}+1)} \\
&= 0.
\end{aligned}$$

(d) $a_n = \dfrac{(2n+3)!}{5n^3(2n)!}$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{(2n+3)!}{5n^3(2n)!} &= \lim_{n\to\infty} \frac{(2n+3)(2n+2)(2n+1)(2n)!}{5n^3(2n)!} \\
&= \lim_{n\to\infty} \frac{(2n+3)(2n+2)(2n+1)}{5n^3} \\
&= \frac{8}{5} \quad \textit{Compare the coefficients of the leading } n^3 \textit{terms}
\end{aligned}$$

(e) $a_n = (2^n + 3^n)^{\frac{1}{n}}$

Hint: $a_n \geq (0+3^n)^{\frac{1}{n}} = 3$ *and* $a_n \leq (2 \cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3$

(f) $a_n = (2^n + 3^n)^{\frac{1}{n}}$
Hint: $a_n \geq (0+3^n)^{\frac{1}{n}} = 3$ *and*
$a_n \leq (2 \cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3$

Solution: *From the hint*

$$3 = (0 + 3^n)^{\frac{1}{n}} \leq a_n \leq (2 \cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3.$$

So by the squeeze theorem, we have that

$$\begin{aligned} \lim_{n\to\infty} 3 \leq \lim_{n\to\infty} a_n &\leq \lim_{n\to\infty} 2^{\frac{1}{n}} \cdot 3 \\ \Longrightarrow \qquad 3 \leq \lim_{n\to\infty} a_n &\leq 1 \cdot 3 = 3. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} a_n = 3.$$

(g) $a_n = \dfrac{n^{365} + 5^n}{8^n + n^3}$

Solution:

$$\begin{aligned} \lim_{n\to\infty} \frac{n^{365} + 5^n}{8^n + n^3} &= \lim_{n\to\infty} \frac{n^{365} + 5^n}{8^n + n^3} \cdot \frac{\frac{1}{8^n}}{\frac{1}{8^n}} \\ &= \lim_{n\to\infty} \frac{\frac{n^{365}}{8^n} + \left(\frac{5}{8}\right)^n}{1 + \frac{n^3}{8^n}} \\ &= \frac{0+0}{1+0} = 0. \quad \textit{due to growth rates, } \lim_{n\to\infty} \frac{n^k}{a^n} = 0. \end{aligned}$$