Recitation #13: An overview of sequences and series

Warm up:

For each of the following sequences, list the first four terms (start each with $n = 1$).

(a) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, $a_1 = 1$.

Solution: **n=1:** $a_1 = 1$.

n=2: $a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$.

n=3: $a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = \frac{17}{12}$.

n=4: $a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\frac{17}{12}}\right) = \frac{577}{408}$.

(b) $a_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(2n)! \cdot 2n!}$, Recall that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1) \cdot n$.

Solution: **n=1:** $a_1 = \frac{1}{2!2!} = \frac{1}{4}$.

n=2: $a_2 = \frac{1 \cdot 3}{(2 \cdot 2)! \cdot 2 \cdot 2!} = \frac{3}{96} = \frac{1}{32}$.

n=3: $a_3 = \frac{1 \cdot 3 \cdot 5}{6! \cdot 2 \cdot 3!}$.

n=4: $a_4 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{8! \cdot 2 \cdot 4!}$.

Group work:

Problem 1 *Give an explicit formula for each of the following sequences:*

Learning outcomes:

(a) $\frac{2}{3}, \frac{-2}{7}, \frac{2}{11}, \frac{-2}{15}, \ldots$

Solution: $a_n = \dfrac{(-1)^{n+1} \cdot 2}{-1+4n}$, *starting at* $n = 1$.

(b) $-2, 6, -24, 120, -720, \ldots$

Solution: $a_n = (-1)^n (n+1)!$, *starting at* $n = 1$.

(c) $2, 8, 26, 80, 242, \ldots$

Solution: $a_n = 3^n - 1$, *starting at* $n = 1$.

Problem 2 *For the sequence* $a_k = (2-k)^k$

(a) *calculate and list* a_0, a_1, a_2, a_3, *and* a_4.

Solution: $a_0 = (2-0)^0 = 1$.
$a_1 = (2-1)^1 = 1$.
$a_2 = (2-2)^2 = 0$.
$a_3 = (2-3)^3 = -1$.
$a_4 = (2-4)^4 = 16$.

(b) *Starting with* $k = 0$, *calculate and list* $S_0 = \sum_{k=0}^{0} a_k$, $S_1 = \sum_{k=0}^{1} a_k$, $S_2 = \sum_{k=0}^{2} a_k$, $S_3 = \sum_{k=0}^{3} a_k$, *and* $S_4 = \sum_{k=0}^{4} a_k$. *Write* S_n *in summation form and write* S_∞ *in summation form.*

Solution: $S_0 = \sum_{k=0}^{0} a_k = a_0 = 1$.

$$S_1 = \sum_{k=0}^{1} a_k = a_0 + a_1 = 1 + 1 = 2.$$

$$S_2 = \sum_{k=0}^{2} a_k = a_0 + a_1 + a_2 = 2 + 0 = 2.$$

$$S_3 = \sum_{k=0}^{3} a_k = a_0 + a_1 + a_2 + a_3 = 2 + (-1) = 1.$$

$$S_4 = \sum_{k=0}^{4} a_k = a_0 + a_1 + a_2 + a_3 + a_4 = 1 + 16 = 17.$$

$$S_n = \sum_{k=0}^{n} a_k.$$
$$S_\infty = \lim_{n\to\infty} \sum_{k=0}^{n} a_k.$$

Problem 3 *For each of the following, write the series in the form* $\sum_{k=0}^{\infty} a_k$.

(a) $0.2 + 0.06 + 0.018 + 0.0054 + \ldots$

Solution: $\displaystyle\sum_{k=0}^{\infty} 0.2\left(\frac{3}{10}\right)^k.$

(b) $\frac{1}{2} + \frac{1}{3^2} - \frac{1}{3} - \frac{1}{4^2} + \frac{1}{4} + \frac{1}{5^2} - \frac{1}{5} - \frac{1}{6^2} \ldots$

Solution: $\displaystyle\sum_{k=1}^{\infty} \left[(-1)^k \left(\frac{1}{k+2} + \frac{1}{(k+3)^2}\right)\right].$

Problem 4 *Reindex the series*

$$\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)}$$

in the form $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=-4}^{\infty} c_k$.

Solution: *For the first series, we let* $i = k+1$. *Then* $k = i-1$ *and, when* $k = 0$, $i = 1$. *So we have that*

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)} &= \sum_{i=1}^{\infty} \frac{5}{(i-1+2)(i-1+1)} \\
&= \sum_{i=1}^{\infty} \frac{5}{i(i+1)} \\
&= \sum_{k=1}^{\infty} \frac{5}{k(k+1)}. \qquad \textit{Resubstituting k=i.}
\end{aligned}$$

For the second series, we let $i = k - 4$. Then $k = i + 4$ and, when $k = 0$, $i = -4$. So we have that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{5}{(k+2)(k+1)} &= \sum_{i=-4}^{\infty} \frac{5}{(i+4+2)(i+4+1)} \\
&= \sum_{i=-4}^{\infty} \frac{5}{(i+5)(i+6)} \\
&= \sum_{k=-4}^{\infty} \frac{5}{(k+5)(k+6)}. \qquad \textit{Resubstituting k=i.}
\end{aligned}$$

Problem 5 *If* $\displaystyle\sum_{k=0}^{\infty} a_k = 6$ *and* $a_n = \dfrac{3}{2^n}$, *what is* $\displaystyle\sum_{k=4}^{\infty} a_k$*?*

Solution:

$$\begin{aligned}
6 &= \sum_{k=0}^{\infty} a_k \\
&= a_0 + a_1 + a_2 + a_3 + \sum_{k=4}^{\infty} a_k \\
&= \frac{3}{1} + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \sum_{k=4}^{\infty} a_k \\
&= \frac{45}{8} + \sum_{k=4}^{\infty} a_k.
\end{aligned}$$

Thus,

$$\sum_{k=4}^{\infty} a_k = 6 - \frac{45}{8} = \frac{3}{8}.$$