Recitation #18: Comparison Tests and Alternating Series - Full

Warm up:

For each of the following, answer **True** or **False**, and explain why.

(a) If $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n^2$ converges.

(b) If $a_n, b_n \geq 0$ and both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, then $\sum_{n=0}^{\infty} a_n b_n$ converges.

Solution: (a) **True**

Since $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. So, in particular, there exists an integer N such that $a_k < 1$ for all $k \geq N$. Then for all $k \geq N$, $a_k^2 < a_k$, and therefore we have that

$$\sum_{n=N}^{\infty} a_n^2 < \sum_{n=N}^{\infty} a_n.$$

Thus, by the Comparison Test, $\sum_{n=0}^{\infty} a_n^2$ is convergent.

(b) **True**

Just as in part (a) there exists an integer N such that $a_k < 1$ for all $k \geq N$. Then

$$\sum_{n=N}^{\infty} a_n b_n < \sum_{n=N}^{\infty} b_n$$

and thus, by the Comparison Test, $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

Instructor Notes: Show that these series converge formally using the comparison test.

Learning outcomes:

Group work:

Problem 1 (a) *Why can we not use the Comparison test with* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges?*

(b) *Adjust* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges via the Comparison Test.*

(c) *Give a convergent series we can use in the Limit Comparison Test to show that* $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$ *converges.*

Solution: (a) *We cannot use the Comparison Test here because*

$$\frac{1}{k^2} < \frac{1}{k^2-5}$$

for all $k \geq 1$. *So we would just be showing the the series in question is greater than a series which converges, which does not give us any information.*

(b) *Notice that*

$$\frac{2}{k^2} > \frac{1}{k^2-5}$$

for all $k \geq 4$. *Since* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges,* $\sum_{k=1}^{\infty} \frac{2}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$. *Thus,* $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *converges.*

Therefore, the Comparison Test with $\sum_{k=1}^{\infty} \frac{2}{k^2}$ *shows that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

(c) *For the Limit Comparison Test, we* **can** *use* $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$\begin{aligned} \lim_{k\to\infty} \frac{\frac{1}{k^2-5}}{\frac{1}{k^2}} &= \lim_{k\to\infty} \frac{k^2}{k^2-5} \\ &= 1. \end{aligned}$$

Thus, since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, by the Limit Comparison Test we know that* $\sum_{k=0}^{\infty} \frac{1}{k^2-5}$ *converges.*

Instructor Notes: This problem may be done as a quick whole class discussion. For (b), use something like $\frac{2}{k^2}$. Be sure to determine for which k the inequality will hold.

Problem 2 *Determine if the following series converge or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^2 e^{-n}\right]$

Solution: (a) *Use the* ***Limit Comparison Test*** *with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^3+1} \cdot \frac{n}{1} \\ &= \frac{1}{3}.\end{aligned}$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, by the Limit Comparison Test we know that* $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^3+1}$ *diverges.*

(b) *Use the* ***Limit Comparison Test*** *with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^4+1} \cdot \frac{n^2}{1} \\ &= \frac{1}{3}.\end{aligned}$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges, by the Limit Comparison Test we know that* $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$ *converges.*

(c) *Use the **Comparison Test** with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Since we always have that $0 < \cos^2(n) < 1$, *we know that*

$$\frac{\cos^2(n)}{n^3+1} \le \frac{1}{n^3+1} < \frac{1}{n^3}.$$

Therefore, by the Comparison Test, we have that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^2 n}{n^3+1}$ *converges.*

(d) *Use the **Comparison Test** with* $\displaystyle\sum_{n=1}^{\infty} 2 \cdot e^{-n}$.

First, notice that for all $n \ge 3$,

$$\left(1 + \frac{1}{n}\right)^2 < 2.$$

Also, notice that $\displaystyle\sum_{n=1}^{\infty} e^{-n}$ *is a convergent geometric series. Therefore* $\displaystyle\sum_{n=1}^{\infty} 2 \cdot e^{-n}$ *converges, and so* $\displaystyle\sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^2 e^{-n}\right]$ *converges by the Comparison Test.*

Instructor Notes: These are all done using either the Comparison Test or the Limit Comparison Test. Parts (a) and (b) should be done with the limit comparison test (explain why). It is important to compare and contrast these two problems. Parts (c) and (d) should be done with the Comparison Test (again, explain why). The e^{-n} should be treated as a geometric series.

Problem 3 *Determine if the following series absolutely converge, conditionally converge, or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$

(e) $\displaystyle\sum_{n=4}^{\infty} \frac{(-2)^n}{n}$

Solution: (a) *Since* $\frac{1}{n+3}$ *is positive and decreasing, the Alternating Series Test applies. Thus, this series converges. But we know that* $\sum_{n=1}^{\infty} \frac{1}{n+3}$ *diverges (Harmonic Series) and so* $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ *is conditionally convergent.*

(b) *Let* $a_n = \frac{(n+1)^n}{(2n)^n}$. *Applying the Root Test, we see that*

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \frac{n+1}{2n} \\ &= \frac{1}{2} < 1. \end{aligned}$$

So $\sum_{n=1}^{\infty} \frac{(n+1)^n}{(2n)^n}$ *converges, and therefore* $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$ *is absolutely convergent.*

(c) *Let* $f(x) = x^2 e^{\frac{-x^3}{3}}$. *The function* $f(x)$ *is continuous, positive, and decreasing. So we apply the integral test*

$$\begin{aligned} \int_1^{\infty} x^2 e^{\frac{-x^3}{3}}\,dx &= \lim_{b\to\infty} \int_1^{b} x^2 e^{\frac{-x^3}{3}}\,dx \\ &= \lim_{b\to\infty} \int_{\frac{1}{3}}^{\frac{b^3}{3}} e^{-u}\,du \qquad u = \frac{x^3}{3},\ du = x^2\,dx \\ &= \lim_{b\to\infty} \left[-e^{\frac{-b^3}{3}} + e^{\frac{-1}{3}}\right] \\ &= 0 + e^{\frac{-1}{3}} = e^{\frac{-1}{3}}. \end{aligned}$$

So, by the Integral Test, $\sum_{n=1}^{\infty} n^2 e^{\frac{-n^3}{3}}$ *converges. Therefore,* $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$ *is absolutely convergent.*

(d) *First, notice that*

$$\frac{5}{3^n} > \frac{5}{3^n + 3^{-n}}.$$

Then since $\sum_{n=0}^{\infty} \frac{5}{3^n}$ *converges (geometric series with* $r = \frac{1}{3} < 1$*), we know that* $\sum_{n=0}^{\infty} \frac{5}{3^n + 3^{-n}}$ *converges by the Comparison Test. Thus, the series* $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$ *is absolutely convergent.*

(e) *Since the sequence* $\left\{\frac{(-2)^n}{n}\right\}$ *diverges, the series* $\sum_{n=4}^{\infty}\frac{(-2)^n}{n}$ *diverges by the Divergence Test.*

Instructor Notes: These problems use a mix of tests. One of the main goals is for students to get practice determining which test to use.

(a) Limit Comparison Test with the harmonic series

(b) Root Test

(c) Integral Test

(d) Limit Comparison Test

(e) Divergence Test. Be sure to talk about "pulling out the -1" to get an alternating series in standard form. Talk about how the Alternating Series Test and the Divergence Test will take care of conditional convergence for most (but not all) alternating series that they will see.

Problem 4 (a) *Find an upper bound for how close* $\sum_{k=0}^{4}\frac{(-1)^k k}{4^k}$ *is to the value of* $\sum_{k=0}^{\infty}\frac{(-1)^k k}{4^k}$.

(b) *How many terms are needed to estimate* $\sum_{n=1}^{\infty}\frac{(-1)^n \ln n}{n!}$ *to within* 10^{-6}*?*

Solution: (a) *Recall from the lesson that the remainder* R_n *is given by*

$$R_n = |S - S_n| \leq a_{n+1}.$$

Also notice that

$$\sum_{k=0}^{\infty}\frac{(-1)^k k}{4^k} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)}{4^{k-1}}$$

and

$$\sum_{k=0}^{4}\frac{(-1)^k k}{4^k} = \sum_{k=1}^{5}\frac{(-1)^{k-1}(k-1)}{4^{k-1}}.$$

So

$$
\begin{aligned}
&\left|\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} - \sum_{k=0}^{4} \frac{(-1)^k k}{4^k}\right| \\
=&\left|\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}} - \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}\right| \\
=&\ |S - S_5| \\
=&\ R_6 \leq a_6 = \boxed{\frac{5}{4^5}}
\end{aligned}
$$

(b) *Since $R_n \leq a_{n+1}$, we need to find n so that*

$$
\begin{aligned}
& a_{n+1} \leq 10^{-6} \\
\iff \quad & \frac{\ln(n+1)}{(n+1)!} \leq 10^{-6} \\
\iff \quad & \frac{(n+1)!}{\ln(n+1)} \geq 10^{6}
\end{aligned}
$$

Using a calculator, we see that this inequality holds for $n \approx 8.8$. So for $\boxed{n \geq 9}$, $R_n < 10^{-6}$.

Instructor Notes: Split (a) and (b) among the groups.