Recitation 23: Parametric equations & Polar coordinates - Solutions

Warm up:

Describe the motion given by $x = 8$, $y = 7\sin(t)$ for all t.

Solution: The parametric curve keeps oscillating up and down the vertical line segment between the points $(8, -7)$ and $(8, 7)$. This is because $x = 8$ is fixed, and $y = 7\sin(t)$ oscillates between -7 and 7 as t varies.

Group work:

Problem 1 *Try to figure out the shape of the following curve and then eliminate the parameter and check your intuition.*
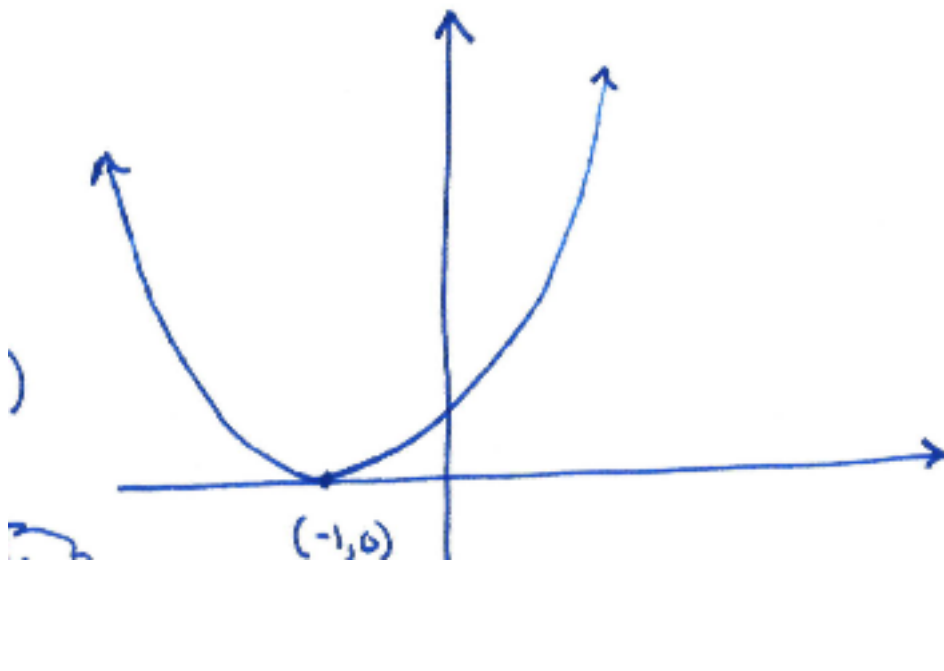
$$x = \ln t - 1 \qquad y = (\ln t)^2$$

Solution: *First, note that these two functions are both only defined when $0 < t < \infty$. Also, we see that $\ln t = x + 1$, and so*

$$y = (\ln t)^2 = (x + 1)^2.$$

So this graph is a parabola that opens up and has vertex $(-1, 0)$.

Learning outcomes:

$(-1, 0)$

Problem 2 *Find parametric equations for the path of a particle moving around the circle*

$$(x-3)^2 + (y+7)^2 = 4$$

(a) *one time around clockwise starting at* $(5, -7)$.

(b) *three times around counterclockwise starting at* $(5, -7)$.

(c) *halfway around clockwise starting at* $(-1, -7)$.

Solution: First, notice that this is the equation of the circle with radius 2 centered at $(3, -7)$.

(a) *The point* $(5, -7)$ *is the "right-most" point on the circle. In order to parameterize the circle one time around going counter-clockwise we use the parameterization*

$$x = 3 + 2\cos t \qquad y = -7 + 2\sin t \qquad 0 \le t < 2\pi.$$

In order to traverse the circle one time clockwise, we just negate the t in the above parameterization. So we get

$$\boxed{x = 3 + 2\cos(-t) \qquad y = -7 + 2\sin(-t) \qquad 0 \le t < 2\pi}.$$

Remark: Since cos is an even function and sin is an odd function, this solution is equivalent to

$$x = 3 + 2\cos t \qquad y = -7 - 2\sin t \qquad 0 \le t < 2\pi$$

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(b) *To traverse the circle three times counter-clockwise, we just triple the domain of t from the parameterization above. So we have that*

$$\boxed{x = 3 + 2\cos t \qquad y = -7 + 2\sin t \qquad 0 \le t < 6\pi}.$$

(c) *Note that this problem starts at $(-1, -7)$, not $(5, -7)$. So this parameterization begins at the "left-most" point of the circle. Therefore, this is just the "second half" of the answer to part (a). So a parameterization for this problem is*

$$\boxed{x = 3 + 2\cos(-t) \qquad y = -7 + 2\sin(-t) \qquad \pi \le t < 2\pi}.$$

Problem 3 *Find the intersection point(s) of the lines*

$$x = -6 + 9t, \qquad y = 3 - 2t \tag{1}$$

and

$$x = 3 + t, \qquad y = -4 - 2t. \tag{2}$$

Do they intersect at the same time?

Solution: Line *1* is the line with slope $-\dfrac{2}{9}$ passing through $(-6, 3)$. So it has equation

$$y - 3 = -\frac{2}{9}(x - (-6))$$

$$\implies \quad y = -\frac{2}{9}x - \frac{4}{3} + 3 = -\frac{2}{9}x + \frac{5}{3}.$$

Line *2* is the line with slope -2 passing through $(3, -4)$. So it has equation

$$y + 4 = -2(x - 3)$$

$$\implies \quad y = -2x + 2.$$

Since these lines have different slopes, they intersect in a single point. To find this point, we set the equations equal to each other:

$$-\frac{2}{9}x + \frac{5}{3} = -2x + 2$$

$$\implies \quad \frac{16}{9}x = \frac{1}{3}$$

$$\implies \quad x = \frac{3}{16}$$

$$\implies \quad y = -2\left(\frac{3}{16}\right) + 2 = \frac{13}{8}.$$

Therefore, the intersection point is

$$\boxed{\left(\frac{3}{16}, \frac{13}{8}\right)}.$$

To see if they intersect in the same point, let us first find the t value which makes the x-coordinate of line 1 equal to $\frac{3}{16}$.

$$\frac{3}{16} = -6 + 9t$$

$$\implies \quad t = \frac{1}{9}\left(\frac{3}{16} + 6\right) = \frac{11}{16}.$$

So now, we plug this into the equation for $x(t)$ for line 2 and see if we get $\frac{3}{16}$.

$$x\left(\frac{11}{16}\right) = 3 + \frac{11}{16}$$

$$= \frac{59}{16} \neq \frac{3}{16}.$$

*Therefore, these lines do **not** intersect at the same time.*

Problem 4 *Consider the curve defined by the parameterization $x = t^2$, $y = t^3 - 3t$. Show that this curve has two tangent lines at $(3,0)$, and find the equations of the tangent lines there.*

Solution: *First, recall that*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Then, since $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3$, we have that

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}.$$

Now,

$$x(t) = 3$$
$$\iff \quad t^2 = 3$$
$$\iff \quad t = \pm\sqrt{3}.$$

Also, notice that both $y(\sqrt{3}) = 0 = y(-\sqrt{3})$. So the given parametric curve intersects the point $(3,0)$ at two times, when $t = \pm\sqrt{3}$. At these two times, the tangent lines have slopes

$$\frac{dy}{dx}\bigg|_{t=\sqrt{3}} = \frac{(3)(3) - 3}{2\sqrt{3}} = \sqrt{3} \qquad and \qquad \frac{dy}{dx}\bigg|_{t=-\sqrt{3}} = \frac{(3)(3) - 3}{-2\sqrt{3}} = -\sqrt{3}.$$

So the equations of the tangent lines are
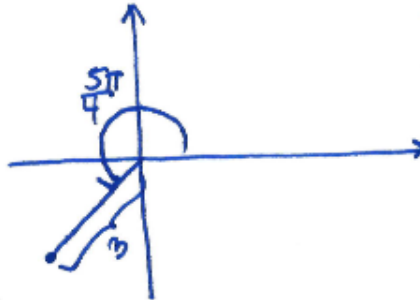
$$\boxed{y = \sqrt{3}(x - 3)}$$

$$\boxed{y = -\sqrt{3}(x - 3)}.$$

Problem 5 *Plot the following (polar) points in the xy-plane and then rewrite them as rectangular coordinates.*

(a) $\left(3, \dfrac{5\pi}{4}\right)$ (b) $\left(3, -\dfrac{5\pi}{4}\right)$ (c) $\left(-3, \dfrac{5\pi}{4}\right)$ (d) $\left(-3, -\dfrac{5\pi}{4}\right)$

Solution: (a)

$$x = 3\cos\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$

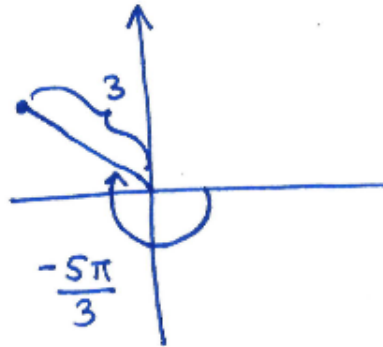$$y = 3\sin\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$

(b)

$$x = 3\cos\left(-\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$

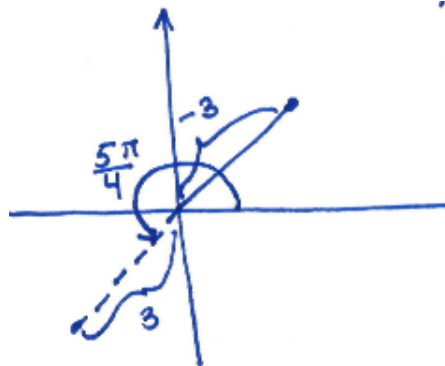$$y = 3\sin\left(-\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

(c)

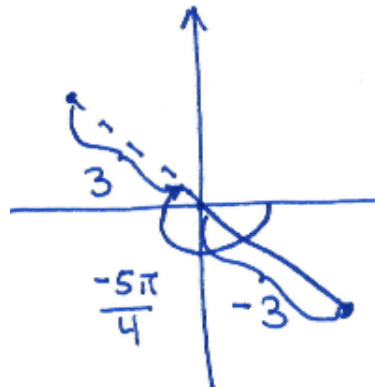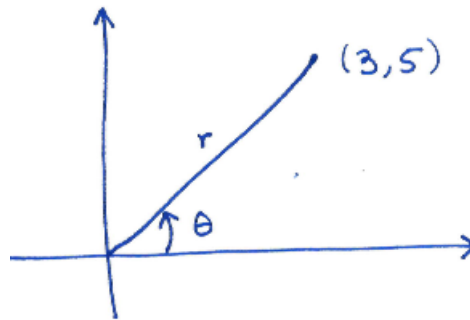$$x = -3\cos\left(\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$y = -3\sin\left(\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

(d)

$$x = -3\cos\left(-\frac{5\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$y = -3\sin\left(-\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$

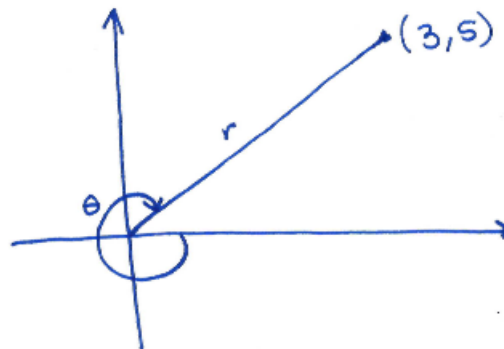**Problem 6** *Rewrite the rectangular point* $(3, 5)$ *in polar coordinates in three different ways.*

Solution: *i.*

$$r = \sqrt{3^2 + 5^2} = \sqrt{34} \qquad \theta = \arctan\left(\frac{5}{3}\right)$$
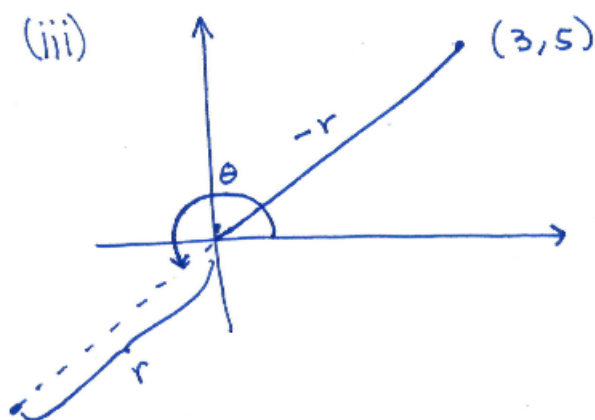

ii.

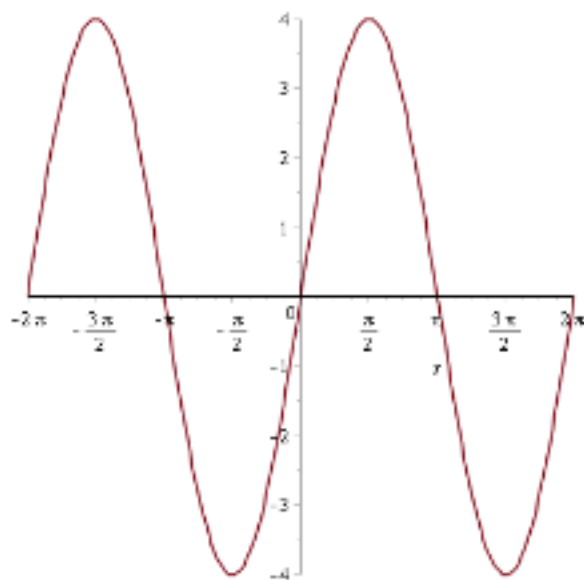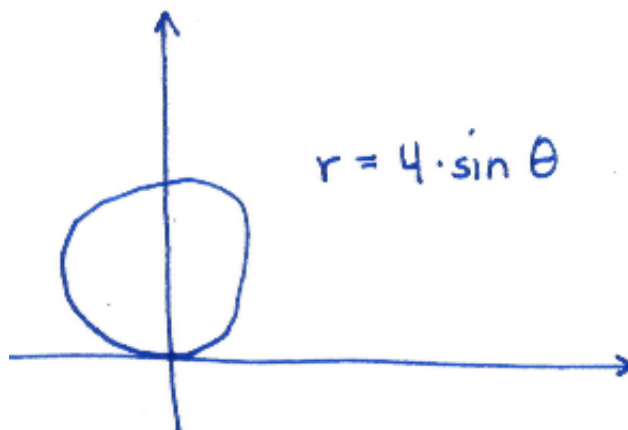$$r = \sqrt{34} \qquad \theta = \arctan\left(\frac{5}{3}\right) - 2\pi$$

iii.

$$r = -\sqrt{34} \qquad \theta = \arctan\left(\frac{5}{3}\right) + \pi$$

(iii)

(3,5)

−r

θ

r

Problem 7 *The graph of the curve $r = 4\sin\theta$ is a circle. Use the graph below to sketch this circle. Can you verify this algebraically? What is the period of the polar curve? Is $0 \le \theta \le 2\pi$ necessary to complete the graph?*

Solution: *Graphing this equation in the picture below, we see that this is a circle with radius 2 and center $(0, 2)$.*

To verify this algebraically,

$$4 = 4\sin\theta \quad \implies \quad r^2 = 4r\sin\theta$$
$$\implies \quad x^2 + y^2 = 4y$$
$$\implies \quad x^2 + y^2 - 4y = 0$$
$$\implies \quad x^2 + y^2 - 4y + 4 = 4$$
$$\implies \quad x^2 + (y-2)^2 = 2^2.$$

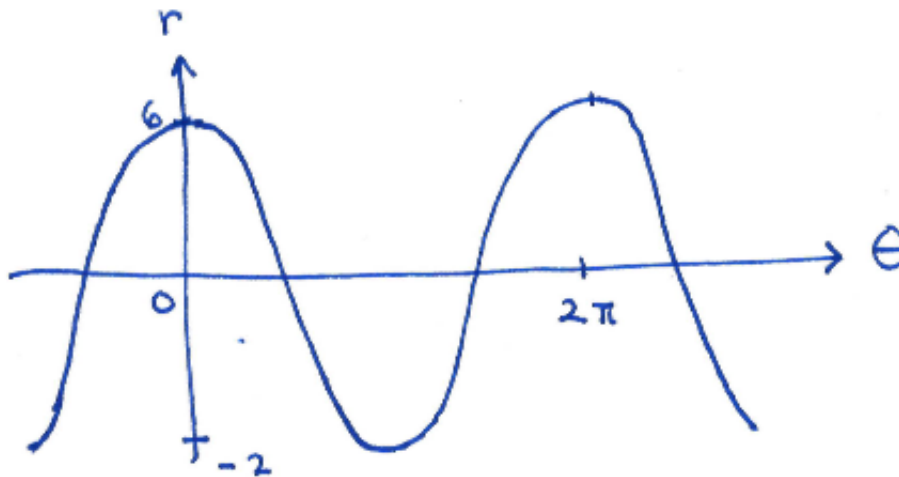To find the period of the polar curve, we convert both x and y into parametric equations with parameter θ.

$$x = r\cos\theta = 4\sin\theta\cos\theta = 2\sin(2\theta)$$
$$y = r\sin\theta = 4\sin^2\theta = 4 \cdot \frac{1}{2}(1 - \cos(2\theta)) = 2(1 - \cos(2\theta)).$$

The period of both $\sin(2\theta)$ and $\cos(2\theta)$ is π. So the entire graph of the equation $r = 4\sin\theta$ is traversed over the region $0 \le \theta < \pi$.
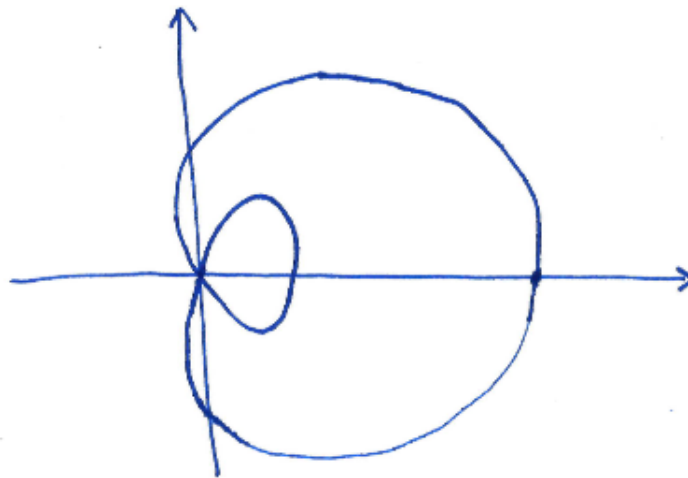

Problem 8 *Graph $r = 2 + 4\cos\theta$ using the "Cartesian-to-Polar" method.*

Solution: First, we graph $r = 2 + 4\cos\theta$ as if r and θ were Cartesian coordinates.

We then use this to draw the following graph in the xy-plane

If it helps, here are the parametric equations for this graph

$$x = r\cos\theta = (2 + 4\cos\theta)\cos\theta = 2\cos\theta + 4\cos^2\theta = 2(\cos\theta + 1 + \cos(2\theta))$$
$$y = 4\sin\theta = (2 + 4\cos\theta)\sin\theta = 2\sin\theta + 2\sin(2\theta).$$