Recitation 28: Cross Products and Lines and Curves in Space

Warm up:

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in 3-space $\mathbb{R}^3$, which of the following make sense?

(a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

(b) $(\vec{a} \cdot \vec{b})\vec{c}$

(c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$

(d) $(\vec{a} \cdot \vec{b}) + \vec{c}$

(e) $(\vec{a} \times \vec{b}) + \vec{c}$

(f) $\vec{a} \cdot (\vec{b} + \vec{c})$

(g) $\vec{a} \cdot (\vec{b} \times \vec{c})$

(h) $\vec{a} \times (\vec{b} \cdot \vec{c})$

(i) $(\vec{a} \times \vec{b})\vec{c}$

Solution: (a) Since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ does **not** make sense.

(b) Now since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b})\vec{c}$ **does** make sense as regular scalar multiplication.

(c) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ **does** make sense.

(d) This is of the form "scalar + vector", which does **not** make sense.

(e) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) + \vec{c}$ **does** make sense.

(f) This is of the form "vector · vector", which **does** make sense.

(g) This is also of the form "vector · vector", which **does** make sense.

(h) This is of the form "vector × scalar", which does **not** make sense.

(i) Since $\vec{a} \times \vec{b}$ is a vector, this does **not** make sense.

Group work:

Problem 1 *Given three dimensional vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, use dot product or cross product notation to describe the following vectors:*

(a) *The vector projection of $\vec{w}$ onto $\vec{u}$.*

Learning outcomes:

(b) *A vector orthogonal to both $\vec{u}$ and $\vec{v}$.*

(c) *A vector with the length of $\vec{v}$ and the direction of $\vec{w}$.*

(d) *A vector orthogonal to $\vec{u} \times \vec{v}$ and $\vec{w}$.*

Solution: (a) *This is the definition of vector projections.*

$$\mathrm{proj}_u w = \boxed{\left(\frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\right) \vec{u}}$$

(b) *There are many such vectors, but one of them is*

$$\boxed{\vec{u} \times \vec{v}}$$

(c) *Note that $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ so that $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.*

$$|\vec{v}| \left(\frac{\vec{w}}{|\vec{w}|}\right) = \boxed{\frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\vec{w} \cdot \vec{w}}} \vec{w}}$$

(d)

$$\boxed{(\vec{u} \times \vec{v}) \times \vec{w}}$$

Problem 2 *Let $\vec{u} = \langle 5, -1, 8 \rangle$ and $\vec{v} = \langle -2, 10, 5 \rangle$.*

(a) *Find a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.*

(b) *Verify that your answer is perpendicular to both $\vec{u}$ and $\vec{v}$*

(c) *Find a vector of length 7 perpendicular to both $\vec{u}$ and $\vec{c}$.*

Solution: (a) *Let $\vec{u} = \langle 5, -1, 8 \rangle$ and $\vec{v} = \langle -2, 10, 5 \rangle$. Then a vector which is perpendicular to both $\vec{u}$ and $\vec{v}$ is $\vec{w} := \vec{u} \times \vec{v}$. So we calculate*

$$\begin{aligned} \vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -1 & 8 \\ -2 & 10 & 5 \end{vmatrix} &= (-5 - 80)\hat{\imath} - (25 + 16)\hat{\jmath} + (50 - 2)\hat{k} \\ &= -85\hat{\imath} - 41\hat{\jmath} + 48\hat{k} \end{aligned}$$

(b) *To verify perpendicularity, we take the dot product.*

$$\vec{u} \cdot \vec{w} = \langle 5, -1, 8 \rangle \cdot \langle -85, -41, 48 \rangle = 5(-85) - 1(-41) + 8(48) = -425 + 41 + 384 = 0$$
$$\vec{v} \cdot \vec{w} = \langle -2, 10, 5 \rangle \cdot \langle -85, -41, 48 \rangle = -2(-85) + 10(-41) + 5(48) = 170 - 410 + 240 = 0$$

(c) *A unit vector in the same direction as $\vec{w}$ is*

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{(-85)^2 + (-41)^2 + 48^2}}\vec{w} = \frac{1}{\sqrt{11210}}\vec{w}.$$

Therefore, a vector with a magnitude of 7 in the same direction as $\vec{w}$ is

$$\vec{t} = \frac{7}{|\vec{w}|}\vec{w} = \boxed{\frac{7}{\sqrt{11210}}\langle -85, -41, 48\rangle}$$

Problem 3 *Find the area of the triangle in $\mathbb{R}^3$ with vertices at $P(2, -1, 0)$, $Q(1, 1, 4)$ and $R(2, -1, 6)$.*

Solution: *The area of the triangle is $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$.*

$$\vec{PR} = \langle 2, -1, 6\rangle - \langle 2, -1, 0\rangle = \langle 0, 0, 6\rangle,$$
$$\vec{PQ} = \langle 1, 1, 4, \rangle - \langle 2, -1, 0\rangle = \langle -1, 2, 4\rangle.$$

So

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= (-\vec{i} + 2\vec{j} + 4\vec{k}) \times 6\vec{k} = -(\vec{i} \times \vec{k}) + 2(\vec{j} \times \vec{k}) + 24(\vec{k} \times \vec{k}) \\ &= -(-\vec{j}) + 2\vec{i} + 0 = \langle 2, 1, 0\rangle.\end{aligned}$$

The area of the triangle is $\frac{1}{2}\sqrt{2^2 + 1^2 + 0^2} = \frac{\sqrt{5}}{2}$.

Problem 4 *Find a vector-valued function for the line segment connecting the points $P = (-3, 7, 6)$ and $Q = (5, -4, 7)$ in such a way that the value at $t = 0$ is P and the value at $t = 1$ is Q. Also, find the point two-thirds of the way from P to Q.*

Solution: *The line segment $\vec{r}(t)$ from P to Q is*

$$\begin{aligned}\vec{r}(t) &= (1 - t)P + tQ \\ &= (1 - t)\langle -3, 7, 6\rangle + t\langle 5, -4, 7\rangle \\ &= \boxed{\langle -3 + 8t, 7 - 11t, 6 + t\rangle \quad \textit{for } 0 \leq t \leq 1}.\end{aligned}$$

The point two-thirds of the way from P to Q is

$$\begin{aligned}\vec{r}\left(\frac{2}{3}\right) &= \left\langle -3 + 8\left(\frac{2}{3}\right), 7 - 11\left(\frac{2}{3}\right), 6 + \frac{2}{3}\right\rangle \\ &= \boxed{\left\langle \frac{7}{3}, -\frac{1}{3}, \frac{20}{3}\right\rangle}\end{aligned}$$

Problem 5 *Find a vector-valued function for the line through the point* $(1, -2, 3)$ *that is perpendicular to the lines*

$$\vec{r}_1(t) = \langle 7, 8, -2\rangle + t\langle 3, 5, 7\rangle \quad \textit{and} \quad \vec{r}_2(s) = \langle 4, -3, -7\rangle + s\langle 4, 9, -1\rangle$$

Solution: *Let* $\vec{v}_1 = \langle 3, 5, 7\rangle$ *and* $\vec{v}_2 = \langle 4, 9, -1\rangle$. *Then* $\vec{v}_1$ *is parallel to the line* $\vec{r}_1$, *and similarly for* $\vec{v}_2$ *and* $\vec{r}_2$. *So a vector perpendicular to both of the lines* $\vec{r}_1$ *and* $\vec{r}_2$ *is*

$$\begin{aligned} \vec{n} = \vec{v}_1 \times \vec{v}_2 &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & 7 \\ 4 & 9 & -1 \end{vmatrix} \\ &= (-5 - 63)\hat{\imath} - (-3 - 28)\hat{\jmath} + (27 - 20)\hat{k} \\ &= \langle -68, 31, 7\rangle. \end{aligned}$$

So the equation of the line through $(1, -2, 3)$ *and perpendicular to both* $\vec{r}_1$ *and* $\vec{r}_2$ *is*

$$\begin{aligned} \vec{r}_3(t) &= \langle 1, -2, 3\rangle + t\langle -68, 31, 7\rangle \\ &= \boxed{\langle 1 - 68t, -2 + 31t, 3 + 7t\rangle \quad \textit{for } -\infty < t < \infty} \end{aligned}$$

Problem 6 *Show that the curve* $\vec{r} = \langle t\cos t, t\sin t, t\rangle$ *lies completely on the cone* $z^2 = x^2 + y^2$.

Solution: *We just need to check that the components of* $\vec{r}$ *satisfies the given equation. So we compute*

$$\begin{aligned} x^2 + y^2 &= (t\cos t)^2 + (t\sin t)^2 \\ &= t^2\cos^2 t + t^2\sin^2 t \\ &= t^2(\cos^2 t + \sin^2 t) \\ &= t^2 \\ &= z^2. \end{aligned}$$

Challenge Problems

Problem 7 *Find the distance from the point* $P(-1, 4, 3)$ *to the line* $\langle 8 + t, 3 - 3t, -26t\rangle$. *Hint: The distance from the point to the line is the distance from the point* P *and the closest point on the line.*

Solution: *Let $P = (-1, 4, 3)$ and, for any time t, let $Q(t) = (8 + t, 3 - 3t, -26t)$. Then the distance from P to $Q(t)$ is given by*

$$\begin{aligned} D(t) &= \sqrt{(8 + t - (-1))^2 + (3 - 3t - 4)^2 + (-26t - 3)^2} \\ &= \sqrt{(9 + t)^2 + (-1 - 3t)^2 + (-3 - 26t)^2}. \end{aligned}$$

Instead of minimizing the distance $D(t)$, we will minimized the square of the distance $D^2(t)$, which leads to the same point. So

$$D^2(t) = (9 + t)^2 + (-1 - 3t)^2 + (-3 - 26t)^2.$$

To find the minimum of this function, we differentiate and find critical points

$$\begin{aligned} \frac{d}{dt}D^2(t) &= 2(9 + t) + 2(-1 - 3t)(-3) + 2(-3 - 26t)(-26) \\ &= (18 + 2t) + (6 + 18t) + (156 + 1352t) \\ &= 180 + 1372t := 0 \\ \implies \qquad t &= -\frac{180}{1372} = -\frac{45}{343}. \end{aligned}$$

Since there is exactly one critical point and since the second derivative is positive (it is the constant 1372), this value of t gives an absolute minimum. Therefore, the distance from P to the line is

$$\boxed{\sqrt{\left(9 - \frac{45}{343}\right)^2 + \left(-1 - 3\left(-\frac{45}{343}\right)\right)^2 + \left(-3 - 26\left(-\frac{45}{343}\right)\right)^2}}$$

Problem 8 *Match each of the following curves to the corresponding vector-valued function.*

(a) $\langle 3, t^2, 5\rangle$

(b) $\langle 3, t^2, t\rangle$

(c) $\langle 3, \sin t, \cos t\rangle$

(d) $\langle 3t, 5\sin t, 5\cos t\rangle$

(e) $\langle \sin t, \cos t, 2\cos t\rangle$

(f) $\langle 2\cos t, \sin t, \cos(3t)\rangle$

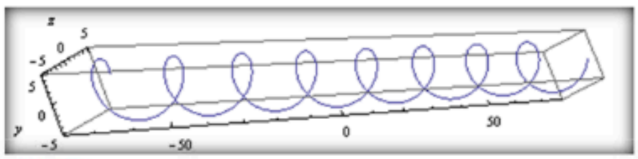

Figure 1

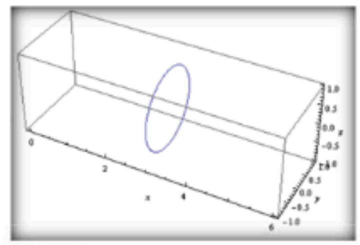

Figure 2

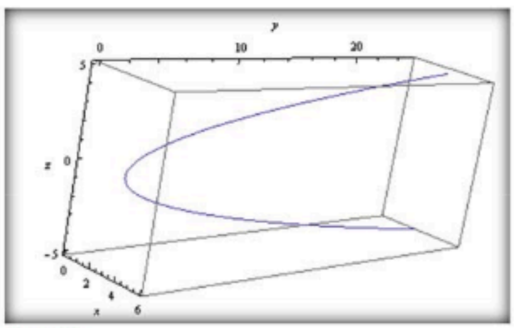

Figure 3

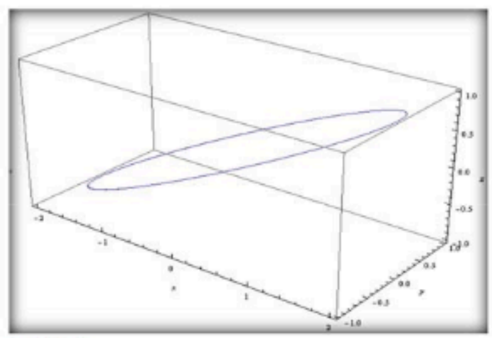

Figure 4

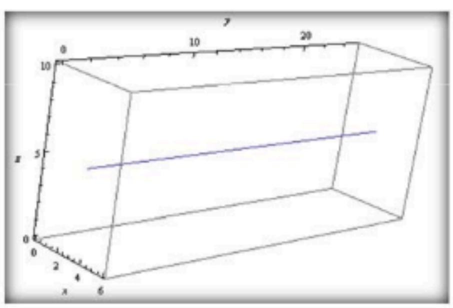

Figure 5

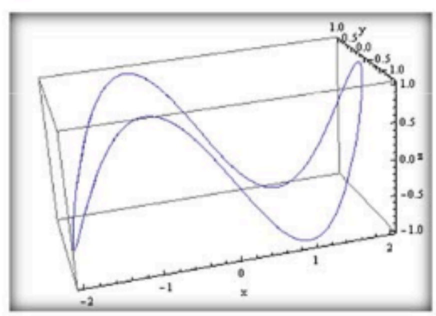

Figure 6

Solution: (a) **Figure 5**. *This is a line with both x and z held fixed.*

(b) **Figure 3**. *This is a parabola parallel to the yz-plane at $x = 3$.*

(c) **Figure 2**. *This is a circle of radius 1 in the plane $x = 3$.*

(d) **Figure 1**. *The x-component is linear, while the projection onto the yz-plane is a circle of radius* 5*. So this looks like a "spring".*

(e) **Figure 4**. *This is a circle with radius* 1 *when projected onto the xy-plane.*

(f) **Figure 6**. *This one is tricky. Maybe the best way to spot it is that it is an ellipse when projected onto the xy-plane, while the z-component varies between* -1 *and* 1*.*