

## Section 7.4: Trig Substitution

### Group work:

**Problem 1** For the integral:

$$\int \frac{27x^2}{(4+9x^2)^{3/2}} dx$$

find an appropriate constant  $C$  and an appropriate trigonometric substitution of one of the forms  $x = C \sin \theta$ ,  $x = C \sec \theta$ , or  $x = C \tan \theta$  to simplify the integral. Then, perform any trigonometric manipulations necessary and evaluate the integral.

**Solution:** There is no “ $\pm$ ” in this expression, so we choose the substitution  $x = C \tan \theta$ .

Here,  $4 + 9x^2 = 4 + 9C^2 \tan^2 \theta$  so we need  $9C^2 = 4$ , and thus  $C = 2/3$ . Now make the trigonometric substitution:

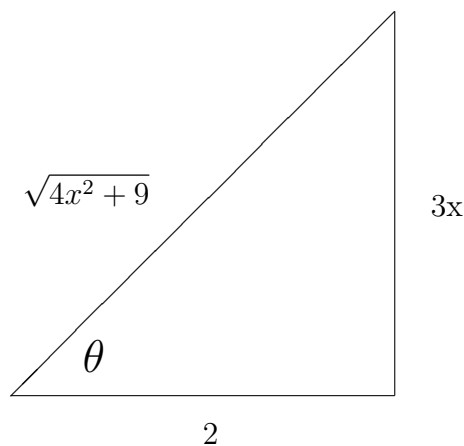
$$\begin{aligned} x &= \frac{2}{3} \tan \theta \\ dx &= \frac{2}{3} \sec^2 \theta d\theta \end{aligned}$$

Making the substitution into the integral gives:

$$\begin{aligned} \int \frac{27x^2}{(4+9x^2)^{3/2}} dx &= \int \frac{27 \cdot \frac{4}{9} \tan^2 \theta}{(4 + 9 \cdot \frac{4}{9} \tan^2 \theta)^{3/2}} \left[ \frac{2}{3} \sec^2 \theta d\theta \right] \\ &= \int \frac{8 \tan^2 \theta}{(4[1 + \tan^2 \theta])^{3/2}} \cdot \sec^2 \theta d\theta \\ &= \int \frac{8 \tan^2 \theta}{8(\sec^2 \theta)^{3/2}} \cdot \sec^2 \theta d\theta \\ &= \int \frac{\tan^2 \theta}{\sec^3 \theta} \cdot \sec^2 \theta d\theta \\ &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \quad \leftarrow \text{neither } u = \sec \theta \text{ nor } u = \tan \theta \text{ will work! So, convert } \tan^2 \theta! \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{\sec^2 \theta + 1}{\sec \theta} d\theta \\
 &= \int \left[ \frac{\sec^2 \theta}{\sec \theta} + \frac{1}{\sec \theta} \right] d\theta \\
 &= \int (\sec \theta + \cos \theta) d\theta \\
 &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \\
 &= \ln \left| \frac{\sqrt{4+9x^2}}{2} + \frac{3x}{2} \right| - \frac{3x}{\sqrt{4x^2+9}} + C
 \end{aligned}$$

where we have used the triangle shown below to evaluate the trigonometric functions:



Since  $x = \frac{2}{3} \tan \theta$ , we have  $\tan \theta = \frac{3x}{2}$ , which allows us to draw this triangle, from which we can read the expressions for the other trigonometric functions.

**Problem 2** Evaluate the following integrals

(a)

$$\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3} dx.$$

**Solution:** First notice that

$$\begin{aligned}\sqrt{36x^2 - 25} &= 5\sqrt{\frac{36x^2}{25} - 1} \\ &= 5\sqrt{\left(\frac{6x}{5}\right)^2 - 1}.\end{aligned}$$

So we substitute

$$\frac{6x}{5} = \sec \theta \quad \implies \quad x = \frac{5}{6} \sec \theta$$

which gives

$$dx = \frac{5}{6} \sec \theta \tan \theta d\theta.$$

Also, notice that

- when  $x = -\frac{5}{3}$ :

$$-\frac{5}{3} = \frac{5}{6} \sec \theta \quad \implies \quad \sec \theta = 2 \quad \implies \quad \theta = \frac{2\pi}{3}$$

- and when  $x = -\frac{5}{6}$ :

$$-\frac{5}{6} = \frac{5}{6} \sec \theta \quad \implies \quad \sec \theta = -1 \quad \implies \quad \theta = \pi.$$

Therefore

$$\begin{aligned}\int_{-\frac{5}{3}}^{-\frac{5}{6}} \frac{\sqrt{36x^2 - 25}}{x^3} dx &= 5 \int_{\frac{2\pi}{3}}^{\pi} \frac{\sqrt{\sec^2 \theta - 1}}{\left(\frac{5}{6} \sec \theta\right)^3} \left(\frac{5}{6} \sec \theta \tan \theta\right) d\theta \\ &= 5 \cdot \left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan \theta| \tan \theta}{\sec^2 \theta} d\theta.\end{aligned}$$

Now, notice that  $\tan \theta < 0$  whenever  $\frac{2\pi}{3} \leq \theta \leq \pi$ . So  $|\tan \theta| = -\tan \theta$ .

We continue:

$$\begin{aligned}
 5 \cdot \left(\frac{6}{5}\right)^2 \int_{\frac{2\pi}{3}}^{\pi} \frac{|\tan \theta| \tan \theta}{\sec^2 \theta} d\theta &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \frac{\tan^2 \theta}{\sec^2 \theta} d\theta \\
 &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{\cos^2 \theta}{1} d\theta \\
 &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \sin^2 \theta d\theta \\
 &= -\frac{36}{5} \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2} (1 - \cos(2\theta)) d\theta \\
 &= -\frac{18}{5} \left[ \theta - \frac{1}{2} \sin(2\theta) \right]_{\frac{2\pi}{3}}^{\pi} \\
 &= -\frac{18}{5} \left[ (\pi - 0) - \left( \frac{2\pi}{3} + \frac{\sqrt{3}}{4} \right) \right] \quad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\
 &= -\frac{18}{5} \left( \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right).
 \end{aligned}$$

(b)

$$\int \frac{dx}{(x^2 - 6x + 11)^2}.$$

**Solution:** We begin by completing the square in the denominator

$$x^2 - 6x - 11 = x^2 - 6x + 9 + 2 = (x - 3)^2 + 2.$$

We then have that

$$\begin{aligned}
 \int \frac{dx}{(x^2 - 6x + 11)^2} &= \int \frac{1}{((x - 3)^2 + 2)^2} dx \\
 &= \frac{1}{4} \int \frac{1}{\left(\frac{(x-3)^2}{2} + 1\right)^2} dx \\
 &= \frac{1}{4} \int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2 + 1\right)^2} dx.
 \end{aligned}$$

So we substitute

$$\frac{x - 3}{\sqrt{2}} = \tan \theta \quad \implies \quad x = \sqrt{2} \tan \theta + 3 \quad (1)$$

and then

$$dx = \sqrt{2} \sec^2 \theta d\theta.$$

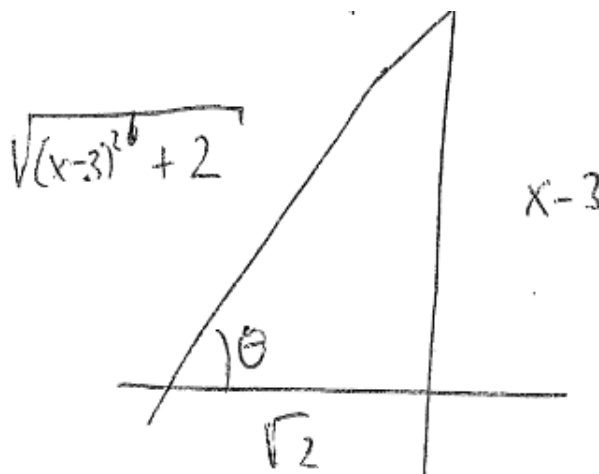
Continuing with the integral

$$\begin{aligned}
 \frac{1}{4} \int \frac{1}{\left(\left(\frac{x-3}{\sqrt{2}}\right)^2 + 1\right)^2} dx &= \frac{1}{4} \int \frac{1}{(\tan^2 \theta + 1)^2} \sqrt{2} \sec^2 \theta d\theta \\
 &= \frac{\sqrt{2}}{4} \int \frac{1}{\sec^2 \theta} d\theta \\
 &= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\
 &= \frac{\sqrt{2}}{4} \int \frac{1}{2} (1 + \cos(2\theta)) d\theta \\
 &= \frac{\sqrt{2}}{8} \left( \theta + \frac{1}{2} \sin(2\theta) \right) + C.
 \end{aligned}$$

Now all that is left to do is to reverse-substitute for  $\theta$ . First, from equation (1) we have that

$$\theta = \arctan\left(\frac{x-3}{\sqrt{2}}\right).$$

Now, we again use equation (1) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta) = 2 \cdot \frac{x-3}{\sqrt{(x-3)^2 + 2}} \cdot \frac{\sqrt{2}}{\sqrt{(x-3)^2 + 2}}.$$

Thus,

$$\int \frac{dx}{(x^2 - 6x + 11)^2} = \frac{\sqrt{2}}{8} \left( \arctan\left(\frac{x-3}{\sqrt{2}}\right) + \frac{\sqrt{2}(x-3)}{(x-3)^2 + 2} \right) + C.$$

**Problem 3** Evaluate the following integrals

(a)

$$\int \frac{x^2}{\sqrt{4x - x^2}} dx.$$

**Solution:** Again, we begin by completing the square in the denominator, and then factoring

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x) \\ &= -(x^2 - 4x + 4) + 4 \\ &= -(x - 2)^2 + 4 \\ &= 4 \left( -\frac{(x - 2)^2}{4} + 1 \right) \\ &= 4 \left( 1 - \left( \frac{x - 2}{2} \right)^2 \right). \end{aligned}$$

So

$$\begin{aligned} \int \frac{x^2}{\sqrt{4x - x^2}} dx &= \int \frac{x^2}{\sqrt{4 \left( 1 - \left( \frac{x-2}{2} \right)^2 \right)}} dx \\ &= \frac{1}{2} \int \frac{x^2}{\sqrt{1 - \left( \frac{x-2}{2} \right)^2}} dx. \end{aligned}$$

We make the substitution

$$\frac{x - 2}{2} = \sin \theta \quad \implies \quad x = 2 \sin \theta + 2 \quad (2)$$

which gives

$$dx = 2 \cos \theta d\theta.$$

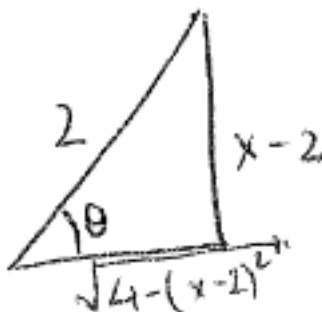
Continuing with the integral, we have that

$$\begin{aligned}
 \frac{1}{2} \int \frac{x^2}{\sqrt{1 - \left(\frac{x-2}{2}\right)^2}} dx &= \frac{1}{2} \int \frac{(2 \sin \theta + 2)^2}{\sqrt{1 - \sin^2 \theta}} \cdot 2 \cos \theta d\theta \\
 &= \int (2 \sin \theta + 2)^2 d\theta \\
 &= \int (4 \sin^2 \theta + 8 \sin \theta + 4) d\theta \\
 &= \int (2(1 - \cos(2\theta)) + 8 \sin \theta + 4) d\theta \\
 &= \int (6 + 8 \sin \theta - 2 \cos(2\theta)) d\theta \\
 &= 6\theta - 8 \cos \theta - \sin(2\theta) + C.
 \end{aligned}$$

Now all that is left to do is to reverse-substitute for  $\theta$ . First, from equation (2) we have that

$$\theta = \arcsin\left(\frac{x-2}{2}\right).$$

Now, we again use equation (2) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\begin{aligned}
 \cos \theta &= \frac{\sqrt{4 - (x-2)^2}}{2} \\
 \sin(2\theta) &= 2 \sin \theta \cos \theta = 2 \cdot \frac{x-2}{2} \cdot \frac{\sqrt{4 - (x-2)^2}}{2}.
 \end{aligned}$$

Thus

$$\int \frac{x^2}{\sqrt{4x - x^2}} dx = 6 \arcsin\left(\frac{x-2}{2}\right) - 4\sqrt{4 - (x-2)^2} - \frac{(x-2)\sqrt{4 - (x-2)^2}}{2}.$$

(b)

$$\int \frac{e^x}{\sqrt{e^{2x} + 9}} dx.$$

**Solution:** First, notice that

$$\sqrt{e^{2x} + 9} = \sqrt{9 \left( \frac{e^{2x}}{9} + 1 \right)} = 3 \sqrt{\left( \frac{e^x}{3} \right)^2 + 1}.$$

So

$$\int \frac{e^x}{\sqrt{e^{2x} + 9}} dx = \frac{1}{3} \int \frac{e^x}{\sqrt{\left( \frac{e^x}{3} \right)^2 + 1}} dx.$$

We make the substitution

$$\frac{e^x}{3} = \tan \theta \quad \implies \quad 3 \tan \theta = e^x \quad (3)$$

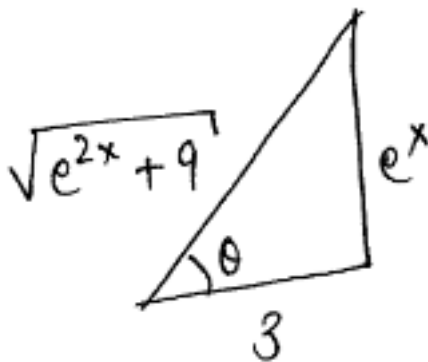
which gives

$$e^x dx = 3 \sec^2 \theta d\theta.$$

Continuing with the integral, we have that

$$\begin{aligned} \frac{1}{3} \int \frac{e^x}{\sqrt{\left( \frac{e^x}{3} \right)^2 + 1}} dx &= \frac{1}{3} \int \frac{1}{\sqrt{\tan^2 \theta + 1}} \cdot 3 \sec^2 \theta d\theta \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

Now all that is left to do is to reverse-substitute for  $\theta$ . We use equation (3) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\begin{aligned} \sec \theta &= \frac{\sqrt{e^{2x} + 9}}{3} \\ \tan \theta &= \frac{e^x}{3}. \end{aligned}$$

Thus

$$\int \frac{e^x}{\sqrt{e^{2x} + 9}} dx = \ln \left( \frac{\sqrt{e^{2x} + 9} + e^x}{3} \right) + C.$$



(c)

$$\int \frac{dx}{x^{\frac{1}{2}} - 9x^{\frac{3}{2}}}.$$

**Solution:**

$$\begin{aligned} \int \frac{dx}{x^{\frac{1}{2}} - 9x^{\frac{3}{2}}} &= \int \frac{1}{x^{\frac{1}{2}}(1 - 9x)} dx \\ &= \int \frac{1}{x^{\frac{1}{2}} \left(1 - \left(3x^{\frac{1}{2}}\right)^2\right)} dx \\ &= \frac{2}{3} \int \frac{1}{1 - u^2} dx \quad \text{where } u = 3x^{\frac{1}{2}} \\ &= \frac{2}{3} \int \frac{1}{1 - \sin^2 \theta} \cos \theta d\theta \quad \text{where } u = \sin \theta \\ &= \frac{2}{3} \int \frac{\cos \theta}{\cos^2 \theta} d\theta \\ &= \frac{2}{3} \int \sec \theta d\theta \\ &= \frac{2}{3} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{2}{3} \ln \left| \frac{1}{\sqrt{1 - u^2}} + \frac{u}{\sqrt{1 - u^2}} \right| + C \\ &= \frac{2}{3} \ln \left( \frac{1 + 3\sqrt{x}}{\sqrt{1 - 9x}} \right) + C \end{aligned}$$

