Recitation #10: Trig Substitution and Partial Fractions

Group work:

Problem 1 *Without determining the coefficients, write the partial fraction decomposition of the following rational function:*

$$\frac{5x^{13} - 6x^{12} + 7x^3 - 5x - 18}{(2x-3)(5x+9)^3(x^2+9x+19)(x^2+9x+21)^2}$$

Solution: *The degree of the numerator is* 13*, whereas the degree of the denominator is* 10*. So if we perform long division, we will get a degree* $13 - 10 = 3$ *polynomial plus partial fractions for the remainder term:*

$$\frac{5x^{13} - 6x^{12} + 7x^3 - 5x - 18}{(2x-3)(5x+9)^3(x^2+9x+19)(x^2+9x+21)^2} = Ax^3 + Bx^2 + Cx + D$$
$$+ \frac{E}{2x-3} + \frac{F}{5x+9} + \frac{G}{(5x+9)^2} + \frac{H}{(5x+9)^3} + \frac{I}{x-i_1} + \frac{J}{x-i_2}$$
$$+ \frac{Kx+L}{x^2+9x+21} + \frac{Mx+N}{(x^2+9x+21)^2}.$$

Explanation of i_1 ***and*** i_2***:*** *The quadratic* $x^2 + 9x + 19$ *can be factored over the real numbers, since the discriminant* $b^2 - 4ac = 81 - 76 > 0$*. The numbers* i_1 *and* i_2 *are the two real roots to this polynomial, ie*

$$i_1 = \frac{-9+\sqrt{5}}{2} \qquad i_2 = \frac{-9-\sqrt{5}}{2}.$$

Note that the polynomial $x^2 + 9x + 21$ *is irreducible (over the real numbers) since its discriminant is less than* 0*.*

Problem 2 *Evaluate:*

$$\int \frac{7x^3 + 18x + 9}{x^4 + 9x^2}\, dx$$

Hint: If $f(x) = 7x^3 + 18x + 9$*, then* $f(2) = 101$*,* $f(1) = 34$*, and* $f(-1) = -16$*.*

Learning outcomes:

Solution: *First factor the denominator*

$$x^4 + 9x^2 = x^2(x^2 + 9).$$

The we can decompose the integrand as a partial fraction

$$\frac{7x^3 + 18x + 9}{x^2(x^2 + 9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9}$$

$$\begin{aligned} \implies \quad 7x^3 + 18x + 9 &= Ax(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2 \\ &= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2 \\ &= (A + C)x^3 + (B + D)x^2 + 9Ax + 9B. \end{aligned}$$

By equating coefficients for powers of x we have that

$$\begin{aligned} &9 = 9B \quad \implies \quad B = 1 \\ &18 = 9A \quad \implies \quad A = 2 \\ &0 = B + D \quad \implies \quad 0 = 1 + D \quad \implies \quad D = -1 \\ &7 = A + C \quad \implies \quad 7 = 2 + C \quad \implies \quad C = 5. \end{aligned}$$

Thus

$$\begin{aligned} \int \frac{7x^3 + 18x + 9}{x^4 + 9x^2}\,dx &= \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{5x - 1}{x^2 + 9}\right) dx \\ &= 2\ln|x| - \frac{1}{x} + 5\int \frac{x}{x^2 + 9}\,dx - \int \frac{1}{x^2 + 9}\,dx \\ &= 2\ln|x| - \frac{1}{x} + \frac{5}{2}\ln(x^2 + 9) - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C. \end{aligned}$$

Note that, in the previous step, we substituted $u = x^2 + 9$ for the first integral and $u = \frac{x}{3}$ in the second integral.

Problem 3 *Evaluate the following integrals*

(a)

$$\int \frac{x^2}{\sqrt{4x - x^2}}\,dx.$$

Solution: *Again, we begin by completing the square in the denomina-*

tor, and then factoring

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x) \\ &= -(x^2 - 4x + 4) + 4 \\ &= -(x-2)^2 + 4 \\ &= 4\left(-\frac{(x-2)^2}{4} + 1\right) \\ &= 4\left(1 - \left(\frac{x-2}{2}\right)^2\right). \end{aligned}$$

So

$$\begin{aligned} \int \frac{x^2}{\sqrt{4x - x^2}}\,dx &= \int \frac{x^2}{\sqrt{4\left(1 - \left(\frac{x-2}{2}\right)^2\right)}}\,dx \\ &= \frac{1}{2}\int \frac{x^2}{\sqrt{1 - \left(\frac{x-2}{2}\right)^2}}\,dx. \end{aligned}$$

We make the substitution

$$\frac{x-2}{2} = \sin\theta \qquad \Longrightarrow \qquad x = 2\sin\theta + 2 \tag{1}$$

which gives

$$dx = 2\cos\theta\,d\theta.$$

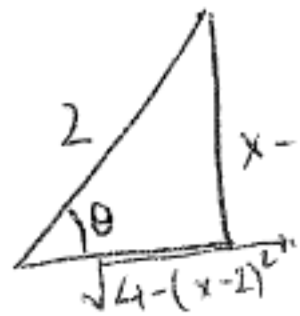
Continuing with the integral, we have that

$$\begin{aligned} \frac{1}{2}\int \frac{x^2}{\sqrt{1 - \left(\frac{x-2}{2}\right)^2}}\,dx &= \frac{1}{2}\int \frac{(2\sin\theta + 2)^2}{\sqrt{1 - \sin^2\theta}} \cdot 2\cos\theta\,d\theta \\ &= \int (2\sin\theta + 2)^2\,d\theta \\ &= \int (4\sin^2\theta + 8\sin\theta + 4)\,d\theta \\ &= \int (2(1 - \cos(2\theta)) + 8\sin\theta + 4)\,d\theta \\ &= \int (6 + 8\sin\theta - 2\cos(2\theta))\,d\theta \\ &= 6\theta - 8\cos\theta - \sin(2\theta) + C. \end{aligned}$$

Now all that is left to do is to reverse-substitute for θ. First, from equation (1) we have that

$$\theta = \arcsin\left(\frac{x-2}{2}\right).$$

Now, we again use equation (1) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\cos\theta = \frac{\sqrt{4-(x-2)^2}}{2}$$
$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{x-2}{2}\cdot\frac{\sqrt{4-(x-2)^2}}{2}.$$

Thus

$$\int \frac{x^2}{\sqrt{4x-x^2}}\,dx = 6\arcsin\left(\frac{x-2}{2}\right) - 4\sqrt{4-(x-2)^2} - \frac{(x-2)\sqrt{4-(x-2)^2}}{2}.$$

(b)

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx.$$

Solution: *First, notice that*

$$\sqrt{e^{2x}+9} = \sqrt{9\left(\frac{e^{2x}}{9}+1\right)} = 3\sqrt{\left(\frac{e^x}{3}\right)^2+1}.$$

So

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx = \frac{1}{3}\int \frac{e^x}{\sqrt{\left(\frac{e^x}{3}\right)^2+1}}\,dx.$$

We make the substitution

$$\frac{e^x}{3} = \tan\theta \qquad \Longrightarrow \qquad 3\tan\theta = e^x \tag{2}$$

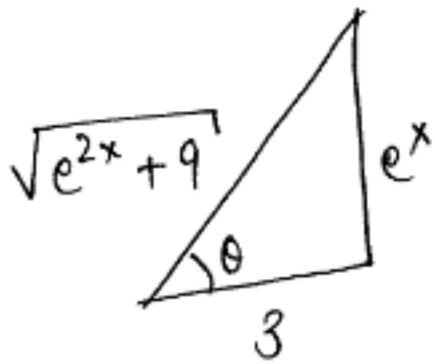
which gives

$$e^x\,dx = 3\sec^2\theta\,d\theta.$$

Continuing with the integral, we have that

$$\begin{aligned}\frac{1}{3}\int \frac{e^x}{\sqrt{\left(\frac{e^x}{3}\right)^2+1}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{\tan^2\theta+1}}\cdot 3\sec^2\theta\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

Now all that is left to do is to reverse-substitute for θ. We use equation (2) along with Pythagorean's Theorem to construct the following triangle.



Then we have that

$$\sec\theta = \frac{\sqrt{e^{2x}+9}}{3}$$
$$\tan\theta = \frac{e^x}{3}.$$

Thus

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx = \ln\left(\frac{\sqrt{e^{2x}+9}+e^x}{3}\right) + C.$$

(c)

$$\int \frac{dx}{x^{\frac{1}{2}} - 9x^{\frac{3}{2}}}.$$

Solution:

$$\begin{aligned}
\int \frac{dx}{x^{\frac{1}{2}} - 9x^{\frac{3}{2}}} &= \int \frac{1}{x^{\frac{1}{2}}(1-9x)}\,dx \\
&= \int \frac{1}{x^{\frac{1}{2}}\left(1-\left(3x^{\frac{1}{2}}\right)^2\right)}\,dx \\
&= \frac{2}{3}\int \frac{1}{1-u^2}\,dx \qquad \textcolor{red}{\text{where } u = 3x^{\frac{1}{2}}} \\
&= \frac{2}{3}\int \frac{1}{1-\sin^2\theta}\cos\theta\,d\theta \qquad \textcolor{red}{\text{where } u = \sin\theta} \\
&= \frac{2}{3}\int \frac{\cos\theta}{\cos^2\theta}\,d\theta \\
&= \frac{2}{3}\int \sec\theta\,d\theta \\
&= \frac{2}{3}\ln|\sec\theta + \tan\theta| + C \\
&= \frac{2}{3}\ln\left|\frac{1}{\sqrt{1-u^2}} + \frac{u}{\sqrt{1-u^2}}\right| + C \\
&= \frac{2}{3}\ln\left(\frac{1+3\sqrt{x}}{\sqrt{1-9x}}\right) + C
\end{aligned}$$