Section 9.4: Divergence and Integral Tests and Conceptual Questions

Warm-Up

Problem 1 *Suppose* $\sum_{k=1}^{\infty} a_k$ *is an infinite series.*

A. *If* $\lim_{k\to\infty} a_k = 0$*, does* $\sum_{k=1}^{\infty} a_k$ *have to converge?*

Solution: *No; consider the harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$*. Here* $a_k = \frac{1}{k}$*, so* $\lim_{k\to\infty} a_k = 0$*, but the series diverges!*

B. *If* $\sum_{k=1}^{\infty} a_k$ *converges, does* $\lim_{k\to\infty} a_k = 0$ *necessarily?*

Solution: *Yes; if* $\lim_{k\to\infty} a_k \neq 0$*, the divergence test guarantees that* $\sum_{k=1}^{\infty} a_k$ *diverges. Since we know the series converges by assumption, we must have* $\lim_{k\to\infty} a_k = 0$*!*

Group Work

Problem 2 *Suppose* $\{a_n\}_{n\geq 1}$ *is a sequence and* $\sum_{n=1}^{\infty} a_n$ *converges to* $L > 0$*. Let* $s_n = \sum_{k=1}^{n} a_k$*. Circle all of the statements that* MUST *be true.*

Learning outcomes:

A. $\lim_{n\to\infty} a_n = L$ *B.* $\lim_{n\to\infty} a_n = 0$ *C.* $\lim_{n\to\infty} s_n = 0$

D. $\lim_{n\to\infty} s_n = L$ *E.* $\sum_{n=1}^{\infty} s_n$ *MUST diverge.* *F.* $\sum_{n=1}^{\infty}(a_n + 1) = L + 1$

G. The divergence test tells us $\sum_{n=1}^{\infty} a_n$ *converges to* L*.*

Solution: *A.* ***False***

Since $\{a_n\}_{n\geq 1} = L$, $\{a_n\}_{n\geq 1}$ *is a* convergent *series, so* $\lim_{n\to\infty} a_n = 0$*. Since* $L > 0$*, there is no way that* $\lim_{n\to\infty} a_n = L$*.*

B. ***True***

If $\lim_{n\to\infty} a_n \neq 0$*, the divergence test implies* $\sum_{n=1}^{\infty} a_n$ *diverges! Anytime a series* $\sum_{n=1}^{\infty} a_n$ *converges, it MUST be true that* $\lim_{n\to\infty} a_n = 0$*.*

C. ***False***

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{n=1}^{\infty} a_n = L > 0$$

D. ***True***

Some essential facts are:

- $\sum_{n=1}^{\infty} a_n$ *converges iff* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ *exists*
- *When* $\lim_{n\to\infty} s_n$ *does exist,* $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$*.*
- *The series* $\sum_{n=1}^{\infty} a_n$ *likewise diverges iff the* $\lim_{n\to\infty} s_n$ *does not exist.*

Here, we are given $\sum_{n=1}^{\infty} a_n$ *converges to* $L > 0$*, which tells us immediately that* $\lim_{n\to\infty} s_n = L$*.*

E. ***True***

Since $\lim_{n\to\infty} s_n = L \neq 0$*, the divergence test tells us immediately that* $\sum_{n=1}^{\infty} s_n$ *MUST diverge.*

F. **False**

Since $\sum_{n=1}^{\infty} a_n$ *converges,* $\lim_{n\to\infty} a_n = 0$. *Thus,* $\lim_{n\to\infty} (a_n + 1) = 1$, *and the divergence test immediately tells us that* $\sum_{n=1}^{\infty} (a_n + 1)$ *MUST diverge!*

G. **False**

The divergence test NEVER *can be used to conclude that a series converges!*

Problem 3 *For each of the following, answer* **True** *or* **False**, *and explain why.*

(a) *If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\sum_{n=0}^{\infty} (a_n + 0.001)$ *converges.*

(b) *Since* $\int_1^{\infty} x \sin(\pi x)\, dx$ *diverges then, by the Integral Test,* $\sum_{n=0}^{\infty} n \sin(\pi n)$ *diverges.*

(c) *Since* $\int_1^{\infty} \frac{1}{x^2}\, dx = 1$ *then, by the Integral Test,* $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1$.

Solution: (a) **False**

Since $\sum_{n=0}^{\infty} a_n$ *converges, we know that* $\lim_{n\to\infty} a_n = 0$. *But then*

$$\lim_{n\to\infty} (a_n + 0.0001) = 0.0001 \neq 0$$

and so $\sum_{n=0}^{\infty} (a_n + 0.001)$ *diverges by the Divergence Test.*

(b) **False**

The Integral Test only holds for positive, decreasing functions. The function $f(x) = x \sin(\pi x)$ *is not always positive, nor is it always decreasing. So the Integral Test does not apply here.*

This problem is simpler than that though. Since $\sin(\pi n) = 0$ *for all integers* n, *we have that* $\sum_{n=0}^{\infty} n \sin(\pi n) = 0$.

(c) **False**

The Integral Test tells us that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges, but it does* **not** *give us the sum (this sum is actually* $\dfrac{\pi^2}{6}$*).*

Problem 4 *Assume* $\displaystyle\sum_{k=0}^{\infty} a_k = L$ *and* $b_k = 8$ *for all* k*.*

(a) *What is* $\displaystyle\lim_{k\to\infty} (a_k + b_k)$*?*

(b) *What is* $\displaystyle\lim_{k\to\infty} \sum_{n=0}^{k} (a_n + b_n)$*?*

(c) *What is* $\displaystyle\lim_{k\to\infty} \sum_{n=0}^{k} (a_{n+1} - a_n)$*?*

Solution: (a) *Since* $\displaystyle\sum_{k=0}^{\infty} a_k$ *converges, we know that* $\displaystyle\lim_{k\to\infty} a_k = 0$*. Therefore,*

$$\lim_{k\to\infty} (a_k + b_k) = 0 + 8 = \boxed{8}.$$

(b) *Since* $\displaystyle\lim_{n\to\infty} (a_n + b_n) = 8$*, the series* $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n)$ *diverges by the Divergence Test. But* $\displaystyle\lim_{k\to\infty} \sum_{n=0}^{k} (a_n + b_n) = \sum_{n=0}^{\infty} (a_n + b_n)$*. Thus*

$$\lim_{k\to\infty} \sum_{n=0}^{k} (a_n + b_n) = \sum_{n=0}^{\infty} (a_n + b_n) = \boxed{\infty}.$$

(c) *Let* $\displaystyle S_k = \sum_{n=0}^{k} (a_{n+1} - a_n)$ *(and recall that* $\{S_k\}$ *is the sequence of partial sums). Then*

$$\begin{aligned} S_k &= \sum_{n=0}^{k} (a_{n+1} - a_n) \\ &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_k - a_{k-1}) + (a_{k+1} - a_k) \\ &= a_{k+1} - a_0. \end{aligned}$$

Thus,

$$\lim_{k\to\infty}\sum_{n=0}^{k}(a_{n+1}-a_n)=\lim_{k\to\infty}S_k=\lim_{k\to\infty}a_{k+1}-a_0=\boxed{-a_0}.$$

Problem 5 *Determine if the following series converge or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2+2n+1}{3n^2+1}$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

Solution: (a) ***Divergence Test***

Notice that

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^2+2n+1}{3n^2+1}=\frac{1}{3}.$$

Therefore, since $\displaystyle\lim_{n\to\infty}a_n\neq 0$*, by the Divergence Test this series* $\boxed{\textit{diverges}}$.

(b) ***Integral Test***

First, notice that $f(x)=\dfrac{1}{x(\ln x)^2}$ *is a decreasing and positive function on* $[2,\infty)$*. Then*

$$\begin{aligned}\int_2^{\infty}f(x)\,dx&=\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx\\&=\lim_{b\to\infty}\int_2^{b}\frac{1}{x(\ln x)^2}\,dx\\&=\lim_{b\to\infty}\int_{\ln 2}^{\ln b}u^{-2}\,du\qquad u=\ln x,\ du=\frac{1}{x}\,dx\\&=\lim_{b\to\infty}\left[\frac{-1}{u}\right]_{\ln 2}^{\ln b}\\&=\lim_{b\to\infty}\left(\frac{-1}{\ln b}+\frac{1}{\ln 2}\right)\\&=0+\frac{1}{\ln 2}=\frac{1}{\ln 2}.\end{aligned}$$

Therefore, since the above integral converges, the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ $\boxed{\textit{converges}}$ *by the Integral Test.*

Problem 6 *For a sequence* $\{a_n\}_{n\geq 1}$ *let* $s_n = \sum_{k=1}^{n} a_k$ *denote its sequence of partial sums. Now, suppose that* $\{a_n\}_{n\geq 1}$ *is a sequence such that* $s_n = \dfrac{4n^2+9}{1-2n}$.

(a) *Find* $a_1 + a_2 + a_3$.

(b) *Find* $a_8 + a_9 + a_{10}$.

(c) *Determine whether* $\sum_{k=1}^{\infty} a_k$ *converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.*

(d) *Determine whether* $\sum_{k=1}^{\infty} s_k$ *converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.*

Solution: (a) *Note by definition that* $a_1+a_2+a_3 = s_3$. *Using the formula given for* s_n *with* $n = 3$ *gives:*

$$a_1 + a_2 + a_3 = \frac{4(3)^2+9}{1-2(3)} = \boxed{-9}.$$

(b) *Note that by definition:*

$$\begin{aligned} s_{10} &= a_1 + \cdots + a_7 + a_8 + a_9 + a_{10} \\ s_7 &= a_1 + \cdots + a_7 \end{aligned}$$

so $a_8 + a_9 + a_{10} = s_{10} - s_7$. *Using the formula for* s_n, *we have:*

$$s_10 = \frac{4(10)^2+9}{1-2(10)} = -\frac{409}{19}, \qquad s_7 = \frac{4(7)^2+9}{1-2(7)} = -\frac{205}{13}$$

Thus, $\boxed{a_8 + a_9 + a_{10} = -\frac{409}{19} + \frac{205}{13}}$.

(c) *To determine this, we note that:*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{4n^2+9}{1-2n} = -\infty.$$

Since $\lim_{n\to\infty} s_n$ *does not exist,* $\boxed{\sum_{k=1}^{\infty} a_k \text{ diverges by the Divergence Test}}$.

(d) *We showed that* $\lim_{n\to\infty} s_n = -\infty$, *so* $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by the Divergence Test}}$.