Section 12.3: Dot Products

Warm up:

Problem 1 *If $\vec{u} = \hat{\imath} - 2\hat{\jmath}$ and $\vec{v} = 3\hat{\imath} + 4\hat{k}$, find $\vec{u} \cdot \vec{v}$.*

Solution: *Note that these vectors are in $\mathbb{R}^3$ and not $\mathbb{R}^2$.*

$$\vec{u} \cdot \vec{v} = (1 \cdot 3) + (-2 \cdot 0) + (0 \cdot 4) = \boxed{3}.$$

Group work:

Problem 2 *Find a vector (in the xy-plane) with length 4 that makes a $\frac{\pi}{3}$ radian angle with the vector $\langle 3, 4\rangle$.*

Solution: *Let $\vec{v} = \langle a, b\rangle$ denote a vector that we are looking for, and let $\vec{u} = \langle 3, 4\rangle$. First note that*

$$|\vec{u}| = \sqrt{9 + 16} = 5.$$

So

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\left(\frac{\pi}{3}\right) = 5 \cdot 4 \cdot \frac{1}{2} = 10.$$

Then we have the following two equations:

$$10 = \vec{u} \cdot \vec{v} = 3a + 4b \tag{1}$$

$$16 = |\vec{v}|^2 = a^2 + b^2. \tag{2}$$

Solving equation (1) for a gives us

$$a = \frac{10 - 4b}{3}.$$

Learning outcomes:

Plugging this into equation (2) yields

$$\begin{aligned}\left(\frac{10-4b}{3}\right)^2 + b^2 &= 16\\ (10-4b)^2 + 9b^2 &= 144\\ 16b^2 - 80b + 100 + 9b^2 &= 144\\ 25b^2 - 80b - 44 &= 0\end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned}b &= \frac{80 \pm \sqrt{(-80)^2 - 4(25)(-44)}}{2(25)}\\ &= \frac{80 \pm \sqrt{10800}}{50}\\ &= \frac{80 \pm 60\sqrt{3}}{50}\\ &= \frac{8 \pm 6\sqrt{3}}{5}.\end{aligned}$$

We can choose either value for b. Choosing $b = \dfrac{8+6\sqrt{3}}{5}$ gives a value of $a = \dfrac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}$. Thus,

$$\vec{v} = \boxed{\left\langle \frac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}, \frac{8+6\sqrt{3}}{5} \right\rangle}$$

Problem 3 *Answer the following questions about $\text{proj}_v u$.*

(a) *Is $\text{proj}_v u$ a vector of the form $c\vec{v}$ or $c\vec{u}$ (where c is a real number)? ie, is $\text{proj}_v u$ parallel to $\vec{u}$ or $\vec{v}$?*

(b) *If $\vec{u} = 5\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$ and $\vec{v} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$, find $\text{proj}_v u$.*

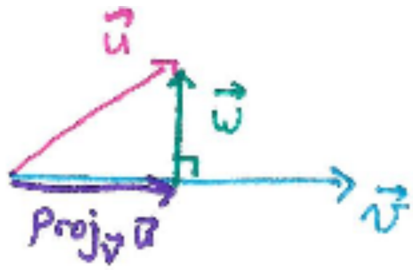
(c) *For $\vec{u}$ and $\vec{v}$ from part (b), write $\vec{u}$ as the sum of two perpendicular vectors, one of which is parallel to $\vec{v}$. Verify that the other vector is perpendicular to $\vec{v}$.*

Solution: (a) $\boxed{c\vec{v}}$

(b)

$$
\begin{aligned}
\text{proj}_v u &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\
&= \frac{10-24-12}{4+16+16}\langle 2,-4,4\rangle \\
&= \boxed{-\frac{13}{18}\langle 2,-4,4\rangle}
\end{aligned}
$$

(c) *A schematic picture of the situation is as follows:*



The vector which is parallel to $\vec{v}$ is

$$\text{proj}_v u = \boxed{\left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9}\right\rangle}$$

The vector which is orthogonal to $\vec{v}$ is

$$
\begin{aligned}
\vec{w} := \vec{u} - \text{proj}_v u = \langle 5,6,-3\rangle - \left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9}\right\rangle \\
= \boxed{\left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9}\right\rangle}
\end{aligned}
$$

And, clearly, $\text{proj}_v u + \vec{w} = \text{proj}_v u + (\vec{u} - \text{proj}_v u) = \vec{u}$.

To verify that $\vec{w}$ is orthogonal to $\vec{v}$, we take the dot product and show we get 0.

$\vec{w}\cdot\vec{v} = \left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9}\right\rangle \cdot \langle 2,4,4\rangle = \frac{58}{9}(2) + \frac{28}{9}(-4) - \frac{1}{9}(4) = \frac{116-112-4}{9} = 0$

Challenge Problem

Problem 4 *Suppose that the deli at the Tiny Sparrow grocery store sells roast beef for \$9 per pound, turkey for \$4 per pound, salami for \$5 per pound, and ham for \$7 per pound. For lunches this week, Sam the sandwhich maker buys 1.5 pounds of roast beef, 2 pounds of turkey, no salami, and half a pound of ham. How can you use a dot product to compute Sam's total bill from the deli?*

Solution: *The cost vector is*

$$\vec{c} = \langle 9, 4, 5, 7 \rangle.$$

The vector for Sam's order is

$$\vec{o} = \left\langle \frac{3}{2}, 2, 0, \frac{1}{2} \right\rangle.$$

Then Sam's bill is

$$\vec{c} \cdot \vec{o} = 9(1.5) + 4(2) + 5(0) + 7(0.5) = 13.5 + 8 + 0 + 3.5 = \boxed{25}.$$