Recitation #18: Comparison Tests and Alternating Series

Problem 1 *Determine if the following series absolutely converge, conditionally converge, or diverge.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$

(e) $\displaystyle\sum_{n=4}^{\infty} \frac{(-2)^n}{n}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 + 2n + 1}{3n^4 + 1}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^2 e^{-n} \right]$

Solution: (a) *Since* $\dfrac{1}{n+3}$ *is positive and decreasing, the Alternating Series Test applies. Thus, this series converges. But we know that* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$ *diverges (Harmonic Series) and so* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ *is conditionally convergent.*

Learning outcomes:

(b) *Let* $a_n = \dfrac{(n+1)^n}{(2n)^n}$. *Applying the Root Test, we see that*

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \frac{n+1}{2n} \\ &= \frac{1}{2} < 1.\end{aligned}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^n}{(2n)^n}$ *converges, and therefore* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)^n}{(2n)^n}$ *is absolutely convergent.*

(c) *Let* $f(x) = x^2 e^{\frac{-x^3}{3}}$. *The function* $f(x)$ *is continuous, positive, and decreasing. So we apply the integral test*

$$\begin{aligned}\int_1^{\infty} x^2 e^{\frac{-x^3}{3}}\,dx &= \lim_{b\to\infty} \int_1^{b} x^2 e^{\frac{-x^3}{3}}\,dx \\ &= \lim_{b\to\infty} \int_{\frac{1}{3}}^{\frac{b^3}{3}} e^{-u}\,du \qquad u = \frac{x^3}{3},\ du = x^2\,dx \\ &= \lim_{b\to\infty} \left[-e^{\frac{-b^3}{3}} + e^{\frac{-1}{3}}\right] \\ &= 0 + e^{\frac{-1}{3}} = e^{\frac{-1}{3}}.\end{aligned}$$

So, by the Integral Test, $\displaystyle\sum_{n=1}^{\infty} n^2 e^{\frac{-n^3}{3}}$ *converges. Therefore,* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{\frac{-n^3}{3}}$ *is absolutely convergent.*

(d) *First, notice that*

$$\frac{5}{3^n} > \frac{5}{3^n + 3^{-n}}.$$

Then since $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n}$ *converges (geometric series with* $r = \dfrac{1}{3} < 1$*), we know that* $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n + 3^{-n}}$ *converges by the Comparison Test. Thus, the series* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{3^n + 3^{-n}}$ *is absolutely convergent.*

(e) *Since the sequence* $\left\{\dfrac{(-2)^n}{n}\right\}$ *diverges, the series* $\displaystyle\sum_{n=4}^{\infty} \frac{(-2)^n}{n}$ *diverges by the Divergence Test.*

(f) *Notice that all the terms of this series are positive. Use the* ***Limit Comparison Test*** *with* $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n+1}{3n^4+1} \cdot \frac{n^2}{1}$$
$$= \frac{1}{3}.$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges, by the Limit Comparison Test we know that* $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3n^4+1}$ *converges and therefore is absolutely convergent.*

(g) *Notice that all the terms of this series are positive. Use the* ***Comparison Test*** *with* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$.

First, notice that for all $n \geq 3$,

$$\left(1+\frac{1}{n}\right)^2 < 2.$$

Also, notice that $\sum_{n=1}^{\infty} e^{-n}$ *is a convergent geometric series. Therefore* $\sum_{n=1}^{\infty} 2 \cdot e^{-n}$ *converges, and so* $\sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^2 e^{-n}\right]$ *converges by the Comparison Test and therefore is absolutely convergent.*

Problem 2 (a) *Find an upper bound for how close* $\sum_{k=0}^{4} \frac{(-1)^k k}{4^k}$ *is to the value of* $\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k}$.

(b) *How many terms are needed to estimate* $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n!}$ *to within* 10^{-6}*?*

Solution: (a) *Recall from the lesson that the remainder* R_n *is given by*

$$R_n = |S - S_n| \leq a_{n+1}.$$

Also notice that

$$\sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}$$

and

$$\sum_{k=0}^{4} \frac{(-1)^k k}{4^k} = \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}}.$$

So

$$\begin{aligned}
&\left| \sum_{k=0}^{\infty} \frac{(-1)^k k}{4^k} - \sum_{k=0}^{4} \frac{(-1)^k k}{4^k} \right| \\
&= \left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)}{4^{k-1}} - \sum_{k=1}^{5} \frac{(-1)^{k-1}(k-1)}{4^{k-1}} \right| \\
&= |S - S_5| \\
&= R_6 \le a_6 = \boxed{\frac{5}{4^5}}
\end{aligned}$$

(b) *Since* $R_n \le a_{n+1}$, *we need to find* n *so that*

$$\begin{aligned}
& a_{n+1} \le 10^{-6} \\
\iff \quad & \frac{\ln(n+1)}{(n+1)!} \le 10^{-6} \\
\iff \quad & \frac{(n+1)!}{\ln(n+1)} \ge 10^{6}
\end{aligned}$$

Using a calculator, we see that this inequality holds for $n \approx 8.8$. *So for* $\boxed{n \ge 9}$, $R_n < 10^{-6}$.