

Recitation #16: The Divergence, Integral, Ratio and Root Tests - Solutions

Group work:

Problem 1 For each of the following, answer **True** or **False**, and explain why.

- (a) If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} (a_n + 0.001)$ converges.
- (b) Since $\int_1^{\infty} x \sin(\pi x) dx$ diverges then, by the Integral Test, $\sum_{n=0}^{\infty} n \sin(\pi n)$ diverges.
- (c) Since $\int_1^{\infty} \frac{1}{x^2} dx = 1$ then, by the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1$.

Solution: (a) **False**

Since $\sum_{n=0}^{\infty} a_n$ converges, we know that $\lim_{n \rightarrow \infty} a_n = 0$. But then

$$\lim_{n \rightarrow \infty} (a_n + 0.0001) = 0.0001 \neq 0$$

and so $\sum_{n=0}^{\infty} (a_n + 0.001)$ diverges by the Divergence Test.

(b) **False**

The Integral Test only holds for positive, decreasing functions. The function $f(x) = x \sin(\pi x)$ is not always positive, nor is it always decreasing. So the Integral Test does not apply here.

This problem is simpler than that though. Since $\sin(\pi n) = 0$ for all integers n , we have that $\sum_{n=0}^{\infty} n \sin(\pi n) = 0$.

Recitation #16: The Divergence, Integral, Ratio and Root Tests - Solutions

(c) **False**

The Integral Test tells us that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, but it does **not** give us the sum (this sum is actually $\frac{\pi^2}{6}$).

Problem 2 Assume $\sum_{k=0}^{\infty} a_k = L$ and $b_k = 8$ for all k .

(a) What is $\lim_{k \rightarrow \infty} (a_k + b_k)$?

(b) What is $\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_n + b_n)$?

(c) What is $\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_{n+1} - a_n)$?

Solution: (a) Since $\sum_{k=0}^{\infty} a_k$ converges, we know that $\lim_{k \rightarrow \infty} a_k = 0$. Therefore,

$$\lim_{k \rightarrow \infty} (a_k + b_k) = 0 + 8 = \boxed{8}.$$

(b) Since $\lim_{n \rightarrow \infty} (a_n + b_n) = 8$, the series $\sum_{n=0}^{\infty} (a_n + b_n)$ diverges by the Divergence

Test. But $\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_n + b_n) = \sum_{n=0}^{\infty} (a_n + b_n)$. Thus

$$\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_n + b_n) = \sum_{n=0}^{\infty} (a_n + b_n) = \boxed{\infty}.$$

(c) Let $S_k = \sum_{n=0}^k (a_{n+1} - a_n)$ (and recall that $\{S_k\}$ is the sequence of partial sums). Then

$$\begin{aligned} S_k &= \sum_{n=0}^k (a_{n+1} - a_n) \\ &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \dots + (a_k - a_{k-1}) + (a_{k+1} - a_k) \\ &= a_{k+1} - a_0. \end{aligned}$$

Recitation #16: The Divergence, Integral, Ratio and Root Tests - Solutions

Thus,

$$\lim_{k \rightarrow \infty} \sum_{n=0}^k (a_{n+1} - a_n) = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} a_{k+1} - a_0 = \boxed{-a_0}.$$

Problem 3 Determine if the following series converge or diverge.

- (a) $\sum_{n=1}^{\infty} \frac{(7n+1)^2 \cdot 2^n}{5^n}$
- (b) $\sum_{n=1}^{\infty} a_n$, where $a_{n+1} = \frac{2n+5}{3n-1} \cdot a_n$ and $a_1 = 1$.
- (c) $\sum_{n=0}^{\infty} \frac{n^2 + 2n + 1}{3n^2 + 1}$
- (d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
- (e) $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$

Solution: (a) **Ratio Test**

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \left[\frac{(7(n+1)+1)^2 \cdot 2^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(7n+1)^2 \cdot 2^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{(7n+8)^2 \cdot 2}{5 \cdot (7n+1)^2} \\ &= \frac{49 \cdot 2}{49 \cdot 5} = \frac{2}{5}. \end{aligned}$$

Thus, since $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$, this series converges.

(b) **Ratio Test**

Even though the terms in this series look a little weird, this is set up perfectly for the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{2n+5}{3n-1} = \frac{2}{3}.$$

Thus, since $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$, this series converges.

(c) **Divergence Test**

Notice that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{3n^2 + 1} = \frac{1}{3}.$$

Therefore, since $\lim_{n \rightarrow \infty} a_n \neq 0$, by the Divergence Test this series diverges.

(d) **Integral Test**

First, notice that $f(x) = \frac{1}{x(\ln x)^2}$ is a decreasing and positive function on $[2, \infty)$. Then

$$\begin{aligned} \int_2^\infty f(x) dx &= \int_2^\infty \frac{1}{x(\ln x)^2} dx \\ &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx \\ &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-2} du \quad u = \ln x, du = \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{-1}{u} \right]_{\ln 2}^{\ln b} \\ &= \lim_{b \rightarrow \infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln 2} \right) \\ &= 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}. \end{aligned}$$

Therefore, since the above integral converges, the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges by the Integral Test.

(e) **Ratio Test**

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= \lim_{k \rightarrow \infty} \left[\frac{((k+1)!)^3}{(3(k+1))!} \cdot \frac{(3k)!}{(k!)^3} \right] \\ &= \lim_{k \rightarrow \infty} \frac{(k+1)^3 (k!)^3}{(3k+3)(3k+2)(3k+1) \cdot (3k)!} \cdot \frac{(3k)!}{(k!)^3} \\ &= \lim_{k \rightarrow \infty} \frac{(k+1)^3}{(3k+3)(3k+2)(3k+1)} \\ &= \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{27}. \end{aligned}$$

Thus, since $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1$, this series converges.

Problem 4 How many terms are needed to estimate $\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$ to within 10^{-4} ?

What is the estimate for the sum of the series?

Solution: Let $f(x) = \frac{1}{x^2 + 1}$. Note that $f(x)$ is continuous, decreasing, and positive for $x \geq 1$, and $a_k = f(k)$.

If $S = \sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$ is the actual value of the sum, and $S_n = \sum_{k=1}^n \frac{1}{k^2 + 1}$ is the n^{th} partial sum, then the remainder $R_n = S - S_n$ can be bounded above by

$$R_n < \int_n^{\infty} f(x) dx.$$

So we need to find n so that

$$\int_n^{\infty} \frac{1}{x^2 + 1} dx < 10^{-4}. \quad (1)$$

We compute

$$\begin{aligned} \int_n^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \int_n^b \frac{1}{x^2 + 1} dx \\ &= \lim_{b \rightarrow \infty} \left[\arctan(x) \right]_n^b \\ &= \lim_{b \rightarrow \infty} (\arctan(b) - \arctan(n)) \\ &= \frac{\pi}{2} - \arctan(n). \end{aligned}$$

Plugging into equation (1) we see that we want

$$\begin{aligned} \frac{\pi}{2} - \arctan(n) &< 10^{-4} \\ \implies \arctan(n) &> \frac{\pi}{2} - 10^{-4} \\ \implies n &> \tan\left(\frac{\pi}{2} - 10^{-4}\right) \approx 9999.999967 \approx 10,000. \end{aligned}$$

Therefore, 10,000 terms will estimate $\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$ to within 10^{-4} . Using a computer, we can also compute

$$\sum_{k=1}^{10,000} \frac{1}{k^2 + 1} \approx 1.07657.$$