Recitation #12: Direction fields and Separable Differential Equations

Warm up:

Which of the following differential equations are separable?

(a) $y' = \dfrac{ty}{t^2+1}$,

(b) $\dfrac{dy}{dx} = x^2 \sin(3y) - x^2$,

(c) $y' = t^2 - y$.

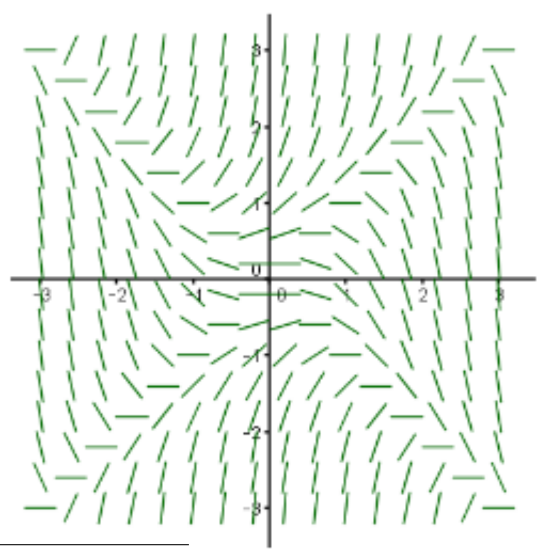
Solution: (a) Yes, it is separable. $y' = y \cdot \dfrac{t}{t^2+1}$.

(b) Yes, it is separable. $\dfrac{dy}{dx} = x^2 (\sin(3y) - 1)$.

(c) No, it is not separable. $t^2 - y$ can not be written in the form $F(t) \cdot G(y)$.

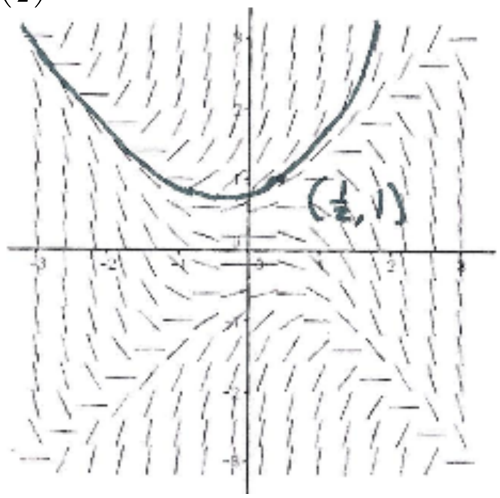
Group work:

Problem 1 *The following is a direction field for the differential equation* $\dfrac{dy}{dx} = y^2 - x^2$.



Learning outcomes:

Sketch the solution such that $y\left(\frac{1}{2}\right) = 1$.

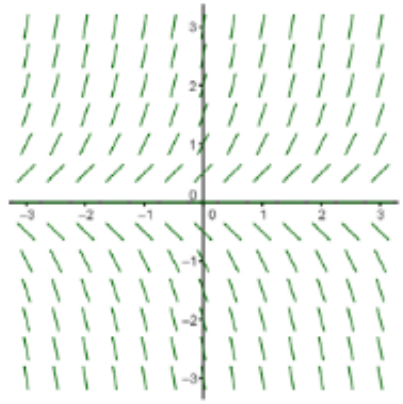


Solution:

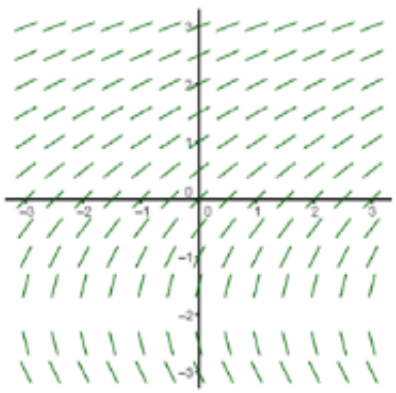
Problem 2 *Which of the following direction fields is the direction field corresponding to the differential equation $y' = 1 + y^2$?*

b.

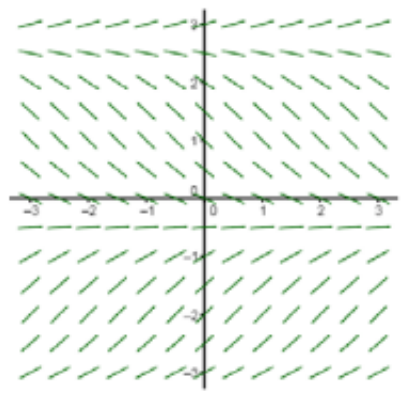
b.



c.



d.



Solution: *Look along the line $t = 0$ (the y-axis).*

Here $y' = 1 + y^2$, so there is no t on the right hand side of the equation. Therefore, y' depends only on y. At $y = 0$ the slope is 1, then as y increases the slopes increase too. Similarly, as y gets more and more negative, the slope gets more and more positive. So it seems as if this direction field is (a).

Problem 3 *Find a specific solution to the differential equation $\dfrac{dy}{dx} = x^{-2}\arctan(x)$ if $y(1) = 5$.*

Solution: *First note that*

$$y = \int x^{-2}\arctan(x)\,dx.$$

To solve this integral, we use integration by parts with

$$u = \arctan(x) \qquad dv = x^{-2}\,dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = -\frac{1}{x}.$$

So

$$\int x^{-2}\arctan(x)\,dx = -\frac{1}{x}\arctan(x) + \int \frac{1}{x(1+x^2)}\,dx.$$

To complete this new integral, we use partial fractions.

$$\begin{aligned} \frac{1}{x(1+x^2)} &= \frac{A}{x} + \frac{Bx+C}{1+x^2} \\ \implies \quad 1 &= A(1+x^2) + (Bx+C)x \\ \implies \quad 1 &= (A+B)x^2 + Cx + A. \end{aligned}$$

Comparing coefficients, we see that $A = 1$, $C + 0$, and $B = -1$.

Thus

$$\begin{aligned} y &= \int x^{-2} \arctan(x)\, dx \\ &= -\frac{1}{x}\arctan(x) + \int \frac{1}{x(1+x^2)}\, dx \\ &= -\frac{1}{x}\arctan(x) + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx \\ &= -\frac{1}{x}\arctan(x) + \ln|x| - \frac{1}{2}\ln(1+x^2) + C. \end{aligned}$$

To finish, we use the initial condition to solve for C.

$$\begin{aligned} 5 = y(1) &= -\frac{\pi}{4} + 0 - \frac{1}{2}\ln(2) + C \\ \implies \quad C &= 5 + \frac{\pi}{4} + \frac{1}{2}\ln(2). \end{aligned}$$

Therefore

$$y(t) = -\frac{1}{x}\arctan(x) + \ln|x| - \frac{1}{2}\ln(1+x^2) + 5 + \frac{\pi}{4} + \frac{1}{2}\ln(2).$$

Problem 4 *Find a specific solution to the initial value problem*

$$\frac{dy}{dx} = x^2\sin(x), \qquad y(0) = 5.$$

Solution: *First, notice that*

$$y = \int x^2 \sin(x)\, dx.$$

To solve this integral, we use integration by parts twice.

$$u = x^2 \qquad dv = \sin(x)\, dx$$

$$du = 2x\, dx \qquad v = -\cos(x).$$

So

$$\int x^2\sin(x)\, dx = -x^2\cos(x) + \int 2x\cos(x)\, dx.$$

Now we use

$$u = 2x \qquad dv = \cos(x)\, dx$$

$$du = 2\, dx \qquad v = \sin(x).$$

Then

$$\begin{aligned}
y &= \int x^2 \sin(x)\, dx \\
&= -x^2 \cos(x) + \int 2x \cos(x)\, dx \\
&= -x^2 \cos(x) + 2x \sin(x) - \int 2 \sin(x)\, dx \\
&= -x^2 \cos(x) + 2x \sin(x) + 2\cos(x) + C.
\end{aligned}$$

Finally, to finish the problem, we solve for C.

$$\begin{aligned}
& 5 = y(0) = 0 + 0 + 2 + C \\
\implies \quad & C = 3.
\end{aligned}$$

Thus,

$$y(t) = -x^2 \cos(x) + 2x \sin(x) + 2\cos(x) + 3.$$

Problem 5 *Solve the following differential equations assuming that $y(4) = 5$.*

(a) $y' = x + xy^2$

Solution:

$$\begin{aligned}
& y' = x + xy^2 \\
\implies \quad & \frac{dy}{dx} = x(1 + y^2) \\
\implies \quad & \frac{dy}{1 + y^2} = x\, dx.
\end{aligned}$$

So this equation ***is*** *separable. To solve, we integrate both sides of the equation:*

$$\begin{aligned}
& \int \frac{1}{1+y^2}\, dy = \int x\, dx \\
\implies \quad & \arctan(y) = \frac{1}{2}x^2 + C \qquad (1) \\
\implies \quad & y = \tan\left(\frac{1}{2}x^2 + C\right).
\end{aligned}$$

To find C, we plug the initial condition $y(4) = 5$ into equation (1) and solve for C.

$$\begin{aligned}
& \arctan(5) = \frac{1}{2}(4)^2 + C = 8 + C \\
\implies \quad & C = \arctan(5) - 8.
\end{aligned}$$

So

$$y = \tan\left(\frac{1}{2}x^2 + \arctan(5) - 8\right).$$

(b) $y' = e^{2x-y}$

Solution:

$$
\begin{aligned}
& y' = e^{2x-y} \\
\implies \quad & \frac{dy}{dx} = \frac{e^{2x}}{e^y} \\
\implies \quad & e^y \, dy = e^{2x} \, dx \qquad (2)
\end{aligned}
$$

and so this ***is*** *a separable equation. To solve, we integrate both sides of equation (2).*

$$
\begin{aligned}
& \int e^y \, dy = \int e^{2x} \, dx \\
\implies \quad & e^y = \frac{1}{2}e^{2x} + C \qquad (3) \\
\implies \quad & y = \ln\left(\frac{1}{2}e^{2x} + C\right).
\end{aligned}
$$

To find C, we plug into equation (3) and solve for C:

$$
\begin{aligned}
& e^5 = \frac{1}{2}e^8 + C \\
\implies \quad & C = e^5 - \frac{1}{2}e^8.
\end{aligned}
$$

Therefore

$$
y = \ln\left(\frac{1}{2}e^{2x} + e^5 - \frac{1}{2}e^8\right).
$$