Dot Products and Cross Products

Warm up:

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in 3-space $\mathbb{R}^3$, which of the following make sense?

(a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

(b) $(\vec{a} \cdot \vec{b})\vec{c}$

(c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$

(d) $(\vec{a} \cdot \vec{b}) + \vec{c}$

(e) $(\vec{a} \times \vec{b}) + \vec{c}$

(f) $\vec{a} \cdot (\vec{b} + \vec{c})$

(g) $\vec{a} \cdot (\vec{b} \times \vec{c})$

(h) $\vec{a} \times (\vec{b} \cdot \vec{c})$

(i) $(\vec{a} \times \vec{b})\vec{c}$

Solution: (a) Since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ does **not** make sense.

(b) Now since $\vec{a} \cdot \vec{b}$ is a scalar, $(\vec{a} \cdot \vec{b})\vec{c}$ **does** make sense as regular scalar multiplication.

(c) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ **does** make sense.

(d) This is of the form "scalar + vector", which does **not** make sense.

(e) Since $\vec{a} \times \vec{b}$ is a vector, $(\vec{a} \times \vec{b}) + \vec{c}$ **does** make sense.

(f) This is of the form "vector · vector", which **does** make sense.

(g) This is also of the form "vector · vector", which **does** make sense.

(h) This is of the form "vector × scalar", which does **not** make sense.

(i) Since $\vec{a} \times \vec{b}$ is a vector, this does **not** make sense.

Group work:

Problem 1 *Answer the following questions about $proj_v u$.*

(a) *Is $proj_v u$ a vector of the form $c\vec{v}$ or $c\vec{u}$ (where c is a real number)? ie, is $proj_v u$ parallel to $\vec{u}$ or $\vec{v}$?*

(b) *If $\vec{u} = 5\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$ and $\vec{v} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$, find $proj_v u$.*

Learning outcomes:

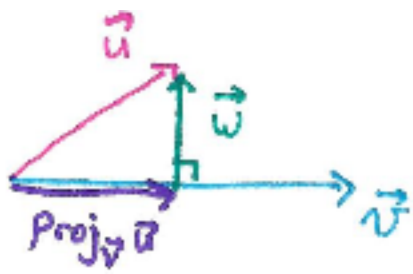
(c) *For $\vec{u}$ and $\vec{v}$ from part (b), write $\vec{u}$ as the sum of two perpendicular vectors, one of which is parallel to $\vec{v}$. Verify that the other vector is perpendicular to $\vec{v}$.*

Solution: (a) $\boxed{c\vec{v}}$

(b)

$$\begin{aligned} \text{proj}_v u &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\ &= \frac{10-24-12}{4+16+16}\langle 2,-4,4\rangle \\ &= \boxed{-\frac{13}{18}\langle 2,-4,4\rangle} \end{aligned}$$

(c) *A schematic picture of the situation is as follows:*



The vector which is parallel to $\vec{v}$ is

$$\text{proj}_v u = \boxed{\left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9}\right\rangle}$$

The vector which is orthogonal to $\vec{v}$ is

$$\begin{aligned} \vec{w} := \vec{u} - \text{proj}_v u = \langle 5,6,-3\rangle - \left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9}\right\rangle \\ = \boxed{\left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9}\right\rangle} \end{aligned}$$

And, clearly, $\text{proj}_v u + \vec{w} = \text{proj}_v u + (\vec{u} - \text{proj}_v u) = \vec{u}$.

To verify that $\vec{w}$ is orthogonal to $\vec{v}$, we take the dot product and show we get 0.

$\vec{w}\cdot\vec{v} = \left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9}\right\rangle\cdot\langle 2,4,4\rangle = \frac{58}{9}(2)+\frac{28}{9}(-4)-\frac{1}{9}(4) = \frac{116-112-4}{9} = 0$

Problem 2 *Given three dimensional vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, use dot product or cross product notation to describe the following vectors:*

(a) *The vector projection of $\vec{w}$ onto $\vec{u}$.*

(b) *A vector orthogonal to both $\vec{u}$ and $\vec{v}$.*

(c) *A vector with the length of $\vec{v}$ and the direction of $\vec{w}$.*

(d) *A vector orthogonal to $\vec{u} \times \vec{v}$ and $\vec{w}$.*

Solution: (a) *This is the definition of vector projections.*

$$\mathrm{proj}_u w = \boxed{\left(\frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\right) \vec{u}}$$

(b) *There are many such vectors, but one of them is*

$$\boxed{\vec{u} \times \vec{v}}$$

(c) *Note that $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ so that $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.*

$$|\vec{v}| \left(\frac{\vec{w}}{|\vec{w}|}\right) = \boxed{\frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\vec{w} \cdot \vec{w}}} \vec{w}}$$

(d)

$$\boxed{(\vec{u} \times \vec{v}) \times \vec{w}}$$

Problem 3 *Let $\vec{u} = \langle 5, -1, 8 \rangle$ and $\vec{v} = \langle -2, 10, 5 \rangle$.*

(a) *Find a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.*

(b) *Verify that your answer is perpendicular to both $\vec{u}$ and $\vec{v}$*

(c) *Find a vector of length 7 perpendicular to both $\vec{u}$ and $\vec{c}$.*

Solution: (a) *Let $\vec{u} = \langle 5, -1, 8 \rangle$ and $\vec{v} = \langle -2, 10, 5 \rangle$. Then a vector which is perpendicular to both $\vec{u}$ and $\vec{v}$ is $\vec{w} := \vec{u} \times \vec{v}$. So we calculate*

$$\begin{aligned} \vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -1 & 8 \\ -2 & 10 & 5 \end{vmatrix} &= (-5 - 80)\hat{\imath} - (25 + 16)\hat{\jmath} + (50 - 2)\hat{k} \\ &= -85\hat{\imath} - 41\hat{\jmath} + 48\hat{k} \end{aligned}$$

(b) *To verify perpendicularity, we take the dot product.*

$$\vec{u}\cdot\vec{w} = \langle 5,-1,8\rangle\cdot\langle -85,-41,48\rangle = 5(-85)-1(-41)+8(48) = -425+41+384 = 0$$
$$\vec{v}\cdot\vec{w} = \langle -2,10,5\rangle\cdot\langle -85,-41,48\rangle = -2(-85)+10(-41)+5(48) = 170-410+240 = 0$$

(c) *A unit vector in the same direction as* $\vec{w}$ *is*

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{(-85)^2+(-41)^2+48^2}}\vec{w} = \frac{1}{\sqrt{11210}}\vec{w}.$$

Therefore, a vector with a magnitude of 7 *in the same direction as* $\vec{w}$ *is*

$$\vec{t} = \frac{7}{|\vec{w}|}\vec{w} = \boxed{\frac{7}{\sqrt{11210}}\langle -85,-41,48\rangle}$$

Problem 4 *Find the area of the triangle in* $\mathbb{R}^3$ *with vertices at* $P(2,-1,0)$, $Q(1,1,4)$ *and* $R(2,-1,6)$.

Solution: *The area of the triangle is* $\frac{1}{2}|\vec{PQ}\times\vec{PR}|$.

$$\vec{PR} = \langle 2,-1,6\rangle - \langle 2,-1,0\rangle = \langle 0,0,6\rangle,$$
$$\vec{PQ} = \langle 1,1,4,\rangle - \langle 2,-1,0\rangle = \langle -1,2,4\rangle.$$

So

$$\begin{aligned}\vec{PQ}\times\vec{PR} &= (-\vec{i}+2\vec{j}+4\vec{k})\times 6\vec{k} = -(\vec{i}\times\vec{k}) + 2(\vec{j}\times\vec{k}) + 24(\vec{k}\times\vec{k})\\ &= -(-\vec{j}) + 2\vec{i} + 0 = \langle 2,1,0\rangle.\end{aligned}$$

The area of the triangle is $\frac{1}{2}\sqrt{2^2+1^2+0^2} = \frac{\sqrt{5}}{2}$.