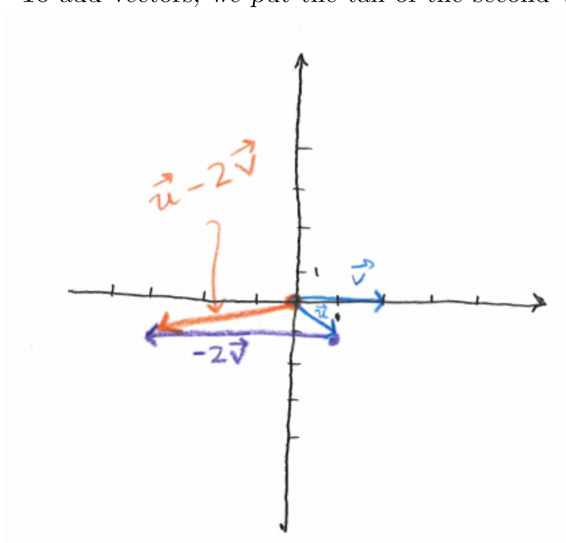
Recitation #27: Vectors in Two and Three Dimensions and Dot Products

Warm up:

Problem 1 *Sketch the vectors* $\mathbf{u} = \langle 1, -1\rangle$ *and* $\mathbf{v} = \langle 2, 0\rangle$. *Now using your sketch of these vectors, sketch* $\mathbf{u} - 2\mathbf{v}$.

Solution: *To add vectors, we put the tail of the second vector on the head of the first.*



Problem 2 *If* $\vec{u} = \hat{\imath} - 2\hat{\jmath}$ *and* $\vec{v} = 3\hat{\imath} + 4\hat{k}$, *find* $\vec{u} \cdot \vec{v}$.

Solution: *Note that these vectors are in* $\mathbb{R}^3$ *and not* $\mathbb{R}^2$.

$$\vec{u} \cdot \vec{v} = (1 \cdot 3) + (-2 \cdot 0) + (0 \cdot 4) = \boxed{3}.$$

Learning outcomes:

Group work:

Problem 3 *Suppose that* $\mathbf{u} = \langle 5, -1\rangle$ *and* $\mathbf{v} = \langle 2, 3\rangle$. *Find the following quantities:*

(a) $-\mathbf{v}$

(b) $3\mathbf{u} - 4\mathbf{v}$

(c) $|\mathbf{u}|$

Solution: (a) $-\mathbf{v} = \langle -2, -3\rangle$

(b) $3\mathbf{u} - 4\mathbf{v} = \langle 15, -4\rangle - \langle 8, 12\rangle = \langle 7, -16\rangle$.

(c) $|\mathbf{u}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$.

Problem 4 *Suppose that* $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ *in a 2-dimensional vector space. Find the following:*

(a) *A unit vector in the same direction of* $\mathbf{u}$.

(b) *All unit vectors parallel to* $\mathbf{u}$. *(How does differ from part (a)?)*

(c) *Two vector parallel to* $\mathbf{u}$ *with length 10.*

(d) *Two non-zero vectors perpendicular to* $\mathbf{u}$.

Solution: (a) $|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = 5$. *A unit vector in the same direction is* $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \langle \dfrac{3}{5}, \dfrac{-4}{5}\rangle$.

(b) *Parallel unit vectors are* $\pm\dfrac{\mathbf{u}}{|\mathbf{u}|}$, *which are* $\langle \dfrac{3}{5}, \dfrac{-4}{5}\rangle$ *and* $\langle \dfrac{-3}{5}, \dfrac{4}{5}\rangle$. *Note that parallel vectors include vectors in the opposite direction.*

(c) *Since* $\mathbf{u}$ *has length 5, two parallel vectors of length 10 are* $\pm 2\mathbf{u}$, *which are* $\langle 6, -8\rangle$ *and* $\langle -6, 8\rangle$.

(d) *In 2 dimensions, we can find a perpendicular vector using what we know about finding a perpedicular line. In particular, we know that two lines are perpendicular if the slope of line 1 is equal to the negative reciprical of the slope of line 2. That is,* $m_1 = \dfrac{-1}{m_2}$. *To find the slope of our vector* $\mathbf{u}$, *we find the slope between the head of the vector at (3,-4) and the* *tail of the vector at (0,0). The slope of* $\mathbf{u}$ *is* $\frac{-4-0}{3-0} = \frac{-4}{3}$ *Therefore, the slope of a perpendicular vector will be* $\frac{3}{4}$*. One vector with this slope is* $\boxed{\mathbf{u_1} = 4\mathbf{i} + 3\mathbf{j}}$ *and another is* $\boxed{\mathbf{u_2} = -4\mathbf{i} - 3\mathbf{j}}$*. Note: In more that 2-dimensions, this technique won't work.*

Problem 5 *Solve the following problems:*

(a) *Which of the points* $(6, 2, 3)$*,* $(-5, -1, 4)$*, and* $(0, 3, 8)$ *is closest to the* xz*-plane? Which point lies on the* yz*-plane?*

(b) *Write an equation of the circle of radius* 2 *centered at* $(-3, 4, 1)$ *that lies in a plane parallel to the* xy*-plane.*

(c) *Describe the sphere* $x^2 + y^2 + z^2 + 6x - 14y - 2z = 5$ *(ie, find its center and radius).*

(d) *Find a vector whose magnitude is* 311 *and is in the same direction as the vector* $\langle 3, -6, 7\rangle$*.*

Solution: (a) *The* xz*-plane has equation* $y = 0$*. The distance from a point* (a, b, c) *to* $y = 0$ *is just* $|b|$*. So*

$$\begin{aligned} &(6, 2, 3) \text{ has distance } 2 \\ &(-5, -1, 4) \text{ has distance } 1 \\ &(0, 3, 8) \text{ has distance } 3 \end{aligned}$$

Therefore, the point $(-5, -1, 4)$ *is closest to the* xz*-plane.*

The yz*-plane is* $x = 0$*, and so the point* $(0, 3, 8)$ *is on the* yz*-plane.*

(b) *A plane parallel to the* xy*-plane has equation* $z = \#$*. We are looking for such a plane containing the point* $(-3, 4, 1)$*, and so the plane is* $z = 1$*. Therefore, the equation is*

$$\boxed{(x+3)^2 + (y-4)^2 = 4\ ,\ z = 1.}$$

(c) *Let* $\vec{v} = \langle 3, -6, 7\rangle$*. Then*

$$\begin{aligned} |\vec{v}| &= \sqrt{3^2 + (-6)^2 + 7^2} \\ &= \sqrt{9 + 36 + 49} \\ &= \sqrt{94}. \end{aligned}$$

So a unit vector in the same direction as $\vec{v}$ *is*

$$\frac{1}{\sqrt{94}}\langle 3, -6, 7\rangle$$

and therefore a vector with magnitude 311 *in the same direction as* v *is*

$$\boxed{\frac{311}{\sqrt{94}}\langle 3, -6, 7\rangle}$$

Problem 6 *Find a vector (in the* xy*-plane) with length* 4 *that makes a* $\frac{\pi}{3}$ *radian angle with the vector* $\langle 3, 4\rangle$.

Solution: *Let* $\vec{v} = \langle a, b\rangle$ *denote a vector that we are looking for, and let* $\vec{u} = \langle 3, 4\rangle$*. First note that*

$$|\vec{u}| = \sqrt{9+16} = 5.$$

So

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\left(\frac{\pi}{3}\right) = 5\cdot 4\cdot\frac{1}{2} = 10.$$

Then we have the following two equations:

$$10 = \vec{u}\cdot\vec{v} = 3a + 4b \tag{1}$$

$$16 = |\vec{v}|^2 = a^2 + b^2. \tag{2}$$

Solving equation (1) *for* a *gives us*

$$a = \frac{10-4b}{3}.$$

Plugging this into equation (2) *yields*

$$\begin{aligned}\left(\frac{10-4b}{3}\right)^2 + b^2 &= 16\\ (10-4b)^2 + 9b^2 &= 144\\ 16b^2 - 80b + 100 + 9b^2 &= 144\\ 25b^2 - 80b - 44 &= 0\end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned}b &= \frac{80 \pm \sqrt{(-80)^2 - 4(25)(-44)}}{2(25)}\\ &= \frac{80\pm\sqrt{10800}}{50}\\ &= \frac{80\pm 60\sqrt{3}}{50}\\ &= \frac{8\pm 6\sqrt{3}}{5}.\end{aligned}$$

We can choose either value for b. Choosing $b = \dfrac{8+6\sqrt{3}}{5}$ *gives a value of* $a = \dfrac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}$*. Thus,*

$$\vec{v} = \boxed{\left\langle \frac{10 - 4\left(\frac{8+6\sqrt{3}}{5}\right)}{3}, \frac{8+6\sqrt{3}}{5} \right\rangle}$$

Problem 7 *Answer the following questions about* $\text{proj}_v u$*.*

(a) *Is* $\text{proj}_v u$ *a vector of the form* $c\vec{v}$ *or* $c\vec{u}$ *(where c is a real number)? ie, is* $\text{proj}_v u$ *parallel to* $\vec{u}$ *or* $\vec{v}$*?*

(b) *If* $\vec{u} = 5\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$ *and* $\vec{v} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$*, find* $\text{proj}_v u$*.*

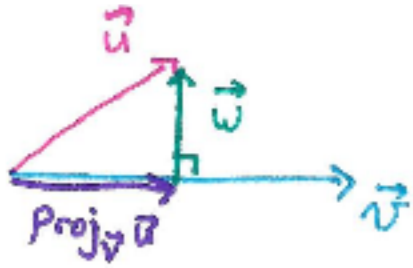
(c) *For* $\vec{u}$ *and* $\vec{v}$ *from part (b), write* $\vec{u}$ *as the sum of two perpendicular vectors, one of which is parallel to* $\vec{v}$*. Verify that the other vector is perpendicular to* $\vec{v}$*.*

Solution: (a) $\boxed{c\vec{v}}$

(b)

$$\begin{aligned}
\text{proj}_v u &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\
&= \frac{10-24-12}{4+16+16}\langle 2, -4, 4\rangle \\
&= \boxed{-\frac{13}{18}\langle 2, -4, 4\rangle}
\end{aligned}$$

(c) *A schematic picture of the situation is as follows:*



The vector which is parallel to $\vec{v}$ *is*

$$\text{proj}_v u = \boxed{\left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9} \right\rangle}$$

The vector which is orthogonal to $\vec{v}$ is

$$\vec{w} := \vec{u} - \text{proj}_v u = \langle 5, 6, -3\rangle - \left\langle -\frac{13}{9}, \frac{26}{9}, -\frac{26}{9} \right\rangle$$
$$= \boxed{\left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9} \right\rangle}$$

And, clearly, $\text{proj}_v u + \vec{w} = \text{proj}_v u + (\vec{u} - \text{proj}_v u) = \vec{u}$.

To verify that $\vec{w}$ is orthogonal to $\vec{v}$, we take the dot product and show we get 0.
$\vec{w} \cdot \vec{v} = \left\langle \frac{58}{9}, \frac{28}{9}, -\frac{1}{9} \right\rangle \cdot \langle 2, 4, 4\rangle = \frac{58}{9}(2) + \frac{28}{9}(-4) - \frac{1}{9}(4) = \frac{116 - 112 - 4}{9} = 0$

Challenge Problem

Problem 8 *Suppose that the deli at the Tiny Sparrow grocery store sells roast beef for \$9 per pound, turkey for \$4 per pound, salami for \$5 per pound, and ham for \$7 per pound. For lunches this week, Sam the sandwhich maker buys 1.5 pounds of roast beef, 2 pounds of turkey, no salami, and half a pound of ham. How can you use a dot product to compute Sam's total bill from the deli?*

Solution: *The cost vector is*

$$\vec{c} = \langle 9, 4, 5, 7\rangle.$$

The vector for Sam's order is

$$\vec{o} = \left\langle \frac{3}{2}, 2, 0, \frac{1}{2} \right\rangle.$$

Then Sam's bill is

$$\vec{c} \cdot \vec{o} = 9(1.5) + 4(2) + 5(0) + 7(0.5) = 13.5 + 8 + 0 + 3.5 = \boxed{25}.$$