Recitation #21: Taylor series - Solutions

Warm up:

Find the Taylor series for:

(a) $27x^2 - 3x + 17$ about $a = 1$.

(b) $\sin(2x)$ about $a = \dfrac{\pi}{8}$.

Solution: (a) Let $f(x) = 27x^2 - 3x + 17$. Then

$$\begin{aligned}
&f(1) = 27 - 3 + 17 = 41 \\
&f'(x) = 54x - 3 \quad \Longrightarrow \quad f'(1) = 54 - 3 = 51 \\
&f''(x) = 54 \quad \Longrightarrow \quad f''(1) = 54 \\
&f^{(3)}(x) = 0 \quad \Longrightarrow \quad f^{(3)}(1) = 0 \\
&\quad\vdots \qquad\qquad\qquad\qquad \vdots \\
&f^{(n)}(x) = 0 \quad \Longrightarrow \quad f^{(n)}(1) = 0.
\end{aligned}$$

So

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \boxed{41 + 51(x-1) + \frac{54}{2!}(x-1)^2}$$

Lastly, note that if you multiply this out then you will get back the original polynomial.

(b) Let $f(x) = \sin(2x)$. Then

$$\begin{aligned}
&f\left(\frac{\pi}{8}\right) = \sin\left(\frac{2\pi}{8}\right) = \frac{\sqrt{2}}{2} \\
&f'(x) = 2\cos(2x) \quad \Longrightarrow \quad f'\left(\frac{\pi}{8}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \\
&f''(x) = -4\sin(2x) \quad \Longrightarrow \quad f''\left(\frac{\pi}{8}\right) = -4 \cdot \frac{\sqrt{2}}{2} = -2\sqrt{2} \\
&f^{(3)}(x) = -8\cos(2x) \quad \Longrightarrow \quad f^{(3)}\left(\frac{\pi}{8}\right) = -8 \cdot \frac{\sqrt{2}}{2} = -4\sqrt{2} \\
&f^{(4)}(x) = 16\cos(2x) \quad \Longrightarrow \quad f^{(4)}\left(\frac{\pi}{8}\right) = 16 \cdot \frac{\sqrt{2}}{2} = 8\sqrt{2}.
\end{aligned}$$

Learning outcomes:

Continuing this pattern, we see that

$$f^{(k)}\left(\frac{\pi}{8}\right) = (-1)^{\lceil \frac{k}{2} \rceil} 2^{k-1}\sqrt{2}$$

where $\left\lceil \frac{k}{2} \right\rceil$ denotes the smallest integer greater than $\frac{k}{2}$. So, for example, $\left\lceil \frac{1}{2} \right\rceil = 1$, $\left\lceil \frac{2}{2} \right\rceil = 1$, $\left\lceil \frac{3}{2} \right\rceil = 2$, and so on.

So from here we have that the Taylor series for $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^{\lceil \frac{k}{2} \rceil} 2^{k-1}\sqrt{2}}{k!}\left(x - \frac{\pi}{8}\right)^k}$$

Group work:

Problem 1 *Find a power series (and interval of convergence) for each of the following functions*

(a) $f(x) = x^3 \sin(x^5)$

(b) $f(x) = \dfrac{1}{(1+x)^4}$

(c) $f(x) = \dfrac{1}{(3-5x^2)^4}$

(d) $f(x) = \sin^{-1}(x^5)$

Solution: (a) *We already know that*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

with interval of convergence $(-\infty, \infty)$. *So we use this to compute*

$$\begin{aligned} x^3 \sin(x^5) &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k (x^5)^{2k+1}}{(2k+1)!} \\ &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k x^{10k+5}}{(2k+1)!} \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{10k+8}}{(2k+1)!}} \end{aligned}$$

with interval of convergence $(-\infty, \infty)$.

(b) *Recall that the Binomial Series has the power series representation*

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$$

with interval of convergence $(-1, 1)$*, and where*

$$\binom{p}{k} = \frac{p(p-1)(p-2)\dots(p-k+1)}{k!}.$$

So

$$\begin{aligned} f(x) &= (1+x)^{-4} \\ &= \boxed{\sum_{k=0}^{\infty} \binom{-4}{k} x^k} \end{aligned}$$

with interval of convergence $(-1, 1)$.

(c) *Just as in part (b), we have that*

$$\begin{aligned} f(x) &= \frac{1}{(3-5x^2)^4} \\ &= (3-5x^2)^{-4} \\ &= 3^{-4}\left(1 - \frac{5}{3}x^2\right)^{-4} \\ &= 3^{-4}\sum_{k=0}^{\infty} \binom{-4}{k} \left(-\frac{5}{3}x^2\right)^k \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k \binom{-4}{k} \frac{5^k x^{2k}}{3^{k+4}}}. \end{aligned}$$

To find the interval of convergence, we know that we must have

$$\left| -\frac{5}{3}x^2 \right| < 1.$$

So we solve

$$\begin{aligned} & \left| -\frac{5}{3}x^2 \right| < 1 \\ \iff \quad & \frac{5}{3}x^2 < 1 \\ \iff \quad & x^2 < \frac{3}{5} \\ \iff \quad & -\sqrt{\frac{3}{5}} < x < \sqrt{\frac{3}{5}}. \end{aligned}$$

Thus, the interval of convergence is $\boxed{\left(-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right)}$.

(d) *Let us first consider* $g(x) = \arcsin(x)$. *Then*

$$\begin{aligned} g'(x) &= \frac{1}{\sqrt{1-x^2}} \\ &= (1-x^2)^{-\frac{1}{2}} \\ &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-x^2)^k \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} x^{2k} \end{aligned}$$

To find the interval of convergence, we need

$$|-x^2| < 1 \qquad \iff \qquad x^2 < 1 \qquad \iff \qquad -1 < x < 1.$$

So the interval of convergence for $g'(x)$ *is* $(-1, 1)$.

Now,

$$\begin{aligned} g(x) &= \int g'(x)\,dx \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1} + C \end{aligned}$$

which also has IOC $(-1, 1)$, *since integrating will not change the IOC of a binomial series. To find* C, *we just evaluate at* $x = 0$:

$$0 = \arcsin(0) = g(0) = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{0^{2k+1}}{2k+1} + C = 0 + C$$

and so $C = 0$. *Therefore, we have that*

$$g(x) = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1}$$

on $(-1, 1)$.

Finally, $f(x) = \arcsin(x^5) = g(x^5)$. *So*

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{(x^5)^{2k+1}}{2k+1} \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{10k+5}}{2k+1}} \end{aligned}$$

with interval of convergence $(-1, 1)$ *(since* $|x^5| < 1$ *if and only if* $|x| < 1$*).*

Problem 2 *Find a function (closed expression) for the following series and the interval on which the function and the series are equal.*

$$x + x^4 + \frac{1}{2}x^7 + \frac{1}{6}x^{10} + \frac{1}{24}x^{13} + \ldots$$

Solution:

$$\begin{aligned} x + x^4 + \frac{1}{2}x^7 + \frac{1}{6}x^{10} + \frac{1}{24}x^{13} + \ldots &= x + x^4 + \frac{1}{2!}x^7 + \frac{1}{3!}x^{10} + \frac{1}{4!}x^{13} + \ldots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}x^{3k+1} \\ &= x\sum_{k=0}^{\infty} \frac{x^{3k}}{k!} \\ &= x\sum_{k=0}^{\infty} \frac{(x^3)^k}{k!} \\ &= \boxed{xe^{x^3}} \end{aligned}$$

which has interval of convergence $(-\infty, \infty)$.

Problem 3 *Compute the sum of the following series (Hint: You should use Taylor series.)*

(a) $1 - \ln 2 + \dfrac{(\ln 2)^2}{2!} - \dfrac{(\ln 2)^3}{3!} + \ldots$

(b) $3 + \dfrac{9}{2!} + \dfrac{27}{3!} + \dfrac{81}{4!} + \ldots$

Solution: (a)

$$\begin{aligned} 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \ldots &= \sum_{k=0}^{\infty} \frac{(-\ln 2)^k}{k!} \\ &= e^{-\ln 2} = e^{\ln 2^{-1}} \\ &= 2^{-1} = \boxed{\frac{1}{2}}. \end{aligned}$$

(b)

$$\begin{aligned}
3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots &= \sum_{k=1}^{\infty} \frac{3^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{3^k}{k!} - \frac{3^0}{0!} \\
&= \boxed{e^3 - 1}.
\end{aligned}$$