Section 6.5: Length of Curves

Group work:

Problem 1 *Find the length of the following curves (length is in feet):*

(a) $y = \frac{4}{3}x^{\frac{3}{2}}$ *from* $(0,0)$ *to* $\left(1, \frac{4}{3}\right)$.

Solution:

$$\begin{aligned}
\textit{Arc Length} &= \int_0^1 \sqrt{1 + y'(x)^2}\, dx \\
&= \int_0^1 \sqrt{1 + \left(2x^{\frac{1}{2}}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1 + 4x}\, dx
\end{aligned}$$

$$\begin{aligned}
u &= 1 + 4x \\
du &= 4dx \\
\frac{du}{4} &= dx
\end{aligned}$$

$$\begin{aligned}
u(0) &= 1 + 4(0) = 1 \\
u(1) &= 1 + 4(1) = 5
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\int_1^5 \sqrt{u}\, du \\
&= \frac{1}{4}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^5 \\
&= \frac{1}{4}\left(\frac{2}{3}(5)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right) = \frac{1}{4}\left(\frac{2}{3}(5)^{\frac{3}{2}} - \frac{2}{3}\right)
\end{aligned}$$

(b) $x = \frac{1}{9}e^{3y} + \frac{1}{4}e^{-3y}$ *from* $\left(\frac{13}{36}, 0\right)$ *to* $\left(\frac{265}{288}, \ln 2\right)$.

Learning outcomes:

Solution:

$$
\begin{aligned}
\text{Arc Length} &= \int_0^{\ln 2} \sqrt{1+x'(y)^2}\,dy \\
&= \int_0^{\ln 2} \sqrt{1+\left(\frac{1}{3}e^{3y}-\frac{3}{4}e^{-3y}\right)^2}\,dy \\
&= \int_0^{\ln 2} \sqrt{1+\left(\frac{1}{9}e^{6y}-\frac{1}{2}+\frac{9}{16}e^{-6y}\right)}\,dy \\
&= \int_0^{\ln 2} \sqrt{\frac{1}{9}e^{6y}+\frac{1}{2}+\frac{9}{16}e^{-6y}}\,dy \\
&= \int_0^{\ln 2} \sqrt{\left(\frac{1}{3}e^{3y}+\frac{3}{4}e^{-3y}\right)^2}\,dy \\
&= \int_0^{\ln 2} \left(\frac{1}{3}e^{3y}+\frac{3}{4}e^{-3y}\right)\,dy \\
&= \left[\frac{1}{9}e^{3y}-\frac{1}{4}e^{-3y}\right]_0^{\ln 2} \\
&\overset{*}{=} \left(\frac{8}{9}-\frac{1}{32}\right)-\left(\frac{1}{9}-\frac{1}{4}\right) \\
&= \frac{7}{9}+\frac{7}{32} = \frac{224+63}{288} = \frac{287}{288}.
\end{aligned}
$$

* *Note that*

$$e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$$

and

$$e^{-3\ln 2} = e^{\ln 2^{-3}} = 2^{-3} = \frac{1}{8}.$$

Problem 2 **Set up** *an integral (or a sum of integrals) to find the perimeter of the region bounded by the curves* $y = 2x^2 - 5x + 13$ *and* $y = x^2 + 6x - 11$.

Solution: *Let* $f(x) = 2x^2 - 5x + 13$ *and* $g(x) = x^2 + 6x - 11$. *We first need to find the points where these two curves intersect. So we solve*

$$
\begin{aligned}
f(x) &= g(x) \\
2x^2 - 5x + 13 &= x^2 + 6x - 11 \\
x^2 - 11x + 24 &= 0 \\
(x-3)(x-8) &= 0 \\
x &= 3, 8.
\end{aligned}
$$

Then the perimeter is $L_1 + L_2$ *where*

$$L_1 = \int_3^8 \sqrt{1 + f'(x)^2}\,dx = \int_3^8 \sqrt{1 + (4x-5)^2}\,dx$$
$$L_2 = \int_3^8 \sqrt{1 + g'(x)^2}\,dx = \int_3^8 \sqrt{1 + (2x+6)^2}\,dx.$$

Problem 3 *A steady wind blows a kite due west. The kite's height above the ground from horizontal position* $x = 0$ *ft. to* $x = 80$ *ft. is given by*

$$y = 150 - \frac{1}{40}(x-50)^2.$$

Set up the integral to find the distance traveled by the kite.

Solution:

$$\text{distance the kite traveled} = \int_0^{80} \sqrt{1 + y'(x)^2}\,dx$$

where $y(x) = 150 - \frac{1}{40}(x-50)^2$*. Then since* $y' = -\frac{1}{20}(x-50)$*, we have that*

$$\text{distance the kite traveled} = \int_0^{80} \sqrt{1 + \frac{(x-50)^2}{400}}\,dx.$$