Recitation # 3: Volume by Slicing & Shells

Group work:

Problem 1 (a) *Consider the region bounded by the curves $y = x^2 + 8$ and $y = 7x - 2$. Set up an integral that will compute the volume of the solid whose base is the region and whose cross sections perpendicular to the region and the x-axis are:*

(i) Equilateral triangles

Solution: *We first set the curves equal to each other to see where they intersect*

$$\begin{aligned} x^2 + 8 &= 7x - 2 \\ x^2 - 7x + 10 &= 0 \\ (x-2)(x-5) &= 0 \\ x &= 2, 5. \end{aligned}$$

By checking the point $x = 3$, we see that $7x - 2 \geq x^2 + 8$ over the interval $[2, 5]$. So our region is bounded

- *from above by $y = 7x - 2$*
- *from below by $y = x^2 + 8$*
- *from the left by $x = 2$*
- *from the right by $x = 5$.*

Now, recall that an equilateral triangle is a triangle where the length of each side is the same. If we let s denote the common side length of an equilateral triangle, then we can use Pythagorean's Theorem to solve for the height h:

$$h^2 + \left(\frac{1}{2}s\right)^2 = s^2 \qquad \Longrightarrow \qquad h = \frac{\sqrt{3}}{2}s.$$

Then the area A of this equilateral triangle is

$$A = \frac{1}{2} \cdot s \cdot h = \frac{\sqrt{3}}{4}s^2.$$

Learning outcomes:

The volume of the solid that we are trying to find is given by the integral

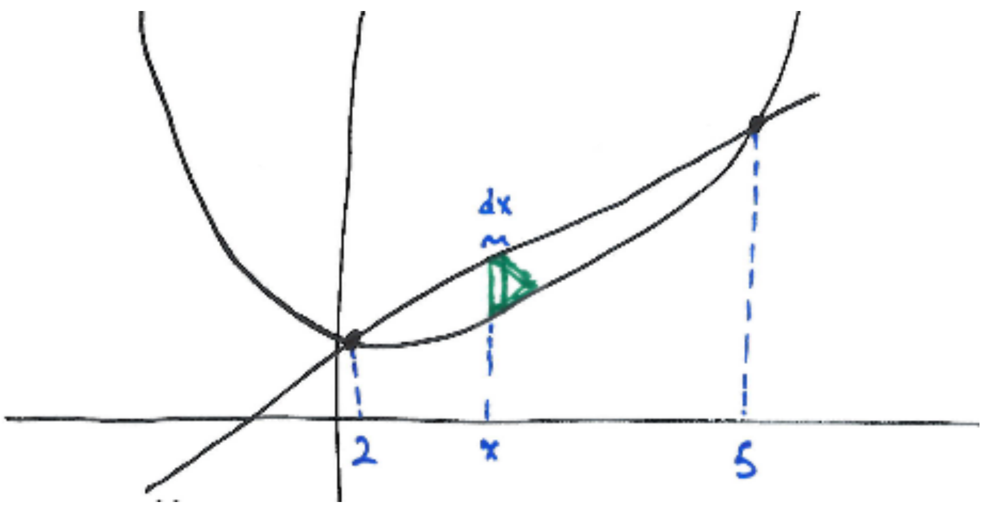
$$\int_2^5 A(x)\,dx$$

where $A(x)$ is the area of an equilateral triangle (within in the region and perpendicular to the x-axis) at a generic point x. The base of such a triangle is given by $s(x) = (7x-2)-(x^2+8) = -x^2+7x-10$ (see the picture below). Thus,

$$A(x) = \frac{\sqrt{3}}{4}s(x)^2 = \frac{\sqrt{3}}{4}(-x^2+7x-10)^2$$

and

$$\text{Volume of region} = \int_2^5 \frac{\sqrt{3}}{4}(-x^2+7x-10)^2\,dx.$$



(ii) *Semicircles*

Solution: *Everything is exactly the same as in part (a), except now each slice is a semicircle instead of an equilateral triangle. Recall that the area of half of a circle is $\frac{\pi}{2}r^2$, and at a generic point x the radius satisfies*

$$2r(x) = -x^2+7x-10 \qquad \Longrightarrow \qquad r^2(x) = \frac{1}{4}\left(-x^2+7x-10\right)^2.$$

Then we have that

$$\begin{aligned}\text{Volume of region} &= \int_2^5 A(x)\,dx \\ &= \int_2^5 \frac{\pi}{2}\cdot\frac{1}{4}\left(-x^2+7x-10\right)^2\,dx \\ &= \frac{\pi}{8}\int_2^5 \left(-x^2+7x-10\right)^2\,dx.\end{aligned}$$

(b) *Do the same as in (a), except that the solid's cross-sections are perpendicular to the region and the y-axis.*

Solution: (i) *Equilateral triangles.*

The structure of this problem is the same as in part (a). We know that the two curves intersect at $x = 2, 5$, and so plugging those into either equation shows that the y-coordinates of these intersection points are $y = 12, 33$. Over the region $12 \le y \le 33$ we can solve both equations for x

$$\begin{aligned}x_1 &= \sqrt{y-8} \\ x_2 &= \frac{1}{7}(y+2).\end{aligned}$$

By either checking a point in the interval $[12, 33]$ or simply by consulting the picture above, we see that $x_1 \ge x_2$ over this region. Then the base of an equilateral triangle at a generic point y is

$$s(y) = \sqrt{y-8} - \frac{1}{7}(y+2)$$

and the volume of the region is

$$\begin{aligned}\text{Volume of region} &= \int_{12}^{33} A(y)\,dy \\ &= \int_{12}^{33} \frac{\sqrt{3}}{4}\left(\sqrt{y-8}-\frac{1}{7}(y+2)\right)^2\,dy.\end{aligned}$$

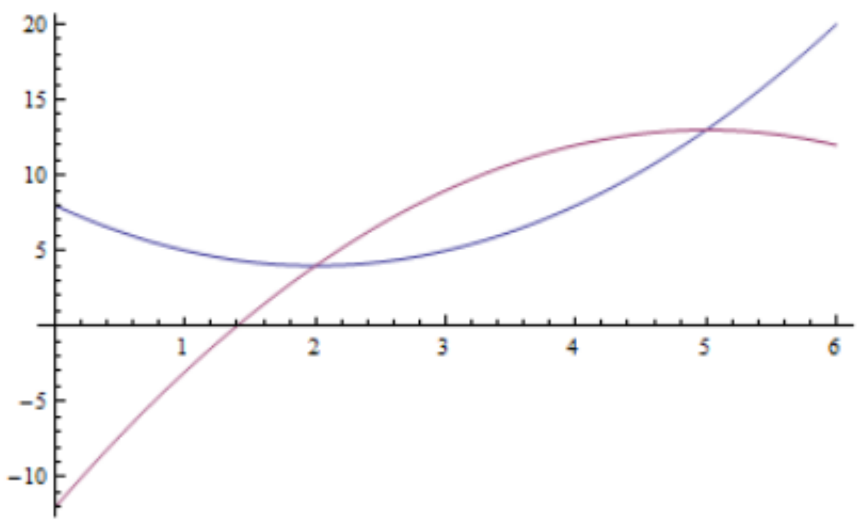
(ii) *Semicircles Again, we proceed in exactly the same way as in part (a). From above, we have that*

$$2r(y) = \sqrt{y-8} - \frac{1}{7}(y+2) \qquad \Longrightarrow \qquad r^2(y) = \frac{1}{4}\left(\sqrt{y-8}-\frac{1}{7}(y+2)\right)^2$$

and

$$\begin{aligned}
\text{Volume of region} &= \int_{12}^{33} A(y)\,dy \\
&= \int_{12}^{33} \frac{\pi}{2} \cdot \frac{1}{4} \left(\sqrt{y-8} - \frac{1}{7}(y+2) \right)^2 dy \\
&= \frac{\pi}{8} \int_{12}^{33} \left(\sqrt{y-8} - \frac{1}{7}(y+2) \right)^2 dy.
\end{aligned}$$

Problem 2 *Set up an integral that will find the volume of the solid formed by revolving the region bounded by the curves* $y = x^2 - 4x + 8$ *and* $y = -x^2 + 10x - 12$ *about:*



(a) *the* x*-axis*

Solution: *First, we need to find where the two curves intersect*

$$\begin{aligned}
x^2 - 4x + 8 &= -x^2 + 10x - 12 \\
2x^2 - 14x + 20 &= 0 \\
x^2 - 7x + 10 &= 0 \\
(x-2)(x-5) &= 0 \\
x &= 2, 5.
\end{aligned}$$

By plugging in the point $x = 3$*, we see that*

$$-x^2 + 10x - 12 \geq x^2 - 4x + 8$$

on the interval $[2, 5]$.

Notice that a cross section at a generic point x *looks like a "washer". Thus, to find the volume of the surface of revolution, we need to compute*

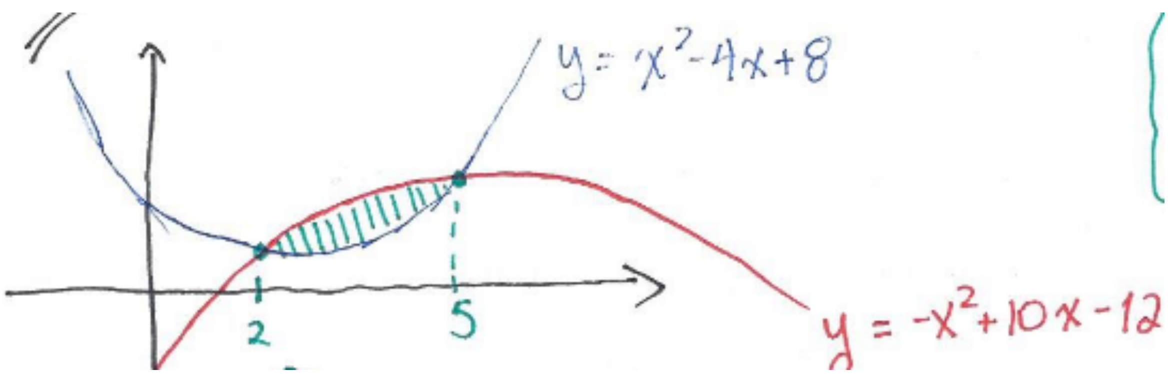
$$\int_2^5 A(x)\,dx$$

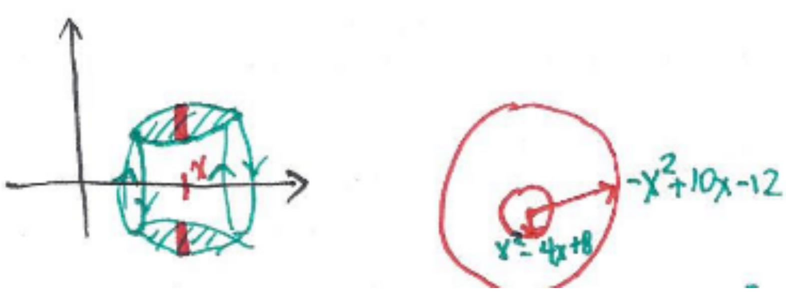
where $A(x)$ *denotes the area of the corresponding washer. Recall that* $A(x) = \pi\left(r_{out}^2 - r_{in}^2\right)$ *where* r_{out} *and* r_{in} *denote the outside and inside radius of the washer, respectively. Consulting the picture, we see that*

$$r_{out} = -x^2 + 10x - 12$$
$$r_{in} = x^2 - 4x + 8$$

and thus

$$\text{Volume of region} = \pi \int_2^5 \left[(-x^2 + 10x - 12)^2 - (x^2 - 4x + 8)^2\right] dx.$$





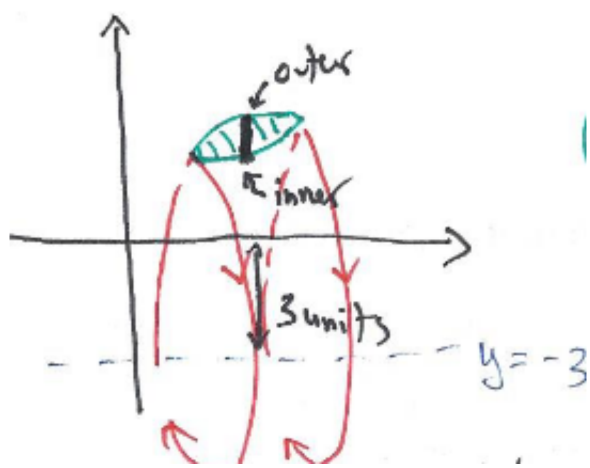
(b) $y = -3$

Solution: *The line $y = -3$ is just the x-axis shifted down 3 units. So both radii just "grow" by 3:*

$$\begin{aligned} r_{out} &= 3 + (-x^2 + 10x - 12) = -x^2 + 10x - 9 \\ r_{in} &= 3 + (x^2 - 4x + 8) = x^2 - 4x + 11. \end{aligned}$$

Then

$$\text{Volume of region} = \pi \int_2^5 \left[(-x^2 + 10x - 9)^2 - (x^2 - 4x + 11)^2\right] dx.$$



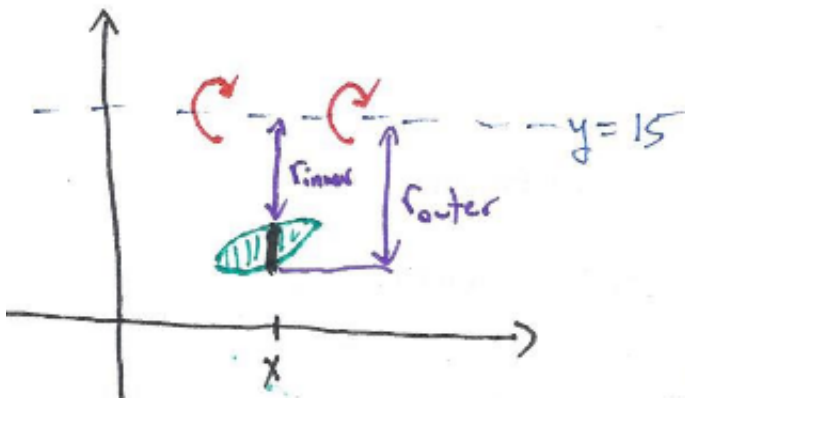
(c) $y = 15$

Solution: *Now, the line $y = 15$ is the x-axis shifted up 15 units. This causes a bigger difference than in part (b), since our axis of rotation has moved to the opposite side of the region between the curves. Consulting the picture below, we see that*

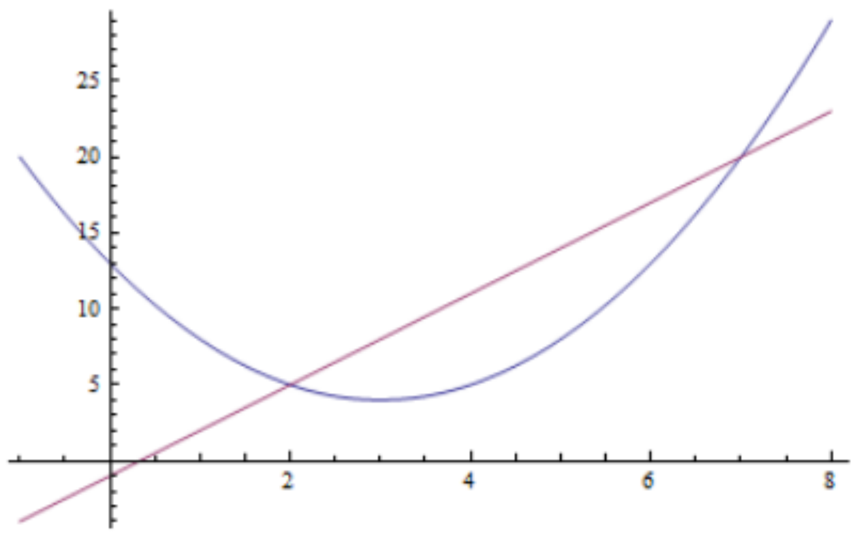
$$\begin{aligned} r_{out} &= 15 - (x^2 - 4x + 8) = -x^2 + 4x + 7 \\ r_{in} &= 15 - (-x^2 + 10x - 12) = x^2 - 10x + 27. \end{aligned}$$

Then

$$\text{Volume of region} = \pi \int_2^5 \left[(-x^2 + 4x + 7)^2 - (x^2 - 10x + 27)^2\right] dx.$$



Problem 3 *Set up an integral that will compute the volume of the solid generated by revolving the region bounded by the curves* $y = x^2 - 6x + 13$ *(i.e.* $x = 3 \pm \sqrt{y-4}$*) and* $y = 3x - 1$ *about:*



Use both the washer method as well as the shell method for each problem. Which method would you prefer for each problem? Why?

(a) *the* x*-axis*

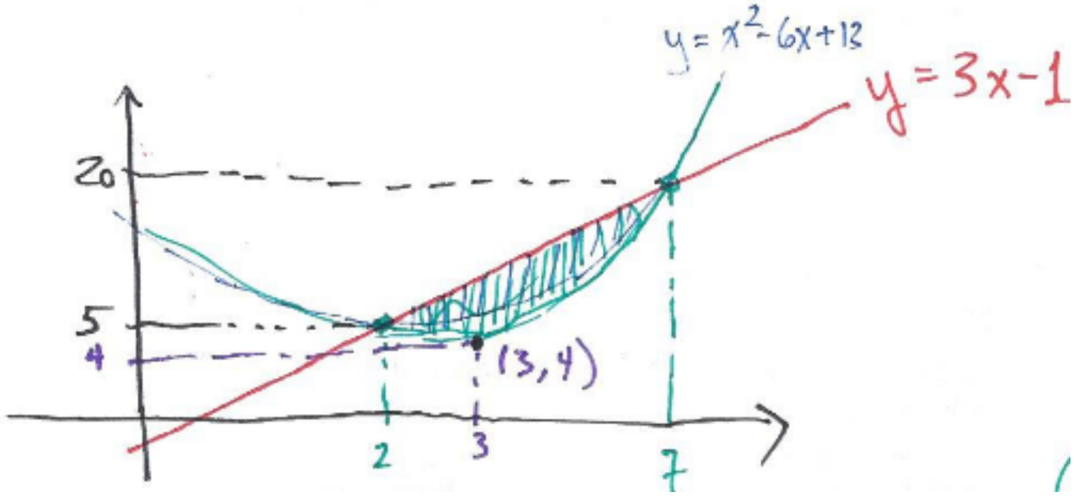
Solution: *First, we need to find the points where the curves intersect*

$$\begin{aligned} x^2 - 6x + 13 &= 3x - 1 \\ x^2 - 9x + 14 &= 0 \\ (x-2)(x-7) &= 0 \\ x &= 2, 7 \\ &(2,5), (7,20). \end{aligned}$$

As we will see later, we also need to locate the vertex of the parabola $x^2 - 6x + 13$*. So we complete the square*

$$\begin{aligned} y &= x^2 - 6x + 13 \\ &= (x^2 - 6x + 9) + 13 - 9 \\ &= (x-3)^2 + 4. \end{aligned}$$

So the vertex of the parabola is $(3, 4)$*.*

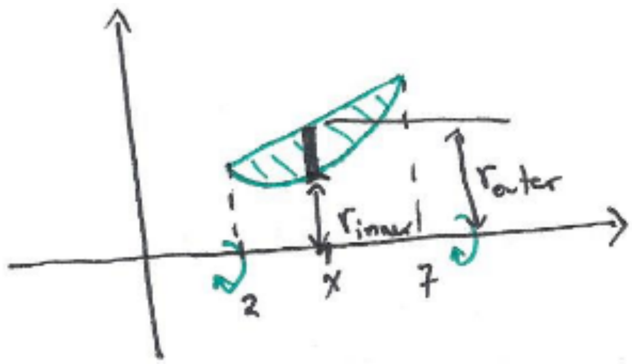


Washers: *For washers, the cross-sections must be* **perpendicular** *to the axis of rotation. So here we integrate along the x-axis. We have that*

$$\begin{aligned} r_{out} &= 3x - 1 \\ r_{in} &= x^2 - 6x + 13 \end{aligned}$$

and

$$\text{Volume of the region} = \pi \int_2^7 \left[(3x-1)^2 - (x^2 - 6x + 13)^2\right] dx.$$



Shells: *For shells, the cross-sections must be* ***parallel*** *to the axis of rotation. So here we integrate along the y-axis, $5 \le y \le 20$. But we have a problem, namely the "bottom" of each cross-section changes at $y = 5$ due to the shape of the region (see the picture below). So we have two different cases:*

(1) *For* $4 \le y \le 5$

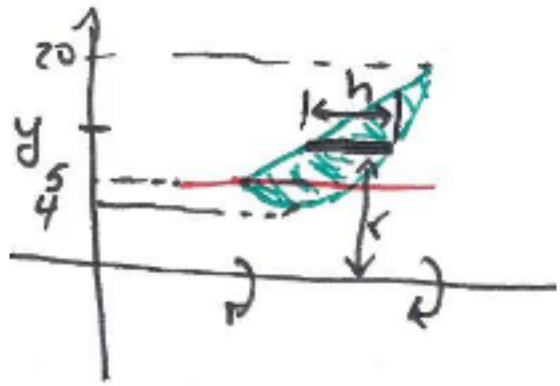
$$h = (3 + \sqrt{y-1}) - (3 - \sqrt{y-1}) = 2\sqrt{y-1}$$
$$r = y$$

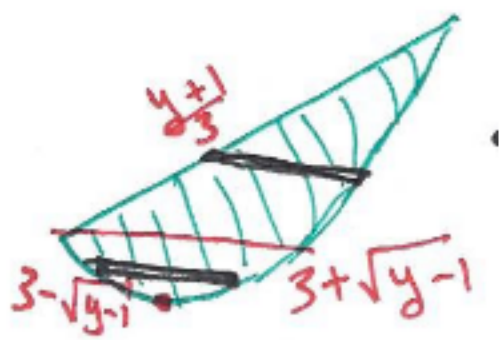
(2) *For* $5 \le y \le 20$

$$h = (3 + \sqrt{y-1}) - \left(\frac{1}{3}(y+1)\right)$$
$$r = y.$$

Thus

$$V = \int_4^{20} 2\pi r h \, dy = 2\pi \left[\int_4^5 y \cdot 2\sqrt{y-1} \, dy + \int_5^{20} y \left((3 + \sqrt{y+1}) - \frac{1}{3}(y+1) \right) dy \right]$$





It is pretty clear in this problem that the washer's method was easier than the shell's method.