Recitation #8 - 3.2 Working with Derivatives (Solutions)

Warm up:

(1) If $f'(2)$ exists, then $\lim_{x \to 2} f(x)$

(a) must exist, but more information is needed if we want to find it.

(b) is equal to $f(2)$.

(c) is equal to $f'(2)$.

(d) need not exist.

Solution: The correct answer is (b). If $f'(2)$ exists, then the function $f(x)$ is differentiable at $x = 2$. There is a theorem that says "if $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$". So $f(x)$ is continuous at $x = 2$, and by definition, this means that $\lim_{x \to 2} f(x) = f(2)$.

(2) The statement $\lim_{x \to 0} \frac{\sin x}{x} = 1$ means

(a) $\frac{0}{0} = 1$.

(b) the tangent line to $y = \sin x$ at $(0, 0)$ has slope 1.

(c) you can cancel the x's.

(d) for all x near 0, $\sin x = x$.

(e) for all x near 0, $\sin x \approx x$.

Solution: The answers are (b) and (e). To see that (b) is true, if we write out the limit definition of the derivative of $f(x) = \sin x$ at $x = 0$, we get that $f'(0) = \lim_{h \to 0} \frac{\sin(h + 0) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h}$. But this equals 1 by the assumption, and so the tangent line to $\sin(x)$ at $(0, 0)$ has slope 1.

For (e), note that this is what it means for $y = x$ to be the tangent line of $y = \sin(x)$ at $x = 0$.

Group work:

Problem 1 *Look at the following work for finding the derivative of* $f(x) = \frac{x}{x - 5}$ *at the point* $x = 3$ *using the definition of the derivative. Adapt the following* *work to find $f'(x)$.*

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{3+h}{(3+h)-5} - \frac{3}{3-5}}{h}$$

$$= \lim_{h\to 0} \left[\frac{3+h}{(3+h)-5} - \frac{3}{3-5}\right] \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \left[\frac{(3+h)(3-5)}{[(3+h)-5](3-5)} - \frac{3[(3+h)-5]}{(3-5)[(3+h)-5]}\right] \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{(3+h)(3-5) - 3[(3+h)-5]}{[(3+h)-5](3-5)h}$$

$$= \lim_{h\to 0} \frac{3\cdot 3 + 3\cdot(-5) + 3h - 5h - [3\cdot 3 + 3h + 3\cdot(-5)]}{[(3+h)-5](3-5)h}$$

$$= \lim_{h\to 0} \frac{\cancel{9} \cancel{-15} + \cancel{3h} - 5h \cancel{-9} \cancel{-3h} \cancel{+15}}{[(3+h)-5](3-5)h}$$

$$= \lim_{h\to 0} \frac{-5\cancel{h}}{[(3+h)-5](3-5)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-5}{[(3+h)-5](3-5)} = \frac{-5}{[(3+0)-5](3-5)}$$

$$= \frac{-5}{(3-5)^2} = \frac{-5}{4}$$

Solution: *Essentially, one just needs to replace all of the 3's with x's. After making this substitution and completing the same algebra, line four becomes*

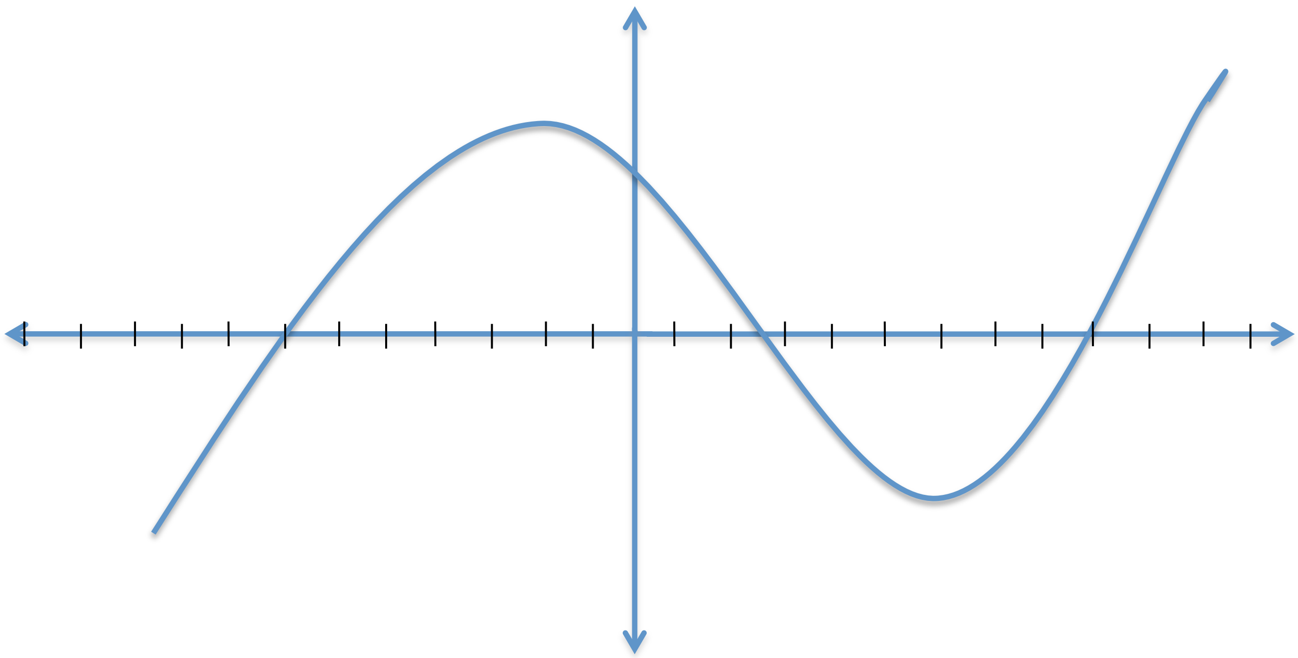
$$f'(x) = \lim_{h\to 0} \frac{(x+h)(x-5) - x[(x+h)-5]}{[(x+h)-5](x-5)h}.$$

Multiplying the numerator out and canceling yields

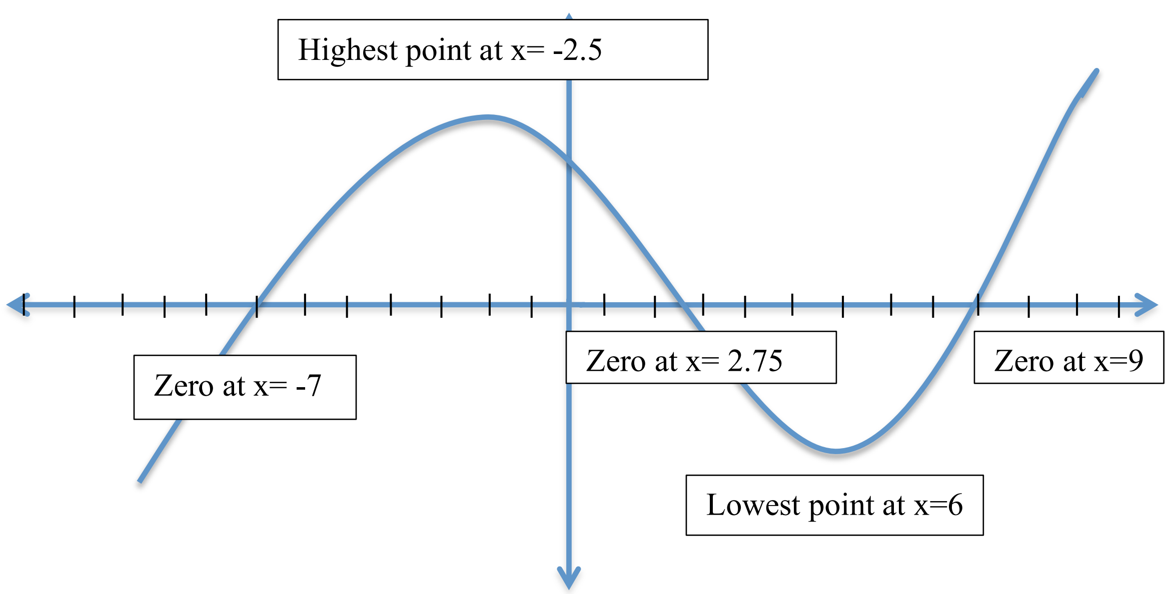
$$f'(x) = \lim_{h\to 0} \frac{-5h}{[(x+h)-5](x-5)h} = \lim_{h\to 0} \frac{-5}{[(x+h)-5](x-5)} = \frac{-5}{(x-5)^2}.$$

Problem 2 *Given the following graph of $f(x)$, find the values where $f(x)$ is zero, positive/negative, increasing/decreasing, and where $f(x)$ has its highest and lowest values. Additionally, find where the graph of $f(x)$ is the steepest. Then, without sketching the graph of $f'(x)$, determine for which values of x is $f'(x)$ zero, positive/negative, and where $f'(x)$ has the highest and lowest values.*

Then sketch $f'(x)$.



Solution:



$f(x)$ *is zero when the function crosses the* x*-axis, so at approximately* $x = -7, 2.75, 9$.

$f(x)$ *is negative when the function is below the* x*-axis, so approximately* $(-\infty, -7) \cup (2.75, 9)$.

$f(x)$ *is positive when the function is above the* x*-axis, so approximately* $(-7, 2.75) \cup (9, \infty)$.

$f(x)$ *is increasing on* $(-\infty, -2.5) \cup (6, \infty)$.

$f(x)$ *is decreasing on* $(-2.5, 6)$.

$f(x)$ *(locally) has its highest point at approximately* $x = -2.5$*. For* $x > 9$*,* $f(x)$ *has no highest point.*

$f(x)$ *(locally) has its lowest point at approximately* $x = 6$*. For* $x < -7$*,* $f(x)$ *has no lowest point.*

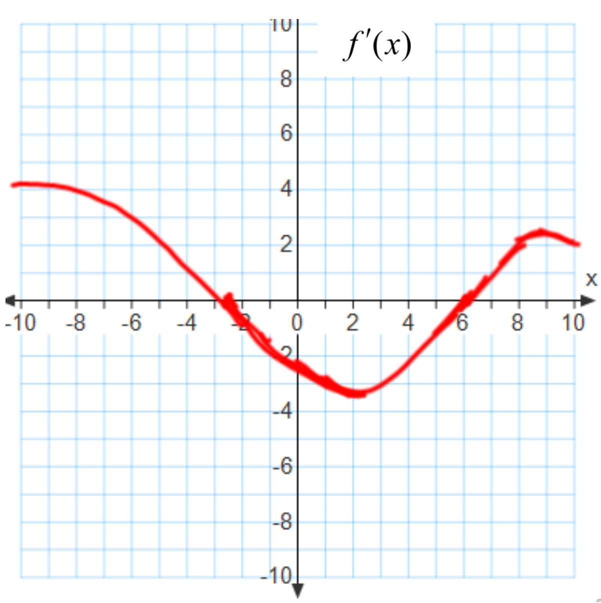
$f(x)$ *is steepest at approximately* $x = 2$ *and* $x = 8.5$.

$f'(x)$ *is zero when the tangent line has a slope of zero, which is approximately at* $x = -2.5$ *and* $x = 6$*. Note, for this question, these are the same answers as the (local) highest and lowest point for* $f(x)$.

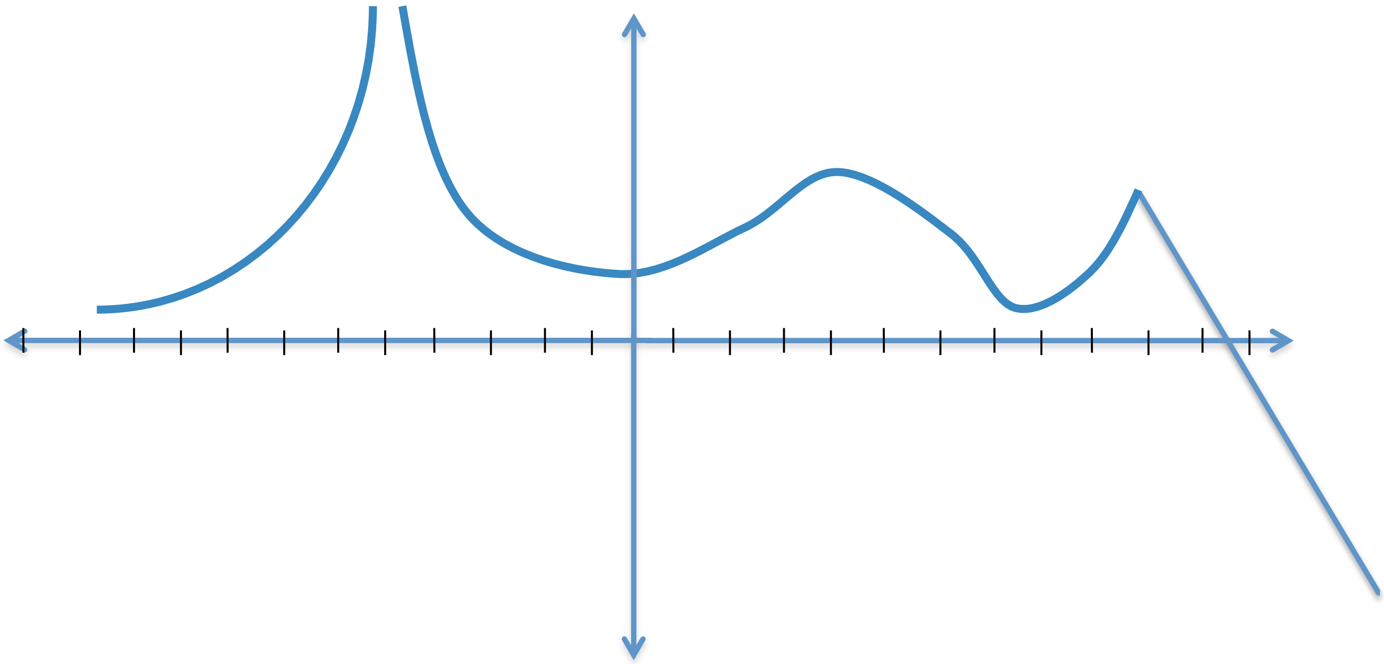
$f'(x)$ *is positive when the slope of the tangent line is positive, so when* $f(x)$ *is increasing which is approximately* $(-\infty, -2.5) \cup (6, \infty)$.

$f'(x)$ *is negative when the slope of the tangent line is negative, so when* $f(x)$*is decreasing which is approximately* $(-2.5, 6)$.

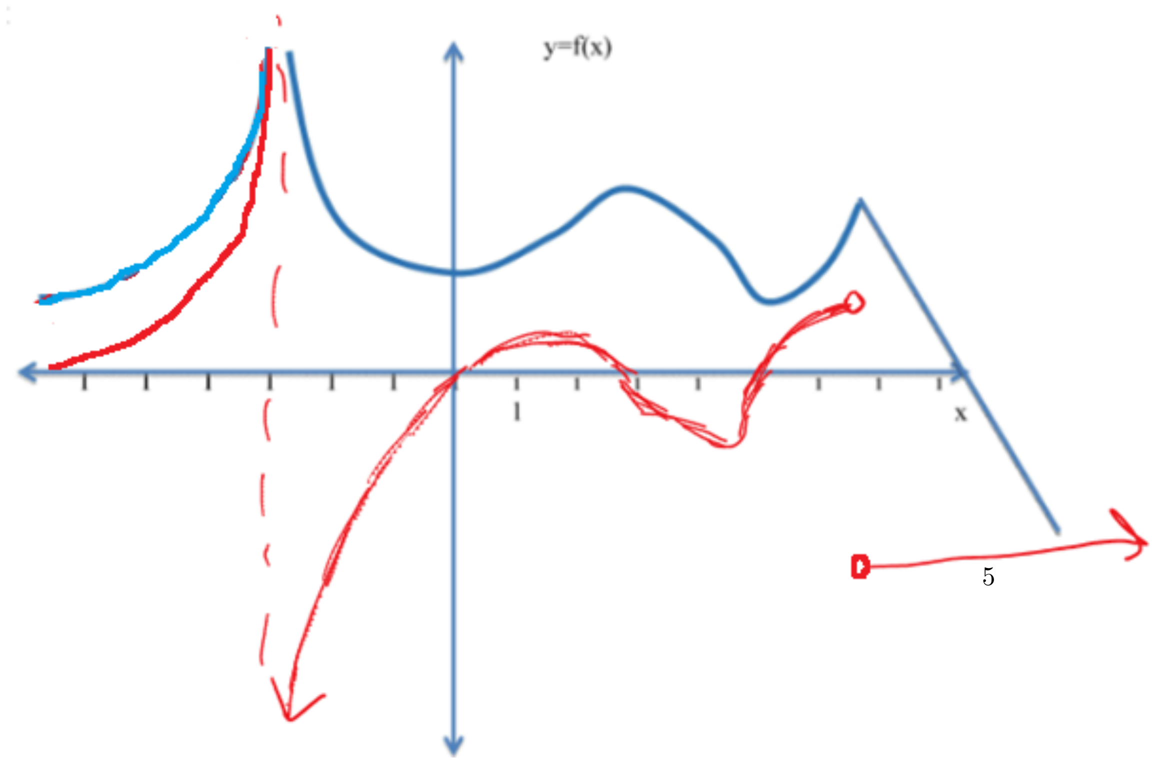
$f'(x)$ *has its highest and lowest values when* $f(x)$ *is steepest, which is approximately when* $x = 2$ *and* $x = 8.5$.



Problem 3 *Given the graph of $y = f(x)$, sketch the graph of the derivative $f'(x)$.*



Solution: *The graph of the derivative is in red.*



Problem 4 *Use the graph of g in the figure to do the following.*

(a) *Find the values of x in $(0, 4)$ at which g is not continuous.*

Solution: *g is not continuous at $x = 1$.*

(b) *Find the values of x in $(0, 4)$ at which g is not differentiable.*

Solution: *g is not differentiable at $x = 1$ and $x = 2$.*