Recitation #4 - 2.3: Limit Laws (Solutions)

Warm up:

Below is a table listing all of the Limit Laws, followed by an argument of what the limit of $\dfrac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}}$ as x approaches 3 must be. State which limit law is used to justify each step.

THEOREM 2.3 Limit Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number and $m > 0$ and $n > 0$ are integers.

1. **Sum** $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference** $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. **Constant multiple** $\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x)$
4. **Product** $\lim_{x \to a} [f(x)\, g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$
5. **Quotient** $\lim_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
6. **Power** $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$
7. **Fractional power** $\lim_{x \to a} [f(x)]^{n/m} = \left[\lim_{x \to a} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms

$$\lim_{x \to 3} \left(\frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}}\right)$$

$$= \frac{\lim_{x \to 3} (5x^3 - 4\sqrt{x})}{\lim_{x \to 3} \sqrt{x^5 - 87}}$$

$$= \frac{5 \lim_{x \to 3} (x^3) - 4 \lim_{x \to 3} \sqrt{x}}{\sqrt{\lim_{x \to 3} (x^5 - 87)}}$$

$$= \frac{5 (\lim_{x \to 3} x)^3 - 4\sqrt{3}}{\sqrt{\lim_{x \to 3} (x^5) - \lim_{x \to 3} (87)}}$$

$$= \frac{5(3)^3 - 4\sqrt{3}}{\sqrt{3^5 - 87}}$$

$$= \frac{135 - 4\sqrt{3}}{\sqrt{156}}$$

Know: $\lim_{x \to a} (x) = a$

and $\lim_{x \to a} (c) = c$, where c is a constant

Solution: Step 1: Limit law 5.

Step 2: Limit laws 2 and 3 in the numerator, limit law 7 in the denominator.

Step 3: In the numerator both limit law 6 as well as the fact that $\lim_{x\to a} x = a$ are used. Limit law 2 is used in the denominator.

Step 4: $\lim_{x\to a} x = a$ is used in the numerator. This same fact, in conjunction with limit law 6, is used in the denominator.

Step 5: This step is just arithmetic.

Group work:

Problem 1 *Evaluate the following limits algebraically using the limit laws.*

(a) $\lim_{x\to 6} \dfrac{4x^2 - 144}{x - 6}$

Solution: $\lim_{x\to 6} \dfrac{4x^2 - 144}{x - 6} = \lim_{x\to 6} \dfrac{4(x-6)(x+6)}{x-6} = \lim_{x\to 6} 4(x+6) = 4(12) = 48$

(b) $\lim_{x\to 6} \dfrac{x - 6}{\sqrt{2x-8} - 2}$

Solution: $\lim_{x\to 6} \dfrac{x-6}{\sqrt{2x-8}-2} \cdot \dfrac{\sqrt{2x-8}+2}{\sqrt{2x-8}+2} = \lim_{x\to 6} \dfrac{(x-6)(\sqrt{2x-8}+2)}{2x-8-4} = \lim_{x\to 6} \dfrac{(x-6)(\sqrt{2x-8}+2)}{2(x-6)} = \lim_{x\to 6} \dfrac{\sqrt{2x-8}+2}{2} = \dfrac{\sqrt{12-8}+2}{2} = \dfrac{4}{2} = 2$

(c) $\lim_{x\to 2} \dfrac{(3x-2)^2 - 16}{x - 2}$

Solution: $\lim_{x\to 2} \dfrac{(3x-2)^2 - 16}{x-2} = \lim_{x\to 2} \dfrac{((3x-2)-4)((3x-2)+4)}{x-2} = \lim_{x\to 2} \dfrac{(3x-6)(3x+2)}{x-2}$
$\lim_{x\to 2} \dfrac{3(x-2)(3x+2)}{x-2} = \lim_{x\to 2} 3(3x+2) = 3(6+2) = 24$

(d) $\lim_{x\to 1} \dfrac{\sqrt{5x-2} - \sqrt{3}}{x - 1}$

Solution: $\lim_{x\to 1} \dfrac{\sqrt{5x-2}-\sqrt{3}}{x-1} \cdot \dfrac{\sqrt{5x-2}+\sqrt{3}}{\sqrt{5x-2}+\sqrt{3}} = \lim_{x\to 1} \dfrac{(5x-2)-3}{(x-1)(\sqrt{5x-2}+\sqrt{3})}$
$= \lim_{x\to 1} \dfrac{5(x-1)}{(x-1)(\sqrt{5x-2}+\sqrt{3})} = \lim_{x\to 1} \dfrac{5}{\sqrt{5x-2}+\sqrt{3}} = \dfrac{5}{\sqrt{5(1)-2}+\sqrt{3}} = \dfrac{5}{2\sqrt{3}}$

Problem 2 *Suppose* $f(x) = \begin{cases} x^2 - ax & \text{if } x < 3 \\ a2^x + 7 + a & \text{if } x > 3 \end{cases}$

Find a *so that* $\lim_{x \to 3} f(x)$ *exists.*

Solution: *We need to find* a *so that* $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$.

- $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - ax) = 9 - 3a.$
- $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} a2^x + 7 + a = a2^3 + 7 + a = 9a + 7$

So we want a *to be such that:*

$9 - 3a = 9a + 7$

$12a = 2$

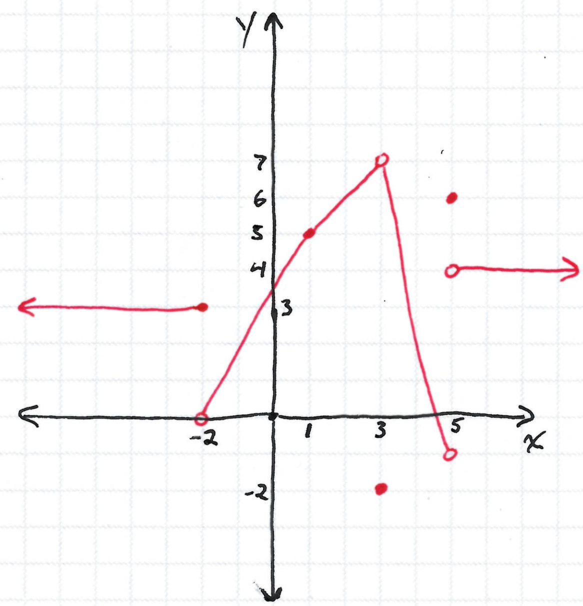
$a = \dfrac{1}{6}$

Problem 3 *Sketch the graph of a function with the given properties. You need not find a formula for the function:*

$f(3) = -2,\ f(-2) = 3,\ f(5) = 6,\ \lim_{x \to 5^-} f(x) = -1,\ \lim_{x \to 5^+} f(x) = 4,\ \lim_{x \to 3} f(x) = 7$

$\lim_{x \to -2^-} f(x) = 3,\ \lim_{x \to -2^+} f(x) = 0,\ \lim_{x \to 1^+} f(x) = 5$

Solution: *There exist infinitely many functions whose graph satisfies the conditions above, but one such graph is the following:*


y
7
6
5
4
3
-2
1
3
5
x
-2