Section - 2.4: Infinite Limits (Solutions)

Warm up:

Come up with your own example of a limit of a function $f(x)$, as x approaches 4, that will be of the following form:

(a) Limit is of the form $\frac{0}{0}$ and the limit exists as a finite number.

Solution: Let $f(x) = \frac{4-x}{x(4-x)}$. Then $\lim_{x\to 4} f(x)$ is of the form $\frac{0}{0}$ and $\lim_{x\to 4} \frac{4-x}{x(4-x)} = \lim_{x\to 4} \frac{1}{x} = \frac{1}{4}$.

(b) Limit is of the form $\frac{0}{0}$ and the limit is infinite.

Solution: Let $f(x) = \frac{4-x}{(4-x)^3}$. Notice that, for $x \neq 4$, $f(x) = \frac{1}{(x-4)^2}$. $\lim_{x\to 4} f(x)$ is of the form $\frac{0}{0}$, and $\lim_{x\to 4} \frac{1}{(x-4)^2}$ is of the form $\frac{1}{0}$. Thus the limit is infinite, and since both 1 and $(x-4)^2$ are always positive (for $x \neq 4$) we have that $\lim_{x\to 4} f(x) = \infty$.

(c) Limit is infinite and approaches positive infinity from both sides of 4.

Solution: The same function as part (b) works.

(d) Limit does not exist (DNE), but approaches positive infinity from the left side of 4 and negative infinity from the right side of 4.

Solution: Let $f(x) = \frac{1}{4-x}$. Since $\lim_{x\to 4} f(x)$ is of the form $\frac{1}{0}$, the limit is either ∞, $-\infty$, or DNE. Since $1 > 0$, $4 - x > 0$ as $x \to 4$ from the left, and $4 - x < 0$ as $x \to 4$ from the right, we have that $\lim_{x\to 4^-} f(x) = \infty$ and $\lim_{x\to 4^+} f(x) = -\infty$.

Group work:

Problem 1 *Determine the following limits:*

(a) $\lim_{x \to 3} \dfrac{x^2 - 3}{x^2 - x - 6}$

Solution: *Notice that as* $x \to 3$, $(x^2 - 3) \to 6$ *and* $(x^2 - x - 6) \to 0$. *Thus, this limit is of the form* $\dfrac{\neq 0}{0}$ *and therefore the answer is one of* ∞, $-\infty$, *or the limit does not exist (DNE). What we need to do is check the limits as* x *approaches* 3 *from both the left and right hand sides. Notice that, in either sided limit, the numerator approaches* $9 > 0$. *Also, the denominator factors as* $(x - 3)(x + 2)$. *As* $x \to 3^-$, $(x - 3)$ *is negative making the denominator and thus the entire fraction negative (note that* $x + 2$ *approaches* 5 *which is positive). As* $x \to 3^+$, $(x - 3)$ *is positive making the entire fraction positive. So we can conclude:*

$$\lim_{x \to 3^-} \frac{x^2 - 3}{x^2 - x - 6} = -\infty$$

$$\lim_{x \to 3^+} \frac{x^2 - 3}{x^2 - x - 6} = \infty$$

Therefore $\lim_{x \to 3} \dfrac{x^2 - 3}{x^2 - x - 6}$ *DNE*

(b) $\lim_{x \to 5} \dfrac{x^2 + 6}{x^2 - 3x - 10}$

Solution: *Just like part (a), this limit is of the form* $\dfrac{\neq 0}{0}$, *and so we need to consider the sided limits. As* x *approaches* 5, *the numerator approaches* 31 *which is positive. The denominator factors as* $(x-5)(x+2)$, *and as* x *approaches* 5 *from eithe side,* $(x + 2)$ *approaches* 7 *which is positive. Now, as* x *approaches* 5 *from the left hand side,* $(x - 5)$ *is negative, making the entire fraction negative. Similarly, as* x *approaches* 5 *from the right,* $(x - 5)$ *is positive and thus the entire fraction is positive. Hence:*

$$\lim_{x \to 5^-} \frac{x^2 + 6}{x^2 - 3x - 10} = -\infty$$

$$\lim_{x \to 5^+} \frac{x^2 + 6}{x^2 - 3x - 10} = \infty$$

Therefore $\lim_{x \to 5} \dfrac{x^2 + 6}{x^2 - 3x - 10}$ *DNE*

(c) $\lim_{x \to 1} \dfrac{4 - x}{x^2 - 2x + 1}$

Solution: *This is exactly like parts (a) and (b) above in that the limit is of the form* $\dfrac{\neq 0}{0}$, *and so we need to consider the sided limits. The function* *can be rewritten as* $\dfrac{4-x}{(x-1)^2}$. *As* $x \to 1$ *(from either side),* $(4-x) \to 3$ *and* 3 *is positive. But now notice that, for* $x \neq 1$, *the denominator* $(x-1)^2$ *is always positive. So as* x *approaches 1 from* **both** *the left and right hand sides, the entire fraction is positive. Thus:*

$$\lim_{x \to 1^-} \frac{4-x}{x^2-2x+1} = \infty$$

$$\lim_{x \to 1^+} \frac{4-x}{x^2-2x+1} = \infty$$

Therefore $\displaystyle\lim_{x \to 1} \frac{4-x}{x^2-2x+1} = \infty$

Problem 2 *Use the Squeeze Theorem to determine the value of* $\displaystyle\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$

Solution: *For all* $x \neq 0$ *we have that*

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1. \tag{1}$$

We now need to split the problem up into two cases, as x *approaches* 0 *from the right and as* x *approaches* 0 *from the left.*

(Case 1: $x \to 0^+$*)*

In this case $x > 0$, *and so multiplying equation (1) by* x *yields*

$$-x \leq x \cos\left(\frac{1}{x}\right) \leq x.$$

But we know that $\displaystyle\lim_{x \to 0^+} (-x) = 0 = \lim_{x \to 0^+} x$, *and thus by the Squeeze Theorem we can conclude that* $\displaystyle\lim_{x \to 0^+} x \cos\left(\frac{1}{x}\right) = 0$.

(Case 2: $x \to 0^-$*)*

In this case $x < 0$, *and so multiplying equation (1) by* x *yields*

$$-x \geq x \cos\left(\frac{1}{x}\right) \geq x.$$

But just like in Case 1 we know that $\displaystyle\lim_{x \to 0^-} x = 0 = \lim_{x \to 0^-} (-x)$, *and thus by the Squeeze Theorem we can conclude that* $\displaystyle\lim_{x \to 0^-} x \cos\left(\frac{1}{x}\right) = 0$.

Therefore, since $\lim_{x\to 0^-} x\cos\left(\frac{1}{x}\right) = 0 = \lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = 0$, *we can conclude that* $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$.