Recitation #4 - 2.3: Limit Laws (Solutions)

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Warm up: Below is a table listing all of the Limit Laws, followed by an argument of what the limit of $\dfrac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}}$ as x approaches 3 must be. State which limit law is used to justify each step.

THEOREM 2.3 Limit Laws

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where c is a real number and $m > 0$ and $n > 0$ are integers.

1. **Sum** $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. **Difference** $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. **Constant multiple** $\lim_{x\to a} [c\, f(x)] = c \lim_{x\to a} f(x)$
4. **Product** $\lim_{x\to a} [f(x)\, g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$
5. **Quotient** $\lim_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
6. **Power** $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. **Fractional power** $\lim_{x\to a} [f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms

$$\lim_{x\to 3}\left(\frac{5x^3-4\sqrt{x}}{\sqrt{x^5-87}}\right)$$

$$= \frac{\lim_{x\to 3}(5x^3-4\sqrt{x})}{\lim_{x\to 3}\sqrt{x^5-87}}$$

$$= \frac{5\lim_{x\to 3}(x^3) - 4\lim_{x\to 3}\sqrt{x}}{\sqrt{\lim_{x\to 3}(x^5-87)}}$$

$$= \frac{5(\lim_{x\to 3} x)^3 - 4\sqrt{3}}{\sqrt{\lim_{x\to 3}(x^5) - \lim_{x\to 3}(87)}}$$

$$= \frac{5(3)^3 - 4\sqrt{3}}{\sqrt{3^5 - 87}}$$

$$= \frac{135 - 4\sqrt{3}}{\sqrt{156}}$$

Know: $\lim_{x\to a}(x) = a$

and $\lim_{x\to a}(c) = c$, where c is a constant

Solution: Step 1: Limit law 5.

Step 2: Limit laws 2 and 3 in the numerator, limit law 7 in the denominator.

Step 3: In the numerator both limit law 6 as well as the fact that $\lim_{x\to a} x = a$ are used. Limit law 2 is used in the denominator.

Step 4: $\lim_{x\to a} x = a$ is used in the numerator. This same fact, in conjunction with limit law 6, is used in the denominator.

Step 5: This step is just arithmetic.

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Group work:

Problem 1 *Evaluate the following limits algebraically using the limit laws.*

(a) $\displaystyle\lim_{x\to 6} \frac{4x^2-144}{x-6}$

Solution:

$$\begin{aligned}\lim_{x\to 6} \frac{4x^2-144}{x-6} &= \lim_{x\to 6} \frac{4(x-6)(x+6)}{x-6} \\ &= \lim_{x\to 6} 4(x+6) \\ &= 4(12) = 48\end{aligned}$$

(b) $\displaystyle\lim_{x\to 6} \frac{x-6}{\sqrt{2x-8}-2}$

Solution:

$$\begin{aligned}\lim_{x\to 6} \frac{x-6}{\sqrt{2x-8}-2} &= \lim_{x\to 6} \frac{x-6}{\sqrt{2x-8}-2} \cdot \frac{\sqrt{2x-8}+2}{\sqrt{2x-8}+2} \\ &= \lim_{x\to 6} \frac{(x-6)(\sqrt{2x-8}+2)}{2x-8-4} \\ &= \lim_{x\to 6} \frac{(x-6)(\sqrt{2x-8}+2)}{2(x-6)} \\ &= \lim_{x\to 6} \frac{\sqrt{2x-8}+2}{2} \\ &= \frac{\sqrt{12-8}+2}{2} \\ &= \frac{4}{2} = 2\end{aligned}$$

(c) $\lim_{x\to 2} \frac{(3x-2)^2 - 16}{x-2}$

Solution:

$$\begin{aligned}
\lim_{x\to 2} \frac{(3x-2)^2 - 16}{x-2} &= \lim_{x\to 2} \frac{((3x-2)-4)((3x-2)+4)}{x-2} \\
&= \lim_{x\to 2} \frac{(3x-6)(3x+2)}{x-2} \\
&= \lim_{x\to 2} \frac{3(x-2)(3x+2)}{x-2} \\
&= \lim_{x\to 2} 3(3x+2) \\
&= 3(6+2) = 24
\end{aligned}$$

(d) $\lim_{x\to 1} \frac{\sqrt{5x-2} - \sqrt{3}}{x-1}$

Solution:

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{5x-2} - \sqrt{3}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{5x-2} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{5x-2} + \sqrt{3}}{\sqrt{5x-2} + \sqrt{3}} \\
&= \lim_{x\to 1} \frac{(5x-2)-3}{(x-1)(\sqrt{5x-2} + \sqrt{3})} \\
&= \lim_{x\to 1} \frac{5(x-1)}{(x-1)(\sqrt{5x-2} + \sqrt{3})} \\
&= \lim_{x\to 1} \frac{5}{\sqrt{5x-2} + \sqrt{3}} \\
&= \frac{5}{\sqrt{5(1)-2} + \sqrt{3}} \\
&= \frac{5}{2\sqrt{3}}
\end{aligned}$$

Problem 2 *Suppose* $f(x) = \begin{cases} x^2 - ax & \text{if } x < 3 \\ a2^x + 7 + a & \text{if } x > 3 \end{cases}$

Find a *so that* $\lim_{x\to 3} f(x)$ *exists.*

Solution: *We need to find* a *so that* $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$.

- $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - ax) = 9 - 3a.$
- $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} a2^x + 7 + a = a2^3 + 7 + a = 9a + 7$

So we want a to be such that:

$$9 - 3a = 9a + 7$$

$$12a = 2$$

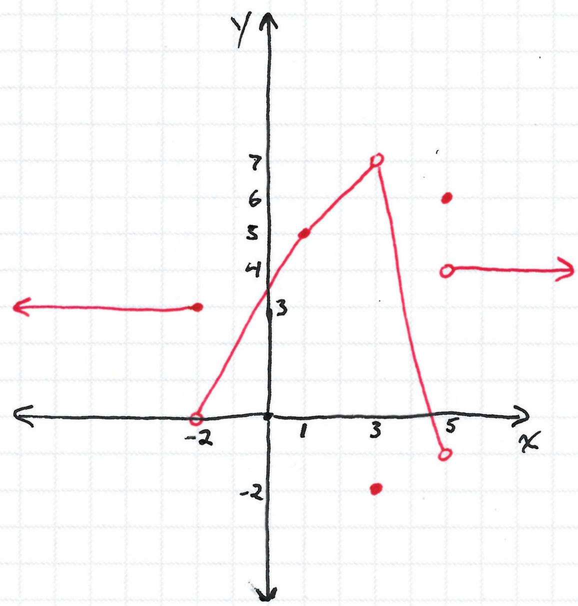
$$a = \frac{1}{6}$$

Problem 3 *Sketch the graph of a function with the given properties. You need not find a formula for the function:*

$f(3) = -2, f(-2) = 3, f(5) = 6, \lim_{x\to 5^-} f(x) = -1, \lim_{x\to 5^+} f(x) = 4, \lim_{x\to 3} f(x) = 7$

$\lim_{x\to -2^-} f(x) = 3, \lim_{x\to -2^+} f(x) = 0, \lim_{x\to 1^+} f(x) = 5$

Solution: *There exist infinitely many functions whose graph satisfies the conditions above, but one such graph is the following:*


y
7
6
5
4
3
-2
-2
1
3
5
x