Recitation #6 - 2.6 Continuity (Solutions)

Warm up:

Explain why the Intermediate Value Theorem does not guarantee a zero for $f(x) = \dfrac{x-1}{x^2-5x}$ on the interval $(2,6)$, even though $f(2) < 0$ and $f(6) > 0$.

Solution: Notice that $f(x) = \dfrac{x-1}{x(x-5)}$. So $f(x)$ is not defined at$x = 5$, and therefore $f(x)$ is not continuous at $x = 5$ which is in the interval $[2,6]$. For the intermediate value theorem to apply, $f(x)$ needs to be continuous on the interval $[2,6]$.

Group work:

Problem 1 *Find the intervals where the following function is continuous. Write your answer as a list of intervals in interval notation, separated by commas.*

$$f(x) = \begin{cases} 5x+7 & \text{if } x < -3 \\ \dfrac{(x-1)(x+2)}{x+2} & \text{if } -3 \leq x < 1 \text{ and } x \neq -2 \\ 4\ln x & \text{if } x \geq 1 \end{cases}$$

Solution: *$f(x)$ is continuous on $(\infty, -3)$ since, in this region, $f(x) = 5x+7$ is a polynomial and therefore continuous on its domain. Note that $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} 5x+7 = 5(-3)+7 = -8$. Also, $f(-3) = \dfrac{(-3-1)(-3+2)}{-3+2} = \dfrac{4}{-1} = -4 \neq -8$. So $f(x)$ is not continuous at $x = -3$ from the left, and therefore one interval of continuity for $f(x)$ is $(-\infty, -3)$.*

For $-3 < x < 1$, $f(x) = \dfrac{(x-1)(x+2)}{x+2}$ is a rational function and therefore continuous on its domain. Since $\dfrac{(x-1)(x+2)}{x+2}$ is undefined only at $x = -2$, $f(x)$ is continuous on the intervals $[-3,-2)$ and $(-2,1)$. Note that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \dfrac{(x-1)(x+2)}{x+2} = \dfrac{0}{3} = 0$. Also, $f(1) = 4\ln(1) = 0$. Thus, $f(x)$ is continuous at $x = 1$ from the left, and hence $f(x)$ is actually continuous on the intervals $[-3,-2)$ and $(-2,1]$.

Finally, the function $4\ln x$ is continuous over the set of positive real numbers.

Thus, $f(x)$ is continuous on $[1, \infty)$. But this interval can be combined with the interval $(-2, 1]$, yielding the final answer:

$$(-\infty, -3), [-3, -2), (-2, \infty).$$

Problem 2 *Find a and b so that $f(x)$ is continuous for all values of x.*

$$f(x) = \begin{cases} ax^2 + 38 & \text{if } x < 3 \\ a + b & \text{if } x = 3 \\ 2bx - a & \text{if } x > 3 \end{cases}$$

Solution: *First, notice that all three of the different expressions that $f(x)$ takes are polynomial functions. Thus, $f(x)$ is automatically continuous at every point except $x = 3$. At $x = 3$, we need that $\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$. Substituting and splitting this into two equalities gives us:*

$$\begin{array}{llll} & \lim_{x\to 3^-} ax^2 + 38 = a + b & \text{and} & a + b = \lim_{x\to 3^+} 2bx - a \\ \Longrightarrow & 9a + 38 = a + b & \text{and} & a + b = 6b - a \\ \Longrightarrow & 8a - b = -38 & \text{and} & 2a - 5b = 0 \\ \Longrightarrow & 8a - b = -38 & \text{and} & -8a + 20b = 0 \end{array}$$

Adding these two equations gives us that $19b = -38$ and therefore $b = -2$. Plugging that into an earlier equation gives us that $2a - 5(-2) = 0 \implies a = -5$.

Problem 3 *Use the Intermediate Value Theorem to find an interval in which you can guarantee that there is a solution to the equation $x^3 = x + \sin x + 1$. Do not use any sort of graphing device to solve this problem.*

Solution: *Let $f(x) = x^3 - x - \sin x - 1$. Since both $x^3 - x - 1$ and $\sin x$ are continuous over the set of all real numbers, and $f(x)$ is the sum (or difference) of these two functions, we have that $f(x)$ is continuous everywhere.*

Now, notice that $f(0) = 0^3 - 0 - 0 - 1 = -1 < 0$ and $f(\pi) = \pi^3 - \pi - \sin(\pi) - 1 = \pi(\pi^2 - 1) - 1 > 3(3^2 - 1) - 1 = 23 > 0$. Thus, by the Intermediate Value Theorem, there exists a number $c \in (0, \pi)$ such that $f(c) = 0$.
