

Recitation #4 - 2.3: Limit Laws (Solutions)

Warm up:

Below is a table listing all of the Limit Laws, followed by an argument of what the limit of $\frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}}$ as x approaches 3 must be. State which limit law is used to justify each step.

THEOREM 2.3 Limit Laws

Assume $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. The following properties hold, where c is a real number and $m > 0$ and $n > 0$ are integers.

1. **Sum** $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. **Difference** $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. **Constant multiple** $\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x)$
4. **Product** $\lim_{x \rightarrow a} [f(x) g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right]$
5. **Quotient** $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, provided $\lim_{x \rightarrow a} g(x) \neq 0$
6. **Power** $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$
7. **Fractional power** $\lim_{x \rightarrow a} [f(x)]^{n/m} = \left[\lim_{x \rightarrow a} f(x) \right]^{n/m}$, provided $f(x) \geq 0$, for x near a , if m is even and n/m is reduced to lowest terms

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \left(\frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}} \right) \\
 &= \frac{\lim_{x \rightarrow 3} (5x^3 - 4\sqrt{x})}{\lim_{x \rightarrow 3} \sqrt{x^5 - 87}} \\
 &= \frac{5 \lim_{x \rightarrow 3} (x^3) - 4 \lim_{x \rightarrow 3} \sqrt{x}}{\sqrt{\lim_{x \rightarrow 3} (x^5 - 87)}} \\
 &= \frac{5(\lim_{x \rightarrow 3} x)^3 - 4\sqrt{3}}{\sqrt{\lim_{x \rightarrow 3} (x^5) - \lim_{x \rightarrow 3} (87)}} \\
 &= \frac{5(3)^3 - 4\sqrt{3}}{\sqrt{3^5 - 87}} \\
 &= \frac{135 - 4\sqrt{3}}{\sqrt{156}}
 \end{aligned}$$

Know: $\lim_{x \rightarrow a} (x) = a$
and $\lim_{x \rightarrow a} (c) = c$, where c is a constant

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Solution: Step 1: Limit law 5.

Step 2: Limit laws 2 and 3 in the numerator, limit law 7 in the denominator.

Step 3: In the numerator both limit law 6 as well as the fact that $\lim_{x \rightarrow a} x = a$ are used. Limit law 2 is used in the denominator.

Step 4: $\lim_{x \rightarrow a} x = a$ is used in the numerator. This same fact, in conjunction with limit law 6, is used in the denominator.

Step 5: This step is just arithmetic.

Group work:

Problem 1 Evaluate the following limits algebraically using the limit laws.

(a) $\lim_{x \rightarrow 6} \frac{4x^2 - 144}{x - 6}$

Solution: $\lim_{x \rightarrow 6} \frac{4x^2 - 144}{x - 6} = \lim_{x \rightarrow 6} \frac{4(x - 6)(x + 6)}{x - 6} = \lim_{x \rightarrow 6} 4(x + 6) = 4(12) = 48$

(b) $\lim_{x \rightarrow 6} \frac{x - 6}{\sqrt{2x - 8} - 2}$

Solution: $\lim_{x \rightarrow 6} \frac{x - 6}{\sqrt{2x - 8} - 2} \cdot \frac{\sqrt{2x - 8} + 2}{\sqrt{2x - 8} + 2} = \lim_{x \rightarrow 6} \frac{(x - 6)(\sqrt{2x - 8} + 2)}{2x - 8 - 4} =$
 $\lim_{x \rightarrow 6} \frac{(x - 6)(\sqrt{2x - 8} + 2)}{2(x - 6)} = \lim_{x \rightarrow 6} \frac{\sqrt{2x - 8} + 2}{2} = \frac{\sqrt{12 - 8} + 2}{2} = \frac{4}{2} = 2$

(c) $\lim_{x \rightarrow 2} \frac{(3x - 2)^2 - 16}{x - 2}$

Solution: $\lim_{x \rightarrow 2} \frac{(3x - 2)^2 - 16}{x - 2} = \lim_{x \rightarrow 2} \frac{((3x - 2) - 4)((3x - 2) + 4)}{x - 2} = \lim_{x \rightarrow 2} \frac{(3x - 6)(3x + 2)}{x - 2}$
 $\lim_{x \rightarrow 2} \frac{3(x - 2)(3x + 2)}{x - 2} = \lim_{x \rightarrow 2} 3(3x + 2) = 3(6 + 2) = 24$

(d) $\lim_{x \rightarrow 1} \frac{\sqrt{5x - 2} - \sqrt{3}}{x - 1}$

Solution: $\lim_{x \rightarrow 1} \frac{\sqrt{5x - 2} - \sqrt{3}}{x - 1} \cdot \frac{\sqrt{5x - 2} + \sqrt{3}}{\sqrt{5x - 2} + \sqrt{3}} = \lim_{x \rightarrow 1} \frac{(5x - 2) - 3}{(x - 1)(\sqrt{5x - 2} + \sqrt{3})}$
 $= \lim_{x \rightarrow 1} \frac{5(x - 1)}{(x - 1)(\sqrt{5x - 2} + \sqrt{3})} = \lim_{x \rightarrow 1} \frac{5}{\sqrt{5x - 2} + \sqrt{3}} = \frac{5}{\sqrt{5(1) - 2} + \sqrt{3}} = \frac{5}{2\sqrt{3}}$

Problem 2 Suppose $f(x) = \begin{cases} x^2 - ax & \text{if } x < 3 \\ a2^x + 7 + a & \text{if } x > 3 \end{cases}$

Find a so that $\lim_{x \rightarrow 3} f(x)$ exists.

Solution: We need to find a so that $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$.

- $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 - ax) = 9 - 3a.$
- $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} a2^x + 7 + a = a2^3 + 7 + a = 9a + 7$

So we want a to be such that:

$$9 - 3a = 9a + 7$$

$$12a = 2$$

$$a = \frac{1}{6}$$

Problem 3 Sketch the graph of a function with the given properties. You need not find a formula for the function:

$$f(3) = -2, f(-2) = 3, f(5) = 6, \lim_{x \rightarrow 5^-} f(x) = -1, \lim_{x \rightarrow 5^+} f(x) = 4, \lim_{x \rightarrow 3} f(x) = 7$$

$$\lim_{x \rightarrow -2^-} f(x) = 3, \lim_{x \rightarrow -2^+} f(x) = 0, \lim_{x \rightarrow 1^+} f(x) = 5$$

Solution: There exist infinitely many functions whose graph satisfies the conditions above, but one such graph is the following:

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