

## Recitation #8 - 3.2 Working with Derivatives (Solutions)

\*

Warm up:

(1) If  $f'(2)$  exists, then  $\lim_{x \rightarrow 2} f(x)$

- (a) must exist, but more information is needed if we want to find it.
- (b) is equal to  $f(2)$ .
- (c) is equal to  $f'(2)$ .
- (d) need not exist.

**Solution:** The correct answer is (b). If  $f'(2)$  exists, then the function  $f(x)$  is differentiable at  $x = 2$ . There is a theorem that says “if  $f(x)$  is differentiable at  $x = a$ , then  $f(x)$  is continuous at  $x = a$ ”. So  $f(x)$  is continuous at  $x = 2$ , and by definition, this means that  $\lim_{x \rightarrow 2} f(x) = f(2)$ .

(2) Assuming that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , we can conclude

- (a)  $\frac{0}{0} = 1$ .
- (b) the tangent line to  $y = \sin x$  at  $(0, 0)$  has slope 1.
- (c) you can cancel the  $x$ 's.
- (d) for all  $x$  near 0,  $\sin x = x$ .
- (e) for all  $x$  near 0,  $\sin x \approx x$ .

**Solution:** The answers are (b) and (e). To see that (b) is true, if we write out the limit definition of the derivative of  $f(x) = \sin x$  at  $x = 0$ , we have that

$$f'(0) = \lim_{h \rightarrow 0} \frac{\sin(h+0) - \sin(0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h}.$$

But this equals 1 by the assumption, and so the tangent line to the graph of  $y = \sin(x)$  at  $(0, 0)$  has slope 1.

For (e), note that this is what it means for  $y = x$  to be the tangent line to the graph of  $y = \sin(x)$  at  $x = 0$ .

Recitation #8 - 3.2 Working with Derivatives (Solutions)

\*

Group work:

**Problem 1** Look at the following work for finding the derivative of  $f(x) = \frac{x}{x-5}$  at the point  $x = 3$  using the definition of the derivative. Adapt the following work to find  $f'(x)$ .

$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3+h}{(3+h)-5} - \frac{3}{3-5}}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{3+h}{(3+h)-5} - \frac{3}{3-5} \right] \cdot \frac{1}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{(3+h)(3-5)}{[(3+h)-5](3-5)} - \frac{3[(3+h)-5]}{(3-5)[(3+h)-5]} \right] \cdot \frac{1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)(3-5) - 3[(3+h)-5]}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{3 \cdot 3 + 3(-5) + 3h - 5h - [3 \cdot 3 + 3h + 3(-5)]}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{9} - 15 + 3h - 5h - \cancel{9} - 3h + 15}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{-5h}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{-5}{[(3+h)-5](3-5)} = \frac{-5}{[(3+0)-5](3-5)} \\
 &= \frac{-5}{(3-5)^2} = \frac{-5}{4}
 \end{aligned}$$

**Solution:** Essentially, one just needs to replace all of the 3's with  $x$ 's. After making this substitution and completing the same algebra, line four becomes

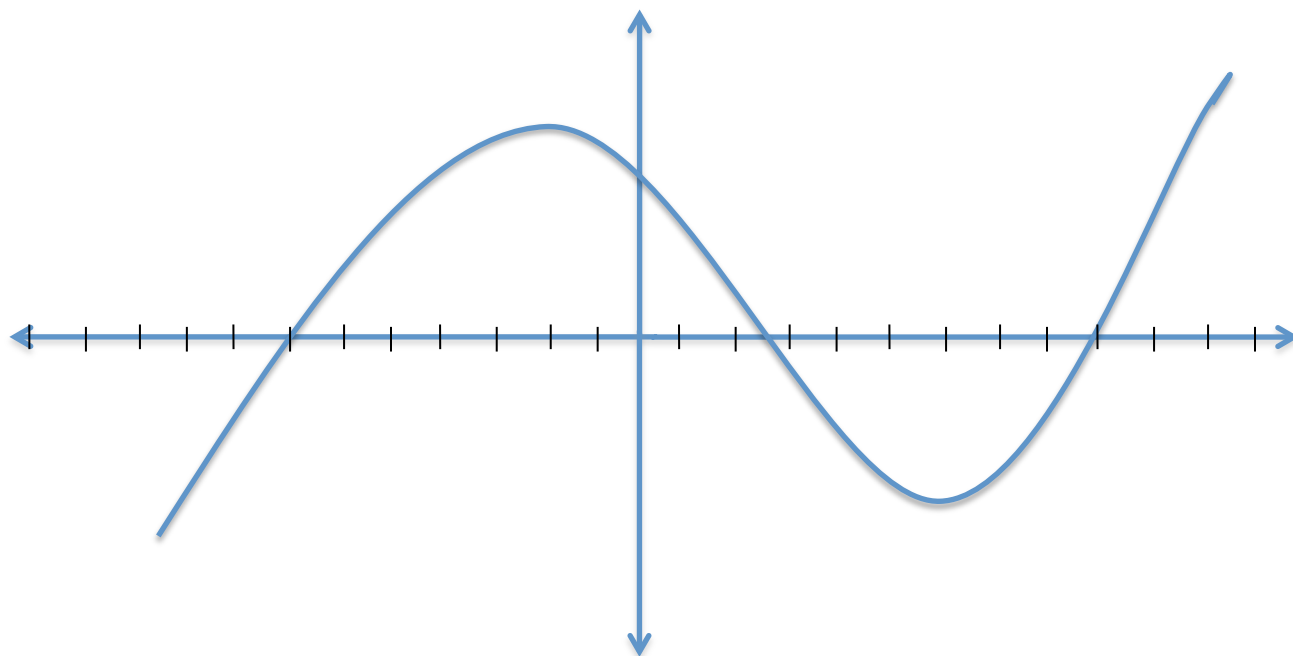
$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)(x-5) - x[(x+h)-5]}{[(x+h)-5](x-5)h}.$$

Multiplying the numerator out and canceling yields

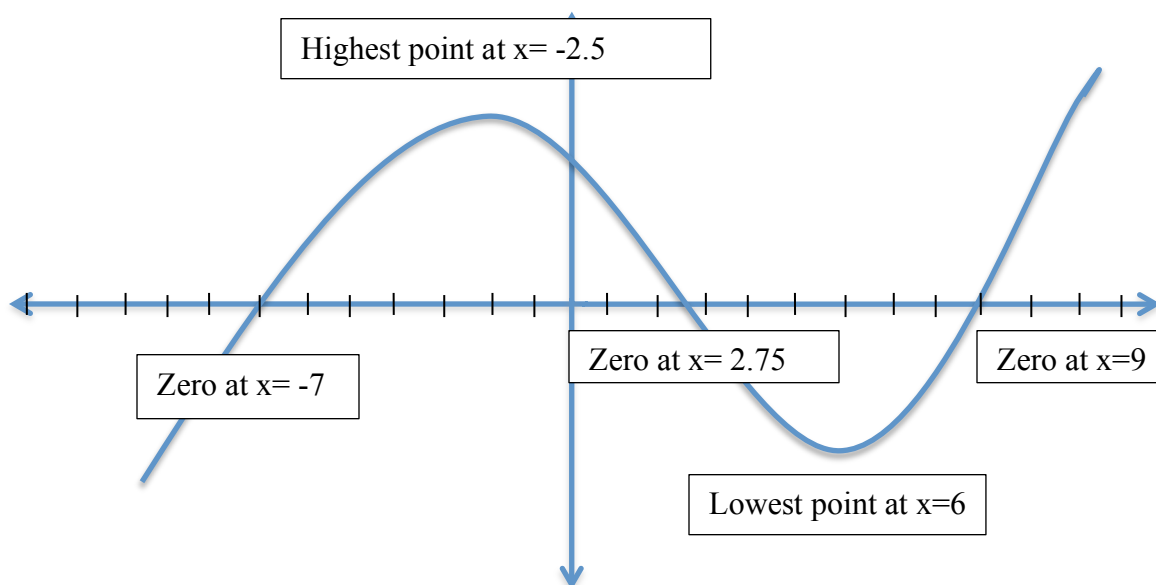
$$f'(x) = \lim_{h \rightarrow 0} \frac{-5h}{[(x+h)-5](x-5)h} = \lim_{h \rightarrow 0} \frac{-5}{[(x+h)-5](x-5)} = \frac{-5}{(x-5)^2}.$$

Recitation #8 - 3.2 Working with Derivatives (Solutions)

**Problem 2** Given the following graph of  $f(x)$ , find the values where  $f(x)$  is zero, positive/negative, increasing/decreasing, and where  $f(x)$  has its highest and lowest values. Additionally, find where the graph of  $f(x)$  is the steepest. Then, without sketching the graph of  $f'(x)$ , determine for which values of  $x$  is  $f'(x)$  zero, positive/negative, and where  $f'(x)$  has the highest and lowest values. Then sketch  $f'(x)$ .



**Solution:**

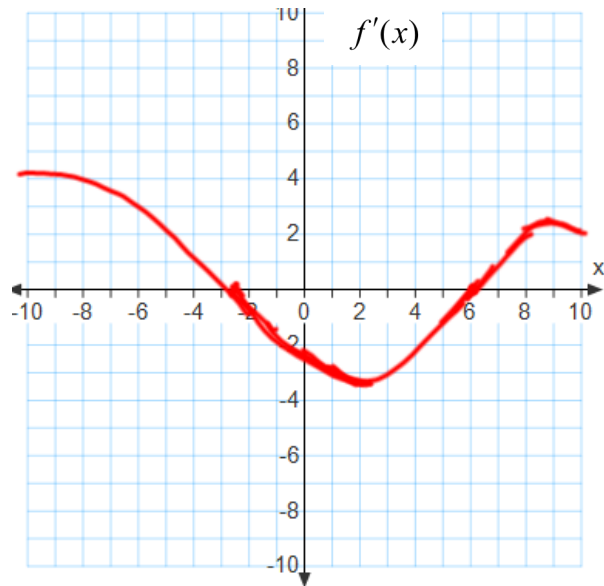


- $f(x)$  is zero when the function crosses the  $x$ -axis, so at approximately  $x = -7, 2.75, 9$ .
- $f(x)$  is negative when the function is below the  $x$ -axis, so approximately  $(-\infty, -7) \cup (2.75, 9)$ .
- $f(x)$  is positive when the function is above the  $x$ -axis, so approximately  $(-7, 2.75) \cup (9, \infty)$ .
- $f(x)$  is increasing on  $(-\infty, -2.5) \cup (6, \infty)$ .
- $f(x)$  is decreasing on  $(-2.5, 6)$ .
- $f(x)$  (locally) has its highest point at approximately  $x = -2.5$ . For  $x > 9$ ,  $f(x)$  has no highest point.
- $f(x)$  (locally) has its lowest point at approximately  $x = 6$ . For  $x < -7$ ,  $f(x)$  has no lowest point.
- $f(x)$  is steepest at approximately  $x = 2$  and  $x = 8.5$ .
- $f'(x)$  is zero when the tangent line has a slope of zero, which is approximately at  $x = -2.5$  and  $x = 6$ . Note, for this question, these are the same answers as the (local) highest and lowest point for  $f(x)$ .
- $f'(x)$  is positive when the slope of the tangent line is positive, so when  $f(x)$  is increasing which is approximately  $(-\infty, -2.5) \cup (6, \infty)$ .
- $f'(x)$  is negative when the slope of the tangent line is negative, so when  $f(x)$  is

Recitation #8 - 3.2 Working with Derivatives (Solutions)

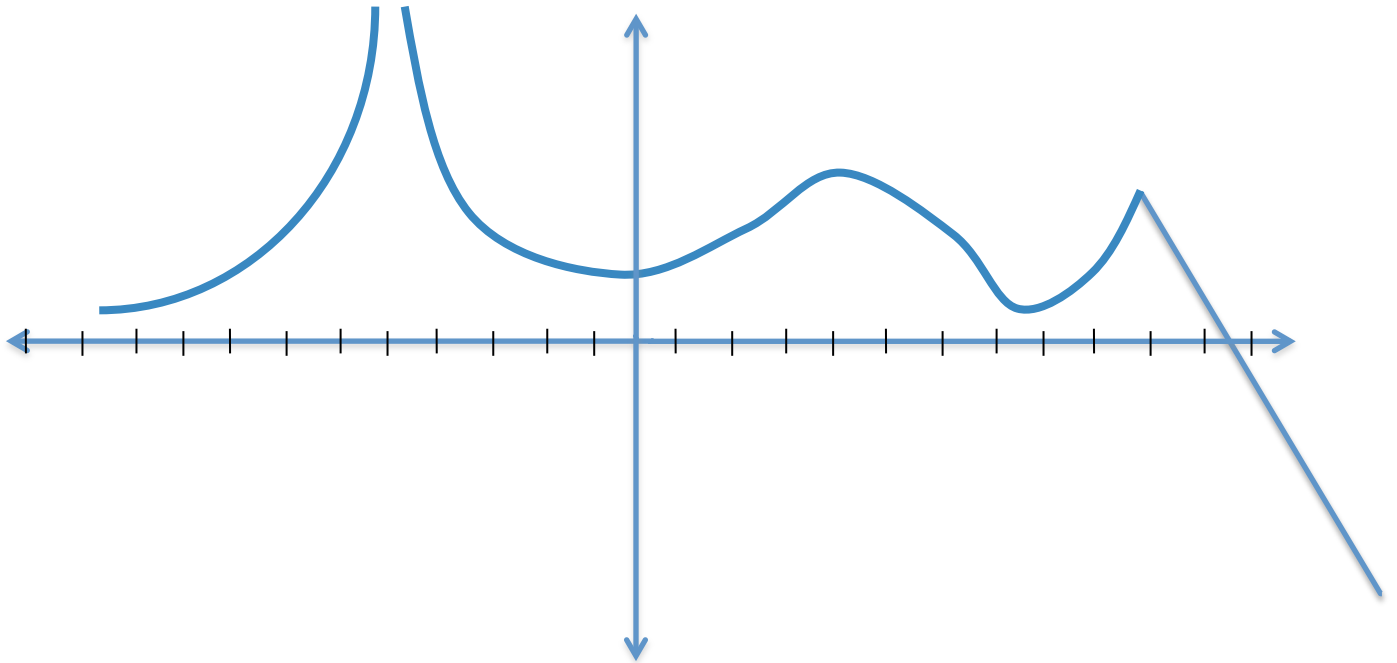
decreasing which is approximately  $(-2.5, 6)$ .

-  $f'(x)$  has its highest and lowest values when  $f(x)$  is steepest, which is approximately when  $x = 2$  and  $x = 8.5$ .

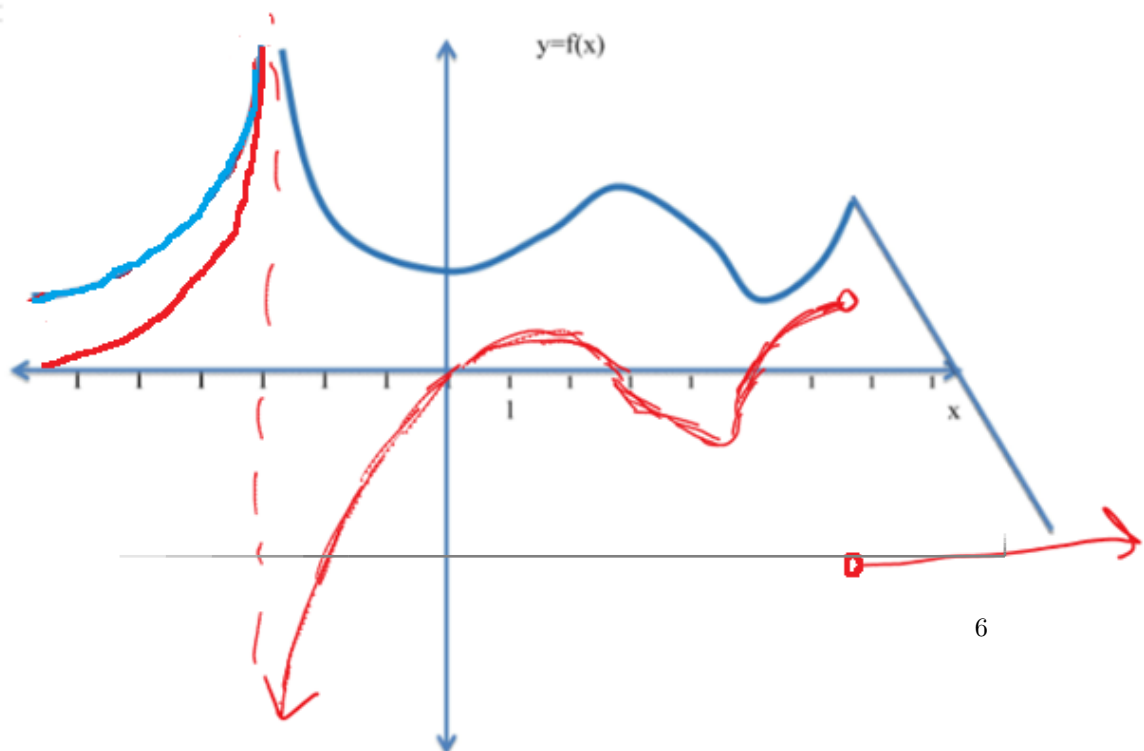


**Problem 3** Given the graph of  $y = f(x)$ , sketch the graph of the derivative  $f'(x)$ .

Recitation #8 - 3.2 Working with Derivatives (Solutions)



**Solution:** The graph of the derivative is in red. Disclaimer: despite being drawn on the same graph, the units for  $f$  and  $f'$  are **not** the same!



**Problem 4** Use the graph of  $g$  in the figure to do the following.

- (a) Find the values of  $x$  in  $(0, 4)$  at which  $g$  is not continuous.

**Solution:**  $g$  is not continuous at  $x = 1$ .

- (b) Find the values of  $x$  in  $(0, 4)$  at which  $g$  is not differentiable.

**Solution:**  $g$  is not differentiable at  $x = 1$  and  $x = 2$ .

