3.4 The Product and Quotient Rules (Solutions)

Warm up:

Differentiate the function $f(x) = \frac{1}{x^8}$ two different ways.

Solution: First way: Since $f(x) = \frac{1}{x^8} = x^{-8}$, $f'(x) = -8x^{-8-1} = -8x^{-9} = -\frac{8}{x^9}$.

Second way: Using the quotient rule, $f'(x) = \frac{(x^8 \cdot \frac{d}{dx}(1)) - (1 \cdot \frac{d}{dx}(x^8))}{(x^8)^2} = \frac{(x^8 \cdot 0) - (1 \cdot 8x^7)}{x^{16}} = \frac{-8x^7}{x^{16}} = -\frac{8}{x^9}$.

Group work:

Problem 1 *Differentiate the following functions:*

(a) $f(x) = (x^2 + 4x - 7)e^{-2x}$

Solution: $f'(x) = \left(\frac{d}{dx}(x^2 + 4x - 7) \cdot e^{-2x}\right) + \left((x^2 + 4x - 7) \cdot \frac{d}{dx}(e^{-2x})\right) = ((2x+4)(e^{-2x}) + (x^2+4x-7)(-2e^{-2x}) = (2x+4-2x^2-8x+14)e^{-2x} = (-2x^2 - 6x + 18)e^{-2x}$.

(b) $g(x) = \frac{x^2 + 4x - 7}{e^{-2x}}$

Solution: $g'(x) = \frac{\left(e^{-2x} \cdot \frac{d}{dx}(x^2 + 4x - 7)\right) - \left((x^2 + 4x - 7) \cdot \frac{d}{dx}(e^{-2x})\right)}{(e^{-2x})^2} = \frac{((e^{-2x}) \cdot (2x+4)) - ((x^2+4x-7) \cdot (-2e^{-2x}))}{e^{-4x}} = \frac{(2x + 4 + 2x^2 + 8x - 14)e^{-2x}}{e^{-4x}} = \frac{2x^2 + 10x - 10}{e^{-2x}}$.

Problem 2 *Suppose that* $f(5) = 7$, $f'(5) = 8$, $g(5) = 3$, *and* $g'(5) = -4$. *Find:*

(a) $(fg)'(5)$.

Solution: $(fg)'(5) = (f'(5) \cdot g(5)) + (f(5) \cdot g'(5)) = (8)(3) + (7)(-4) = 24 - 28 = -4$.

(b) $\left(\frac{f}{g}\right)'(5)$

Solution: $\left(\frac{f}{g}\right)'(5) = \frac{(g(5) \cdot f'(5)) - (f(5) \cdot g'(5))}{(g(5))^2} = \frac{(3)(8) - (7)(-4)}{3^2} = \frac{24+28}{9} = \frac{52}{9}$.

(c) $\left(\frac{g}{f}\right)'(5)$

Solution: $\left(\frac{g}{f}\right)'(5) = \frac{(f(5) \cdot g'(5)) - (g(5) \cdot f'(5))}{(f(5))^2} = \frac{(7)(-4) - (3)(8)}{7^2} = \frac{-28-24}{49} = -\frac{52}{49}$.

Problem 3 *Find the following derivatives:*

(a) *Given $g(x) = x^3 f(x)$, $f(2) = 4$, and $f'(2) = 7$, find the equation of the tangent line to the graph of $g(x)$ at $x = 2$.*

Solution: $g'(x) = 3x^2 f(x) + x^3 f'(x)$. *So* $g'(2) = 12(4) + 8(7) = 48 + 56 = 104$. *Also,* $g(2) = 8f(2) = 32$. *Thus, the equation of the tangent line to the graph of* $g(x)$ *at* $x = 2$ *is*

$y - 32 = 104(x - 2)$ *or* $y = 104x - 176$.

(b) *Given that $h(x) = \frac{xf(x)}{x-3}$, $f(2) = 4$, and $f'(2) = 7$, find the equation of the tangent line to the graph of $h(x)$ at $x = 2$.*

Solution: $h'(x) = \frac{(x-3)\frac{d}{dx}(xf(x)) - xf(x)\frac{d}{dx}(x-3)}{(x-3)^2} = \frac{(x-3)(f(x) + xf'(x)) - xf(x)(1)}{(x-3)^2}$.

So, $h'(2) = \frac{(2-3)(f(2) + 2f'(2)) - 2f(2)}{(2-3)^2} = -(4 + 2(7)) - 2(4) = -18 - 8 = -26$.

Also, $h(2) = \frac{2f(2)}{2-3} = \frac{8}{-1} = -8$. *Thus, the equation of the tangent line to the graph of* $h(x)$ *at* $x = 2$ *is*

$y - (-8) = -26(x - 2)$ *or* $y = -26x + 44$.

(c) *Given the following table, find* $\dfrac{d}{dx}\left(\dfrac{f(x)}{e^x g(x)}\right)$ *at* $x = 2$.

x	1	2	3	4	5
$f(x)$	5	3	0	-4	3
$f'(x)$	-3	-5	-2	6	-4
$g(x)$	6	9	-8	13	15
$g'(x)$	8	5	-10	7	6

Solution: $\dfrac{d}{dx}\left(\dfrac{f(x)}{e^x g(x)}\right) = \dfrac{e^x g(x) f'(x) - f(x)(e^x g(x) + e^x g'(x))}{(e^x g(x))^2}$.

So, $\dfrac{d}{dx}\left(\dfrac{f(x)}{e^x g(x)}\right)\Bigg|_{x=2} = \dfrac{e^2 g(2) f'(2) - f(2)(e^2 g(2) + e^2 g'(2))}{(e^2 g(2))^2} = \dfrac{e^2(9)(-5) - (3)(9e^2 + 5e^2)}{(9e^2)^2} = \dfrac{-45e^2 - 42e^2}{81e^4} = \dfrac{-87e^2}{81e^4} = -\dfrac{87}{81e^2}$.