

Recitation #3 - 2.2: Definition of Limits (Solutions)

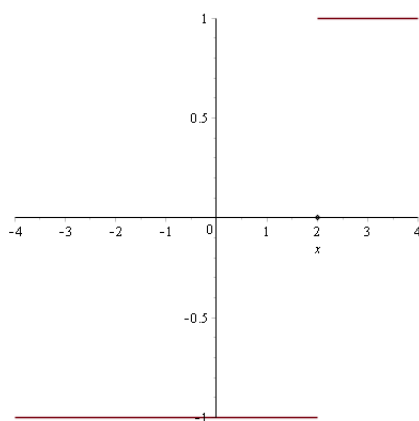
Warm up:

- (a) True or False: To find $\lim_{x \rightarrow 2} f(x)$, it's enough to know $f(2.1)$, $f(2.01)$, $f(2.001)$, etc.

Solution: False, these values will only help us make a guess at $\lim_{x \rightarrow 2^+} f(x)$, the right hand limit of $f(x)$. To determine $\lim_{x \rightarrow 2} f(x)$, we also need to know $\lim_{x \rightarrow 2^-} f(x)$ which we cannot determine from the above values. For example, consider the function

$$f := \begin{cases} 1 & 2 < x \\ 0 & x = 2 \\ -1 & x < 2 \end{cases}$$

Looking at the graph of this function below, we can see that $\lim_{x \rightarrow 2^+} f(x) = 1$ and $\lim_{x \rightarrow 2^-} f(x) = -1$. Thus, since $\lim_{x \rightarrow 2^+} f(x) \neq \lim_{x \rightarrow 2^-} f(x)$, $\lim_{x \rightarrow 2} f(x)$ does not exist.



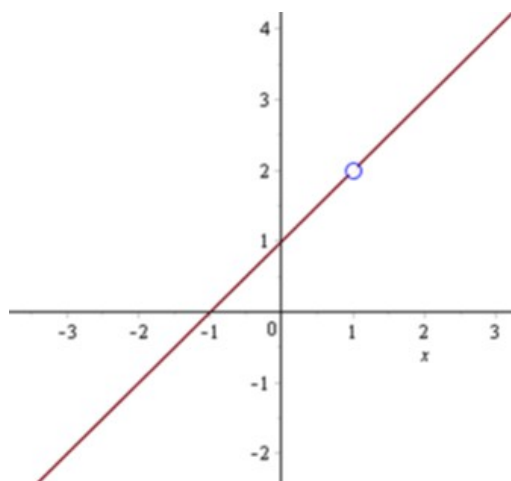
- (b) True or False: If we know $f(2)$, then we know $\lim_{x \rightarrow 2} f(x)$.

Recitation #3 - 2.2: Definition of Limits (Solutions)

Solution: False. In the above example, we have that $f(2) = 0$, however $\lim_{x \rightarrow 2} f(x)$ does not exist.

Group work:

Problem 1 Given the graph of the function, estimate $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$. Then estimate $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ by creating a table of values.



Solution: From the graph and table of values, it appears as though $\lim_{x \rightarrow 1} f(x) = 2$.

x	$f(x)$
.9	$\frac{.9^2 - 1}{.9 - 1} = 1.9$
.99	$\frac{.99^2 - 1}{.99 - 1} = 1.99$
.999	$\frac{.999^2 - 1}{.999 - 1} = 1.999$
1.1	$\frac{1.1^2 - 1}{1.1 - 1} = 2.1$
1.01	$\frac{1.01^2 - 1}{1.01 - 1} = 2.01$
1.001	$\frac{1.001^2 - 1}{1.001 - 1} = 2.001$

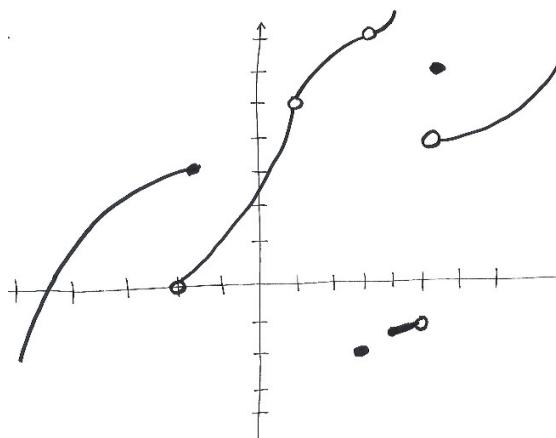
Recitation #3 - 2.2: Definition of Limits (Solutions)

Problem 2 Sketch the graph of a function with the given properties. You need not find a formula for the function.

$$f(3) = -2, f(5) = 6, \lim_{x \rightarrow 5^-} f(x) = -1, \lim_{x \rightarrow 5^+} f(x) = 4, \lim_{x \rightarrow 3} f(x) = 7$$

$$\lim_{x \rightarrow -2^-} f(x) = 3, \lim_{x \rightarrow -2^+} f(x) = 0, \lim_{x \rightarrow 1^+} f(x) = 5$$

Solution: While there are many correct solutions to this problem, one example can be seen below.

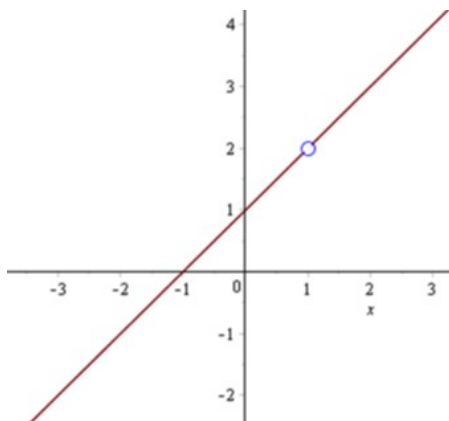


Problem 3 True/False: Give an explanation or counterexample. Assume a and L are finite numbers.

- (a) If $\lim_{x \rightarrow a} f(x) = L$, then $f(a) = L$.

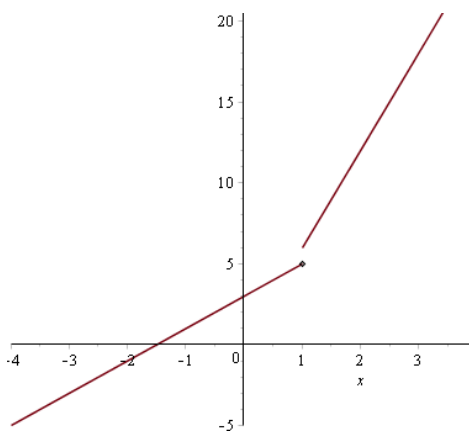
Solution: False, in the graph below $\lim_{x \rightarrow 1} f(x) = 2$, but $f(1)$ does not exist.

Recitation #3 - 2.2: Definition of Limits (Solutions)



(b) If $\lim_{x \rightarrow a^-} f(x) = L$, then $\lim_{x \rightarrow a^+} f(x) = L$.

Solution: False, in the graph below $\lim_{x \rightarrow 1^-} f(x) = 5$ but $\lim_{x \rightarrow 1^+} f(x) = 6$.



(c) If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = L$, then $f(a) = g(a)$.

Solution: False. If we let

$$f := \begin{cases} 3 & x = 1 \\ \frac{x^2 - 1}{x - 1} & \text{otherwise} \end{cases}$$

and

$$g := \begin{cases} 1 & x = 1 \\ x^2 + 1 & \text{otherwise} \end{cases}$$

we see that $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} g(x) = 2$, but $f(1) = 3$ and $g(1) = 1$.

- (d) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ does not exist if $g(a) = 0$.

Solution: False. If $f(x) = x^3$ and $g(x) = x^2$, then $g(0) = 0$ but

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{x^3}{x^2} = \lim_{x \rightarrow 0} x = 0.$$
