Recitation #10 - 3.5 Derivatives of Trig Functions (Solutions)

Warm up:

Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Can you compute $f^{(48)}(x)$ and $g^{(42)}(x)$? Does this remind you of anything that you learned in a high school mathematics course?

Solution: Recall that, for n a nonnegative integer, $f^{(4n)}(x) = f(x) = \sin(x)$ and $g^{(4n)}(x) = g(x) = \cos(x)$. Thus, $f^{(48)}(x) = \sin(x)$ and $g^{(42)}(x) = g''(x) = -\cos(x)$.

This may remind you of the fact that, for n a nonnegative integer, $i^{4n} = i$ where i is the number such that $i^2 = -1$. It is not going to play a role in this class at all, but these two phenomena really are related (which I think is pretty cool haha)!

Group work:

Problem 1 *Find the following limits:*

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(8x)}{x}$

Solution: $\displaystyle\lim_{x\to 0} \frac{\sin(8x)}{x} \cdot \frac{8}{8} = 8 \lim_{x\to 0} \frac{\sin(8x)}{8x} = 8 \lim_{u\to 0} \frac{\sin(u)}{u} = 8 \cdot 1 = 8$ *where* $u = 8x$.

(b) $\displaystyle\lim_{x\to 0} \frac{\cos^2(x) - 1}{4x}$

Solution: $\displaystyle\lim_{x\to 0} \frac{\cos^2(x) - 1}{4x} = \lim_{x\to 0} \frac{\sin^2(x)}{4x} = \lim_{x\to 0} \left(\sin(x)\frac{\sin(x)}{4x}\right) = \frac{1}{4}\left(\lim_{x\to 0} \sin(x)\right) \cdot \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) = \frac{1}{4}(0)(1) = 0$

(c) $\displaystyle\lim_{x\to 0} \frac{x}{\tan(5x)}$

Solution: $\displaystyle\lim_{x\to 0} \frac{x}{\tan(5x)} = \lim_{x\to 0} \frac{x}{\frac{\sin(5x)}{\cos(5x)}} = \lim_{x\to 0} \left(\frac{x}{1} \cdot \frac{\cos(5x)}{\sin(5x)}\right)$

$= \lim_{x\to 0}\left(\cos(5x)\dfrac{x}{\sin(5x)}\cdot\dfrac{5}{5}\right) = \dfrac{1}{5}\left(\lim_{x\to 0}\cos(5x)\right)\left(\lim_{x\to 0}\dfrac{5x}{\sin(5x)}\right) = \dfrac{1}{5}(1)(1) =$
$\dfrac{1}{5}$

Problem 2 *Find the derivative of the following functions:*

(a) $f(x) = \dfrac{x+5}{7x^6+\cot(x)}$

Solution: $f'(x) = \dfrac{(7x^6+\cot(x))(1)-(x+5)(42x^5-\csc^2(x))}{(7x^6+\cot(x))^2} = f'(x) =$
$\dfrac{7x^6+\cot(x)-(x+5)(42x^5-\csc^2(x))}{(7x^6+\cot(x))^2}$.

(b) $f(x) = \sin(x)\cos(x)$

Solution: $f'(x) = (\cos(x))(\cos(x)) + (\sin(x))(-\sin(x)) = \cos^2(x) - \sin^2(x)$.

(c) $f(x) = \dfrac{e^x\tan(x)}{\sec(x)+2}$

Solution: $f'(x) = \dfrac{(\sec(x)+2)(e^x\tan(x)+e^x\sec^2(x)) - e^x\tan(x)(\sec(x)\tan(x))}{(\sec(x)+2)^2} =$
$\dfrac{e^x[(\sec(x)+2)(\tan(x)+\sec^2(x)) - \sec(x)\tan^2(x)]}{(\sec(x)+2)^2}$.

(d) $f(x) = \sin(x)\cos(x)e^{3x}$

Solution: $f'(x) = \dfrac{d}{dx}(\sin(x)\cos(x))e^{3x} + (\sin(x)\cos(x))\dfrac{d}{dx}(e^{3x})$
$= (\cos^2(x)-\sin^2(x))e^{3x} + 3e^{3x}\sin(x)\cos(x)$
$= e^{3x}(\cos^2(x)+3\sin(x)\cos(x)-\sin^2(x))$.

Problem 3 *Find values for a and b so that the following function is both continuous and differentiable everywhere (and where c is an arbitrary constant).*

$$f(x) = \begin{cases} a\sin(x)+b\cos(x) & \text{if } x<0 \\ ax^2+bx+c & \text{if } x\geq 0 \end{cases}$$

Solution: *First, we need that* $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$. *Observe that*

- $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (a\sin(x) + b\cos(x) = b(1) = b.$
- $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} ax^2 + bx + c = c.$

Thus, we must have that $b = c$, *and so*

$$f(x) = \begin{cases} a\sin(x) + c\cos(x) & \text{if } x < 0 \\ ax^2 + cx + c & \text{if } x \geq 0 \end{cases}$$

For $f'(0)$ *to exist, we need the limit* $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ *to exist. So we need*

$$\lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

But notice that the left hand limit is just the derivative of $a\sin(x) + c\cos(x)$ *evaluated at* $x = 0$, *and similarly the right hand limit is just the derivative of* $ax^2 + cx + c$ *evaluated at* $x = 0$. *So we compute*

- $\dfrac{d}{dx}(a\sin(x) + c\cos(x))\Big|_{x=0} = (a\cos(x) - c\sin(x))\Big|_{x=0} = a.$
- $\dfrac{d}{dx}\left(ax^2 + cx + c\right)\Big|_{x=0} = (2ax + c)\Big|_{x=0} = c.$

So, $a = c$ *as well, and thus we can finally conclude that*

$$f(x) = \begin{cases} c\sin(x) + c\cos(x) & \text{if } x < 0 \\ cx^2 + cx + c & \text{if } x \geq 0 \end{cases}$$