Recitation #3 - 2.2: Definition of Limits (Solutions)

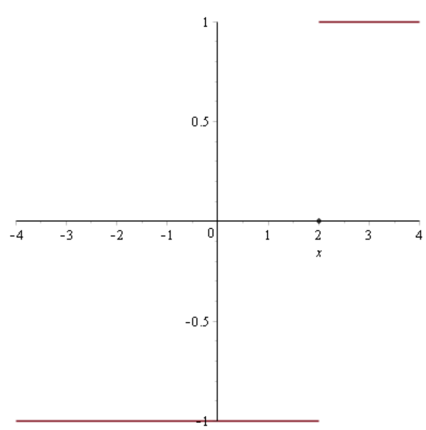
Warm up:

(a) True or False: To find $\lim_{x\to 2} f(x)$, it's enough to know $f(2.1)$, $f(2.01)$, $f(2.001)$, etc.

Solution: False. These values will only help us make a guess at $\lim_{x\to 2^+} f(x)$, the right hand limit of $f(x)$. To determine $\lim_{x\to 2} f(x)$, we also need to know $\lim_{x\to 2^-} f(x)$ which we cannot determine from the above values. For example, consider the function

$$f := \begin{cases} 1 & 2 < x \\ 0 & x = 2 \\ -1 & x < 2 \end{cases}$$

Looking at the graph of this function below, we can see that $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -1$. Thus, since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, $\lim_{x\to 2} f(x)$ does not exist.

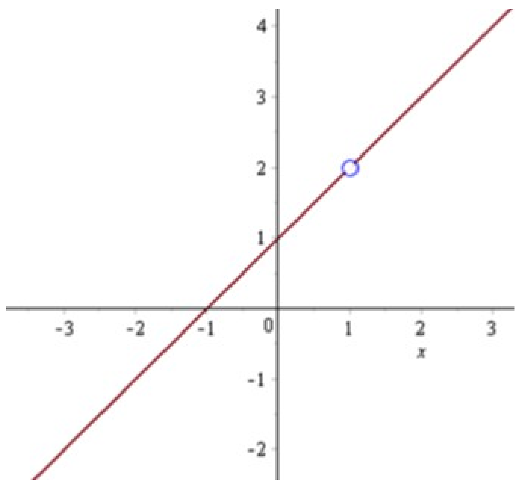


(b) True or False: If we know $f(2)$, then we know $\lim_{x\to 2} f(x)$.

Solution: False. In the above example, we have that $f(2) = 0$, however $\lim_{x\to 2} f(x)$ does not exist.

Group work:

Problem 1 *Given the graph of the function, estimate* $\lim_{x\to 1} \frac{x^2-1}{x-1}$. *Then estimate* $\lim_{x\to 1} \frac{x^2-1}{x-1}$ *by creating a table of values.*



Solution: *From the graph and table of values, it appears as though* $\lim_{x\to 1} f(x) = 2$.

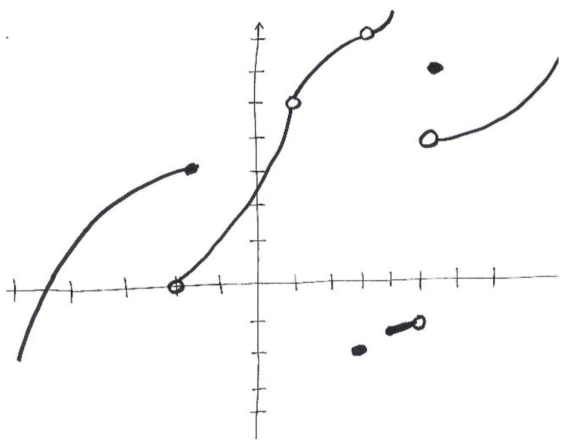
x	$f(x)$
.9	$\frac{.9^2-1}{.9-1} = 1.9$
.99	$\frac{.99^2-1}{.99-1} = 1.99$
.999	$\frac{.999^2-1}{.999-1} = 1.999$
1.1	$\frac{1.1^2-1}{1.1-1} = 2.1$
1.01	$\frac{1.01^2-1}{1.01-1} = 2.01$
1.001	$\frac{1.001^2-1}{1.001-1} = 2.001$

Problem 2 *Sketch the graph of a function with the given properties. You need not find a formula for the function.*

$$f(3) = -2, f(5) = 6, \lim_{x\to 5^-} f(x) = -1, \lim_{x\to 5^+} f(x) = 4, \lim_{x\to 3} f(x) = 7$$

$$\lim_{x\to -2^-} f(x) = 3, \lim_{x\to -2^+} f(x) = 0, \lim_{x\to 1^+} f(x) = 5$$

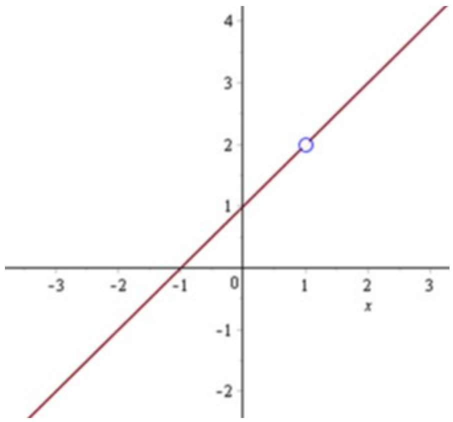
Solution: *While there are many correct solutions to this problem, one example can be seen below.*



Problem 3 *True/False: Give an explanation or counterexample. Assume a and L are finite numbers.*

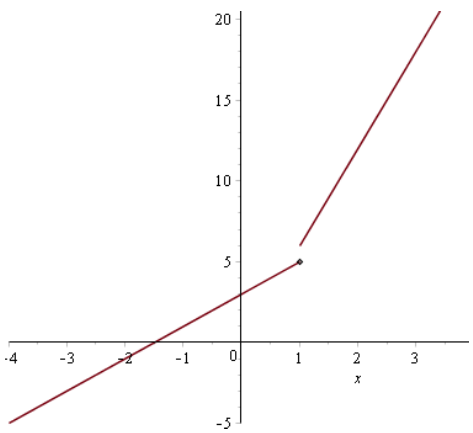
(a) *If $\lim_{x\to a} f(x) = L$, then $f(a) = L$.*

Solution: *False. In the graph below $\lim_{x\to 1} f(x) = 2$, but $f(1)$ does not exist.*



(b) *If* $\lim_{x\to a^-} f(x) = L$, *then* $\lim_{x\to a^+} f(x) = L$.

Solution: *False. In the graph below* $\lim_{x\to 1^-} f(x) = 5$ *but* $\lim_{x\to 1^+} f(x) = 6$.



(c) *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = L$, *then* $f(a) = g(a)$.

Solution: *False. If we let*

$$f := \begin{cases} 3 & x = 1 \\ \dfrac{x^2 - 1}{x - 1} & \textit{otherwise} \end{cases}$$

and

$$g := \begin{cases} 1 & x = 1 \\ x^2 + 1 & \textit{otherwise} \end{cases}$$

we see that $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = 2$, *but* $f(1) = 3$ *and* $g(1) = 1$.

(d) $\lim_{x \to a} \frac{f(x)}{g(x)}$ *does not exist if* $g(a) = 0$.

Solution: *False. If* $f(x) = x^3$ *and* $g(x) = x^2$, *then* $g(0) = 0$ *but* $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^3}{x^2} = \lim_{x \to 0} x = 0$.