

Recitation #8 - 3.2 Working with Derivatives (Solutions)

Warm up:

(1) If $f'(2)$ exists, then $\lim_{x \rightarrow 2} f(x)$

- (a) must exist, but more information is needed if we want to find it.
- (b) is equal to $f(2)$.
- (c) is equal to $f'(2)$.
- (d) need not exist.

Solution: The correct answer is (b). If $f'(2)$ exists, then the function $f(x)$ is differentiable at $x = 2$. There is a theorem that says “if $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$ ”. So $f(x)$ is continuous at $x = 2$, and by definition, this means that $\lim_{x \rightarrow 2} f(x) = f(2)$.

(2) Assuming that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, we can conclude

- (a) $\frac{0}{0} = 1$.
- (b) the tangent line to $y = \sin x$ at $(0, 0)$ has slope 1.
- (c) you can cancel the x 's.
- (d) for all x near 0, $\sin x = x$.
- (e) for all x near 0, $\sin x \approx x$.

Solution: The answers are (b) and (e). To see that (b) is true, if we write out the limit definition of the derivative of $f(x) = \sin x$ at $x = 0$, we get that $f'(0) = \lim_{h \rightarrow 0} \frac{\sin(h+0) - \sin(0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$. But this equals 1 by the assumption, and so the tangent line to the graph of $y = \sin(x)$ at $(0, 0)$ has slope 1.

For (e), note that this is what it means for $y = x$ to be the tangent line to the graph of $y = \sin(x)$ at $x = 0$.

Group work:

Problem 1 Look at the following work for finding the derivative of $f(x) = \frac{x}{x-5}$ at the point $x = 3$ using the definition of the derivative. Adapt the following work to find $f'(x)$.

$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3+h}{(3+h)-5} - \frac{3}{3-5}}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{3+h}{(3+h)-5} - \frac{3}{3-5} \right] \cdot \frac{1}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{(3+h)(3-5)}{[(3+h)-5](3-5)} - \frac{3[(3+h)-5]}{(3-5)[(3+h)-5]} \right] \cdot \frac{1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)(3-5) - 3[(3+h)-5]}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{3 \cdot 3 + 3(-5) + 3h - 5h - [3 \cdot 3 + 3h + 3(-5)]}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{9 - 15 + 3h - 5h - 9 + 3h + 15}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{-5h}{[(3+h)-5](3-5)h} \\
 &= \lim_{h \rightarrow 0} \frac{-5}{[(3+h)-5](3-5)} = \frac{-5}{[(3+0)-5](3-5)} \\
 &= \frac{-5}{(3-5)^2} = \frac{-5}{4}
 \end{aligned}$$

Solution: Essentially, one just needs to replace all of the 3's with x 's. After making this substitution and completing the same algebra, line four becomes

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)(x-5) - x[(x+h)-5]}{[(x+h)-5](x-5)h}.$$

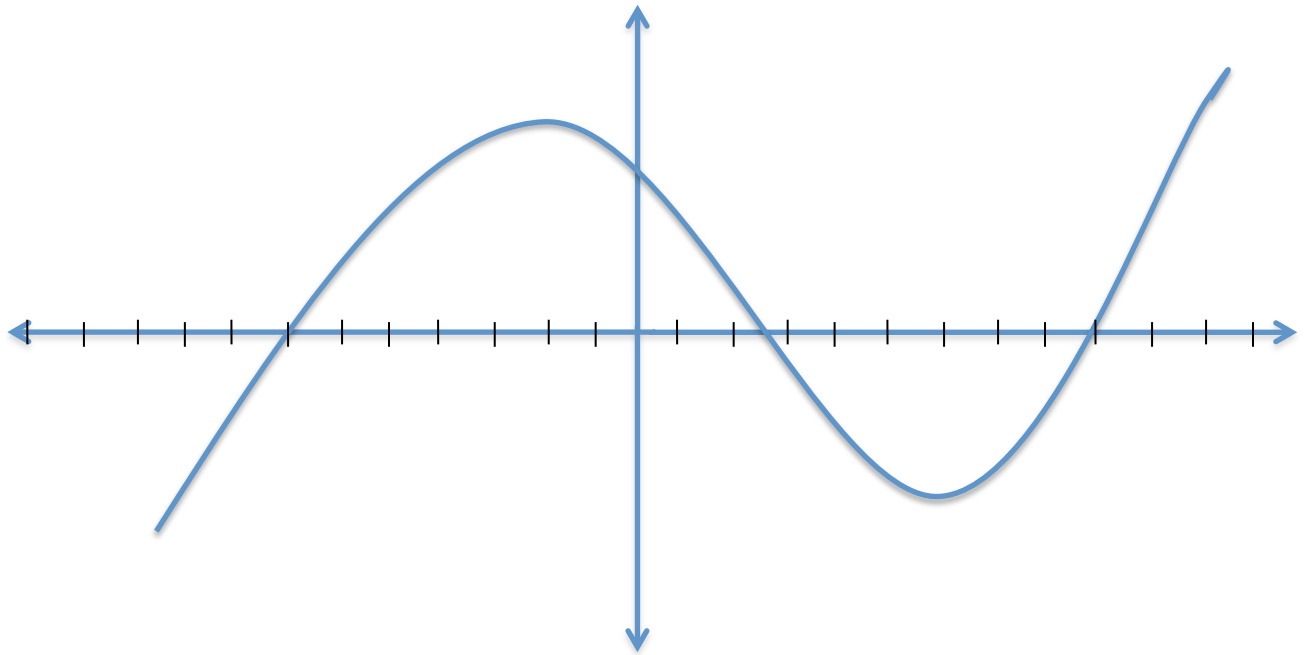
Multiplying the numerator out and canceling yields

$$f'(x) = \lim_{h \rightarrow 0} \frac{-5h}{[(x+h)-5](x-5)h} = \lim_{h \rightarrow 0} \frac{-5}{[(x+h)-5](x-5)} = \frac{-5}{(x-5)^2}.$$

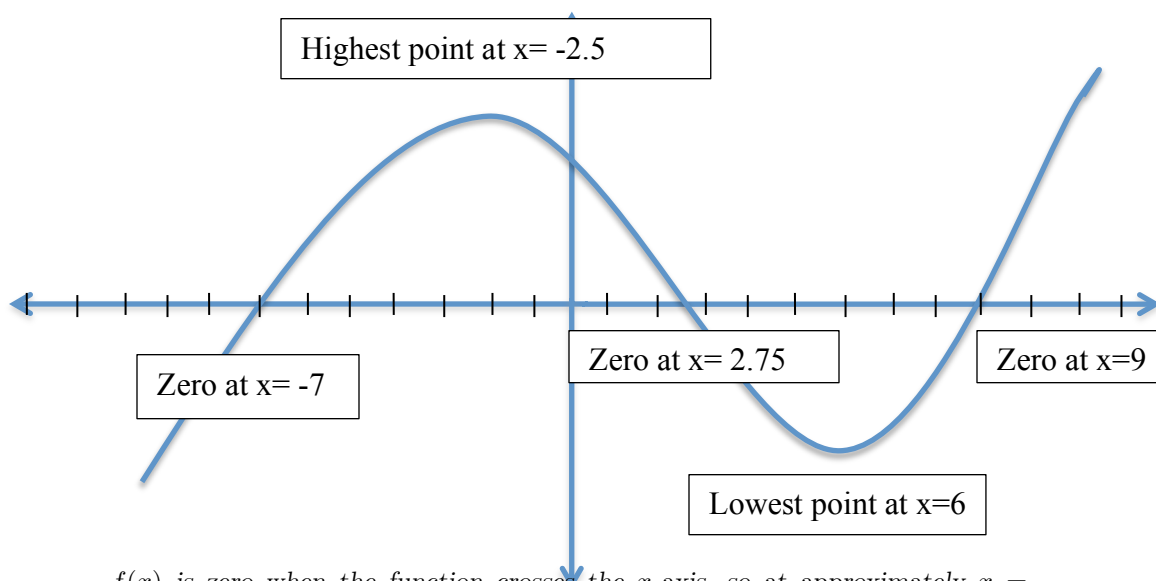
Problem 2 Given the following graph of $f(x)$, find the values where $f(x)$ is zero, positive/negative, increasing/decreasing, and where $f(x)$ has its highest

Recitation #8 - 3.2 Working with Derivatives (Solutions)

and lowest values. Additionally, find where the graph of $f(x)$ is the steepest. Then, without sketching the graph of $f'(x)$, determine for which values of x is $f'(x)$ zero, positive/negative, and where $f'(x)$ has the highest and lowest values. Then sketch $f'(x)$.



Solution:



$f(x)$ is zero when the function crosses the x -axis, so at approximately $x = -7, 2.75, 9$.

$f(x)$ is negative when the function is below the x -axis, so approximately $(-\infty, -7) \cup (2.75, 9)$.

$f(x)$ is positive when the function is above the x -axis, so approximately $(-7, 2.75) \cup (9, \infty)$.

$f(x)$ is increasing on $(-\infty, -2.5) \cup (6, \infty)$.

$f(x)$ is decreasing on $(-2.5, 6)$.

$f(x)$ (locally) has its highest point at approximately $x = -2.5$. For $x > 9$, $f(x)$ has no highest point.

$f(x)$ (locally) has its lowest point at approximately $x = 6$. For $x < -7$, $f(x)$ has no lowest point.

$f(x)$ is steepest at approximately $x = 2$ and $x = 8.5$.

$f'(x)$ is zero when the tangent line has a slope of zero, which is approximately at $x = -2.5$ and $x = 6$. Note, for this question, these are the same answers as the (local) highest and lowest point for $f(x)$.

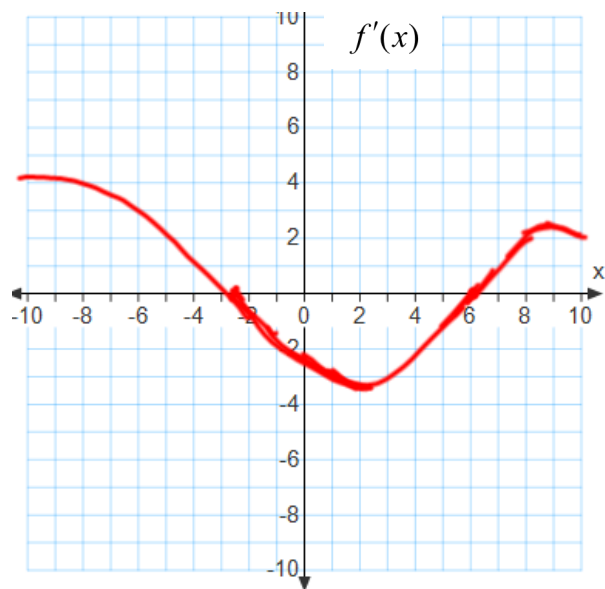
$f'(x)$ is positive when the slope of the tangent line is positive, so when $f(x)$ is increasing which is approximately $(-\infty, -2.5) \cup (6, \infty)$.

$f'(x)$ is negative when the slope of the tangent line is negative, so when $f(x)$ is decreasing which is approximately $(-2.5, 6)$.

$f'(x)$ has its highest and lowest values when $f(x)$ is steepest, which is approxi-

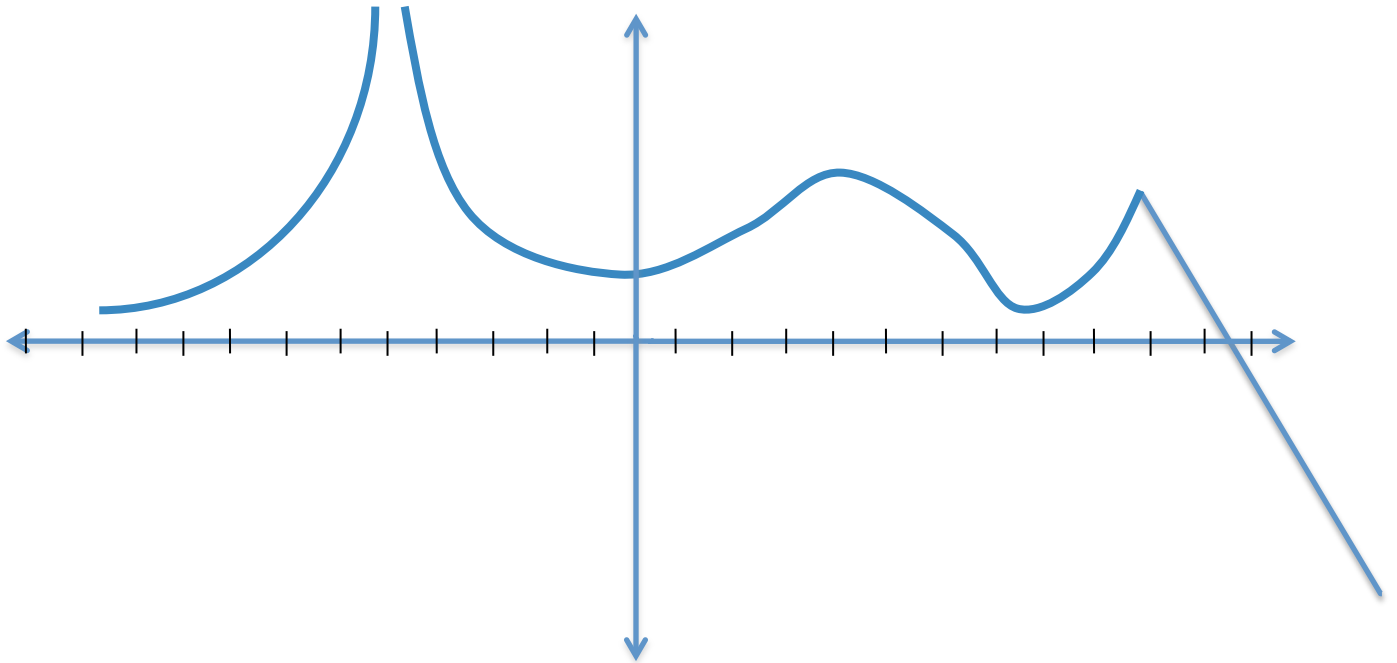
Recitation #8 - 3.2 Working with Derivatives (Solutions)

mately when $x = 2$ and $x = 8.5$.

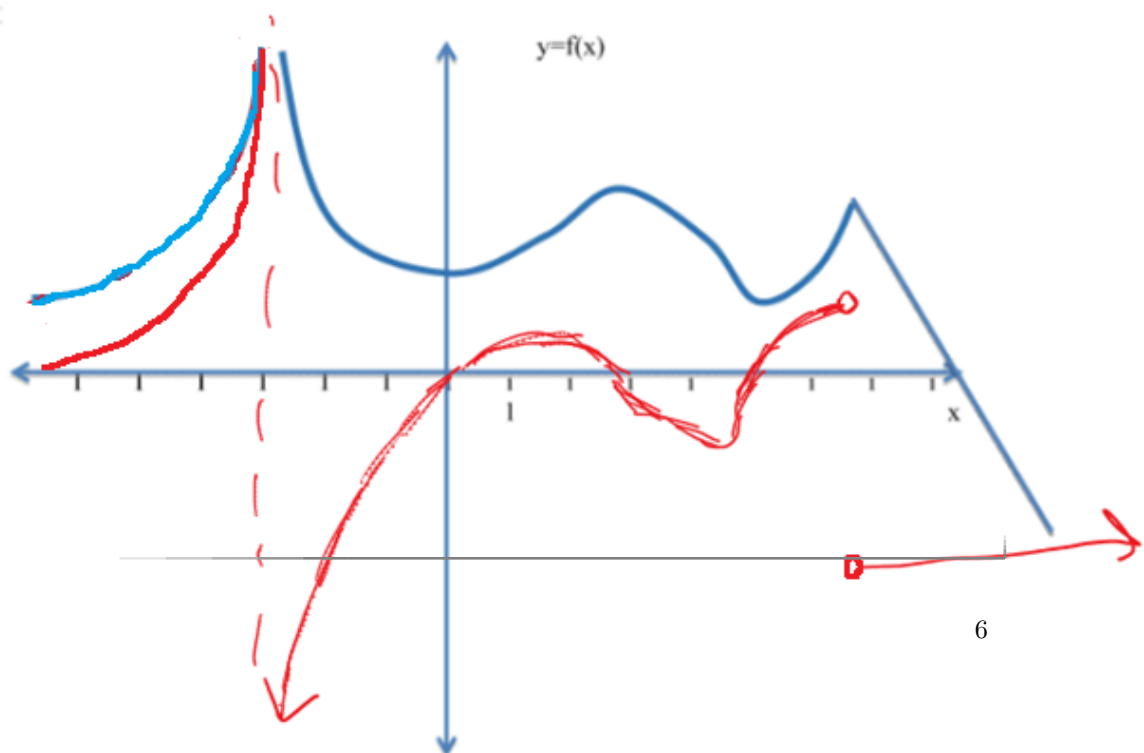


Problem 3 Given the graph of $y = f(x)$, sketch the graph of the derivative $f'(x)$.

Recitation #8 - 3.2 Working with Derivatives (Solutions)



Solution: The graph of the derivative is in red. Disclaimer: despite being drawn on the same graph, the units for f and f' are **not** the same!



Problem 4 Use the graph of g in the figure to do the following.

- (a) Find the values of x in $(0, 4)$ at which g is not continuous.

Solution: g is not continuous at $x = 1$.

- (b) Find the values of x in $(0, 4)$ at which g is not differentiable.

Solution: g is not differentiable at $x = 1$ and $x = 2$.

