Optical trimer,
A theoretical physics approach to waveguide couplers

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We study electromagnetic field propagation through an ideal, passive, triangular three-waveguide coupler using a symmetry based approach to take advantage of the underlying $SU(3)$ symmetry. The planar version of this platform has proven valuable in photonic circuit design providing optical sampling, filtering, modulating, multiplexing, and switching. We show that a group-theory approach can readily provide a starting point for design optimization of these devices. We also try to present our analysis as a practical tutorial on the use of group theory to study photonic lattices for those not familiar with abstract algebra methods.

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1. INTRODUCTION

The planar three-waveguide coupler [1] has proven a reliable platform for optical devices. It has been shown to provide tunable sampling, filtering [2], modulation [3] and power coupling [4] in voltage driven systems, as well as power dividers and combiners in passive devices [5–8] that have allowed efficient signal referencing for integrated optical biosensors [9].

In most of the reported literature, optimization seems the standard approach favored by the optics community to design waveguide couplers [10, 11] but, recently, analogies with quantum mechanical systems have provided an alternative complementary approach [12, 13]. This has also impacted the design of planar three-waveguide couplers that, for example, have provided fast, robust directional beam coupling designed either by standard optimization [14–16] or by quantum analogies [17–20].

Here, our aim is to motivate photonic designers to go beyond analogies between photonic lattices and quantum systems. We will try our best to bridge the gap between theoretical physics and optics to show how the underlying symmetries of a photonic lattice can shed light into the design process. For this, we will use a general version of the three-waveguide coupler. In the next section, we will introduce the mode-coupling model and expose its underlying $SU(3)$ symmetry. Then, we will show how to construct a propagator for any given physical configuration using a Gilmore-Perelomov coherent state approach [21]. In order to provide practical examples, we will focus on arrays of identical waveguides with three identical couplings, which are related to the discrete Fourier transform, an two identical couplings, which are related to the golden ratio and allow devices with a single stable output. Finally, we will present a summary with possible extensions allowed by linear and nonlinear three-waveguide couplers.

2. THREE-WAVEGUIDE COUPLER

Light propagating through an ideal, general three-waveguide coupler can be described by coupled mode theory, c.f. [22] and references therein,

$$-i\partial_z \begin{pmatrix} \mathcal{E}_0(z) \\ \mathcal{E}_1(z) \\ \mathcal{E}_2(z) \end{pmatrix} = \begin{pmatrix} \omega_0(z) & g_{01}(z) & g_{02}(z) \\ g_{01}(z) & \omega_1(z) & g_{12}(z) \\ g_{02}(z) & g_{12}(z) & \omega_2(z) \end{pmatrix} \begin{pmatrix} \mathcal{E}_0(z) \\ \mathcal{E}_1(z) \\ \mathcal{E}_2(z) \end{pmatrix}. \quad (1)$$

Here, the complex field amplitude at the jth waveguide is given by $\mathcal{E}_j(z)$, the effective refractive index at the jth waveguide is $\omega_j(z)$, and the effective coupling between the jth and kth waveguides is $g_{jk}(z)$. These complex field equations can be cast in a Schrödinger-like form [22],

$$-i\partial_z|\mathcal{E}(z)\rangle = \hat{H}(z)|\mathcal{E}(z)\rangle, \quad (2)$$

where kets and operators in Dirac notation represent column vectors and square matrices, in that order. We can normalize the intensity, $\sum_j |\mathcal{E}_j(z)|^2 = 1$, as we are dealing with an ideal lossless device. Experimental realization of this model include, but are not limited, to laser inscribed photonic waveguides [23] and multicore optical fibers [], Fig. **??**(a), whispering-mode cavities [], Fig. **??**(b), or microwave resonators [24] and standard

RLC-circuits [] for electromagnetic radiation outside the visible spectrum.

The formal solution,

$$|\mathcal{E}(z)\rangle = \hat{U}(z)|\mathcal{E}(0)\rangle, \tag{3}$$

to this ordinary linear differential equation is provided by an ordered exponential [25, 26],

$$\hat{U}(z) = \text{Texp}\left[\int_0^z \hat{H}(x)dx\right]. \tag{4}$$

Usually, it is not straightforward to calculate the propagator, but underlying symmetries simplify this endeavor [27, 28]. While group theory is extensively used in mathematical optics [29, 30], it may be possible that the standard Lie algebra approach may look more complicated than it actually is for those outside that field, we hope that the following can help vanishing that feeling.

3. GROUP THEORY APPROACH

Group theory, as an instrument to explore the underlying structure of mathematical models describing the physical world, brings a layer of abstraction into physics that allows deeper insight. As such, it has become an essential tool in quantum mechanics. Coupled mode theory delivers a Schrödinger-like form describing light propagating through arrays of coupled waveguides, thus, the use of group theory to calculate propagation in these systems seems like a natural step.

The mode-coupling matrix $\hat{H}$ for our triangular three-waveguide coupler is a unitary matrix of rank three, so, it can be constructed with elements from the special unitary group $SU(3)$ which is a a household name in physics often related to the work of Gell-Mann [31] and Ne'eman [32],

$$\begin{aligned} \hat{H}(z) &= \frac{1}{3}\sum_{j=0}^{3}\omega_j(z)\hat{\mathbb{1}} + \frac{1}{2}\left[\delta_0(z)+\delta_1(z)\right]\hat{Y} \\ &+ \left[\delta_0(z)-\delta_1(z)\right]\hat{I}_0 + g_{01}(z)\left(\hat{I}_+ + \hat{I}_-\right) \\ &+ g_{12}(z)\left(\hat{U}_+ + \hat{U}_-\right) + g_{02}(z)\left(\hat{V}_+ + \hat{V}_-\right). \end{aligned} \tag{5}$$

Here, we have defined the effective refractive index differences $\delta_j(z) = \omega_j(z) - \omega_2(z)$, the unit matrix $\hat{\mathbb{1}}$ of dimension three, and used the following representation for the $SU(3)$ group [33],

$$\begin{gathered} \hat{Y} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \hat{I}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{I}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{I}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{U}_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{U}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \hat{V}_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{V}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{gathered} \tag{6}$$

due to the fact that we can understand matrices $\hat{I}_\pm$, $\hat{U}_\pm$ and $\hat{V}_\pm$ as those describing the coupling of the electromagnetic field between waveguides zero and one, one and two, and zero and two, in that order.

At this point, our original Schrödinger-like equation is written in terms of a linear combination of Lie group generators for $SU(3)$. Wei and Norman demonstrated that any such equation can be treated by an algebraic method that provides the following propagator [34],

$$\hat{U}(z) = \prod_{j=1}^{9} e^{i\theta_j(z)\hat{X}_j}, \tag{7}$$

where the $su(3)$ algebra elements, $e^{i\theta_j(z)\hat{X}_j}$, are just the exponential map of the group generators, $\{\hat{Y}, \hat{I}_0, \hat{I}_\pm, \hat{U}_\pm, \hat{V}_\pm\}$ plus the identity $\hat{\mathbb{1}}$, and the functions $\theta_j(z)$ are complex functions ruled by the dynamics provided by the mode-coupling matrix, $\hat{H}$. Note, there is no apriori ordering of $su(3)$ elements to write the propagator. However, the values of the $\theta_j(z)$ functions do depend on the chosen order [35, 36]. We will choose a particular ordering,

$$\begin{aligned} \hat{U}(z) &= e^{i\phi(z)\mathbb{1}} e^{i\iota_+(z)\hat{I}_+} e^{i\mu_+(z)\hat{U}_+} e^{i\nu_+(z)\hat{V}_+} e^{\iota_0(z)\hat{I}_0} \\ &\quad \times e^{iy_0(z)\hat{Y}} e^{i\nu_-(z)\hat{V}_-} e^{i\mu_-(z)\hat{U}_-} e^{i\iota_-(z)\hat{I}_-}, \end{aligned} \tag{8}$$

that keeps us in line with the idea of understanding propagation through waveguide lattices as generalized Gilmore-Perelomov coherent states [21].

The next step is straightforward but cumbersome, we substitute the formal solution $|\mathcal{E}(z)\rangle$, using the propagator above, being careful in keeping the ordering through the derivation process. Then, we use the actions of elements of the $su(3)$ algebra on elements of the $SU(3)$ group [37] to find the differential equation set for the auxiliary functions,

$$\begin{aligned}
\phi' &= \frac{1}{3}\left(\omega_0+\omega_1+\omega_2\right), && (9)\\
\iota_+' &= \left[g_{01}\iota_+ + g_{02}\nu_+ + i\left(\omega_0-\omega_1\right)\right]\iota_+ + \\
&\quad -ig_{12}\nu_+ + g_{01}, && (10)\\
\mu_+' &= \left[\left(g_{12}-g_{02}\right)\mu_+ + g_{02}\left(\nu_+ - \iota_+\right) + i\left(\omega_1-\omega_2\right)\right]\mu_+ \\
&\quad +ig_{01}\nu_+ + g_{12} && (11)\\
\nu_+' &= \left[g_{01}\iota_+ + g_{02}\nu_+ + i\left(\omega_0-\omega_2\right)\right]\nu_+ \\
&\quad -ig_{12}\iota_+ + g_{02}, && (12)\\
\iota_0' &= \omega_0-\omega_1 + i\left(g_{12}\mu_+ - g_{02}\nu_+\right) \\
&\quad -\left(g_{02}\mu_+ + 2ig_{01}\right)\iota_+, && (13)\\
y_0' &= -i\frac{3}{2}\left[g_{02}\nu_+ + \left(g_{12}+ig_{02}\iota_+\right)\mu_+\right] \\
&\quad +\frac{1}{2}\left(\omega_0+\omega_1-2\omega_2\right), && (14)\\
\nu_-' &= e^{i\left(y_0-\frac{1}{2}\iota_0\right)}g_{02} - e^{i\iota_0}\left(g_{02}\mu_+ + ig_{01}\right)\mu_-, && (15)\\
\mu_-' &= e^{i\left(y_0-\frac{1}{2}\iota_0\right)}\left(g_{12}+ig_{02}\iota_+\right), && (16)\\
\iota_-' &= e^{i\iota_0}\left(g_{01}-ig_{02}\mu_+\right), && (17)
\end{aligned}$$

where, for the sake of space, we have used $f \equiv f(z)$ and $f' \equiv \partial_z f(z)$ for all propagation dependent auxiliary functions and couplings. As expected, the overall phase is related to the effective refractive indices,

$$e^{i\phi(z)\hat{\mathbb{1}}} = e^{\frac{i}{3}\int_0^z(\omega_0(\zeta)+\omega_1(\zeta)+\omega_2(\zeta))d\zeta}\,\hat{\mathbb{1}}. \tag{18}$$

Non-linear differential equations are known to be hard to solve and finding a solution often requires intuition and knowledge of the system being analyzed. Before delving into details, we would like to point out a key feature of passive, lossless optical models, their mode-coupling matrices are real symmetric, $\hat{H}^T(z) = \hat{H}(z)$ where the operation O^T stands for transposition, and, as a direct consequence, the propagator shares the same property,

$$\hat{U}^T(z) = \hat{U}(z). \tag{19}$$

This feature allows us to conclude that the propagator functions are symmetric,

$$\xi_+(z) = \xi_-(z), \quad \xi = \iota, \nu, \mu. \tag{20}$$

Furthermore, we observe that two equations, namely Eq.(10) and Eq.(12) only include terms of $\iota_+(z)$ and $\nu_+(z)$ and their derivatives. Therefore, they are decoupled from the rest. Nonetheless, these two equations prove intractable and we will pursue a different route to finding a solution. For reasons that will become apparent in a moment, we introduce a set of five auxiliary functions,

$$\begin{aligned}
\Gamma(z) &= e^{i\phi(z)} e^{-i\frac{3}{2}y_0(z)}, & (21)\\
\Delta(z) &= i\Gamma(z)\mu_+(z), & (22)\\
\Theta(z) &= \Gamma(z)\left[-\iota_+(z)\mu_+(z) + i\nu_+(z)\right], & (23)\\
\Pi(z) &= \Gamma(z)\left[e^{iy_0(z)} e^{-i\frac{1}{2}\iota_0(z)} - \mu_+(z)^2\right], & (24)\\
\Sigma(z) &= i\iota_\pm(z)\Pi(z) - \Gamma(z)\mu_\pm(z)\nu_\pm(z)), & (25)
\end{aligned}$$

such that,

$$\begin{aligned}
\iota_+(z) &= i\frac{\Gamma(z)\Sigma(z) - \Delta(z)\Theta(z)}{\Delta^2(z) - \Gamma(z)\Pi(z)}, & (26)\\
\mu_+(z) &= -i\frac{\Delta(z)}{\Gamma(z)}, & (27)\\
\nu_+(z) &= i\frac{\Pi(z)\Theta(z) - \Delta(z)\Sigma(z)}{\Delta^2(z) - \Gamma(z)\Pi(z)}, & (28)\\
\iota_0(z) &= i2\log\frac{\Gamma(z)\Pi(z) - \Delta^2(z)}{\Gamma^{\frac{1}{2}}(z)e^{i\frac{3}{2}\phi(z)}}, & (29)\\
y_0(z) &= i\frac{3}{2}\log\Gamma(z)e^{-i\phi(z)}. & (30)
\end{aligned}$$

Note that the phase functions $y_0(z)$ and $i_0(z)$ are of logarithmic nature and the rest are quotients of the products of the solution basis.

These equations can be further decoupled,

$$\begin{pmatrix}\Theta'(z)\\ \Delta'(z)\\ \Gamma'(z)\end{pmatrix} = iH \begin{pmatrix}\Theta(z)\\ \Delta(z)\\ \Gamma(z)\end{pmatrix} \tag{31}$$

$$\begin{pmatrix}\Sigma'(z)\\ \Pi'(z)\\ \Delta'(z)\end{pmatrix} = iH \begin{pmatrix}\Sigma(z)\\ \Pi(z)\\ \Delta(z)\end{pmatrix} \tag{32}$$

via the identity,

$$g_{01}(z)\Theta(z) + g_{12}(z)\Gamma(z) = g_{02}(z)\Sigma(z) + g_{12}(z)\Pi(z), \tag{33}$$

which is a direct consequence of $\mu_+(z) = \mu_-(z)$. These two sets of differential equations can be readily solved with an additional set of initial values,

$$\begin{aligned}
\Theta(z) &= 0, \quad \Delta(z) = 0, \quad \Sigma(z) = 0, & (34)\\
\Gamma(z) &= 1, \quad \Pi(z) = 1. & (35)
\end{aligned}$$

The two coupled differential sets, Eq. (31) and Eq.(32), in conjunction the initial values, Eq. (34) and Eq.(35), unequivocally determine the five auxiliary functions that return,

$$\begin{aligned}
\mu_\pm(z) &= \frac{-i\Delta(z)}{\Gamma(z)} & (36)\\
\iota_\pm(z) &= \frac{i(-\Gamma(z)\Sigma(z) + \Delta(z)\Theta(z))}{\Delta(z)^2 - \Gamma(z)\Pi(z)} & (37)\\
\nu_\pm(z) &= \frac{-i(\Delta(z)\Sigma(\zeta) - \Pi(z)\Theta(z))}{\Delta(z)^2 - \Gamma(z)\Pi(z)} & (38)\\
y_0(z) &= \frac{3}{2}i\log(\Gamma(z)) & (39)\\
\iota_0(z) &= 2i\log(\frac{\Gamma(z)\Pi(z) - \Delta(z)^2}{\sqrt{\Gamma(z)}}) & (40)
\end{aligned}$$

Coherent states for $SU(3)$ [38]
$SU(3)$ intelligents states on terms of harmonic oscillators [39]

A. Trimer with constant couplings

While we have provided a formal solution to propagation through the optical trimer, considering a specific solution may help build further intuition. Thus, in this section, we will discuss the special case where all the couplings are independent of the propagation distance. For the sake of simplicity, we will introduce the dimensionless propagation parameter $\zeta = g_{01}z$, such that the mode-coupling differential equation becomes

$$-i\partial_\zeta|\mathcal{E}(\zeta)\rangle = \hat{H}|\mathcal{E}(\zeta)\rangle, \tag{41}$$

with the mode-coupling matrix,

$$\hat{H} = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \beta \\ \alpha & \beta & 0 \end{pmatrix}, \tag{42}$$

given in terms of the dimensionless parameters,

$$\alpha = \frac{g_{02}}{g_{01}}, \quad \beta = \frac{g_{12}}{g_{01}}. \tag{43}$$

Under this constant mode-coupling matrix, we could use the results presented in the past section to build a particular solution, but it is well known that a set of linear first order differential equations is equivalent to a single linear differential equation of higher order. It seems worthwhile deriving such an higher order differential equation for $\Delta(\zeta)$, which is the auxiliary function that connects the two sets of differential equations, Eq.(31) and Eq.(32). After some algebra, it is possible to write,

$$\Delta'''(\zeta) + i(1 + \alpha^2 + \beta^2)\Delta'(\zeta) - 2\alpha\beta\Delta(\zeta) = 0, \tag{44}$$

with initial values,

$$\begin{aligned}
\Delta(0) &= 0, & (45)\\
\Delta'(0) &= i\beta, & (46)\\
\Delta''(0) &= -\alpha. & (47)
\end{aligned}$$

It is easy to see that $\Delta(\zeta)$ has the following solution,

$$\Delta(\zeta) = \delta_1 \, e^{i\gamma_1 \zeta} + \delta_2 \, e^{i\gamma_2 \zeta} + \delta_3 \, e^{i\gamma_3 \zeta}, \tag{48}$$

where constant parameters γ_j are the eigenvalues of the mode-coupling matrix determined by the characteristic polynomial, a reduced cubic,

$$\gamma_j^3 - (1 + \alpha^2 + \beta^2)\gamma_j - 2\alpha\beta = 0. \tag{49}$$

It is straightforward to notice that there are three different real eigenvalues for real, positive, non-zero coupling parameters, $\alpha, \beta > 0$. These proper values can be writen in a closed but non-compact form, so we will not write them explicitly. Furthermore, the coefficients are given by

$$\delta_1 = \frac{\alpha - \beta(\gamma_2 + \gamma_3)}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \tag{50}$$

$$\delta_2 = \frac{\alpha - \beta(\gamma_1 + \gamma_3)}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)}, \tag{51}$$

$$\delta_3 = \frac{\alpha - \beta(\gamma_1 + \gamma_2)}{(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)}. \tag{52}$$

The remaining auxiliary functions are straightforward to calculate,

$$\Theta(\zeta) = \frac{\beta(1+\beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha(1-\beta^2)} \tag{53}$$

$$\Gamma(\zeta) = \frac{-(1+\beta^2)\Delta(\zeta) + i\alpha\beta\Delta'(\zeta) - \Delta''(\zeta)}{\alpha(1-\beta^2)} \tag{54}$$

$$\Sigma(\zeta) = \frac{\beta(\alpha^2+\beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha^2 - \beta^2} \tag{55}$$

$$\Pi(\zeta) = \frac{-\alpha(\alpha^2+\beta^2)\Delta(\zeta) + i\beta\Delta'(\zeta) - \alpha\Delta''(\zeta)}{\alpha^2 - \beta^2}. \tag{56}$$

Thus, the propagator functions, $\iota_\pm(z)$, $\mu_\pm(z)$, $\nu_\pm(z)$, $\iota_0(z)$ and $y_0(z)$, will effectively contain terms involving the three eigenvalues as well as sums and differences thereof.

4. APPLICATIONS

Let us present some practical examples of optical trimers that may have a feasible experimental realization.

A. Identical couplings and the discrete Fourier transform

The mode-coupling matrix for three-identical waveguides distributed in an equilateral triangle configuration, $\alpha = \beta = 1$,

$$\hat{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{57}$$

is related to the cyclic group in dimension three,

$$\hat{H} = \hat{Z}_3 + \hat{Z}_3^2, \tag{58}$$

where the generator of the cyclic group,

$$\hat{Z}_3 = \hat{I}_+ + \hat{U}_+ + \hat{V}_-, = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{59}$$

It is well known that the cyclic group is diagonalized by the discrete Fourier transform, $\hat{\Lambda} = \hat{F}_n \hat{Z}_n \hat{F}_n^\dagger$, where the discrete Fourier transform of rank n is given by the operator $\hat{F}_n$, in the case of $n = 3$,

$$\hat{F}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}, \tag{60}$$

and $\hat{\Lambda}$ is a diagonal rank n matrix containing the roots of unity, $\hat{\Lambda}_{mn} = \delta_{m,n} e^{i\frac{2\pi}{n} i}$ with $m, n = 0, 1, 2$. In this particular case, it is possible to compose a propagator,

$$U(\zeta) = \hat{F}_3^\dagger e^{i\hat{\Lambda}_3 \zeta} e^{i\hat{\Lambda}_3^2 \zeta} \hat{F}_3, = \frac{1}{3} \begin{pmatrix} 2 + e^{3i\zeta} & -1 + e^{3i\zeta} & -1 + e^{3i\zeta} \\ -1 + e^{3i\zeta} & 2 + e^{3i\zeta} & -1 + e^{3i\zeta} \\ -1 + e^{3i\zeta} & -1 + e^{3i\zeta} & 2 + e^{3i\zeta} \end{pmatrix} e^{-i\zeta} \tag{61}$$

where we have used the fact that the elements of the cyclic group of rank 3 commute between them, $[\hat{Z}_3, \hat{Z}_3^2] = 0$ because $\hat{Z}_3^3 = \mathbb{1}_3$.

Figure 1 shows the trajectories described by the absolute value of the field amplitudes, $|\mathcal{E}_j(z)|$, as they propagate. All of trajectories will lie over the surface of an octant of the sphere due to unitary propagation. Figure 1(a) shows the response to impulses, $\mathcal{E}_j = \delta_{j,k}$ with $j = 0, 1, 2$ and a fixed $k = 0, 1, 2$. Figure 1(b) shows the trajectories given by initial field superpositions of the more general form: $\mathcal{E}_j = \alpha_j e^{i\phi t}$ with $\alpha \in \mathbb{R}$ and $\sum_j |\alpha_j|^2 = 1$. From the propagator, it is possible to see that only two commensurate frequencies are involved in the propagation of initial fields, thus, the trajectories will be closed and well defined.

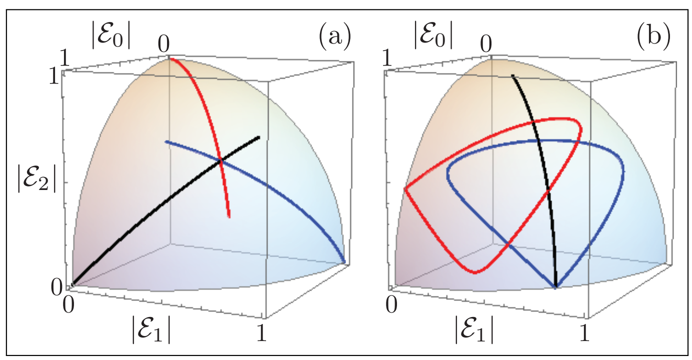

Fig. 1. (Color online) Absolute amplitude trajectories, $(|E_0(z)|, |E_1(z)|, |E_2(z)|)$, for initial fields, $(|E_0(0)|, |E_1(0)|, |E_2(0)|)$, impinging (a) only the zeroth (black), $(1, 0, 0)$, first (blue), $(0, 1, 0)$, and second (red), $(0, 0, 1)$, waveguides and (b) initial fields impinging two waveguides at a time with and without a relative phase, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ (black), $(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0)$ (blue), and $(\frac{2}{\sqrt{5}}, 0, \frac{i}{\sqrt{5}})$ (red).

B. Two identical couplings and the golden ratio

In the case of two equal coupling parameters the unitless Hamiltonian becomes

$$H = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix} \tag{62}$$

Note that the Hamiltonian is $\hat{Z}_2$-invariante, i.e. it is invariant under xchanging the first and the second waveguide. This symmetry also is reflected by the eigenvalues which are $\{-1, \bar{\varphi}, \varphi\}$ with

$$\bar{\varphi} = \frac{1}{2}\left(1 - \sqrt{8\alpha^2 + 1}\right), \quad (63)$$
$$\varphi = \frac{1}{2}\left(1 + \sqrt{8\alpha^2 + 1}\right). \quad (64)$$

It is simple to see that $\bar{\varphi} + \varphi = 1$ and the second and third eigenvalue do transform into each other upon applying the complex conjugate. Note that the latter eigenvalue assumes the golden ratio for $\alpha = \sqrt{1/2}$. Due to the $\hat{Z}_2$-symmetry the propagator can be calculated directly from the Hamiltonian and can be written as:

$$U(\zeta) = \frac{1}{\varphi - \bar{\varphi}} \begin{pmatrix} f(\zeta) & g(\zeta) & h(\zeta) \\ g(\zeta) & f(\zeta) & h(\zeta) \\ h(\zeta) & h(\zeta) & i(\zeta) \end{pmatrix} \quad (65)$$

with

$$f(\zeta) = \frac{\varphi}{2} e^{i\zeta\varphi} - \frac{\bar{\varphi}}{2} e^{i\zeta\bar{\varphi}} + \frac{1}{2}(\varphi - \bar{\varphi}) e^{-i\zeta}, \quad (66)$$
$$g(\zeta) = \frac{\varphi}{2} e^{i\zeta\varphi} - \frac{\bar{\varphi}}{2} e^{i\zeta\bar{\varphi}} - \frac{1}{2}(\varphi - \bar{\varphi}) e^{-i\zeta}, \quad (67)$$
$$h(\zeta) = \alpha\left(e^{i\zeta\varphi} - e^{i\zeta\bar{\varphi}}\right), \quad (68)$$
$$i(\zeta) = \varphi e^{i\zeta\bar{\varphi}} - \bar{\varphi} e^{i\zeta\varphi} \quad (69)$$

In operator form the propagator yields

$$U(\zeta) = \frac{g(\zeta)}{\varphi - \bar{\varphi}}(I_+ + I_-) + \frac{h(\zeta)}{\varphi - \bar{\varphi}}(U_+ + U_- + V_+ + V_-) + \frac{2}{3}\frac{f(\zeta) - i(\zeta)}{\varphi - \bar{\varphi}}(U_0 + V_0) + \frac{1}{3}\frac{i(\zeta) + 2f(\zeta)}{\varphi - \bar{\varphi}}\mathbb{1} \quad (70)$$

Obviously the three eigenvalues are stationary point of the system. However,the particular symmetry of the system might allow for further interesting states.

Looking at the propagator functions (66) to (69) reveals that two of the equations, namely $h(\zeta)$ and $i(\zeta)$ depend on only two of the three eigenfrequencies, φ and $\bar{\varphi}$. Equation (65) further reveals that the evolution of the third component only depends on those two equations $h(\zeta)$ and $i(\zeta)$. Now one can hope that if we chose the initial eigenstate accordingly one of the two eigenfrequencies in the evolution of the third component will cancel out. That is, if the evolution only depends on one frequency, it will only change in phase and the amplitude will remain immutable. Hence, lets look for such a state where no energy transfer into or out of the third component occurs. Without loss of generality we can assume that the state-vector at a given time bears following form:

$$\mathcal{E}(\zeta = 0) = \begin{pmatrix} 0 \\ \mathcal{E}_2(0) \\ \mathcal{E}_3(0) \end{pmatrix} \quad (71)$$

Which implies that the third component evolves as:

$$(\varphi - \bar{\varphi})\mathcal{E}_3(\zeta) = h(\zeta)\mathcal{E}_2(0) + i(\zeta)\mathcal{E}_3(0) \quad (72)$$

As argued above the third waveguide will not change in amplitude if one of the two eigenfrequencies cancel out. This is fulfilled if either of the two following conditions is met:

$$\mathcal{E}_2(0) = \mathcal{E}_3(0)\frac{\varphi}{\alpha} \quad or \quad \mathcal{E}_2(0) = \mathcal{E}_3(0)\frac{\bar{\varphi}}{\alpha} \quad (73)$$

Consequently we obtain two solutions for a state with stationary amplitude in the third waveguide. Without loss of generality we can normalise the initial state such that the amplitude of the third component is one, ie $\mathcal{E}_3(0) = 1$. Then the time evolution of the system is given by:

$$\mathcal{E}_\varphi(\zeta) = \begin{pmatrix} \frac{g(\zeta)}{\varphi - \bar{\varphi}}\frac{\varphi}{\alpha} + \frac{h(\zeta)}{\varphi - \bar{\varphi}} \\ \frac{f(\zeta)}{\varphi - \bar{\varphi}}\frac{\varphi}{\alpha} + \frac{h(\zeta)}{\varphi - \bar{\varphi}} \\ e^{i\zeta\varphi} \end{pmatrix} \quad (74)$$

and

$$\mathcal{E}_{\bar{\varphi}}(\zeta) = \begin{pmatrix} \frac{g(\zeta)}{\varphi - \bar{\varphi}}\frac{\bar{\varphi}}{\alpha} + \frac{h(\zeta)}{\varphi - \bar{\varphi}} \\ \frac{f(\zeta)}{\varphi - \bar{\varphi}}\frac{\bar{\varphi}}{\alpha} + \frac{h(\zeta)}{\varphi - \bar{\varphi}} \\ e^{i\zeta\bar{\varphi}} \end{pmatrix} \quad (75)$$

5. CLASSICAL MECHANICS AND ENVELOPES

An lossless waveguide is an optical cavity, where In quantum mechanics, the Hamiltonian for three coupled harmonic oscillators,

$$\hat{H}(t) = \sum_{j=0}^{2} \omega_j(t)\hat{a}_j^\dagger \hat{a}_j + \sum_{j \neq k=0}^{2} g_{jk}(t)\hat{a}_j^\dagger \hat{a}_k, \quad (76)$$

in the single photon limit,

$$|\psi(t)\rangle = \mathcal{E}_0(t)|1,0,0\rangle + \mathcal{E}_1(t)|0,1,0\rangle + \mathcal{E}_2(t)|0,0,1\rangle, \quad (77)$$

with $\sum_j |\mathcal{E}_j(z)|^2 = 1$, yields a differential equation set equivalent to the mode coupling theory result, Eq.(44) up to substitution of backwards time by propagation distance, or, better,

$$i\partial_t \begin{pmatrix} \mathcal{E}_0(t) \\ \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{pmatrix} = \begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} \mathcal{E}_0(t) \\ \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{pmatrix}. \quad (78)$$

This is just off-topic, we really will focus on this in the next section, right now we want to focus on something else.

We can use quadratures,

$$\hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right), \; \hat{p} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right), \quad (79)$$

and consider classical variables, where we have accounted for the change in the time propagation,

$$i\partial_t \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix}, \quad (80)$$

$$i\partial_t \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix} = - \begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{pmatrix}, \quad (81)$$

This can also be done with the idea of Euler angles for $SU(3)$ in [37]

In the classical limit, the canonical pair provided by the creation and annihilation operators can be replaced by the classical canonical pair of intensity and phase, $\left\{n_j, \phi_j\right\}$, $\hat{a}_j \rightarrow \sqrt{n_j} e^{i\phi_j}$. This delivers a classical Hamiltonian,

$$H(t) = \sum_{j=0}^{2} \omega_j(t) n_j + \sum_{j\neq k=0}^{2} g_{jk} \sqrt{n_j n_k} \cos\left(\phi_j - \phi_k\right), \quad (82)$$

the equations of motion for the canonical pairs are

$$\partial_t n_j = \frac{\partial H}{\partial \phi_j} \quad (83)$$

$$\partial_t \phi_j = -\frac{\partial H}{\partial n_j} \quad (84)$$

and these are the coupled mode equations describing the evolution of intensity and phase.

Classicam mechanics: if the three normal modes frequencies, e.g. eigenvalues of the mode-coupling matrix in Eq.(9), are commesurate then the propagated complex fields will be periodic [40].

In order to visualize the dynamics of the system we will choose a (pseudo?) Poincaré phase-space given by the square roots of the three waveguide intensities $(|\mathcal{E}_0|, |\mathcal{E}_1|, |\mathcal{E}_2|)$. Here normal mode frequencies that are rational multiples of each other, commesurate, will translate into well-defined closed trajectories. For incommesurate normal frequencies the trajectories will be ergodic and fill a region of phase space defined by the energy of the motion [40].

Note that requiring a normalized intensity in the optical system translates into $\sum_j n_j = 1$, this obviates the use of $\partial_t n_2$ and suggest taking the phase at the $j = 2$ as reference as $\partial_t \phi_2 = 0$. This reduces the problem by one degree of freedom to two degrees of freedom.

Numerically, it makes no difference to solve Eq.(9) or use Eq.(11)-Eq.(12). Figure 2 shows the squared root intensities plot for the numerical propagation of random parameter sets and initial conditions. The absolute value of the difference between the numerical propagation from the complex field or the intensity-phase picture was always below 10^{-5}. We are ploting the absolute amplitudes $(|E_0(z)|, |E_1(z)|, |E_2(z)|)$, thus the trajectories will are on an octant of the unit sphere.

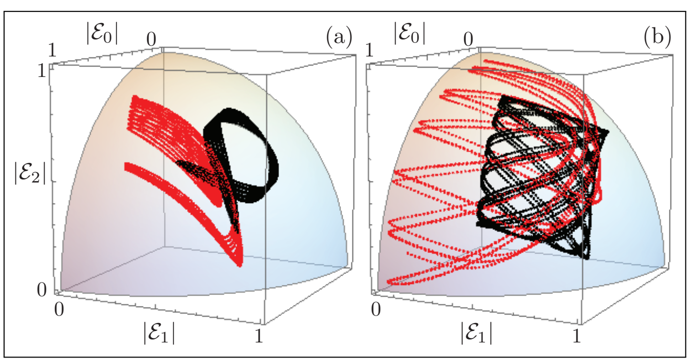

Fig. 2. (Color online) Absolute amplitude trajectories, $(|E_0(z)|, |E_1(z)|, |E_2(z)|)$, for initial random fields impinging random optical trimmers with constant parameters.

Dealing with the problem in the intensity-phase picture does not make it easier to solve but it provides insight. Given a trimmer parameter set, $S(z) = \left\{\omega_j(z), g_{jk}(z)\right\}$, and an initial field configuration, $\mathcal{E}(0) = \left\{\mathcal{E}_j(0)\right\}$, it is possible to calculate the energy of the orbit, $H(0) = \epsilon\left(S(0), \mathcal{E}(\prime)\right)$. Then, it is possible to find the conditions that define the envelope of such orbit in the absolute value of the field amplitude plots via simultaneous elimination of the phase variables with $H(z) - \epsilon = 0$ and $\partial_t \phi_j = 0$ for $j = 0, 1$.

6. ENVELOPE

We revisit the classical Hamiltonian for that governs the equations of motion for the canonical pair of intentsity and phase:

$$H = \sum_{j=0}^{2} \omega_j(t) n_j + \sum_{j\neq k=0}^{2} g_{jk} \sqrt{n_j n_k} \cos\left(\phi_j - \phi_k\right), \quad (85)$$

It proves viabable to notice that for a given Energy and given set of parameters there is an infinity number of trajectories. Here we will first be concerned with finding the outer boundary, or envelope, of the union of all trajectories pertaining to Energy E. Equation ([**?**]) reveals that for given intensities n_1, n_2 and n_3 the energy is maximised when all three phases are zero. Consequently the envelope is defined by all pairs n_1, n_2 and n_3 which fulfil following condition

$$H_{Bndry}(n_1, n_2, n_3) := \sum_{j=0}^{2} \omega_j(t) n_j + \sum_{j\neq k=0}^{2} g_{jk} \sqrt{n_j n_k} = E, \quad (86)$$

Due to energy conservation, the three intensities are not independent and we can safely omit the third intensity in the following discussion, ie. $H_{Bndry}(n_1, n_2, n_3) \rightarrow H_{Bndry}(n_1, n_2)$. Numerically, once a given set of initial intensities and/or energy was chosen $H_{Bndry}(n_1, n_2) \rightarrow E$ the trajectory of the envelope is characterised by the following differential equation:

$$\frac{\partial H_{Bndry}}{\partial n_1} \delta n_1 + \frac{\partial H_{Bndry}}{\partial n_2} \delta n_2 = 0 \quad (87)$$

Having obtained the envelope of the union of all trajectories we can turn the case of single trajectories. We recall that we focus our interest on parameters such that the corresponding trajectories are incommensurate, ie. they fill all the area inside a closed loop. To find the envelope of a certain trajectory, that is the aforementioned loop, we assume we already have obtained the explicit solution for the trajectory. Our solution is most easily expressed as solution to equation () which is given by:

$$\mathcal{E} = a_1 e_1 e^{i\omega_1 t} + a_2 e_2 e^{i\omega_2 t} + a_3 e_3 e^{i\omega_3 t} \quad (88)$$

where are e_1, e_2, e_3 are the eigenvectors, $\omega_1, \omega_2, \omega_3$ are the corresponding eigenfrequencies and $a1, a2, a3$ are parameters in accordance with the initial values n_1 and n_2. Note that $n_j = \mathcal{E}\mathcal{E}^*$ The following is a bit handwaving, we probably need to explain it better Incommensurate trajectories are characterised by broken periodicity and we can introduce a phase relevant parameter which controls the incommensurability:

$$\mathcal{E}(t, \phi) = a_1 e_1 e^{i\omega_1 t + \phi} + a_2 e_2 e^{i\omega_2 t} + a_3 e_3 e^{i\omega_3 t} \quad (89)$$

The parameter ϕ acts such that a given trajectory or more presily a section of the trajectory is offset in a direction pernedicular to the trajectory. This howerver can not hold true in at the boundery or envelope and hence we establish following condition for the envelope:

$$\frac{\left(\frac{\partial n_1}{\partial t}\right)}{\left(\frac{\partial n_2}{\partial t}\right)} = \frac{\left(\frac{\partial n_1}{\partial \phi}\right)}{\left(\frac{\partial n_2}{\partial \phi}\right)} \tag{90}$$

which is equal to:

$$D(t,\phi) := Det[\begin{pmatrix} \frac{\partial n_1}{\partial t} & \frac{\partial n_2}{\partial t} \\ \frac{\partial n_1}{\partial \phi} & \frac{\partial n_2}{\partial \phi} \end{pmatrix}] = 0 \tag{91}$$

Accordingly, the envelope can numerically calculated from following differential equation:

$$\frac{\partial D(t,\phi)}{\partial t}\delta t + \frac{\partial D(t,\phi)}{\partial \phi}\delta\phi = 0 \tag{92}$$

7. CONCLUSION

We have shown that it is possible to solve the light evolution equations in a coupled three-core waveguide, in terms of the Lie group generators of su(3). We focused our attention on a reduced class of structures where the coupling constants are constant in position......... As an example we consider the case of equal coupling constants. we found that the dynamics of such a system i governed by a linear third order differential equation and the corresponding five auxiliary functions can be expressed in term of its solution. we also established a connection between a waveguide cluster with equal couplings and the well known Fourier transform, possibly opening a path to realise quantum Fourier transformations. furthermore we studies an isosceles waveguide triangle. We showed that two equal couplings corresponds to a Z2 symmetry of the Hamiltonian. We also showed that the symmetry allows for two interesting stated characterised by absence of energy transfer into the third core.

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