Understanding of Dynamical System by Time Series Analysis

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Certificate

This is to certify that the project entitled *Understanding of Dynamical System by Time Series Analysis* submitted by Biswajit Mohanty to P.G. Department of Physics, Utkal University under my supervision and I consider it worthy of consideration for M.Sc. semester examination in Physics department.

Dr. Jagadish Kumar

Date:

Declaration

I certify that

The work contained in the project is original and has been done by myself under the general supervision of my supervisor. The work has not been submitted to any other Institute for any degree. I have followed the guidelines provided by the Institute in writing the project.

Biswajit Mohanty

Acknowledgment

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Biswajit Mohanty

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Overview

What do you mean by dynamical system? A system which evolves with time and depends on the initial state. Dynamical systems are mathematical objects used to model physical phenomena whose state changes over time. These models are used in financial and economic forecasting, environmental modeling, medical diagnosis, industrial equipment diagnosis; and a host of other applications. It can be either linear or nonlinear. Linear systems typically exhibit features and properties that are much simpler than the general, nonlinear case. So one of the best example of nonlinear system is study of evolution of double pendulum. The phase-space orbit of this system spirals in to origin, as the motion damped away towards. Due the driving force given the pendulum is pushed away from origin. And some strange happens, if we plot the time series of double pendulum. Here u can see the irregular motion but not chaotic because the motion is still predictable after some initial transient motion. But for large motion it is chaotic and for small motions it is a simple linear system. The dictionary meaning of chaos is associated with disorder and unpredictability. But the mathematical definition of chaotic motion is sensitivity to initial conditions that means a small change in initial condition leads to large change in the results. Here also a small change in initial condition i.e initial parameters leads large change in trajectory. But, after a certain amount of time, we lose essentially all predictive power. Such systems are known as chaotic. In other words chaos is the long term unpredictability of deterministic, nonlinear dynamical systems due to sensitivity to initial condition can be observed in the time series.

What do u mean by time series- It is a sequence of data points, typically consisting of successive measurements made over a time interval. A question may arises in your mind Why we prefer time series analysis to describe a nonlinear system? Because time series analysis comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data. Time series forecasting is also useful for a model to predict future values based on previously observed values. In time series analysis we are analyzing one variable let's say 'y' over definite time intervals, in fact we are collecting data very regularly with time points which could be centuries for collecting geological data, every seconds if you are collecting some biological data. If the data is regular and periodic then linear method may explain all regular structures. If the data is irregular then nonlinear time series analysis is essential to study the behavior of the system.

Then comes to the model on which i experimented i.e Lorenz model. The Lorenz system is a system of ordinary differential equations rst studied by Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. Lorenz was running a computer program that modeled weather patterns and created forecasts. When he noticed, by chance, that the system he had programmed was severely sensitive to initial conditions. He accidentally changed an initial condition by what could be compared to the effect of butterfly's wings on the global weather pattern. He then saw his system diverge slowly from what it had initially been until it bore (passage or tunnel) no resemblance to the original at all. Lorenz was fascinated with the phenomenon. A while later, while attempting to model the convection patterns of a fluid being heated in a container, Lorenz produced the first ever chaotic dynamical system. He stripped Navier-Stokes equations down to their bones and came up with the following system of first order differential equations:Lorenz was fascinated with the phenomenon. A while later, while attempting to model the convection patterns of a fluid being heated in a container, Lorenz produced the first ever chaotic dynamical system. He stripped Navier-Stokes equations down to their bones and came up with the following system of first order differential equations. The three equations are so codependent that their trajectories orbit back and forth (to and fro motion) between two centers but never cross. That's Why it is called as Lorenz attractor. Such properties (combined with the sensitivity to initial conditions) are what make systems chaotic. Lorenz water wheel A Lorenzian (or "chaotic") waterwheel is a physical model that perfectly corresponds to the Lorenz equations. A chaotic waterwheel is just like a normal waterwheel except for the facts that he buckets leak. Water pours into the top bucket at a steady rate and gives the system energy while water leaks out of each bucket at a steady rate and removes energy from the system. If the parameters of the wheel are set correctly, the wheel will exhibit chaotic motion: rather than spinning in one direction at a constant speed, the wheel will speed up, slow down, stop, change directions, and oscillate back and forth between combinations of behaviors in an unpredictable manner. But due time restriction I cant explain Lorenz water wheel in details, if u want to know about this u can ask me after the end of seminar.

In this study, the classical fourth-order Runge-Kutta method is modified to obtain new methods which are of order five. The methods are tested on the Lorenz system which involves the chaotic and nonchaotic characteristics.

The modified fifth-order methods using arithmetic mean are compared with another method with the same order. Comparisons between the methods are made in two step sizes and the accuracy of all the methods are discussed, ode45() the best function for Lorenz system but the results are very poor with RK4 even with one million sub-interval. Mathematica and Matlab software's are used to solve the Lorenz system and sketch the graphical solutions of the chaotic and non-chaotic system.

TISEAN is a software project for the analysis of time series with methods based on the theory of nonlinear deterministic dynamical systems, or chaos theory.You may use and refer to the programs as you do with any scientific publication. Tisean described the implementation of methods of nonlinear time series analysis which are based on the paradigm of deterministic chaos. A variety of algorithms for data representation, prediction, noise reduction, dimension and Lyapunov estimation, and nonlinearity testing are discussed with particular emphasis on issues of implementation and choice of parameters. Computer programs that implement the resulting strategies are publicly available as the TISEAN software package. The use of each algorithm will be illustrated with a typical application.

Chapter 1

Introduction

1.1 Dynamical System

In mathematics, a dynamical system is a set of relationships among two or more measurable quantities, in which a fixed rule describes how the quantities evolve over time in response to their own values. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe.

At any given time a dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold). The evolution rule of the dynamical system is a function that describes what future states follow from the current state. Often the function is deterministic; in other words, for a given time interval only one future state follows from the current state; however, some systems are stochastic, in that random events also affect the evolution of the state variables. A means of describing how one state develops into another state over the course of time. Technically, a dynamical system is a smooth action of the reals or the integers on another object (usually a manifold). When the reals are acting, the system is called a continuous dynamical system, and when the integers are acting, the system is called a discrete dynamical system.

So, as far as we are concerned, the characteristics of dynamical systems are the characteristics of mathematical models, e.g., linear, nonlinear, deterministic, stochastic, discrete, continuous. One of the most important characteristics of dynamical systems concerns the notion of observability.

The concept of a dynamical system has its origins in Newtonian mechanics. The evolution rule of dynamical systems is an implicit relation that gives the state of the system for only a short time into the future. (The relation is either a differential equation, difference equation or other time scale.) To determine the state for all future times requires iterating the relation many timeseach advancing time a small step. The iteration procedure is referred to as solving the system or integrating the system. If the system can be solved, given an initial point it is possible to determine all its future positions, a collection of points known as a trajectory or orbit.

Before the advent of computers, finding an orbit required sophisticated mathematical techniques and could be accomplished only for a small class of dynamical systems. Numerical methods implemented on electronic computing machines have simplified the task of determining the orbits of a dynamical system. For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated to be understood in terms of individual trajectories. The difficulties arise because:

(i) The systems studied may only be known approximatelythe parameters of the system may not be known precisely or terms may be missing from the equations. The approximations used bring into question the validity of numerical solutions. To address these questions several notions of stability have been introduced in the study of dynamical systems, such as Lyapunov stability or structural stability. The stability of the dynamical system implies that there is a class of models or initial conditions for which the trajectories would be equivalent. The operation for comparing orbits to establish their equivalence changes with the different notions of stability.

(ii) The type of trajectory may be more important than one particular trajectory. Some trajectories may be periodic, whereas others may wander through many different states of the system. Applications often require enumerating these classes or maintaining the system within one class. Classifying all possible trajectories has led to the qualitative study of dynamical systems, that is, properties that do not change under coordinate changes. Linear dynamical systems and systems that have two numbers describing a state are examples of dynamical systems where the possible classes of orbits are understood.

(iii) The behavior of trajectories as a function of a parameter may be what is needed for an application. As a parameter is varied, the dynamical systems may have bifurcation points where the qualitative behavior of the dynamical system changes. For example, it may go from having only periodic motions to apparently erratic behavior, as in the transition to turbulence of a fluid.

(iv) The trajectories of the system may appear erratic, as if random. In these cases it may be necessary to compute averages using one very long trajectory or many different trajectories. The averages are well defined for ergodic systems and a more detailed understanding has been worked out for hyperbolic systems.

Why do we care about dynamical systems?

Dynamical systems are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time. These models are used in financial and economic forecasting, environmental modeling, medical diagnosis, industrial equipment diagnosis; and a host of other applications.

For the most part, applications fall into three broad categories: predictive (also referred to as generative), in which the objective is to predict future states of the system from observations of the past and present states of the system, diagnostic, in which the objective is to infer what possible past states of the system might have led to the present state of the system (or observations leading up to the present state) and finally, applications in which the objective is neither to predict the future nor explain the past but rather to provide a theory for the physical phenomena These three categories correspond roughly to the need to predict, explain, and understand physical phenomena.

Not all physical phenomena can be easily predicted or diagnosed. Some phenomena appear to be highly stochastic in the sense that the evolution of the system state appears to be governed by influences similar to those governing the role of dice or the decay of radioactive material. Other phenomena may be deterministic but the equations governing their behavior are so complicated or so critically dependent on accurate observations of the state that accurate long-term observations are practically impossible.

Some of the good examples of Dyanmical Sytsems are: Logistic map, Tent map, Double pendulum, Arnolds cat map, Horseshoe map, Bakers map is an example of a chaotic piecewise linear map, Billiards and outer billiards, Hnon map, Lorenz system, Circle map, Rssler map, Kaplan-Yorke map, List of chaotic maps, Swinging Atwoods machine.

1.2 Time Series Analysis:

A time series is a sequence of data points, typically consisting of successive measurements made over a time interval. Examples of time series are ocean tides, counts of sunspots. Time series are very frequently plotted via line charts. Time series are used in statistics, signal processing, pattern recognition, econometrics, mathematical finance, weather forecasting, intelligent transport and trajectory forecasting, earthquake prediction, electroencephalography, control engineering, astronomy, communications engineering, and largely in any domain of applied science and engineering which involves temporal measurements.

Time series analysis comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data. Time series forecasting is the use of a model to predict future values based on previously observed values.

Time series data have a natural temporal ordering. Time series analysis is also distinct from spatial data analysis where the observations typically relate to geographical locations. A stochastic model for a time series will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In addition, time series models will often make use of the natural one-way ordering of time so that values for a given period will be expressed as deriving in some way from past values, rather than from future values.

Time series analysis can be applied to real-valued, continuous data, discrete numeric data, or discrete symbolic data (i.e. sequences of characters, such as letters and words in the English language).

In time series analysis we are analyzing one variable let's say 'y' over definite time intervals, in fact we are collecting data very regularly with time points which could be centuries for collecting geological data, every seconds if you are collecting some biological data, it can be every quarter if you are collecting economical data and the y variable can be anything-the price of the stock, economic rate, your blood pressure, geological data.(give some graphical examples); Time series analysis is a method that is used to understand the underlying mechanism generating the data points. If the data is regular and periodic then linear method may explain all regular structures. If the data is irregular then nonlinear time series method is essential to study the behavior of the system.

My focus is now to study the behavior of nonlinear systems, so before i talk about nonlinear system we need to first have some understanding of the general concept of what a system defines and some idea about linear system.

1.3 Linear System

A system you may define as an arrangement of things or a group of related thing work towards a common goal or a group of related parts that work together or a set of parts that are interconnected in resulting some joint outcome.(give some examples with fig). In the world of science there fundamentally two different type of systems: what are called linear system and nonlinear system. Now i will talk about the each separately and the distinction between them. Firstly i'm going to give some basic understanding of what a linear system is?

A linear system is a mathematical model of a system based on the use of a linear operator. Linear systems typically exhibit features and properties that are much simpler than the general, nonlinear case. As a mathematical abstraction or idealization, linear systems find important applications in automatic control theory, signal processing, and telecommunications. For example, the propagation medium for wireless communication systems can often be modeled by linear systems.

They are defined by superposition principles,tha is homogeneity and additive principle. Additive principle states that we can add the effect or output of two systems together and the resulting combined system will be nothing more but the simple combination of each systems output in isolation. Whereas homogeneity principle states the output to a linear system is always directly proportional to the input, so if we put twice as much into the system it will return twice as much in output. Now i am going to explain what a nonlinear system is and how it differs from linear model?

1.4 Nonlinear System

In physics and other sciences, a nonlinear system, in contrast to a linear system, is a system which does not satisfy the superposition principle meaning that the output of a nonlinear system is not directly proportional to the input.

Nonlinear systems describes the vast majority phenomena in our world.

They have unfortunately being designated as alternatives being defined by what they are not. Let's start a discussion by asking why this is so?

The real world of living is inherently complex and nonlinear. But for scientific perspective all we have is our models has enables to started simple developed to become more complex and sophisticated representation. The simple in this case means things that are the product of direct cause and effect interactions. The century of science and mathematics has been focused upon the simple linear interaction and geometric forms. They are the easiest phenomena to describe. It is only in past few decades the scientists had begun to approach the world of systems that are nonlinear.

With nonlinear phenomena the principle of homogeneity and additive breaks down. Let us take a close look why this is so. The additive principle breaks down in nonlinear systems because the way we put things together and the type of things put together matters the effect and the interaction to make the overall product of the component combination more or less than a simple additive function,thus defies the non- additive principle, we call it as nonlinear. Non-linearity arises from the non-additive nature to the interaction between things when combine them. Then come to homogeneity principle, the linear model does not gives feedback loops i.e inputs and outputs appears and disappears without any relation between them. This means as soon as we deal with real world the things start get nonlinear. The more interaction we incorporate into our model thus making the more robust and realistic, the more nonlinear are likely the things become. So superposition principle breaks down and nonlinearity arises whenever we are taking into account the nature of the interaction with in a system.

1.5 Nonlinear Time Series Analysis

Nonlinear time series analysis is a practical spin off(incidental result) from complex dynamical systems theory and chaos theory. It allows one to characterize dynamical systems in which non-linearity give rise to a complex temporal evolution. Importantly, this concept allows extracting information that cannot be resolved using classical linear techniques such as the power spectrum or spectral coherence. In this chapter, we show how a nonlinear prediction error can be used to attempt to distinguish between purely stochastic, purely deterministic, and deterministic dynamics superimposed with noise. The framework of nonlinear time series analysis comprises a wide variety of measures that allow one to extract different characteristic features of a dynamical system underlying some measured signal. These include the correlation dimension as an estimate of the number of independent degrees of freedom, the lyapunov exponent as a measure for the divergence of similar system states in time, prediction errors as detectors for characteristic traits of deterministic dynamics, or different information theory measures. The nonlinear time series measures are uni-variate, i.e., they are applied to single signals measured from individual dynamics. In contrast, bi-variate measures are used to analyze pairs of signals measured simultaneously from two dynamics. Such bi-variate time series analysis measures aim to distinguish whether the two dynamics are independent or interacting through some coupling. Some of these bi-variate measures aim to extract not only the strength, but also the direction of these couplings

Importance Of Nonlinear Time series analysis

Nonlinear time series analysis is becoming a more and more reliable tool for the study of complicated dynamics from measurements. The concept of deterministic low-dimensional chaos has proven to be fruitful in the understanding of many complex phenomena, but even where processes are obviously non-deterministic nonlinear methods allow for improved data analysis. Despite wide interest from many disciplines, many potential users of nonlinear time series analysis are not sufficiently familiar with nonlinear methods of data analysis as to guarantee a successful application of these methods.It allows one to determine the number of degrees of freedom involved for that nonlinear system.

1.6 Chaos Theory

Simple nonlinear dynamical systems and even piece-wise linear systems can exhibit a completely unpredictable behavior, which might seem to be random, despite the fact that they are fundamentally deterministic. This seemingly unpredictable behavior has been called chaos. This branch of mathematics deals with the long-term qualitative behavior of dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system (which is often hopeless), but rather to answer questions like Will the system settle down to a steady state in the long term, and if so, what are the possible attractors? or Does the long-term behavior of the system depend on its initial condition? Note that the chaotic behavior of complex systems is not the issue. Chaos theory has been so surprising because chaos can be found within almost trivial systems. The logistic map is only a second-degree polynomial; the horseshoe map is piece-wise linear.

Indeed the most direct link between chaos theory and the real world is through the analysis of time series data in terms of nonlinear dynamics. Although this has been known to physicists since the beginning of the 20th century, its importance has only become clear in the context of modern meteorology and the computer simulation of weather patterns. The sensitive dependence of chaotic systems upon their initial conditions has become known to the general public in the butterfly effect: in principle, the beat of a butterfly's wing can set off a hurricane at another place on the globe. Chaos theory has found applications in very different contexts and disciplines, such as in astronomy (the movement of planets), physics (turbulence), chemistry (the BelousovZhabotinsky reaction), biology (prey-and-predator systems, neural networks), meteorology, medicine (cardiology), sociology (traffic flow), economics (finance markets).

Chaos theory is concerned with non-linear systems, the dynamics of which depend extremely sensitively upon the starting conditions. The predictability of such systems is highly restricted, even in cases where the system's development is subject to strict laws or regularities. This is because the smallest fluctuations in the initial conditions can be reinforced exponentially, while the initial conditions themselves can only be determined experimentally with a finite degree of accuracy, so that fundamental limitations are placed upon the predictability of such systems. All dynamic systems with more than two degrees of freedom can display chaotic behavior.Chaos theory stated that such irregular signals can arise from purely deterministic dynamics.In experiments, uncertainty in trajectories in case of chaotic system arises due to variety of reasons. Indeed, most experimental signals from chaotic systems are actually corrupted by noise. The main sources of noise may be due to the lack of control over external or internal parameters.So there may be limitation in the accuracy of measurement both in terms of the frequency of measurement and the precision in sampling the data.(Sampling allows us to work with a small, manageable amount of data in order to build and run analytic models more quickly. Sampling can be particularly useful with data sets that are too large to efficiently analyze in full.In some cases, a very small sample can tell all of the most important information about a data set).It is therefore necessary to measure the extent of the component(degrees of freedom) of the signals arising from intrinsic non-linearity as against that due to random noise. There are several methods for distinguishing a chaotic signal from a stochastic one. Some of them are rather simplistic but are quite useful in giving a clue about the possibility of chaos. Such methods are easy to implement or to visualize in terms of plots.Often, using one quantitative estimate may not be adequate. For this reason, it is frequently necessary to use several complimentary methods till one can have high level of belief in the end results of the analysis.There are methods which allow elimination of the noise component. One method is called the singular value decomposition method,this method, in addition to curing the time series of noise, also provides an estimate of the dimension of the attractor as well as its visualization. Furthermore, there is a issue that the time series are invariably short. Thus it requires modifications of the methods developed, for ideal systems for which the length of the time series has no limitation.

Chaotic maps

In mathematics, a chaotic map is a map (= evolution function) that exhibits some sort of chaotic behavior. Maps may be parameterized by a discrete-time or a continuous-time parameter. Discrete maps usually take the form of iterated functions. Chaotic maps often occur in the study of dynamical systems.

Chaotic maps often generate fractals. Although a fractal may be constructed by an iterative procedure, some fractals are studied in and of themselves, as sets rather than in terms of the map that generates them. This is often because there are several different iterative procedures to generate the same fractal. A list of chaotic maps are Logistic map, Tent map, Arnolds cat map, Horseshoe map, Bakers map, Hnon map, Lorenz attractor etc.

Chapter 2

Background Materials On Nonlinear Dynamical Methods

Next,we will discuss the tools necessary to identify and characterize the chaotic system

2.1 Visual Inspection

Visual inspection of the data in many instance gives hint about the nature of the dynamics. Very often experimental data contains some fault which can be detected by visual inspection. A plot of the signal as a function of time gives the first hint of possible presence or absence of stationarity, drifts, systematically varying amplitudes or time scales and the presence of rare events. Enlarged view of different portions of the time series can also give a hint about the nature of time series. Visual inspection of the data can also reveal symmetries in the data or can guide us to more useful representation of the data. It also allows us to select parts of the series that appear more stationary (search more on it in tisean or google). Visual inspection can also reveal symmetries in the data or can guide us to a more useful representation of the data. Sometimes one finds exact symmetries in the data, e.g., under change of sign of the observable. In this case, one can enlarge the data base for a purely geometric analysis by just replicating every data point by the symmetry operation. For an analysis of the dynamics one can apply the symmetry operation to the time series as a whole.

2.2 Stationarity

A time series is called stationary if its statistical properties such as mean, variance, autocorrelation are constant over time. For instance, given a time series $X(t_i), i = 1, 2, 3, \ldots,$ the n point correlation function is invariant with respect to time translation i.e.,

$$< X(t_1 + \tau)X(t_2 + \tau) \ldots X(t_n + \tau) >=< X(t_1)X(t_2) \ldots X(t_n) > \quad (1.0)$$

for all n, all t and all $t_1, t_2, \ldots, t_n$. Here $< \cdots >$ represents the average value (time or ensemble average). This is called strict sense stationarity. However, stationary processes as defined by Equation-2.22 is never used in practice and is discussed only for their mathematical properties. For most practical applications one uses a much simpler expression that involves only two points, i.e. two point correlation functions. If $< X >, < X(t)X(t + \tau) >$ and the variance $\sigma^2 = < X^2 > - < X >^2$ are independent of time, then the time series is stationary in respect to statistical characteristics. The first requirement for stationarity of a time series is that it is measured over a time period much longer than the longest characteristic time scale of the system under consideration. Usual method adopted is to calculate the running variance of the time series. For a stationary time series the running variance should remain constant within the acceptable error limits for the entire time series. Stationarity can also be tested by calculating the statistical quantities like, transition probabilities, correlations etc. for different sections of the time series. Another working principle is to calculate correlation dimension for different section of the time series if the length of the time series is long and if the results remain the same within the acceptable error limit, one can be certain about the stationarity of the time series.

Stationarity requires that all parameters that are relevant(closely connected) for a systems dynamics have to be fixed and constant during the measurement period (and these parameters should be the same when the experiment is reproduced). This is a requirement to be fulfilled not only by the experimental setup but also by the process taking place in this fixed environment. For the moment this might be puzzling since one usually expects that constant external parameters induce a stationary process. If the process under observation is a probabilistic one, it will be characterized by probability distributions for the variables involved. For a stationary process, these probabilities may not depend on time. If the calibration(correlate the readings with standard data) of the measurement apparatus drifts, for example, we can try to rescale the data continuously in order to keep the mean and variance constant. we are sure that only the measurement scale and not the dynamics is drifting. A strictly periodic modulation of a parameter can be interpreted as a dynamical variable rather than a parameter and does not necessarily destroy stationarity.

Unfortunately, in most cases we do not have direct access to the system which produces a signal and we cannot establish evidence that its parameters arc indeed constant. Thus we have to formulate a second concept of stationarity which is based on the available data itself. This concept has to be different since there are many processes which are formally stationary when the limit of infinitely long observation times can be taken but which behave effectively like non-stationary processes when studied over finite times. A prominent phenomenon belonging to this class is called intermittency.

Test of stationarity

After such emphasis has been put on the stationarity problem we have to address the question how non-stationarity can be detected for a given data set. Obviously stationarity is a property which can never be positively established. It turns out that the matter is even more complicated because the stationarity requirement differs depending on the application. As a first requirement, the time series should cover a stretch of time which is much longer than the longest characteristic time scale that is relevant for the evolution of the system. For instance, the concentration of sugar in the blood of a human is driven by the consumption of food and thus roughly follows a 24 hour cycle. If this quantity is recorded over 24 hours or less, the process must be considered non-stationary no matter how many data points have been taken during that time. A time series can be considered stationary only on much larger time scales.

2.3 Auto-Correlation

The correlation of a variable with itself over successive time intervals is called autocorrelation. This is defined as,

$$C(\tau) = \frac{[< x_i \cdot x_{i+\tau} > \ < xi >^2]}{\sigma^2} \qquad (1.1)$$

where $\tau = n\delta t$ is in units of the time interval of measurement δt and $\sigma^2 = [< x_i^2 > - < x_i >^2]$ is the variance of the time sequence. $C(\tau)$ gives the information about how much the variable x at a given time is correlated to its value at a later time. If the time series is periodic, then $C(\tau)$ will be periodic in τ . For a stochastic time series, $C(\tau)$ decays rapidly with the rate of decay depending on the process. For chaotic systems, it is known that $C(\tau)$ decays exponentially with oscillations at longer times. Obviously, C(0)= 1, and for some $\tau = \tau^*$, $C(\tau)$=0. Thus, at τ^*, x_i and $x_{i+\tau^*}$ become statistically independent. However, the autocorrelation alone may not be able to distinguish chaotic signals from stochastic noise.

The estimation of the autocorrelations from a time series is straightforward as long as the lag τ is small compared to the total length of the time series. Therefore estimates $C(\tau)$ are only reasonable for $\tau < N$. If we plot values x_i versus the corresponding values a fixed lag τ earlier $x_{i+\tau}$ the autocorrelation $C(\tau)$ quantifies how these points are distributed. If they spread out evenly over the plane, then $C(\tau) = 0$. If they tend to crowd (move too close to other pts.) along the diagonal $x_i = x_{i+\tau}$, then $C > 0$, and if they are closer to the line $x_i = -x_{i+\tau}$, we have $c < 0$. The latter two cases reflect some tendency of x_i and $x_{i+\tau}$ to be proportional to each other, which makes it possible that the autocorrelation function reflects only linear correlations. If the signal is observed over continuous time, one can introduce the autocorrelation function $C(\tau)$. and the correlations of Eq.(1.1) are estimates of $C(\tau = \tau\delta t)$. Obviously, $C(\tau) = C(-\tau)$ and C(0)=1. Obviously if a signal is periodic in time, then the autocorrelation function is periodic in the lag τ. Stochastic processes have decaying autocorrelations but the rate of decay depends on the properties of the process. Autocorrelations of signals from deterministic chaotic systems typically also decay exponentially with increasing lag. Autocorrelations are not characteristic enough to distinguish random from deterministic chaotic signals.Instead of describing the statistical properties of a signal in real space one can ask about its properties in Fourier space. The Fourier transform establishes a one-to-one correspondence between the signal at certain times (time domain) and how certain frequencies contribute to the signal and how the phases of the oscillations are related to the phases of other oscillations (frequency domain).

2.4 Phase Space

The nonlinear time series methods based on the theory of dynamical systems; that is, the time evolution is defined in some phase space. Since such nonlinear systems can exhibit deterministic chaos, this is a natural starting point when irregularity is present in a signal. Eventually, one might think of incorporating a stochastic component into the description as well. So far, however, we have to assume that this stochastic component is small and essentially does not change the nonlinear properties. Thus all the successful approaches we are aware of either assume the nonlinearity to be a small perturbation of an essentially linear stochastic process, or they regard the stochastic element as a small contamination of an essentially deterministic nonlinear process. Consider for a moment a purely deterministic system. Once its present state is fixed, the states at all future times are determined as well. Thus it will be important to establish a vector space (called a state space or phase space) for the system, such that specifying a point in this space specifies the state of the system, and viceversa. Then we can study the dynamics of the system by studying the dynamics of the corresponding phase space points. In theory, dynamical systems are usually defined by a set of first-order ordinary differential equations (see below) acting on a phase space. The mathematical theory of ordinary differential equations ensures the existence and uniqueness of the trajectories, if certain conditions are met. We will not hold up any academic distinction between the state and the phase space, but we remark that except for mathematical dynamical models with given equations of motion, there will not be a unique choice of what the phase space of a system can be. Let us now introduce some notation for deterministic dynamical systems in phase space. For simplicity we will restrict ourselves to the case where the phase space is a finite dimensional vector space R^n. A state is specified by a vector xϵR. Then we can describe the dynamics either by an m dimensional map or by an explicit system of m first-order ordinary differential equations. In the first case, the time is a discrete variable:

$$x_n + 1 = F(x_n), n\epsilon Z \qquad (1.2)$$

and in the second case it is a continuous one:

$$\frac{dx(t))}{dt} = f(x(t)), t\epsilon R \qquad (1.3)$$

The second situation is usually referred to as a flow. The vector field f in Eq. (1.3) is defined not to depend explicitly on time, and thus is called autonomous. If f contains an explicit time dependence, e.g., through some external driving term, the mathematical literature does not consider this a dynamical system any more since time translation invariance is broken. The state vector alone (i.e., without the information about the actual time t) does not define the evolution uniquely. In many cases such as periodic driving forces, the system can be made autonomous by the introduction of additional degrees of freedom (e.g., a sinusoidal driving can be generated by an additional autonomous harmonic oscillator with a unidirectional coupling). Then, one can typically define an extended phase space in which the time evolution is again a unique function of the state vectors, even without introducing auxiliary degrees of freedom; just by introducing, e.g., a phase angle of the driving force. In the autonomous case, the solution of the initial value problem of Eq. (1.3) is known to exist and to be unique if the vector field f is Lipshitz continuous. A sequence of points x_n or x(t), solving the above equations is called a trajectory of the dynamical system, with X_0 or x(0), respectively, the initial condition. Typical trajectories will either run away to infinity as time proceeds or stay in a bounded area forever, which is the case we are interested in here. The observed behavior depends both on the form of F (or respectively f) and on the initial condition; many systems allow for both types of solution. The set of initial conditions leading to the same asymptotic behavior of the trajectory is called the basin of attraction for this particular motion.

2.5 Power Spectrum

The power spectrum is another quantity useful in identifying periodic or quasi-periodic nature of the time series since they give sharp peaks at the dominant frequencies and their harmonics. Purely random process gives flat power spectrum without any peak. The power spectrum of a chaotic process that exhibits a broad band spectrum may sometimes resemble that of stochastic process (colored noise). Thus, there remains an element of ambiguity(uncertainty). The power spectrum is usually obtained by the conventional fast Fourier transform (FFT)algorithms.

The power spectrum of a process is defined to be the squared modulus of the continuous Fourier transform. $S(f)$=$|s(f)|^2$. It is the square of the amplitude, by which the frequency f contributes to the signal. If we want to estimate it from a finite-discrete series, we can use the periodogram $S_k = |s_k|^2$.

This is not the most useful estimator for S(f) for two reasons. First, the finite frequency resolution of the discrete Fourier transform leads to leakage into adjacent frequency bins. For example, in the limit of a pure harmonic signal, where the continuous transform yields a δ distribution, the discrete transform yields only a finite peak which becomes higher for longer time series. Second, its statistical fluctuations are of the same order as S(f) itself. This can be remedied either by averaging over adjacent frequency bins or more efficiently by averaging over running windows in the time domain. The power spectrum is particularly useful for studying the oscillations of a system. There will be sharper or broader peaks at the dominant frequencies and at their integer multiples, the harmonics. Purely periodic or quasi-periodic signals show sharp spectral lines; measurement noise adds a continuous floor to the spectrum. Thus in the spectrum signal and noise are readily distinguished. Deterministic chaotic signals may also have sharp spectral lines but even in the absence of noise there will be a continuous part of the spectrum. This is an immediate consequence of the exponentially decaying autocorrelation function. Without additional information it is impossible to infer from the spectrum whether the continuous part is due to noise on top of a (quasi-)periodic signal or to chaoticity. Since it is unlikely that a parameter change has such a dramatic influence on the noise in the system, the change was attributed to the transition from quasi-periodic motion to chaos.

Fast Fourier Transform

Virtually everything in the world can be described via a waveform - a function of time, space or some other variable. For instance, sound waves, electromagnetic fields, the elevation of a hill versus location, a plot of VSWR versus frequency, the price of your favorite stock versus time, etc. The Fourier Transform gives us a unique and powerful way of viewing these waveforms. "All waveforms, no matter what you scribble or observe in the universe, are actually just the sum of simple sinusoids of different frequencies." The Fourier Transform decomposes a waveform - basically any real world waveform, into sinusoids. That is, the Fourier Transform gives us a function derived from a given function can be represented by a series of sinusoidal functions. It shows how any waveform/function can be re-written as the sum simple sine and cosine functions.

The Fourier transform decomposes a function of time (a signal) into the frequencies that make it up, similarly to how a musical chord can be expressed as the amplitude (or loudness) of its constituent notes. The Fourier transform of a function of time itself is a complex-valued function of frequency, whose absolute value represents the amount of that frequency present in the original function. The Fourier transform is called the frequency domain representation of the original signal. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time. For many functions of practical interest one can define an operation that reverses this: the inverse Fourier transformation, also called Fourier synthesis, of a frequency domain representation combines the contributions of all the different frequencies to recover the original function of time.

The fourier transform of an integrable function f(x) is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \exp^{-2\pi i x \xi} dx \tag{1.4}$$

for any real number ξ where the independent variable x represents time, the transform variable ξ represents frequency.

A fast Fourier transform (FFT) algorithm computes the discrete Fourier transform (DFT) of a sequence, or its inverse. Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa.

The DFT is obtained by decomposing a sequence of values into components of different frequencies. This operation is useful in many fields (see discrete Fourier transform for properties and applications of the transform) but computing it directly from the definition is often too slow to be practical. An FFT is a way to compute the same result more quickly: computing the DFT of N points in the naive way, using the definition, takes $0(N^2)$ arithmetical operations, while an FFT can compute the same DFT in only 0(NlogN) operations. The difference in speed can be enormous, especially for long data sets where N may be in the thousands or millions. In practice, the computation time can be reduced by several orders of magnitude in such cases, and the improvement is roughly proportional to $\frac{N}{log(N)}$. This huge improvement made the calculation of the DFT practical; FFTs are of great importance to a wide variety of applications, from digital signal processing and solving partial differential equations to algorithms for quick multiplication of large integers.

Fast Fourier Transform - Algorithms and Applications presents an introduction to the principles of the fast Fourier transform (FFT). It covers FFTs, frequency domain filtering, and applications to video and audio signal processing. As fields like communications, speech and image processing, and related areas are rapidly developing, the FFT as one of the essential parts in digital signal processing has been widely used.

2.6 Embedding

In most experiments, only one variable can be monitored. However, on the basis of some elementary checks mentioned above, if there is enough clue that the dynamics of the system is controlled by many degrees of freedom, then attempts can be made to reconstruct the original attractor since the asymptotic dynamics of the system is confined to the attractor. This is done by embedding the scalar time series in a higher dimensional space. In the process, it is possible to obtain information about the number of degrees of freedom required for a dynamical description of the system. In a nonlinear process, all variables are inter-related and the observed time series is considered as a projection of all the dynamical variables on the real line. If the time series x(t) is an outcome of the evolution of a dynamical system, all other dynamical variables of the system affect the evolution of variable x. So $x(t+\tau)$ is generally a nonlinear function of all the variables of the system at time t. Then, it is possible to construct a d-dimensional vector using d such time delayed variables to represent d active variables of the system under observation. Thus by embedding in higher dimensions, we expect to reconstruct an equivalent of the original attractor. Such embedding procedures have been built on a sound mathematical framework. Whit-ney has shown that in general a D dimensional manifold can always be embedded in a phase-space of dimension d=2D +1. The extension of this theorem to strange attractors is due to Takens and Mane. They have shown that the multivariate dynamics of the attractor can be unfolded from the scalar time series in a higher dimensional space under quite general conditions. Packard et al. have demonstrated numerically that in the case of chaotic systems the multi-dimensional phase space can be reconstructed from a single scalar time series. They use the value of a given scalar time series and its derivatives up to order $d_E - 1$ as the components of the reconstructed $d_E - 1$ dimensional vector space, i.e.,

$\vec{\xi_i} = \{x_i, x'_i, x''_i, ..., x_i^{(d_E-1)\prime\prime}\}$, where x'_i represents the derivative with respect to time. Since finding higher order derivatives from a discrete time series necessarily contributes increasing levels of errors, this method is not suitable for experimental time series sampled at finite intervals of time. Following Ruelle, Packard et al. also suggested the method of delay coordinates for the reconstruction of the phase-space. Takens showed that a d_E dimensional attractor reconstructed using d delay coordinates with $d_E > 2D_{f+1}$ with D_f referring to the fractal dimension of the attractor, will ensure a one-to-one representation. Later, it has been shown that the number of degrees of freedom of a dynamical system is given by the value of embedding dimension $d_E > D_f$. That is, we can reconstruct the d_E dimensional attractor from the experimental data as $\vec{\xi_i} = \{x_i, x_{i+\tau}, x_{i+2\tau}, ...x_{i+(d_E-1)\tau}\}$, with the delay time τ chosen appropriately for an infinitely long and noise free data. The time-delay method is also easy to implement. Also, most of the algorithms developed in the literature are based on the method of embedding. We will be using time-delay embedding for the analysis of the time series. It is clear that the geometrical nature of the attractor is preserved(From figure 1).

2.7 Correlation Dimension

A direct evidence of chaos is the self-similar nature of the associated strange attractor characterized by non-integer dimension. Since fractal dimension does not give accurate results for attractors reconstructed from experimental data, Grassberger and Procaccia(GP) introduced the idea of correlation dimension as a generalization of fractal dimension. The correlation integral is the fraction of the number of pairs of the vectors $(\vec{\xi_i}, \vec{\xi_j})$ whose distance is less than r that gives an idea about the correlation. The correlation integral C(r)is defined as

$$C_2(r, d_E) = \frac{2}{N_T(N_T-1)} \sum_{i=1}^{N_T} \sum_{j=i+1}^{N_T} \Theta(r - |\vec{\xi_i} - \vec{\xi_j}|) \tag{1.5}$$

where $\ominus(..)$ is the Heaviside step function, $N_T = N(d_E-1)\tau$ is the number of available vectors, N is the length of the time series, d_E the embedding dimension and τ the delay time. Thus, $C_2(r, d_E)$ represents the fraction of the number of pairs of the vectors $(\vec{\xi_i}, \vec{\xi_j})$ whose distance is less than r.

These vectors may refer to the d_E dimensional vectors either obtained by time delay embedding of the time series Due to the fractal nature of the attractor, correlation integral $C_2(r, d_E)$ in the limit of small r behaves as

$$C_2(r, d_E) \sim \lim_{r \to 2} r^{D_2} \tag{1.6}$$

where the exponent D_2 is termed as the correlation dimension. Correlation dimension is shown to be a lower bound to the fractal dimension D_f [26]. Theiler [27] has found that the correlation of points on the reference trajectory lead to wrong conclusions in the calculation of correlation dimension. This is because points on the same trajectory cannot diverge. This problem can be rectified by ignoring points within a window w on either side of the reference point. Usually, the window chosen is a few times the autocorrelation time. Thus,the expression for calculating the correlation dimension becomes:

$$C_2(r, d_E) = \frac{2}{(N_T - \omega)(N_T - \omega - 1)} \sum_{i=1}^{N_T} \sum_{j=i+\omega}^{N_T} \Theta(r - |\vec{\xi_i}, \vec{\xi_j}|) \tag{1.7}$$

Correlation dimension can be obtained from the slope of the double logarithmic plot of $C_2(r, d_E)$ versus r. One looks for the convergence of the local slope D_2 as the embedding dimension is increased. The idea of increasing the embedding dimension is to eliminate the effect of false neighbors. In a reconstructed attractor, two points may appear to be neighboring if the embedding dimension is lower than the actual dimension. Upon increasing the embedding dimension, these false neighbors do not appear as neighbors. Beyond a particular dimension their number reduces to zero and the influence of the false neighbors vanishes beyond this embedding dimension. Hence, the necessity for convergence of the slope with the embedding dimension. In practice, it is found that for short time series, the scaling regime is seen for intermediate values of r (not for small r). A better way is to plot the local slope $dlnC_2(r, dE)/dlnr$ as a function of dlnr from which we can obtain directly the value of correlation dimension. This is particularly recommended since embedding in higher and higher dimensions give the impression of converging slope while in reality there would be no convergence when the local slope is plotted. Another observation is that larger scaling region can be obtained for small delay time τ, but then $C_2(r, d_E)$ saturates only for larger d_E.Thus, it is found that more than d_E, the slope of $C_2(r, d_E)$ saturates as function of $(d_E - 1)\tau$.

There are several ways to quantify the self-similarity of a geometrical object by a dimension. Of course we require the definition to coincide with the usual notion of dimension when applied to nonfractal objects: a finite collection of points is zero dimensional, lines have dimension one, surfaces two, etc. But let us for the moment cut a long story short and propose a definition which is of particular interest in practical applications where the geometrical object has to be reconstructed from a finite sample of data points which are most likely to contain some errors as well. This notion, called the correlation dimension, was introduced by Grassberger Procaccia (1983) and (1983a). Let us first define the correlation sum for a collection of points x_n in some vector space to be the fraction of all possible pairs of points which are closer than a given distance r in a particular norm, $\theta(x) = O$ if $\xi \leq O$ and $\theta(x) = 1$ for $x > 0$.The sum just counts the pairs $(\vec{\xi_i}, \vec{\xi_j})$ whose distance is smaller then r. In the limit of an infinite amount of data $(N \to \infty)$ and for small r, we expect C to scale like a power law,

$$C(r, d_E) \propto r_2^D \tag{1.8}$$

and we can define the correlation dimension D by

$$d(r, N) = \frac{t}{dlnC(r, N)} lnr \tag{1.9}$$

$$D = \lim_{r \to 0} \lim_{N \to 0} d(N, \epsilon) \tag{2.0}$$

We can easily verify that this definition yields the correct integral dimensions for regular geometrical objects. It is equally obvious that the two limits we have to take will get us into trouble whenever we have a finite sample instead of a full distribution: N is limited by the sample size, and the range of meaningful choices for ϵ limited from below by the finite accuracy of the data and by the inevitable lack of near neighbors at small length scales.

2.8 Largest Lyapunov Exponents

In mathematics the Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories.The usual test for chaos is calculation of the largest Lyapunov exponent. A positive largest Lyapunov exponent indicates chaos. When one has access to the equations generating the chaos, this is relatively easy to do. When one only has access to an experimental data record, such a calculation is difficult and that case will not be considered here. The general idea is to follow two nearby orbits and to calculate their average logarithmic rate of separation. Whenever they get too far apart, one of the orbits has to be moved back to the vicinity of the other along the line of separation. A conservative procedure is to do this at each iteration.

To estimate the uncertainty in your calculated Lyapunov exponent, you can repeat the calculation for many different initial conditions (within the basin of attraction) and perturbation directions. For a chaotic system, the initial condition need only be changed slightly since orbits quickly become uncorrelated due to the sensitive dependence on initial conditions. You can then calculate a mean and standard deviation of the calculated values so as to avoid the all too common mistake of quoting more digits than are significant.

The d_E-dimensional space constructed by embedding procedure can be used for the calculation of the Lyapunov exponents. In a chaotic time series, points on two neighboring trajectories diverge exponentially in time. The divergence can be measured by calculating the distance between them at t=0 and at later time say, after k time steps. In a short time the difference vector corresponding to the two points aligns itself in the direction of maximum stretching. Consider two vectors $\vec{\xi_i} and \vec{\xi_j}$ in d_E dimension, let $d_ij(k) = ||\vec{\xi_i} + k - \vec{\xi_j} + k||$, where $||.....||$ is the Euclidean distance between the two vectors at time k between $\vec{\xi_i} and \vec{\xi_j}$, Let $d_{ij}(0)$ be the initial distance. Since for a chaotic system, $d_{ij}(k)$ should be larger than $d_{ij}(0)$, a measure of the divergence is the Lvapunov exponent,

$$\Lambda = \frac{1}{k\tau < ln[d_ij(k)/d_ij(0)] >} \tag{2.1}$$

where $< .. >$ refers to the average over all pairs of points for which $d_{ij}(0)$ is less than a chosen d_0.

The rate of separation can be different for different orientations of initial separation vector. Thus, there is a spectrum of Lyapunov exponentsequal in number to the dimensionality of the phase space. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive Maximal Lyapunov exponent is usually taken as an indication that the system is chaotic (provided some other conditions are met, e.g., phase space compactness). Note that an arbitrary initial separation vector will typically contain some component in the direction associated with the MLE, and because of the exponential growth rate, the effect of the other exponents will be obliterated over time.

A.M. Lyapunov proved that if the system of the first approximation is regular (e.g., all systems with constant and periodic coefficients are regular) and its largest Lyapunov exponent is negative, then the solution of the original system is asymptotically Lyapunov stable. Later, it was stated by O. Perron that the requirement of regularity of the first approximation is substantial.

In 1930 O. Perron constructed an example of the second-order system, the first approximation of which has negative Lyapunov exponents along a zero solution of the original system but, at the same time, this zero solution of original nonlinear system is Lyapunov unstable. Furthermore, in a certain neighborhood of this zero solution almost all solutions of original system have positive Lyapunov exponents. Also it is possible to construct reverse example when first approximation has positive Lyapunov exponents along a zero solution of the original system but, at the same time, this zero solution of original nonlinear system is Lyapunov stable. The effect of sign inversion of Lyapunov exponents of solutions of the original system and the system of first approximation with the same initial data was subsequently called the Perron effect.

Perron's counterexample shows that negative largest Lyapunov exponent does not, in general, indicate stability, and that positive largest Lyapunov exponent does not, in general, indicate chaos. If the system is conservative (i.e. there is no dissipation), a volume element of the phase space will stay the same along a trajectory. Thus the sum of all Lyapunov exponents must be zero. If the system is dissipative, the sum of Lyapunov exponents is negative. If the system is a flow and the trajectory does not converge to a single point, one exponent is always zerothe Lyapunov exponent corresponding to the eigenvalue of 1 with an eigenvector in the direction of the flow. The Lyapunov spectrum can be used to give an estimate of the rate of entropy production and of the fractal dimension of the considered dynamical system.

In particular from the knowledge of the Lyapunov spectrum it is possible to obtain the so-called KaplanYorke dimension D_{KY} , which is defined as follows:

$$D_{KY} = k + \sum_{i=1}^{k} \frac{\lambda_i}{|\lambda_{k+1}|} \tag{2.2}$$

where k is the maximum integer such that the sum of the k largest exponents is still non-negative. D_{KY} represents an upper bound for the information dimension of the system. Moreover, the sum of all the positive Lyapunov exponents gives an estimate of the KolmogorovSinai entropy accordingly to Pesin's theorem.

The multiplicative inverse of the largest Lyapunov exponent is sometimes referred in literature as Lyapunov time, and defines the characteristic e-folding time. For chaotic orbits, the Lyapunov time will be finite, whereas for regular orbits it will be infinite. For the calculation of Lyapunov exponents from limited experimental data, various methods have been proposed. However, there are many difficulties with applying these methods and such problems should be approached with care. The main difficulty is that the data does not fully explore the phase space, rather it is confined to the attractor which has very limited (if any) extension along certain directions. These thinner or more singular directions within the data set are the ones associated with the more negative exponents. The use of nonlinear mappings to model the evolution of small displacements from the attractor has been shown to dramatically improve the ability to recover the Lyapunov spectrum, provided the data has a very low level of noise. The singular nature of the data and its connection to the more negative exponents has also been explored.

Chapter 3

Study of Lorenz Model

3.1 Introduction

Edward Norton Lorenz, (19172008), American mathematician, meteorologist, and MIT professor. Edward Lorenz studied mathematics at Dartmouth College in the town of Hanover in western New Hampshire. He graduated from Dartmouth College with a bachelor's degree in 1938, then went to Harvard where he studied for a Master's Degree in mathematics. He received the A.M. degree in 1940 and submitted his first mathematical paper for publication. The paper A generalization of the Dirac equations appeared in the Proceeding of the National Academy of Sciences in 1941.

After the award of his master's degree, Lorenz undertook war service with the United States Army Air Corps. His work with the Army Air Corps involved applying his mathematical skills to weather forecasting and soon he was looking to undertake further study of meteorology. He did a second master's degree, this time an S.M. in meteorology, at the Massachusetts Institute of Technology graduating in 1943. Then, after World War II ended, he continued to study for his doctorate at the Massachusetts Institute of Technology with James Austin as his thesis advisor. He was awarded his Sc.D. in 1948 after submitting the dissertation A Method of Applying the Hydrodynamic and Thermodynamic Equations to Atmospheric Models.

Lorenz had been employed as an assistant meteorologist at the Massachusetts Institute of Technology from 1946, but when he was awarded his doctorate in 1948 he was promoted to meteorologist. He held this post until 1954 and during these eight years he published some major works. He submitted Dynamic Models Illustrating the Energy Balance of the Atmosphere to the Journal of the Atmospheric Sciences in June 1949 and it was published in February 1950. The paper Seasonal and Irregular Variations of the Northern Hemisphere Sea-Level Pressure Profile appeared a year later in the same journal. In 1952 he published Flow of Angular Momentum as a Predictor for the Zonal Westerlies.

Lorenz spent his whole career at the Massachusetts Institute of Technology, being appointed assistant professor in 1954, then promoted to associate professor before becoming Professor of Meteorology in 1962. He served as Head of Department from 1977 to 1981, retiring in 1987 when he was made professor emeritus.

Discovery of Chaos

It was a simple event in 1961 which led Lorenz to results which brought him worldwide fame. He was using a computer to investigate models of the atmosphere which he had devised involving twelve differential equations. Having obtained results from running his program, he decided that he would like to carry the calculations further. Rather than start the whole program again (for although he was using an up-to-date computer, it was painfully slow by later standards), he started the program in the middle of the calculation inputting the data as calculated by the machine at that middle point. After going away for a cup of coffee, he came back to see how the computer was getting on with the calculations. He was surprised to see that the computer had found significantly different answers over the range that it had calculated before. At first he thought it must be a hardware problem, since the software should come up with the same answer every time when given the same input data. Eventually he discovered that the data he had input to begin the second run had not been printed out to the same number of decimal places as the machine had stored, so the initial data was slightly different for the second run (differing in the fourth decimal place). He then set about trying to understand how a tiny change in the initial data could have such a major effect on the calculations. Lorenz had discovered chaos.

Lorenz was not the first person to discover chaos. Poincar had discovered chaos in the 1880s when studying the 3-body problem. However, Poincar's discovery had not led to any significant developments. Now that Lorenz understood the significance of his discovery he wrote it up in the paper

”Deterministic Non-periodic Flow” and it appeared in the Journal of the Atmospheric Sciences in 1963. The abstract begins:-

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

The paper, like Poincar’s work 80 years earlier, had relatively little impact in the time immediately after it appeared. However, it has gone on to become one of the most quoted papers of all time. The set of equations and the attractors described by this set of equations are now called the ’Lorenz equations’ and ’Lorenz attractors’, respectively. It would be fair to say that Lorenz began a scientific revolution with this paper which he and many others developed over the following years.According to Lorenz, when he failed to provide a title for a talk he was to present at the 139th meeting of the American Association for the Advancement of Science in 1972, Philip Merilees concocted ”Does the Flap of a Butterfly’s Wings in Brazil Set off a Tornado in Texas?”. By encapsulating the essence of chaos theory in the title of his talk, Lorenz succeeded in capturing the public’s imagination and the term ”butterfly effect” was soon the popular term for chaos.

3.2 Butterfly Effect

In chaos theory, the butterfly effect is the sensitive dependence on initial conditions in which a small change in one state of a deterministic nonlinear system can result in large differences in a later state. The name of the effect, coined by Edward Lorenz, is derived from the metaphorical example of the details of a hurricane (exact time of formation, exact path taken) being influenced by minor perturbations such as the flapping of the wings of a distant butterfly several weeks earlier. Lorenz discovered the effect when he observed that runs of his weather model with initial condition data that was rounded in a seemingly inconsequential manner would fail to reproduce the results of runs with the unrounded initial condition data. A very small change in initial conditions had created a significantly different outcome.

The butterfly effect is exhibited by very simple systems. For example, the

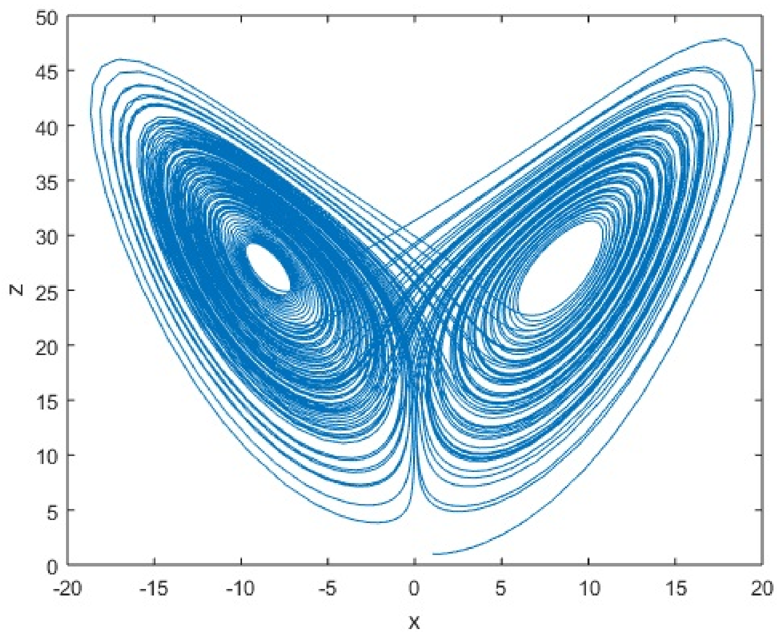

Figure 3.1: Butterfly Structure of Lorenz attractor

randomness of the outcomes of throwing dice depends on this characteristic to amplify small differences in initial conditions-the precise direction, thrust, and orientation of the throw-into significantly different dice paths and outcomes, which makes it virtually impossible to throw dice exactly the same way twice.

3.3 The Model

The Lorenz system is a system of ordinary differential equations first studied by Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system which, when plotted, resemble a butterfly or figure eight.

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection. The model is a system of three ordinary differential equations now known as the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x), \tag{2.3}$$

$$\frac{dy}{dt} = x(\rho - z) - y, \tag{2.4}$$

$$\frac{dz}{dt} = xy - \beta z \tag{2.5}$$

Here x, y, and z make up the system state, t is time, and σ, ρ, β are the system parameters. The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions and forward osmosis. From a technical standpoint, the Lorenz system is nonlinear, three-dimensional and deterministic.

Computer Simulation

In this study, the classical fourth-order Runge-Kutta method is modified to obtain new methods which are of order five. The methods are tested on the Lorenz system which involves the chaotic and nonchaotic characteristics. The modified fifth-order methods using arithmetic mean are compared with another method with the same order. Comparisons between the methods are made in two step sizes and the accuracy of all the methods are discussed, ode45() the best function for Lorenz system but the results are very poor with RK4 even with one million sub-interval. Mathematica and Matlab softwares are used to solve the Lorenz system and sketch the graphical solutions of the chaotic and non-chaotic system.

I have taken $\sigma = 10$,

$\rho = 28$, and $\beta = 8/3$ as well as the initial condition x(0)=1, y(0)=1, z(0)=1 and solved the Lorenz equation, where 'f' is the set of differential equations and 'a' is an array containing values of x,y and z variables, 't' is the time variable with interval 0 to 100.

3.4 Time Series Analysis Prediction

The qualitative and quantitative features of chaotic systems are suitable only when the equations of motion are known. However, most of the time, the equation of motions are not known, for example of an experimental system. Furthermore, very often we measure only a scalar time series that appears quite irregular. Having known that deterministic nonlinear systems with few degrees of freedom can lead to irregular time series, a natural question is whether the underlying mechanisms of such measured irregular time series are an outcome of deterministic dynamics or is it of a stochastic origin. In present section, we will discuss the tools that are useful in answering this question of a model that i have simulated and got the results i.e Lorenz model.

Time series is a sequence of data points measured typically at regular intervals of time. The frequency of measurement of the data points is called sampling rate. Time series analysis is a method that is used to understand the underlying mechanism generating the data points. If the data is regular and periodic then linear method may interpret all regular structures. The most direct link between chaos theory and the real world is through the analysis of time series data in terms of nonlinear dynamics. Non-linear time series analysis provides tools to analyze the underlying dynamics. Here, we will discuss the tools necessary to identify and characterize an irregular time series of Lorenz system.

The behavior of all dynamical systems depends on the initial conditions. In the case of chaotic system sensitive dependence on initial conditions is the most important feature. This means that there is a loss of memory of the initial state. In experiments, uncertainty in trajectories arises due to variety of reasons. Indeed, most experimental signals from chaotic systems are actually corrupted by noise. The main sources of noise may be due to the lack of control over external or internal parameters.

There are several qualitative and quantitative methods for distinguishing a chaotic signal from a stochastic one. Some of them are rather simplistic but are quite useful in giving a clue about the possibility of chaos. Such methods are easy to implement or to visualize in terms of plots. Often, using one quantitative estimate may not be adequate. For this reason, it is often necessary to use several complimentary methods till one can have high level of confidence in the end results of the analysis.

Now i'm going to discuss several methods necessary that i have discussed earlier to identify the chaotic behavior and find out the no. of degrees of freedom to describe the Lorenz system.

3.4.1 Visual Inspection

From this figures u can visualize the nonlinear picks are due motion of the particle in Lorenz attractor trajectory i.e the particle rotating around a attracting fixed point in one side then it goes to another attracting fixed point rotate around this like back and forth motion. Here visual inspection of the data shows the nonlinearity of the system.

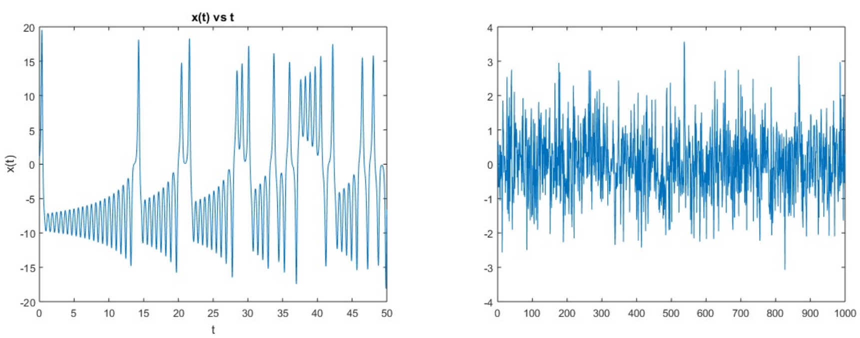

Figure 3.2: Comparison of Chaotic and Random data by Time series Analysis respectively

You can compare both these figures and see which one is chaotic data and which one is random data.

3.4.2 Sensitivity to initial Condition

The mathematical definition of chaotic motion is sensitivity to initial conditions that means a small change in initial condition leads to large change in the results. Here also a small change in initial condition i.e initial parameters

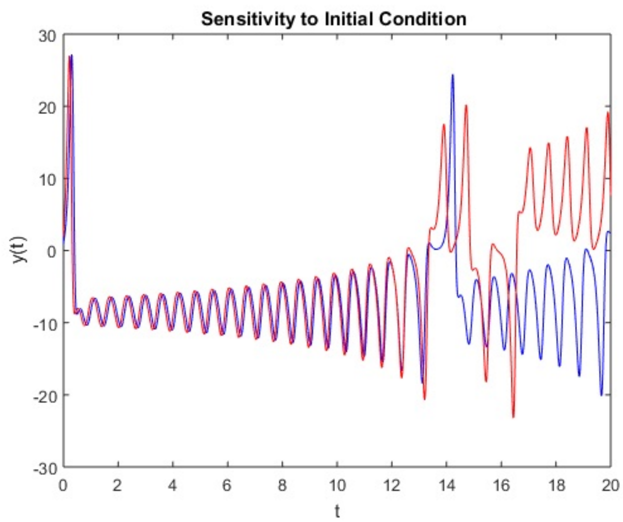

Figure 3.3: Time Series of two Trajectories with slight variation of Initial Condition

leads large change in trajectory. But, after a certain amount of time, we lose essentially all predictive power. Such systems are known as chaotic. In other words chaos is the long term unpredictability of deterministic, nonlinear dynamical systems due to sensitivity to initial condition can be observed in the time series. This figure reveals all you question about how it depends on initial condition.

3.4.3 Auto-Correlation

Autocorrelation function that describes the system is random or chaotic. Auto-correlation function is a function that describes a relation of 1st data point to 2nd , 1st to 3rd, 1st to 4th, and so on . Here also I have simulated random data points and chaotic data points to find autocorrelation function plot. So by comparing these plots u can see how the a.c.f of suddenly falls down in case of random data that means the random data points are highly uncorrelated i.e the points has no correlation bet. them. But in case of chaotic system the a.c.f decreases exponentially upto definite initial time interval i.e it follow aperiodic

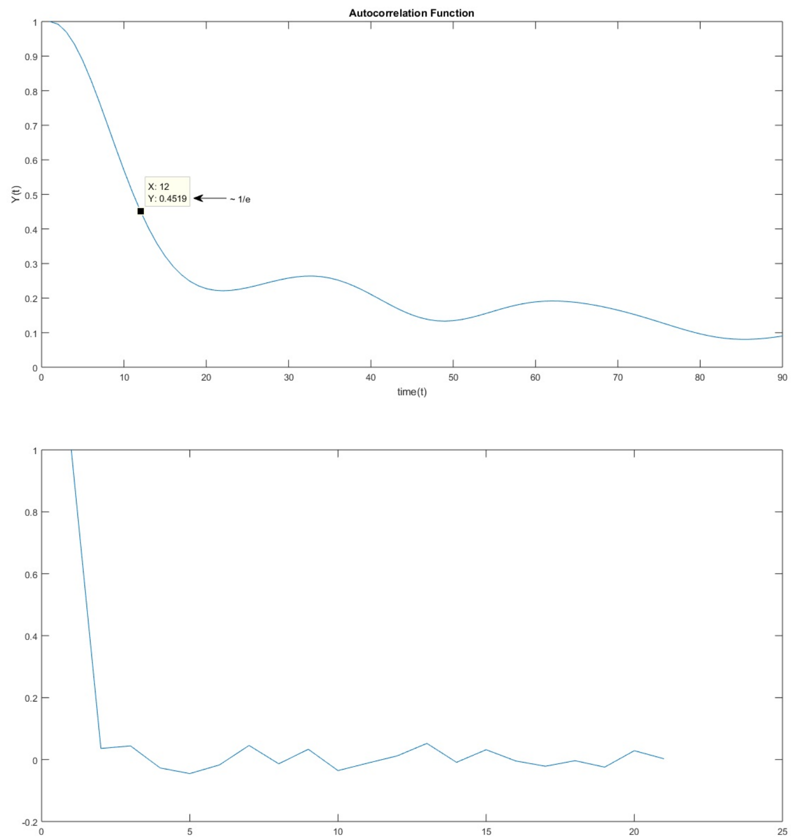

Figure 3.4: Comparison of Auto-correlation Function of Chaotic and Random Data Respectively

path but periodic curve now becomes aperiodic after some time interval. In details curve is periodic until data points are correlated and after it loses the correlation so the curve becomes aperiodic which is shown in the figure. This a tool to identify a chaotic system from nonchaotic system. The below figures diffrentiates which one is chaotic and which one random.

3.4.4 Phase Space

Here the phase space plots contain graph between x, y, z, t variable 2-D and 3-D plots. The first plot is in between the x-variables (x_i) with its next values(x_{i+1}), which gives a sign of 2-d structure of the Lorenz model.

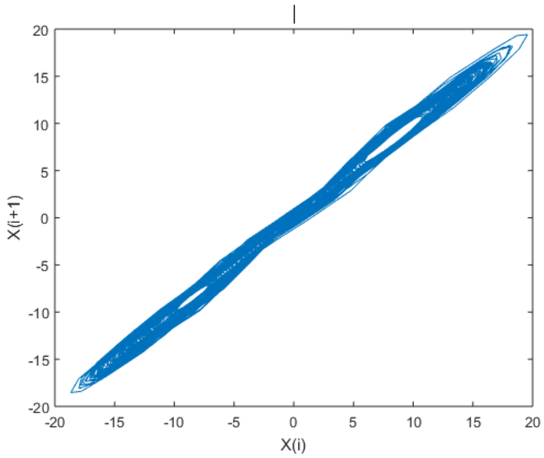

Figure 3.5: sign of 2-d Structure by plotting $x_i vs x_{i+1}$

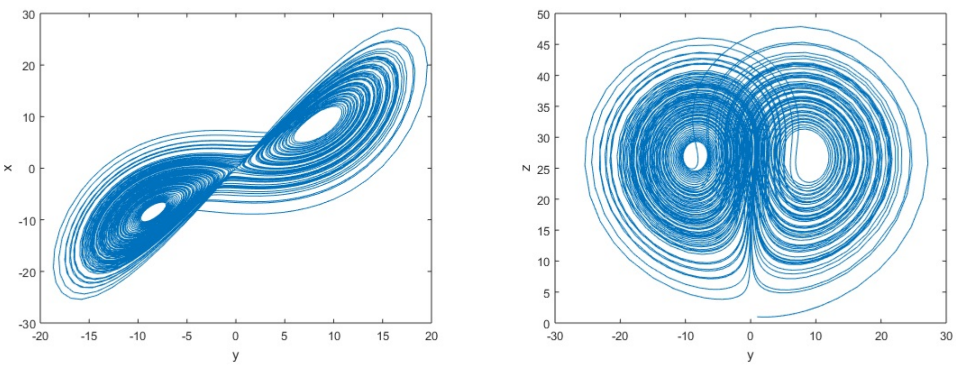

Figure 3.6: 2-D Structures of x(t), y(t), z(t) Variables

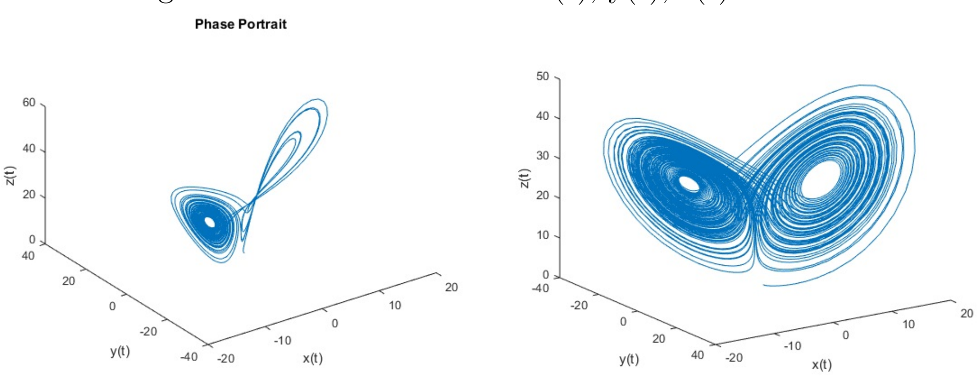

Figure 3.7: 3-D Phase Portraits in different View

3.4.7 Correlation Dimension

Due to the fractal nature of the attractor, correlation integral $C_2(r)$ in the limit of small r behaves as

$$C_2(r, d_E) \sim \lim_{r \to 2} r^{D_2}$$

Where the exponent D_2 is termed as the correlation dimension. Then u have to take logarithm on both the sides and the Correlation dimension can be obtained from the slope of the double logarithmic plot of $C_2(r)$ versus r. I have also find the correlation dimension of Lorenz attractor from the slope of double logarithm plot i.e $D_2 = 2.20$. Thus the minimum number of variables required to describe system completely can be obtained by adding 1 to the correlation dimension $=\tau+1= \quad 3$, where τ=2.20.

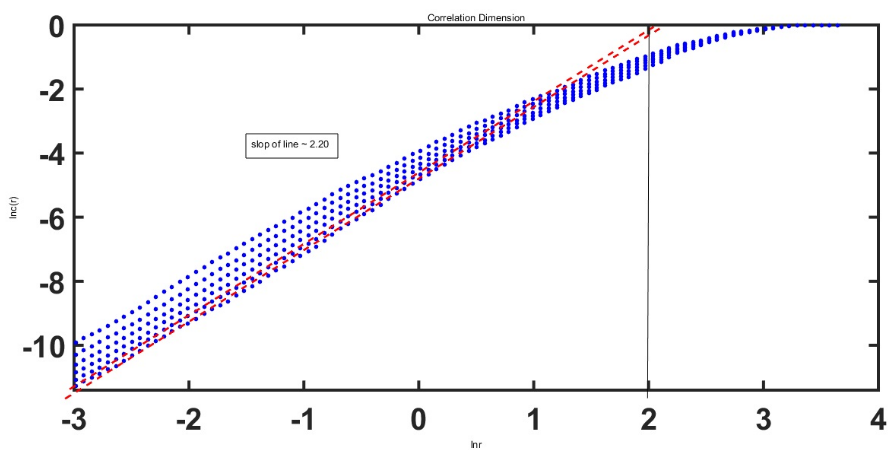

Figure 3.8: Correlation Dimension of Lorenz Attractor

3.4.8 Maximal Lyapunov Exponent

One of the best tool is largest lyapunov exponent. To describe this 1st consider the two trajectories of Lorenz system by slight change in initial condition. U can see with increase in time the separation between the two points increases i.e. Two trajectories in phase space with initial separation l(0) diverge at a rate given by, $l(t) \sim l(0)exp()$, where is the Lyapunov exponent, which is given by,

$$\lambda \sim ln[\frac{l(0)}{l(0)} l(t)/l(0)]/t$$

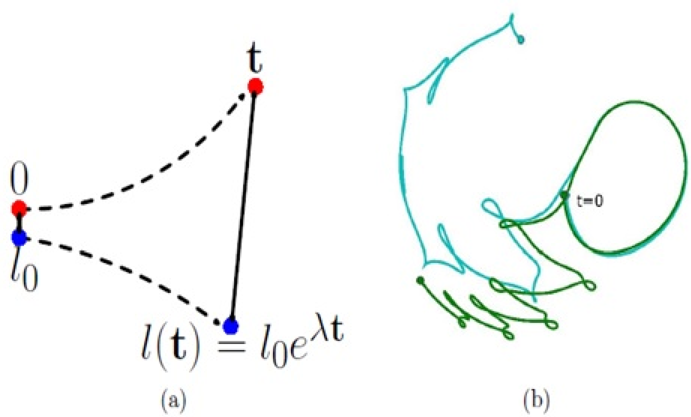

Figure 3.9: Two trajectories in Phase space by slight Change in Initial Condition

It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic (provided some other conditions are met, e.g., phase space compactness). If the system is conservative (i.e. there is no dissipation), a volume element of the phase space will stay the same along a trajectory. Thus the sum of all Lyapunov exponents must be zero, that means the separation bet the two trajectories l(t) is constant over time (l(t)=l(0)). If the system is dissipative, the sum of Lyapunov exponents is negative, i.e means the separation between the two trajectories l(t) decreases after some time steps. Anyone can also use matlab to find lyapunov exponent by finding the difference between the two trajectories x-variable datas and then put it in equation and check the value is +ve or -ve or 0. But in case of Lorenz system the MLE is found to be +ve which determines that the system is chaotic.

Chapter 4

Summery and Conclusions

The data sets we are obtained are nonlinear and the Lorenz system is a nonlinear system which is proved now from this time series plot which you may called as visual inspection of time series data. From the figures represented in visual inspection can observed the nonlinear picks are due to motion of the particle in Lorenz attractor trajectory i.e the particle rotating around a attracting fixed point in one side then it goes to another attracting fixed point rotate around this like back and forth motion. Here visual inspection of the data shows the nonlinearity of the system. Here I have taken simulated data of Lorenz equation of x-variable. And on simulating the x-data points with its next data points in matlab. I have found a phase plot which confirms that the Lorenz system is having at least two degrees of freedom or having minimum 2-dimension. Another tool for analyzing the model is autocorrelation function that describes the system is random or chaotic. Auto-correlation function is a function that describes a relation of 1st data point to 2nd , 1st to 3rd, 1st to 4th, and so on... . Here also I have simulated random data points (which are coming from different sources or systems) and chaotic data points to find autocorrealation function plot. So by comparing these plots u can see how the a.c.f of suddenly falls down in case of random data that means the random data points are highly uncorrelated i.e the points has no correlation between them. But in case of chaotic system the a.c.f decreases exponentially upto definite initial time interval i.e it follow aperiodic path but periodic curve now becomes aperiodic after some time interval. In details curve is periodic until data points are correlated and after it loses the correlation so the curve becomes aperiodic. This a tool to identify a chaotic system from nonchaotic one. This method only gives us the possibility of chaos, but there many more tools that will give us certainty of chaos; one of the best tool is largest lyapunov exponent. To describe this 1st consider the two trajectories of Lorenz system by slight change in initial condition. You can see in figure of lyapunov exponet with increase in time the separation between the two points increases. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic (provided some other conditions are met, e.g., phase space compactness). A direct evidence of chaos is the self-similar nature of the associated strange attractor which is characterized by non-integer dimension or fractal dimension. Self-similarity refers to the reiteration of a specific pattern where a fragment of the object, figure or illustration appears similar to the whole. Which u can seen in case of Lorenz attractor phase plot, here the particle reiterating the specific pattern in which a fragment of the figure appears similar to the whole, Hence Lorenz system has fractal dimension. Since fractal dimension does not give accurate results for attractors reconstructed from experimental data, Grassberger and Procaccia (GP) introduced the idea of correlation dimension as a generalization of fractal dimension. Then i have also find the correlation dimension of Lorenz attractor from the slope of double logarithm plot i.e $D_2 = 2.20$. Thus the minimum number of variables required to describe system completely can be obtained by adding 1 to the correlation dimension $=tau{+}1 \quad 3$, where $\tau = 2.20$. The TISEAN project makes available a number of algorithms of nonlinear time series analysis to find the autocorrelation function, correlation dimension of data sets of Lorenz model. You can also use the Tisean package to find other aspects of Nonlinear Time series analysis like largest lyapunov exponent.

After all the discussion i concluded that the Lorenz system is Chaotic that is described by the a.c.f which gave the possibility of chaos and +ve value of MLE confirms the system as chaotic. And there are 3 variables required to describe Lorenz system which is find out by using Correlation integral.Finally i want explain interesting fact that comes out from this model i.e Butterfly effect. It's a belief that the flap of a butterfly's wing in Brazil can set off a cascade of atmospheric events that, weeks later, spurs(prompt or stimulate) the formation of a tornado in Texas. This so called "butterfly effect" is used to explain why chaotic systems like the weather can't be predicted more than a few days in advance. One can't know every little factor affecting the atmosphere - every flutter of every butterfly in Brazil, so there's little hope of foreseeing the exact time and place a storm will touch down weeks later. The butterfly effect is all the more pleasing because the computer model that led to its discovery resembles a butterfly. The mathematician Edward Lorenz created the model, called a strange attractor.

But later many scientists argued that the energy from the wing will dissipate, rather than build. In short, butterflies can't muster up storms. If a butterfly flaps its wings the effect really just gets damped out. Therefore butterfly effect proven to be incorrect.

4.1 Applications of Chaos and time series analysis

1. Chaos exist in many natural systems, such as weather and climate. This behavior can be studied through analysis of a chaotic mathematical model or through time series analysis of chaotic mathematical model like lorenz model or through analytical techniques such as recurrence plots and Poincare maps.

2. Chaos theory has applications in several disciplines, including meterology for weather forecasting, sociology, physics to study different dynamical systems and to model or construct it, computer science, engineering, economics, biology and philosophy.

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