STRATIFICATIONS OF TOPOLOGICAL SPACES AND ∞-CATEGORIES

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ABSTRACT. I am interested in Stratifications in general. There does not seem to be an exceeding amount of literature on the subject, so I am interested in compiling notes on the subject as I learn. The hope is to compile notes on stratifications of topological spaces, conical stratifications, and stratifications of stable infinity categories.

CONTENTS

1. Posets 1
1.1. Posets as Sets 1
1.2. Posets as Categories 2
1.3. Posets as Topological Spaces 2
1.4. The category of Posets 2

1. POSETS

1.1. Posets as Sets.

Definition 1.1.1 (Posets)**.** A poset is

+) A set P

+) A binary relation $\leq$

such that the binary relation satisfies the following properties. Let

-) (Reflexive)-$p \leq p$

-) (Antisymmetry)-If $p \leq p'$ and $p' \leq p$, then $p = p'$

-) (Transitive)-$p \leq p'$ and $p' \leq p''$ then $p \leq p''$

Example 1.1.2. The most basic example of a poset is a finite set, with a linear ordering. A linear ordering is a binary operation such that for all p and p' in P, either $p \leq p'$ or $p' \leq p$. Since every set is isomorphic to a set $\{0, \ldots, n\}$, we defined the following poset $[n]$, which gives the usual ordering to the set $\{0, \ldots, n\}$

$$[n] := \{0 \leq \cdots \leq n\}$$

Example 1.1.3. Let X be a topological space. The set of closed subsets $\mathsf{cls}(X)$ forms a poset, with the ordering given by inclusion of closed subsets.

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Example 1.1.4. The product of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ is given by taking the product of $P \times Q$ as the set, with relation $\leq_{P\times Q}$ given by $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $q \leq q'$.

Definition 1.1.5. A map between posets P and P'

+) A map $f : \mathsf{P} \to \mathsf{P}'$

such that

-) The map f induces a map on the binary relation $\leq_P$ to $\leq_{\mathsf{P}'}$

1.2. **Posets as Categories.**

Definition 1.2.1 (Poset)**.** A poset is

+) A category P

such that

-) Either $\mathsf{Hom}(p, p') = \emptyset$ or $\mathsf{Hom}(p, p') \simeq *$

-) If $\mathsf{Hom}(p, p') = *$ and $\mathsf{Hom}(p', p) \simeq *$ then $p = p'$

Remark 1.2.2. The transitive condition of a poset P is the condition that there is a composition rule for the category, namely if $\mathsf{Hom}(p, p') \simeq \mathsf{Hom}(p', p'') \simeq *$, then it must be the case that $\mathsf{Hom}(p, p'') \simeq *$.

Definition 1.2.3. A map between posets regarded as categories is

+) A functor $f : \mathsf{P} \to \mathsf{P}'$

1.3. **Posets as Topological Spaces.**

Definition 1.3.1 (Downwards Closed)**.** A subset $\mathsf{S} \subset \mathsf{P}$ is downwards closed if for all $s \in \mathsf{S}$, if $s' \leq s \in \mathsf{P}$ then $s' \in \mathsf{S}$.

Definition 1.3.2. The poset topology on P is the topology such that a subset S is closed if and only if it is downwards closed.

Remark 1.3.3. This topology is sometimes known as the Alexandroff Topology.

Example 1.3.4. The poset topology on [1] is the simplest example of a non Hausdorff space.

Notation 1.3.5. Each element $p \in \mathsf{P}$ determines a closed subset

$$U_p := \{q \in P \mid q \leq p\}$$

This gives a map of posets from P into the set of closed subsets of P

$$P \hookrightarrow \mathsf{cls}(P)$$

Proposition 1.3.6. *A map $f : \mathsf{P} \to \mathsf{P}'$ is continuous if and only if it is a map of posets.*

Proof. First assume that $f : \mathsf{P} \to \mathsf{P}'$ is continuous. Let $p \leq q \in P$. We seek to show that $f(p) \leq f(q)$. This is equivalent to showing that $f(p) \in U_{f(q)}$. since $f(q) \in U_{f(q)}$ then $q \in f^{-1}(U_{f(q)})$. Since f is continuous, then $p \in f^{-1}(U(q))$. Therefore $f(p) \in U_{f(q)}$.

Now assume that f is a map of posets. We seek to show that the preimage of a closed set $U \subset_{\mathsf{cls}} P'$ is closed. Let $q \in f^{-1}(U)$. For $p \leq q$, we know that $f(p) \leq f(q)$. Since we know that $f(q)$ is in U, and U is closed, then $f(p) \in U$. Therefore $p \in f^{-1}(U)$. □

1.4. **The category of Posets.**

Definition 1.4.1. The category of Poset is the following category.

+) The objects are posets P

+) The morphisms are maps between posets.

Remark 1.4.2. The category of Poset embeds fully faithfully into Top by Proposition 1.3.6

Remark 1.4.3. The category of Poset embeds fully faithfully into Cat.

Let P be a poset. The product of two elements in P, if it exists, is given by

$$p \times q = \mathsf{max}\{a \mid a \leq p, q\}$$

ie, the product is the greatest lower bound of p and q.

Proof. Recall the universal property of products $p \times q$ in a given category P.

$$\begin{tikzcd}
 & & p \\
m \arrow[urr] \arrow[drr] \arrow[r, dashed] & p \times q \arrow[ur] \arrow[dr] & \\
 & & q
\end{tikzcd}$$

Note that existence of the arrow $m \to p \times q$ implies that it is unique since we are working in a poset. Recall now that there is an arrow from $p \times q \to p$ and $p \times q \to q$ if and only if $p \times q \leq p$ and $p \times q \leq q$. Therefore $p \times q \in \{a \mid a \leq p \times q\}$. Finally we know that this must be the maximum element of this set, since for $m \in \{a \mid a \leq p, q\}$, we need that $a \leq p \times q$ by again looking at the universal property. Therefore we have that

$$p \times q = \mathsf{max}\{a \mid \leq p \times q\}$$

This can also be said as the product $p \times q$ is the final) object in the following pullback.

$$\begin{tikzcd}
P_{\leq p,q} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\leq q} \arrow[d, hook] \\
P_{\leq p} \arrow[r, hook] & P
\end{tikzcd}$$

□