# 5. The Standard Model

After having developed the theoretical foundations of QFTs, we are now in the position to introduce the Standard Model (SM) of particle physics. The SM is a relativistic and renormalizable QFT that describes all known elementary particles and their interactions, except gravity.

The particles in the SM have spin 0, 1/2 or 1, and they thus belong to the class of particles that we considered in the previous chapters. It turns out that massless spin-1 particles like photons or gluons play a prominent role in nature. As we have discussed towards the end of chapter 3, massless spin-1 particles naturally lead to the concept of <u>gauge symmetries</u>, and the SM is indeed heavily based on this principle.

Whereas the gauge group associated with the electromagnetic interaction is well familiar from classical electrodynamics, the

QFT of the strong interaction is based on a more complex non-abelian gauge group, which as we will see leads to a completely different characteristics of the strong interaction. The particles that mediate the weak interaction - the W and Z bosons - are on the other hand not massless, but the underlying QFT is nevertheless based on a gauge symmetry, which is spontaneously broken via the Higgs mechanism.

The SM has been tested with ever increasing precision over the past decades, but it is nevertheless commonly thought to be incomplete. The main goal of today's particle accelerators like the Large Hadron Collider (LHC) at CERN therefore consists in revealing some phenomena that cannot be explained within the SM, which would give us a hint clue about the theory that completes it.

The outline of this chapter is as follows. We will first introduce in turn the QFTs of the electromagnetic, strong and weak interactions, before we summarise the Lagrangian and the particle content of the SM. We finally conclude this chapter with a brief discussion about the shortcomings and the open questions of the SM.

A final remark about the intention of this chapter. Our goal consists in giving a basic introduction to the SM of particle physics, without explaining the new theoretical concepts – like non-abelian gauge invariance, spontaneous symmetry breaking or renormalisation – in detail. A more careful and elaborate discussion of these concepts will be given in TPP 2.

## 5.1 Electromagnetic interactions

Quantum Electrodynamics (QED) is the QFT of photons and electrically charged (massive) spin-1/2 particles. For concreteness, we will restrict our attention to electrons here with mass $m = 0{,}51$ MeV and charge $-e$ (in natural units one has $\alpha \equiv \frac{e^2}{4\pi} \simeq \frac{1}{137}$).

As we learned in the previous chapters, electrons are described by a four-component Dirac field

$$\psi(x) = \sum_{s=\pm 1/2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big( u(p,s)\, e^{-ipx}\, \underbrace{a(p,s)}_{\text{annihilates electrons}} + v(p,s)\, e^{ipx}\, \underbrace{b^+(p,s)}_{\text{creates positrons}} \Big)$$

with coefficients $u(p,s)$ and $v(p,s)$ that we derived in detail in section 3.4. There we also saw that the free propagation of electrons (and positrons) is described by the Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}\,(i\not{\partial} - m)\,\psi$$

Photons are neutral, massless spin-1 particles that are described by a real vector field

$$A^\mu(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left( \varepsilon^\mu(p,\sigma)\, e^{-ipx}\, a(p,\sigma) + \varepsilon^\mu(p,\sigma)^*\, e^{ipx}\, a^\dagger(p,\sigma) \right)$$

Although photons only have two physical (transverse) polarisations, it is convenient to work with a field operator that describes additional unphysical polarisations to formulate a Lorentz-covariant theory. As we discussed in section 3.5 this can be achieved in Lorenz gauge $\partial_\mu A^\mu(x) = 0$, which can however only be implemented on the level of the physical states. The Gupta-Bleuler condition then ensures that the unphysical polarisations do not contribute to physical observables. The free propagation of photons in Lorenz gauge is then described by the Lagrangian

$$\mathcal{L}_{\text{vector}} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{Lorentz invariant}} \underbrace{- \frac{1}{2\xi} (\partial_\mu A^\mu)^2}_{\substack{\text{gauge-fixing term} \\ \text{(refers to Lorenz gauge)}}}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor and $\xi$ an unphysical gauge parameter.

We also argued in section 3.5 that massless spin-1 particles need to be coupled to conserved currents. The Dirac Lagrangian indeed has an internal symmetry, which provides such a current. In the tutorials we showed that the invariance under (global) phase transformations

$$\psi'(x) = e^{ie\omega}\,\psi(x) \qquad \omega \in \mathbb{R}$$

leads to a Noether current

$$j^\mu = -e\,\bar{\psi}\gamma^\mu\psi$$

which transforms as a 4-vector under Lorentz transformations. Although the vector field $A^\mu$ does <u>not</u> transform as a 4-vector in the massless case, the term $j^\mu A_\mu$ then transforms as a Lorentz scalar <u>if</u> the current $j^\mu$ is conserved (see page 210). We may therefore add the following interaction term to the Lagrangian

$$\mathcal{L}_{int} = e\,\bar{\psi}\gamma^\mu\psi A_\mu$$

where the electric charge $e$ plays the role of a coupling constant. Given the mass dimensions of the fields (~ page 228) $[\psi] = 3/2$ and $[A^\mu] = 1$, we see that $[e] = 0$, i.e. the QED interaction is renormalisable.

all couplings have non-negative mass dimension

Leaving the issues of gauge fixing aside for the moment, the Lagrangian of QED becomes

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{vector} + \mathcal{L}_{int}$$

$$= \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

which is Lorentz invariant*, renormalisable and invariant under global $U(1)$ transformations $\psi'(x) = e^{ie\omega}\psi(x)$.

As anticipated in section 3.5, the QED Lagrangian has in fact another important symmetry; it is invariant under local $U(1)$ transformations of the form

$$\psi'(x) = e^{ie\omega(x)}\psi(x)$$

$$A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}\omega(x)$$

Let us verify explicitly that the Lagrangian is invariant under these gauge transformations.

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* More precisely, the action $S_{QED} = \int d^4x\, \mathcal{L}_{QED}$ is Lorentz invariant.

We first note that the field-strength tensor is gauge invariant

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

$$= \partial_\mu A_\nu + \partial_\mu \partial_\nu \omega - \partial_\nu A_\mu - \partial_\nu \partial_\mu \omega$$

$$= F_{\mu\nu}$$

It is furthermore easy to see that the Dirac mass term is invariant

$$m\bar{\psi}'\psi' = m\bar{\psi}e^{-ie\omega}e^{ie\omega}\psi = m\bar{\psi}\psi$$

but the kinetic term of the Dirac field is not

$$\bar{\psi}' i\not{\partial}\psi' = \bar{\psi}e^{-ie\omega} i\not{\partial} e^{ie\omega}\psi$$

$$= \bar{\psi} i\not{\partial}\psi - e\bar{\psi}(\not{\partial}\omega)\psi$$

The second term is, however, exactly canceled by the transformation law of the photon field in the interaction term

$$e\bar{\psi}'\gamma^\mu\psi' A'_\mu = e\bar{\psi}e^{-ie\omega}\gamma^\mu e^{ie\omega}\psi(A_\mu + \partial_\mu\omega)$$

$$= e\bar{\psi}\gamma^\mu\psi A_\mu + e\bar{\psi}(\not{\partial}\omega)\psi$$

and the QED Lagrangian is thus indeed gauge invariant.

The above discussion motivates the definition of a covariant derivative

$$D_\mu \equiv \partial_\mu - ieA_\mu$$

Whereas the ordinary derivative $\partial_\mu \psi$ transforms non-trivially under gauge transformations, the covariant derivative $D_\mu \psi$ transforms like the Dirac field itself

$$\begin{aligned} D'_\mu \psi' &= (\partial_\mu - ieA'_\mu)\psi' \\ &= (\partial_\mu - ieA_\mu - ie(\partial_\mu \omega))\, e^{ie\omega} \psi \\ &= e^{ie\omega}\partial_\mu \psi + ie(\partial_\mu \omega)\, e^{ie\omega}\psi - e^{ie\omega} ieA_\mu \psi - ie(\partial_\mu \omega) e^{ie\omega}\psi \\ &= e^{ie\omega} D_\mu \psi \end{aligned}$$

The covariant derivative can therefore easily be used as a building block to construct gauge-invariant terms like $\bar{\psi} D_\mu \psi$.

In terms of the covariant derivative, the QED Lagrangian (still before gauge fixing) takes a particular compact form

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

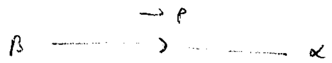
Remarks:

- The QED Lagrangian can actually be directly obtained from the free Lagrangian $\bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ by replacing the ordinary derivative by a covariant one. This procedure - known as minimal coupling - holds in the general case: Whenever a theory is invariant under global $U(1)$ transformations, it will automatically be invariant under local $U(1)$ transformations if one replaces $\partial_\mu \to D_\mu$.
- The QED Lagrangian is the most general renormalisable Lagrangian that is compatibel with Lorentz, gauge and parity invariance.
- As we discussed in Section 3.5, the quantisation of the electromagnetic field requires to add a gauge-fixing term to the Lagrangian, which obviously breaks gauge invariance since one has chosen to work in a specific gauge. In Lorentz gauge this term is

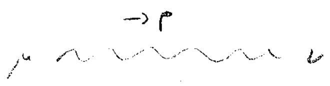
$$\mathcal{L}_{\text{gauge-fix}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Starting from the QED Lagrangian, we can derive the momentum-space Feynman rules for the computation of scattering matrix elements in analogy to Yukawa theory (see page 299 - 303).

1) For each internal line



$$= \frac{i(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}$$



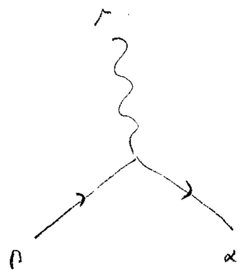
$$= \frac{i}{p^2+i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\varepsilon}\right]$$

general Lorentz gauge

2) For each vertex

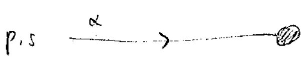
$\exp\left(-i\int d^4x\, \mathcal{H}_I(x)\right)$

$\mathcal{H}_I(x) = -e\bar\psi\gamma^\mu\psi A_\mu$

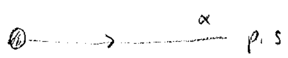


$$= ie\gamma^\mu_{\alpha\beta}$$

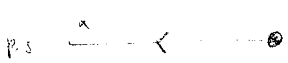
3) For each external line



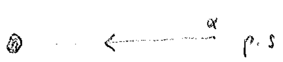
$= u_\alpha(p,s)$ incoming electron



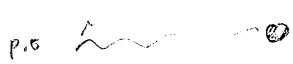
$= \bar u_\alpha(p,s)$ outgoing electron



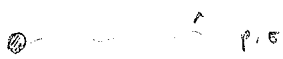
$= \bar v_\alpha(p,s)$ incoming positron



$= v_\alpha(p,s)$ outgoing positron



$= \varepsilon^\mu(p,\sigma)$ incoming photon



$= \varepsilon^\mu(p,\sigma)^*$ outgoing photon

4) Impose momentum conservation at each vertex and integrate over all undetermined momenta $\int \frac{d^4p}{(2\pi)^4}$.

5) Multiply with the fermion sign of the diagram.

Remarks:

- Similar to Yukawa theory, there are no symmetry factors in QED and the fermion signs again arise from the interchange of identical external fermions and from closed fermion loops.

- The photon propagator takes a particular simple form in Feynman gauge with $\xi = 1$

$$= \frac{i}{p^2+i\varepsilon}\left(-g^{\mu\nu}\right)$$

As discussed in section 3.5, it describes the propagation of a virtual photon with two physical (transverse) and two unphysical (time-like and longitudinal) polarisations.

- As one is only interested in scattering processes of physical photons, the external photon states are on the other hand always transverse.

The polarisation sum that is needed to compute spin-averaged squared transition matrix elements is then given by (see page 217)

$$\sum_{\sigma=\pm 1} \varepsilon^\mu(k,\sigma)\,\varepsilon^\nu(k,\sigma)^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} & \\ 0 & & & \end{pmatrix}^{\mu\nu}$$

which can be written in terms of an auxiliary vector $n^\mu = (1,0,0,0)$ in the form

$$\sum_{\sigma=\pm 1} \varepsilon^\mu(k,\sigma)\,\varepsilon^\nu(k,\sigma)^* = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n} - \frac{k^\mu k^\nu}{(k\cdot n)^2}$$

$\mu=0, \nu=0$ : $\quad -1 + \frac{k^0 + k^0}{k^0} - \frac{(k^0)^2}{(k^0)^2} = -1+2-1 = 0$ ✓

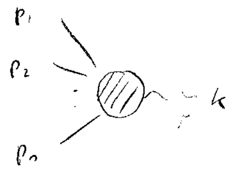
$\mu=0, \nu=i$ : $\quad \frac{k^i}{k^0} - \frac{k^0 k^i}{(k^0)^2} = 0$ ✓ $\quad$ (similar for $\mu=i, \nu=0$)

$\mu=i, \nu=j$ : $\quad \delta^{ij} - \frac{k^i k^j}{(k^0)^2} = \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2}$ ✓

since $k^2 = (k^0)^2 - |\vec{k}|^2 = 0$

The terms proportional to $k^\mu$ or $k^\nu$ in the polarisation sum do not contribute, however, in the calculation of scattering matrix elements as a consequence of gauge invariance.

More specifically, we will show in TPP2 that a transition matrix element* with $n$ external fermions of momenta $p_i$ ($p_i^2 = m_i^2$) and an external photon of momentum $k$ ($k^2 = 0$)



$$\mathcal{M}(k, p_1, \dots, p_n) = \varepsilon^\mu(k,\sigma)^* \mathcal{M}_\mu(k, p_1, \dots p_n)$$

fulfills the exact relation

$$k^\mu \mathcal{M}_\mu(k, p_1, \dots, p_n) = 0$$

This relation – known as <u>Ward identity</u> – is a consequence of gauge invariance, and it can most easily be derived in the path-integral formulation. Instead of presenting an alternative (and much more tedious) proof here, we will simply verify the Ward identity for a specific example in the tutorials.

The Ward identity thus ensures that the terms proportional to $k^\mu$ or $k^\nu$ do not contribute to S-matrix elements, and one may therefore simply replace the polarisation sum by

$$\sum_{\sigma=\pm 1} \varepsilon^\mu(k,\sigma)\, \varepsilon^\nu(k,\sigma)^* \longrightarrow -g^{\mu\nu}$$

in practical calculations.

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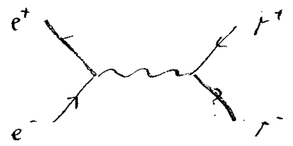
* The matrix element may actually involve an arbitrary number of additional external photons.

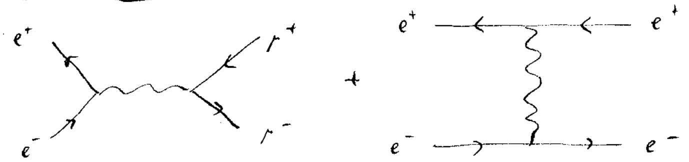
With the Feynman rules at hand, we are now in the position to calculate scattering cross sections of elementary QED processes to leading order in the perturbative expansion. Some prominent examples are

- $e^+e^- \to \mu^+\mu^-$



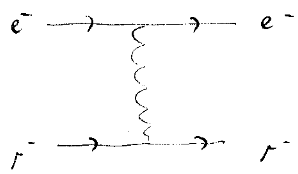

electron-positron annihilation into a muon pair

- $e^+e^- \to e^+e^-$




Bhabha scattering

- $e^-\mu^- \to e^-\mu^-$

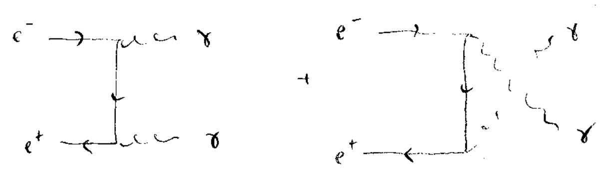



electron-muon scattering

- $e^-\gamma \to e^-\gamma$


Compton scattering

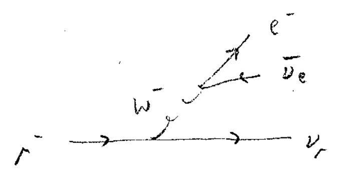
- $e^+e^- \to \gamma\gamma$




photon pair production

As an example, we consider the process $e^+e^- \to \mu^+\mu^-$ in detail here, which is an important reference process for $e^+e^-$ colliders.

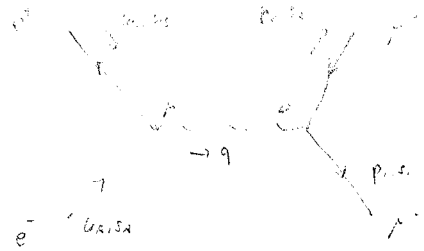
A few words about muons, which we have not introduced yet. Muons are very similar to electrons (spin-1/2, same electric charge) except that they are much heavier with $m_\mu \simeq 106$ MeV. They are moreover unstable since they can decay via the weak interaction




with a mean lifetime $\tau_\mu \simeq 2\,\mu s$.

As the center-of-mass energy required to produce a muon pair at an $e^+e^-$ collider, $E_{cm} \geq 2m_\mu \gg m_e$, we will neglect the electron mass in the following. This approximation is valid up to corrections of order $\frac{m_e}{m_\mu} \approx \frac{1}{200}$, which is comparable to the expected size of the perturbative corrections of order $\alpha \simeq \frac{1}{137}$.

We first assign momenta and spin configurations to the external particles and apply the Feynman rules to write down the transition matrix element (in a general Lorenz gauge)




$$q = k_A + k_B = p_1 + p_2$$

$$i\mathcal{M} = \bar{v}(k_B, s_B)\, ie\gamma^\mu\, u(k_A, s_A)\; \bar{u}(p_1, s_1)\, ie\gamma^\nu\, v(p_2, s_2)$$

$$\frac{i}{q^2 + i\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2 + i\varepsilon}\right]$$

We can then use the relations (~ page 183)

$$(\not{p} - m)\, u(p,s) = 0$$

$$(\not{p} + m)\, v(p,s) = 0$$

to show that the gauge-parameter dependent term of the photon propagator does not contribute

$$\bar{u}(p_1)\, \not{q}\, v(p_2) = \bar{u}(p_1)(\not{p}_1 + \not{p}_2)\, v(p_2)$$

$$= \bar{u}(p_1)(m_\mu - m_\mu)\, v(p_2) = 0$$

and a similar relation holds for the electron string.

We are thus left with

$$i\mathcal{M} = \frac{ie^2}{q^2}\, \bar{v}(k_B, s_B)\,\gamma^\nu\, u(k_A, s_A)\; \bar{u}(p_1, s_1)\,\gamma_\nu\, v(p_2, s_2)$$

For the complex conjugate expression, we need

$$[\bar{u}(p_1,s_1)\,\gamma_\nu\, v(p_2,s_2)]^*$$

since the result is a number

$$= [u^\dagger(p_1,s_1)\,\gamma^0\gamma_\nu\, v(p_2,s_2)]^\dagger$$

$$= v^\dagger(p_2,s_2)\,\underbrace{\gamma_\nu^\dagger}_{\gamma^0\gamma_\nu\gamma^0}\,\underbrace{\gamma^{0\dagger}}_{\gamma^0}\, u(p_1,s_1) = \bar{v}(p_2,s_2)\,\gamma_\nu\, u(p_1,s_1)$$

and similarly for the second Dirac string. It follows

$$-i\mathcal{M}^* = -\frac{ie^2}{q^2}\, \bar{u}(k_A, s_A)\,\gamma^\nu\, v(k_B, s_B)\; \bar{v}(p_2, s_2)\,\gamma_\nu\, u(p_1, s_1)$$

As in Yukawa theory (→ pages 305-310), we only consider the spin-averaged squared transition matrix element here with

$$|\overline{\mathcal{M}}|^2 \equiv \frac{1}{2\cdot 2}\sum_{s_A, s_B}\sum_{s_1, s_2} |\mathcal{M}|^2$$

which after using the spin sums

$$\sum_s u(p,s)\,\bar{u}(p,s) = \not{p} + m$$

$$\sum_s v(p,s)\,\bar{v}(p,s) = \not{p} - m$$

leads to a trace of Dirac matrices for each Dirac string.

We obtain

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\,\frac{e^4}{q^4}\,\underbrace{\mathrm{tr}\left(\not{k}_A\,\gamma^\nu\,\not{k}_B\,\gamma^\mu\right)}_{\text{electron trace}}\;\underbrace{\mathrm{tr}\left((\not{p}_1+m_\mu)\,\gamma_\mu\,(\not{p}_2-m_\mu)\,\gamma_\nu\right)}_{\text{muon trace}}$$

To evaluate the traces, we use the identities

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = 4\,g^{\mu\nu}$$

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = 0$$

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

$$\Rightarrow\quad \mathrm{tr}\left(\not{k}_A\,\gamma^\nu\,\not{k}_B\,\gamma^\mu\right) = 4\left(k_A^\nu k_B^\mu - (k_A\cdot k_B)\,g^{\mu\nu} + k_A^\mu k_B^\nu\right)$$

$$\mathrm{tr}\left((\not{p}_1+m_\mu)\,\gamma_\mu\,(\not{p}_2-m_\mu)\,\gamma_\nu\right)$$

$$= 4\left(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \left[(p_1\cdot p_2)+m_\mu^2\right]g_{\mu\nu}\right)$$

$\Rightarrow$

$$\overline{|\mathcal{M}|^2} = \frac{4e^4}{q^4}\Big\{2\,(k_A\cdot p_2)(k_B\cdot p_1) + 2\,(k_A\cdot p_1)(k_B\cdot p_2)$$

$$- 2\,(k_A\cdot k_B)\left[(p_1\cdot p_2)+m_\mu^2\right] - 2\,(k_A\cdot k_B)(p_1\cdot p_2)$$

$$+ 4\,(k_A\cdot k_B)\left[(p_1\cdot p_2)+m_\mu^2\right]\Big\}$$

$$= \frac{8e^4}{q^4}\Big\{(k_A\cdot p_2)(k_B\cdot p_1) + (k_A\cdot p_1)(k_B\cdot p_2) + m_\mu^2\,(k_A\cdot k_B)\Big\}$$

which may also be expressed in terms of Mandelstam variables

$$s = (k_A + k_B)^2 = 2k_A k_B = (p_1 + p_2)^2 = 2m_f^2 + 2p_1 p_2$$

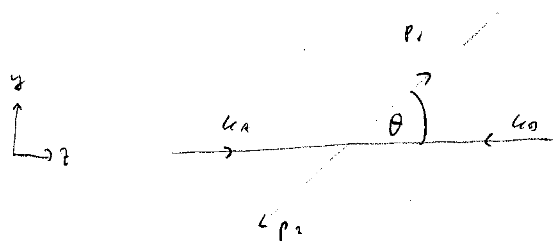
$$t = (k_A - p_1)^2 = m_f^2 - 2k_A p_1 = (p_2 - k_B)^2 = m_f^2 - 2k_B p_2$$

$$u = (k_A - p_2)^2 = m_f^2 - 2k_A p_2 = (p_1 - k_B)^2 = m_f^2 - 2k_B p_1$$

with $s + t + u = 2m_f^2$ as

$$|\overline{\mathcal{M}}|^2 = \frac{2e^4}{s^2} \left\{ (m_f^2 - u)^2 + (m_f^2 - t)^2 + 2m_f^2 s \right\}$$

In the center-of-mass frame the two independent kinematic variables are the center-of-mass energy $E_{cm}$ and the scattering angle $\theta$, which we introduce as




$$k_A^\mu = \left(\frac{E_{cm}}{2}, 0, 0, \frac{E_{cm}}{2}\right)$$

$$k_B^\mu = \left(\frac{E_{cm}}{2}, 0, 0, -\frac{E_{cm}}{2}\right)$$

$$p_1^\mu = \left(\frac{E_{cm}}{2}, 0, |\vec{p}|\sin\theta, |\vec{p}|\cos\theta\right)$$

$$p_2^\mu = \left(\frac{E_{cm}}{2}, 0, -|\vec{p}|\sin\theta, -|\vec{p}|\cos\theta\right)$$

such that $k_A^2 = k_B^2 = 0$ and $p_1^2 = p_2^2 = m_f^2$

$$\rightarrow \quad |\vec{p}|^2 = \frac{E_{cm}^2}{4} - m_f^2$$

$$\Rightarrow \quad s = 2k_A k_B = E_{cm}^2$$

$$t = m_f^2 - 2k_A p_1 = m_f^2 - E_{cm}\left(\frac{E_{cm}}{2} - |\vec{p}|\cos\theta\right)$$

$$u = m_f^2 - 2k_A p_2 = m_f^2 - E_{cm}\left(\frac{E_{cm}}{2} + |\vec{p}|\cos\theta\right)$$

In terms of these variables, the squared transition matrix element becomes

$$|\bar{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left\{ s\left(\frac{\sqrt{s}}{2} + |\vec{p}|\cos\theta\right)^2 + s\left(\frac{\sqrt{s}}{2} - |\vec{p}|\cos\theta\right)^2 + 2m_l^2 s \right\}$$

$$= \frac{2e^4}{s}\left\{ \frac{s}{2} + 2|\vec{p}|^2\cos^2\theta + 2m_l^2 \right\}$$

$$= e^4\left\{ 1 + \frac{4}{s}\left(\frac{s}{4} - m_l^2\right)\cos^2\theta + \frac{4m_l^2}{s} \right\}$$

$$= e^4\left\{ 1 + \cos^2\theta + \frac{4m_l^2}{s}\left(1 - \cos^2\theta\right) \right\}$$

In the tutorials we furthermore showed that the differential cross section of a $2 \to 2$ scattering process can be written in the form

$$\frac{d\sigma}{d\cos\theta} = \frac{\lambda(s, m_1^2, m_2^2)}{32\pi s\, \lambda(s, m_A^2, m_B^2)}\, |\mathcal{M}(k_A k_B \to p_1 p_2)|^2$$

where $\theta$ is the scattering angle in the center-of-mass frame (defined between $k_A$ and $p_1$) and

$$\lambda(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$$

is the Källén function. In our case, we have

$$\lambda(s, 0, 0) = s$$

$$\lambda(s, m_l^2, m_l^2) = \sqrt{s^2 + 2m_l^4 - 4sm_l^2 - 2m_l^4}$$

$$= s\sqrt{1 - \frac{4m_l^2}{s}}$$

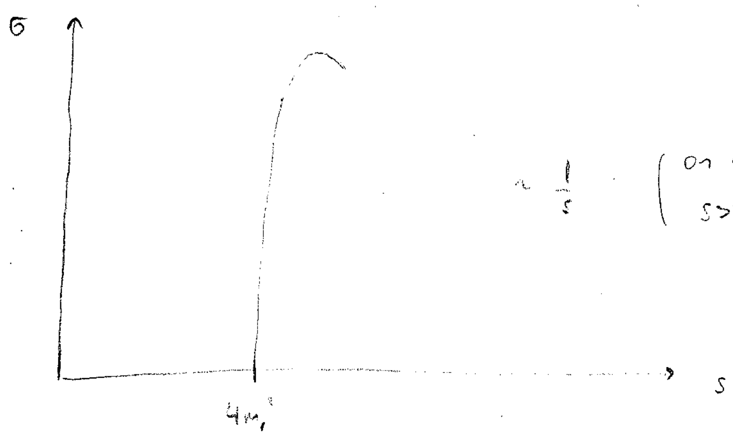
The differential cross section for muon pair production at $e^+e^-$ colliders finally becomes

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}\sqrt{1-\frac{4m_\mu^2}{s}}\left\{1+\cos^2\theta+\frac{4m_\mu^2}{s}\left(1-\cos^2\theta\right)\right\} \qquad \alpha=\frac{e^2}{4\pi}$$

which entails a characteristic angular dependence for an annihilation (s-channel) process into a massless spin-1 particle.

For the total cross section, we then obtain

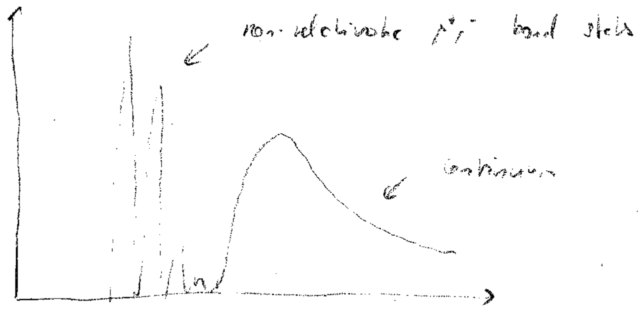
$$\sigma = \int_{-1}^{1} d\cos\theta\,\frac{d\sigma}{d\cos\theta} = \frac{4\pi\alpha^2}{3s}\underbrace{\sqrt{1-\frac{4m_\mu^2}{s}}}_{\text{from phase space}}\left\{1+\underbrace{\frac{2m_\mu^2}{s}}_{\text{from matrix element}}\right\}$$



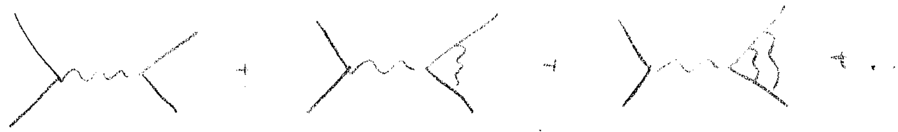

$\sim \frac{1}{s}$ (on dimensional grounds since for $s \gg m_\mu^2$ this is the only physical scale)

threshold for muon pair production

For $s \simeq 4m_\mu^2$ the muons move non-relativistically and they may thus form $\mu^+\mu^-$ "atoms". These bound-state contributions lead to a pronounced resonance structure just below the $\mu^+\mu^-$ threshold (since the non-relativistic binding energies are negative)




Whereas our leading-order calculation gave a good description of the continuum contribution, the bound states are only found after summing up an infinite number of Feynman diagrams



$$1 \quad + \quad \frac{\alpha}{v} \quad + \quad \frac{\alpha^2}{v^2} \quad + \ldots$$

For small muon velocities $v \ll 1$ (in the centre-of-mass frame), it turns out that each term in the perturbative expansion is of similar size in this particular region of phase space.

## 5.2 Strong interactions

Quantum Chromodynamics (QCD) is the QFT of gluons (massless spin-1 particles) and quarks (massive spin-1/2 particles). As in QED the quantisation of the gluon field calls for a gauge symmetry, but the underlying gauge group $SU(3)$ is more complex, which leads to a richer structure of the gauge theory.

But before we start deriving the QCD Lagrangian, we first review some basic properties of the group $SU(3)$ from chapter 1:

- The dimension of the group $SU(3)$ is $3^2-1=8$.
- An arbitrary group element in the vicinity of the identity element can be represented via the exponential map

$$U = e^{i\varepsilon^A T^A} \in SU(3) \qquad A=1,\dots,8$$

where $\varepsilon^A$ are real coefficients and $T^A$ hermitian and traceless generators that satisfy the <u>Lie algebra</u>

$$[T^A, T^B] = i f^{ABC}\, T^C$$

The structure constants $f^{ABC}$ of $su(3)$ are totally antisymmetric and real*, and they fulfill the Jacobi identity

$$f^{ABD} f^{CDE} + f^{CAD} f^{BDE} + f^{BCD} f^{ADE} = 0$$

- In the fundamental representation, the generators of $su(3)$ are given by the Gell-Mann matrices $\lambda^A$ (→ page 31)

$$T_F^A = \frac{\lambda^A}{2}$$

One usually refers to the fundamental representation by its dimension as 3.

- In contrast to $SU(2)$, the fundamental representation 3 and its complex conjugate representation $\bar{3}$ are not equivalent.

- Another important representation is the adjoint representation with

$$\left(T_{adj}^B\right)_{AC} = i f^{ABC}$$

The adjoint representation of $SU(3)$ is a real representation and its dimension is equal to the dimension of the group, i.e. 8.

---

* The explicit values of the structure constants can be found on page 31.

- The group $SU(3)$ has two Casimir operators, the most important one being the <u>quadratic Casimir operator</u> $T_R^A T_R^A$ (in a representation $R$). According to the Lemma of Schur (~ page 52), the Casimir operators are proportional to the identity operator in an irreducible representation

$$T_R^A \, T_R^A = C_R \, \mathbb{1}_R$$

One often refers to the constant of proportionality $C_R$ itself as the quadratic Casimir operator.

- The <u>Dynkin index</u> $T_R$ of an irreducible representation $R$ is defined as

$$\mathrm{Tr}\,[\underbrace{T_R^A \, T_R^B}_{\text{generators in representation } R}] \equiv \underbrace{T_R}_{\text{Dynkin index of representation } R} \delta^{AB}$$

We aim at constructing a Lagrangian that is invariant under

<u>local</u> SU(3) transformations

$$\psi'(x) = e^{i\varepsilon^A(x)T^A}\,\psi(x)$$

where the quark field is supposed to belong to the fundamental representation of SU(3). For a given quark species of mass $m$, the field operator thus has three components according to its SU(3) "colour" charge

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \psi_{red}(x) \\ \psi_{blue}(x) \\ \psi_{green}(x) \end{pmatrix}$$

each of these entries has 4 spinor components

Making the SU(3) indices explicit, the infinitesimal transformation law of the quark field can then be written in the form

$$\psi'_a(x) = \psi_a(x) + i\,\varepsilon^A(x)\,T^A_{ab}\,\psi_b(x)$$

with $A = 1,\dots,\dim G = 8$ and $a,b = 1,\dots,\dim R = 3$.

As in QED one further introduces the <u>covariant derivative</u>

$$\mathcal{D}_{\mu,ab} \equiv \partial_\mu \delta_{ab} - i g_s A_\mu^A T^A_{ab}$$

which is a 3×3 matrix in colour space and depends on eight gluon fields $A_\mu^A$.

The covariant derivative of the quark field should then again transform like the quark field itself, i.e. we impose the transformation law

$$\mathcal{D}'_\mu \psi' \equiv U \mathcal{D}_\mu \psi$$

with $U = e^{i\varepsilon^A T^A}$. It is furthermore convenient to introduce the shorthand notation $A_\mu(x) \equiv A_\mu^A(x) T^A$, which is again a 3×3 matrix. We then obtain

$$\begin{aligned}
\mathcal{D}'_\mu \psi' &= (\partial_\mu - i g_s A'_\mu) U \psi \\
&= U \partial_\mu \psi + (\partial_\mu U) \psi - i g_s A'_\mu U \psi \\
&= U \mathcal{D}_\mu \psi + \underbrace{i g_s U A_\mu \psi - i g_s A'_\mu U \psi + (\partial_\mu U) \psi}_{\stackrel{!}{=} 0}
\end{aligned}$$

$$\Rightarrow \quad A'_\mu = U A_\mu U^+ - \frac{i}{g_s} (\partial_\mu U) U^+$$

$U^{-1} = U^+$

For infinitesimal transformations this implies

$$A'_\mu = A'^A_\mu T^A$$

$$= A^A_\mu T^A + i\varepsilon^B T^B A^A_\mu T^A - i A^A_\mu T^A \varepsilon^B T^B - \frac{i}{g_s} i\partial_\mu \varepsilon^A T^A$$

$$= A^A_\mu T^A + i\varepsilon^B A^A_\mu \, i f^{BAC} T^C + \frac{1}{g_s}\partial_\mu \varepsilon^A T^A$$

$$= \left(A^A_\mu + \frac{1}{g_s}\partial_\mu \varepsilon^A + f^{ABC} A^B_\mu \varepsilon^C\right) T^A$$

$$\Rightarrow \quad A'^A_\mu = A^A_\mu + \frac{1}{g_s}\partial_\mu \varepsilon^A + f^{ABC} A^B_\mu \varepsilon^C$$

Notice that the last term contributes even under <u>global</u> SU(3) transformations, and it therefore gives an additional contribution to the Noether current (→ tutorials).

One easily verifies that the above relations reduce to the familiar QED results for $U = e^{ie\omega}$, $g_s = e$, $f^{ABC} = 0$ and $\varepsilon^A = e\omega$.

We finally need to construct the non-abelian field-strength tensor. To do so, we first note that in QED

$$\frac{i}{e}\left[D_\mu, D_\nu\right]\psi$$

$$= \frac{i}{e}\left\{ \underbrace{\left[\partial_\mu, \partial_\nu\right]}_{=0} - ie\left[A_\mu, \partial_\nu\right] - ie\left[\partial_\mu, A_\nu\right] - e^2 \underbrace{\left[A_\mu, A_\nu\right]}_{=0} \right\}\psi$$

$$= A_\mu \partial_\nu \psi - \partial_\nu (A_\mu \psi) + \partial_\mu (A_\nu \psi) - A_\nu \partial_\mu \psi$$

$$= \left\{ -\partial_\nu A_\mu + \partial_\mu A_\nu \right\}\psi = F_{\mu\nu}\psi$$

$$\Longrightarrow \quad F_{\mu\nu} = \frac{i}{e}\left[D_\mu, D_\nu\right]$$

In QCD we define in analogy

$$G_{\mu\nu} \equiv \frac{i}{g_s}\left[D_\mu, D_\nu\right]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - i g_s \left[A_\mu, A_\nu\right]$$

Notice that the non-abelian field-strength tensor is <u>not</u> gauge invariant but transforms as

$$G'_{\mu\nu} = \frac{i}{g_s}\left[D'_\mu, D'_\nu\right]$$

$$= \frac{i}{g_s}\left\{ U D_\mu \underbrace{U^+ U}_{=1\!\!1} D_\nu U^+ - U D_\nu \underbrace{U^+ U}_{=1\!\!1} D_\mu U^+ \right\}$$

$$= U G_{\mu\nu} U^+$$

The combination $\mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$ is, however, gauge invariant.

Writing $\underbrace{G_{\mu\nu}(x)}_{3\times 3 \text{ matrix}} = G^A_{\mu\nu}(x)\, T^A$, we further have

$$G_{\mu\nu} = \left( \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g_s f^{BCA} A_\mu^B A_\nu^C \right) T^A$$

$$\Rightarrow \quad G^A_{\mu\nu} = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g_s f^{ABC} A_\mu^B A_\nu^C$$

which transforms as

$$\begin{aligned} G'_{\mu\nu} &= U\, G_{\mu\nu}\, U^+ \\ &= G^A_{\mu\nu} T^A + i \varepsilon^B T^B G^A_{\mu\nu} T^A - i G^A_{\mu\nu} T^A \varepsilon^B T^B \\ &= \left( G^A_{\mu\nu} + f^{ABC} G^B_{\mu\nu} \varepsilon^C \right) T^A \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad G'^A_{\mu\nu} &= G^A_{\mu\nu} + f^{ABC} G^B_{\mu\nu} \varepsilon^C \\ &= G^A_{\mu\nu} + i \varepsilon^C \underbrace{(T^C_{ad})_{AB}}_{i f^{ACB}} G^B_{\mu\nu} \end{aligned}$$

i.e. the field-strength tensor transforms in the adjoint representation of $SU(3)$ (which is not true for the gauge field because of the inhomogeneous term $\frac{1}{g} \partial_\mu \varepsilon^A$ in the transformation law).

Notice also that

$$\mathrm{Tr}\left[ G_{\mu\nu} G^{\mu\nu} \right] = G^A_{\mu\nu} G^{\mu\nu,B}\, \mathrm{Tr}\left( T^A T^B \right) = \underbrace{T_R}_{\text{Dynkin index in the representation of the fermions}} G^A_{\mu\nu} G^{\mu\nu,A}$$

We are now in the position to write down a gauge-invariant Lagrangian in analogy to QED

$$\boxed{\begin{aligned}\mathcal{L}_{QCD} &= \bar{\psi}(i\not{D}-m)\psi - \frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} \\ &= \bar{\psi}(i\not{\partial}-m)\psi - \frac{1}{2}\left(\partial_\mu A^A_\nu \partial^\mu A^{A,\nu} - \partial_\mu A^A_\nu \partial^\nu A^{A,\mu}\right) \\ &\quad + g_s \bar{\psi}\gamma^\mu T^A \psi A^A_\mu - g_s f^{ABC} (\partial_\mu A^A_\nu) A^{B,\mu} A^{C,\nu} \\ &\quad - \frac{g_s^2}{4} f^{ABC} f^{ADE} A^B_\mu A^C_\nu A^{D,\mu} A^{E,\nu}\end{aligned}}$$

<u>Remarks:</u>

- In contrast to QED, the QCD Lagrangian contains cubic and quartic self-interaction terms of the gauge bosons. As the strength of these interactions is constrained by gauge invariance, there exists however a single coupling constant in QCD. On dimensional grounds, one finds $[g_s]=0$, i.e. the QCD interactions are renormalisable

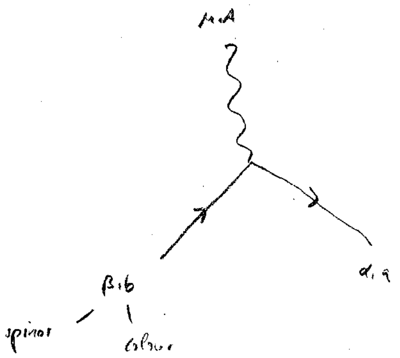
- In QED one has the freedom to rescale the generator $Q \to e_f Q$, and each fermion may therefore have a different electromagnetic charge, e.g. $e_e = -1$ for electrons, $e_u = +\frac{2}{3}$ for up quarks and $e_d = -\frac{1}{3}$ for down quarks. There is, however, no such freedom in non-abelian gauge theories since $[T^A, T^B] = i f^{ABC} T^C$.

- As in QED the quantisation of the gluon field requires to add a gauge-fixing term to the Lagrangian. In Lorenz gauge this term becomes

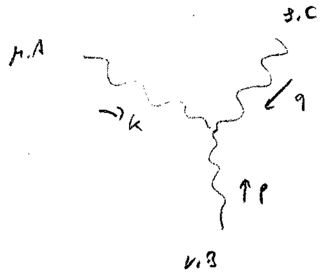
$$\mathcal{L}_{gauge-fix} = -\frac{1}{2\xi} (\partial^\mu A^A_\mu)^2$$

Due to the self-interactions of the gauge bosons, however the Gupta-Bleuler procedure cannot be applied in this case, and we will introduce an alternative method for the quantisation of non-abelian gauge fields in TPP2 using path-integral methods. There we will see that one has to add further degrees of freedom to the theory ("Faddeev-Popov ghosts") to ensure that the unphysical polarisations of the non-abelian gauge field do not contribute to S-matrix elements.

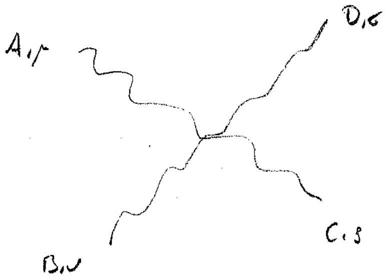
We postpone the derivation of the QCD Feynman rules to TPP2, and instead only quote the results here. For the vertices, one obtains




$$= i g_s \gamma^\mu_{\alpha\beta} T^A_{ab}$$

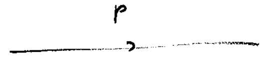


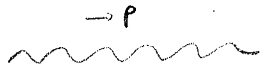

$$= g_s f^{ABC} \left[ g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu \right]$$




$$= -i g_s^2 \left[ f^{ABE} f^{CDE} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \right.$$
$$+ f^{ACE} f^{BDE} \left( g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$
$$\left. + f^{ADE} f^{BCE} \left( g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) \right]$$

The quark and gluon propagators are, on the other hand, diagonal in colour space and they can hence be directly taken over from QED.

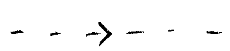
$\beta, b$  $\alpha, a$ $\quad = \dfrac{i(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}\,\delta_{ab}$

$\mu, A$  $\nu, B$ $\quad = \dfrac{i}{p^2 + i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\dfrac{p^\mu p^\nu}{p^2 + i\varepsilon}\right]\delta^{AB}$ general Lorenz gauge

The Feynman rules for external lines, loop integrations and fermion signs are also similar to the ones of QED.

Remarks:

- The above Feynman rules can only be used for the calculation of S-matrix elements to lowest order in perturbation theory. At higher orders, Faddeev-Popov ghosts appear in loop diagrams, and one needs additional Feynman rules to describe their propagation as well as their interaction with the gauge bosons

The Faddeev-Popov ghosts have another odd property: They have spin 0, but obey Fermi statistics ($\rightsquigarrow$ unphysical degrees of freedom, more in TPP 2).

- In contrast to QED, there are non-trivial symmetry factors in QCD due to the gluon self interactions. As in $\phi^4$-theory the symmetry factors can always be derived from the underlying Wick contractions. One finds e.g.

$s = 2$

$s = 2$

Notice that one has to divide the expressions for these diagrams by the symmetry factors.

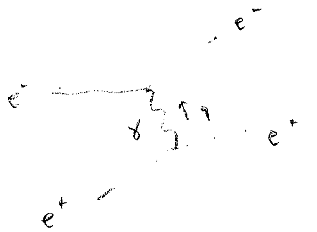
- As in QED, one can substitute the gluon polarisation sum by

$$\sum_{\sigma=\pm 1} \varepsilon_\mu^A(k,\sigma)\,\varepsilon_\nu^B(k,\sigma)^* \longrightarrow -g_{\mu\nu}\,\delta^{AB}$$

in the calculation of S-matrix elements.

The main difference between QED and QCD thus consists in the gluon self interactions. This has in fact an important physical consequence, which we can only appreciate in detail once we have discussed renormalisation in TPP 2.

At this level we instead have to satisfy ourselves with a qualitative discussion. To this end, we consider electron-positron scattering to lowest order in QED




$$i\mathcal{M} \sim \frac{e^2}{q^2}$$

Corresponds to $q^0 \sim mv^2$ and $|\vec{q}| \sim mv$ for a non-relativistic velocity $v \ll 1$

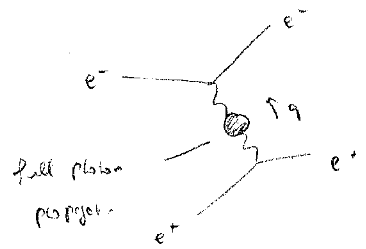
In the non-relativistic limit with $q^0 \ll |\vec{q}| \ll m$, this corresponds to a static Coulomb interaction, which can be verified by taking the Fourier transform

$$V(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\vec{x}}\, \frac{e^2}{-\vec{q}^{\,2}}$$

$$= -\frac{4\pi\alpha}{(2\pi)^3} \int_0^\infty dq \int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta} \int_0^{2\pi} d\varphi = -\frac{\alpha}{r}$$

At higher orders in perturbation theory, the photon propagator receives quantum corrections of the form

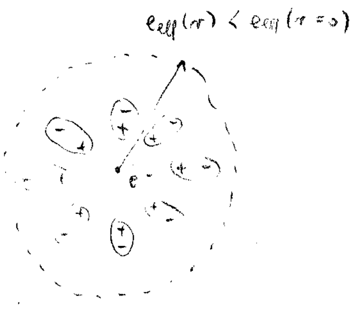
which modify the scattering process according to




$$iM \sim \frac{e^2_{eff}(q^2)}{q^2}$$

i.e. the quantum effects in the photon propagator can be absorbed into an <u>effective $q^2$-dependent coupling constant</u> $\alpha_{eff}(q^2) \equiv \frac{e^2_{eff}(q^2)}{4\pi}$. It turns out that the effective charge increases as $\vec{q}^{\,2}$ increases.

Intuitively, this can be understood as follows

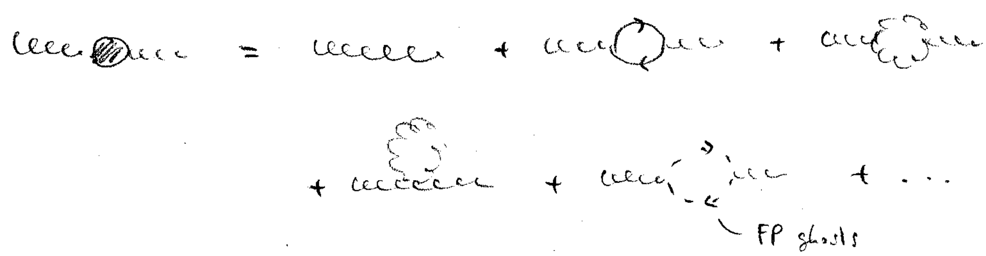



The electron polarises the vacuum, which acts as a dielectric medium, and the virtual pairs of charged particles <u>screen</u> the charge of that electron.

As $\vec{q}^{\,2}$ increases the photon probes more and more deeply into the cloud of virtual particles that surrounds the electron and the effective charge increases.

This effect has been confirmed experimentally. Whereas at low energies one obtains the familiar result from quantum mechanics with $\alpha_{eff}(q^2 \approx 0) = \alpha \sim \frac{1}{137}$, at LEP energies the effective coupling constant is slightly larger with $\alpha_{eff}(M_Z^2) \sim \frac{1}{128}$. $(M_Z \sim 90\ \text{GeV})$

A similar effect arises in QCD in quark-antiquark scattering, but the quantum corrections to the gluon propagator are more complex due to the gluon self interactions




The effective coupling constant $\alpha_s^{eff}(q^2) \equiv \frac{g_{eff}^2(q^2)}{4\pi}$ then again becomes $q^2$-dependent, but it turns out that it <u>decreases</u> as $\vec{q}^{\,2}$ increases.

In terms of the intuitive picture from above, this means that gluons give an additional contribution to the vacuum polarisation since they are themselves charged under the $SU(3)$ symmetry. It furthermore turns out that gluons have the opposite effect than quark-antiquark pairs; i.e. they <u>antiscreen</u> the colour charge of the original quark. On the quantitative level, one then finds that the antiscreening from virtual gluons dominates over the screening from virtual quark-antiquark pairs with the net effect that the effective strong coupling decreases when a gluon probes more deeply into the cloud of virtual particles.

The "running" of the effective strong coupling is moreover found to be more pronounced numerically. Whereas $\alpha_s^{eff}(M_Z^2) \sim 0.12$ at LEP energies, one has $\alpha_s^{eff}(M_b^2) \sim 0.2$ and $\alpha_s^{eff}(1\,GeV^2) = O(1)$. $(M_b \sim 5\,GeV)$

At high enough energies, quarks and gluons are thus only weakly coupled and one can perform perturbative calculations in QCD as in QED. The very fact that QCD is <u>asymptotically free</u>, i.e. $\alpha_s^{eff}(Q^2 \to \infty) \to 0$, is a characteristic property of non-abelian gauge theories. The discovery of asymptotic freedom in 1973 by Gross, Wilczek and Politzer was awarded with the Nobel Prize in Physics in 2004.

At low energies $Q \lesssim 2$ GeV, on the other hand, the strong interactions indeed become very strong and the perturbative expansion in $\alpha_s^{eff}(Q^2)$ breaks down. It therefore becomes very difficult to make quantitative predictions in QCD at low energies. The strength of the QCD interaction makes it however plausible that quarks and gluons form bound states at low energies. It is indeed not possible to directly observe free quarks or gluons in nature since they are always confined in colour-neutral hadrons. These hadrons either have the quantum numbers of $q\bar{q}$ or $qqq$ states.* The $q\bar{q}$ states are called mesons

---

* Notice that both tensor product representations

$$3 \otimes \bar{3} = 1 \oplus 8$$

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

induce a singlet representation, to which the colour-neutral hadron states belong.

and examples include pions, kaons or rho mesons. The qqq states, on the other hand, are called <u>baryons</u> of which the most prominent examples are protons or neutrons. One should not forget, however, that this "naive quark model" greatly oversimplifies the picture; the dynamics of these bound states is actually extremely complex and their particle content involves an arbitrary number of additional quark-antiquark and gluon fluctuations.

## 5.3. Weak interactions

The third fundamental interaction in nature is the weak interaction, which is described by a QFT of W- and Z-bosons (massive spin-1 particles) and (massive) spin-1/2 particles. For concreteness, we will consider up quarks (u), down quarks (d), electrons (e) and electron-neutrinos ($\nu_e$) in the following.

The starting point for the construction of the weak interactions is an SU(2) gauge theory with Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu}$$

where the index A runs from $A = 1, \dots, \dim G = 2^2 - 1 = 3$.

Remarks:

- The fermions again transform under the fundamental representation of SU(2) with

$$\psi'(x) = e^{i\varepsilon^A(x) T^A_F} \psi(x)$$

where $T^A_F = \frac{\sigma^A}{2}$ are given by the Pauli matrices.

The fermions are thus arranged in "flavour" doublets

$$\psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \qquad \text{or} \qquad \psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

each entry has 4 spinor components

3 colours and 4 spinor components

for the leptons and the quarks, respectively.

- The covariant derivative is a 2x2 matrix in flavour space

$$D_{\mu,ab} = \partial_\mu \delta_{ab} - ig W_\mu^A T_{ab}^A$$

which depends on the weak coupling constant $g$ and on three gauge bosons $W_\mu^1, W_\mu^2, W_\mu^3$.

- The non-abelian field-strength tensor

$$G_{\mu\nu}^A = \partial_\mu W_\nu^A - \partial_\nu W_\mu^A + g f^{ABC} W_\mu^B W_\nu^C$$

$\varepsilon^{ABC}$ in $SU(2)$ (→ page 301)

then again transforms under the adjoint representation of $SU(2)$

$$G'^A_{\mu\nu} = G^A_{\mu\nu} + f^{ABC} G^B_{\mu\nu} \varepsilon^C$$

The physical implications of this theory are, however, not in agreement with experimental observations

- In 1957 it was found that the weak interactions violate parity (→ Wu experiment). The above Lagrangian has, however, been constructed in analogy to the electromagnetic and strong interactions which are invariant under parity transformations.

- The above theory assumes that electrons and neutrinos have the same mass

$$-m\bar{\psi}\psi = -m\,\bar{\nu}_e\nu_e - m\,\bar{e}\,e$$

  Similarly, up and down quarks are assumed to have the same mass, which is not what is realised in nature.

- As in any gauge theory, the $SU(2)$ gauge bosons $W^1, W^2, W^3$ are massless, but the physical $W^+, W^-$ and $Z$-bosons are found to be massive with

$$M_{W^\pm} = 80.4 \text{ GeV} \quad \text{and} \quad M_Z = 91.2 \text{ GeV}.$$

The Lagrangian of the weak interactions is therefore not just a copy of the QED and QCD Lagrangians. Instead one has to implement the following modifications:

a) In order to account for parity violation, the theory should be formulated as a <u>chiral gauge theory</u> that treats left-handed and right-handed Weyl spinors on a different footing

b) The gauge boson masses will be generated by an elegant formalism known as the <u>Higgs mechanism</u>. In this setup the Lagrangian will still be invariant under the SU(2) gauge symmetry, which is however spontaneously broken by the vacuum state of the theory. As fermion masses are forbidden in chiral theories, they will also be generated by the Higgs mechanism.

c) The weak interactions cannot be discussed independently of the electromagnetic interactions, since their underlying gauge groups are mixed. One therefore refers to the <u>electroweak interactions</u> in this context. The mixing of the gauge groups actually explains why the masses of the W- and the Z-bosons are different.

Let us address these modifications in turn:

a) <u>Chiral gauge theory</u>

In section 3.3 we learned that field operators transform under irreducible representations of the homogeneous Lorentz group, which are characterised by two numbers $(a,b)$. In particular, we saw that left-handed Weyl spinors $\psi_L$ transform under the $(0,\frac{1}{2})$ representation, and right-handed Weyl spinors $\psi_R$ under the $(\frac{1}{2},0)$ representation. The Dirac spinor $\psi = \binom{\psi_L}{\psi_R}$ then transforms under the direct sum representation $(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$, which is irreducible only if one demands that the theory is invariant under parity transformations.

Once we give up parity invariance, it is obvious that the theory should be formulated in terms of two-component Weyl spinors. Instead of using an explicit two-component notation, it is however more convenient to introduce projection operators *

$$P_L = \frac{1}{2}(1\!\!1 - \gamma_5) \qquad P_R = \frac{1}{2}(1\!\!1 + \gamma_5)$$

such that – with a slight abuse of notation – one has

$$P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \equiv \psi_L$$

$$P_R \psi = \underbrace{\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}}_{2\text{ components}} \equiv \underbrace{\psi_R}_{4\text{ components}}$$

chiral representation

$$\gamma_5 = \begin{pmatrix} -1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

and $\psi = \psi_L + \psi_R$.

One easily verifies that the projection operators fulfill the relations

$$P_L + P_R = 1\!\!1 \qquad (P_L)^2 = P_L$$

$$P_L^+ = P_L \qquad (P_R)^2 = P_R$$

$$P_R^+ = P_R \qquad P_L P_R = P_R P_L = 0$$

---

* Recall that $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ satisfies

$$\{\gamma_5, \gamma^\mu\} = 0$$

$$(\gamma_5)^2 = 1\!\!1$$

$$\gamma_5^+ = \gamma_5$$

in any representation.

We may then construct a chiral gauge theory by simply imposing different transformation laws for left-handed and right-handed fields.

We may e.g. require that the left-handed fields

$$L_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \qquad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

transform in the fundamental representation

$$L_L' = e^{i\varepsilon^A T_F^A} L_L \qquad Q_L' = e^{i\varepsilon^A T_F^A} Q_L$$

whereas the right-handed fields $e_R, \nu_{eR}, u_R, d_R$ transform in the trivial representation*

$$e_R' = e_R \qquad \nu_{eR}' = \nu_{eR} \qquad u_R' = u_R \qquad d_R' = d_R$$

The Lagrangian of this $SU(2)_L$ gauge theory can then be written in the form

$$\mathcal{L} = \sum_\psi \bar{\psi}\, i \not{D} \psi - \frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu}$$

where the sum runs over all fermion fields $\psi = \{L_L, Q_L, e_R, \nu_{eR}, u_R, d_R\}$

---

* In its original version, the SM did not contain a right-handed neutrino field.

and the covariant derivative

$$D_\mu = \partial_\mu - ig W_\mu^A T^A$$

refers to the corresponding representation of the fermion species, i.e. $T_F^A = \frac{\sigma^A}{2}$ for left-handed fields, and $T_1^A = 0$ for right handed fields. In other words, the right-handed fields do not interact with the $SU(2)_L$ gauge bosons at all!

<u>Remarks:</u>

- Notice that the kinetic term of a Dirac field $\psi = \psi_L + \psi_R$ splits into a sum of left- and right-handed fields

$$\bar\psi \not{D} \psi = (\bar\psi_L + \bar\psi_R) \not{D} (\psi_L + \psi_R)$$
$$= \bar\psi_L \not{D} \psi_L + \bar\psi_R \not{D} \psi_R$$

$\psi_L = P_L \psi \rightarrow \bar\psi_L = \bar\psi P_R$

$\bar\psi_L \not{D} \psi_R = \bar\psi P_R \not{D} P_R \psi$

$= \bar\psi \not{D} \underbrace{P_L P_R}_{=0} \psi = 0$

The Dirac mass term, on the other hand, mixes left- and right-handed fields

$$m \bar\psi \psi = m (\bar\psi_L + \bar\psi_R)(\psi_L + \psi_R)$$
$$= m \bar\psi_L \psi_R + m \bar\psi_R \psi_L$$

and it is therefore <u>not</u> $SU(2)_L$ invariant

$\Rightarrow$ Fermions are necessarily massless in chiral gauge theories!

- It is important to distinguish the concepts of chirality and helicity
  - <u>Chirality</u> is a formal concept, which refers to the irreducible representation of the Lorentz group under which the field operator transforms.
  - <u>Helicity</u> is a physical observable that is defined as the projection of the particle spin onto its direction of motion.

Note that for massive particles, helicity is not a Lorentz-invariant concept, since one can always find a boost to overtake the particle such that the direction of motion and hence the helicity flips in the new frame. For massless particles, on the other hand, there is no such boost and the concepts of chirality and helicity are equivalent in this case.

## b) Higgs mechanism

For massive spin-1 particles, one may actually wonder why one should start from a gauge theory at all, since the gauge symmetry anyway one arose as a method for quantising massless spin-1 particles (see the discussion on page 214-224). In section 3.5 we argued that the free propagation of a massive spin-1 particle is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu$$

which gives rise to the Feynman propagator (→ page 212)

$$\tilde{\Delta}_F^{\mu\nu}(p) = \frac{i}{p^2 - m_A^2 + i\varepsilon} \left[ -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_A^2} \right]$$

The point to note is that this propagator has a bad ultraviolet (UV) behaviour

$$\tilde{\Delta}_F^{\mu\nu}(p) \xrightarrow{p^\mu \to \infty} \frac{i}{m_A^2} \frac{p^\mu p^\nu}{p^2} \sim (p)^0$$

which is to be contrasted with

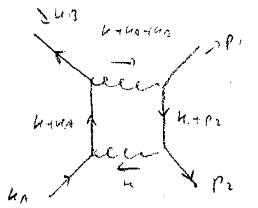
$$\tilde{\Delta}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon} \xrightarrow{p^\mu \to \infty} \frac{i}{p^2} \sim \frac{1}{(p)^2}$$

$$\tilde{S}_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} \xrightarrow{p^\mu \to \infty} \frac{i\not{p}}{p^2} \sim \frac{1}{(p)^1}$$

$$\tilde{D}_F^{\mu\nu}(p) = \frac{i}{p^2 + i\varepsilon} \left[ -g^{\mu\nu} + (1-\xi) \frac{p^\mu p^\nu}{p^2} \right] \sim \frac{1}{(p)^2}$$

for spin-0, spin-1/2 and massless spin-1 particles, respectively.

Loop diagrams with massive vector bosons are therefore often found to be UV-divergent, consider e.g.



$$\xrightarrow{k^\mu \to \infty} \int d^4k \underbrace{\frac{1}{k}\,\frac{1}{k}}_{\text{fermions}} \underbrace{k^0\, k^0}_{\text{vector bosons}} = \int^\infty dk \, \frac{k^3}{k^2} \to \text{UV-divergent}$$

which is to be compared with

$$\int d^4k \underbrace{\frac{1}{k}\,\frac{1}{k}}_{\text{fermions}} \underbrace{\frac{1}{k^2}\,\frac{1}{k^2}}_{\text{gauge bosons}} = \int^\infty dk \, \frac{k^3}{k^6} \to \text{UV finite}$$

in the massless case. Due to the bad UV behaviour of the Feynman propagator, a theory of interacting, massive spin-1 particles will therefore in general not be renormalisable.*

---

* There are, however, some exceptions; most notably an abelian theory with explicit mass term ("massive photon") turns out to be renormalisable. This is, however, not true for a non-abelian theory with explicit mass term.

Instead of adding a mass term to the Lagrangian that explicitly breaks the gauge invariance, there exists however a more subtle way of breaking a symmetry known as <u>spontaneous symmetry breaking</u> (SSB). Here one assumes that the equations that govern the dynamics are symmetric, and that the theory has a degenerate ground state. By choosing a specific ground state, the system then "breaks the symmetry itself", and by doing so it generates a mass term for the gauge bosons.

In the context of gauge symmetries SSB is usually referred to as the Higgs mechanism. We will see later on that the propagator of a massive vector boson in a spontaneously broken gauge theory becomes

$$\tilde{\Delta}_F^{\mu\nu}(p) = \frac{i}{p^2 - M_A^2 + i\varepsilon} \left[ -g^{\mu\nu} + (1-\xi) \frac{p^\mu p^\nu}{p^2 - \xi M_A^2 + i\varepsilon} \right] \sim \frac{1}{(p)^2} \quad \text{for } p^2 \to \infty$$

which for any value of the gauge parameter $\xi$ has a better UV behaviour. It has indeed been shown that spontaneously broken gauge theories are renormalizable ('t Hooft 1971).

In order to introduce the concept of SSB, we first consider a scalar toy theory that is invariant under <u>global</u> $U(1)$ transformations

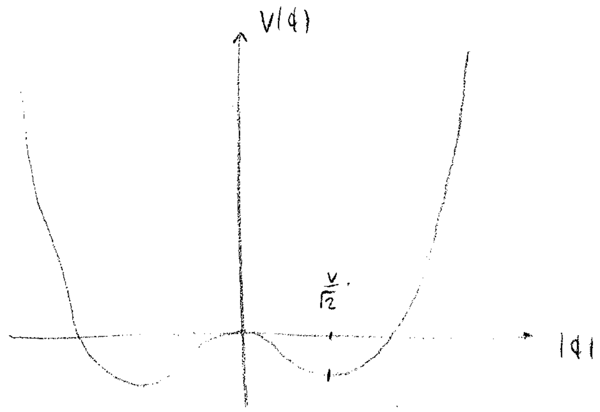
$$\phi'(x) = e^{ie\omega} \phi(x)$$

Specifically, we consider a theory

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi)$$

with potential $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$, where $\mu^2$ and $\lambda$ are real parameters with $\mu^2, \lambda > 0$.

For the following discussion, it is sufficient to consider the theory on the classical level (we will examine the role of quantum corrections in TPP 2). The classical ground states of the theory can then be found by minimizing the potential $V(\phi)$, which due to the $U(1)$ symmetry is a function of $|\phi|^2 = \phi^\dagger \phi$.




minimum at

$$\phi^\dagger \phi = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}$$

$$\hookrightarrow \quad \phi(x) = \frac{v}{\sqrt{2}} e^{\frac{i}{v}\sigma(x)}$$

The theory thus has an infinite number of classical ground states, which by themselves are not invariant under the $U(1)$ symmetry $\phi'(x) = e^{i\alpha}\,\phi(x)$ (which rather transforms one ground state into another). The system will thus necessarily choose one particular ground state, e.g. $\phi(x) = \frac{v}{\sqrt{2}}$, and by doing so it breaks the $U(1)$ symmetry spontaneously.

It is instructive to rewrite the Lagrangian in terms of variables that parametrise the fluctuations around the chosen ground state

$$\phi(x) = \frac{1}{\sqrt{2}}\,(v + \rho(x))\,e^{\frac{i}{v}\sigma(x)}$$

such that $\rho(x) = \sigma(x) = 0$ correspond to $\phi(x) = \frac{v}{\sqrt{2}}$.

$$\Rightarrow \quad \partial_\mu \phi = \frac{1}{\sqrt{2}}\left(\partial_\mu \rho + \frac{i}{v}(v+\rho)\,\partial_\mu \sigma\right) e^{\frac{i}{v}\sigma}$$

$$\partial_\mu \phi^* \partial^\mu \phi = \frac{1}{2}\,\partial_\mu \rho\, \partial^\mu \rho + \frac{1}{2v^2}\,(v+\rho)^2\, \partial_\mu \sigma\, \partial^\mu \sigma$$

$$\phi^\dagger \phi = \frac{1}{2}\,(v+\rho)^2$$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\,\partial_\mu \rho\, \partial^\mu \rho + \frac{1}{2}\,\partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{v}\,\rho\, \partial_\mu \sigma\, \partial^\mu \sigma$$

$$+ \frac{1}{2v^2}\,\rho^2\, \partial_\mu \sigma\, \partial^\mu \sigma + \frac{\lambda v^4}{4} - \lambda v^2 \rho^2 - \lambda v \rho^3 - \frac{\lambda}{4}\rho^4$$

mass term

$$-\frac{M_\rho^2}{2}\rho^2$$

We thus see that the kinetic terms are properly normalized, and that the field $g(x)$ has a mass $m_g = \sqrt{2\lambda v^2} = \sqrt{2\mu^2}$.

The field $\sigma(x)$, on the other hand, is massless, $m_\sigma = 0$.

$\Rightarrow$ A symmetry that is spontaneously broken leads to a massless particle in the spectrum (a <u>Goldstone boson</u>).

In classical field theory this is easy to understand. Whenever the ground state is not invariant under a symmetry transformation, the potential must have a flat direction, which corresponds to a massless excitation. There are actually always as many Goldstone bosons in the spectrum as there are spontaneously broken symmetry generators. This is a consequence of <u>Goldstone's theorem</u>, which we will prove in TPP 2 ($\rightsquigarrow$ see also tutorials).

After having clarified what SSB is and what it implies, let us now turn to the Higgs mechanism. To this end, we consider the same toy theory as before, but we now request that the theory is invariant under <u>local</u> $U(1)$ transformations

$$\phi'(x) = e^{ie\omega(x)} \phi(x)$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \omega(x)$$

According to the minimal coupling prescription, the Lagrangian of this theory is then given by

$$\mathcal{L} = (D_\mu \phi)^+ (D^\mu \phi) - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field-strength tensor.

As the potential $V(\phi)$ has not changed, we can closely follow the previous discussion, writing

$$\phi(x) = \frac{1}{\sqrt{2}} (v + s(x))\, e^{\frac{i}{v} \sigma(x)}$$

to parametrise the fluctuations around the ground state

$$\phi(x) = \frac{v}{\sqrt{2}}.$$

In contrast to the global $U(1)$ symmetry, however, the fluctuations in $\sigma(x)$ are not physical anymore since they correspond to a gauge transformation! In other words, we may choose e.g. $\omega(x) = -\frac{\sigma(x)}{ev}$ such that the Goldstone boson disappears completely

$$\phi'(x) = e^{ie\left(-\frac{\sigma(x)}{ev}\right)}\,\phi(x)$$

$$= \frac{1}{\sqrt{2}}\left(v + \rho(x)\right)$$

which corresponds to setting $\sigma(x) = 0$. This particular choice is called <u>unitary gauge</u>.

In unitary gauge, one obtains

$$\mathcal{D}_\mu \phi = \frac{1}{\sqrt{2}}\left(\partial_\mu \rho - ieA_\mu (v+\rho)\right)$$

$$(\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho + \frac{1}{2}e^2 A_\mu A^\mu (v+\rho)^2$$

$$\phi^\dagger \phi = \frac{1}{2}(v+\rho)^2$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho + \underbrace{\frac{1}{2}e^2 v^2 A_\mu A^\mu}_{\frac{1}{2}m_A^2 A_\mu A^\mu} + e^2 v \rho A_\mu A^\mu$$

$$+ \frac{1}{2}e^2\rho^2 A_\mu A^\mu + \frac{\lambda}{4}v^4 \underbrace{- \lambda v^2 \rho^2}_{-\frac{m_\rho^2}{2}\rho^2} - \lambda v \rho^3 - \frac{\lambda}{4}\rho^4$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

We thus again obtain a massive scalar field with mass $m_\sigma = \sqrt{2\lambda v^2}$, but in addition we see that the gauge boson has acquired a mass $m_A = ev$! This is called the Higgs mechanism.

Remarks:

- As the Goldstone boson has disappeared from the spectrum and the gauge boson has become massive, one sometimes says that the gauge boson has "eaten" the Goldstone boson.

  Notice also that the number of degrees of freedom match

  massless gauge boson $A^\mu$ (2) $\longrightarrow$ massive vector boson $A^\mu$ (3)
  massless real scalar $\sigma$ (1)
  ↑ physical polarisations

- In unitary gauge it turns out that the vector boson propagator becomes

$$\tilde{\Delta}_F^{\mu\nu} = \frac{i}{p^2 - m_A^2 + i\varepsilon} \left[ -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_A^2} \right]$$

  with $m_A = ev$. The propagator thus has the same bad UV behaviour as the one that we discussed at the beginning of this section. So what did we gain?

It turns out that spontaneously broken gauge theories are indeed renormalisable. As the gauge symmetry is still intact on the level of the Lagrangian, the potentially dangerous UV-divergent terms all conspire and drop out at the end of the calculation. The fact that a spontaneously broken gauge theory has a better UV behaviour can be seen, however, more clearly in a different gauge, called <u>$R_\xi$-gauge</u>. We will not enter the details here, but one finds that the vector boson propagator in $R_\xi$-gauge takes the form

unphysical gauge parameter

$$\tilde{\Delta}_F^{\mu\nu}(p) = \frac{i}{p^2 - m_A^2 + i\varepsilon} \left[ -g^{\mu\nu} + (1-\xi) \frac{p^\mu p^\nu}{p^2 - \xi m_A^2 + i\varepsilon} \right]$$

which indeed falls off as $\frac{1}{(p)^2}$ as $p^\mu \to \infty$. Whereas renormalisability becomes more apparent in $R_\xi$-gauge, this gauge has however other drawbacks since it does not only describe physical degrees of freedom (massive vector boson $A^\mu$ and Higgs boson $\xi$), but also unphysical degrees of freedom (would-be Goldstone boson $\sigma$, ghost fields, ...) as indicated by the artificial pole in the propagator at $p^2 = \xi m_A^2$. We will discuss $R_\xi$-gauge in more detail in TPP2.

## c) Electroweak unification

In the SM the electromagnetic and the weak interactions are entangled, and the W- and Z-bosons acquire their masses through the Higgs mechanism. The electroweak sector of the SM is based on the gauge group $\underbrace{SU(2)_L}_{\text{weak isospin}} \otimes \underbrace{U(1)_Y}_{\text{weak hypercharge}}$ with Lagrangian

$$\mathcal{L} = \sum_{\psi} \bar{\psi} i \not{D} \psi \underbrace{- \frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu}}_{SU(2)_L} \underbrace{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{U(1)_Y}$$

where the sum again runs over $\psi = \{L_L, Q_L, e_R, \nu_{eR}, u_R, d_R\}$ and the covariant derivative is now given by

$$D_\mu = \partial_\mu - i g W_\mu^A T^A - i g' Y B_\mu$$

with three $SU(2)_L$ gauge bosons $W_\mu^1, W_\mu^2, W_\mu^3$ and one $U(1)_Y$ gauge boson $B_\mu$. The corresponding coupling constants are denoted by $g$ and $g'$, respectively.

As we discussed on page 343, each fermion may have its own charge under the $U(1)_Y$ symmetry, and one chooses

$$Y_{L_L} = -1/2 \qquad Y_{e_R} = -1 \qquad Y_{u_R} = \frac{2}{3}$$

$$Y_{Q_L} = 1/6 \qquad Y_{\nu_{eR}} = 0 \qquad Y_{d_R} = -\frac{1}{3}$$

The fermions thus transform under the $SU(2)_L \otimes U(1)_Y$ symmetry as

$$\psi'(x) = \underbrace{e^{i\varepsilon^A(x) T^A}}_{SU(2)_L} \underbrace{e^{i\varepsilon(x) Y_\psi}}_{U(1)_Y} \psi(x)$$

with the corresponding representation (fundamental or trivial) for the $SU(2)_L$ generators and the corresponding values of the $U(1)_Y$ hypercharge.

The gauge boson masses are then generated via the Higgs mechanism according to the SSB pattern

$$SU(2)_L \otimes U(1)_Y \xrightarrow{SSB} U(1)_Q$$

where $U(1)_Q$ is the familiar gauge group from electrodynamics, which is not broken by the Higgs mechanism. By counting the dimensions of the group

$$\dim \left( SU(2)_L \otimes U(1)_Y \right) = 3 + 1 = 4$$
$$\dim \left( U(1)_Q \right) = 1$$

we see that three symmetry generators are broken by the Higgs mechanism, which - according to Goldstone's theorem - implies that there are three (would-be) Goldstone bosons in the theory. This is indeed what is required to make three vector bosons massive $(W^\pm, Z)$, whereas one of them stays massless (photon).

The Higgs mechanism is furthermore driven by the standard "mexican-hat" potential

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

with $\mu^2, \lambda > 0$, where $\phi(x)$ is now a complex scalar field that transforms as a doublet under $SU(2)_L$ transformations. The Higgs field has furthermore $U(1)_Y$ hypercharge $Y_\phi = 1/2$, and it transforms as

$$\phi'(x) = e^{i\varepsilon^a(x) T^a} e^{i\varepsilon(x) Y_\phi} \phi(x)$$

with $T_F^a = \frac{\sigma^a}{2}$ in the fundamental representation.

The potential is then again minimised by $\phi^\dagger \phi = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}$, and the standard choice that is consistent with the considered SSB pattern is

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Let us check explicitly which symmetries are broken by this ground state. To this end, we consider infinitesimal transformations

$$\phi'(x) = \left(\mathbb{1} + i\varepsilon^A T^A + i\varepsilon Y_\phi\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

and we identify the symmetries that do not leave the ground state invariant

$$\Rightarrow \quad \varepsilon^1 T^1 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^1}{2} \begin{pmatrix} v \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \underline{\text{broken}}$$

$$\varepsilon^2 T^2 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^2}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^2}{2} \begin{pmatrix} -iv \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \underline{\text{broken}}$$

$$\left.\begin{aligned} \varepsilon^3 T^3 \begin{pmatrix} 0 \\ v \end{pmatrix} &= \frac{\varepsilon^3}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^3}{2} \begin{pmatrix} 0 \\ -v \end{pmatrix} \\ \varepsilon Y_\phi \begin{pmatrix} 0 \\ v \end{pmatrix} &= \frac{\varepsilon}{2} \begin{pmatrix} 0 \\ v \end{pmatrix} \end{aligned}\right\} \quad \text{invariant for } \varepsilon^3 = \varepsilon \,!$$

$\Rightarrow$ Transformations with

$$e^{i\varepsilon(T^3 + Y_\phi)} = e^{i\varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \equiv e^{i\varepsilon Q_\phi} \in U(1)_Q$$

are thus still a symmetry after SSB !

We next parametrise the fluctuations around the ground state as

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} e^{\frac{i}{v} T^a \sigma^a(x)}$$

Goldstone bosons live in coset space $SU(2) \otimes U(1) / U(1) \simeq SU(2)$

and in unitary gauge one again chooses $\sigma^a(x) = 0$. The Higgs boson $h(x)$ then again acquires a mass $m_h = \sqrt{2\lambda v^2}$ as in the abelian toy model, and it has electromagnetic charge

$$h: \; Q = T^3 + Y = -\frac{1}{2} + \frac{1}{2} = 0$$

lower component of $SU(2)_L$ doublet

Let us also evaluate the charges of the fermions

$\nu_{eL}$ : $Q = \frac{1}{2} - \frac{1}{2} = 0$ $\qquad$ $\nu_{eR}$ : $Q = 0 + 0 = 0$

$e_L$ : $Q = -\frac{1}{2} - \frac{1}{2} = -1$ $\qquad$ $e_R$ : $Q = 0 - 1 = -1$

$u_L$ : $Q = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ $\qquad$ $u_R$ : $Q = 0 + \frac{2}{3} = \frac{2}{3}$

$d_L$ : $Q = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$ $\qquad$ $d_R$ : $Q = 0 - \frac{1}{3} = -\frac{1}{3}$

which should, however, not be viewed as a prediction of the SM since we have chosen the $U(1)_Y$ hypercharges accordingly.

We next work out the gauge boson masses induced by the Higgs mechanism. To this end, we consider the covariant derivative of the Higgs field

$$\mathcal{D}_\mu \phi = \left(\partial_\mu - i g W_\mu^A \frac{\sigma^A}{2} - i g' \frac{1}{2} B_\mu\right) \phi$$

and we introduce the physical gauge bosons as linear combinations of the $SU(2)_L$ and $U(1)_Y$ gauge fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2\right) \qquad W^\pm\text{-bosons}$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}} \left(g W_\mu^3 - g' B_\mu\right) \qquad Z\text{-boson}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}} \left(g' W_\mu^3 + g B_\mu\right) \qquad \text{photon}$$

The kinetic term of the Higgs field then gives rise to the following mass terms (~ see tutorials)

$$(\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) = \underbrace{\frac{g^2 v^2}{4}}_{M_W^2} W_\mu^- W^{\mu +} + \underbrace{\frac{v^2}{8}(g^2+g'^2)}_{\frac{1}{2} M_Z^2} Z_\mu Z^\mu + \dots$$

from which we read off

$$m_W = \frac{1}{2} g v$$

$$m_Z = \frac{1}{2} \sqrt{g^2 + g'^2}\, v$$

as well as $m_A = 0$, which corresponds to the unbroken $U(1)_Q$ gauge symmetry.

The mixing of the $SU(2)_L$ and the $U(1)_Y$ gauge groups can be quantified in terms of a weak-mixing (or Weinberg) angle $\theta_W$, which is defined as $\tan\theta_W \equiv \frac{g'}{g}$.

$$\Rightarrow \quad Z_\mu = \cos\theta_W\, W_\mu^3 - \sin\theta_W\, B_\mu$$

$$A_\mu = \sin\theta_W\, W_\mu^3 + \cos\theta_W\, B_\mu$$

such that the Z-boson / photon coincides with the $SU(2)_L$ / $U(1)_Y$ gauge boson, respectively, in the limit $\theta_W \to 0$. Notice that the Z-boson and the W-boson masses are degenerate in this limit

$$m_Z = \frac{g v}{2\cos\theta_W} = \frac{m_W}{\cos\theta_W} \xrightarrow{\theta_W \to 0} m_W$$

In order to connect to our results from section 5.1, we examine the covariant derivative more closely. First, we note that

$$W_\mu^3 = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$$

$$B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu$$

It follows

$$\begin{aligned} D_\mu &= \partial_\mu - ig W_\mu^3 T^3 - ig' Y B_\mu + \dots \\ &= \partial_\mu - ig \sin\theta_W A_\mu T^3 - ig \tan\theta_W Y \cos\theta_W A_\mu + \dots \\ &= \partial_\mu - ig \sin\theta_W \underbrace{(T^3 + Y)}_{Q} A_\mu + \dots \\ &\equiv \partial_\mu - ie Q A_\mu + \dots \end{aligned}$$

We thus recover the QED coupling constant as

$$e = g \sin\theta_W = g' \cos\theta_W$$

At this stage, the W- and Z-boson have acquired their masses, but the fermions are still massless. How can we introduce a fermion mass term without breaking the chiral gauge symmetry explicitly?

The idea consists in coupling the fermions to the Higgs field, which then generates a mass term for the fermions when it acquires a non-zero vacuum expectation value $v$. To see how this works, we first group the left-handed and right-handed fermions with the Higgs field into gauge-invariant combinations. We may consider e.g. the terms

$$\bar{L}_L \phi\, e_R \qquad\qquad \bar{Q}_L \phi\, d_R$$

which are obviously $SU(2)_L$ and $SU(3)^{QCD}$ invariant. In order to verify if these terms are also invariant under the $U(1)_Y$ symmetry, we need to add up the corresponding hypercharges

$$\bar{L}_L \phi\, e_R : \quad \frac{1}{2} + \frac{1}{2} - 1 = 0 \quad ✓$$

$$\bar{Q}_L \phi\, d_R : \quad -\frac{1}{6} + \frac{1}{2} - \frac{1}{3} = 0 \quad ✓$$

One similarly verifies that the terms $\bar{L}_L \phi\, u_R$ and $\bar{Q}_L \phi\, u_R$ are, however, <u>not</u> $U(1)_Y$ invariant.

One therefore introduces a "conjugate" Higgs field

$$\tilde{\phi} \equiv i\sigma^2 \phi^*$$

which also transforms as an $SU(2)_L$ doublet, but has hypercharge $Y_{\tilde{\phi}} = -1/2$. Let us briefly convince ourselves that $\tilde{\phi}$ transforms in the fundamental representation of $SU(2)_L$

$$\tilde{\phi}' = i\sigma^2 (\phi')^*$$

$$= i\sigma^2 \left( e^{i\varepsilon^A \frac{\sigma^A}{2}} \phi \right)^*$$

$$= i\sigma^2 e^{-i\varepsilon^A \frac{\sigma^{A*}}{2}} \underbrace{\sigma^2 \sigma^2}_{=\mathbb{1}} \phi^*$$

$$= e^{i\varepsilon^A \frac{\sigma^A}{2}} i\sigma^2 \phi^* = e^{i\varepsilon^A \frac{\sigma^A}{2}} \tilde{\phi}$$

where we have used $\sigma^2 \sigma^{A*} \sigma^2 = -\sigma^A$.

With the help of $\tilde{\phi}$, we can then construct similar gauge-invariant combinations for "up-type" fermions

$\bar{L}_L \tilde{\phi} \nu_R$ : $\frac{1}{2} - \frac{1}{2} + 0 = 0$ ✓

$\bar{Q}_L \tilde{\phi} u_R$ : $-\frac{1}{6} - \frac{1}{2} + \frac{2}{3} = 0$ ✓

which are obviously also $SU(2)_L$ and $SU(3)$ invariant.

We can thus write down gauge-invariant Yukawa interactions

$$\mathcal{L}_{\text{Yukawa}} = -\lambda_e \bar{L}_L \phi e_R - \lambda_{\nu_e} \bar{L}_L \tilde{\phi} \nu_{eR}$$

$$- \lambda_d \bar{Q}_L \phi d_R - \lambda_u \bar{Q}_L \tilde{\phi} u_R + \underbrace{h.c.}_{\text{hermitian conjugate}}$$

with dimensionless ($\rightarrow$ renormalizable) coupling constants $\lambda_e$, $\lambda_{\nu_e}$, $\lambda_d$ and $\lambda_u$.

In unitary gauge

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

$$\Rightarrow \tilde{\phi}(x) = i\sigma^2 \phi^*(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}$$

we see that the Yukawa interactions describe both fermion masses and couplings of the fermions to the Higgs boson. In particular, one obtains e.g.

$$\mathcal{L}_{\text{Yukawa}} = -\underbrace{\frac{\lambda_e v}{\sqrt{2}}}_{m_e} (\bar{e}_L e_R + \bar{e}_R e_L) + \ldots$$

from which we read off $m_e = \frac{\lambda_e v}{\sqrt{2}}$ and similarly

$$m_{\nu_e} = \frac{\lambda_{\nu_e} v}{\sqrt{2}}, \quad m_d = \frac{\lambda_d v}{\sqrt{2}} \quad \text{and} \quad m_u = \frac{\lambda_u v}{\sqrt{2}}.$$

## 5.4 SM Lagrangian

We close this lecture with a brief summary of the SM Lagrangian.

The SM is a chiral gauge theory in which the particle masses are generated via the Higgs mechanism

$$\underbrace{SU(3)_c}_{QCD} \otimes \underbrace{SU(2)_L \otimes U(1)_Y}_{\text{electroweak}} \xrightarrow{SSB} \underbrace{SU(3)_c}_{QCD} \times \underbrace{U(1)_Q}_{QED}$$

The SM thus contains a variety of spin-1 gauge bosons

- 8 gluons $\quad G_\mu^A \quad m_g = 0$
- photon $\quad A_\mu \quad m_A = 0$
- $W^\pm$ bosons $\quad W_\mu^\pm \quad m_W = 80.4\ \text{GeV}$
- $Z$ boson $\quad Z_\mu \quad m_Z = 91.2\ \text{GeV}$

that mediate the strong and electroweak interactions.

There in addition exist three copies of matter particles in nature, which have the same quantum numbers but differ in their masses.

The <u>spin-1/2 fermion fields</u> in the SM are

- <u>left-handed lepton doublets</u>

generation index $I = 1,2,3$

$$L_L^I = \left\{ \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \right\} \sim (1_c, 2_L)_{-1/2}$$

weak hypercharge

$SU(3)_c$ singlet $\quad SU(2)_L$ doublet

with $m_e = 0.51$ MeV (electron)

$m_\mu = 105.7$ MeV (muon)

$m_\tau = 1777$ MeV (tauon)

The neutrinos are, on the other hand, very light ($< 2$ eV), but their precise values have not been measured so far (only their mass differences).

- <u>left-handed quark doublets</u>

$$Q_L^I = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\} \sim (3_c, 2_L)_{1/6}$$

3 colours

| | | | |
|---|---|---|---|
| with $m_u \simeq 2$ MeV | (up) | $m_c = 1.3$ GeV | (charm) |
| $m_d \simeq 5$ MeV | (down) | $m_b = 4.2$ GeV | (bottom) |
| $m_s \simeq 95$ MeV | (strange) | $m_t = 173$ GeV | (top) |
| light quarks | | heavy quarks | |

As the quarks cannot be observed as free particles, there exists however some ambiguity in defining their particle masses (→ TPP2).

- <u>right-handed charged leptons</u>

$$E_R^I = \{ e_R, \mu_R, \tau_R \} \sim (1_c, 1_L)_{-1}$$

- <u>right-handed neutrinos</u>

$$\nu_R^I = \{ \nu_{eR}, \nu_{\mu R}, \nu_{\tau R} \} \sim (1_c, 1_L)_0$$

The right-handed neutrinos thus have no SM interactions at all, and in the massless case there would be no need for introducing right-handed neutrinos. As the neutrinos are, however, known to have non-zero masses, one introduces right-handed neutrino fields since the Dirac mass term couples left- and right-handed components (but there exists an alternative mechanism for generating neutrino masses, see below).

- right-handed up quarks

$$U_R^I = \{u_R, c_R, t_R\} \sim (3_c, 1_L)_{2/3}$$

- right-handed down quarks

$$D_R^I = \{d_R, s_R, b_R\} \sim (3_c, 1_L)_{-1/3}$$

Finally, the SM contains a neutral spin-0 particle, which preferably couples to heavy particles

- Higgs field

$$\underbrace{\phi}_{\text{unitary gauge}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \underbrace{h}_{\text{Higgs boson}} \end{pmatrix} \sim (1_c, 2_L)_{1/2}$$

The Higgs boson was discovered in 2012 at the LHC and its mass was measured to be

$$m_h = 125 \text{ GeV}$$

In 2013 the Nobel Prize in Physics was awarded to Peter Higgs and Francois Englert for the theoretical discovery of the Higgs mechanism.

After having summarised the particle content of the SM, let us now discuss its Lagrangian, which we can split into four terms

$$\boxed{\mathcal{L}_{SM} = \mathcal{L}_{\text{gauge bosons}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}}$$

where the first term contains the kinetic terms and the self interactions of the gauge bosons

$$\mathcal{L}_{\text{gauge bosons}} = \underbrace{-\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu}}_{SU(3)_c} \underbrace{- \frac{1}{4} W^A_{\mu\nu} W^{A,\mu\nu}}_{SU(2)_L} \underbrace{- \frac{1}{4} B_{\mu\nu} B^{\mu\nu}}_{U(1)_Y}$$

$$+ \mathcal{L}_{\text{gauge-fix}} + \mathcal{L}_{\text{FP ghosts}}$$

with

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu + g_s f^{ABC} G^B_\mu G^C_\nu$$

$$W^A_{\mu\nu} = \partial_\mu W^A_\nu - \partial_\nu W^A_\mu + g \, \varepsilon^{ABC} W^B_\mu W^C_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

The physical $W^\pm$, $Z$ and photon fields are then linear combinations of the $SU(2)_L$ and $U(1)_Y$ gauge fields, as specified on page 377.

The second term contains the kinetic terms of the fermions as well as their gauge interactions

$$\mathcal{L}_{fermion} = \sum_{\psi} \bar{\psi}^I i \not{D} \psi^I$$

with $\psi^I = \{ L_L^I, Q_L^I, E_R^I, \nu_R^I, U_R^I, D_R^I \}$ and

$$D_\mu = \partial_\mu \underbrace{- i g_s G_\mu^A T^A}_{SU(3)_c} \underbrace{- i g W_\mu^a T^a}_{SU(2)_L} \underbrace{- i g' Y B_\mu}_{U(1)_Y}$$

The kinetic terms can always be chosen to be diagonal in generation space.

Explicitly, the covariant derivative becomes

$$Q_L^I : \quad D_\mu = \partial_\mu - i g_s G_\mu^A \frac{\lambda^A}{2} - i g W_\mu^a \frac{\sigma^a}{2} - i g' \left(\frac{1}{6}\right) B_\mu$$

$$L_L^I : \quad D_\mu = \partial_\mu - i g W_\mu^a \frac{\sigma^a}{2} - i g' \left(-\frac{1}{2}\right) B_\mu$$

$$E_R^I : \quad D_\mu = \partial_\mu - i g' (-1) B_\mu$$

$$\nu_R^I : \quad D_\mu = \partial_\mu$$

$$U_R^I : \quad D_\mu = \partial_\mu - i g_s G_\mu^A \frac{\lambda^A}{2} - i g' \left(\frac{2}{3}\right) B_\mu$$

$$D_R^I : \quad D_\mu = \partial_\mu - i g_s G_\mu^A \frac{\lambda^A}{2} - i g' \left(-\frac{1}{3}\right) B_\mu$$

The third term contains the kinetic term of the Higgs boson, the gauge and Higgs boson masses, the gauge interactions of the Higgs boson and the Higgs self interactions

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^+ (D^\mu \phi) - V(\phi)$$

with $V(\phi) = -\mu^2 \phi^+\phi + \lambda (\phi^+\phi)^2$ and

$$\phi: \quad D_\mu = \partial_\mu - ig W_\mu^a \frac{\sigma^a}{2} - ig' \left(\frac{1}{2}\right) B_\mu$$

Finally, the last term contains the fermion mass terms and the fermion - Higgs interactions

$$\mathcal{L}_{Yukawa} = -\lambda_E^{IJ} \bar{L}_L^I \phi E_R^J - \lambda_\nu^{IJ} \bar{L}_L^I \tilde{\phi} \nu_R^J$$
$$- \lambda_D^{IJ} \bar{Q}_L^I \phi D_R^J - \lambda_U^{IJ} \bar{Q}_L^I \tilde{\phi} U_R^J + h.c.$$

where the Yukawa couplings are now $3 \times 3$ matrices in generation space and $\tilde{\phi} = i\sigma^2 \phi^*$. The Yukawa matrices seem to introduce a huge number of parameters

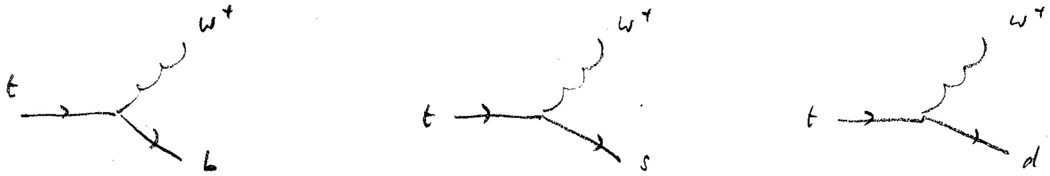
$$\underbrace{4}_{\lambda\text{'s}} \cdot \underbrace{(3\cdot 3)}_{3\times 3 \text{ matrices}} \cdot \underbrace{2}_{\text{complex entries}} = 72 \text{ real parameters}$$

but one can show that not all of these parameters are physical.

In the quark sector, there are 10 physical parameters: six quark masses and four parameters that describe quark mixing in terms of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [2008 Nobel Prize for M. Kobayashi and T. Maskawa]

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

As a consequence, the W-boson interactions allow for transitions between different generations ("flavour-changing charged currents")



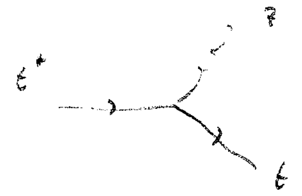

$\sim V_{tb}$ $\sim V_{ts}$ $\sim V_{td}$

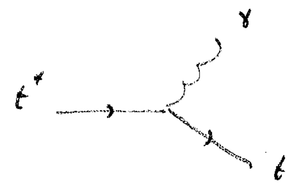
The CKM matrix is found to have a hierarchical structure

$$|V_{CKM}| \simeq \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.974 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

and the transitions between the same generation are therefore the most probable.

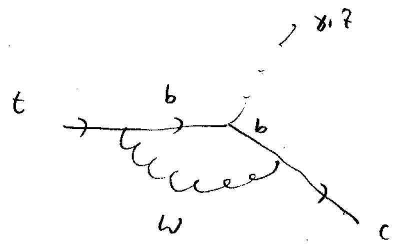
The Z-boson and photon interactions are, on the other hand, diagonal in generation space




There thus exist no <u>flavour-changing neutral currents (FCNCs)</u> at tree level in the SM.

While FCNCs are very rare in the SM, they are not completely forbidden since they may be induced at the loop level by diagrams like




$\sim V_{tb} V_{cb}^*$

If the neutrinos were massless, the Yukawa matrix $\lambda_\nu^{IJ}$ would be absent and the neutrino sector would only depend on three parameters $m_e, m_\mu, m_\tau$. As the neutrinos have, however, tiny masses, one is essentially in the same situation as in the quark sector with six lepton masses and four parameters that describe lepton mixing. The corresponding mixing matrix is called <u>Pontecorvo-Maki-Nagakawa-Sakata (PMNS) matrix</u>.

The role of the neutrinos is, however, special since the right-handed neutrinos do not carry any SM charges and they may therefore be described by a Majorana instead of a Weyl spinor (we did not discuss Majorana fields that are needed to describe massive, <u>neutral</u> spin-1/2 particles in section 3). In this case, it turns out that the lepton-mixing matrix must be parametrised by six instead of four parameters.

The advantage of a Majorana mass term is that it provides a natural explanation why the observed neutrino masses are so small via the Seesaw-mechanism.

We are now in the position to count the SM parameters

| | | |
|---|---|---|
| · gauge couplings | 3 | $g_s, g, g'$ |
| · Higgs potential | 2 | $\mu, \lambda$ |
| · quark sector | $6 + 4$ | |
| · lepton sector | $\underbrace{6}_{\text{masses}} + \underbrace{4}_{\text{mixing}}$ | (+2 if right-handed neutrinos are Majorana particles) |
| | 25 (+2) | |

Except for a few parameters related to the PMNS-matrix, all of these parameters have been measured to date.

Consider e.g. the weak-mixing angle

$$\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2} \simeq 0.22$$

or $\theta_W \simeq 28°$. Together with $\alpha = \frac{e^2}{4\pi}$, this then determines $g$ and $g'$ via $e = g\sin\theta_W = g'\cos\theta_W$.

From the W- and Higgs boson masses, one can furthermore extract the parameters of the Higgs potential

$$m_W = \frac{1}{2} g v = \frac{e v}{2 \sin\theta_W} \qquad \longrightarrow \qquad v \simeq 245 \text{ GeV}$$

$$m_h = \sqrt{2\lambda v^2} \qquad \qquad \lambda \simeq 0.15$$

The fact that $\lambda \ll 1$ implies that the Higgs self-interactions are perturbative (at any reasonable scale).

The SM is an extremely successful theory that describes all experimental measurements at particle colliders to date. There are, however, several reasons why the SM is commonly thought to be only a low-energy approximation of a more complete theory that we hope to unravel in the future.

Without going into the details here, the most prominent problems of the SM are

- that it does not explain gravity
- that it does not contain a particle that could make up dark matter
- that it cannot explain why there is so much more matter in the universe than antimatter (although the SM contains all ingredients to produce a matter-antimatter asymmetry, the quantitative prediction falls short by many orders of magnitude).

While the SM is clearly in conflict with these astronomical observations, it also has some theoretical issues, which are not problems per se, but imply a lack of understanding

- At the loop level the Higgs boson propagator receives quantum corrections that drive the Higgs mass to very large scales. The fact that the observed Higgs mass is of the order of the electroweak scale ($v \simeq 245$ GeV) requires a huge amount of fine-tuning which seems unnatural. New particles in the quantum loops with masses of a few TeV would significantly weaken the fine-tuning problem. This is often called the <u>hierarchy problem</u>.

- The SM does not explain why there are three generations of matter particles in nature and why their masses range over many orders of magnitude. The Yukawa sector, in particular, resembles a mere parametrisation, lacking a deeper theoretical understanding.

- More generally the SM contains a large number of parameters, whose numerical values are not determined but have to be extracted from experimental measurements. One would hope to find a more complete theory in which at least some these parameters can be directly computed. The three gauge couplings, for instance, could be related to one gauge coupling of a grand-unified theory (GUT), which unifies the strong and the electroweak interactions.