5. Spontaneous symmetry breaking

SSB is an elegant theoretical concept that finds application in different branches of theoretical physics. Here we are mainly intersted in electroweak symmetry breaking, which is the mechanism that generates the masses of the fundamental particles as e.g. the W or Z bosons. But how can we implement a gauge boson mass term in a gauge theory?

We may simply opt for giving up the concept of gauge symmetry and add an explicit mass term $\frac{1}{2}m_A^2 A_\mu A^\mu$ to the Lagrangian. In the previous sections, we anyway learned that massless vector bosons lead to many complications (redundant description, unphysical degrees of freedom, ...), which are not present in the massive case. The problem is that the propagator of a massive vector boson

$$\frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_A^2}\right] \xrightarrow{k\to\infty} \frac{i}{m_A^2}\,\frac{k^\mu k^\nu}{k^2}$$

has a bad UV behaviour. Unless specific cancellations occur at every order in perturbation theory, the theory will in general not be renormalisable and can only be viewed as an effective low-energy theory with intrinsic cutoff $k^\mu < \Lambda$. (*)

There exists, however, a more subtle way of breaking a symmetry known as SSB. Here one assumes that the equations that govern the dynamics <u>are</u> symmetric, and that the theory has a degenerate vacuum state. By choosing a specific vacuum, the system then "breaks the symmetry itself", and by doing so it generates a mass term for the gauge bosons.

(*) It turns out that an abelian theory with explicit mass term is renormalisable since the theory is still BRST invariant. This is, however, not true for non-abelian theories with explicit mass term. For details cf. Collins, chapter 12.9.

As in the massless case, the theory is then built on a redundant description and contains unphysical degrees of freedom. We will see later on that the propagator of a massive vector boson then becomes

$$\frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi m_A^2}\right] \xrightarrow{k\to\infty} \frac{i}{k^2}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$$

which for any finite value of ξ has a better UV behaviour (the expression actually coincides with the one for massless vector bosons). It has indeed been shown that spontaneously broken gauge theories are renormalisable ('t Hooft 1971).

In the following, we will first consider SSB of <u>global</u> symmetries. We will later on apply the concept to gauge theories, and finally discuss electroweak symmetry breaking.

5.1 Emergence of Goldstone bosons

We will start with an analysis of SSB in classical field theory. Consider the Lagrangian

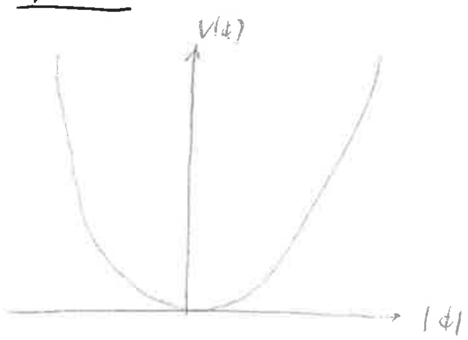
$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi)$$

with potential $V(\phi) = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$, which is invariant under global $U(1)$ transformations

$$\phi'(x) = e^{i\omega} \phi(x)$$

In order to have a stable vacuum state, we will assume that $\lambda > 0$. We then distinguish between two situations:

i) $\mu^2 \geq 0$



minimum at $\phi(x) = 0$

(classical ground state)

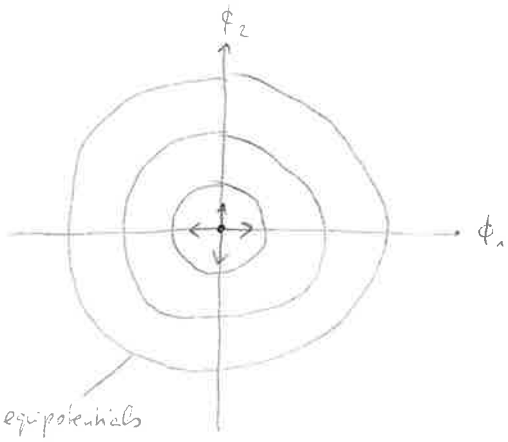
The ground state is invariant under $\phi'(x) = e^{i\omega} \phi(x)$

$\Rightarrow$ the symmetry is manifest

In this case we write

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\,\phi_2(x)\right)$$

in terms of two real scalar fields $\phi_{1,2}(x)$.



symmetry transformation

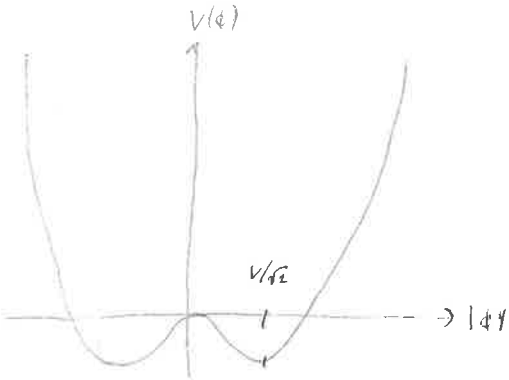
→ rotation in ϕ_1-ϕ_2 plane

small oscillations around ground state

→ two independent modes with frequencies μ

We thus see that the symmetry leads to a <u>degenerate spectrum</u>!

ii) <u>$\mu^2 < 0$</u>



minimum at $\phi^+\phi = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}$

infinitely many classical ground states with $\phi(x) = \frac{v}{\sqrt{2}}\,e^{\frac{i}{v}\sigma(x)}$

The ground states are <u>not</u> invariant under $\phi'(x) = e^{i\omega}\phi(x)$, which rather transforms one ground state into another

⇒ the symmetry is <u>spontaneously broken</u>

We now write

$$\phi(x) = \frac{1}{\sqrt{2}}\, \rho(x)\, e^{\frac{i}{v}\sigma(x)}$$

$$\Rightarrow \quad \partial_\mu \phi = \frac{1}{\sqrt{2}}\left(\partial_\mu \rho + \frac{i}{v}\rho\, \partial_\mu \sigma\right) e^{\frac{i}{v}\sigma}$$

$$\partial_\mu \phi^\dagger \partial^\mu \phi = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho + \frac{\rho^2}{2v^2}\partial_\mu \sigma\, \partial^\mu \sigma$$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho + \frac{1}{2v^2}\rho^2 \partial_\mu \sigma\, \partial^\mu \sigma + \frac{\lambda v^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

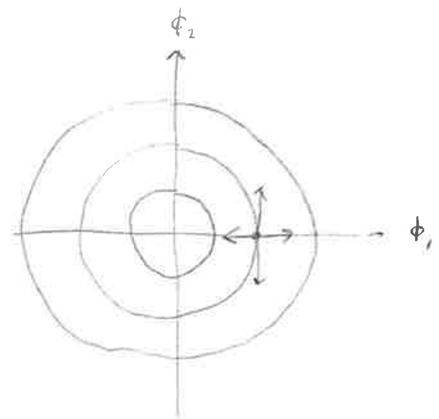
In these coordinates, the symmetry transformation acts as

$$\sigma'(x) = \sigma(x) + \omega v$$

and the Lagrangian can therefore only depend on σ through <u>derivatives</u> of σ.

We next expand around the classical ground state, choosing

$$\phi(x) = \frac{v}{\sqrt{2}}$$



Writing $\rho(x) = v + \rho'(x)$ and dropping the prime for convenience, we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho - \frac{\lambda}{4}\rho^4 - \lambda v \rho^3 - \lambda v^2 \rho^2 + \frac{\lambda v^4}{4}$$

$$+ \frac{1}{2}\partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{v}\rho\, \partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{2v^2}\rho^2 \partial_\mu \sigma\, \partial^\mu \sigma$$

We thus find a massive excitation with $m_\xi = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2}$, and a massless excitation with $m_\sigma = 0$.

$\Rightarrow$ a symmetry that is spontaneously broken does not lead to a degenerate spectrum, but to a massless particle (a <u>Goldstone boson</u>). The symmetry further implies that Goldstone bosons have <u>derivative interactions</u>.

In classical field theory this is easy to understand. Whenever the ground state is not invariant under a symmetry transformation, the potential must have a flat direction which corresponds to a massless excitation. There are actually always as many Goldstone bosons as there are spontaneously broken symmetries (Goldstone theorem, see below).

But how do quantum fluctuations modify the classical picture?

5.2 Effective action

We will now introduce a formalism that is useful for the study of SSB, but it is also important for renormalisation theory.

We consider the generating functional of a scalar field theory

$$Z[J] = N \int \mathcal{D}\phi \, e^{i\int d^4x \, (\mathcal{L} + J\phi)}$$

which generates the Green functions via

$$\langle \Omega | T \phi(x_1) \dots \phi(x_n) | \Omega \rangle$$

$$= \frac{1}{i} \frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i} \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0}$$

On page 45, we introduced the generating functional of connected Green functions via

$$W[J] = -i \ln Z[J]$$

We now introduce the <u>effective action</u> by taking the Legendre transform

$$\Gamma[\varphi] = W[J] - \int d^4x \; \varphi(x) J(x)$$

with

$$\varphi(x) = \frac{\delta W[J]}{\delta J(x)} = \langle \Omega | \phi(x) | \Omega \rangle_J$$

Note that we do <u>not</u> set $J = 0$ in this expression. $\varphi(x)$ is called the <u>classical field</u>.

The Legendre transform implies

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = \int d^4y \; \underbrace{\frac{\delta W[J]}{\delta J(y)}}_{\varphi(y)} \frac{\delta J(y)}{\delta \varphi(x)} - J(x) - \int d^4y \; \varphi(y) \frac{\delta J(y)}{\delta \varphi(x)}$$

$$= - J(x)$$

which vanishes when we set the external sources to zero. In other words, for $J = 0$ the classical field $\varphi(x) = \langle \Omega | \phi(x) | \Omega \rangle_0$ fulfills

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = 0$$

which is of the same form as the classical equation of motion $\frac{\delta S[\phi]}{\delta \phi(x)} = 0$

Let us now consider the classical limit $\hbar \to 0$.

Starting from

$$Z[J] = N \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}\left[S + \int d^4x \, J\phi\right]}$$

we read off that propagators $\sim \hbar$ (→ inverse of quadratic terms)

vertices $\sim \frac{1}{\hbar}$

A connected diagram with P propagators and V vertices furthermore fulfills the topological relation (→ page 125)

$$P - V + 1 = L \text{ loops}$$

$\Rightarrow$ a connected diagram gives a contribution $\hbar^{P-V} \sim \hbar^{L-1}$,

i.e. the loop expansion is an expansion in $\hbar$!

We now further write

$$Z[J] = N \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}\tilde{S}[\phi]}$$

with $\tilde{S}[\phi] = S[\phi] + \int d^4x \, J\phi$. In the limit $\hbar \to 0$, the integral is dominated by the stationary point $\phi_0(x)$ with

$$\left.\frac{\delta \tilde{S}[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} = 0 \quad \Rightarrow \quad \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} = -J(x)$$

We can therefore approximate

$$\tilde{S}(\phi) = \tilde{S}(\phi_0) + \frac{1}{2}\int d^4x_1\, d^4x_2 \left.\frac{\delta^2 \tilde{S}(\phi)}{\delta\phi(x_1)\,\delta\phi(x_2)}\right|_{\phi=\phi_0} \delta\phi(x_1)\,\delta\phi(x_2)$$

$$+ \frac{1}{6}\int d^4x_1\, d^4x_2\, d^4x_3 \left.\frac{\delta^3 \tilde{S}(\phi)}{\delta\phi(x_1)\,\delta\phi(x_2)\,\delta\phi(x_3)}\right|_{\phi=\phi_0} \delta\phi(x_1)\,\delta\phi(x_2)\,\delta\phi(x_3) + \dots$$

with $\delta\phi(x) = \phi(x) - \phi_0(x)$. It follows

$$Z[J] \simeq N \int \mathcal{D}\phi\; e^{\frac{i}{\hbar}\left[\tilde{S}(\phi_0) + \frac{1}{2}\int d^4x_1\, d^4x_2 \left.\frac{\delta^2\tilde{S}}{\delta\phi(x_1)\,\delta\phi(x_2)}\right|_{\phi=\phi_0} \delta\phi(x_1)\,\delta\phi(x_2)\right]}$$

$$\left(1 + \frac{i}{\hbar}\,\frac{1}{6}\int d^4x_1\, d^4x_2\, d^4x_3 \left.\frac{\delta^3\tilde{S}}{\delta\phi(x_1)\,\delta\phi(x_2)\,\delta\phi(x_3)}\right|_{\phi=\phi_0} \delta\phi(x_1)\,\delta\phi(x_2)\,\delta\phi(x_3) + \dots\right)$$

and we are thus left with Gaussian integrals, which are weighted by polynomial factors. We recall that

$$\int d^N x\; e^{-\frac{1}{2}x_i A_{ij} x_j} = \left(\det \frac{A}{2\pi}\right)^{-1/2}$$

$$\int d^N x\; e^{-\frac{1}{2}x_i A_{ij} x_j}\, x_k x_e = \left(\det\frac{A}{2\pi}\right)^{-1/2} \underbrace{(A^{-1})_{ke}}_{O(\hbar)}$$

where in our case $A = O(\frac{1}{\hbar})$. The higher order terms in the parenthesis therefore only generate terms that are suppressed in the classical limit as e.g.

$$\frac{1}{\hbar}\,\delta^4\tilde{S} \to \frac{1}{\hbar}\,\phi\phi\phi\phi \to \frac{1}{\hbar}\,\hbar^2 = \hbar$$

$$\frac{1}{\hbar}\,\delta^3\tilde{S}\;\frac{1}{\hbar}\,\delta^3\tilde{S} \to \frac{1}{\hbar^2}\,\phi\phi\phi\phi\phi\phi \to \frac{1}{\hbar^2}\,\hbar^3 = \hbar$$

We thus arrive at

$$Z[J] = e^{\frac{i}{\hbar} W[J]} \simeq e^{\frac{i}{\hbar} \bar{S}(\phi_0)} + \mathcal{O}(\hbar)$$

$$\Rightarrow \quad W[J] \approx \bar{S}(\phi_0) = S(\phi_0) + \int d^4x \, J(x) \, \phi_0(x)$$

with $\left.\frac{\delta S(\phi)}{\delta \phi(x)}\right|_{\phi=\phi_0} = - J(x)$, but this precisely corresponds to the expressions that we found for the effective action. We therefore identify

$$\Gamma[\varphi] = S[\varphi] + \mathcal{O}(\hbar)$$

i.e. the effective action is a generalisation of the classical action that includes all quantum corrections.

Can we write down a theory that contains the effective action as classical action? To do so, we start from the generating functional

$$Z_\Gamma[J;g] = e^{\frac{i}{g} W_\Gamma(J;g)}$$
$$\equiv N \int \mathcal{D}\varphi \, e^{\frac{i}{g} [\Gamma + \int d^4x \, J \varphi]}$$

where g is an arbitrary constant that plays the role of $\hbar$ in the previous analysis. We may thus perform an expansion in g, which corresponds to a loop expansion in a theory that contains the effective action as classical action. In the limit $g \to 0$, we then obtain as before

$$e^{\frac{i}{g} W_\Gamma [J;0]} \approx e^{\frac{i}{g} (\Gamma + \int d^4x \, J \varphi)} \Big|_{\varphi = \varphi_0}$$

where φ_0 is the stationary point with

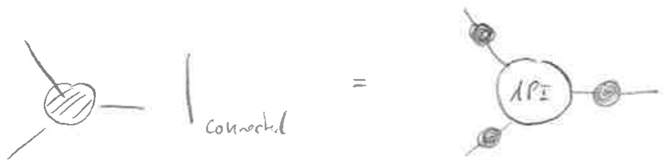
$$\frac{\delta \Gamma(\varphi)}{\delta \varphi(x)} \Big|_{\varphi = \varphi_0} = - J(x)$$

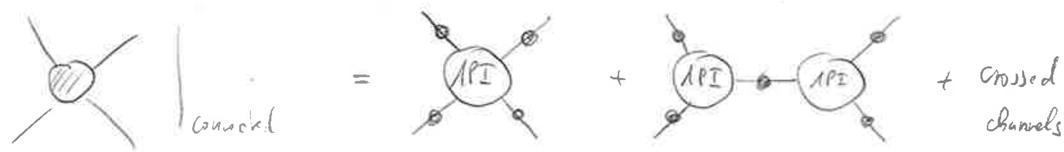
We thus read off

$$W_\Gamma [J;0] = \Gamma + \int d^4x \, J\varphi \Big|_{\varphi = \varphi_0} = W[J]$$

which implies that the connected diagrams of a theory with classical action S are given by the <u>tree diagrams</u> of a theory with effective action Γ as classical action!

But a connected diagram can always be represented as a tree diagram with full propagators and 1PI vertices, e.g.





We therefore conclude that the theory with effective action Γ as classical action has full propagators and 1PI diagrams as Feynman rules.

This suggests that the effective action is the generating functional of 1PI Green functions. In order to verify this for a few examples, it will be convenient to introduce a shorthand notation

$$\varphi(x) \equiv \varphi_x \qquad \int d^4x = \int_x \qquad \delta^{(4)}(x-y) = \delta_{xy}$$

Starting from

$$\frac{\delta\Gamma}{\delta\varphi_x} = -J_x$$

$$\Rightarrow \frac{\delta}{\delta J_y}\frac{\delta\Gamma}{\delta\varphi_x} = \int_z \frac{\delta^2\Gamma}{\delta\varphi_x\,\delta\varphi_z}\frac{\delta\varphi_z}{\delta J_y} = \int_z \frac{\delta^2\Gamma}{\delta\varphi_x\,\delta\varphi_z}\underbrace{\frac{\delta^2 W}{\delta J_z\,\delta J_y}}_{i\Delta_{zy}\ \text{(full propagator)}}$$

$$= -\delta_{xy}$$

and hence

$$\frac{\delta^2\Gamma}{\delta\varphi_x\,\delta\varphi_z} = i\,(\Delta^{-1})_{xz}\,.$$

In order to address the 3-point function, we will need

i) $$\frac{\delta}{\delta J_x} = \int_y \frac{\delta \varphi_y}{\delta J_x}\frac{\delta}{\delta \varphi_y} = \int_y \frac{\delta^2 W}{\delta J_x\, \delta J_y}\frac{\delta}{\delta \varphi_y} = i \int_y \Delta_{xy}\frac{\delta}{\delta \varphi_y}$$

ii) the derivative of the inverse matrix

$$\frac{\delta}{\delta \alpha}(M M^{-1}) = \frac{\delta M}{\delta \alpha} M^{-1} + M \frac{\delta M^{-1}}{\delta \alpha} = 0$$

$$\Rightarrow \quad \frac{\delta M^{-1}}{\delta \alpha} = - M^{-1}\frac{\delta M}{\delta \alpha} M^{-1}$$

We thus obtain

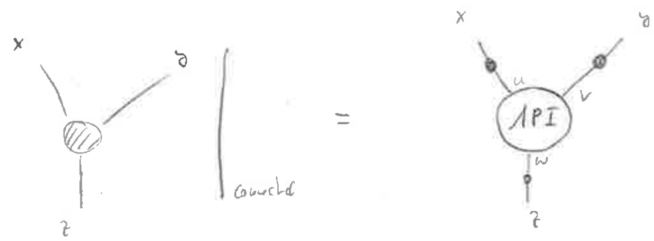
$$\frac{1}{i^3}\frac{\delta^3 iW}{\delta J_x\, \delta J_y\, \delta J_z} \overset{(i)}{=} \frac{1}{i}\int_u \Delta_{xu}\frac{\delta}{\delta \varphi_u}\frac{\delta^2 W}{\delta J_y\, \delta J_z}$$

$$= -\frac{1}{i}\int_u \Delta_{xu}\frac{\delta}{\delta \varphi_u}\left[\frac{\delta^2 \Gamma}{\delta \varphi_y\, \delta \varphi_z}\right]^{-1}$$

$$\overset{(ii)}{=} \frac{1}{i}\int_u \Delta_{xu}\int_v\int_w \underbrace{\left[\frac{\delta^2 \Gamma}{\delta \varphi_y\, \delta \varphi_v}\right]^{-1}}_{-i\Delta_{yv}}\frac{\delta^3 \Gamma}{\delta \varphi_v\, \delta \varphi_w\, \delta \varphi_u}\underbrace{\left[\frac{\delta^2 \Gamma}{\delta \varphi_w\, \delta \varphi_z}\right]^{-1}}_{-i\Delta_{wz}}$$

$$= i \int_u\int_v\int_w \Delta_{xu}\Delta_{yv}\Delta_{wz}\frac{\delta^3 \Gamma}{\delta \varphi_u\, \delta \varphi_v\, \delta \varphi_w}$$

which diagrammatically corresponds to



and hence $\dfrac{\delta^3 \Gamma}{\delta \varphi_u\, \delta \varphi_v\, \delta \varphi_z}$ generates the 1PI 3-point function.

One can proceed similarly for higher derivatives.

Let us summarise what we have learned so far

i) the effective action generalises the classical action and includes all quantum corrections

$$\Gamma[\varphi] = S[\varphi] + O(\hbar)$$

ii) in the absence of external sources, the classical field $\varphi(x) = \langle \Omega | \phi(x) | \Omega \rangle_0$ fulfills $\frac{\delta \Gamma(\varphi)}{\delta \varphi(x)} = 0$, which is of the same form as the classical equation of motion.

iii) the effective action is the generating functional of 1PI Green functions (and it is therefore important in renormalisation theory since the 1PI diagrams contain the full information on the loops).

In the context of SSB, we typically assume that $\varphi(x) = \langle \Omega | \phi(x) | \Omega \rangle \equiv v$ is a constant. The effective action then becomes

$$\Gamma(\varphi = v) = \int d^4x \, [-V_{eff}(\varphi)] = -VT \, V_{eff}(\varphi)$$

where $V_{eff}(\varphi)$ is the <u>effective potential</u>. For $J=0$, we now obtain

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = 0 \quad \Rightarrow \quad \frac{\delta V_{eff}(\varphi)}{\delta \varphi}\Big|_{\varphi = v} = 0$$

and one can indeed show that $V_{eff}(v)$ is the <u>minimum</u> of the expectation value of the energy density for all states constrained by $\varphi(x) = \langle \Omega | \phi(x) | \Omega \rangle$ (for details cf. e.g. Weinberg II, chapter 16.3).

But how can we calculate the effective potential? This is in general complicated, but we may again resort to the semiclassical expansion $\hbar \to 0$. In a scalar theory with

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

we have

$$\tilde{S} = S + \int d^4x \, J \phi$$

$$\frac{\delta^2 \tilde{S}[\phi]}{\delta \phi(x_1) \, \delta \phi(x_2)}\Big|_{\phi = \phi_0} = - \left[\partial^2 + V''(\phi_0) \right] \delta^{(4)}(x_1 - x_2)$$

and our earlier result from page 239 becomes

$$Z[J] = e^{\frac{i}{\hbar} W(J)}$$

$$\simeq N\, e^{\frac{i}{\hbar} \tilde{S}(\phi_0)} \int \mathcal{D}\phi\; e^{-\frac{i}{2\hbar} \int d^4x\, \delta\phi(x) \left[\partial^2 + V''(\phi_0)\right] \delta\phi(x)}$$

$$= e^{\frac{i}{\hbar} \tilde{S}(\phi_0)} \det\left[\partial^2 + V''(\phi_0)\right]^{-1/2}$$

Using $\det A = e^{\operatorname{tr} \ln A}$, we obtain

$$W[J] = \tilde{S}(\phi_0) - i\hbar \left(-\tfrac{1}{2}\right) \operatorname{tr} \ln\left[\partial^2 + V''(\phi_0)\right]$$

$$= S(\phi_0) + \int d^4x\, J(x)\, \phi_0(x) + \frac{i\hbar}{2} \operatorname{tr} \ln\left(\partial^2 + V''(\phi_0)\right)$$

$$\Rightarrow \quad \varphi = \frac{\delta W}{\delta J} = \underbrace{\frac{\delta S(\phi_0)}{\delta \phi_0}}_{=0 \text{ since } \phi_0 \text{ is a stationary point}} \frac{\delta \phi_0}{\delta J} + \phi_0(x) + \mathcal{O}(\hbar)$$

$$= \phi_0 + \mathcal{O}(\hbar)$$

The effective action then becomes

$$\Gamma[\varphi] = W[J] - \int d^4x\, \varphi(x)\, J(x)$$

$$= S(\phi_0) + \frac{i\hbar}{2} \operatorname{tr} \ln\left(\partial^2 + V''(\phi_0)\right)$$

Setting $\phi_0 = v = \text{const}$, we get

$$V_{eff}(v) = V(v) - \frac{i\hbar}{2} \frac{1}{VT} \operatorname{tr} \ln\left[\partial^2 + V''(v)\right]$$

We can evaluate the trace as a sum over eigenvalues

$$\mathrm{tr}\ \ln\left(\partial^2 + V''(v)\right) = \int d^4x \ \langle x | \ln(\partial^2 + V''(v)) | x \rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \ \langle x | k \rangle \langle k | \ln(\partial^2 + V''(v)) | k' \rangle \langle k' | x \rangle$$

$$= \underbrace{\int d^4x}_{VT} \int \frac{d^4k}{(2\pi)^4} \ \ln\left(-k^2 + V''(v)\right)$$

and we obtain

$$V_{eff}(v) = V(v) - \frac{i\hbar}{2} \int \frac{d^4k}{(2\pi)^4} \ \ln\left(-k^2 + V''(v)\right) + O(\hbar^2)$$

i.e we have computed the one-loop correction to the classical potential! The result is actually UV-divergent and one needs to impose some renormalisation conditions to obtain a finite (scheme-dependent) expression

5.3 Goldstone theorem

The formalism that we have developed in the last section allows us to lift the classical analysis of SSB to the quantum level. In a classical theory, we determine the ground state of a system by minimising its potential energy. Whenever the classical ground state is not invariant under a symmetry transformation, we say that the symmetry is spontaneously broken. There then exists a flat direction in the potential which corresponds to a massless excitation.

In the last section we saw that the effective potential is the quantum generalisation of the classical potential. The effective potential includes quantum corrections, and it is minimised by the vacuum expectation value of the scalar field operator. We will now prove the Goldstone theorem, which says that there exists a massless Goldstone boson in the spectrum for each spontaneously broken symmetry.

We consider a theory with N real scalar fields that is invariant under a global, continous symmetry transformation

$$\delta\phi_n(x) = \phi_n'(x) - \phi_n(x) = i\varepsilon^A T^A_{nm}\phi_m(x)$$

where $n,m = 1,\dots,N$ and $A = 1,\dots,\dim G$. There then exist conserved Noether currents j^A_μ with charges Q^A that generate the symmetry transformation

$$[Q^A, \phi_n] = T^A_{nm}\phi_m$$

We say that the symmetry is <u>manifest</u> if

$$\langle 0|[Q^A, \phi_n]|0\rangle = \langle 0|T^A_{nm}\phi_m|0\rangle = 0$$

and it is <u>spontaneously broken</u> if

$$\langle 0|T^A_{nm}\phi_m|0\rangle \neq 0$$

It will be convenient to divide the generators into two subsets

$$T^A \rightsquigarrow \begin{cases} Y^i & i=1,\dots,\dim H \quad \text{unbroken} \quad \langle 0|Y^i\phi|0\rangle = 0 \\ X^j & j=1,\dots,\dim G-\dim H \quad \text{broken} \quad \langle 0|X^j\phi|0\rangle \neq 0 \end{cases}$$

Some linear combinations

We will now first show that the effective action is invariant under the symmetry transformation. To this end, we will exploit the associated Ward identity. We thus start from the generating functional

$$Z[J_n] = N \int \mathcal{D}\phi_n \, e^{i\left(S(\phi_n) + \int d^4x \, J_n \phi_n\right)}$$

and shift the integration variables according to

$$\phi_n'(x) = \phi_n(x) + i\varepsilon^A T^A_{nm} \phi_m(x)$$

We further assume that the path integral measure is invariant under this transformation (i.e. that the symmetry is not anomalous). As the action is also invariant, we obtain

$$Z[J_n] = N \int \mathcal{D}\phi_n \, e^{i\left(S(\phi_n) + \int d^4x \, J_n\left(\phi_n - i\varepsilon^A T^A_{nm}\phi_m\right)\right)}$$

$$= N \int \mathcal{D}\phi_n \, e^{i\left[S(\phi_n) + \int d^4x \, J_n \phi_n\right]}$$

$$\left(1 + \int d^4x \, J_n \varepsilon^A T^A_{nm} \phi_m + \dots\right)$$

$\hookrightarrow Z[J_n]$

It follows

$$\int \mathcal{D}\phi_n \, e^{i[S(\phi_n) + \int d^4x \, J_n \phi_n]} \int d^4x \, J_n \, i\varepsilon^A T^A_{nm} \phi_m = 0$$

which is the desired Ward identity.

We next recall that

$$J_n = - \frac{\delta \Gamma[\varphi]}{\delta \varphi_n}$$

and

$$\int \mathcal{D}\phi_n \, e^{i[S[\phi_n] + \int d^4x \, J_n \phi_n]} \phi_k = \langle \Omega | \phi_k | \Omega \rangle_J = \varphi_k$$

which yields

$$- \int d^4x \frac{\delta \Gamma[\varphi]}{\delta \varphi_n} i\varepsilon^A T^A_{nm} \varphi_m = 0$$

$$= - \int d^4x \frac{\delta \Gamma[\varphi]}{\delta \varphi_n(x)} \delta \varphi_n(x) = - \delta \Gamma[\varphi]$$

$\Rightarrow$ the effective action is invariant under the symmetry transformation

One can actually show that the effective action inherits the symmetry of the classical action, whenever the symmetry transformation is linear in the fields.

For constant φ_n with

$$\Gamma[\varphi] = - VT \; V_{eff}(\varphi)$$

this then implies that the effective ~~action~~ potential is also invariant under the symmetry transformation. As

$$\delta V_{eff}(\varphi) = \frac{\partial V_{eff}(\varphi)}{\partial \varphi_n} \delta \varphi_n = \frac{\partial V_{eff}(\varphi)}{\partial \varphi_n} i \varepsilon^A T^A_{nm} \varphi_m = 0$$

holds for arbitrary ε^A, we have

$$\frac{\partial V_{eff}(\varphi)}{\partial \varphi_n} T^A_{nm} \varphi_m = 0$$

for <u>each</u> symmetry, and hence

$$\frac{\partial^2 V_{eff}}{\partial \varphi_n \partial \varphi_k} T^A_{nm} \varphi_m + \frac{\partial V_{eff}}{\partial \varphi_n} T^A_{nk} = 0$$

We are interested here in the vacuum state $\varphi = v$ that minimises the effective potential

$$\rightarrow \quad \left.\frac{\partial V_{eff}}{\partial \varphi_n}\right|_{\varphi=v} = 0$$

$$\left.\frac{\partial^2 V_{eff}}{\partial \varphi_n \partial \varphi_k}\right|_{\varphi=v} \equiv M^2_{kn}$$

and M^2 is a positive semi-definite $N \times N$ matrix.

We thus obtain

$$\frac{\partial^2 V_{eff}}{\partial \varphi_n \partial \varphi_k} T^A_{nm} \varphi_m \Big|_{\varphi=v} = M^2_{kn} T^A_{nm} v_m = 0$$

For the unbroken generators Y^i with $Y^i_{nm} v_m = 0$ this is trivially fulfilled. For the broken generators X^j with $X^j_{nm} v_m \neq 0$, however, this implies that $X^j v$ is an eigenvector of the mass matrix with eigenvalue 0. As there are dim G - dim H broken generators, there thus exist dim G - dim H massless Goldstone bosons in the theory.

We further note that the unbroken generators form a subgroup H, and that the number of Goldstone bosons is equal to the dimension of the coset space G/H.

5.4 Chiral symmetry breaking

Before discussing SSB in the context of gauge symmetries, let us address the question if there are any Goldstone bosons, i.e massless spin-0 particles, realised in nature.

Although not exactly massless, it happens that the pions, and to some extent also the kaons, are significantly lighter than the other hadrons:

$$m_\pi \simeq 140 \text{ MeV} \quad m_K \simeq 500 \text{ MeV} \quad \Big] \; 0^- \text{ mesons}$$

$$m_\rho \simeq 770 \text{ MeV} \quad 1^- \text{ meson}$$

$$m_{p,n} \simeq 940 \text{ MeV} \quad 1/2^+ \text{ baryon}$$

and there are many more resonances above 1 GeV. Are they the Goldstone bosons of an approximate symmetry, i.e would they be exactly massless if the symmetry was exact?

In order to identify the underlying symmetry, we consider the QCD Lagrangian for <u>massless</u> quarks

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} + \sum_{f=1}^{n_f} \bar{q}_f \, i \not{D} \, q_f$$

In terms of the chiral fields $q_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$, we obtain

$$\sum_{f=1}^{n_f} \bar{q}_f \, i \not{D} \, q_f = \sum_{f=1}^{n_f} (\bar{q}_{fL} + \bar{q}_{fR}) \, i \not{D} \, (q_{fL} + q_{fR})$$

$$= \sum_{f=1}^{n_f} \left(\bar{q}_{fL} \, i \not{D} q_{fL} + \bar{q}_{fR} \, i \not{D} \, q_{fR} \right)$$

$\Rightarrow$ the Lagrangian is invariant under the global symmetry group

$$U(n_f)_L \otimes U(n_f)_R = SU(n_f)_L \otimes SU(n_f)_R \otimes U(1)_B \otimes U(1)_A$$

$SU(n_f)_L$: $q_L \to V_L q_L$ $\quad V_L = e^{i\varepsilon_L^A T^A} \in SU(n_f)$

$q_R \to q_R$

$SU(n_f)_R$: $q_L \to q_L$ $\quad V_R = e^{i\varepsilon_R^A T^A} \in SU(n_f)$

$q_R \to V_R q_R$

} chiral symmetry

$U(1)_B$: $q_{L/R} \to e^{i\varepsilon_B} q_{L/R}$

[or $q \to e^{i\varepsilon_B} q$ with $q = \begin{pmatrix} q_L \\ q_R \end{pmatrix}$]

} baryon number

$U(1)_A$: $q_{L/R} \to e^{\mp i \varepsilon_A} q_{L/R}$

[or $q \to e^{i\varepsilon_A \gamma_5}$ with $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$]

} axial symmetry (→ anomalous, i.e. broken by quantum corrections)

We will focus here on the chiral symmetry $SU(n_f)_L \otimes SU(n_f)_R$,

which can also be formulated as a $SU(n_f)_V \otimes SU(n_f)_A$

transformation with

$$q \to e^{i \varepsilon_V^A T^A + i \varepsilon_A^A T^A \gamma_5} q \qquad \varepsilon_{V/A}^A = \frac{1}{2} (\varepsilon_R^A \pm \varepsilon_L^A)$$

The associated Noether currents are

$$j_V^{A,\mu} = i \bar{q} \gamma^\mu T^A q$$
$$j_A^{A,\mu} = i \bar{q} \gamma^\mu \gamma_5 T^A q$$

Empirically one observes that

* hadrons can be classified according to $SU(3)_V$ definite parity multiplets ("eightfold way")

mesons : $q\bar{q} \sim 3 \otimes \bar{3} = 1 \oplus 8$

e.g. 0^--octet : $\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta_8$

[η and η' are admixtures of η_0 and η_8]

baryons : $qqq \sim 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

e.g. $1/2^+$-octet : $n, p, \Sigma^0, \Sigma^\pm, \Xi^0, \Xi^-, \Lambda$

$3/2^+$-decuplet : $\Delta^0, \Delta^\pm, \Delta^{++}, \Sigma^{*0}, \Sigma^{*\pm}, \Xi^{0*}, \Xi^{-*}, \Omega^-$

* there do not exist approximately degenerate states with opposite parity

$\Rightarrow$ is the $SU(3)_A$ spontaneously broken ?

In the following we assume that the $SU(3)_A$ is spontaneously broken. The pattern

$$SU(3)_L \otimes SU(3)_R \xrightarrow{SSB} SU(3)_V$$

then gives rise to 8 Goldstone bosons, which we identify with the 0^--octet mesons (they are pseudoscalars since the broken symmetry groups $SU(3)_A$ has negative parity). The 0^--mesons are not exactly massless since the chiral symmetry is only an approximate symmetry of QCD in the limit $m_q \to 0$.

One can formulate a Goldstone theorem for approximate symmetries whenever the symmetry-breaking terms are only a small perturbation. This gives rise to pseudo-Goldstone bosons that become massless in the exact symmetry limit.

In our example, one indeed finds

$$m_\pi^2 \sim (m_u + m_d)\,\Lambda_{had}$$

$$\Lambda_{had} = -\frac{\langle\Omega|q\bar{q}|\Omega\rangle}{f_\pi^2}$$

$$m_K^2 \sim (m_u + m_s)\,\Lambda_{had}$$

and in particular $\dfrac{m_\pi^2}{m_K^2} = \dfrac{m_u + m_d}{m_u + m_s}$.

But which field mediates the SSB? As there is no fundamental scalar field in QCD, the chiral symmetry must be broken by a composite operator. It turns out that the <u>chiral condensate</u>

$$\langle\Omega|\,\bar{q}_{Li}\, q_{Rj}\,|\Omega\rangle = -v^3\,\delta_{ij}$$

obtains a non-zero vacuum expectation value. The details of the symmetry breakdown are complicated and lie in the domain of non-perturbative QCD. The low-energy <u>consequences</u> of the symmetry breakdown are, however, independent of the non-perturbative dynamics and follow from symmetry considerations alone.

[In our abelian example, the precise form of the potential did not matter either. It was only important that the Lagrangian is invariant under global U(1) transformations, and that the potential has a degenerate ground state.]

Under $SU(3)_L \otimes SU(3)_R$ transformations, the chiral condensate transforms as

$$\langle \bar{q}_{Li} q_{Ri} \rangle \longrightarrow \langle \bar{q}_{Lk} V^+_{L\,ki} V_{R\,je} q_{Re} \rangle$$

$$= V^+_{L\,ki} V_{R\,je} \, (-v^3) \, \delta_{ke}$$

$$= -v^3 \, (V_R V_L^+)_{ji}$$

and it is thus invariant under the $SU(3)_V$ subgroup with $V_L = V_R$, which is left unbroken.

As the Goldstone bosons are massless (or light) and have derivative interactions, we can formulate an <u>effective field theory</u> that describes their self-interactions at low energies. The effective Lagrangian will be formulated in terms of fundamental pseudoscalar fields with interaction terms that are constrained by the chiral symmetry. In other words, the pions, kaons etc will be treated as <u>pointlike</u> particles, which is a valid approximation as long as the scattering energy $E \ll R^{-1}$, where $R \sim 1$ fm is the typical size of the 0^--mesons (we will specify the cutoff more precisely below).

The Goldstone bosons lie in the coset space

$SU(3)_L \otimes SU(3)_R / SU(3)_V = SU(3)_A$. We therefore parametrise

$$\Sigma(\phi) = e^{\frac{2i}{F} T^A \phi^A}$$

in terms of a parameter F with dimension mass, and

$$T^A \phi^A = \frac{1}{2} \begin{pmatrix} \phi^3 + \frac{\phi^8}{\sqrt{3}} & \phi^1 - i\phi^2 & \phi^4 - i\phi^5 \\ \phi^1 + i\phi^2 & -\phi^3 + \frac{\phi^8}{\sqrt{3}} & \phi^6 - i\phi^7 \\ \phi^4 + i\phi^5 & \phi^6 + i\phi^7 & -\frac{2}{\sqrt{3}}\phi^8 \end{pmatrix} \rightsquigarrow \text{Gell-Mann matrices}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}$$

The field Σ is constrained by the SSB pattern. In particular, it reflects the transformation law of the chiral condensate

$$\Sigma \longrightarrow V_L^+ \Sigma V_R$$

and obtains a non-zero vacuum expectation value

$$\langle \Omega | \Sigma_{ij}(0) | \Omega \rangle = \delta_{ij}$$

which is invariant under $SU(3)_V$ transformations. As $\Sigma \in SU(3)_A$, we have $\Sigma \Sigma^+ = \mathbb{1}$. This implies that we cannot construct any $SU(3)_L \otimes SU(3)_R$ invariant operator without derivatives, which is consistent with our argument that Goldstone bosons have derivative interactions.

There are infinitely many operators that are invariant under the chiral symmetry, and since we work in an effective theory there is no constraint on their mass dimension. At low energies, however, the operators with lowest dimension, i.e with lowest number of derivatives, give the dominant contribution. It turns out that there is only one independent operator with two derivatives

$$\mathrm{Tr}\left[\partial_\mu \Sigma^+ \partial^\mu \Sigma\right]$$

$$\mathrm{Tr}\left(\partial^2 \Sigma^+ \Sigma\right] \quad \longrightarrow \text{ related by partial integration}$$

$$\mathrm{Tr}\left(\partial_\mu \Sigma \Sigma^+ \partial^\mu \Sigma \Sigma^+\right] = - \mathrm{Tr}\left(\partial_\mu \Sigma^+ \partial^\mu \Sigma\right]$$

where we used

$$\Sigma \Sigma^+ = \mathbb{1} \quad \rightarrow \quad \partial_\mu \Sigma \, \Sigma^+ = - \Sigma \, \partial_\mu \Sigma^+$$

and similar arguments can be applied to other terms with two derivatives. The effective Lagrangian at leading power in the chiral expansion is therefore given by

$$\boxed{\mathcal{L}_{eff}^{(2)} = \frac{1}{4} F^2 \, \mathrm{Tr}\left[\partial_\mu \Sigma^+ \partial^\mu \Sigma\right]}$$

The Lagrangian takes a highly non-trivial form when it is expressed in terms of the field $\phi = T^A \phi^A$

$$\Sigma = e^{\frac{2i}{F}\phi} = 1 + \frac{2i}{F}\phi - \frac{2}{F^2}\phi^2 - \frac{4i}{3F^3}\phi^3 + \dots$$

$$\partial_\mu \Sigma = \frac{2i}{F}\partial_\mu \phi - \frac{2}{F^2}\left[\partial_\mu \phi\, \phi + \phi\, \partial_\mu \phi\right]$$
$$- \frac{4i}{3F^3}\left[\partial_\mu \phi\,\phi\,\phi + \phi\,\partial_\mu \phi\,\phi + \phi\,\phi\,\partial_\mu \phi\right] + \dots$$

$$\Rightarrow \mathcal{L}_{eff}^{(2)} = \frac{F^2}{4}\,\mathrm{Tr}\Big[\frac{4}{F^2}\partial_\mu \phi\, \partial^\mu \phi + \frac{4}{F^4}\left[\partial_\mu \phi\,\phi + \phi\,\partial_\mu\phi\right]^2$$
$$- \frac{16}{3F^4}\partial^\mu_\mu \phi\left[\partial_\mu \phi\,\phi\,\phi + \phi\,\partial_\mu\phi\,\phi + \phi\,\phi\,\partial_\mu\phi\right] + \mathcal{O}(\phi^6)\Big]$$

Note that the higher order terms in ϕ are <u>not</u> suppressed since they all contain two derivatives, which is the relevant power counting parameter here. In the following, we focus on $\pi\pi \to \pi\pi$ scattering at tree level, and we therefore do not need ~~to~~ consider terms with more than four powers of the field ϕ.

As long as we focus on terms that contain pions only, we can use a compact 2×2 notation with

$$\phi\phi = \frac{1}{2}\begin{pmatrix} \pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 & \frac{1}{\sqrt{2}}\left(\pi^0\pi^+ - \pi^+\pi^0\right) \\ \frac{1}{\sqrt{2}}\left(\pi^-\pi^0 - \pi^0\pi^-\right) & \pi^-\pi^+ + \frac{1}{2}\pi^0\pi^0 \end{pmatrix}$$

[the pions are scalars and commute, but we write the product in this form since we will later discuss the term with derivatives]

As the Lagrangian involves a trace, we only need the diagonal elements of

$$\phi\phi\phi\phi = \frac{1}{4}\begin{pmatrix} (\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0)^2 + \frac{1}{2}(\pi^0\pi^+ - \pi^+\pi^0)(\pi^-\pi^0 - \pi^0\pi^-) & \cdots \\ \cdots & (\pi^-\pi^+ + \frac{1}{2}\pi^0\pi^0)^2 + \frac{1}{2}(\pi^-\pi^0 - \pi^0\pi^-)(\pi^0\pi^+ - \pi^+\pi^0) \end{pmatrix}$$

The terms with at most four pions are then given by

$$\begin{aligned} \mathcal{L}_{eff}^{(2)} = {} & \frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0 + \partial_\mu\pi^+\partial^\mu\pi^- \\ & + \frac{1}{3F^2}\Big[\frac{1}{2}\pi^-\pi^-\partial_\mu\pi^+\partial^\mu\pi^+ + \frac{1}{2}\pi^+\pi^+\partial_\mu\pi^-\partial^\mu\pi^- \\ & \quad - \pi^0\pi^0\partial_\mu\pi^+\partial^\mu\pi^- - \pi^-\pi^+\partial_\mu\pi^0\partial^\mu\pi^0 \\ & \quad + \pi^0\pi^-\partial_\mu\pi^0\partial^\mu\pi^+ + \pi^0\pi^+\partial_\mu\pi^0\partial^\mu\pi^- \\ & \quad - \pi^-\pi^+\partial_\mu\pi^-\partial^\mu\pi^+\Big] + \ldots \end{aligned}$$

Notice that the pions are massless and that they have derivative interactions. There is furthermore a single parameter F that controls the strength of the various interaction terms due to the underlying chiral symmetry.

The Lagrangian can be used to compute $\pi\pi \to \pi\pi$ scattering at tree level at low energies. (→ tutorial).

We conclude with a few more remarks:

- The formalism can be extended to account for an explicit breaking of the chiral symmetry due to non-zero quark masses, and to include electromagnetic and weak interactions.

- The parameter F can be identified with the pion decay constant, which is defined by

$$\langle \Omega | \bar{d} \gamma^\mu \gamma_5 u | \pi(p) \rangle = i \sqrt{2} f_\pi p^\mu \qquad f_\pi \simeq 92 \text{ MeV}$$

- There are two types of corrections at $O(p^4)$

  i) one-loop diagrams with two vertices from $\mathcal{L}_{eff}^{(2)}$

  ii) tree diagrams from terms in $\mathcal{L}_{eff}^{(4)}$ with <u>four</u> derivatives

  $\Rightarrow$ the corrections are suppressed by $\frac{p^2}{(4\pi f_\pi)^2}$, where

  $4\pi f_\pi \simeq 1.2$ GeV is the cutoff scale of the EFT

- The procedure is completely general and applies similarly for a low-energy description of other (pseudo) Goldstone bosons. Remarkably, the effective Lagrangian is completely determined by the symmetry breaking pattern $G \xrightarrow{SSB} H$.

5.5. Higgs mechanism

We return to the abelian theory from chapter 5.1 with

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi)$$

and $V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$. We now assume that $\mu^2 < 0$ and that the underlying $U(1)$ symmetry is gauged.

Following the minimal coupling prescription, we can construct an invariant Lagrangian by replacing $\partial_\mu \phi$ with the covariant derivative

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$$

The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu + ieA_\mu)\phi^\dagger (\partial^\mu - ieA^\mu)\phi - V(\phi)$$

is then invariant under <u>local</u> $U(1)$ transformations

$$\phi'(x) = e^{ie\omega(x)} \phi(x)$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \omega(x)$$

We first study this theory on the classical level. As in chapter 5.1, we change variables to

$$\phi(x) = \frac{1}{\sqrt{2}} \rho(x)\, e^{\frac{i}{v}\sigma(x)}$$

$$\Rightarrow (\partial^\mu - ieA^\mu)\phi = \frac{1}{\sqrt{2}}\left(\partial^\mu \rho + \frac{i}{v}\rho\,\partial^\mu\sigma - ieA^\mu \rho\right) e^{\frac{i}{v}\sigma}$$

$$(\partial_\mu + ieA_\mu)\phi^* (\partial^\mu - ieA^\mu)\phi$$

$$= \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho + \frac{\rho^2}{2v^2}(\partial^\mu\sigma - ev A^\mu)^2$$

$\mu^2 = -\lambda v^2 < 0$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho + \frac{1}{2v^2}\rho^2(\partial^\mu\sigma - evA^\mu)^2$$

$$+ \frac{\lambda v^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

In these coordinates, the gauge transformation acts as

$$\sigma'(x) = \sigma(x) + ev\,\omega(x)$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \omega(x)$$

and so $D_\mu\sigma \equiv \partial_\mu\sigma - evA_\mu$ is invariant. The Lagrangian can therefore depend on σ only through covariant derivatives of σ.

In this model the ground state is still given by $\sigma(x) = v$. But in contrast to the global symmetry, the fluctuations in σ are not physical, they are just gauge transformations. By gauging the symmetry, we thus have removed the Goldstone boson from the spectrum! This is called the Higgs mechanism.

Strictly speaking in a gauge theory there exists no SSB at all. Let us contrast the following situations:

* in a theory with a global symmetry, different values of σ represent distinct but equivalent classical ground states (global symmetry: different points in configuration space that have the same physical properties).
* in a gauge theory, different values of σ are different mathematical descriptions of the same unique ground state (local symmetry: apparently different points in configuration space that are physically identical). As the ground state is not degenerate, there is no SSB (→ misnomer).

Despite the fact that there is no SSB in gauge theories, there is a qualitative difference between the models with $\mu^2 \geq 0$ and $\mu^2 < 0$. To illustrate this point, it will be convenient to fix a gauge. In the following, we choose the unitary gauge with $\sigma = 0$.

In unitary gauge, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \, \partial^\mu \rho + \frac{e^2}{2} \rho^2 A_\mu A^\mu + \frac{\lambda v^2}{2} \rho^2 - \frac{\lambda}{4} \rho^4$$

We next expand around the ground state, writing

$$\rho(x) \to v + \rho(x)$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \, \partial^\mu \rho - \lambda v^2 \rho^2 - \lambda v \rho^3 - \frac{\lambda}{4} \rho^4$$
$$+ \frac{1}{2} e^2 v^2 A_\mu A^\mu + e^2 v \rho A_\mu A^\mu + \frac{1}{2} e^2 \rho^2 A_\mu A^\mu$$

We thus again obtain a massive scalar field with mass $m_\rho = \sqrt{2 \lambda v^2}$, but in addition the gauge boson has acquired a mass $m_A = ev$! As the Goldstone boson has disappeared from the spectrum and the gauge boson has become massive, one sometimes says that the gauge boson has "eaten" the Goldstone boson.

It is easy to see that the gauge boson stays massless for $\mu^2 \geq 0$. In this case one finds two degenerate spin-0 particles with mass $m_\phi = \mu$ (as for the global symmetry). The two scalar particles represent particle and antiparticle solutions with opposite charges $\pm e$.

Notice that the gauge boson mass $m_A = ev$ vanishes in the limit $e \to 0$, in which the gauge boson decouples from the matter particles. We thus obtain the following pattern:

$\underline{\mu^2 \geq 0}$		$\underline{\mu^2 < 0}$	
massless gauge boson	[2]	massive gauge boson	[3]
massive complex scalar	[2]	massive real scalar	[1]
(scalar QED)		(abelian Higgs model)	
$\downarrow e \to 0$		$\downarrow e \to 0$	
massless gauge boson	(2)	massless gauge boson	[2]
massive complex scalar	(2)	massless real scalar	(1)
(ϕ^4-theory and free photon)		massive real scalar	(1)
		(abelian Goldstone model + free photon)	

For all values of e and μ^2 there are four helicity states. For $\mu^2 < 0$ one of the scalar particles becomes the helicity $\lambda = 0$ component of the gauge field. As $e \to 0$ the $\lambda = 0$ component decouples from the $\lambda = \pm 1$ states and becomes the Goldstone boson of the abelian Goldstone model.

5.6. Renormalisation

One may wonder if after SSB the theory with

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^2 v^2 A_\mu A^\mu + \frac{1}{2} \partial_\mu s \, \partial^\mu s - d v^2 s^2$$

$$+ e^2 v s A_\mu A^\mu + \frac{1}{2} e^2 s^2 A_\mu A^\mu - d v s^3 - \frac{d}{4} s^4$$

is renormalisable, given that there are only five renormalisation constants associated with A_μ, s, e, d, v. One may choose e.g. to fix Z_A and Z_e by absorbing the divergences of the gauge boson two-point function (since $m_A = ev$), Z_s and Z_v with the scalar two-point function (since $m_s = \sqrt{2dv^2}$) and Z_d with the s^4-vertex. But does this guarantee that the sAA, $ssAA$ and sss vertices are finite to all orders in perturbation theory?

We encountered a similar situation when we discussed renormalisation of non-abelian gauge theories. There it was gauge invariance (encoded in the Slavnov-Taylor identities), which guaranteed that all divergences can be absorbed by the renormalisation constants. In the present case, one can show that the effective action inherits the symmetry of the classical action (since gauge transformations are linear in the fields). There therefore exist Ward identities that relate the various counterterms, and one finally expands around the minimum of the renormalised effective potential to discuss the consequences of SSB. In simple terms, the UV behaviour of the theory is unaffected by SSB!

But there is another aspect of renormalisation in spontaneously broken gauge theories. As emphasised at the beginning of this section, the gauge boson propagator in unitary gauge

$$\frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_A^2}\right] \xrightarrow{k\to\infty} \frac{i}{m_A^2}\,\frac{k^\mu k^\nu}{k^2}$$

has a bad UV behaviour. In a spontaneously broken theory, the symmetry is still intact on the level of the Lagrangian, which ensures that the dangerous terms cancel for arbitrary S matrix elements. But one may wonder if there exists a different gauge choice, in which the good UV behaviour of the gauge boson propagator is manifest.

In order to study the UV behaviour, it is convenient to work with "cartesian coordinates". Starting from

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu + ieA_\mu)\phi^\dagger (\partial^\mu - ieA^\mu)\phi - V(\phi)$$

with $V(\phi) = \mu^2 \phi^\dagger\phi + \lambda (\phi^\dagger\phi)^2$, we now substitute

$$\phi(x) = \frac{1}{\sqrt{2}}\Big(v + h(x) + i\varphi(x)\Big)$$

$$\Rightarrow \quad D^\mu\phi = \frac{1}{\sqrt{2}}\Big(\partial^\mu h + e\varphi A^\mu + i(\partial^\mu\varphi - e(v+h)A^\mu)\Big)$$

$$(D_\mu\phi)^\dagger (D^\mu\phi) = \frac{1}{2}(\partial^\mu h + e\varphi A^\mu)^2 + \frac{1}{2}(\partial^\mu\varphi - e(v+h)A^\mu)^2$$

$$= \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}e^2v^2 A_\mu A^\mu \underline{- ev\partial_\mu\varphi A^\mu}$$

$$+ e(\varphi\partial_\mu h - h\partial_\mu\varphi)A^\mu + e^2 v h A_\mu A^\mu + \frac{1}{2}e^2(h^2+\varphi^2)A_\mu A^\mu$$

$$\phi^\dagger\phi = \frac{1}{2}\left[(v+h)^2 + \varphi^2\right]$$

$$V(\phi) = \lambda v^2 h^2 + \lambda v h (h^2+\varphi^2) + \frac{\lambda}{4}(h^2+\varphi^2)^2 \qquad \mu^2 = -\lambda v^2$$

Notice the presence of a mixed quadratic term $-ev\partial_\mu\varphi A^\mu$. While this term does not pose a problem per se, it is akward since we would have to diagonalise the quadratic terms.

A particular convenient gauge choice therefore consists in

$$\partial_\mu A^\mu(x) + \xi e v \varphi(x) = \alpha(x) \qquad \underline{\text{t'Hooft or } R_\xi\text{-gauge}}$$

which leads to

$$\mathcal{L}_{gauge-fix} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi e v \varphi)^2$$

$$= -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \underline{- e v \varphi \partial_\mu A^\mu} - \frac{1}{2}\xi e^2 v^2 \varphi^2$$

After integrating by parts, the second term then cancels the unwanted non-diagonal term. Notice that the last term corresponds to a mass term of the would-be Goldstone boson with

$m_\varphi = \sqrt{\xi}\, e v = \sqrt{\xi}\, m_A$.

We next have to work out the Faddeev-Popov determinant in this gauge. To this end, we need the transformation laws of the h and φ fields

$$\delta\phi = \frac{1}{\sqrt{2}}(\delta h + i\delta\varphi) \qquad\qquad \delta h = -e\omega\varphi$$

$$= ie\omega\phi = \frac{ie\omega}{\sqrt{2}}(v + h + i\varphi) \quad \Rightarrow \quad \delta\varphi = e\omega(v+h)$$

along with $\delta A_\mu = \partial_\mu \omega$.

We thus obtain

$$G[A^\omega] = \partial_\mu A^\mu + \partial^2 \omega + 3ev\varphi + 3e^2 v\omega(v+h) - \alpha$$

$$\Rightarrow \frac{\delta G[A^\omega]}{\delta \omega} = \left[\partial^2 + 3e^2 v(v+h)\right] \delta^{(4)}(x-y)$$

which does not depend on A_μ, but on h! As in the non-abelian case, we therefore rewrite the Faddeev-Popov determinant in terms of a path integral over ghost fields (cf page 202)

$$\det\left(\left[\partial^2 + 3e^2 v(v+h)\right] \delta^{(4)}(x-y)\right)$$

$$= \int \mathcal{D}\bar{c} \int \mathcal{D}c \; e^{-\int d^4x \, \bar{c}(x)\left[\partial^2 + 3e^2 v(v+h)\right] c(x)}$$

which after rescaling $c \to ic$ leads to

$$\mathcal{L}_{ghost} = -\bar{c}\left[\partial^2 + 3e^2 v(v+h)\right] c$$

$$\overset{p.I.}{=} \partial_\mu \bar{c}\, \partial^\mu c - 3e^2 v^2 \bar{c} c - 3e^2 v h \bar{c} c$$

The ghost field thus obtains the same mass as the would-be Goldstone boson with $m_{ghost} = \sqrt{3}\, ev = \sqrt{3}\, m_A$.

Putting everything together, we arrive at

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu - \frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

$$+ \frac{1}{2} \partial_\mu h \, \partial^\mu h - \frac{1}{2} m_h^2 h^2 + \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{1}{2} \xi m_A^2 \varphi^2$$

$$+ \partial_\mu \bar{c} \, \partial^\mu c - \xi m_A^2 \bar{c} c$$

$$+ \text{interactions} \; (\varphi h A, \; hAA, \; hhAA, \; \varphi\varphi AA, \; h\bar{c}c, \; h\varphi^2, \; h^2\varphi^2, \; h^3, \; h^4, \; \varphi^4)$$

with $m_A = ev$ and $m_h = \sqrt{2\lambda v^2}$.

Inverting the quadratic terms of the gauge field, we obtain the gauge boson propagator in R_ξ-gauge

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{i}{k^2 - m_A^2 + i\varepsilon} \left[-g^{\mu\nu} + (1-\xi) \frac{k^\mu k^\nu}{k^2 - \xi m_A^2 + i\varepsilon} \right]$$

which is indeed of the form that we anticipated at the beginning of this section. The propagator in R_ξ-gauge thus falls off as $\frac{1}{k^2}$ for $k \to \infty$, and it has an artificial pole at $k^2 = \xi m_A^2$.

It is instructive to isolate the artificial pole, writing

$$\tilde{\Delta}^{\mu\nu}(k) = \underbrace{\frac{i}{k^2 - m_A^2 + i\varepsilon}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_A^2}\right]}_{\substack{\text{3 physical polarisations} \\ \text{(same expression as in unitary gauge)}}} - \underbrace{\frac{i}{k^2 - 3m_A^2 + i\varepsilon}\,\frac{k^\mu k^\nu}{m_A^2}}_{\substack{\text{unphysical scalar polarisation} \\ \text{with mass } \sqrt{3}\,m_A}}$$

We conclude that the propagator in R_ξ-gauge has a good UV behaviour and renormalisability is manifest. But in addition to the physical degrees of freedom (massive vector field and Higgs field), the theory contains various unphysical modes with mass $\sqrt{3}\,m_A$ (scalar polarisation of vector field, would-be Goldstone boson, ghost field). Gauge invariance ensures that the contributions of the unphysical modes cancel out in S-matrix elements.

As in the class of generalized covariant gauges for (unbroken) gauge theories, the parameter ξ is unphysical and one is free to choose a specific value. Convenient choices are

- $\underline{\xi = 0}$

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right] \qquad \text{(transverse)}$$

in which the unphysical degrees of freedom become massless. This is called <u>t'Hooft - Landau gauge.</u>

- $\underline{\xi = 1}$

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{-i}{k^2 - m_A^2} g^{\mu\nu} \qquad \text{(simplest choice)}$$

in which the unphysical modes have the same mass as the vector boson. This is the <u>t'Hooft - Feynman gauge.</u>

- $\underline{\xi \to \infty}$

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_A^2} \right]$$

in which the unphysical degrees of freedom disappear from the spectrum. We thus recover <u>unitary gauge.</u>

The formalism can be generalised to non-abelian gauge theories (cf. e.g. Peskin / Schröder, chapter 21.1 or Srednicki, chapter 86). We will not go into the details here, but we instead simply count the degrees of freedom of a theory that is invariant under a gauge group G, which is spontaneously broken to a subgroup H. We further assume that the theory consists of N real-valued scalar fields.

Before SSB:	dof	1 complex scalar ↓ $U(1)$	1 complex doublet ↓ $SU(2) \times U(1) \to U(1)$
· $\dim G$ massless gauge bosons	$2 \dim G$	2	8
· N real scalars	N	2	4
	$2 \dim G + N$	4	12

After SSB:			
· $\dim H$ massless gauge bosons	$2 \dim H$	0	2
· $(\dim G - \dim H)$ massive vectors	$3(\dim G - \dim H)$	3	9
· $N - (\dim G - \dim H)$ real scalars	$N - (\dim G - \dim H)$	1	1
	$2 \dim G + N$	4	12

In addition there are a number of <u>unphysical modes</u>:

- dim H massless scalar + long. gauge bosons
- dim H massless ghost + antighosts

cancel for S-matrix elements

- $(\dim G - \dim H)$ massive scalar gauge bosons
- $(\dim G - \dim H)$ massive would-be Goldstone bosons
- $(\dim G - \dim H)$ massive ghosts and antighosts

cancel for S-matrix elements (common mass $\sqrt{\xi}\, m_A$)

5.7. Electroweak symmetry breaking (→ chapter 5.3 of TPP1)

In the SM the W and Z bosons acquire their masses through the Higgs mechanism. The SM is based on the symmetry pattern

$$SU(2)_L \otimes U(1)_Y \xrightarrow{SSB} U(1)_Q$$

and it contains a complex scalar field which transforms as a doublet under $SU(2)_L$ transformations. The scalar has hypercharge $Y_\phi = 1/2$ and transforms as

$$\phi'(x) = e^{i\varepsilon^A T^A} \; e^{i\varepsilon Y_\phi} \; \phi(x)$$

Assuming the standard mexican hat potential

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

with $\mu^2 < 0$, the scalar field acquires a non-zero vacuum expectation value which we choose as

$$\langle 0 | \phi(x) | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Let us check which of the symmetries are broken

$\varepsilon^i \in \mathbb{R}$

$$\varepsilon^1 T^1 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^1}{2} \begin{pmatrix} v \\ 0 \end{pmatrix} \quad \longrightarrow \quad \text{broken}$$

$$\varepsilon^2 T^2 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^2}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^2}{2} \begin{pmatrix} -iv \\ 0 \end{pmatrix} \quad \longrightarrow \quad \text{broken}$$

$$\left.\begin{aligned} \varepsilon^3 T^3 \begin{pmatrix} 0 \\ v \end{pmatrix} &= \frac{\varepsilon^3}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{\varepsilon^3}{2} \begin{pmatrix} 0 \\ -v \end{pmatrix} \\ \varepsilon Y_4 \begin{pmatrix} 0 \\ v \end{pmatrix} &= \frac{\varepsilon}{2} \begin{pmatrix} 0 \\ v \end{pmatrix} \end{aligned}\right\} \quad \text{invariant for } \varepsilon^3 = \varepsilon$$

Transformations with

$$e^{i\varepsilon (T^3 + Y_4)} = e^{i\varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \equiv e^{i\varepsilon Q} \in U(1)_Q$$

are thus still a symmetry after SSB.

We next parametrise the four components of the scalar doublet as

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} e^{\frac{i}{v} T^A \sigma^A(x)}$$

[Goldstone bosons live in coset space $SU(2) \times U(1) / U(1) = SU(2)$]

and in <u>unitary gauge</u> we simply set $\sigma^A(x) = 0$. The Higgs boson then acquires a mass $m_h = \sqrt{2\lambda v^2}$ as in the abelian model, and it has charge

$$Q = T_3 + Y = -\frac{1}{2} + \frac{1}{2} = 0$$

We next work out the gauge boson masses. The covariant derivatives now becomes

$$D_\mu \phi = \left(\partial_\mu - i g W_\mu^A T^A - i g' \frac{1}{2} B_\mu \right) \phi$$

As long as we focus on the mass terms, we may simply replace $\phi \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$, which yields

$$D_\mu \phi \to \frac{v}{2\sqrt{2}} \begin{pmatrix} -ig W_\mu^1 - g W_\mu^2 \\ ig W_\mu^3 - ig' B_\mu \end{pmatrix} \equiv \frac{v}{2\sqrt{2}} \begin{pmatrix} -ig\sqrt{2} W_\mu^+ \\ i\sqrt{g^2+g'^2}\, Z_\mu \end{pmatrix}$$

where we introduced

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp i W_\mu^2)$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}} (g W_\mu^3 - g' B_\mu)$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}} (g' W_\mu^3 + g B_\mu)$$

which yields kinetic terms with the correct canonical normalisation.

It follows

$$(D_\mu \phi)^\dagger (D^\mu \phi) \to \frac{g^2 v^2}{4} W_\mu^- W^{\mu +} + \frac{v^2}{8} (g^2 + g'^2) Z_\mu Z^\mu$$

from which we read off

$$m_W = \frac{1}{2} g v$$

$$m_Z = \frac{1}{2} \sqrt{g^2+g'^2}\, v$$

as well as $m_A = 0$, which corresponds to the unbroken $U(1)_Q$.

In a gauge theory a fermion mass term $m\bar{\psi}\psi$ is not forbidden since the fermions transform under unitary transformations. In a <u>chiral</u> gauge theory, however, left- and right handed fields transform differently and a mass term

$$m\bar{\psi}\psi = m\,(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$$

is not gauge invariant. In the SM the fermions therefore also have to acquire their masses through the Higgs mechanism.

In order to illustrate this idea, we will focus on one generation of fermions. The left-handed fermions are grouped into SU(2)-doublets and the right-handed fermions are singletts. We write

	$L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$	$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	e_R	u_R	d_R
Y_f :	$-1/2$	$1/6$	-1	$2/3$	$-1/3$

where the hypercharges have been chosen such that the known charges $Q = T_3 + Y$ are reproduced.

We want to group left- and right-handed fields into gauge-invariant combinations. We first note that

	$\bar{L} e_R$	$\bar{Q} u_R$	$\bar{Q} d_R$
Y:	$-1/2$	$+1/2$	$-1/2$

are $SU(2)$-doublets (and $SU(3)$-singlets) with non-vanishing hypercharge. In addition to the Higgs field $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ with $Y_\phi = +1/2$, we also need

$$\tilde{\phi} = i\sigma^2 \phi^* = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix}$$

which also transforms as an $SU(2)$-doublet and has hypercharge $Y_{\tilde{\phi}} = -1/2$.

Let us check if $\tilde{\phi}$ transforms as a $SU(2)$-doublet

$$\tilde{\phi}' = i\sigma^2 (\phi')^*$$

$$= i\sigma^2 \left(e^{i\varepsilon^A \frac{\sigma^A}{2}} \phi \right)^*$$

$$= i\sigma^2 e^{-i\varepsilon^A \frac{\sigma^{A*}}{2}} \underbrace{\sigma^2 \sigma^2}_{=\mathbb{1}} \phi^*$$

$$= e^{i\varepsilon^A \frac{\sigma^A}{2}} i\sigma^2 \phi^* = e^{i\varepsilon^A \frac{\sigma^A}{2}} \tilde{\phi} \qquad \checkmark$$

where we used $\sigma^2 \sigma^{A*} \sigma^2 = -\sigma^A$.

We can thus write down gauge-invariant Yukawa interactions

$$\mathcal{L}_{\text{Yukawa}} = - d_e \, \bar{L} \, \phi \, e_R - d_u \, \bar{Q} \, \tilde{\phi} \, u_R - d_d \, \bar{Q} \, \phi \, d_R + h.c.$$

which after SSB with $\phi \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ and $\tilde{\phi} \to \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}$

generates fermion mass terms

$$\mathcal{L}_{\text{Yukawa}} \to - \frac{v d_e}{\sqrt{2}} \bar{e}_L e_R - \frac{v d_u}{\sqrt{2}} \bar{u}_L u_R - \frac{v d_d}{\sqrt{2}} \bar{d}_L d_R + h.c.$$

from which we can read off $m_e = \frac{v d_e}{\sqrt{2}}$, $m_u = \frac{v d_u}{\sqrt{2}}$ and $m_d = \frac{v d_d}{\sqrt{2}}$ along with $m_\nu = 0$ (since we did not introduce a right-handed neutrino field ν_R).

There is now convincing experimental evidence that the neutrinos are <u>not</u> massless. One should therefore add a right-handed field ν_R with hypercharge $Y_{\nu_R} = 0$ to construct a similar (Dirac) mass term

$$- d_\nu \bar{L} \tilde{\phi} \nu_R + h.c.$$

$$Y: \quad \tfrac{1}{2} - \tfrac{1}{2} + 0 = 0 \ \checkmark$$

$$\to \quad m_\nu = \frac{v d_\nu}{\sqrt{2}}$$

The right-handed neutrinos are, however, special since they are not charged under the SM gauge group. They can therefore be their own antiparticles, and one can hence also construct a Majorana mass term

$$-\frac{M_\nu}{2}\left(\nu_R^T \nu_R + h.c\right)$$

which could explain why the neutrino masses are so small via the Seesaw mechanism.

In nature there are three generations of matter particles, and so the Yukawa couplings become 3×3 matrices, which need to be diagonalised to determine the mass eigenstates. This transformation leads to a non-diagonal structure in the charged current interactions, and hence to non-trivial quark and lepton mixing matrices.

In the quark sector, the Cabibbo-Kobayashi-Maskawa (CKM) matrix can be parametrised by three angles and one phase. Similarly, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix parametrises the lepton mixing in terms of three angles and one (three) phases for Dirac (Majorana) neutrinos. The presence of non-trivial physical phases lead to an asymmetry between the interactions of particles and antiparticles (CP violation), which is one of the ingredients that is needed to explain the observed matter-antimatter asymmetry in the universe.

It is remarkable that the SM contains full generations of matter particles. This is not an accident, but it is required for consistency since the SM gauge symmetry would otherwise be anomalous (i.e. broken by quantum corrections).

The SM is an extremely successful theory, but we know that it is incomplete. It does not contain e.g. a dark matter candidate, it does not explain gravity and the amount of CP violation in the SM is insufficient to explain the baryon asymmetry. There are also theoretical considerations (→ hierarchy problem) why we believe that new physical effects should show up at TeV energy scales. Many experiments around the world are scrutinising the SM, looking for hints of new heavy particles.