0. Recap TTP I

QFT describes the quantum theory of systems with an infinite number of degrees of freedom. In particle physics we are in particular interested in relativistic QFTs in which - due to the equivalence of energy and mass - the particle number is not conserved.

Elementary particles are associated with irreducible representations of the Poincaré group. These are characterised by two numbers, which we identify with the mass and the spin of the particle (or helicity for massless particles).

For each particle species in the theory we introduce a field operator. The simplest example

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[e^{-ipx} a(p) + e^{ipx} a^+(p) \right]$$

with $p^0 = \sqrt{\vec{p}^2 + m^2}$ describes neutral spin-0 particles.

The annihilation and creation operators act on the Fock space as

$$a(p)\,|0\rangle = 0$$

$$a^+(p)\,|0\rangle = |p\rangle$$

They further fulfil canonical commutation relations

$$[a(p), a^{+}(p')] = (2\pi)^3 \, 2p^0 \, \delta^{(3)}(\vec{p} - \vec{p}\,')$$

$$[a(p), a(p')] = [a^{+}(p), a^{+}(p')] = 0$$

We work in the <u>Heisenberg picture</u> in which the particle states are time-independent and the field operators are time-dependent. Their time evolution is governed by the Hamiltonian (for a free particle)

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \, p^0 \, a^{+}(p) \, a(p)$$

with

$$\phi(t, \vec{x}) = e^{iHt} \, \phi(0, \vec{x}) \, e^{-iHt}$$

We often start from the Lagrange formulation in which Lorentz-invariance is manifest. For a free theory we construct the most general Lorentz-invariant Lagrangian that is quadratic in the fields ($\rightarrow$ linear equations of motion). For neutral spin-0 particles this yields

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

[We can always remove a linear term by a field redefinition (by "completing the square"). Other invariant operators with derivatives are not independent since they are related by a partial integration ($\rightarrow$ adds irrelevant surface term to the action)]

The principle of least action

$$S = \int d^4x \;\; \mathcal{L}(x)$$

leads to the Euler - Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi}$$

$$= \left(\partial^2 + m^2 \right) \phi(x) = 0$$ <u>Klein - Gordon equation</u>

The Lagrangian density $\mathcal{L}$ and the Hamiltonian density $\mathcal{H}$ with $H = \int d^3x \;\; \mathcal{H}(x)$ are related by a Legendre transformation

$$\mathcal{H}(x) = \pi(x) \, \dot{\phi}(x) - \mathcal{L}(x)$$

where $\pi(x)$ is the <u>conjugate field</u>

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$$

The fields obey <u>equal-time</u> commutation relations

$$\left[\phi(t, \vec{x}) \, , \; \pi(t, \vec{x}') \right] = i \, \delta^{(3)} (\vec{x} - \vec{x}')$$

$$\left[\phi(t, \vec{x}) \, , \; \phi(t, \vec{x}') \right] = \left[\pi(t, \vec{x}) , \, \pi(t, \vec{x}') \right] = 0$$

One further introduces a time-ordering prescription

$$T\,\phi(x)\,\phi(y) = \theta(x^0-y^0)\,\phi(x)\,\phi(y) + \theta(y^0-x^0)\,\phi(y)\,\phi(x)$$

The vacuum matrix element of this object then describes the propagation of a particle from $\vec{y}$ at time y^0 to $\vec{x}$ at time x^0 (and vice versa). This is called the Feynman propagator

$$\langle 0|\,T\,\phi(x)\,\phi(y)\,|0\rangle \equiv \Delta_F(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2-m^2+i\varepsilon}$$

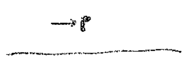
which is a Green function of the Klein-Gordon operator

$$(\partial_x^2+m^2)\,\Delta_F(x-y) = -i\,\delta^{(4)}(x-y)$$

We obtain our first Feynman rule

$$= \Delta_F(x-y)$$

or in momentum space



$$= \tilde{\Delta}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$$

Charged spin-0 particles are described by a <u>complex scalar field</u>

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[e^{-ipx} a(p) + e^{ipx} b^+(p) \right]$$

where $b(p)$ and $b^+(p)$ are an independent set of annihilation and creation operators that are associated with the antiparticle states. The dynamics of the free theory is now governed by

$$\mathcal{L} = \partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi$$

which is invariant under $U(1)$ (or phase) transformations

$$\phi'(x) = e^{i\alpha} \phi(x)$$

According to the Noether theorem this gives rise to a conserved current and a conserved Noether charge.

We can now define several two-point functions

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \langle 0 | T \phi^+(x) \phi^+(y) | 0 \rangle = 0$$

since they are not invariant under the $U(1)$ symmetry and

$$\langle 0 | T \phi(x) \phi^+(y) | 0 \rangle = \Delta_F(x-y) \qquad x \longleftarrow y$$

$$\langle 0 | T \phi^+(x) \phi(y) | 0 \rangle = \Delta_F(x-y) \qquad x \longrightarrow y$$

where the arrow indicates the direction of the particle flow (which is opposite to the antiparticle flow).

[Notice that ϕ acting on $|0\rangle$ creates an antiparticle and ϕ^+ creates a particle.]

For particles with spin, the field operator transforms non-trivially under a finite-dimensional representation of the Lorentz group. Massive, neutral spin-1/2 particles are described by a two-component Majorana spinor and massive, charged spin-1/2 particles are associated with a four-component <u>Dirac spinor</u>

$$\psi_\alpha(x) = \sum_{s=\pm 1/2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(u_\alpha(p,s)\, e^{-ipx} a(p,s) + v_\alpha(p,s)\, e^{ipx} b^+(p,s) \right)$$

Consistency requires to postulate <u>anticommutation relations</u> for the creation and annihilation operators associated with fermionic states

$$\{ a(p,s), a^+(p',s') \} = (2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p}-\vec{p}')\, \delta_{ss'} \quad \text{etc.}$$

The Lagrangian of the free theory now becomes

$$\mathcal{L} = \bar{\psi}\, (i\not{\partial} - m)\, \psi$$

with $\bar{\psi}(x) = \psi^+(x)\,\gamma^0$. The theory is again invariant under $U(1)$ transformations ($\rightarrow$ Noether charge). The equations of motion read

$$(i\not{\partial} - m)\,\psi(x) = 0 \qquad \underline{\text{Dirac equation}}$$

which implies

$$(\not{p} - m)\, u(p,s) = 0 \qquad (\not{p} + m)\, v(p,s) = 0$$

Notice that the time-ordering now accounts for the anticommutating nature of the fields

$$T\,\psi_\alpha(x)\,\bar\psi_\beta(y) = \theta(x^0-y^0)\,\psi_\alpha(x)\,\bar\psi_\beta(y) - \theta(y^0-x^0)\,\bar\psi_\beta(y)\,\psi_\alpha(x)$$

The Feynman propagator is now given by

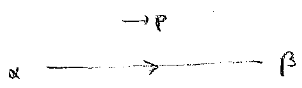
$$\langle 0|\,T\,\psi_\alpha(x)\,\bar\psi_\beta(y)\,|0\rangle \equiv S_F(x-y)_{\alpha\beta}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i\,(\not p + m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$

which is a Green function of the Dirac operator

$$(i\not\partial_x - m)\, S_F(x-y) = i\,\delta^{(4)}(x-y)$$

The momentum-space Feynman rule now becomes



$$= \frac{i\,(\not p + m)_{\beta\alpha}}{p^2-m^2+i\varepsilon}$$

where the arrow again indicates the direction of the particle flow.

As a last example we consider massive, neutral spin-1 particles that are associated with a vector field

$$A^\mu(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[e^{-ipx} \varepsilon^\mu(p,s)\, a(p,s) + e^{ipx} \varepsilon^\mu(p,s)^* a^+(p,s) \right]$$

with $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$ and bosonic annihilation and creation operators that fulfil the usual commutation relations. Note that the vector field describes four degrees of freedom, but a massive spin-1 particle has only three physical polarisations. This is related to the fact that the vector representation of the Lorentz-group is not irreducible under rotations, but splits as

$$4 = \underbrace{1}_{\text{spin-0}} + \underbrace{3}_{\text{spin-1}}$$

One therefore needs to impose a constraint on the vector field that removes the spin-0 component. We will see in the tutorials that this can be achieved by requiring

$$\partial_\mu A^\mu(x) = 0 \qquad \longrightarrow \qquad p_\mu \varepsilon^\mu(p,s) = 0$$

(see also TPP1, page 200 - 205)

The description of massless spin-1 particles with two physical polarisations is more complicated and leads to the concept of <u>gauge symmetries</u> (the gauge-fixing condition then gives another constraint on the vector field). In particular, one finds that massless spin-1 particles have to be coupled to <u>conserved currents</u>.

So far we were only dealing with free theories. Interacting theories are much more complicated and, apart from a few exceptions, they cannot be solved exactly. For weakly-interacting theories, on the other hand, one can apply time-dependent <u>perturbation theory</u>. Here one assumes that the Hamiltonian can be written in the form (at the time $t=0$)

$$H = \underbrace{H_0}_{\substack{\text{free theory} \\ (\to \text{ quadratic in the fields})}} + \underbrace{H_{int}}_{\substack{\text{small perturbation} \\ (\to \text{ higher-dimensional operators})}}$$

One further switches to the <u>interaction picture</u> with

$$\phi_I(t,\vec{x}) = e^{iH_0 t}\,\phi(0,\vec{x})\,e^{-iH_0 t}$$

free Hamiltonian!

The field operators in the interaction picture therefore admit the usual Fourier-decomposition in terms of creation and annihilation operators and they fulfil the Euler-Lagrange equations of the free theory (→ same Feynman rules for particle propagators).

The non-trivial time dependence is encoded in the time-evolution operator

$$U_I(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)$$

H_{int} expressed in terms of fields in the interaction picture

which is the basis of the perturbative expansion (→ new Feynman rules associated with vertices).

The two-point function in the interacting theory ("full propagator")

is then given by

Heisenberg operators

vacuum of the interacting theory

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$$

vacuum of the free theory

interaction picture

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T \phi_I(x) \phi_I(y) U_I(T,-T) | 0 \rangle}{\langle 0 | U_I(T,-T) | 0 \rangle}$$

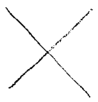
required to project out the vacuum state

and similarly for higher n-point functions.

As an example we consider ϕ^4-theory with interaction Hamiltonian

$$H_I(t) = \frac{\lambda}{4!} \int d^3x \, [\phi_I(x)]^4$$

and momentum-space Feynman rule

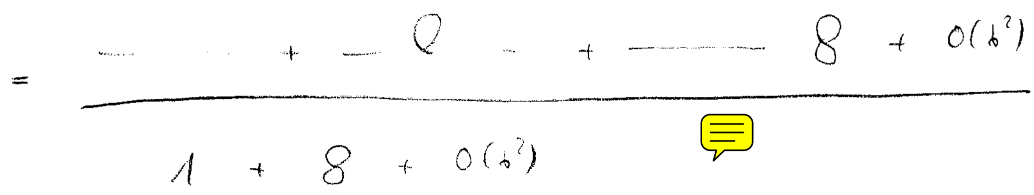


$$= -i\lambda$$

Wick's theorem tells us that we have to connect all points with propagators (only "full contractions" survive in the correlation functions)

It follows

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$$



The "vacuum bubbles" thus drop out in the ratio.

In order to connect to experimental observations, we have to consider cross sections and decay rates. To do so, one introduces the <u>S-matrix</u>

asymptotic states with definite momentum

(eigenstates of H)

$$_{out}\langle p_1 p_2 \ldots | k_A k_B \rangle_{in}$$

eigenstate of H_0

$$= \lim_{T\to\infty} \langle p_1 p_2 \ldots | U_I(T, -T) | k_A k_B \rangle$$

$$\equiv \langle p_1 p_2 \ldots | S | k_A k_B \rangle$$

Writing $S = \mathbb{1} + iT$ one finds

no interaction

$$\langle p_1 p_2 \ldots | iT | k_A k_B \rangle$$

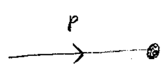
$$= (2\pi)^4 \, \delta^{(4)}\Big(k_A + k_B - \sum_i p_i\Big) \, i\mathcal{M}(k_A k_B \to p_1 p_2 \ldots)$$

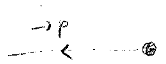
with

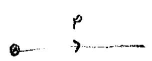
$$i\mathcal{M}(k_A k_B \to p_1 p_2 \ldots) = \left(\sqrt{Z}\right)^{n+2} \left(\begin{array}{l} \text{sum of all connected and amputated diagrams} \\ \text{with } k_A, k_B \text{ incoming and } p_1 p_2 \ldots \text{ outgoing} \end{array} \right)$$

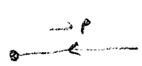
self interactions modify normalization of the particle states (→ wave function renormalization)

and we get additional Feynman rules associated with the external states, e.g. in momentum space

incoming fermion  $= u(p,s)$

incoming antifermion  $= \bar{v}(p,s)$

outgoing fermion  $= \bar{u}(p,s)$

outgoing antifermion  $= v(p,s)$

The master formula for the computation of cross section then becomes

$$d\sigma(k_A k_B \to p_1 \ldots p_n)$$

flux factor — phase space

$$= \frac{1}{2k_A^0\, 2k_B^0\, |\vec{v}_A - \vec{v}_B|} \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0} \cdots \frac{d^3p_n}{(2\pi)^3}\frac{1}{2p_n^0}$$

$$(2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum_{i=1}^{n} p_i\Big)\, |\mathcal{M}(k_A k_B \to p_1 \ldots p_n)|^2$$

overall momentum conservation — square of Lorentz-invariant transition matrix element

and similarly for the decay rate

$$d\Gamma(k_A \to p_1 \ldots p_n)$$

$$= \frac{1}{2k_A^0} \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0} \cdots \frac{d^3p_n}{(2\pi)^3}\frac{1}{2p_n^0}\, (2\pi)^4 \delta^{(4)}\Big(k_A - \sum_{i=1}^{n} p_i\Big)$$

$$|\mathcal{M}(k_A \to p_1 \ldots p_n)|^2$$

Note that one has to add a factor of $\frac{1}{n!}$ when one integrates over the phase space of n identical particles.

TPP I further covered the basic concepts and the particle content of the SM. There are, however, some open questions that we are going to address in the current lecture. Most importantly:

* We have so far only considered processes to leading order in the perturbative expansion. At higher orders one encounters a different topological class of Feynman diagrams ("loop diagrams") which often lead to divergent integrals. The correct interpretation of these divergences leads to the concept of <u>renormalisation</u>.

* In QED the electromagnetic field can be quantised using the Gupta-Bleuler method. The Gupta-Bleuler method cannot be applied to non-abelian gauge theories, and we will instead introduce a more general formalism based on the <u>path-integral formulation</u>.

We will also discuss <u>anomalous symmetries</u> and we consider more advanced topics and several key processes within the SM of particle physics.

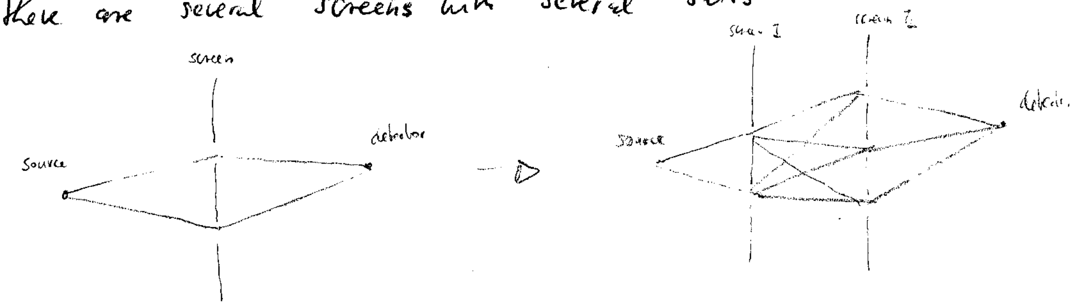
1. Path-integral formulation

We will now introduce a completely different formulation of quantum field theories that is based on path-integral methods. Path-integral quantisation is equivalent to canonical quantisation, but has several advantages. As it is based on the Lagrangian rather than the Hamiltonian formulation, Lorentz-invariance is manifest. The derivation of Feynman rules is also often simpler in the path-integral approach. The new method will provide insights into the classical limit $\hbar \to 0$, and it will become particularly important when we discuss non-abelian gauge theories.

The path integral is also the starting point for non-perturbative studies like lattice gauge theory in which spacetime is discretised on a 4-dimensional lattice.

1.1. Path integrals in quantum mechanics

The basic idea of the path-integral formulation is simple. When we discussed the double-slit experiment in QM, we learned that we have to sum up the amplitudes for a "particle" passing through each of the slits. Adding the two amplitudes and taking the square then yields the probability including quantum interference effects. Whenever there are several screens with several slits



it is obvious that we have to sum up all the individual amplitudes coherently. But in the continuum limit # screens → ∞ and # slits → ∞, the screens disappear and we learn that the quantum mechanical amplitude can be represented as the sum over all possible paths.

Let us now see how we can use this intuition to calculate a probability amplitude in non-relativistic quantum mechanics.

We start with the simplest quantum mechanical system with one dynamical variable Q, conjugate momentum P and Hamiltonian $H(Q,P)$.

The operators Q and P have complete sets of eigenstates

$$\underbrace{Q}_{\text{operator (Schrödinger)}} |q\rangle = \underbrace{q}_{\text{eigenvalue}} |q\rangle \qquad P|p\rangle = p|p\rangle$$

with

$$\langle q'|q\rangle = \delta(q-q') \qquad \langle p'|p\rangle = 2\pi\, \delta(p-p')$$

$$\int dq\, |q\rangle\langle q| = 1 \qquad \int \frac{dp}{2\pi} |p\rangle\langle p| = 1$$

From $[Q,P] = i$ we know that $P = \frac{1}{i}\frac{\partial}{\partial q}$ and

$$\langle q|P|p\rangle = p\langle q|p\rangle = \frac{1}{i}\frac{\partial}{\partial q}\langle q|p\rangle$$

$$\Rightarrow \quad \langle q|p\rangle = e^{iqp}$$

In the following we will work in the Heisenberg picture with

$$Q(t) = e^{iHt}\, Q\, e^{-iHt}$$

$$|q,t\rangle = e^{iHt}\,|q\rangle$$

$$\Rightarrow \quad Q(t)\,|q,t\rangle = e^{iHt}\, Q\,|q\rangle = q\,|q,t\rangle$$

and similar relations hold for the momentum eigenstates $|p,t\rangle$.

The states $|q,t\rangle$ and $|p,t\rangle$ thus form complete sets of instantaneous eigenstates with

$$\langle q',t|q,t\rangle = \langle q'|q\rangle = \delta(q-q')$$

$$\int dq\, |q,t\rangle\langle q,t| = e^{iHt}\int dq\, |q\rangle\langle q|\, e^{-iHt} = 1 \qquad \text{etc.}$$

We wish to calculate the probability amplitude for a transition between a position eigenstate $|q_i,t_i\rangle$ with eigenvalue q_i at time t_i and a position eigenstate $|q_f,t_f\rangle$ with eigenvalue q_f at time t_f, i.e.

$$\langle q_f,t_f|q_i,t_i\rangle = \langle q_f|\, e^{-iH(t_f-t_i)}\, |q_i\rangle$$

In the following, we need to specify a convention for the ordering of the Q's and P's in $H(Q,P)$. The differences between the various conventions are not important, and we will assume here that all Q's are ordered to the left of all P's in $H(Q,P)$.

It follows

$$\langle q', t+\Delta t \,|\, q, t \rangle$$

$$= \langle q' | \, e^{-iH(Q,P)\Delta t} \, | q \rangle$$

$$= \int \frac{dp}{2\pi} \, \langle q' | \, e^{-iH(Q,P)\Delta t} \, | p \rangle \langle p | q \rangle$$

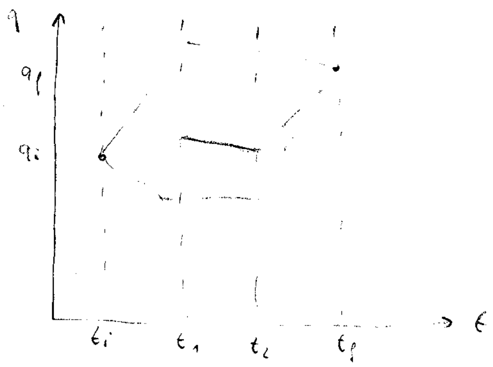
$$\simeq \int \frac{dp}{2\pi} \, \langle q' | \, 1 - iH(Q,P)\Delta t + \ldots \, | p \rangle \langle p | q \rangle$$

$$= \int \frac{dp}{2\pi} \, \left(1 - iH(q',p)\,\Delta t + \ldots \right) \langle q' | p \rangle \langle p | q \rangle$$

$$\simeq \int \frac{dp}{2\pi} \, e^{-iH(q',p)\Delta t} \, e^{i(q'-q)p}$$

which is valid to linear order in Δt.

We next partition the time interval $[t_i, t_f]$ into $N+1$ equal segments of width



$$\Delta t = \frac{t_f - t_i}{N+1}$$

with $t_i \equiv t_0 < t_1 < \ldots < t_N < t_{N+1} \equiv t_f$.

Inserting a complete set of position eigenstates at each intermediate time t_k results in

$$\langle q_f, t_f | q_i, t_i \rangle$$

$$= \int dq_1 \dots \int dq_N$$

$$\langle q_f, t_f | q_N, t_N \rangle \langle q_N, t_N | \dots | q_1, t_1 \rangle \langle q_1, t_1 | q_i, t_i \rangle$$

$$\simeq \int dq_1 \dots \int dq_N \int \frac{dp_1}{2\pi} \dots \int \frac{dp_{N+1}}{2\pi}$$

$$\exp \left\{ i \sum_{k=1}^{N+1} \left((q_k - q_{k-1}) p_k - H(q_k, p_k) \Delta t \right) \right\} \qquad (*)$$

In the limit $N \to \infty$, we have

$$q_k \to q(t)$$

$$p_k \to p(t)$$

$$q_k - q_{k-1} \longrightarrow \dot{q}(t)\, dt$$

$$\sum_{k=1}^{N+1} \Delta t \longrightarrow \int_{t_i}^{t_f} dt$$

$$\int \prod_{k=1}^{N} dq_k \prod_{k=1}^{N+1} \frac{dp_k}{2\pi}$$

$$\longrightarrow \int \prod_t dq(t) \prod_t \frac{dp(t)}{2\pi} \equiv \int \mathcal{D}q(t)\, \mathcal{D}p(t)$$

We obtain

$$\boxed{\begin{aligned}&\langle q_f, t_f \mid q_i, t_i \rangle \\ &= \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q(t)\, \mathcal{D}p(t)\; \exp\left\{ i \int_{t_i}^{t_f} dt \left(\dot{q}(t)\, p(t) - H(q(t), p(t)) \right) \right\}\end{aligned}}$$

i.e. an infinite-dimensional integral over all phase-space trajectories with fixed endpoints at $q(t_i)$ and $q(t_f)$ (there is no boundary condition on $p(t)$). From the above limiting procedure we further learn that the integral measure $\mathcal{D}q(t)\, \mathcal{D}p(t)$ has dimension of p, which is consistent with eg. $\langle q' | q \rangle = \delta(q-q')$. Notice also that $\dot{q}p - H \neq L$ since q and p are independent integration variables here.

Whenever the Hamiltonian is quadratic in momentum, the integral over $\mathcal{D}p(t)$ can be performed explicitly. To this end, we recall the following relations for Gaussian integrals

$$\int_{-\infty}^{\infty} dz \; e^{-\frac{az^2}{2}} = \sqrt{\frac{2\pi}{a}}$$

$$\int_{-\infty}^{\infty} dz \; e^{-\frac{1}{2}az^2 - bz - c} = \sqrt{\frac{2\pi}{a}} \; e^{\frac{b^2}{2a} - c}$$

where the last equation can easily be verified by shifting the integration variable to complete the square.

Assuming that the Hamiltonian is of the form

$$H = \frac{P^2}{2m} + V(Q)$$

the discretized version of the path integral $(*)$ becomes

$$\int \frac{dp_k}{2\pi} \exp\left\{ i\Delta t \left(\dot{q}_k p_k - \frac{p_k^2}{2m} - V(q_k) \right) \right\}$$

$a = \frac{i\Delta t}{m}$

$b = -i\Delta t \, \dot{q}_k$

$c = i\Delta t \, V(q_k)$

$$= \frac{1}{\sqrt{2\pi i \Delta t/m}} \exp\left\{ i\Delta t \left(\frac{m}{2} \dot{q}_k^2 - V(q_k) \right) \right\}$$

where we now indeed obtain the Lagrange function

$$L(q,\dot{q}) = \dot{q}p - H(q,p)$$

$$= \frac{m}{2}\dot{q}^2 - V(q)$$

with $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$.

In the limit $N \to \infty$, we now absorb the constant prefactors into the definition of the path integral measure

$$\frac{1}{\sqrt{2\pi i \Delta t/m}} \int \prod_{k=1}^{N} \frac{dq_k}{\sqrt{2\pi i \Delta t/m}} \longrightarrow \int \mathcal{D}q(t)$$

We thus arrive at

$$\langle q_f, t_f \,|\, q_i, t_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q(t)\, e^{iS[q]}$$

where

$$S[q] = \int_{t_i}^{t_f} dt\; L(q(t), \dot{q}(t))$$

is the classical action.

The result is Feynman's famous "sum over histories" formula of quantum mechanics, which tell us that the probability amplitude can be represented as the coherent sum over all possible paths with fixed endpoints, and each path is weighted by a pure phase factor which is determined by the classical action.

The path integral representation sheds light onto the <u>classical limit</u> $\hbar \to 0$. So far we have set $\hbar = 1$, but we can easily restore the factors of $\hbar$ on dimensional grounds. In particular we obtain

$$e^{\frac{i}{\hbar} S(y)}$$

In the classical limit $\hbar \to 0$ (i.e. when the action is large in units of $\hbar$) the phase factor oscillates very rapidly and the contributions from neighbouring paths completely cancel out (destructive interference). The only non-vanishing contribution then stems from the region where $\delta S = 0$ ("stationary point") since in this case the neighbouring paths have essentially the same action and their phases add up constructively. But the path with $\delta S = 0$ is just the classical trajectory and we thus obtain the classical result directly from the quantum mechanical superposition principle in the limit $\hbar \to 0$.

Whenever the action is <u>not</u> large in units of $\hbar$, the "quantum fluctuations" around the classical trajectory are <u>not</u> suppressed and quantum mechanical interference effects cannot be neglected.

Before we turn to the field theory formulation, let us consider expressions of the form

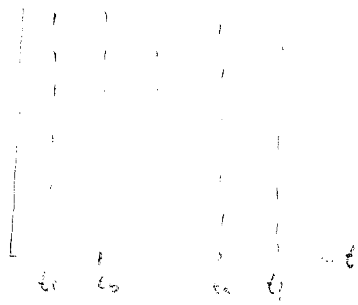
$$\langle q_f, t_f \,|\, Q(t_a)\, Q(t_b)\,|\, q_i, t_i\rangle$$

$$= \int dq_1 \dots dq_N$$

$$\langle q_f, t_f \,|\, q_N, t_N\rangle \langle q_N, t_N \,|\, q_{N-1}, t_{N-1}\rangle \dots \langle q_{j+1}, t_{j+1} \,|\, Q(t_a) \,|\, q_j, t_j\rangle$$

$$\dots \langle q_{k+1}, t_{k+1} \,|\, Q(t_b) \,|\, q_k, t_k\rangle \dots \langle q_1, t_1 \,|\, q_i, t_i\rangle$$

where we assumed that $t_a > t_b$. W.l.o.g. we can always choose our lattice such that $t_a = t_j$ and $t_b = t_k$. We thus obtain matrix elements of the form



eigenstate at $t_j = t_a$

$$\langle q_{j+1}, t_{j+1} \,|\, Q(t_a) \,|\, q_j, t_j\rangle \qquad (t_a = t_j)$$

$$= q_a \,\langle q_{j+1}, t_{j+1} \,|\, q_j, t_j\rangle$$

We therefore obtain the same expression as before with an additional factor

$$q_a \cdot q_b \to q(t_a)\, q(t_b)$$

For $t_b > t_a$ the same arguments apply and we obtain

$$q(t_b)\, q(t_a) = q(t_a)\, q(t_b)$$

since numbers, not operators

In total we find an expression for the time-ordered product

$$\boxed{\begin{aligned}&\langle q_f, t_f \,|\, T\, Q(t_a)\, Q(t_b) \,|\, q_i, t_i \rangle \\ &\quad = \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q(t)\; e^{iS[q]}\; q(t_a)\, q(t_b)\end{aligned}}$$

which is valid as long as $t_a, t_b \in [t_i, t_f]$.

1.2. Scalar field theory

The formalism can be generalised to systems with more than one degree of freedom. Denoting the dynamical variables by Q_n and their conjugate momenta by P_n, we have

$$[Q_n, Q_m] = [P_n, P_m] = 0$$

$$[Q_n, P_m] = i\,\delta_{nm}$$

The Heisenberg operators then fulfil equal-time commutation relations

$$[Q_n(t), Q_m(t)]$$

$$= e^{iHt} Q_n e^{-iHt} e^{iHt} Q_m e^{-iHt} - (n \leftrightarrow m)$$

$$= e^{iHt} [Q_n, Q_m] e^{-iHt} = 0$$

$$[P_n(t), P_m(t)] = 0$$

$$[Q_n(t), P_m(t)] = i\,\delta_{nm}$$

We can therefore find a simultaneous set of eigenstates with

$$Q_n(t)\,|q,t\rangle = q_n\,|q,t\rangle$$

and

$$\langle q', t | q, t\rangle = \prod_n \delta(q_n - q_n')$$

$$\int \prod_n dq_n\, |q,t\rangle\langle q,t| = 1$$

and similar generalisations hold for the momentum eigenstates $|p,t\rangle$.

Proceeding along the lines of the previous calculation, one finds

$$\langle q_f, t_f | q_i, t_i \rangle$$

$$= \int_{\substack{q_n(t_i) = q_{i,n} \\ q_n(t_f) = q_{f,n}}} \mathcal{D}q_n(t)\, \mathcal{D}p_n(t)\, \exp\left\{ i \int_{t_i}^{t_f} dt \left(\sum_n \dot{q}_n(t) p_n(t) - H(q_n(t), p_n(t)) \right) \right\}$$

A quantum field theory, on the other hand, is formulated in terms of a Heisenberg operator $\phi(x) = \phi(t, \vec{x})$ with conjugate field $\pi(t, \vec{x})$, which fulfil similar equal-time commutation relations

$$[\phi(t,\vec{x}), \phi(t,\vec{x}')] = [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0$$

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\, \delta^{(3)}(\vec{x} - \vec{x}')$$

Once we discretize space as well as time, we can apply the above formula with the replacements

$$Q_n \rightarrow \phi(\vec{x})$$

$$P_n \rightarrow \pi(\vec{x})$$

In the continuum limit, we therefore obtain

$$\langle \phi_f, t_f | \phi_i, t_i \rangle$$

$$= \int_{\substack{\phi(t_i, \vec{x}) = \phi_i(\vec{x}) \\ \phi(t_f, \vec{x}) = \phi_f(\vec{x})}} \mathcal{D}\phi(x)\, \mathcal{D}\pi(x)\, \exp\left\{ i \int_{t_i}^{t_f} dt \int d^3x \left(\dot{\phi}(x)\, \pi(x) - \mathcal{H}(\phi(x), \pi(x)) \right) \right\}$$

where $\mathcal{H}(\phi(x), \pi(x))$ is the Hamiltonian density with

$$H = \int d^3x\, \mathcal{H}(x).$$

The states $|\phi,t\rangle$ are simultaneous eigenstates of the generalised "coordinates" $\phi(\vec{x})$ with

$$\phi(t,\vec{x})\,|\phi,t\rangle = \phi(\vec{x})\,|\phi,t\rangle$$

Heisenberg operator   eigenvalue

Whenever the Hamiltonian is quadratic in $\pi(x)$ and the coefficient of the quadratic term is independent of $\phi(x)$, we can proceed further along the lines of the QM calculation. This yields

Heisenberg operators

$$\boxed{\begin{aligned}&\langle \phi_f,t_f|\,T\,\phi(x_1)\ldots\phi(x_n)\,|\phi_i,t_i\rangle\\ &= \int\limits_{\substack{\phi(t_i,\vec{x})=\phi_i(\vec{x})\\ \phi(t_f,\vec{x})=\phi_f(\vec{x})}} \mathcal{D}\phi(x)\; e^{iS(\phi)}\;\phi(x_1)\ldots\phi(x_n)\end{aligned}}$$

real-valued functions

Although this representation is often called a path integral, we should keep in mind that we now integrate over field configurations rather than "paths". One therefore also refers to the above representation simply as a functional integral.

In quantum field theory, we are typically interested in correlation functions rather than transitions between eigenstates of the field operator. These can be represented as

$$\langle \Omega | T \phi(x_1) \dots \phi(x_n) | \Omega \rangle$$

$$= \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \int \prod_{\vec{x}} d\phi_i(\vec{x}) \prod_{\vec{x}} d\phi_f(\vec{x})$$

$$\langle \Omega | \phi_f, t_f \rangle \langle \phi_f, t_f | T \phi(x_1) \dots \phi(x_n) | \phi_i, t_i \rangle \langle \phi_i, t_i | \Omega \rangle$$

$$= \int \mathcal{D}\phi(x) \quad \langle \Omega | \phi, \infty \rangle \langle \phi, -\infty | \Omega \rangle$$

$$e^{iS(\phi)} \quad \phi(x_1) \dots \phi(x_n)$$

where now

$$S(\phi] = \int_{-\infty}^{\infty} dt \int d^3x \; \mathcal{L}(x) = \int d^4x \quad \mathcal{L}(x)$$

and the path integral is no longer constrained by any type of boundary conditions since we have summed over all possible field configurations in the infinite past / future.

The above formula involves the vacuum functionals

$$\langle \Omega | \phi, \infty \rangle \langle \phi, -\infty | \Omega \rangle$$

which can be viewed as the position representation of the vacuum wave function (cf. $\psi(x) = \langle x | \psi \rangle$ in QM).

In the tutorials we will show that

$$\langle \phi, \mp\infty | \Omega \rangle = \underset{\uparrow}{N} \exp\left\{ -\frac{1}{2} \int d^3x \, d^3y \, k(\vec{x}, \vec{y}) \, \phi(\mp\infty, \vec{x}) \, \phi(\mp\infty, \vec{y}) \right\}$$

irrelevant normalization

with kernel

$$k(\vec{x}, \vec{y}) = \int \frac{d^3p}{(2\pi)^3} \, e^{i\vec{p}(\vec{x}-\vec{y})} \, p^0$$

and $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$.

We can therefore write

$$\langle \Omega | \phi, \infty \rangle \langle \phi, -\infty | \Omega \rangle$$

$$= |N|^2 \exp\left\{ -\frac{1}{2} \int d^3x \, d^3y \, k(\vec{x}, \vec{y}) \Big(\phi(+\infty, \vec{x}) \, \phi(+\infty, \vec{y}) + \phi(-\infty, \vec{x}) \, \phi(-\infty, \vec{y}) \Big) \right\}$$

which we would like to absorb into an additional contribution to the action.

We can introduce a time integral as follows

$$\lim_{\varepsilon\to 0+} \varepsilon \int_{-\infty}^{\infty} dt\, f(t)\, e^{-\varepsilon|t|}$$

$$= \lim_{\varepsilon\to 0+} \left[\int_0^\infty dt\, f(t)\, \varepsilon e^{-\varepsilon t} + \int_{-\infty}^0 dt\, f(t)\, \varepsilon e^{\varepsilon t} \right]$$

$$= \lim_{\varepsilon\to 0+} \left[\int_0^\infty dt\, f(t) \frac{d}{dt}\left(-e^{-\varepsilon t}\right) + \int_{-\infty}^0 dt\, f(t) \frac{d}{dt}\left(e^{\varepsilon t}\right) \right]$$

$$= \lim_{\varepsilon\to 0+} \left[-f(t)\, e^{-\varepsilon t}\Big|_0^\infty + \int_0^\infty dt\, \frac{df(t)}{dt}\, e^{-\varepsilon t} + f(t)\, e^{\varepsilon t}\Big|_{-\infty}^0 - \int_{-\infty}^0 dt\, \frac{df(t)}{dt}\, e^{\varepsilon t} \right]$$

$$= \int_0^\infty dt\, \frac{df(t)}{dt} - \int_{-\infty}^0 dt\, \frac{df(t)}{dt} + 2 f(0)$$

$$= f(\infty) + f(-\infty)$$

Reading this relation backwards, we obtain (surpressing the $\varepsilon\to 0+$ prescription from now on)

$$\langle \Omega | \phi, \infty\rangle \langle \phi, -\infty | \Omega\rangle$$

$$= |N|^2 \exp\left\{ -\frac{\varepsilon}{2} \int d^3x\, d^3y\, k(\vec{x},\vec{y}) \int_{-\infty}^{\infty} dt\, e^{-\varepsilon|t|}\, \phi(t,\vec{x})\, \phi(t,\vec{y}) \right\}$$

We can further simplify

$$\varepsilon\, k(\vec{x},\vec{y}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}(\vec{x}-\vec{y})} \underbrace{\varepsilon p^0}_{\approx \varepsilon} \quad \text{since } p^0 > 0$$

$$\simeq \varepsilon\, \delta^{(3)}(\vec{x}-\vec{y})$$

and $e^{-\varepsilon|t|} \simeq 1 + O(\varepsilon)$.

It follows

$$\langle \Omega | \phi, \infty \rangle \langle \phi, -\infty | \Omega \rangle$$

$$= |N|^2 \exp\left\{ -\frac{\varepsilon}{2} \int d^4x \, (\phi(x))^2 \right\}$$

which can be view as an additional contribution to the Lagrangian

$$\mathcal{L} + \frac{i\varepsilon}{2} \phi^2$$

$$= \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} (m^2 - i\varepsilon) \, \phi^2$$

and we will see later that this corresponds to the $i\varepsilon$-prescription of the Feynman propagator.

The normalisation $|N|^2$ is, on the other hand, irrelevant since we know that $\langle \Omega | \Omega \rangle = 1$. Our final result therefore becomes

$$\boxed{\langle \Omega | T \phi(x_1) \ldots \phi(x_n) | \Omega \rangle = \frac{\int \mathcal{D}\phi(x) e^{iS(\phi)} \phi(x_1) \ldots \phi(x_n)}{\int \mathcal{D}\phi(x) e^{iS(\phi)}}}$$

where the $i\varepsilon$-terms are implicit in $S(\phi)$.

1.3 Generating functional

We have found a new representation of correlation functions in terms of path integrals. But does the new expression reproduce the results that we obtained in the canonical quantisation approach?

An efficient way to calculate correlation functions in the path-integral formulation starts from the generating functional

$$\boxed{Z[J] = N \int \mathcal{D}\phi(x)\, e^{i\int d^4x\,(\mathcal{L} + J(x)\phi(x))}}$$

Here $J(x)$ is an external (classical) source and the normalisation is chosen to satisfy $Z[0] = 1$

$$\Rightarrow \quad N^{-1} = \int \mathcal{D}\phi(x)\, e^{i\int d^4x\, \mathcal{L}}$$

By taking functional derivatives wrt the source $J(x)$ and setting the sources to zero, we generate all correlation functions directly from $Z[J]$.

This can be seen as follows. First recall that

$$\frac{\delta}{\delta J(y)} J(x) = \delta^{(4)}(x-y) \qquad \frac{\partial}{\partial x_j} x_i = \delta_{ij}$$

$$\frac{\delta}{\delta J(y)} \int d^4x \, J(x)\, \phi(x) = \phi(y) \qquad \frac{\partial}{\partial x_j} \sum_i x_i k_i = k_j$$

and we will prove the product and the chain rules for functional derivatives in the tutorials. (→ TPP1)

We therefore obtain e.g.

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)} \frac{1}{i}\frac{\delta}{\delta J(x_2)} Z[J] \Big|_{J=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta J(x_1)} N \int \mathcal{D}\phi(x)\, e^{i\int d^4x\,(\mathcal{L}+J\phi)} \phi(x_2) \Big|_{J=0}$$

$$= N \int \mathcal{D}\phi(x)\, e^{i\int d^4x\,(\mathcal{L}+J\phi)} \phi(x_1)\,\phi(x_2) \Big|_{J=0}$$

$$= \frac{\int \mathcal{D}\phi(x)\, e^{i\int d^4x\,\mathcal{L}} \phi(x_1)\,\phi(x_2)}{\int \mathcal{D}\phi(x)\, e^{i\int d^4x\,\mathcal{L}}}$$

$$= \langle \Omega | T \phi(x_1)\, \phi(x_2) | \Omega \rangle$$

and the procedure can easily be generalised to arbitrary n-point function with the dictionary

$$\frac{1}{i}\frac{\delta}{\delta J(x)} \longleftrightarrow \phi(x)$$

Notice also that the normalisation implies

$$Z[J]\Big|_{J=0} = \langle \Omega | \Omega \rangle = 1$$

We will now compute the generating functional of a free, real scalar field explicitly. Starting from the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} (m^2 - i\varepsilon) \phi^2$$

we obtain

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi(x) \, e^{i \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}(m^2 - i\varepsilon)\phi^2 + J\phi \right)}$$

p.I.

$$= \mathcal{N} \int \mathcal{D}\phi(x) \, e^{\frac{i}{2} \int d^4x \, \phi(x) \left(-\partial^2 - m^2 + i\varepsilon \right) \phi(x) + i \int d^4x \, J(x) \phi(x)}$$

$$= \mathcal{N} \int \mathcal{D}\phi(x) \, e^{-\frac{1}{2} \int d^4x \, d^4y \, \phi(x) D(x,y) \phi(y) + i \int d^4x \, J(x) \phi(x)}$$

where we introduced

$$D(x,y) \equiv i \, \delta^{(4)}(x-y) \left(\partial_y^2 + m^2 - i\varepsilon \right)$$

We are thus left with a Gaussian path integral that can be solved exactly.

In the tutorials we will prove the relation

$$\int_{-\infty}^{\infty} dz_1 \ldots dz_N \, e^{-\frac{1}{2} \sum_{i,j} z_i A_{ij} z_j - \sum_i B_i z_i - C}$$

$$= \left(\det \frac{A}{2\pi} \right)^{-1/2} e^{\frac{1}{2} \sum_{i,j} B_i A^{-1}_{ij} B_j - C}$$

where A is a (real-valued) symmetric $N \times N$ matrix.

---

cf. PS 2.2

$$\int_{-\infty}^{\infty} dz \, e^{-\frac{1}{2} a z^2 - bz - c} = \sqrt{\frac{2\pi}{a}} \, e^{\frac{b^2}{2a} - c}$$

In the continuum limit $N \to \infty$, this implies

$$\int \mathcal{D}z(t)\; e^{-\frac{1}{2}\int dt\, dt'\, z(t)\, A(t,t')\, z(t') - \int dt\, B(t)\, z(t) - C}$$

$$= \left(\det \frac{A}{2\pi}\right)^{-1/2} e^{\frac{1}{2}\int dt\, dt'\, B(t)\, A^{-1}(t,t')\, B(t') - C}$$

where A is now an infinite-dimensional matrix with inverse A^{-1} that satisfies

$$\int dt''\, A^{-1}(t,t'')\, A(t'',t') = \delta(t-t') \qquad A^{-1}_{ik} A_{kj} = \delta_{ij}$$

The determinant of A is defined as usual

$$\det A = \prod_n d_n$$

where d_n are the eigenvalues that are determined by

$$\int dt'\, A(t,t')\, z(t') = d_n\, z(t) \qquad A_{ij} z_j = d_n z_i$$

Notice also that the precise normalization of the path integral measure is irrelevant here, since it drops out in the ratio that defines the generating functional.

Applying our formula for Gaussian path integrals, we obtain

$A(x,x') = D(x,y)$

$\eta(x) = -i J(x)$

$C = 0$

$$Z[J] = \mathcal{N} \left(\det \frac{D(x,y)}{2\pi} \right)^{-1/2} e^{-\frac{1}{2} \int d^4x \, d^4y \, J(x) \, D^{-1}(x,y) \, J(y)}$$

$Z[0] = 1$

$$= e^{-\frac{1}{2} \int d^4x \, d^4y \, J(x) \, \Delta_F(x-y) \, J(y)}$$

where we defined $\Delta_F(x-y) \equiv D^{-1}(x,y)$, and we anticipated the answer by our notation since we know that the Feynman propagator is a Green function of the Klein-Gordon operator.

Let us nevertheless derive this explicitly here

$$\int d^4z \, D(x,z) \, \Delta_F(z-y) = i \, (\partial_x^2 \ldots - i\varepsilon) \, \Delta_F(x-y) \overset{!}{=} \delta^{(4)}(x-y)$$

which in Fourier space becomes (TTP 1 ...)

$$\int d^4x \, e^{ip(x-y)} \, (\partial_x^2 + m^2 - i\varepsilon) \, i \, \Delta_F(x-y) = 1$$

p.I.

$$= \int d^4x \, (-p^2 + m^2 - i\varepsilon) \, e^{ip(x-y)} \, i \, \Delta_F(x-y)$$

$$\Rightarrow \Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip(x-y)} \, \frac{i}{p^2 - m^2 + i\varepsilon}$$

which indeed corresponds to the Feynman propagator, including the correct $i\varepsilon$-prescription.

Having an explicit expression for the generating functional, we can calculate all correlation functions in the free theory. Consider e.g.

$$\langle \Omega | \phi(x) | \Omega \rangle = \frac{1}{i} \frac{\delta}{\delta J(x)} Z[J] \Big|_{J=0}$$

$$= -i \, e^{-\frac{1}{2} \int d^4x' d^4y' J(x') \Delta_F(x'-y') J(y')}$$

$$\left\{ -\frac{1}{2} \int d^4y' \Delta_F(x-y') J(y') - \frac{1}{2} \int d^4x' J(x') \Delta_F(x'-x) \right\}_{J=0}$$

$\Delta_F(x-y) = \Delta_F(y-x)$

$$= -i \, e^{-\frac{1}{2} \int d^4x' d^4y' J(x') \Delta_F(x'-y') J(y')} \left\{ -\int d^4y' \Delta_F(x-y') J(y') \right\}_{J=0}$$

$$= 0$$

Moreover

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$$

$$= \frac{1}{i} \frac{\delta}{\delta J(x)} \frac{1}{i} \frac{\delta}{\delta J(y)} Z[J] \Big|_{J=0}$$

$$= - e^{-\frac{1}{2} \int d^4x' d^4y' J(x') \Delta_F(x'-y') J(y')}$$

$$\left\{ \int d^4x' \Delta_F(x-x') J(x') \int d^4y' \Delta_F(y-y') J(y') - \Delta_F(x-y) \right\}_{J=0}$$

$$= \Delta_F(x-y)$$

and it is easy to show that all odd n-point functions vanish.

Let us also consider the four-point function explicitly. To this end, we introduce a shorthand notation

$$\phi(x) = \phi_x \qquad \int d^4x\, A(x)\, B(x) = A_x B_x \qquad \delta^{(4)}(x-y) = \delta_{xy} \qquad \Delta_F(x-y) = \Delta_{xy}$$

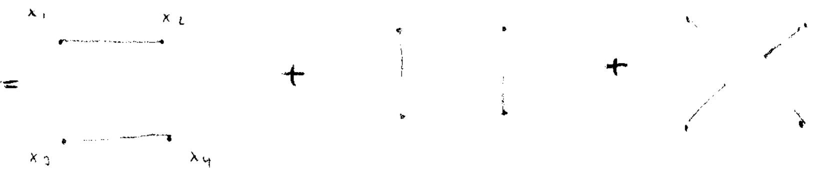
$$\Rightarrow \langle \Omega | T \phi_{x_1} \phi_{x_2} \phi_{x_3} \phi_{x_4} | \Omega \rangle$$

$$= \frac{1}{i^4} \frac{\delta}{\delta J_{x_1}} \frac{\delta}{\delta J_{x_2}} \frac{\delta}{\delta J_{x_3}} \frac{\delta}{\delta J_{x_4}} e^{-\frac{1}{2} J_{x'} \Delta_{x'y'} J_{y'}} \Big|_{J=0}$$

$$= \frac{\delta}{\delta J_{x_1}} e^{-\frac{1}{2} J_{x'} \Delta_{x'y'} J_{y'}} \Big\{ - \Delta_{x_2 y} J_y \Delta_{x_3 y'} J_{y'} \Delta_{x_4 y''} J_{y''}$$

$$+ \Delta_{x_2 y} J_y \Delta_{x_3 x_4} + \Delta_{x_2 x_3} \Delta_{x_4 y} J_y + \Delta_{x_3 y} J_y \Delta_{x_2 x_4} \Big\}_{J=0}$$

$$= \Delta_{x_1 x_2} \Delta_{x_3 x_4} + \Delta_{x_1 x_3} \Delta_{x_2 x_4} + \Delta_{x_1 x_4} \Delta_{x_2 x_3}$$

$=$ 

which again agrees with the result that we obtained in the canonical quantisation approach. For arbitrary even n-point functions, this generalises to the sum of all "full contractions" in agreement with the prediction from Wick's theorem. (TEIL page 245 – 248)

We next consider interacting theories with

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$\mathcal{L}_0$: quadratic in the fields ($\to$ Gaussian path integral)

$\mathcal{L}_{int}$: higher-dimensional operators

and generating functional

$$Z[J] = N \int \mathcal{D}\phi(x)\, e^{i\int d^4x\,(\mathcal{L}_0 + \mathcal{L}_{int} + J\phi)}$$

which is normalised to $Z[0] = 1$.

As the path integral for interacting theories cannot be solved exactly in general, one typically refers to one of the following approximations:

1) One can go back to the discretised version of the path integral to compute the multi-dimensional integral explicitly (recall that we integrate over classical field configurations that correspond to simple numbers). Here one needs to make sure that discretisation effects as well as finite volume effects are under control. There are a couple of complications when dealing with a gauge theory on the <u>lattice</u>, but to date this is without doubt the most efficient method for studying e.g. non-perturbative effects in QCD. Typical lattice sizes for QCD calculations are 48×24^3 or 128×64^3 depending on the considered observable.

2) For weakly-interacting theories, $\mathcal{L}_{int}$ can be considered as a small perturbation and so we can expand

$$e^{i\int d^4x\, \mathcal{L}_{int}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\int d^4x\, \mathcal{L}_{int}\right)^n$$

Truncating the perturbative expansion at some (low) order n_0, we obtain Gaussian integrals with polynomial prefactors which can again be calculated explicitly. We obtain

$$Z[J] = N \int \mathcal{D}\phi(x) \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\int d^4z\, \mathcal{L}_{int}(\phi(z))\right)^n e^{i\int d^4x\, (\mathcal{L}_0 + J\phi)}$$

$$= N \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\int d^4z\, \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right)^n \int \mathcal{D}\phi(x)\, e^{i\int d^4x\, (\mathcal{L}_0 + J\phi)} \quad \leftarrow \text{Gaussian path integral}$$

$$= N' \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\int d^4z\, \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right)^n e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)}$$

As an example, let us consider ϕ^4-theory with

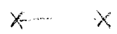
$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

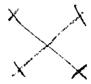
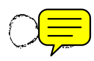
To lowest non-trivial order, we find (adopting the shorthand notation from above)

$$Z[J]\Big|_{O(\lambda)} = N' \left(-\frac{i\lambda}{4!}\right) \int d^4 z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 e^{-\frac{1}{2} J_x \Delta_{xy} J_y}$$

$$= N' \left(-\frac{i\lambda}{4!}\right) \int d^4 z \; e^{-\frac{1}{2} J_x \Delta_{xy} J_y}$$

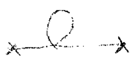
$$\left\{ (\Delta_{zz'} J_{z'})^4 - 6 (\Delta_{zz'} J_{z'})^2 \Delta_{zz} + 3 (\Delta_{zz})^2 \right\}$$

$= N' \left(-\frac{i\lambda}{4!}\right) e^{-\frac{1}{2}}$ 

$\Big\{$  $- 6$  $+ 3$  $\Big\}$

where we visualized a propagator with a line and an external source with a cross (notice that we integrate over the spacetime positions of the sources).

As the generating functional is normalized to $Z[0]=1$, the vacuum bubbles drop out and we obtain

$Z[J] = \Big\{ 1 - \frac{i\lambda}{4!} \Big($  $- 6$  $\Big) + O(\lambda^2) \Big\}$

$e^{-\frac{1}{2}}$ 

Given this expression, we can now compute <u>all</u> correlation functions in ϕ^4-theory to $O(\lambda)$. Consider e.g. the two-point function

$$\langle \Omega | \, T \phi(x) \phi(y) \, | \Omega \rangle$$

$$= \frac{1}{i} \frac{\delta}{\delta J(x)} \frac{1}{i} \frac{\delta}{\delta J(y)} Z[J] \Big|_{J=0}$$

$$= - \frac{\delta}{\delta J_x} \frac{\delta}{\delta J_y} \left\{ 1 - \frac{i\lambda}{4!} \int d^4z \left((\Delta_{zz'} J_{z'})^4 - 6 (\Delta_{zz'} J_{z'})^2 \Delta_{zz} \right) + \cdots \right\} e^{-\frac{1}{2} J_{z'} \Delta_{z'z''} J_{z''}}$$

$$= \Delta_{xy} + \frac{i\lambda}{4!} \left(-12 \, \Delta_{zx} \Delta_{zy} \Delta_{zz} \right) + O(\lambda^2)$$

$=$ $+$  $+ \; O(\lambda^2)$

and we obtain the correct Feynman rules with

 $= -i\lambda \int d^4z$

and the correct symmetry factor $S = \frac{1}{2}$ for the one-loop graph.

The procedure generalises to arbitrary n-point functions in interacting theories and so we conclude that the path-integral formulation and the canonical quantisation approach are equivalent.

(we will consider another non-trivial example in the tutorials)

Before we turn to the path-integral formulation of higher-spin fields, we introduce the generating functional of <u>connected</u> Green functions (→ relevant for S-matrix)

$$\boxed{W[J] = -i \ln Z[J]} \qquad \text{or} \qquad Z[J] = e^{iW[J]}$$

with $W[0] = 0$.

The functional derivatives now give

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)} W[J]\Big|_{J=0}$$

$$= (-i)\ \langle \Omega | T\phi(x_1) \ldots \phi(x_n) | \Omega \rangle_{\text{connected}}$$

Explicitly

$$\frac{1}{i}\frac{\delta}{\delta J(x)} W[J]\Big|_{J=0}$$

$$= -\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}\Big|_{J=0}$$

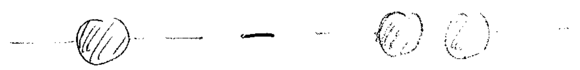
$$= (-i)\ \langle \Omega | \phi(x) | \Omega \rangle = (-i) \ —$$

$$\frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)} W(J)\Big|_{J=0}$$

$$= i \frac{\delta}{\delta J(x)} \frac{1}{Z(J)}\frac{\delta Z(J)}{\delta J(y)}\Big|_{J=0}$$

$$= i\left\{ -\frac{1}{Z(J)^2}\frac{\delta Z(J)}{\delta J(x)}\frac{\delta Z(J)}{\delta J(y)} + \frac{1}{Z(J)}\frac{\delta^2 Z(J)}{\delta J(x)\,\delta J(y)}\right\}_{J=0}$$

$$= i\left\{\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle - \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle\right\}$$

$= (-i)\ \{$  $\}$

where we see that we indeed subtract the disconnected diagrams from the 2-point function. We will consider another non-trivial example in the tutorials.

1.4 Bosonic path integrals

The extension of the path-integral formalism to other bosonic fields is straight-forward. Consider e.g. the complex scalar field, for which ϕ and $\phi^\dagger$ can be treated as independent variables with individual source terms. In the free theory with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - (m^2 - i\varepsilon)\,\phi^\dagger \phi$$

the generating functional becomes

$$Z[J, J^\dagger] = \mathcal{N} \int \mathcal{D}\phi(x)\, \mathcal{D}\phi^\dagger(x)\; e^{\,i\int d^4x\, (\mathcal{L} + J\phi + J^\dagger \phi^\dagger)}$$

where the normalisation is fixed by $Z[0,0] = 1$.

We can rewrite this expression in terms of two real scalar fields $\phi_{1,2}$ with sources $J_{1,2}$, substituting

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad J = \frac{1}{\sqrt{2}}(J_1 - iJ_2)$$

$$\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \qquad J^\dagger = \frac{1}{\sqrt{2}}(J_1 + iJ_2)$$

This yields

$$Z[J, J^*] = N \int \mathcal{D}\phi_1(x)\, \mathcal{D}\phi_2(x)\; e^{i\int d^4x \sum_{i=1}^{2} \left(\frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2}(m^2 - i\varepsilon)\phi_i^2 + J_i \phi_i \right)}$$

$J_1 = \frac{1}{\sqrt{2}}(J + J^*)$

$J_2 = \frac{i}{\sqrt{2}}(J - J^*)$

$$= e^{-\frac{1}{2}\int d^4x\, d^4y \sum_{i=1}^{2} J_i(x)\, \Delta_F(x-y)\, J_i(y)}$$

$$= e^{-\frac{1}{2}\int d^4x\, d^4y \left(J(x)\, \Delta_F(x-y)\, J^*(y) + J^*(x)\, \Delta_F(x-y)\, J(y) \right)}$$

$$= e^{-\int d^4x\, d^4y\; J(x)\, \Delta_F(x-y)\, J^*(y)}$$

By taking functional derivatives wrt

$$\frac{1}{i}\frac{\delta}{\delta J(x)} \rightarrow \phi(x) \qquad\qquad \frac{1}{i}\frac{\delta}{\delta J^*(x)} \rightarrow \phi^*(x)$$

and setting the sources to zero, we generate the Green functions of the free theory. We obtain e.g.

$$\langle 0 | T\, \phi(x)\, \phi(y) | 0 \rangle$$

$$= \frac{1}{i^2} \frac{\delta}{\delta J_x} \frac{\delta}{\delta J_y} e^{-J_{x'} \Delta_{x'y'} J^*_{y'}} \Big|_{J=J^*=0}$$

$$= - e^{-J_{x'} \Delta_{x'y'} J^*_{y'}} \left\{ \Delta_{xx'} J^*_{x'}\, \Delta_{yy'} J^*_{y'} \right\}_{J=J^*=0}$$

$$= 0$$

$$\langle 0 | T\, \phi(x)\, \phi^*(y) | 0 \rangle$$

$$= \frac{1}{i^2} \frac{\delta}{\delta J_x} \frac{\delta}{\delta J^*_y} e^{-J_{x'} \Delta_{x'y'} J^*_{y'}} \Big|_{J=J^*=0}$$

$$= - e^{-J_{x'} \Delta_{x'y'} J^*_{y'}} \left\{ \Delta_{xx'} J^*_{x'}\, \Delta_{y'y} J_{y'} - \Delta_{xy} \right\}_{J=J^*=0}$$

$$= \Delta_{xy}$$

Let us also consider the massive vector field with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ explicitly. In chapter 1 we argued that the constraint

$$\partial_\mu A^\mu(x) = 0$$

is needed to remove the spin-0 component of the vector representation of the Lorentz group. How can we formulate a path-integral formulation in the presence of this constraint?

The path-integral formulation always starts from the Hamiltonian version. The conjugate field is now given by (TPP1, page 207)

$$\begin{aligned}
\Pi_\mu(x) &= \frac{\partial \mathcal{L}}{\partial_0 A^\mu(x)} \\
&= \frac{\partial \mathcal{L}}{\partial F^{\rho\sigma}(x)} \frac{\partial F^{\rho\sigma}(x)}{\partial_0 A^\mu(x)} \\
&= -\frac{1}{2} F_{\rho\sigma} \left(g^\rho{}_0 \, g^\sigma{}_\mu - g^\sigma{}_0 \, g^\rho{}_\mu \right) \\
&= F_{\mu 0}
\end{aligned}$$

which implies $\Pi_0(x) = 0$.

The field $A^0(x)$ is therefore not a dynamical variable, but it is constrained by the other field components. This can be seen most easily from the equations of motion (TFFA, pg 208 - 209)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu}$$

$$= - \partial_\mu F^{\mu\nu} - m^2 A^\nu = 0$$

which for $\nu = 0$ gives

$$A^0 = - \frac{1}{m^2} \partial_\mu \underbrace{F^{\mu 0}}_{\pi^\mu}$$

$$= - \frac{1}{m^2} \vec{\nabla} \cdot \vec{\pi}$$

recall that
$\partial_\mu = (\partial_0, + \vec{\nabla})$
$\pi^\mu = (0, \vec{\pi})$

The Hamiltonian is now given by

$$\mathcal{H} = \pi_\mu \dot{A}^\mu - \mathcal{L}$$

$$= - \pi^i \dot{A}^i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu$$

in which we need to express A^0 and $\dot{A}^i$ in terms of the conjugate field $\vec{\pi}$.

We first note that

$$\pi^i = F^{i0} = \partial^i A^0 - \dot{A}^i$$

from which we obtain

$$\dot{A}^i = \partial^i \left(-\frac{1}{m^2} \vec{\nabla} \cdot \vec{\pi} \right) - \pi^i$$

$$= \frac{1}{m^2} \nabla^i \nabla^j \pi^j - \pi^i$$

Moreover

$$+\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = +\frac{1}{4} \left(F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} \right)$$

$$= -\frac{1}{2} F^{i0} F^{i0} + \frac{1}{2} \left(\partial^i A^j \partial^i A^j - \partial^i A^j \partial^j A^i \right)$$

$$= -\frac{1}{2} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2$$

since

$$(\vec{\nabla} \times \vec{A})^2 = \varepsilon^{kij} \partial^i A^j \varepsilon^{klm} \partial^l A^m$$

$$= (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}) \partial^i A^j \partial^l A^m$$

$$= \partial^i A^j \partial^i A^j - \partial^i A^j \partial^j A^i$$

The Hamiltonian then follows as

$$\mathcal{H} = -\frac{1}{m^2} \pi^i \overset{\swarrow \text{P.I}}{\nabla^i} \nabla^j \pi^j + \vec{\pi}^2 - \frac{1}{2} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2$$

$$- \frac{1}{2} m^2 \left(-\frac{1}{m^2} \vec{\nabla} \cdot \vec{\pi} \right)^2 + \frac{1}{2} m^2 \vec{A}^2$$

$$= \frac{1}{2} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 + \frac{1}{2m^2} (\vec{\nabla} \cdot \vec{\pi})^2 + \frac{1}{2} m^2 \vec{A}^2$$

The <u>Hamiltonian</u> version of the path integral reads (cf. page 21)

$$\int \mathcal{D}A^i(x)\, \mathcal{D}\pi^i(x)\; e^{\,i\int d^4x\,(-\pi^i \dot{A}^i - \mathcal{H})}$$

this is <u>not</u> $\mathcal{L}$ since A^i and π_i are independent integration variables

As $\mathcal{H}$ is quadratic in the conjugate field $\vec{\pi}$, we can perform the associated Gaussian integral explicitly, which yields an expression of the form

$$\int \mathcal{D}A^i(x)\; e^{\,i\int d^4x\, \mathcal{L}'}$$

with an unusual, non-covariant expression for $\mathcal{L}' \neq \mathcal{L}$ since we essentially have integrated over $A^0 = -\frac{1}{m^2}\vec{\nabla}\cdot\vec{\pi}$.

It is therefore more convenient to introduce an artificial integration over $A^0(x)$ in the form

$$\int \mathcal{D}A^0(x)\; e^{\,i\frac{m^2}{2}\int d^4x\,\left(A^0 + \frac{1}{m^2}\vec{\nabla}\cdot\vec{\pi}\right)^2}$$

$$= \text{const}$$

which is <u>independent</u> of $\vec{\pi}$ as can easily be seen by replacing $A^0 \to A^{0\prime} = A^0 + \frac{1}{m^2}(\vec{\nabla}\cdot\vec{\pi})$.

---

or use master formula for Gaussian path integrals from page 37 with

$A = -im^2\,\delta^{(4)}(x-y)$

$B = -i\,(\vec{\nabla}\cdot\vec{\pi})$

$C = \frac{-i}{2m^2}\int d^4x\,(\vec{\nabla}\cdot\vec{\pi})^2$

$$\Rightarrow \frac{1}{2}\int d^4x\, d^4y\; B(x)\, A^{-1}(x-y)\, B(y) - C$$

$$= -\frac{i}{2m^2}\int d^4x\, d^4y\,(\vec{\nabla}\cdot\vec{\pi})(x)\,\delta^{(4)}(x-y)\,(\vec{\nabla}\cdot\vec{\pi})(y) + \frac{i}{2m^2}\int d^4x\,(\vec{\nabla}\cdot\vec{\pi})(x)^2$$

$$= 0$$

We then obtain

$$-\pi^i \dot{A}^i - \mathcal{H} + \frac{m^2}{2}\left(A^0 + \frac{1}{m^2}\vec{\nabla}\cdot\vec{\pi}\right)^2$$

$$= -\pi^i \dot{A}^i - \frac{1}{2}\vec{\pi}^2 - \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 - \frac{1}{2m^2}(\vec{\nabla}\cdot\vec{\pi})^2 - \frac{1}{2}m^2\vec{A}^2$$

$$+ \frac{m^2}{2}(A^0)^2 + \underbrace{A^0\,\vec{\nabla}\cdot\vec{\pi}}_{\text{p.I} \to -\pi^i\nabla^i A^0} + \frac{1}{2m^2}(\vec{\nabla}\cdot\vec{\pi})^2$$

$$= -\frac{1}{2}\vec{\pi}^2 - \pi^i\underbrace{(\dot{A}^i + \nabla^i A^0)}_{\partial^0 A^i - \partial^i A^0 = F^{0i}} - \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}m^2 A_\mu A^\mu$$

[illegible]

$$= -\frac{1}{2}\vec{\pi}^2 - \pi^i F^{0i} - \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}m^2 A_\mu A^\mu$$

and the Gaussian integral over the conjugate field gives

$$\int \mathcal{D}A^\mu(x)\,\mathcal{D}\pi^i(x)\; e^{i\int d^4x\left(-\frac{1}{2}\vec{\pi}^2 - \pi^i F^{0i} - \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}m^2 A_\mu A^\mu\right)}$$

$$= N \int \mathcal{D}A^\mu(x)\; e^{i\int d^4x\left(\frac{1}{2}F^{0i}F^{0i} - \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}m^2 A_\mu A^\mu\right)}$$

$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (TPPM, page 202)

[illegible] conventions for p. 24 [illegible]

$$\begin{cases} A = i\,\delta^{(4)}(x-y) \\ B = i\,F^{0i} \end{cases}$$

$$= N \int \mathcal{D}A^\mu(x)\; e^{i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu\right)}$$

$$= N \int \mathcal{D}A^\mu(x)\; e^{i\int d^4x\,\mathcal{L}}$$

i.e. we obtain the standard Lagrange version of the path integral only when we integrate over all four components of the vector field. The generating functional then becomes

$$Z[J^\mu] = N \int \mathcal{D}A^\mu(x)\; e^{i\int d^4x\,(\mathcal{L} + J_\mu A^\mu)}$$

with $Z[0] = 1$, and we generate Green functions via $\frac{1}{i}\frac{\delta}{\delta J_\mu(x)} \to A^\mu(x)$.

1.5. Fermionic path integrals

We would like to formulate a similar path-integral representation for fermion fields. As fermion fields obey canonical anticommutation relations, we cannot get along with integration variables (i.e. classical field configurations) that commute.

We need a new concept: anticommuting numbers!

Grassmann numbers:

Two Grassmann numbers θ and η anticommute

$$\theta\eta = -\eta\theta$$

$\Rightarrow$ the square of any Grassmann number vanishes, $\theta^2 = 0$.

Notice that the product of two Grassmann numbers gives an ordinary number

$$(\theta_1\theta_2)\,\theta_3 = -\theta_1\theta_3\theta_2 = \theta_3\,(\theta_1\theta_2)$$

Addition of Grassmann numbers and multiplication with ordinary numbers as usual

$$a(\theta + b\eta) = a\theta + ab\eta \qquad a,b \in \mathbb{R}$$

Functions of Grassmann numbers have a <u>finite</u> Taylor expansion

$$f(\theta) = f_0 + \theta f_1 \qquad (\text{since } \theta^2 = 0)$$

Notice that if

$$f \begin{cases} \text{Grassmann} \\ \text{ordinary} \end{cases} \rightarrow f_0 \begin{cases} \text{Grassmann} \\ \text{ordinary} \end{cases} \text{ and } f_1 \begin{cases} \text{ordinary} \\ \text{Grassmann} \end{cases}$$

There is an ambiguity in defining the derivative

$$f(\theta + d\theta) = \begin{cases} f(\theta) + d\theta\, f'(\theta) & \text{left-derivative} \\ f(\theta) + f'(\theta)\, d\theta & \text{right-derivative} \end{cases}$$

We will use the <u>left-derivative</u> here with

$$\frac{d}{d\theta}(\theta\eta) = \eta \qquad \frac{d}{d\eta}(\theta\eta) = \frac{d}{d\eta}(-\eta\theta) = -\theta$$

and

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}(f_0 + \theta f_1) = f_1$$

In the path-integral formalism, we will also integrate over Grassmann numbers (we only need the analog of $\int_{-\infty}^{\infty} dx$).

We demand that the integration acts linearly

$$\int d\theta\, f(\theta) = \int d\theta\, (f_0 + \theta f_1) = \int d\theta\, f_0 + \int d\theta\, \theta\, f_1$$

and that the integration is invariant under a shift $\theta \to \theta + \eta$

$$\int d\theta\, f(\theta) = \int d\theta\, f(\theta + \eta) = \int d\theta\, [f_0 + (\theta + \eta) f_1]$$

$$= \int d\theta\, (f_0 + \eta f_1) + \int d\theta\, \theta\, f_1$$

$\Rightarrow$ we need to impose $\int d\theta = 0$

The remaining integral can be defined as $\int d\theta\, \theta \equiv 1.$

$$\Rightarrow \int d\theta\, f(\theta) = \int d\theta\, (f_0 + \theta f_1) = f_1$$

and $\int d\theta\, f(\theta) = \frac{d}{d\theta} f(\theta)$

Notice that Grassmann integration acts as ordinary derivation.

For functions that depend on N Grassmann variables we write

$$f(\theta) = f_0 + \theta_i f_1^i + \theta_i \theta_j f_2^{ij} + \ldots + \theta_1 \ldots \theta_N f_N$$

where the Grassmann variables are in <u>ascending order</u> in each term.

We next define the N-dimensional integration measure

$$d^N\theta = d\theta_N \ldots d\theta_1$$

in <u>descending order</u>. We then have

$$\int d^N\theta\, f(\theta) = \int d\theta_N \ldots d\theta_1 \left[f_0 + \theta_i f_1^i + \ldots + \theta_1 \ldots \theta_N f_N \right]$$

$$= \int d\theta_N \ldots \underbrace{d\theta_1\, \theta_1}_{1} \ldots \theta_N f_N = f_N$$

We also need the Jacobian $\left|\frac{d\theta}{d\theta'}\right|$ of a linear transformation

$$\theta_i' = \underbrace{A_{ij}}_{\text{ordinary numbers}} \theta_j$$

$$\int d^N\theta\, f(A\theta) = \int d^N\theta' \underbrace{\left|\frac{d\theta}{d\theta'}\right|}_{\in \mathbb{R}} f(\theta') = \left|\frac{d\theta}{d\theta'}\right| f_N$$

$$= \int d^N\theta\, (A\theta)_1 \ldots (A\theta)_N f_N$$

$$= A_{1 i_1} \ldots A_{N i_N} \int d\theta_N \ldots d\theta_1 \underbrace{\theta_{i_1} \ldots \theta_{i_N}}_{\varepsilon_{i_1 \ldots i_N}} f_N = \det A\, f_N$$

$$\Rightarrow \quad \left|\frac{d\theta}{d\theta'}\right| = \det A$$

whereas for ordinary variables with $x_i' = A_{ij} x_j$, one obtains

$$\left|\frac{dx}{dx'}\right| = \det A^{-1} = \frac{1}{\det A}$$

We also need the concept of complex Grassmann numbers, which we introduce via

$$\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \qquad \theta^* = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$$

$$\Rightarrow \theta\theta^* = \frac{1}{2}(i\theta_2\theta_1 - i\theta_1\theta_2) = i\theta_2\theta_1 = -\theta^*\theta$$

Moreover

$$1 = \int d\theta_2 \, d\theta_1 \, \theta_1 \theta_2 = \int d\theta^* d\theta \left| \frac{d(\theta_1, \theta_2)}{d(\theta, \theta^*)} \right| i\theta\theta^*$$

$$= \int d\theta^* d\theta \, \theta\theta^* \quad \checkmark$$

We then obtain (for $a \in \mathbb{C}$)

$$\int d\theta^* d\theta \, e^{-\theta^* a \theta} = \int d\theta^* d\theta \, (1 - \theta^* a \theta) = +a$$

which is again to be compared with the ordinary case

$$\int dx \, dx^* \, e^{-x^* a x} = \int dx_1 \int dx_2 \, e^{-\frac{a}{2}(x_1^2 + x_2^2)}$$

$$= \sqrt{\frac{2\pi}{a}} \sqrt{\frac{2\pi}{a}} = \frac{2\pi}{a}$$

(generalisation to N-dimensional case …)

For N complex Grassmann numbers we define

$$d^N\theta^* d^N\theta = d\theta_N^* d\theta_N \dots d\theta_1^* d\theta_1$$

and one finds ($\rightarrow$ tutorials)

$$\boxed{\begin{aligned} \int d^N\theta^* d^N\theta \, e^{-\theta_i^* A_{ij} \theta_j} &= \det A \\ \int d^N\theta^* d^N\theta \, \theta_k \theta_l^* \, e^{-\theta_i^* A_{ij} \theta_j} &= \det A \, (A^{-1})_{kl} \end{aligned}}$$

which is to be compared with

$$\left(\det \frac{A}{2\pi}\right)^{-1} \quad \text{and} \quad \left(\det \frac{A}{2\pi}\right)^{-1} (A^{-1})_{kl}$$

in the bosonic case

$\Rightarrow$ except for the determinant, Gaussian integrals over Grassmann variables behave exactly as Gaussian integrals over ordinary variables!

Having developed the formalism of anticommuting numbers, we may now formulate a path-integral representation of fermion fields. For concreteness, we will consider the Dirac field here.

We first introduce the notion of a "classical" Dirac field via

$$\psi_\alpha(x) = \sum_i \psi_i \, \phi_{i\alpha}(x)$$

Grassmann-valued coefficients

basis functions (spinor-valued, ordinary numbers)

It will be convenient to use $\bar\psi(x) = \psi^\dagger(x)\,\gamma^0$ instead of $\psi^\dagger(x)$.

Instead of repeating the path-integral derivation for anticommuting operators, we will follow a pragmatic approach here. We have just seen that the structure of Gaussian integrals over Grassmann variables is very similar to the one of ordinary Gaussian integrals. We will therefore simply <u>define</u> the generating functional for fermion fields, and we will verify if the definition is consistent with the results that we obtained in the canonical quantisation approach.

We thus start from

$$Z[\eta,\bar\eta] \equiv N \int \mathcal{D}\bar\psi(x)\,\mathcal{D}\psi(x)\; e^{i\int d^4x\,(\mathcal{L}+\bar\eta\psi+\bar\psi\eta)}$$

where $\eta(x)$ and $\bar\eta(x)$ are Grassmann-valued sources ($\rightarrow$ classical Dirac fields) and the normalisation is fixed as usual by $Z[0,0]=1$.

Notice that the functional derivatives now act as

$$\frac{1}{i}\frac{\delta}{\delta\bar\eta(x)} \longrightarrow \psi(x) \qquad\qquad -\frac{1}{i}\frac{\delta}{\delta\eta(x)} \longrightarrow \bar\psi(x)$$

since we use the left-derivative here.

Let us now calculate the generating functional for the free Dirac theory with Lagrangian

$$\mathcal{L} = \bar\psi(i\not\partial - m + i\varepsilon)\psi$$

follows again from the vacuum functionals in the limit $t_{i,f}\to\mp\infty$ (will not show this here)

and hence

$$Z[\eta,\bar\eta] = N\int\mathcal{D}\bar\psi(x)\,\mathcal{D}\psi(x)\; e^{i\int d^4x\,(\bar\psi(i\not\partial-m)\psi+\bar\eta\psi+\bar\psi\eta)}$$

In the following, we will suppress the $i\varepsilon$-prescription, which can easily be restored by taking $m\to m-i\varepsilon$.

We now use the shift-invariance of Grassmann integrals and complete the square via the substitution

$$\psi(x) = \psi'(x) - (i\not\partial - m)^{-1}\eta(x)$$
$$\bar\psi(x) = \bar\psi'(x) - \bar\eta(x)(i\not\partial - m)^{-1}$$

with Jacobian = 1.

We then obtain

$$\bar\psi(i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta$$
$$= \bar\psi'(i\not\partial - m)\psi' - \bar\eta\psi' - \bar\psi'\eta + \bar\eta(i\not\partial - m)^{-1}\eta$$
$$+ \bar\eta\psi' - \bar\eta(i\not\partial - m)^{-1}\eta + \bar\psi'\eta - \bar\eta(i\not\partial - m)^{-1}\eta$$
$$= \bar\psi'(i\not\partial - m)\psi' - \bar\eta(i\not\partial - m)^{-1}\eta$$

and the generating functional becomes

$$Z[\eta,\bar\eta] = N\int \mathcal{D}\bar\psi'(x)\,\mathcal{D}\psi'(x)\, e^{i\int d^4x\,(\bar\psi'(i\not\partial - m)\psi' - \bar\eta(i\not\partial - m)^{-1}\eta)}$$

$$= \underbrace{N\,\det\big(-i\overbrace{(i\not\partial - m)}^{\delta^{(4)}(x-y)}\big)}_{Z[0,0]=1}\, e^{-i\int d^4x\,\bar\eta(i\not\partial - m)^{-1}\eta}$$

$$= e^{-\int d^4x\, d^4y\,\bar\eta(x)\,S_F(x-y)\,\eta(y)}$$

where we defined

$$S_F(x-y) \equiv i\,\delta^{(4)}(x-y)\,(i\not\partial_y - m + i\varepsilon)^{-1} \qquad \text{(TPP1, page 197)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\,\frac{i}{\not p - m + i\varepsilon}$$

The structure of the generating functional is identical to the one of the complex scalar field. The only difference here are the additional fermion signs, and so let us consider a few examples to check if they work out correctly.

For the 2-point function we obtain

implicit spinor indices

$$\langle 0 | T \psi_1 \bar{\psi}_2 | 0 \rangle$$

$$= \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_1} \frac{1}{i} \frac{\delta}{\delta \eta_2} e^{-\bar{\eta}_x S_{xy} \eta_y} \Big|_{\eta = \bar{\eta} = 0}$$

$$= \frac{\delta}{\delta \bar{\eta}_1} e^{-\bar{\eta}_x S_{xy} \eta_y} \left\{ + \bar{\eta}_x S_{x2} \right\}_{\eta = \bar{\eta} = 0}$$

$$= S_{12} \quad \checkmark$$

Moreover

$$\langle 0 | T \psi_1 \psi_2 | 0 \rangle = \langle 0 | T \bar{\psi}_1 \bar{\psi}_2 | 0 \rangle = 0 \quad \checkmark$$

$$\langle 0 | T \bar{\psi}_1 \psi_2 | 0 \rangle$$

$$= \frac{-1}{i} \frac{\delta}{\delta \eta_1} \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_2} e^{-\bar{\eta}_x S_{xy} \eta_y} \Big|_{\eta = \bar{\eta} = 0}$$

$$= \frac{\delta}{\delta \eta_1} e^{-\bar{\eta}_x S_{xy} \eta_y} \left\{ - S_{2y} \eta_y \right\}_{\eta = \bar{\eta} = 0}$$

$$= - S_{12} \quad \checkmark$$

which is indeed consistent with the contraction

$$\overline{\psi(x) \bar{\psi}(y)} = S_F(x-y)$$

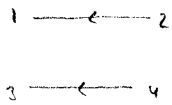
Let us also consider the 4-point function

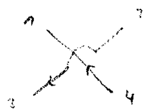
$$\langle 0 | T \psi_1 \bar\psi_2 \psi_3 \bar\psi_4 | 0 \rangle$$

$$= \frac{1}{i}\frac{\delta}{\delta\bar\eta_1}\,\frac{-1}{i}\frac{\delta}{\delta\eta_2}\,\frac{1}{i}\frac{\delta}{\delta\bar\eta_3}\,\frac{-1}{i}\frac{\delta}{\delta\eta_4}\, e^{-\bar\eta_x S_{xy}\eta_y}\Big|_{\eta=\bar\eta=0}$$

$$= \frac{\delta}{\delta\bar\eta_1}\frac{\delta}{\delta\eta_2}\, e^{-\bar\eta_x S_{xy}\eta_y}\Big\{ - S_{3y'}\eta_{y'}\bar\eta_{x'}S_{x'4} + S_{34}\Big\}_{\eta=\bar\eta=0}$$

$$= \frac{\delta}{\delta\bar\eta_1}\, e^{-\bar\eta_x S_{xy}\eta_y}\Big\{ -\bar\eta_{z'}S_{z'2}\,S_{3y'}\eta_{y'}\bar\eta_{x'}S_{x'4} + \bar\eta_{z'}S_{z'2}\,S_{34} - S_{32}\,\bar\eta_{x'}S_{x'4}\Big\}_{\eta=\bar\eta=0}$$

$$= S_{12}S_{34} - S_{14}S_{32} =$$ 

$-$ 

(TPPI, page 293)

and we thus obtain the full set of contractions with the correct fermion signs.

For weakly-coupled interacting theories, we then again apply the perturbative expansion which generates Gaussian integrals that are multiplied with polynomial prefactors. As the structure of these integrals is again the same for ordinary and Grassmann integrals (cf. page 53), the results for the bosonic theories immediately carry over to fermionic theories.