3. Quantum Electrodynamics

3.1. Gauge symmetry

QED is the quantum field theory of a charged, massive spin-1/2 field (→ Dirac spinor), which interacts with a massless, neutral spin-1 field (→ real vector field).

The Lagrangian of QED reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the <u>field-strength tensor</u> and

$$D_\mu = \partial_\mu - ieA_\mu$$

is the <u>covariant derivative</u>.

The QED Lagrangian is invariant under <u>global</u> $U(1)$ transformations

$$\psi'(x) = e^{ie\omega}\,\psi(x)$$

$$= \psi(x) + ie\omega\,\psi(x) + \mathcal{O}(\omega^2)$$

which gives rise to a conserved Noether current

$$j^\mu(x) = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}}_{i\bar\psi\gamma^\mu}\underbrace{\delta\psi}_{ie\psi} + \underbrace{\frac{\delta \mathcal{L}}{\partial(\partial_\mu \bar\psi)}}_{0}\,\delta\bar\psi = -e\,\bar\psi\gamma^\mu\psi(x)$$

with $\partial_\mu j^\mu(x) = 0$. The vector field thus couples to a conserved current as required for a consistent formulation of a quantum field theory for massless spin-1 particles.

The QED Lagrangian is also invariant under <u>local</u> $U(1)$ transformations of the form

$$\psi'(x) = e^{ie\omega(x)}\,\psi(x)$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\,\omega(x)$$

It follows

$$\bar{\psi}'(i\cancel{D}'-m)\psi'$$

$$= \bar{\psi}'(i\cancel{\partial} + e\cancel{A}' - m)\psi'$$

$$= \bar{\psi} e^{-ie\omega} (i\cancel{\partial} + e\cancel{A} + e\cancel{\partial}\omega - m)\, e^{ie\omega}\psi$$

$$= \bar{\psi}(i\cancel{\partial} - \cancel{e\cancel{\partial}\omega} + e\cancel{A} + \cancel{e\cancel{\partial}\omega} - m)\psi$$

$$= \bar{\psi}(i\cancel{\partial} - m)\psi$$

and

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

$$= \partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \omega} - \partial_\nu A_\mu - \cancel{\partial_\nu \partial_\mu \omega}$$

$$= F_{\mu\nu}$$

The local $U(1)$ symmetry is called a <u>gauge symmetry</u>, and it implies that the theory does not have a well-defined initial-value problem. Suppose that one fixes the fields on an initial time surface (by specifying $\omega(x^0 = t_0, \vec{x})$), the fields are then <u>not</u> uniquely determined at later times (since we still have the freedom to choose $\omega(x^0 > t_0, \vec{x})$).

A unique solution requires to impose a <u>local</u> condition on $A_\mu(x)$ that specifies $\omega(x)$ for <u>all</u> x. This <u>gauge condition</u> reduces the number of dynamical degrees of freedom from three to two, in accordance with the number of physical polarisation states of a massless vector boson.

<u>Example</u>: $A_3(x) = 0$ (axial gauge)

as for the massive vector field, $A_0(x)$ is not a dynamical variable, and so we impose canonical commutation relations on $[A_i(x), \pi_i(x)]$ with $i = 1,2$.

Notice also that the massive theory does not show the local symmetry, since the mass term $\frac{1}{2} m^2 A_\mu A^\mu$ is not gauge invariant.

It is important to distinguish the concepts of global and local symmetries. A global symmetry tells us that there are <u>different</u> points in the configuration space that have the same physical properties ($\rightsquigarrow$ conservation law). A local symmetry, on the other hand, implies that there are <u>apparently</u> different points in the configuration space that are physically <u>identical</u>. Gauge symmetry therefore describes a redundancy ($\rightsquigarrow$ gauge fixing) rather than a "true" symmetry.

Gauge symmetries play a central role in particle physics, and one therefore often takes a different viewpoint and imposes gauge invariance on the Lagrangian. The QED Lagrangian is then the most general, renormalisable Lagrangian that is gauge and Lorentz invariant.

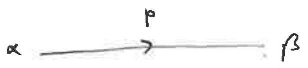
3.2. Faddeev - Popov quantisation

From the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D} - m)\psi$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi + e\,\bar{\psi} \not{A} \psi$$

we can read off the Feynman rules



$$= \frac{i(\not{p}+m)_{\beta\alpha}}{p^2 - m^2 + i\varepsilon}$$

$$= i e \gamma^\mu_{\beta\alpha}$$

The derivation of the gauge-boson propagator turns out, however, to be problematic. This can already be seen from our results for the massive vector field, for which one obtained

$$Z[J] = N \int \mathcal{D}A^\mu(x)\, e^{i\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(m^2 - i\varepsilon) A_\mu A^\mu - J_\mu A^\mu\right)}$$

$$= e^{-\frac{1}{2}\int d^4x\, d^4y\, J_\mu(x)\, \Delta_F^{\mu\nu}(x-y)\, J_\nu(y)}$$

The Feynman propagator $\Delta_F^{\mu\nu}(x-y)$ then fulfils (TTPA, page 210-213)

$$\left[(\partial^2 + m^2 - i\varepsilon) g_{\mu\nu} - \partial_\mu \partial_\nu \right] \Delta_F^{\nu\lambda}(x-y) = i\, \delta^{(4)}(x-y)\, g_\mu{}^\lambda$$

which can be inverted to give

$$\Delta_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon} \left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right]$$

The expression diverges in the limit $m \to 0$, which is however not surprising since a massless vector field has only two physical polarisations, whereas a massive vector field has three.

Technically speaking, the differential operator in the massless case $(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu)$ cannot be inverted since some of its eigenvalues are zero. Specifically, for the "pure gauges" [= gauge orbit of zero field configuration]

$$A^\nu(x) = \partial^\nu \Lambda(x)$$

we obtain

$$\left[\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu \right] \partial^\nu \Lambda(x)$$

$$= \partial^2 \partial_\mu \Lambda(x) - \partial_\mu \partial^2 \Lambda(x) = 0$$

The problem thus appears to be related to the gauge redundancy. As you have already learned in TPP 1, there exist various solutions; one may e.g. quantise the electromagnetic field in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$ (in which manifest Lorentz invariance is lost). Or one uses the Gupta-Bleuler method, in which one imposes the Lorenz condition $\partial_\mu A^\mu = 0$ on the states of the physical Hilbert space. We will now develop a third method using path-integral techniques, which has the advantage that it can be generalised to non-abelian gauge theories.

The starting point of our analysis is the generating functional

$$Z[J] = N \int \mathcal{D}A^\mu(x)\, e^{i\int d^4x\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu\right)}$$

with $Z[0] = 1$.

Notice that due to gauge invariance, this involves an integration over infinitely many equivalent configurations! We therefore have to single out the part of the path integral that counts each physical configuration exactly once.

To do so, we will follow the method invented by Faddeev and Popov. We first write the gauge-fixing condition in the form

$$G[A] = 0$$

and insert a trivial factor under the path integral

$$1 = \int \mathcal{D}\omega(x) \; \delta[G[A^\omega]] \; \det\left(\frac{\delta G[A^\omega]}{\delta\omega}\right)$$

which is the continuum generalisation of

$$1 = \int d\omega_1 \ldots d\omega_N \; \delta^{(N)}(f(\omega)) \; \det\left|\frac{\partial f_i}{\partial \omega_j}\right|$$

Here $A_\mu^\omega(x)$ is the gauge-transformed field

$$A_\mu^\omega(x) = A_\mu(x) + \partial_\mu \omega(x)$$

(gauge orbit of $A_\mu(x)$)

As an example consider Lorenz gauge $G[A] = \partial_\mu A^\mu$

$\hookrightarrow \quad G[A^\omega] = \partial_\mu A^\mu + \partial^2 \omega$

$$\hookrightarrow \quad \det\left(\frac{\delta G(A^\omega)(x)}{\delta \omega(y)}\right) = \det(\partial^2)\, \delta^{(4)}(x-y)$$

In the following we will not yet specify the gauge-fixing condition, but we will assume that the determinant does not depend on A^μ or ω (which is always true in a linear gauge). We thus obtain

$$Z[J] = N \det\left(\frac{\delta G(A^\omega)}{\delta \omega}\right) \int \mathcal{D}\omega(x)$$

$$\int \mathcal{D}A^\mu(x)\, e^{i\int d^4x\, (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu)}\, \delta(G(A^\omega))$$

We now shift the integration variable to $A_\mu^\omega = A_\mu + \partial_\mu \omega$

$$\Rightarrow \quad \mathcal{D}[A^\mu(x)] = \mathcal{D}[A^{\omega\mu}(x)] \qquad \text{(just a constant shift)}$$

$$\int d^4x\, (-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu)$$

$$= \int d^4x\, (-\tfrac{1}{4}F_{\mu\nu}^\omega F^{\omega\mu\nu} - J^\mu A_\mu^\omega + J^\mu \partial_\mu \omega)$$

) P.I.

$$= \int d^4x\, (-\tfrac{1}{4}F_{\mu\nu}^\omega F^{\omega\mu\nu} - J^\mu A_\mu^\omega - \underbrace{\partial_\mu J^\mu}_{=0}\, \omega)$$

conserved current !

Renaming A_μ^ω by A_μ, we finally arrive at

$$Z[J] = N \det\left(\frac{\delta G(A^\omega)}{\delta \omega}\right) \int \mathcal{D}\omega(x)$$

field-independent, diverging prefactor that is irrelevant for Green functions

$$\underbrace{\int \mathcal{D}A^\mu(x)\, e^{i\int d^4x\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)}\, \delta[G(A)]}_{\text{integration over physically inequivalent configurations!}}$$

We would like to bring the generating functional into a form that is suited for perturbative calculations. To do so, we now specify the gauge-fixing condition, choosing

$$G(A) = \partial_\mu A^\mu(x) - \alpha(x)$$

which is called the generalised Lorenz gauge. We now have $\det\left(\frac{\delta G(A^\omega)}{\delta\omega}\right) = \det\left(\partial^2\right)$, and arrive at

$\delta^{(4)}(x-z)$ $\delta^{(4)}(x-z)$

$$Z[J] = N \det(\partial^2) \int \mathcal{D}\omega(x)$$

$$\int \mathcal{D}A^\mu(x)\, e^{i\int d^4x\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)}\, \delta[\partial_\mu A^\mu - \alpha]$$

Note that this expression does not depend on the particular form of the gauge-fixing condition, and therefore it is also independent of $\alpha(x)$.

(same $Z[J]$ in all gauges)

We may therefore once more introduce a trivial factor

$$1 = N(\xi) \int \mathcal{D}\alpha(x)\, e^{-i\int d^4x \, \frac{\alpha^2}{2\xi}}$$

where ξ is an arbitrary parameter and the factor $N(\xi)$ ensures that the Gauß integral is normalised to 1. Interchanging the order of the integrations, we then obtain

$$Z[J] = N \det\left(\partial^2 \delta^{(4)}(x-y)\right) N(\xi) \int \mathcal{D}\omega(x) \int \mathcal{D}A^\mu(x)\, e^{i\int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)}\, e^{-i\int d^4x \, \frac{(\partial_\mu A^\mu)^2}{2\xi}}$$

The effect of the gauge-fixing thus is to add a term $-\frac{(\partial_\mu A^\mu)^2}{2\xi}$ to the Lagrangian! The gauge parameter ξ is unphysical and can be chosen arbitrarily.

We are now in the position to derive the free propagator for a massless vector field. We start from

$$Z[J] = N' \int \mathcal{D}A^\mu(x)\, e^{i\int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{(\partial_\mu A^\mu)^2}{2\xi} - \frac{i\varepsilon}{2} A_\mu A^\mu - J_\mu A^\mu\right)}$$

same prescription as in massive case

where the normalisation is fixed as usual by $Z[0] = 1$.

Proceeding as in the massive case, we get

$$Z[J] = e^{-\frac{1}{2}\int d^4x\, d^4y\, J_\mu(x)\, \Delta_F^{\mu\nu}(x-y)\, J_\nu(y)}$$

where the Feynman propagator $\Delta_F^{\mu\nu}(x-y)$ is now determined by

$$\left[(\partial^2 - i\varepsilon) g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu \partial_\nu\right] \Delta_F^{\nu\sigma}(x-y) = i\,\delta^{(4)}(x-y)\, g_\mu{}^\sigma$$

Notice that there is no problem with pure gauges anywhere since

$$\left[\partial^2 g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]\partial^\nu \Lambda(x) = \frac{1}{\xi}\,\partial_\mu \partial^2 \Lambda(x) \neq 0$$

The differential operator can now indeed be inverted, and gives (TPP1, page 222-224)

$$\Delta_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2+i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\varepsilon}\right]$$

which diverges in the limit $\xi \to \infty$, i.e. when we switch the gauge-fixing term off.

Check that this is a Green function of the differential operator

$$\int (d^4p)\, e^{-ip(x-y)}\, \frac{i}{p^2+i\varepsilon}\left[-p^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)p_\mu p_\nu\right]\left[-g^{\nu\sigma} + (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right]$$

$$= \int (d^4p)\, e^{-ip(x-y)}\, \frac{i}{p^2+i\varepsilon}\left[p^2 g_\mu{}^\sigma + p_\mu p^\sigma\left(\cancel{-1} + \cancel{\tfrac{1}{\xi}} \cancel{-1} + \cancel{\xi} \cancel{+1} \cancel{-\xi} \cancel{-\tfrac{1}{\xi}} \cancel{+1}\right)\right]$$

$$= \int (d^4p)\, e^{-ip(x-y)}\, i\, g_\mu{}^\sigma = i\,\delta^{(4)}(x-y)\, g_\mu{}^\sigma \quad \checkmark$$

Physical quantities are independent of the gauge parameter ξ. We are therefore free to choose a specific value for ξ in practical calculations. Convenient choices are

$\underline{\xi = 1}$: Feynman gauge $\quad \dfrac{-ig^{\mu\nu}}{p^2+i\varepsilon}$ $\quad$ (the simplest choice)

$\underline{\xi = 0}$: Landau gauge $\quad \dfrac{-i}{p^2+i\varepsilon}\left[g^{\mu\nu} - \dfrac{p^\mu p^\nu}{p^2} \right]$ $\quad$ (transverse)

Alternatively, one may work with arbitrary ξ and use the cancellation of ξ in physical observables as a powerful cross-check of the calculation.

As elementary QED processes have already been considered at tree level in the TPP1 lecture, we will not review them here. Instead we will focus on radiative corrections and the renormalisation of QED.

3.3. Ward - Takahashi identities

Gauge invariance implies certain exact relations between Green functions that are known as Ward - Takahashi identities. They can be viewed as the quantum generalisation of Noether's theorem, and they will become important when we discuss the renormalisation of QED.

The Ward - Takahashi identities can be derived using path integral methods. We start from the generating functional

$$Z[\eta, \bar{\eta}, J] = N \int \mathcal{D}\bar{\psi}(x)\, \mathcal{D}\psi(x)\, \mathcal{D}A^\mu(x)\; e^{\,i\int d^4x \left(\mathcal{L} - \frac{(\partial_\mu A^\mu)^2}{2\xi} + \bar{\eta}\psi + \bar{\psi}\eta + J_\mu A^\mu \right)}$$

with

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi$$

We now shift the integration variables to

$$\psi'(x) = \psi(x) + ie\,\omega(x)\,\psi(x)$$

$$\bar{\psi}'(x) = \bar{\psi}(x) - ie\,\omega(x)\,\bar{\psi}(x)$$

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \omega(x)$$

which is the infinitesimal version of a gauge transformation.

The path integral measure is invariant under this transformation (it is a constant shift for A^μ and a unitary transformation with $UU^\dagger = 1$ for the fermion field). As $\mathcal{L}$ is also invariant under this transformation, we pick up a factor

$$e^{i\int d^4x \left(-\frac{1}{\xi}(\partial_\mu A^\mu)\underbrace{\partial^2\omega}_{\text{p.I}} + ie\omega\bar{\eta}\psi - ie\omega\bar{\psi}\eta + \underbrace{J_\mu\partial^\mu\omega}_{\text{p.I}} + O(\omega^2)\right)}$$

$$= 1 + i\int d^4x \left[-\frac{1}{\xi}\partial^2(\partial_\mu A^\mu) + ie\bar{\eta}\psi - ie\bar{\psi}\eta - \partial^\mu J_\mu\right]\omega(x) + O(\omega^2)$$

under the path integral. The first term gives $Z[\eta,\bar{\eta},J]$, and so the second term must vanish

$$\int \mathcal{D}\bar{\psi}(x)\,\mathcal{D}\psi(x)\,\mathcal{D}A^\mu(x)\, e^{i\int d^4x\left(\mathcal{L} - \frac{(\partial_\mu A^\mu)^2}{2\xi} + \bar{\eta}\psi + \bar{\psi}\eta + J_\mu A^\mu\right)}$$

$$\times \int d^4x \left[-\frac{1}{\xi}\partial^2\partial_\mu A^\mu + ie\bar{\eta}\psi - ie\bar{\psi}\eta - \partial^\mu J_\mu\right]\omega(x) = 0$$

We next rewrite the fields by their functional derivatives

$$\psi(x) \to \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)} \qquad \bar{\psi}(x) \to -\frac{1}{i}\frac{\delta}{\delta\eta(x)} \qquad A^\mu(x) \to \frac{1}{i}\frac{\delta}{\delta J_\mu(x)}$$

and use the fact that $\omega(x)$ is an arbitrary function.

This yields the differential equation

$$i\frac{1}{\xi}\partial^2\partial_\nu \frac{\delta Z}{\delta J_\nu} + e\bar{\eta}\frac{\delta Z}{\delta \bar{\eta}} + e\frac{\delta Z}{\delta \eta}\eta - \partial_\nu J^\nu Z = 0$$

It is convenient to rewrite this equation in terms of $Z = e^{iW}$, where iW is the generating functional of

→ page 45, 46

connected Green functions. This gives

$$\boxed{\frac{1}{\xi}\partial^2\partial_\nu \frac{\delta W}{\delta J_\nu} - ie\bar{\eta}\frac{\delta W}{\delta \bar{\eta}} - ie\frac{\delta W}{\delta \eta}\eta + \partial_\nu J^\nu = 0}$$

which summarises an infinite number of relations between connected Green functions, the Ward-Takahashi identities (in the class of covariant gauges).

As an example, let us take the derivative with respect to $J_\mu(y)$ and set all sources to zero. This yields

$$\frac{1}{\xi}\partial^2_{(x)}\partial^{(x)}_\nu \underbrace{\left.\frac{\delta^2 W}{\delta J_\nu(x)\,\delta J_\mu(y)}\right|_{\eta=\bar{\eta}=J=0}}_{= i\langle\Omega|T A^\nu(x) A^\mu(y)|\Omega\rangle} + \partial^\mu_{(x)}\,\delta^{(4)}(x-y) = 0$$

(note that iW is the generating functional of connected GFs)

The connected two-point function is the __full__ photon propagator, which we write in the form

$$\langle \Omega | T A^\mu(x) A^\nu(y) | \Omega \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} D^{\mu\nu}(k)$$

Using

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}$$

the WT identity takes the form *

$$k_\mu D^{\mu\nu}(k) = -i \xi \frac{k^\nu}{k^2}$$

Decomposing

$$D^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu) D_T(k^2) + k^\mu k^\nu D_L(k^2)$$

we obtain

$$\boxed{D_L(k^2) = -i \frac{\xi}{k^4}}$$

Gauge invariance thus constraints the longitudinal part of the photon propagator to its leading order expression, which is not altered by higher order corrections.

Recall that for the free propagator

$$\tilde{\Delta}_F^{\mu\nu}(k) = \frac{i}{k^2}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$$

$$= \frac{i}{k^2}\left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right) - \xi \frac{k^\mu k^\nu}{k^2}\right] \rightsquigarrow D_L(k^2) = -i\frac{\xi}{k^4}$$

$$\frac{i}{\xi} k^2 k_\mu i D^{\mu\nu}(k) - ik^\nu = 0$$

Let us consider another example and differentiate with respect to $\frac{\delta}{\delta\bar{\eta}(y)}\,\frac{\delta}{\delta\eta(z)}$

$$\Rightarrow \frac{1}{\xi}\,\partial^2_{(x)}\,\partial^{(x)}_{\mu}\,\overbrace{\frac{\delta^3 W}{\delta\bar{\eta}(y)\,\delta\eta(z)\,\delta J_\mu(x)}}^{\langle\Omega|T\,\psi(y)\,\bar{\psi}(z)\,A^\mu(x)|\Omega\rangle}$$

$$+\,ie\,\delta^{(4)}(y-x)\,\underbrace{\frac{\delta^2 W}{\delta\eta(z)\,\delta\bar{\eta}(x)}}_{+i\langle\Omega|T\,\psi(x)\,\bar{\psi}(z)|\Omega\rangle}\,+\,ie\,\underbrace{\frac{\delta^2 W}{\delta\bar{\eta}(y)\,\delta\eta(x)}}_{-i\langle\Omega|T\,\psi(y)\,\bar{\psi}(x)|\Omega\rangle}\,\delta^{(4)}(z-x)\,\Bigg|_{\eta=\bar{\eta}=J=0} = 0$$

The connected two-point function is the full fermion propagator

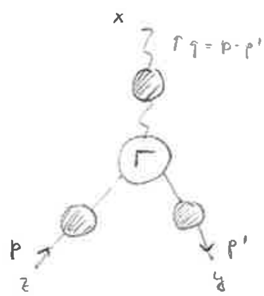
$$\langle\Omega|T\,\psi(x)\,\bar{\psi}(y)|\Omega\rangle = \int\frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\,S(p)$$ (we suppress spinor indices)

and the connected three-point function can be written as

$$\langle\Omega|T\,\psi(y)\,\bar{\psi}(z)\,A^\mu(x)\,|\Omega\rangle$$

$$= \int\frac{d^4p}{(2\pi)^4}\,\frac{d^4p'}{(2\pi)^4}\,e^{ip(z-x)}\,e^{-ip'(y-x)}\,S(p')\,ie\,\Gamma_\nu(p,p')\,S(p)\,D^{\nu\mu}(p-p')$$

where $\Gamma^\mu(p,p') = \gamma^\mu + \mathcal{O}(e^2)$ is the amputated three-point function.



We then obtain

$$\frac{1}{\xi}\underbrace{\partial^2_{(x)}}_{-q^2}\underbrace{\partial^{(x)}_\mu}_{-iq_\mu}\int\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\,e^{ip(z-x)}\,e^{-ip'(y-x)}\,S(p')\,ie\Gamma_\nu(p,p')\,S(p)\,D^{\nu\mu}(p-p')$$

$$-e\int\frac{d^4p'}{(2\pi)^4}e^{-ip'(y-x)}\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-z)}\,S(p)$$

$$+e\int\frac{d^4p'}{(2\pi)^4}e^{-ip'(y-x)}\,S(p')\int\frac{d^4p}{(2\pi)^4}e^{-ip(\cancel{x}-\cancel{z})}\quad 0$$

$$\rightarrow\ \frac{i}{\xi}q^2\,S(p')\,ie\Gamma_\nu(p,p')\,S(p)\,\underbrace{q_\mu D^{\nu\mu}(q)}_{-i\xi\frac{q^\nu}{q^2}} = e\,S(p) - e\,S(p')$$

$$\rightarrow\ i\,S(p')\,q^\nu\,\Gamma_\nu(p,p')\,S(p) = S(p) - S(p')$$

$$\rightarrow\ \boxed{q_\mu\Gamma^\mu(p,p') = i\left[\frac{1}{S(p)} - \frac{1}{S(p')}\right]}$$

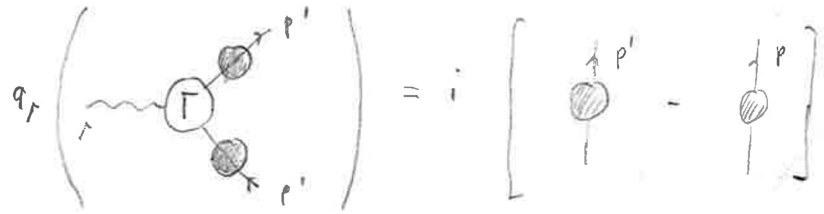
where $q = p - p'$. Gauge invariance thus provides an exact relation between the amputated vertex function and the full fermion propagators.

We can easily verify the WT identity at tree level

$$q_\mu\gamma^\mu = i\left[\frac{\not p - m}{i} - \frac{\not p' - m}{i}\right] = \not p - \not p' = \not q \quad ✓$$

We may rewrite the WT identity in the form

$$S(p')\, q_\mu \Gamma^\mu(p,p')\, S(p) = i\,[\,S(p') - S(p)\,]$$



We then extract S-matrix elements with external fermions from the double pole in $\frac{1}{\not p - m}\,\frac{1}{\not p' - m}$. The right-hand side, however, does not contribute in this case since each of the terms only has one of the poles. In other words

$$q_\mu \Gamma^\mu(p,p')\Big|_{p^2=p'^2=m^2} = 0$$

The discussion can be generalised to arbitrary Green functions. For an amplitude $M^\mu(k; p_1 \dots, p_n)$ that involves a photon with momentum k^μ, one finds

$$k_\mu M^\mu(k; p_1 \dots, p_n)\Big|_{p_i^2=m_i^2} = 0$$

and one again obtains non-vanishing contact terms on the right-hand side when some of the fermions are off-shell.

(see also the discussion in TPP1, page 323-324)

The result provides insight why the ξ-dependent terms in the photon propagator do not contribute to S-matrix elements. We argued before that the gauge boson couples to a conserved current, and so

$$\Big(-g^{\mu\nu} + (1-\xi) \underbrace{\frac{k^\mu k^\nu}{k^2} \Big) \tilde{j}_\nu(k)}_{=0}$$

Current conservation is, however, a classical argument since in the derivation of $\partial_\mu j^\mu = 0$, we have used the equations of motion. The WT identity provides the quantum generalisation of this statement.

3.4. Renormalisation

One can easily verify that the coupling constant e in QED is dimensionless (in 4 dimensions), and so we conclude that QED is a renormalisable theory. The renormalisation of QED proceeds along the same lines as for scalar theories, but there is one new aspect: gauge invariance. We saw in the last chapter that gauge invariance implies certain exact relations between Green functions, but what does this mean for the renormalisation program?

There is another aspect related with gauge invariance, namely that one needs to make sure that the chosen UV regulator does not spoil manifest gauge invariance. This happens e.g. in a cutoff regularisation scheme, in which radiative corrections generate a photon mass term (which we then need to renormalise to zero by hand). A cutoff regularisation scheme is therefore extremely unpleasant for gauge theories.

Dimensional regularisation (DR), on the other hand, obviously preserves manifest gauge invariance (which does not rely on the fact that the theory is formulated in 4 dimensions). The photon therefore automatically stays massless to all orders in perturbation theory in DR. The only drawback of DR is that the Dirac algebra needs to be formulated in $d \neq 4$ dimensions, which is however merely a calculational complication rather than a conceptual problem.

In d dimensions, we start from

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

where γ^μ are 4×4-dimensional matrices, and the indices μ,ν now run formally from 1 to d. This implies that contractions as

$$\gamma^\mu\gamma_\mu = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = g^\mu{}_\mu = d = 4-2\varepsilon$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = \gamma^\mu\{\gamma^\nu, \gamma_\mu\} - \gamma^\mu\gamma_\mu\gamma^\nu$$

$$= 2\gamma^\nu - d\gamma^\nu = (-2+2\varepsilon)\gamma^\nu$$

are modified, whereas traces of γ-matrices are not changed

$$\mathrm{Tr}(\gamma^\mu) = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4 g^{\mu\nu}$$

etc.

A difficulty arises in defining $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, which intrinsically is a four-dimensional object. As QED does not contain chiral interactions, we disregard the problem of defining γ_5 in DR for the moment.

We will now discuss QED in renormalised perturbation theory. First of all, we need to make sure that the renormalised coupling constant is dimensionless, and so we start with the usual dimensional analysis. As $[\mathcal{L}] = d$, we have

$$\mathcal{L} \sim \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \longrightarrow \quad [\psi] = \frac{d-1}{2}, \quad [m] = 1$$

$$\mathcal{L} \sim -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \longrightarrow [A] = \frac{d-2}{2}, \quad [\xi] = 0$$

$$\mathcal{L} \sim e\bar{\psi}A\!\!\!/\psi \quad \longrightarrow \quad [e] = d - (d-1) - \frac{d-2}{2} = 2 - \frac{d}{2} = \varepsilon$$

We then introduce renormalised parameters as follows

$$\boxed{\begin{array}{ll} \psi_0 = \sqrt{Z_\psi}\,\psi & A_0^\mu = \sqrt{Z_A}\,A^\mu \\ m_0 = Z_m\, m & \xi_0 = Z_\xi\,\xi \\ e_0 = \tilde{\mu}^\varepsilon Z_e\, e & \end{array}}$$

such that $[e] = 0$.

We now split the Lagrangian into $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$ with

$$\mathcal{L}_r = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \bar{\psi} (i\not{\partial} - m) \psi$$
$$+ e \tilde{\mu}^{\varepsilon} \bar{\psi} \not{A} \psi$$

$$\mathcal{L}_{ct} = -\frac{1}{4} (Z_A - 1) F_{\mu\nu} F^{\mu\nu} - \left(\frac{Z_A}{Z_\xi} - 1\right) \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + (Z_\psi - 1) \bar{\psi} i \not{\partial} \psi$$
$$- (Z_\psi Z_m - 1) m \bar{\psi} \psi + (Z_e Z_\psi \sqrt{Z_A} - 1) e \tilde{\mu}^{\varepsilon} \bar{\psi} \not{A} \psi$$

The Feynman rules in renormalised perturbation theory are

$$= \frac{i}{q^2 + i\varepsilon} \left[-g^{\mu\nu} + (1-\xi) \frac{q^\mu q^\nu}{q^2 + i\varepsilon} \right]$$

$$= \frac{i (\not{p} + m)_{\beta\alpha}}{p^2 - m^2 + i\varepsilon}$$

$$= i e \tilde{\mu}^{\varepsilon} \gamma^\mu_{\beta\alpha}$$

$$= (-i) \left[(Z_A - 1)(q^2 g^{\mu\nu} - q^\mu q^\nu) + \left(\frac{Z_A}{Z_\xi} - 1\right) \frac{1}{\xi} q^\mu q^\nu \right]$$

$$= i \left[(Z_\psi - 1) \not{p} - (Z_\psi Z_m - 1) m \right]_{\beta\alpha}$$

$$= i (Z_e Z_\psi \sqrt{Z_A} - 1) e \tilde{\mu}^{\varepsilon} \gamma^\mu_{\beta\alpha}$$

We further have to specify renormalisation conditions that define the renormalised parameters. In QED one often adopts the <u>on-shell scheme</u>, which gives rise to the following conditions:

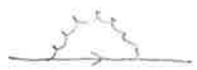
i) <u>Z_ψ, Z_m</u>

We have to make sure that the full fermion propagator

$$S(p) = \int d^4x \, e^{ipx} \langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle$$

has a one-particle pole at the physical mass m_p with residue 1. To do so, we introduce the 1PI two-point function

$-i\,\Sigma(p) \equiv$ 

$=$  $+$  $+ \; O(e^4)$

and we perform the usual resummation

$S(p) =$ 

$$= \frac{i}{\not{p} - m_p} + \frac{i}{\not{p} - m_p}\left(-i\,\Sigma(\not{p})\right)\frac{i}{\not{p} - m_p} + \dots$$

$$= \frac{i}{\not{p} - m_p} \, \frac{1}{1 - \frac{\Sigma(\not{p})}{\not{p} - m_p}} = \frac{i}{\not{p} - m_p - \Sigma(\not{p})}$$

The requirement that the full propagator has a pole at $p^2 = m_r^2$ thus implies

$$\boxed{\Sigma(\not p)\Big|_{\not p = m_r} = 0} \quad \longrightarrow \quad Z_m$$

In order to extract the residue, we expand

$$\frac{i}{\not p - m_r - \Sigma(\not p)} = \frac{i}{\underbrace{m_r - m_r - \Sigma(m_r)}_{=0} + \left(1 - \frac{\partial \Sigma}{\partial \not p}\right)_{\not p = m_r}(\not p - m_r) + \dots}$$

$$\stackrel{!}{=} \frac{i}{\not p - m_r} + \dots$$

$$\Rightarrow \quad \boxed{\frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m_r} = 0} \quad \longrightarrow \quad Z_2$$

[cf. page 108 for scalar field in renormalized perturbation theory]

ii) Z_A, Z_3

We proceed similarly with the full photon propagator

$$D^{\mu\nu}(k) = \int d^4x \, e^{ikx} \langle \Omega | T A^\mu(x) A^\nu(0) | \Omega \rangle$$

and we define

$i\Pi^{\mu\nu}(k) \equiv$ 

$=$  $+ \; \mathcal{O}(e^4)$

In order to disentangle the tensor structure, we decompose

$$\Pi^{\lambda\nu}(k) = (k^2 g^{\lambda\nu} - k^\lambda k^\nu)\,\Pi_T(k^2) + k^\lambda k^\nu\,\Pi_L(k^2)$$

and we introduce the projectors

$$P_T^{\lambda\nu}(k) = g^{\lambda\nu} - \frac{k^\lambda k^\nu}{k^2} \qquad\qquad P_T^{\lambda\nu}(k)\,P_{T\nu}{}^{\rho}(k) = P_T^{\lambda\rho}(k)$$

$$\longrightarrow \qquad P_L^{\lambda\nu}(k)\,P_{L\nu}{}^{\rho}(k) = P_L^{\lambda\rho}(k)$$

$$P_L^{\lambda\nu}(k) = \frac{k^\lambda k^\nu}{k^2} \qquad\qquad P_T^{\lambda\nu}(k)\,P_{L\nu}{}^{\rho}(k) = 0$$

We then obtain

$D^{\lambda\nu}(k) =$ 

$$= -\frac{i}{k^2}\left[P_T^{\lambda\nu} + \xi P_L^{\lambda\nu}\right] + \frac{-i}{k^2}\left[P_T^{\lambda\sigma} + \xi P_L^{\lambda\sigma}\right]$$

$$\times \underbrace{i k^2\left[P_{T\sigma\rho}\,\Pi_T + P_{L\sigma\rho}\,\Pi_L\right]\frac{-i}{k^2}\left[P_T^{\rho\nu} + \xi P_L^{\rho\nu}\right]}_{P_{T\sigma}{}^{\nu}\,\Pi_T \,+\, P_{L\sigma}{}^{\nu}\,\xi\,\Pi_L} + \dots$$

$$= \frac{-i}{k^2}\left[P_T^{\lambda\nu}\left(1 + \Pi_T + \Pi_T^2 + \dots\right) + \xi P_L^{\lambda\nu}\left(1 + \xi\Pi_L + \xi^2\Pi_L^2 + \dots\right)\right]$$

$$= \frac{-i}{k^2}\left[P_T^{\lambda\nu}\,\frac{1}{1-\Pi_T(k^2)} + P_L^{\lambda\nu}\,\frac{\xi}{1-\xi\,\Pi_L(k^2)}\right]$$

which has a pole at $k^2 = 0$ (as long as $\Pi_{T/L}(0)$ are finite).

We need to make sure that the residue is trivial at $k^2=0$

$$-\frac{i}{k^2}\left[P_T^{\mu\nu} \frac{1}{1-\Pi_T(0)} + P_L^{\mu\nu} \frac{\mathfrak{Z}}{1-\mathfrak{Z}\,\Pi_L(0)} \right] \overset{!}{=} \frac{-i}{k^2}\left[P_T^{\mu\nu} + \mathfrak{Z}\, P_L^{\mu\nu} \right]$$

$$\Rightarrow \quad \boxed{\Pi_T(k^2=0) = 0} \quad \rightarrow Z_A$$

$$\Pi_L(k^2=0) = 0$$

In the last section we found, however, that gauge invariance completely determines the longitudinal part of the photon propagator. In terms of the bare parameters, we found

$$D_0^{\mu\nu}(k) = P_T^{\mu\nu} \cdots + P_L^{\mu\nu} \frac{(-i)\,\mathfrak{Z}_0}{k^2}$$

where $D_0 \sim \langle A_0 A_0 \rangle \sim Z_A \langle A A \rangle \sim Z_A D$. The longitudinal part of the full propagator thus fulfills

$$\frac{1}{Z_A} \frac{(-i)}{k^2} \underbrace{Z_{\mathfrak{Z}}\, \mathfrak{Z}}_{\mathfrak{Z}_0} = \frac{-i}{k^2} \frac{\mathfrak{Z}}{1-\mathfrak{Z}\,\Pi_L(k^2)} \longrightarrow \frac{Z_{\mathfrak{Z}}}{Z_A} = \frac{1}{1-\mathfrak{Z}\,\Pi_L(k^2)}$$

As the left-hand side is independent of k^2, $\Pi_L(k^2)$ must be constant with

$$\Pi_L(k^2) = \Pi_L(k^2=0) = 0$$

$$\Rightarrow \quad \boxed{Z_{\mathfrak{Z}} = Z_A}$$

which is a consequence of gauge invariance.

iii) $\underline{Z_e}$

We finally need to define the renormalized coupling e.
To this end, we typically consider a scattering matrix element at some reference scale, which is however equivalent to putting a constraint on the computed vertex function $\Gamma^\mu(p,p')$. The vertex function for on-shell fermions ($p^2 = p'^2 = m^2$) and arbitrary $q^\mu = p^\mu - p'^\mu$ has the following general decomposition (→ cf. problem sheet)

$$\boxed{\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m} F_2(q^2)}$$

in terms of two scalar form factors with

$$F_1(q^2) = 1 + \mathcal{O}(e^2)$$
$$F_2(q^2) = \mathcal{O}(e^2)$$

In order to connect to classical electrodynamics, we now <u>define</u> the electromagnetic charge as the strength of the electron-photon interaction at vanishing momentum transfer.
We thus demand

$$ie\,\Gamma^\mu(p,p') \overset{q^\mu \to 0}{\equiv} ie\gamma^\mu \quad \longrightarrow \quad F_1(q^2=0) = 1$$

to all orders in perturbation theory.

(there is no constraint on $F_2(0)$ since the prefactor automatically vanishes).

In the last section we derived an exact relation between the amputated vertex function and the full fermion propagator

$$q_\mu \Gamma_0^\mu(p,p') = i \left[\frac{1}{S_0(p)} - \frac{1}{S_0(p')} \right]$$

where the index "0" reminds us that we were dealing with bare parameters and Green functions in the last section. In the limit $q^\mu = p^\mu - p'^\mu \to 0$, this yields

$$q_\mu \Gamma_0^\mu(p,p') = i \left[\frac{1}{S_0(p)} - \left(\frac{1}{S_0(p)} + \frac{\partial}{\partial p'_\mu} \frac{1}{S_0(p')} \Big|_{p'=p} (p'-p)_\mu + \dots \right) \right]$$

$$= i \frac{\partial}{\partial p'_\mu} \frac{1}{S_0(p')} \Big|_{p'=p} q_\mu + \dots$$

In terms of the renormalised quantities, we thus obtain

$$\frac{1}{Z_e Z_\psi \sqrt{Z_A}} \Gamma^\mu(p,p') = \frac{1}{Z_\psi} \frac{\partial}{\partial p'_\mu} \frac{i}{S(p')} \Big|_{p'=p}$$

$$\overset{p^2=m^2}{\simeq} \frac{1}{Z_\psi} \frac{\partial}{\partial p'_\mu} (\not{p}'-m) \Big|_{p'=p}$$

$$= \frac{1}{Z_\psi} \gamma^\mu$$

$S_0 \sim \langle \psi_0 \bar\psi_0 \rangle \sim Z_\psi \langle \psi \bar\psi \rangle \sim Z_\psi S$

$\langle \psi_0 \bar\psi_0 A_0 \rangle \sim e_0 S_0 \Gamma_0 S_0 D_0$

$\sim Z_e Z_\psi^2 Z_A \, e S \Gamma_0 S D$

$\sim Z_\psi \sqrt{Z_A} \langle \psi \bar\psi A \rangle \sim Z_\psi \sqrt{Z_A} \, e S \Gamma S D$

$\to \Gamma \sim Z_e Z_\psi \sqrt{Z_A} \, \Gamma_0$

The renormalisation condition $\Gamma^\mu(p,p') \overset{q^\mu \to 0}{\equiv} \gamma^\mu$ thus implies

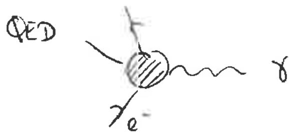
$$Z_e \sqrt{Z_A} = 1 \quad \longrightarrow \quad \boxed{Z_e = \frac{1}{\sqrt{Z_A}}}$$

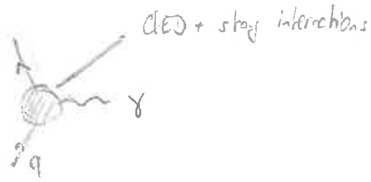
which is again fixed by gauge invariance.

We have specified all renormalisation conditions in the on-shell scheme. In particular, we found that the renormalisation constants associated with the photon are fixed by gauge invariance

$$Z_A = \frac{1}{Z_e^2} \qquad Z_3 = Z_A = \frac{1}{Z_e^2}$$

This implies that the renormalisation of the electromagnetic charge is intimately related to the photon field, and in particular <u>independent</u> of the fermion species. This provides a clue to understanding why the electron and proton charges seem to be <u>exactly</u> opposite, despite the fact that the proton receives radiative corrections from strong interactions





If for some reason the <u>bare</u> electron and proton charges are exactly opposite, this will not change under renormalisation since gauge invariance protects the electromagnetic current to be renormalised under strong interactions ("the vector current is conserved").

3.5 Quantum effects

Having discussed the formal aspects of the renormalisation program, we will now discuss two interesting physical effects that arise at the quantum level: the anomalous magnetic moment and the screening of the electromagnetic charge due to vacuum fluctuations.

Anomalous magnetic moment

On page 165 we introduced two form factors $F_1(q^2)$ and $F_2(q^2)$, which contain the complete information about the electron's response to an electromagnetic field. We have already seen that $F_1(0)$ is related to the charge of an electron, and we will now show that $F_2(0)$ contributes to its magnetic moment.

We consider an electron in an external magnetic field, which we assume to be time-independent. The external field will be treated as a classical background field.

It is of the form

$$A^{\mu}_{cl}(x) = (0, \vec{A}_{cl}(\vec{x}))$$

$$\rightarrow \quad B^{i}_{cl}(\vec{x}) = \varepsilon^{ijk}\, \partial^{j} A^{k}_{cl}(\vec{x})$$

or $$\tilde{B}^{i}_{cl}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\vec{x}}\, \varepsilon^{ijk}\, \partial^{j} A^{k}_{cl}(\vec{x})$$

$$\overset{p.I.}{=} + i\, \varepsilon^{ijk}\, q^{j}\, \tilde{A}^{k}_{cl}(\vec{q})$$

The coupling of the electron to the external field reads

$$i\mathcal{A} = ie\, \bar{u}(p', s') \left[\gamma^{\mu} F_1(q^2) + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m} F_2(q^2) \right] u(p,s)\, \tilde{A}_{cl,\mu}(q)$$

Using the Gordon identity (→ problem sheet)

$$\bar{u}(p', s')\, \gamma^{\mu}\, u(p,s) = \bar{u}(p', s') \left[\frac{p^{\mu} + p'^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m} \right] u(p,s)$$

this can be cast into the form

$$i\mathcal{A} = -\, ie\, \bar{u}(p', s') \left[\frac{p^{i} + p'^{i}}{2m} F_1(q^2) \right.$$

from raising the space-like index i

$$\left. + \frac{i\sigma^{i\nu}(p'_\nu - p_\nu)}{2m} \left(F_1(q^2) + F_2(q^2) \right) \right] u(p,s)\, \tilde{A}^{i}_{cl}(\vec{q})$$

For $\vec{q} \ll m$ the external field varies only slowly compared to the Compton wavelength of the electron. We moreover assume that the electron moves non-relativistically, $\vec{p}, \vec{p}' \ll m$, and approximate

rest frame

$$u(p,s) \simeq u(\cancel{k}, s) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}$$

where $\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

we use the Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and the non-relativistic normalisation of the particle states (→ next page)

We thus obtain

$$\bar{u}(p',s')\, u(p,s) \simeq (\chi_{s'}^+, 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = \chi_{s'}^+ \chi_s$$

$$[\sigma^i, \sigma^j] = 2 i \varepsilon^{ijk} \sigma^k$$

$$\bar{u}(p',s')\, \sigma^{0i} u(p,s) \simeq (\chi_{s'}^+, 0)\, \frac{i}{2} \begin{pmatrix} 0 & 2\sigma^i \\ -2\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = 0$$

$$\bar{u}(p',s')\, \sigma^{ij} u(p,s) \simeq (\chi_{s'}^+, 0)\, \frac{i}{2} \begin{pmatrix} -\sigma^i\sigma^j + \sigma^j\sigma^i & 0 \\ 0 & +\sigma^i\sigma^j - \sigma^j\sigma^i \end{pmatrix} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = \chi_{s'}^+ \varepsilon^{ijk} \sigma^k \chi_s$$

It follows

$$i\mathcal{M} = -ie\, \chi_{s'}^+ \Big[\underbrace{\frac{p^i + p'^i}{2m}}_{\text{spin-independent}} F_1(0) + \underbrace{\frac{i \varepsilon^{ijk} \sigma^k (p'^j - p^j)}{2m} (F_1(0) + F_2(0))}_{\text{spin-dependent}} \Big] \chi_s \tilde{A}^i_{cl}(\vec{q})$$

from raising j

Dirac representation [TPP1, tutorials]

$$\gamma^0_{Dirac} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i_{Dirac} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma'_{Dirac} = U \gamma'_{chiral} U^+ \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

free Dirac equation

$$(i\not{\partial} - m)\psi = 0$$

rest frame:

- chiral representation (TPP1, p. 182)

$$u(k, \tfrac{1}{2}) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u(k, -\tfrac{1}{2}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v(k, \tfrac{1}{2}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad v(k, -\tfrac{1}{2}) = \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- Dirac representation

$$U(i\underbrace{\not{\partial}}_{\text{chiral}} - m) U^+ U \psi = (i\underbrace{\not{\partial}'}_{\text{Dirac}} - m)\psi' = 0 \quad \rightarrow \quad \psi' = U\psi$$

$$u'(k, \tfrac{1}{2}) = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u'(k, -\tfrac{1}{2}) = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v(k, \tfrac{1}{2}) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad v(k, -\tfrac{1}{2}) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

In the following, assume non-relativistic normalization of states

$$\hookrightarrow \quad u'(k, s) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \qquad \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In terms of the magnetic field $\tilde{B}^i_{cl}(\vec{q}) = i\varepsilon^{ijk} q^j \tilde{A}^k_{cl}(\vec{q})$,

the spin-dependent term becomes $\qquad \vec{q} = \vec{p} - \vec{p}'$

$$i\mathcal{A}_{spin} = \frac{ie}{2m} \chi^\dagger_{s'} \frac{\sigma^i}{2} \tilde{B}^i_{cl}(\vec{q}) \chi_s 2(\underbrace{F_1(0)}_{=1 \text{ (by the definition of the el. charge)}} + F_2(0))$$

This can be compared to the matrix element

$$\langle p', s' | -iH_{int} | p, s \rangle$$

of a Hamiltonian $H_{int} = -\vec{\mu} \cdot \vec{B}$ with magnetic moment

$$\vec{\mu} = g \frac{e}{2m} \frac{\vec{\sigma}}{2}$$

where $\boxed{g = 2(1 + F_2(0))}$ is the Landé g-factor. As

$F_2(0) = \mathcal{O}(e^2)$, we recover Dirac's prediction $g = 2$ at

tree level. Radiative corrections generate, however, a deviation

encoded in the <u>anomalous magnetic moment</u>

$$a \equiv \frac{g-2}{2} = F_2(0) = \mathcal{O}(e^2)$$

In the 1940s experimental measurements showed a discrepancy from Dirac's prediction, which could later be explained by the one-loop QED correction. The famous Schwinger calculation will be part of the problem sheets.

The anomalous magnetic moment of the electron has to date been measured to a truly impressive precision

$$a_e\big|_{exp} = 0.001\ 159\ 652\ 180\ 73(28) \qquad \text{[Gabrielse et al, 0801.1134]}$$

and is currently being used to extract the fine structure constant $\alpha = \frac{e^2}{4\pi}$.

The anomalous magnetic moment of the muon has also been measured precisely

$$a_\mu\big|_{exp} = 0.001\ 165\ 920\ 91\ (63) \qquad \text{[PDG 2013]}$$

and provides a high precision test of the Standard Model. At higher orders, it receives corrections from

- QED  known to 5 loops!
- electroweak interactions   known to 2 loops
- QCD   complicated since strong interactions are non-perturbative for $k^2 \sim m_\mu$

The latest theoretical prediction

$$a_\mu\big|_{theo} = 0.001\ 165\ 918\ 03(49) \qquad \text{[PDG 2013]}$$

has a similar uncertainty, but is about $3.6\,\sigma$ below the experimental value.

New heavy particle beyond the SM could produce such a deviation. They generically give a contribution of order

$$\frac{\alpha}{4\pi}\,\frac{m_e^2}{M^2}$$

$\bar\Psi\sigma^{\mu\nu}\Psi$ is a dim-5 operator, but one needs in addition a helicity flip

Assuming $\alpha \simeq 1$, the current experimental uncertainties translate into a sensitivity of ~ 150 GeV (a_e) !

~ 700 GeV (a_μ)

Screening of the electromagnetic charge

We will now compute the one-loop corrections to the 1PI part of the photon two-point function, the so-called <u>vacuum polarisation</u>

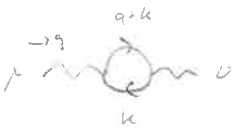
$$i\,\Pi^{\mu\nu}(q) = i\,(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)$$

=  $+\ O(e^4)$

and we recall that the longitudinal part $\Pi_L(q^2) = 0$ as a consequence of gauge invariance.

We anticipate that the vacuum polarisation is UV-divergent (and IR-finite), and we will apply DR with $d = 4-2\varepsilon$ in the following.

A straight-forward application of the QED Feynman rules gives



Fermion loop

$$= (-1) \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[ie\tilde{\mu}^{\varepsilon} \gamma^{\nu}\, i(\not{q}+\not{k}+m)\, ie\tilde{\mu}^{\varepsilon}\gamma^{\mu}\, i(\not{k}+m)\right]}{[(q+k)^2-m^2]\,[k^2-m^2]}$$

where we suppressed the $i\varepsilon$-prescription, which can be restored by $m^2 \to m^2 - i\varepsilon$.

We first combine the propagators with a Feynman parameter

$$= -e^2\tilde{\mu}^{2\varepsilon} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^{\nu}(\not{k}+\not{q}+m)\gamma^{\mu}(\not{k}+m)\right]}{\left[k^2+2xkq+xq^2-m^2\right]^2}$$

We next complete the square in the denominator via $k^{\mu} \to k^{\mu} - xq^{\mu}$

$$= -e^2\tilde{\mu}^{2\varepsilon} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^{\nu}(\not{k}+\bar{x}\not{q}+m)\gamma^{\mu}(\not{k}-x\not{q}+m)\right]}{(k^2-\Delta)^2}$$

with $\Delta = m^2 - x\bar{x}q^2$ and $\bar{x} = 1-x$.

We can use d-dimensional rotational invariance to simplify the tensor integrals

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2-\Delta)^n} = 0$$

in 3 dimensions: $\int d^3x \, x_i f(r) = 0$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2-\Delta]^n} = \int \frac{d^dk}{(2\pi)^d} \frac{\frac{1}{d} k^2 g^{\mu\nu}}{[k^2-\Delta]^n}$$

$\sim g^{\mu\nu}$ since integral vanishes for $\mu \neq \nu$. Contract with $g_{\mu\nu}$ to get coefficient.

We are thus left with

$$= -e^2 \tilde{\mu}^{2\varepsilon} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\frac{1}{d} k^2 \, \mathrm{Tr}[\gamma^\nu \gamma^\rho \gamma^\mu \gamma_\rho] - \mathrm{Tr}[\gamma^\nu (\bar{x}\not{q}+m)\gamma^\mu (x\not{q}-m)]}{[k^2-\Delta]^2}$$

We need to evaluate the traces in d dimensions (→ tutorials)

$$\mathrm{Tr}(\gamma^\nu \gamma^\rho \gamma^\mu \gamma_\rho) = (2-d)\, \mathrm{Tr}(\gamma^\nu \gamma^\mu) = 4(2-d)\, g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\nu (\bar{x}\not{q}+m) \gamma^\mu (x\not{q}-m)]$$

$$= 4x\bar{x}\,[q^\nu q^\mu - q^2 g^{\mu\nu} + q^\nu q^\mu] - 4m^2 g^{\mu\nu}$$

$$= 8x\bar{x}\, q^\mu q^\nu - 4(x\bar{x} q^2 + m^2)\, g^{\mu\nu}$$

We recall the master formula for one-loop integrals (→ page 94)

$$A(n,\Delta) \equiv \tilde{\mu}^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-\Delta)^n}$$

$$= \frac{i}{16\pi^2} (\mu^2 e^\gamma)^\varepsilon \frac{(-1)^n}{\Gamma(n)} \Gamma(n-d/2)\, \Delta^{\frac{d}{2}-n}$$

Here we in addition need

$$B(n,\Delta) \equiv \tilde{\mu}^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^n} = \tilde{\mu}^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{k^2-\Delta+\Delta}{(k^2-\Delta)^n}$$

$$= A(n-1,\Delta) + \Delta\, A(n,\Delta) \qquad x\Gamma(x) = \Gamma(1+x)$$

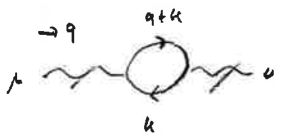
$$= \frac{i}{16\pi^2} (\mu^2 e^\gamma)^\varepsilon \frac{(-1)^{n-1}}{\Gamma(n)} \frac{d}{2} \Gamma(n-1-d/2)\, \Delta^{\frac{d}{2}-n+1}$$

Putting everything together, we arrive at

$$= -\frac{2i\alpha}{\pi} (q^2 g^{\lambda\nu} - q^\lambda q^\nu)\, \Gamma(\varepsilon) \int_0^1 dx\, x\bar{x} \left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{\mu^2 e^\gamma} \right)^{-\varepsilon}$$

where we introduced the fine structure constant $\alpha = \frac{e^2}{4\pi}$

and we restored the $i\varepsilon$-prescription.



$$= (-1)\, e^2 \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left(\gamma^\nu(\not q+\not k+m)\gamma^\mu(\not k+m)\right)}{[(q+k)^2-m^2+i\varepsilon](k^2-m^2+i\varepsilon)}$$

$$= -e^2\mu^{2\varepsilon}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left(\gamma^\nu(\not k+\not q+m)\gamma^\mu(\not k+m)\right)}{[k^2+2xkq+xq^2-m^2+i\varepsilon]^2} \qquad k=k'-xq$$

$$= -e^2\mu^{2\varepsilon}\int_0^1 dx\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left(\gamma^\nu(\not k+\bar x\not q+m)\gamma^\mu(\not k - x\not q+m)\right)}{[k^2-\Delta]^2} \qquad \Delta=-x\bar x q^2+m^2-i\varepsilon$$

$$= -e^2\mu^{2\varepsilon}\int_0^1 dx\int \frac{d^dk}{(2\pi)^d}\frac{\frac{1}{d}k^2\, \mathrm{Tr}(\gamma^\nu\gamma^\lambda\gamma^\mu\gamma_\lambda) + \mathrm{Tr}\left(\gamma^\nu(\bar x\not q+m)\gamma^\mu(-x\not q+m)\right)}{(k^2-\Delta)^2}$$

$$= -4e^2\mu^{2\varepsilon}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{\frac{2-d}{d}k^2 g^{\mu\nu} + 2x\bar x\, q^\mu q^\nu + (-x\bar x q^2 - m^2)\, g^{\mu\nu}}{[k^2-\Delta]^2}$$

$$= -4e^2\mu^{2\varepsilon}\frac{i}{(4\pi)^{d/2}}\int_0^1 dx \left[\underbrace{\frac{d-2}{2}\Gamma(-1+\varepsilon)}_{-\Gamma(\varepsilon)}(\cancel{m^2}-x\bar x q^2)\,g^{\mu\nu} + \left(2x\bar x\, q^\mu q^\nu+(-x\bar x q^2 - \cancel{m^2})\,g^{\mu\nu}\right)\Gamma(\varepsilon)\right]\Delta^{-\varepsilon}$$

$$= -\frac{8ie^2}{(4\pi)^{d/2}}\,\Gamma(\varepsilon)\,\mu^{2\varepsilon}\,(q^2g^{\mu\nu}-q^\mu q^\nu)\int_0^1 dx\; x\bar x\; \Delta^{-\varepsilon}$$

As the integral is finite, we may expand the integrand

$$= -\frac{2i\alpha}{\pi}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\left(\frac{1}{\varepsilon} - \gamma + O(\varepsilon)\right)$$

$$\int_0^1 dx\; x\bar{x}\left(1 - \varepsilon \ln\left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{\mu^2 e^\gamma}\right) + O(\varepsilon^2)\right)$$

$$= -\frac{2i\alpha}{\pi}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$\left[\frac{1}{6\varepsilon} - \int_0^1 dx\; x\bar{x}\,\ln\left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{\mu^2}\right) + O(\varepsilon)\right]$$

The UV-divergence will be absorbed by the counterterm

(page 160 with $Z_3 = Z_A$)

$$= -i\,(Z_A - 1)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

We thus obtain

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\left[\frac{1}{6\varepsilon} - \int_0^1 dx\; x\bar{x}\,\ln\left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{\mu^2}\right)\right] - (Z_A - 1)$$

In the on-shell scheme, we demand $\Pi(q^2 = 0) = 0$ (→ page 164)

$$\rightarrow \Pi(0) = -\frac{2\alpha}{\pi}\left[\frac{1}{6\varepsilon} - \int_0^1 dx\; x\bar{x}\,\ln\left(\frac{m^2}{\mu^2}\right)\right] - (Z_A^{os} - 1) \overset{!}{=} 0$$

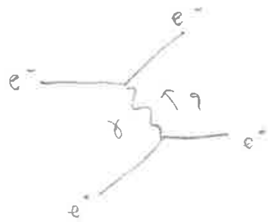
$$\rightarrow Z_A^{os} = 1 - \frac{2\alpha}{\pi}\left[\frac{1}{6\varepsilon} - \int_0^1 dx\; x\bar{x}\,\ln\frac{m^2}{\mu^2}\right] + O(\alpha^2)$$

The vacuum polarization is thus finite as $\varepsilon \to 0$

$$\Pi^{os}(q^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x\bar{x}\, \ln\left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{m^2}\right) + O(\alpha^2)$$

and it is obviously renormalization-scheme dependent (which is not a problem since it is not a physical observable). In a physical scheme like the on-shell scheme, it has however physical implications which we are going to address in the following.

Consider elastic electron-positron scattering (→ TPP1, page 347-349)



$$i\mathcal{A} \sim \frac{e^2}{q^2}$$

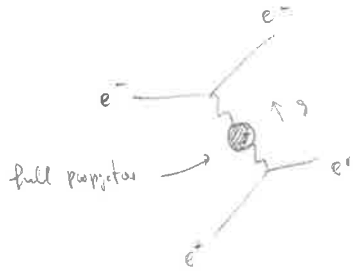
In the non-relativistic limit $\vec{q} \ll m$, this corresponds to a static Coulomb interaction, which can be verified by taking the Fourier transform

$$V(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\vec{x}}\, \frac{e^2}{-\vec{q}^2}$$

$q^2 \simeq -\vec{q}^2$ since $q^0 = O(mv^2)$ and $\vec{q} = O(mv)$ with $v \ll 1$

$$= -\frac{4\pi\alpha}{(2\pi)^3}\int_0^\infty dq \int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta} \int d\varphi = -\frac{\alpha}{r}$$

The quantum fluctuations encoded in the vacuum polarisation modify this picture according to



Dyson resummation from page 163

$$iA \sim \frac{e^2}{q^2}\,\frac{1}{1-\Pi^{os}(q^2)} \equiv \frac{e^2_{eff}(q^2)}{q^2}$$

where we absorbed the vacuum polarisation into an <u>effective q^2-dependent coupling constant.</u> As $\Pi(-\vec{q}^2) > 0$ is a monotonically increasing function with $\vec{q}^2$, the effective charge increases as $\vec{q}^2$ increases.

Intuitively, this can be understood as follows

The electron polarises the vacuum which acts as an dielectric medium, and the virtual pairs of charged particles <u>screen</u> the charge of that electron.

As $\vec{q}^2$ increases the photon probes more and more deeply into the "vacuum polarisation cloud" that surrounds the electron and the effective charge increases.

The effect has been verified experimentally. In the opposite high-energy limit $-q^2 \gg m^2$, we obtain

$$\Pi^{(2)}(q^2) \simeq \frac{2\alpha}{\pi} \int_0^1 dx\, x\bar{x}\, \ln\left(\frac{-x\bar{x}q^2 - i\varepsilon}{m^2}\right)$$

$$= \frac{\alpha}{3\pi}\left[\ln\frac{-q^2-i\varepsilon}{m^2} - \frac{5}{3}\right]$$

The effective coupling constant then becomes

$$\alpha_{eff}(q^2) = \frac{\alpha}{1-\Pi^{(2)}(q^2)}$$

$$\simeq \frac{\alpha}{1-\frac{\alpha}{3\pi}\ln\frac{-q^2-i\varepsilon}{cm^2}} \qquad \text{with } c = e^{5/3}$$

When the effect of other charged particles (muon, tau, quarks) is taken into account, this increases from $\alpha_{eff}(q^2 \simeq 0) = \alpha \simeq \frac{1}{137}$ to $\alpha_{eff}(M_Z^2) \simeq \frac{1}{128}$.

We already encountered the concept of a running coupling constant.

In the $\overline{MS}$ scheme, we have

$$\bar{Z}_A = 1 - \frac{\bar{\alpha}}{3\pi\varepsilon} + O(\bar{\alpha}^2)$$

$$\rightarrow \quad \bar{Z}_\alpha = \bar{Z}_e^2 = \frac{1}{\bar{Z}_A} = 1 + \frac{\bar{\alpha}}{3\pi\varepsilon} + O(\bar{\alpha}^2)$$

Ward-identities also hold in the $\overline{MS}$ scheme

Repeating the analysis of ϕ^4-theory, cf. page 121, we now obtain

$$\beta_0 = \frac{4}{3}$$

$$\bar{\alpha}(\mu) = \frac{\bar{\alpha}(\mu_0)}{1 - \frac{\bar{\alpha}(\mu_0)}{3\pi}\ln\frac{\mu^2}{\mu_0^2}}$$

which is of a similar form as the expression that we found for $\alpha_{eff}(q^2)$

$\rightarrow$ the scale-dependence of the $\overline{MS}$ parameters reflects a momentum dependence of a <u>physical</u> quantity!