3. Free fields

QFTs are formulated in terms of field operators, which are linear combinations of creation and annihilation operators in position space.

One may wonder why fields are regarded as the fundamental objects in a relativistic quantum theory whereas particles arise from the quantisation of these fields.

Perhaps the best answer to this question is that QFT solves a fundamental conflict between the uncertainty principle and causality. In special relativity we learned that events which are separated by a space-like distance

$$(x-y)^2 = (x^0-y^0)^2 - (\vec{x}-\vec{y})^2 < 0$$

cannot influence each other. But the uncertainty principle in a quantum theory implies, on the other hand, that (free) particles cannot be localised and their wave functions are non-zero everywhere and so they overlap. But what then prevents information from leaking out of the light cone?

We will see that this conflict is solved in QFT by a subtle interference effect between two amplitudes, which describe the propagation of a particle across a space-like distance and an antiparticle that propagates into the opposite direction.

In order to understand this, we will first introduce the basic concepts of QFT by considering the simplest example, which is a theory of free spin-0 particles. We will then learn that field operators themselves have non-trivial transformation properties under Lorentz-transformations, and we will study the two most important examples with non-zero spin – the Dirac field for spin-$\frac{1}{2}$ and the vector field for spin-1 particles – in detail. But before doing so, we first review some elements of classical field theory.

3.1. Classical field theory

In classical mechanics a system of N pointlike particles is described by a Lagrangian $L(q_n, \dot{q}_n)$, which is a function of generalized coordinates q_n and generalized velocities $\dot{q}_n$ (we focus here on systems without explicit time dependence).

The equations of motion can then be derived via a variational principle. To this end, one defines the action

$$S[q_n] = \int_{t_1}^{t_2} dt \; L(q_n, \dot{q}_n)$$

which is a functional of the q_n, i.e. it associates to the functions $q_n(t)$ a number $S[q_n]$. The principle of least action then tells us that the action is stationary for the physical trajectories, i.e.

$$\delta S[q_n] = S[q_n + \delta q_n] - S[q_n] \overset{!}{=} 0$$

where $\delta q_n(t)$ are small variations around $q_n(t)$, which vanish at the endpoints, $\delta q_n(t_1) = \delta q_n(t_2) = 0$.

It follows

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_n} \delta q_n + \frac{\partial L}{\partial \dot q_n} \delta \dot q_n \right)$$

implicit sum over $n = 1, \dots, N$

$\delta \dot q_n = \frac{d}{dt} \delta q_n$

$$= \underbrace{\frac{\partial L}{\partial \dot q_n} \delta q_n \Big|_{t_1}^{t_2}}_{=0} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot q_n} \right) \delta q_n \overset{!}{=} 0$$

and since the variations δq_n are arbitrary, we obtain the

<u>Euler-Lagrange equations</u>

$$\frac{d}{dt} \frac{\partial L}{\partial \dot q_n} - \frac{\partial L}{\partial q_n} = 0$$

One further introduces generalized (or conjugate) momenta

$$p_n \equiv \frac{\partial L}{\partial \dot q_n}$$

and defines the <u>Hamiltonian</u> via a Legendre transformation

$$H(q_n, p_n) = \dot q_n p_n - L(q_n, \dot q_n)$$

In terms of the Poisson bracket

$$[A, B]_P = \frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q_n}$$

the canonical coordinates satisfy

$$[q_k, q_\ell]_P = [p_k, p_\ell]_P = 0$$

$$[q_k, p_\ell]_P = \delta_{k\ell}$$

In the canonical formulation, the equations of motion then take the form

$$\dot{q}_n = [q_n, H]_P = \frac{\partial H}{\partial p_n}$$

$$\dot{p}_n = [p_n, H]_P = -\frac{\partial H}{\partial q_n}$$

In a quantum theory the canonical coordinates are promoted to operators on the Hilbert space and the Poisson brackets are substituted by commutators

$$[A, B]_P \longrightarrow \frac{1}{i}[A, B]$$

The fundamental commutation relations then become

$$[q_k, q_\ell] = [p_k, p_\ell] = 0$$

$$[q_k, p_\ell] = i\delta_{k\ell}$$

whereas

$$\dot{q}_n = \frac{1}{i}[q_n, H]$$

$$\dot{p}_n = \frac{1}{i}[p_n, H]$$

are the Heisenberg equations for the operators q_n and p_n.

We next want to extend this formalism to field theory. We thus consider N fields with amplitudes $\phi_n(t,\vec{x})$, which for each point $\vec{x}$ can be considered as a set of independent generalised coordinates. If we imagine space to be discretised on a lattice, we can thus use the results from above, and the transition to field theory then simply corresponds to taking the continuum limit

$$q_n(t) \longrightarrow \phi_{n,\vec{x}}(t) \longrightarrow \phi_n(t,\vec{x}) = \phi_n(x)$$

$$\sum_n \longrightarrow \sum_n \sum_{\vec{x}} \longrightarrow \sum_n \int d^3x$$

$$\delta_{nm} \longrightarrow \delta_{nm}\,\delta_{\vec{x}\vec{y}} \longrightarrow \delta_{nm}\,\delta^{(3)}(\vec{x}-\vec{y})$$

In a local field theory the Lagrangian furthermore only depends on products of fields at the same point $\vec{x}$. We can therefore introduce a Lagrangian density $\mathcal{L}$ by

$$L(q_n,\dot{q}_n) \longrightarrow L(\phi_n,\dot{\phi}_n) = \int d^3x \; \mathcal{L}(\phi_n, \underbrace{\dot{\phi}_n, \vec{\nabla}\phi_n}_{\partial_\mu\phi_n})$$

In the following we almost always assume that $t_1 = -\infty$ and $t_2 = +\infty$ such that

$$S[\phi_n] = \underbrace{\int_{-\infty}^{\infty} dt \int d^3x}_{\int d^4x} \; \mathcal{L}(\phi_n, \partial_\mu\phi_n)$$

The principle of least action then implies

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_n} \delta\phi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \underbrace{\delta(\partial_\mu \phi_n)}_{= \partial_\mu \delta\phi_n} \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right) \right) \delta\phi_n$$

where we assumed that the surface terms vanish since the fields and their variations fall off sufficiently fast at infinity. Imposing $\delta S \overset{!}{=} 0$ then yields the Euler-Lagrange equations in field theory

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_n} = 0}$$

We then proceed as before and introduce conjugate fields

$$\Pi_n \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_n)}$$

as well as the Hamiltonian density

$$\mathcal{H}(\phi_n, \Pi_n, \vec{\nabla}\phi_n) = (\partial_0 \phi_n)\, \Pi_n - \mathcal{L}(\phi_n, \partial_\mu \phi_n)$$

In a quantum theory we are going to impose equal-time commutation relations

$$[\phi_a(t,\vec{x}), \phi_b(t,\vec{y})] = [\pi_a(t,\vec{x}), \pi_b(t,\vec{y})] = 0$$

$$[\phi_a(t,\vec{x}), \pi_b(t,\vec{y})] = i\delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$$

Our goal thus consists in building field operators out of creation and annihilation operators that fulfill these relations (the quantisation of fields is sometimes called second quantisation). This then implies, however, that there exist similar anticommutation relations for fermionic fields, which have no analog in quantum mechanics. The forthcoming analysis will actually clarify the relation between spin and statistics, i.e. we will learn that bosons (→ symmetric states under the exchange of identical particles) have integer and fermions (→ antisymmetric states under exchange of identical particles) have half-integer spin.

We close this review with a derivation of Noether's theorem in field theory.

A continuous transformation of the fields

$$\phi_n'(x) = \phi_n(x) + \varepsilon F_n(\phi_n(x), \partial_\mu \phi_n(x)) + \mathcal{O}(\varepsilon^2)$$

is called a <u>symmetry</u> if it leaves the action invariant

$$S'[\phi_n'] = S[\phi_n]$$

<u>Noether's theorem</u> then states that there exists a conserved current and a conserved charge for each continuous symmetry.

This can be seen as follows. The invariance of the action implies, on the one hand, that the Lagrangian may change by a total derivative (assuming again that surface terms vanish)

$$\delta\mathcal{L} = \mathcal{L}'(\phi_n', \partial_\mu \phi_n') - \mathcal{L}(\phi_n, \partial_\mu \phi_n) \equiv \varepsilon\, \partial_\mu K^\mu(x) \qquad (1)$$

But the variation of $\mathcal{L}$ gives on the other hand

$$\delta\mathcal{L} = \underbrace{\frac{\partial\mathcal{L}}{\partial\phi_n}}_{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\right)} \underbrace{\delta\phi_n}_{\varepsilon F_n} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \underbrace{\delta(\partial_\mu\phi_n)}_{\varepsilon\partial_\mu F_n}$$

$$= \varepsilon\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\right) F_n + \varepsilon \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \partial_\mu F_n$$

$$= \varepsilon\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} F_n\right) \qquad (2)$$

By taking the difference of (1) and (2), we see that the current

$$j^\mu(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} F_n(x) - k^\mu(x)$$

is conserved, $\partial_\mu j^\mu(x) = 0$, and hence the charge

$$Q \equiv \int d^3x \; j^0(x)$$

is conserved, since

$$\frac{dQ}{dt} = \int d^3x \;\; \partial_0 j^0(x) = - \int d^3x \;\; \vec{\nabla} \cdot \vec{j}(x) = 0$$

where we assumed again that the surface terms vanish at infinity.

Notice that the invariance of the action holds for arbitrary field configurations, whereas the conservation of the current $j^\mu(x)$ only holds for those configurations, which satisfy the equations of motion, since we have explicitly used the Euler-Lagrange equations in the above derivation.

We will encounter several applications of Noether's theorem in the context of internal symmetries below. A special role is played, on the other hand, by Poincaré invariance which we will analyse next.

We first consider space-time translations with

$$\phi_n'(x) = \phi_n(x+\varepsilon)$$
$$= \phi_n(x) + \varepsilon_\nu \partial^\nu \phi_n(x) + \mathcal{O}(\varepsilon^2)$$

which corresponds to four independent symmetries with $F_n^\nu = \partial^\nu \phi_n$.

The Lagrangian transforms similarly under translations with

$$\mathcal{L}'(x) = \mathcal{L}(x+\varepsilon)$$
$$= \mathcal{L}(x) + \varepsilon_\nu \partial^\nu \mathcal{L}(x) + \mathcal{O}(\varepsilon^2)$$

$$\Rightarrow \quad \varepsilon_\nu \partial^\nu \mathcal{L} = \varepsilon_\nu \partial_\mu g^{\mu\nu} \mathcal{L}$$

$$\Rightarrow \quad K^{\mu\nu} = g^{\mu\nu} \mathcal{L}$$

where μ refers to the index μ in (1) and ν corresponds to the symmetry in ν-direction as specified by ε_ν. The conserved current in this case is the <u>canonical energy-momentum tensor</u>

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial^\nu \phi_n - g^{\mu\nu} \mathcal{L}$$

which fulfills $\partial_\mu T^{\mu\nu} = 0$.

The associated charges are

$$P^0 = \int d^3x \; T^{00} = \int d^3x \left(\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_n)}}_{\Pi_n} \partial^0 \phi_n - \mathcal{L} \right)$$

$$= \int d^3x \; \mathcal{H} = H$$

which is the Hamiltonian, and

$$P^i = \int d^3x \; T^{0i} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_n)} \partial^i \phi_n \right)$$

$$= - \int d^3x \; \Pi_n \vec{\nabla}_i \phi_n$$

which - as we will see - is the 3-momentum of the system.

The invariance under space-time translations thus leads to the familiar result of 4-momentum conservation.

<u>Further remarks:</u>

- In general $T^{\mu\nu}$ is not symmetric in μ and ν. It is, however, possible to define a symmetric energy-momentum tensor with

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} + \partial_\rho f^{\rho\mu\nu}$$

which is conserved if $f^{\rho\mu\nu}$ is antisymmetric in ρ and μ

$$\partial_\mu \Theta^{\mu\nu} = \underbrace{\partial_\mu T^{\mu\nu}}_{\substack{=0 \\ \text{Noether current}}} + \underbrace{\partial_\mu \partial_\rho f^{\rho\mu\nu}}_{\substack{=0 \\ \text{antisymmetry of } f}} = 0$$

The corresponding conserved charges are still H and P^i since the additional terms yield irrelevant surface terms

$$\int d^3x\, \theta^{00} = \int d^3x\, (T^{00} + \partial_i f^{i00}) = \int d^3x\, T^{00}$$

The symmetrized Belinfante energy-momentum tensor enters the Einstein equation of general relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G\, \theta_{\mu\nu}$$

- The current associated with homogeneous Lorentz transformations is

$$M^{\lambda\nu\beta} \equiv x^{\nu}\theta^{\beta\lambda} - x^{\beta}\theta^{\nu\lambda}$$

which is conserved since

$$\partial_\lambda M^{\lambda\nu\beta} = g_\lambda{}^{\nu}\theta^{\beta\lambda} + x^{\nu}\underbrace{\partial_\lambda\theta^{\beta\lambda}}_{=0} - g_\lambda{}^{\beta}\theta^{\nu\lambda} - x^{\beta}\underbrace{\partial_\lambda\theta^{\nu\lambda}}_{=0}$$

$$= \theta^{\beta\nu} - \theta^{\nu\beta} = 0$$

The corresponding conserved charges are the angular momentum J^i as well as K^i associated with the Lorentz boosts. When promoted to operators in a quantum theory, H, P^i, J^i and K^i defined here indeed fulfill the Poincaré algebra (for details, see chapter 7.4 of Weinberg I).

3.2. Scalar field

We start with the simplest QFT of free spin-0 particles that are not charged under any internal symmetry. The particles are thus completely characterized by their mass m, and we denote the corresponding one-particle states by $|p\rangle$ and the respective creation and annihilation operators by $a^\dagger(p)$ and $a(p)$.

We will see in the next section that these particles can be described by a real scalar field $\phi(x)$. Here "real" implies that the field operator is hermitean, $\phi^\dagger(x) = \phi(x)$, and "scalar" refers to the transformation properties under Lorentz transformations that we are going to analyze in the next section.

We aim at constructing a field operator out of creation and annihilation operators that fulfills the equal-time commutation or anticommutation relations from page 125. As we do not know yet if this requires $a(p)$ and $a^\dagger(p)$ that obey Bose or Fermi statistics, we will consider both cases simultaneously in the following, writing $[A, B]_\mp = AB \mp BA$.

We first define annihilation and creation fields

$$\phi^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\;u(x,p)\;a(p)$$

$$\phi^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\;v(x,p)\;a^+(p)$$

where $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$ and the coefficients $u(x,p)$ and $v(x,p)$ are completely determined by the transformation properties under spacetime translations. To see this, we recall that (see page 116)
\ chapter 2.5

$$U(\Lambda,b)\;a^+(p)\;U^{-1}(\Lambda,b) = e^{ipb}\;a^+(p)$$

$$U(\Lambda,b)\;a(p)\;U^{-1}(\Lambda,b) = e^{-ipb}\;a(p)$$

whereas the field operators transform as (see page 128)
\ chapter 3.1

$$U(\Lambda,b)\;\phi^{(\pm)}(x)\;U^{-1}(\Lambda,b) = \phi^{(\pm)}(x+b)$$

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\;u(x+b,p)\;a(p) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\;u(x,p)\;e^{-ipb}\;a(p)$$

As b is arbitrary, we obtain $u(x,p) = e^{-ipx}$ and similarly $v(x,p) = e^{ipx}$. The annihilation and creation fields thus satisfy $\left(\phi^{(+)}(x)\right)^+ = \phi^{(-)}(x)$.

As the fields $\phi^{(\pm)}(x)$ are not hermitean, we consider the linear combination

$$\phi(x) \equiv \phi^{(+)}(x) + \phi^{(-)}(x)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(e^{-ipx} a(p) + e^{ipx} a^+(p) \right)$$

where we have implicitly fixed the normalization and a phase convention of the field operator.

Notice that the factors $e^{\pm ipx}$ contain a time dependence $e^{\pm ip^0 t}$, i.e. the field operator is to be understood as a time-dependent Heisenberg operator

$$\underbrace{\phi(t,\vec{x})}_{\text{time-dependent field operator in Heisenberg picture}} = e^{iHt} \underbrace{\phi(t=0,\vec{x})}_{\text{time-independent field operator in Schrödinger picture}} e^{-iHt}$$

which satisfies the Heisenberg equation

$$i\partial_0 \phi = [\phi, H]$$

with a Hamiltonian that is yet to be determined. In the Heisenberg picture the states are, on the other hand, considered to be time-independent.

The field operator furthermore satisfies the Klein-Gordon (KG) equation

$$\boxed{(\partial^2 + m^2)\,\phi(x) = 0}$$

as is easily verified by noting that $\partial^2 = \partial_\mu \partial^\mu$ yields factors of $(\pm ip)^2 = -p^2 = -m^2$. The KG equation thus simply reflects the relation $p^2 = m^2$ on the operator level.

The field operator $\phi(x)$ can be understood as a superposition of plane wave solutions with dispersion relation $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$. The two signs in $e^{\pm ipx}$ originally caused confusion when the KG equation was interpreted as a relativistic wave equation since it involves negative energies with $E = \pm\sqrt{\vec{p}^{\,2} + m^2}$ *. In contrast $\phi(x)$ is considered to be an operator here, which acts on the particle states in the Fock space, which all have positive energies $p^0 > 0$

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* Compare with a time-dependent wave function

$$\Psi(\vec{x}, t) = e^{-iEt}\,\psi(\vec{x})\,.$$

Can we find a Lagrangian that yields the KG equation as its Euler-Lagrange equation?

We will learn how to systematically construct Lagrangians that are Lorentz-invariant in the next chapter, so let us anticipate the result here

$$\boxed{\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2}$$

$$\Rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi}$$

$$= \partial_\mu (\partial^\mu \phi) + m^2 \phi = (\partial^2 + m^2) \phi = 0 \quad \checkmark$$

Starting from the Lagrangian, we may then derive the conjugate field

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi$$

$$\Rightarrow \quad \pi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{(-i)}{2} \left(e^{-ipx} a(p) - e^{ipx} a^\dagger(p) \right)$$

which is again a hermitean Heisenberg operator that satisfies $i\partial_0 \pi = [\pi, H]$.

We are now in the position to verify if the field operators obey equal-time commutation or anticommutation relations, i.e. if spin-0 particles are bosons or fermions. We first consider

$$p^0 = \sqrt{\vec{p}^2 + m^2}$$
$$q^0 = \sqrt{\vec{q}^2 + m^2}$$

$$[\phi(t,\vec{x}), \phi(t,\vec{y})]_\mp$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0}$$
$$\left[e^{-i(p^0t - \vec{p}\vec{x})} a(p) + e^{i(p^0t - \vec{p}\vec{x})} a^+(p) \,,\, e^{-i(q^0t - \vec{q}\vec{y})} a(q) + e^{i(q^0t - \vec{q}\vec{y})} a^+(q) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0}$$
$$\Big\{ e^{-i(p^0t - \vec{p}\vec{x})} e^{i(q^0t - \vec{q}\vec{y})} \underbrace{[a(p), a^+(q)]_\mp}_{(2\pi)^3 2q^0 \delta^{(3)}(\vec{p}-\vec{q})}$$
$$+ e^{i(p^0t - \vec{p}\vec{x})} e^{-i(q^0t - \vec{q}\vec{y})} \underbrace{[a^+(p), a(q)]_\mp}_{\mp (2\pi)^3 2q^0 \delta^{(3)}(\vec{p}-\vec{q})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big\{ e^{i\vec{p}(\vec{x}-\vec{y})} \mp \underbrace{e^{-i\vec{p}(\vec{x}-\vec{y})}}_{\vec{p} \to -\vec{p}} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big\{ e^{i\vec{p}(\vec{x}-\vec{y})} \mp e^{i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

which vanishes, as required by the relations on page 125, for the commutator.

We next verify

$$[\varphi(t,\vec{x}), \pi(t,\vec{y})]_{\mp}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{(-i)}{2}$$

$$\left[e^{-i(p^0t - \vec{p}\vec{x})} a(p) + e^{i(p^0t-\vec{p}\vec{x})} a^\dagger(p) , e^{-i(q^0t - \vec{q}\vec{y})} a(q) - e^{i(q^0t-\vec{q}\vec{y})} a^\dagger(q) \right]_{\mp}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{(-i)}{2} \left\{ -e^{i\vec{p}(\vec{x}-\vec{y})} \mp e^{-i\vec{p}(\vec{x}-\vec{y})} \right\}$$

$$= \frac{(-i)}{2} \left\{ -\delta^{(3)}(\vec{x}-\vec{y}) \mp \delta^{(3)}(\vec{x}-\vec{y}) \right\} = \begin{cases} i\,\delta^{(3)}(\vec{x}-\vec{y}) \\ 0 \end{cases}$$

which again gives the desired result for the commutator (and the fields are thus properly normalised).

One finally verifies that

$$[\pi(t,\vec{x}), \pi(t,\vec{y})]_{\mp} = \int \frac{d^3p}{(2\pi)^3} \frac{(-p^0)}{2} \left\{ -e^{-i\vec{p}(\vec{x}-\vec{y})} \pm e^{-i\vec{p}(\vec{x}-\vec{y})} \right\}$$

which vanishes for the commutator. The real scalar field thus describes spin-0 particles, which are bound to obey Bose statistics!

$$[\pi(t,\vec{x}), \pi(t,\vec{y})]_{\mp}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{(-i)}{2} \int \frac{d^3q}{(2\pi)^3} \frac{(-i)}{2}$$

$$\left[e^{-i(p^0t - \vec{p}\vec{x})} a(p) - e^{i(p^0t - \vec{p}\vec{x})} a^\dagger(p) \,,\, e^{-i(q^0t - \vec{q}\vec{y})} a(q) - e^{i(q^0t - \vec{q}\vec{y})} a^\dagger(q) \right]_{\mp}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{(-p^0)}{2} \left\{ - e^{-i\vec{p}(\vec{x}-\vec{y})} \pm \underbrace{e^{i\vec{p}(\vec{x}-\vec{y})}}_{\vec{p} \to -\vec{p}} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{(-p^0)}{2} \left\{ - e^{-i\vec{p}(\vec{x}-\vec{y})} \pm e^{-i\vec{p}(\vec{x}-\vec{y})} \right\}$$

We next compute the Hamiltonian density

$$\mathcal{H}(\phi, \pi, \vec{\partial}\phi) = (\partial_0 \phi)\, \pi - \mathcal{L}(\phi, \partial_\mu \phi)$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\partial}\phi)^2 + \frac{\mu^2}{2}\phi^2$$

By expressing the field operators in terms of creation and annihilation operators, we will show in the tutorials that the Hamiltonian becomes

$$H = \int d^3x \; \mathcal{H}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \; p^0 \left\{ a^\dagger(p)\, a(p) + \frac{1}{2}\left[a(p), a^\dagger(p)\right] \right\}$$

which is the analog of a familiar result from quantum mechanics

$$H = \hbar\omega \left\{ a^\dagger a + \frac{1}{2}\left[a, a^\dagger\right] \right\}$$

for the one-dimensional harmonic oscillator. The second term in H can therefore similarly be interpreted as the sum over all zero-point energies, which turns out however to diverge in QFT since $[a(p), a^\dagger(p)] = \delta^{(3)}(0)$! As experiments can, however, only measure energy differences, the absolute scale of the energy is irrelevant and we will therefore simply drop the second (constant) term in the following. The vacuum state then satisfies

$$H\,|0\rangle = 0$$

One formally introduces a normal ordering prescription, which consists in commuting / anticommuting all creation operators to the left, but disregarding the corresponding $\delta(p-p')$ terms. One writes e.g.

$$: a^+(p)\, a(p') : \;=\; a^+(p)\, a(p')$$

since the product is already in normal order, whereas

$$: a(p)\, a^+(p') : \;=\; \pm\, a^+(p')\, a(p) \qquad \begin{matrix} + & \text{bosons} \\ - & \text{fermions} \end{matrix}$$

The Hamiltonian can then be written as

$$H = \int d^3x \; : \mathcal{H} :$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\; p^0\, a^+(p)\, a(p)$$

which is the result that we anticipated on page 114.

Applying the Hamiltonian on an N-particle state then yields

$$H\, |p_1 \ldots p_N\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\; p^0\, a^+(p)\, a(p)\; |p_1 \ldots p_N\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} p^0 a^+(p) \sum_{i=1}^{N} (\pm 1)^{i+1} (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}-\vec{p}_i) |p_1 \dots p_{i-1} p_{i+1} \dots p_N\rangle$$

$$= \sum_{i=1}^{N} (\pm 1)^{i+1} p_i^0 a^+(p_i) |p_1 \dots p_{i-1} p_{i+1} \dots p_N\rangle$$

$$= \sum_{i=1}^{N} (\pm 1)^{i+1} p_i^0 \underbrace{|p_i p_1 \dots p_{i-1} p_{i+1} \dots p_N\rangle}_{(\pm 1)^{i-1} |p_1 \dots p_N\rangle}$$

$$= \left(\sum_{i=1}^{N} p_i^0 \right) |p_1 \dots p_N\rangle$$

i.e. $|p_1 \dots p_N\rangle$ is indeed an eigenstate of H with energy (eigenvalue) given by the sum over all one-particle energies, which is what one would expect for a system of non-interacting particles.

We will further show in the tutorials that $\vec{P}$, which we defined on page 125 as the conserved charge of spacial translation symmetry, becomes with the appropriate normal-ordering prescription

$$\vec{P} = - \int d^3x :\pi \vec{\nabla} \phi:$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \vec{p}\, a^+(p) a(p)$$

In analogy to the above derivation, one has

$$\vec{P} |p_1 \dots p_N\rangle = \left(\sum_{i=1}^{N} \vec{p}_i \right) |p_1 \dots p_N\rangle$$

which is consistent with the interpretation of $\vec{P}$ as the 3-momentum operator.

Now that we have formulated our first QFT of free, neutral spin-0 particles, let us come back to the question about the compatibility of the uncertainty principle and causality that we raised at the beginning of this chapter.

As the field operator $\phi(x)$ is hermitian, it represents itself an observable. We would like to understand under which circumstances a measurement of the field at the point $x = (x^0, \vec{x})$ influences a measurement at another point $y = (y^0, \vec{y})$.

In the quantum mechanics course we learned that two observables can be measured independently if their commutator vanishes. The measurements at the points x and y are therefore uncorrelated and cannot influence each other if $\phi(x)$ and $\phi(y)$ commute.

The commutator of the fields at different times $x^0 \neq y^0$ can be computed along the lines of the calculation on page 136.

We obtain

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left\{ e^{-ip(x-y)} - e^{ip(x-y)} \right\}$$

$$= \Delta(x-y) - \Delta(y-x) \qquad (*)$$

where we introduced the function

$$\Delta(z) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipz}$$

which is frame-independent since it involves the Lorentz-invariant integration measure.

At equal times $x^0 = y^0$, we found on page 136 that the two terms in $(*)$ cancel exactly. As the result is, however, Lorentz-invariant, this is true for all events with a space-like separation $(x-y)^2 < 0$ (for which one can always find a boost such that the events occur at the same time). For time-like events with $(x-y)^2 > 0$, on the other hand, the commutator is in general non-zero (the explicit result in terms of Bessel functions is not needed here, but can be found e.g. in Schwartz, chapter 12.6).

We conclude that the uncertainty principle forbids in general to measure the field at different points to arbitrary precision. For events that are not causally connected, however, the measurements are independent and cannot influence each other. In QFT causality is guaranteed by an intricate cancellation between the two terms in (*). Can we understand the physical meaning of this cancellation?

To answer this question, we consider the expression

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0}$$

$$\langle 0 | \left(e^{-ipx} a(p) + \cancel{e^{ipx} a^+(p)} \right) \left(e^{-iqy} a(q) + e^{iqy} a^+(q) \right) | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} e^{-ipx} e^{iqy} \underbrace{\langle 0 | a(p) a^+(q) | 0 \rangle}_{(2\pi)^3 2q^0 \delta^{(3)}(\vec{p}-\vec{q})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ip(x-y)} = \Delta(x-y)$$

The first term in (*) may thus be interpreted as the amplitude for a particle created at $\vec{y}$ at time y^0 to propagate to $\vec{x}$ at time x^0. One can show that $\Delta(x-y)$ is indeed non-zero outside the light cone, and the particle therefore has a non-zero amplitude to propagate faster than light (and to propagate backwards in time for $y^0 > x^0$).

But causality only requires that information cannot propagate faster than light (such that space-like events cannot influence each other). This brings us back to the expression (*), which is the difference of two amplitudes that describe the propagation of a particle from y to x and from x to y. Although the individual amplitudes are non-zero outside the light cone, the two amplitudes cancel out exactly in their difference (outside the light cone) and causality is restored. We will see later that this cancellation requires the existence of antiparticles (as neutral spin-0 particles are their own antiparticles, we cannot appreciate the role of antiparticles here).

We next are going to derive our first Feynman rule, which is a pictorial representation of a certain mathematical expression that we encounter when we discuss interacting theories. In interacting theories the equations of motion are non-linear and they cannot be solved exactly (except for a very few cases). As long as the theory is weakly coupled, one can however construct a perturbative solution as an expansion in the coupling constant. In this context the method of Green's functions turns out to be very useful.*

The method of <u>Green's functions</u> is a tool to solve inhomogeneous, linear, partial differential equations. Consider e.g.

$$(\partial^2 + m^2)\, \phi(x) = j(x)$$

where $j(x)$ represents an external, classical source that the field $\phi(x)$ is coupled to. If we find the Green's function $G(x-y)$ that satisfies

$$(\partial^2 + m^2)\, G(x-y) = \underbrace{-i}_{\text{convention}}\, \delta^{(4)}(x-y)$$

---

* For an example in classical field theory see Schwartz, chapter 3.5.

The solution to the inhomogeneous differential equation is readily found as

$$\phi(x) = \underbrace{\phi_{hom}(x)}_{\substack{\text{general solution to}\\ \text{homogeneous diff eq.}}} + \underbrace{\phi_{part}(x)}_{\substack{\text{particular solution to}\\ \text{inhomogeneous diff eq}}}$$

with

$$\phi_{part}(x) = i \int d^4y \; G(x-y)\, j(y)$$

The Green's function to the KG operator can be found by a Fourier transformation

$$\int d^4x \; e^{ip(x-y)} \left(\partial_x^2 + m^2\right) G(x-y)$$

$$\overset{\text{p.I.}}{=} \int d^4x \left(-p^2 + m^2\right) e^{ip(x-y)} \; G(x-y)$$

$$= \int d^4x \; e^{ip(x-y)} \, (-i)\, \delta^{(4)}(x-y) = -i$$

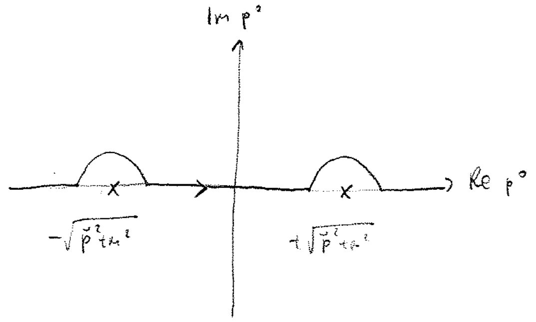
$$\Rightarrow \quad G(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip(x-y)} \, \frac{i}{p^2 - m^2}$$

Notice that this involves a four-dimensional momentum integral, i.e.

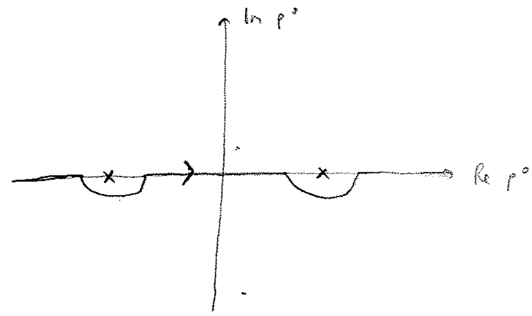
p^0 is <u>not</u> fixed to $\sqrt{\vec{p}^{\,2} + m^2}$ here!

The integration over p^0 actually diverges since the integrand has poles at $p^0 = \pm\sqrt{\vec{p}^2+m^2}$. The integral is therefore ill-defined without a prescription that tells us how to avoid the poles in the complex p^0-plane. There are four possibilities that correspond to different boundary conditions of the Green's functions

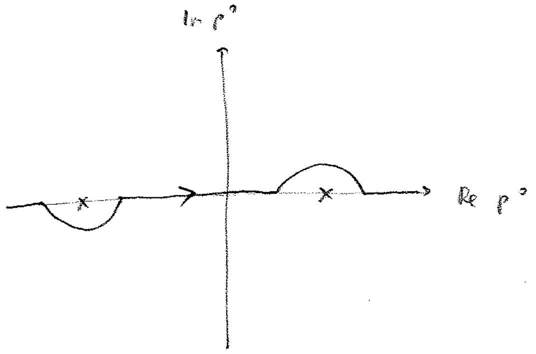
Retarded Green's function



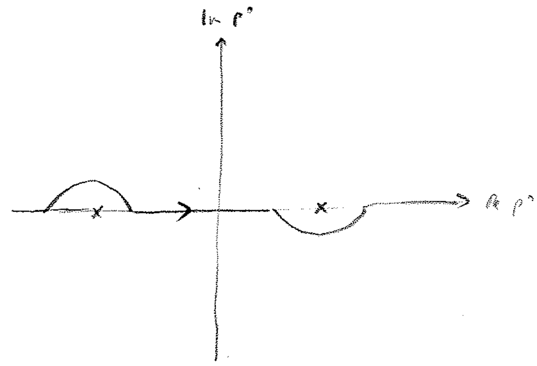
Advanced Green's function



Feynman prescription



Dyson prescription



We will compare these prescriptions in detail in the tutorials. In order to discuss scattering reactions, it turns out that the Feynman prescription is of particular interest.

The Feynman Green's function – or <u>Feynman propagator</u> – is usually written as

$$\Delta_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \frac{i}{p^2-m^2+i\varepsilon}$$

where the $i\varepsilon$-prescription reminds us of the corresponding contour in the complex p^0-plane. As

$$p^2-m^2+i\varepsilon = (p^0-\sqrt{\vec{p}^2+m^2}+i\varepsilon)(p^0+\sqrt{\vec{p}^2+m^2}-i\varepsilon)$$

the poles lie at $p^0 = \pm\sqrt{\vec{p}^2+m^2} \mp i\varepsilon$, which indeed corresponds to the Feynman contour from above.

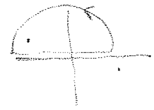
In order to perform the p^0-integration with Cauchy's theorem, we have to make sure that the contribution from the integration over the half circle at infinity vanishes. To this end, we consider

$$e^{-ip^0(x^0-y^0)} = \underbrace{e^{-i\,\mathrm{Re}\,p^0(x^0-y^0)}}_{\text{oscillating factor}} \cdot \underbrace{e^{\mathrm{Im}\,p^0(x^0-y^0)}}_{\substack{\text{damping factor if contour}\\ \text{is closed appropriately}}}$$

We therefore close the contour in the upper half plane for $x^0<y^0$ and in the lower half plane for $x^0>y^0$.

This yields

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{\left(p^0-\sqrt{\vec{p}^{\,2}+m^2}+i\varepsilon\right)\left(p^0+\sqrt{\vec{p}^{\,2}+m^2}-i\varepsilon\right)}$$



$$= \frac{i}{2\pi}\int\frac{d^3p}{(2\pi)^3}\left\{\theta(y^0-x^0)\, 2\pi i\, \frac{1}{-2\sqrt{\vec{p}^{\,2}+m^2}}\, e^{i\sqrt{\vec{p}^{\,2}+m^2}\,(x^0-y^0)}\, \underbrace{e^{i\vec{p}(\vec{x}-\vec{y})}}_{\vec{p}\to-\vec{p}}\right.$$

$$\left. +\ \theta(x^0-y^0)\,(-2\pi i)\,\frac{1}{2\sqrt{\vec{p}^{\,2}+m^2}}\, e^{-i\sqrt{\vec{p}^{\,2}+m^2}\,(x^0-y^0)}\, e^{i\vec{p}(\vec{x}-\vec{y})}\right\}$$

$$= \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\sqrt{\vec{p}^{\,2}+m^2}}\left\{\theta(y^0-x^0)\, e^{ip(x-y)} + \theta(x^0-y^0)\, e^{-ip(x-y)}\right\}_{p^0=\sqrt{\vec{p}^{\,2}+m^2}}$$

$$= \theta(y^0-x^0)\,\Delta(y-x) + \theta(x^0-y^0)\,\Delta(x-y)$$

$$= \theta(y^0-x^0)\,\langle 0|\,\phi(y)\,\phi(x)\,|0\rangle + \theta(x^0-y^0)\,\langle 0|\,\phi(x)\,\phi(y)\,|0\rangle$$

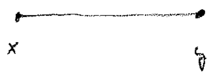
$$= \langle 0|\,T\,\phi(x)\,\phi(y)\,|0\rangle$$

where we introduced the <u>time-ordering prescription</u>

$$T\,\phi(x)\,\phi(y) \equiv \theta(x^0-y^0)\,\phi(x)\,\phi(y) + \theta(y^0-x^0)\,\phi(y)\,\phi(x)$$

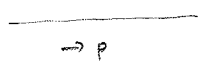
The Feynman propagator thus describes the propagation of a particle from x to y if $y^0 > x^0$ and from y to x if $x^0 > y^0$, i.e. it picks out the amplitude for propagation from earlier to later times.

The Feynman propagator is usually visualised as follows



$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2-m^2+i\varepsilon}$$

in position space and



$$\tilde{\Delta}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$$

in momentum space.

<u>Remarks:</u>

- Feynman diagrams are not to be understood as classical spacetime diagrams. In a quantum theory, we do not know what happens between x and y as long as we do not perform any measurements.
- The particle that propagates between x and y is called a <u>virtual particle</u>, which in contrast to a real particle, does not satisfy the mass-shell condition $p^0 = \sqrt{\vec{p}^{\,2}+m^2}$. Virtual particles do not exist as asymptotic states (for $t \to \pm\infty$) in a scattering reaction; they rather describe temporal fluctuations of the underlying quantum fields.

- In contrast to

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \Delta(x-y) - \Delta(y-x)$$

the Feynman propagator

$$\langle 0 | T\phi(x)\phi(y) | 0 \rangle = \theta(x^0-y^0)\,\Delta(x-y) + \theta(y^0-x^0)\,\Delta(y-x)$$

does not vanish outside the light cone. But this is not a problem since causality only requires – as we have seen above – that <u>commutators</u> of observables vanish outside the light cone.

Before we turn to the complex scalar field, we note that the limit $m \to 0$ for spin-0 particles is smooth (cf. page 33). For $m=0$ the field operator $\phi(x)$ with $p^0 = |\vec{p}|$ therefore satisfies the massless KG equation

$$\partial^2 \phi(x) = 0$$

and it describes neutral, massless particles with helicity $\sigma = 0$.

The real scalar field cannot describe particles that are charged under internal symmetries. One therefore introduces the complex scalar field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(e^{-ipx} a(p) + e^{ipx} b^+(p) \right)$$

which depends on two independent sets of creation and annihilation operators that satisfy

$$[a(p), a(p')] = 0 \qquad [a(p), a^+(p')] = \delta(p-p')$$

$$[b(p), b(p')] = 0 \qquad [b(p), b^+(p')] = \delta(p-p')$$

$$[a(p), b(p')] = 0 \qquad [a(p), b^+(p')] = 0$$

and the remaining commutators are fixed by the adjoint of these relations.

Notice that the complex scalar field is not hermitian

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(e^{-ipx} b(p) + e^{ipx} a^+(p) \right) \neq \phi(x)$$

The complex scalar field can be constructed from two real scalar fields $\phi_{1,2}(x)$ that describe its real and imaginary part

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\,\phi_2(x)\right)$$

It is, however, more convenient to consider $\phi(x)$ and $\phi^+(x)$ as two independent variables.

The complex scalar field allows to construct a Lagrangian density that is invariant under phase transformations

$$\phi'(x) = e^{i\alpha}\,\phi(x) \qquad \alpha \in \mathbb{R}$$

if the Lagrangian only depends on products of $\phi^+\phi$. The free theory Lagrangian is then given by

$$\boxed{\mathcal{L}(\phi, \phi^+, \partial_\mu\phi, \partial_\mu\phi^+) = \partial_\mu\phi^+\partial^\mu\phi - m^2\phi^+\phi}$$

which yields the KG equations

$$(\partial^2 + m^2)\,\phi(x) = 0 \qquad\qquad (\partial^2 + m^2)\,\phi^+(x) = 0$$

We will compute the conserved Noether charge associated with the $U(1)$ symmetry in the tutorials.

The particles created and annihilated by $a(p), a^\dagger(p)$ and $b(p), b^\dagger(p)$ have the same mass m. The creation and annihilation operators furthermore transform under the internal symmetry as (see page 116)

$$\mathcal{U}(g)\, a^\dagger(p)\, \mathcal{U}^{-1}(g) = e^{-i\alpha}\, a^\dagger(p)$$

$$\mathcal{U}(g)\, b^\dagger(p)\, \mathcal{U}^{-1}(g) = e^{i\alpha}\, b^\dagger(p)$$

i.e. $b(p), b^\dagger(p)$ transform under the complex conjugate representation of $a(p), a^\dagger(p)$. The particles created by $b^\dagger(p)$ are therefore the antiparticles of the particles created by $a^\dagger(p)$. The field operator $\phi(x)$ thus creates antiparticles (and annihilates particles), whereas $\phi^\dagger(x)$ creates particles (and annihilates antiparticles).

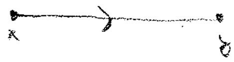
There now exist various two-point functions like

$$\langle 0 |\, T \phi(x)\, \phi(y)\, | 0 \rangle = \langle 0 |\, T\, \phi^\dagger(x)\, \phi^\dagger(y)\, | 0 \rangle = 0$$

which vanish since they are not invariant under the $U(1)$ symmetry.

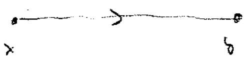
One further has

$$\langle 0 | T\, \phi^\dagger(x)\, \phi(y) | 0 \rangle$$



which describes the propagation of a particle from x to y (for $y^0 > x^0$) and the propagation of an antiparticle from y to x (for $x^0 > y^0$). One therefore adds an arrow to the Feynman rule, which illustrates the direction of the particle flow (which is opposite to the antiparticle flow).

We will show in the tutorials that the Feynman rule for the free propagation of a charged, spin-0 particle is the same as the one of a neutral, spin-0 particle, i.e.



$$= \Delta_F(x-y)$$

with $\Delta_F(x-y)$ from page 148.

3.3. Non-trivial spin

So far we only considered real and complex scalar fields associated with spin-0 particles. For particles with non-zero spin, one introduces creation and annihilation fields as

$$\phi_\alpha^{(+)}(x) \equiv \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\; u_\alpha(p,s)\; e^{-ipx}\; a(p,s)$$

$$\phi_\alpha^{(-)}(x) \equiv \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\; v_\alpha(p,s)\; e^{ipx}\; a^\dagger(p,s)$$

where $p^0 = \sqrt{\vec{p}^{\,2}+m^2}$ and we assume for the moment that the particles are not charged under any internal symmetry ($\rightarrow$ one set of creation and annihilation operators).

The generalized field operators have various components (indicated by α), and the coefficients $u_\alpha(p,s)$ and $v_\alpha(p,s)$ provide the connection between these components and the spin configurations of the particle states $|p,s\rangle$.

The x-dependence of the field operators is, on the other hand, again fixed by the transformation properties under spacetime translations (cf. page 132). Each component of the field operator therefore satisfies the KG equation

$$(\partial^2+m^2)\,\phi_\alpha^{(s)}(x) = 0$$

We will see in the following that some fields obey additional field equations, depending on whether or not the number of degrees of freedom of the field operator and the particle states are the same. The Dirac field e.g. has four complex-valued components ($\sim$ 8 real degrees of freedom), but the associated charged spin-1/2 particles only have two spin configurations for each the particle and the antiparticle states ($\sim$ 4 real degrees of freedom). The field operator therefore satisfies four additional relations that are encoded in the Dirac equation.

We next address an aspect that we have disregarded so far. We learned in chapter 2 that the particle states transform under unitary representations of the homogeneous Lorentz group (or more generally the Poincaré group). But how do the field operators transform?

As its name suggests, the scalar field transforms trivially with

$$\mathcal{U}(\Lambda)\, \phi^{(\pm)}(x)\, \mathcal{U}^{-1}(\Lambda) = \phi^{(\pm)}(\Lambda x)$$

(unitary infinite-dimensional representation of the proper, orthochronous Lorentz group (see chapter 2)

The Lagrangians that we constructed for the real and complex scalar fields thus transform as

$$\mathcal{U}(\Lambda)\, \mathcal{L}(x)\, \mathcal{U}^{-1}(\Lambda) = \mathcal{L}(\Lambda x)$$

and the action

$$S[\phi] = \int d^4x\ \mathcal{L}(x)$$

is invariant.

For particles with non-zero spin, one generalizes the transformation law to

$$\boxed{\mathcal{U}(\Lambda)\, \phi_\alpha^{(\pm)}(x)\, \mathcal{U}^{-1}(\Lambda) = \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1})\, \phi_{\alpha'}^{(\pm)}(\Lambda x)}$$

where the matrix $D(\Lambda^{-1})$ does not depend on x. One can then construct an invariant action through

$$\mathcal{L}(x) = \sum_{n,m} \sum_{\alpha_1 \dots \alpha_n} \sum_{\beta_1 \dots \beta_m} c_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}\, \phi_{\alpha_1}^{(-)}(x) \dots \phi_{\alpha_n}^{(-)}(x)\, \phi_{\beta_1}^{(+)}(x) \dots \phi_{\beta_m}^{(+)}(x)$$

if the coefficients $c_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}$ compensate for the factors of $D(\Lambda^{-1})$ (and one in addition has to make sure that the Lagrangian density constructed this way is hermitian).

By performing two subsequent Lorentz transformations Λ_1 and Λ_2, we get

$$\mathcal{U}(\Lambda_2\Lambda_1)\, \phi_\alpha^{(\pm)}(x)\, \mathcal{U}^{-1}(\Lambda_2\Lambda_1) = \sum_\gamma D_{\alpha\gamma}((\Lambda_2\Lambda_1)^{-1})\, \phi_\gamma^{(\pm)}(\Lambda_2\Lambda_1 x)$$

$$= \mathcal{U}(\Lambda_2)\,\mathcal{U}(\Lambda_1)\, \phi_\alpha^{(\pm)}(x)\, \mathcal{U}^{-1}(\Lambda_1)\,\mathcal{U}^{-1}(\Lambda_2)$$

$$= \sum_\beta D_{\alpha\beta}(\Lambda_1^{-1})\, \mathcal{U}(\Lambda_2)\, \phi_\beta^{(\pm)}(\Lambda_1 x)\, \mathcal{U}^{-1}(\Lambda_2)$$

$$= \sum_{\beta,\gamma} D_{\alpha\beta}(\Lambda_1^{-1})\, D_{\beta\gamma}(\Lambda_2^{-1})\, \phi_\gamma^{(\pm)}(\Lambda_2\Lambda_1 x)$$

$$\Rightarrow \quad D(\Lambda_1^{-1}\Lambda_2^{-1}) = D(\Lambda_1^{-1})\, D(\Lambda_2^{-1})$$

(which explains why the argument of D is Λ^{-1}).

The matrices $D(\Lambda^\mu)$ thus furnish a <u>finite-dimensional</u> representation of the proper, orthochronous Lorentz group. As they do not act on the particle states, the matrices $D(\Lambda^\mu)$ do not have to be unitary (there actually does not exist a non-trivial finite-dimensional representation of the homogeneous Lorentz group that is unitary).

In order to formulate QFTs for particles with non-zero spin, we thus have to identify all finite-dimensional irreducible representations of the homogeneous Lorentz group. We will see that this turns out to be a fairly straightforward generalisation of the representation theory of the angular momentum algebra that we discussed at the end of chapter 1.4 (see pages 53-57).

We start from the algebra (~ page 72)

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\mu\sigma} J^{\nu\rho} + g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho}\right)$$

which we reexpressed in terms of the generators of 3-dimensional rotations

$J^i = \frac{1}{2} \varepsilon^{ijk} J^{jk}$ and the Lorentz boosts $K^i = J^{i0}$ as

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

$$[J^i, K^j] = i\varepsilon^{ijk} K^k$$

$$[K^i, K^j] = -i\varepsilon^{ijk} J^k$$

We now define

$$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K})$$

$$\vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$$

$$\Rightarrow [A^i, A^j] = \frac{1}{4}\left\{[J^i, J^j] + i[J^i, K^j] + i[K^i, J^j] - [K^i, K^j]\right\}$$

$$= \frac{i}{4}\varepsilon^{ijk}\left\{J^k + iK^k + iK^k + J^k\right\} = i\varepsilon^{ijk} A^k$$

$$[B^i, B^j] = \frac{i}{4}\varepsilon^{ijk}\left\{J^k - iK^k - iK^k + J^k\right\} = i\varepsilon^{ijk} B^k$$

$$[A^i, B^j] = \frac{i}{4}\varepsilon^{ijk}\left\{J^k - iK^k + iK^k - J^k\right\} = 0$$

We thus obtain two independent angular momentum algebras!

The irreducible representations of the Lorentz algebra are thus characterised by two numbers (a,b), which correspond to the eigenvalues of $\vec{A}^2$ $a(a+1)$ and $\vec{B}^2$ $b(b+1)$. The representations of the Lorentz group then follow as usual via the exponential map

$$D(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \qquad (\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \dots)$$

where

$$J^{ij} = \varepsilon^{ijk} J^k = \varepsilon^{ijk}(A^k + B^k)$$
$$J^{i0} = K^i = -i(A^i - B^i)$$

Similar to the rotation group, the representations of the homogeneous Lorentz group are projective for half-integer values of a and b, and one in this case considers its universal covering group $SL(2,C)$ (this is analogous to the relation between $SO(3)$ and $SU(2)$).

But before we start constructing the irreducible representations of the Lorentz group, we recall that the generators transform under a parity transformation as (see page 98)

$$U_P \vec{J} U_P^{-1} = \vec{J} \qquad\qquad U_P \vec{A} U_P^{-1} = \vec{B}$$
$$\Longrightarrow$$
$$U_P \vec{K} U_P^{-1} = -\vec{K} \qquad\qquad U_P \vec{B} U_P^{-1} = \vec{A}$$

i.e. the representations (a,b) and (b,a) are related by a parity transformation.

<u>Trivial representation $(0,0)$</u>

By combining the trivial representations of A- and B-spin, we obtain

$$A^i = 0 \qquad\qquad J^{ij} = \varepsilon^{ijk}(A^k + B^k) = 0$$
$$\Longrightarrow$$
$$B^i = 0 \qquad\qquad J^{i0} = -i(A^i - B^i) = 0$$

and hence

$$D_{(0,0)}(\Lambda) = 1\!\!1$$

This is the representation of the scalar field (see page 158).

Spinor representation $(1/2, 0)$

We next combine the fundamental representation of A-spin and the trivial representation for B-spin

$$A^i = \frac{\sigma^i}{2} \qquad \Rightarrow \qquad J^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma^k$$

$$B^i = 0 \qquad\qquad J^{i0} = -\frac{i}{2}\sigma^i$$

where σ^i are the Pauli matrices that satisfy

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$$

By defining

$$\sigma^\mu \equiv (\mathbb{1}, \vec{\sigma}) \qquad\qquad \sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$$

$$\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma}) \qquad\qquad \bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$$

the generators can be written more compactly as

$$J^{\mu\nu} = \bar{\sigma}^{\mu\nu}$$

Check:

- $\bar{\sigma}^{\mu\nu}$ is antisymmetric ✓
- $\bar{\sigma}^{i0} = \frac{i}{4}(\bar{\sigma}^i\sigma^0 - \bar{\sigma}^0\sigma^i)$

  $= \frac{i}{4}(-\sigma^i - \sigma^i) = -\frac{i}{2}\sigma^i$ ✓
- $\bar{\sigma}^{ij} = \frac{i}{4}(\bar{\sigma}^i\sigma^j - \bar{\sigma}^j\sigma^i)$

  $= -\frac{i}{4}[\sigma^i, \sigma^j] = \frac{1}{2}\varepsilon^{ijk}\sigma^k$ ✓

We thus obtain

$$\mathcal{D}_{(1/2,0)}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}$$

and the objects that transform under this representation are called <u>right-handed Weyl spinors</u> (the notation refers to the helicity of the states that are annihilated by these fields).

As

$$(\bar{\sigma}^{\mu\nu})^{+} = -\frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}) = \sigma^{\mu\nu} \neq \bar{\sigma}^{\mu\nu}$$

we can see explicitly that the representation $\mathcal{D}_{(1/2,0)}(\Lambda)$ is not unitary.

<u>Spinor representation $(0, 1/2)$</u>

We now exchange the roles of $\vec{A}$ and $\vec{B}$

$$\begin{aligned} A^i &= 0 \\ B^i &= \frac{\sigma^i}{2} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \mathcal{J}^{ij} &= \frac{1}{2}\varepsilon^{ijk}\sigma^k \\ \mathcal{J}^{i0} &= \frac{i}{2}\sigma^i \end{aligned}$$

and hence $\mathcal{J}^{\mu\nu} = \sigma^{\mu\nu}$.

Check:

- $\sigma^{\mu\nu}$ antisymmetric ✓
- $\sigma^{i0} = \frac{i}{4}(\sigma^i + \sigma^i) = \frac{i}{2}\sigma^i$ ✓
- $\sigma^{ij} = -\frac{i}{4}(\sigma^i, \sigma^j) = \frac{1}{2}\varepsilon^{ijk}\sigma^k$ ✓

We now get

$$D_{(0,1/2)}(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}$$

which is again not unitary. The objects that transform under this representation are called <u>left-handed Weyl spinors</u>.

Notice that the representations $(1/2, 0)$ and $(0, 1/2)$ are not equivalent, i.e. there exists no matrix S with

$$D_{(0,1/2)}(\Lambda) = S\, D_{(1/2,0)}(\Lambda) S^{-1} \qquad \forall \Lambda$$

But the matrices of the two representations fulfill

(1) $$\left[D_{(0,1/2)}(\Lambda) \right]^{+} = \left[D_{(1/2,0)}(\Lambda) \right]^{-1}$$

(2) $$D_{(0,1/2)}(\Lambda) = \varepsilon \left[D_{(1/2,0)}(\Lambda) \right]^{*} \varepsilon^{-1}$$

with $\varepsilon \equiv -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

<u>Proof</u>:

(1) $$\left[D_{(0,1/2)}(\Lambda) \right]^{+} = e^{\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})^{+}}$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}} = \left[D_{(1/2,0)}(\Lambda) \right]^{-1} \quad \checkmark$$

(2) One first verifies that

$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\varepsilon\, \sigma^{i*}\, \varepsilon^{-1} = -\sigma^i$$

$$\varepsilon\, \sigma^{0*}\, \varepsilon^{-1} = +\sigma^0$$

$$\Rightarrow \varepsilon \left(\bar\sigma^{\mu\nu}\right)^* \varepsilon^{-1} = -\frac{i}{4}\left(\varepsilon\, \bar\sigma^{\mu *} \varepsilon^{-1} \varepsilon\, \sigma^{\nu *} \varepsilon^{-1} - \varepsilon\, \bar\sigma^{\nu *} \varepsilon^{-1} \varepsilon\, \sigma^{\mu *} \varepsilon^{-1}\right)$$

$$= -\frac{i}{4}\left(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu\right) = -\sigma^{\mu\nu}$$

$$\Rightarrow \varepsilon\, \left[D_{(1/2,0)}(\Lambda)\right]^* \varepsilon^{-1}$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}\, \varepsilon\, \bar\sigma^{\mu\nu *} \varepsilon^{-1}} = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}} = D_{(0,1/2)}(\Lambda)$$ ✓

One further has (without proof)

(3) If ψ transforms as a left-handed (right-handed) spinor, $(\varepsilon\psi)^*$ transforms as a right-handed (left-handed) spinor.

(4)
$$\left[D_{(0,1/2)}(\Lambda)\right]^{-1} \sigma^\mu \left[D_{(1/2,0)}(\Lambda)\right] = \Lambda^\mu{}_\nu\, \sigma^\nu$$

$$\left[D_{(1/2,0)}(\Lambda)\right]^{-1} \bar\sigma^\mu \left[D_{(0,1/2)}(\Lambda)\right] = \Lambda^\mu{}_\nu\, \bar\sigma^\nu$$

As the representations $(1/2, 0)$ and $(0, 1/2)$ transform into each other under a parity transformation, they cannot be used individually to construct a parity-invariant theory.

Direct sum representation $(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$

The representation

$$D_{(0,\frac{1}{2})\oplus(\frac{1}{2},0)}(\Lambda) = \begin{pmatrix} D_{(0,\frac{1}{2})}(\Lambda) & 0 \\ 0 & D_{(\frac{1}{2},0)}(\Lambda) \end{pmatrix}$$

2×2 blocks

is reducible under homogeneous LT, but it is irreducible if one demands in addition invariance under parity transformations.

The four-component objects $\psi \sim \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ that transform under this representation are called <u>Dirac spinors</u>.

We now define <u>Dirac matrices</u>

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$ (4-vector of 4×4 matrices)

in the chiral or Weyl representation such that

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$= \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}$$

$$\Rightarrow D_{(0,\frac{1}{2})\oplus(\frac{1}{2},0)}(\Lambda) = \begin{pmatrix} e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}} & 0 \\ 0 & e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}} \end{pmatrix}$$

$$= e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$$

The Dirac matrices transform under homogeneous LT as

$$[D_{(0,1/2)\oplus(1/2,0)}(\Lambda)]^{-1}\,\gamma^\mu\,[D_{(0,1/2)\oplus(1/2,0)}(\Lambda)]$$

$$= \begin{pmatrix} [D_{(0,1/2)}(\Lambda)]^{-1} & 0 \\ 0 & (D_{(1/2,0)}(\Lambda))^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} (D_{(0,1/2)}(\Lambda)) & 0 \\ 0 & (D_{(1/2,0)}(\Lambda)) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (D_{(0,1/2)}(\Lambda))^{-1}\,\sigma^\mu\,(D_{(1/2,0)}(\Lambda)) \\ [D_{(1/2,0)}(\Lambda)]^{-1}\,\bar\sigma^\mu\,[D_{(0,1/2)}(\Lambda)] & 0 \end{pmatrix}$$

relations (4) on page 167

$$= \begin{pmatrix} 0 & \Lambda^\mu{}_\nu\,\sigma^\nu \\ \Lambda^\mu{}_\nu\,\bar\sigma^\nu & 0 \end{pmatrix} = \Lambda^\mu{}_\nu\,\gamma^\nu$$

One can introduce the Dirac matrices formally as the set of complex 4x4 matrices that satisfy the Clifford algebra

$$\boxed{\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}}$$

The explicit representation of the Dirac matrices is actually not unique. Apart from the chiral representation introduced above, one finds e.g. the Dirac representation with

$$\gamma^0 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

in the literature. The two representations are related by a unitary transformation

$$\gamma^\mu_{\text{Dirac}} = U\,\gamma^\mu_{\text{chiral}}\,U^+ \qquad U = \frac{1}{\sqrt2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and they both fulfill the defining relation $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$.

Fundamental representation $(1/2, 1/2)$

The tensor product representation

$$D_{(1/2,1/2)}(\Lambda) = D_{(1/2,0)}(\Lambda) \otimes D_{(0,1/2)}(\Lambda)$$

is 4-dimensional, and one can show that it is equivalent to the fundamental representation

$$D_F(\Lambda) = \Lambda$$

This representation is thus again not unitary, but it transforms into itself under a parity transformation and it can hence be used to construct parity-invariant theories. The objects that transform under this representation are usual 4-vectors.

General representation (a,b)

One can proceed similarly and construct higher-dimensional representations by taking direct products of the $(1/2,0)$ and $(0,1/2)$ representations. According to $\vec{J} = \vec{A} + \vec{B}$, the representation (a,b) then contains representations of spin $j = |a-b|, \dots, a+b$.

For $(1/2, 1/2)$ this corresponds e.g. to the components of a 4-vector V^μ

- V^0 is invariant under rotations (singlet $\rightarrow$ $j=0$)
- $\vec{V}$ transforms as 3-vector under rotations (triplet $\rightarrow$ $j=1$)

Now that we have derived the irreducible representations of the homogeneous Lorentz group, let us come back to the transformation law of the field operators (~ page 159)

$$\mathcal{U}(\Lambda)\, \phi_\alpha^{(\pm)}(x)\, \mathcal{U}^{-1}(\Lambda) = \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1})\, \phi_{\alpha'}^{(\pm)}(\Lambda x)$$

which provides a non-trivial constraint on the coefficients $u_\alpha(p,s)$ and $v_\alpha(p,s)$ in the decomposition of the field operators $\phi_\alpha^{(\pm)}(x)$.

This can be seen as follows. In the massive case, the relations from page 116 imply

$$\mathcal{U}(\Lambda)\, \phi_\alpha^{(+)}(x)\, \mathcal{U}^{-1}(\Lambda)$$

$$= \sum_{s'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, u_\alpha(p,s')\, e^{-ipx} \sum_s \underbrace{D_{ss'}^{(s)}(R)^*}_{\text{Wigner function}}\, a(\Lambda p, s)$$

The right hand side yields

$$\sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1})\, \phi_{\alpha'}^{(+)}(\Lambda x)$$

$$= \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1}) \sum_s \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2(p')^0}\, u_{\alpha'}(p',s)\, e^{-ip'\Lambda x}\, a(p',s)$$

$p' = \Lambda p$

$\Rightarrow p'\Lambda x = (\Lambda p)(\Lambda x) = p\cdot x$

$$= \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1}) \sum_s \underbrace{\int \frac{d^3(\Lambda p)}{(2\pi)^3} \frac{1}{2(\Lambda p)^0}}_{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \;\; \text{(Lorentz-invariant)}}\, u_{\alpha'}(\Lambda p,s)\, e^{-ipx}\, a(\Lambda p,s)$$

By comparing both sides of this equation, we read off

$$\sum_{s'} u_\alpha(p,s')\, D^{(j)}_{ss'}(R)^* = \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1})\, u_{\alpha'}(\Lambda p, s)$$

which can be inverted by multiplying the equation with $D_{\rho\alpha}(\Lambda)$ from the left and with $D^{(j)}_{st}(R)$ from the right

$$\Rightarrow \sum_{s'}\sum_{\alpha} D_{\rho\alpha}(\Lambda)\, u_\alpha(p,s') \underbrace{\sum_{s} D^{(j)}_{ss'}(R)^* \, D^{(j)}_{st}(R)}_{= \delta_{s't} \text{ since } D^{(j)}(R) \text{ is unitary}}$$

$$= \sum_{\alpha'}\sum_{s} \underbrace{\sum_{\alpha} D_{\rho\alpha}(\Lambda)\, D_{\alpha\alpha'}(\Lambda^{-1})}_{=\delta_{\rho\alpha'}}\, u_{\alpha'}(\Lambda p, s)\, D^{(j)}_{st}(R)$$

which after reshuffling indices becomes

$\alpha \to \alpha'$
$s \to s'$
$\rho \to \alpha$
$t \to s$

$$\sum_{\alpha'} D_{\alpha\alpha'}(\Lambda)\, u_{\alpha'}(p,s) = \sum_{s'} u_\alpha(\Lambda p, s')\, D^{(j)}_{s's}(R)$$

For $\phi^{(-)}_\alpha(x)$ one obtains similarly

$$\sum_{\alpha'} D_{\alpha\alpha'}(\Lambda)\, v_{\alpha'}(p,s) = \sum_{s'} v_\alpha(\Lambda p, s')\, D^{(j)}_{s's}(R)^*$$

These relations hold for arbitrary momenta p and LT Λ in the massive case with $R = L^{-1}(\Lambda p)\,\Lambda\, L(p)$.

page 85

We now evaluate these relations for the specific case in which $p' \to k' = (m, \vec{0})$ refers to the rest frame and $\Lambda \to L(p)$ is the standard boost from page 82 with $L(p)\,k = p$.

$$\Rightarrow \quad R = L^{-1}(\underbrace{L(p)k}_{p})\, L(p)\, \underbrace{L(k)}_{\mathbb{1}} = L^{-1}(p)\, L(p) = \mathbb{1}$$

and hence $D^{(s)}_{s's}(R) = \delta_{s's}$.

$$\Rightarrow \quad \boxed{\begin{aligned} u_\alpha(p,s) &= \sum_{\alpha'} D_{\alpha\alpha'}(L(p))\, u_{\alpha'}(k,s) \\ v_\alpha(p,s) &= \sum_{\alpha'} D_{\alpha\alpha'}(L(p))\, v_{\alpha'}(k,s) \end{aligned}} \qquad (I)$$

For a given representation $D(\Lambda)$ of the homogeneous Lorentz group, it is thus sufficient to determine the coefficients $u_\alpha(k,s)$ and $v_\alpha(k,s)$ in the rest frame of the massive particle.

As another specific case we consider a rotation $\Lambda \to \mathcal{R}$ in the rest frame with $p' \to k' = (m, \vec{0})$ such that $\mathcal{R}k = k$.

$$\Rightarrow \quad R = \underbrace{L^{-1}(\mathcal{R}k)}_{= \mathbb{1}}\, \mathcal{R}\, \underbrace{L(k)}_{= \mathbb{1}} = \mathcal{R}$$

$$\Rightarrow \boxed{\begin{aligned} \sum_{\alpha'} D_{\alpha\alpha'}(R)\, u_{\alpha'}(k,s) &= \sum_{s'} u_\alpha(k,s')\, \mathcal{D}^{(j)}_{s's}(R) \\ \sum_{\alpha'} D_{\alpha\alpha'}(R)\, v_{\alpha'}(k,s) &= \sum_{s'} v_\alpha(k,s')\, \mathcal{D}^{(j)}_{s's}(R)^* \end{aligned}} \qquad (\mathrm{II})$$

Wigner functions

What do these relations imply for the real scalar field, which transforms under $D_{(0,0)}(\Lambda) = \mathbb{1}$?

$$\mathrm{I}: \quad u(p,s) = u(k,s)$$
$$v(p,s) = v(k,s)$$

$$\mathrm{II}: \quad u(k,s) = \sum_{s'} u(k,s')\, \mathcal{D}^{(j)}_{s's}(R) \qquad \forall R$$
$$v(k,s) = \sum_{s'} v(k,s')\, \mathcal{D}^{(j)}_{s's}(R)^*$$

The latter relations are only satisfied for $j=0$, i.e. we learn that the scalar field indeed describes particles with spin 0 !

We are furthermore free to fix the normalisation of the coefficients to $u(k)=1$ and $v(k)=1$ such that with I we obtain

$$u(p) = u(k) = 1$$
$$v(p) = v(k) = 1$$

which reproduces the results from section 3.2.

For the complex scalar field, on the other hand, the relations I and II hold separately for the coefficients of the creation and annihilation operators of particle and antiparticle states, and we therefore again reproduce the results from the previous section.

Let us finally consider the massless case for which we have to start from the relations on page 117. We now obtain

$$\sum_{\alpha'} D_{\alpha\alpha'}(\Lambda)\, u_{\alpha'}(p,\sigma) = u_\alpha(\Lambda p,\sigma)\, e^{-i\theta\sigma}$$

$$\sum_{\alpha'} D_{\alpha\alpha'}(\Lambda)\, v_{\alpha'}(p,\sigma) = v_\alpha(\Lambda p,\sigma)\, e^{i\theta\sigma}$$

where σ is the helicity of the massless particle and θ is determined by the Wigner transformation $W = L^{-1}(\Lambda p)\,\Lambda\, L(p)$ as described in section 2.2.

We next specify to the reference frame with $p' \to k' = (\kappa,0,0,\kappa)$ and $\Lambda \to L(p)$ is the standard boost from page 89 with $L(p)\,k = p$.

$$\Rightarrow W = L^{-1}(\underbrace{L(p)k}_{p})\, L(p)\, \underbrace{L(k)}_{=\mathbb{1}} = L^{-1}(p)\, L(p) = \mathbb{1}$$

and hence $\theta = 0$.

$$\Rightarrow \boxed{\begin{aligned} u_\alpha(p,\sigma) &= \sum_{\alpha'} D_{\alpha\alpha'}(L(p))\, u_{\alpha'}(k,\sigma) \\ v_\alpha(p,\sigma) &= \sum_{\alpha'} D_{\alpha\alpha'}(L(p))\, v_{\alpha'}(k,\sigma) \end{aligned}} \qquad (\mathrm{I}')$$

For a rotation $\Lambda \to R$ around the z-axis that leaves $p^\mu \to k^\mu = (\kappa,0,0,\kappa)$ invariant, we now obtain with $Rk = k$

$$\Rightarrow W = \underbrace{L^{-1}(Rk)}_{=1} R \underbrace{L(k)}_{=1} = R \overset{!}{=} S(\alpha,\beta)\, \bar{R}(\theta) \qquad (\text{page } 90)$$

and hence $\alpha = \beta = 0$ and θ is the angle of the rotation matrix R

$$\Rightarrow \boxed{\begin{aligned} \sum_{\alpha'} D_{\alpha\alpha'}(R)\, u_{\alpha'}(k,\sigma) &= u_\alpha(k,\sigma)\, e^{-i\theta\sigma} \\ \sum_{\alpha'} D_{\alpha\alpha'}(R)\, v_{\alpha'}(k,\sigma) &= v_\alpha(k,\sigma)\, e^{i\theta\sigma} \end{aligned}} \qquad (\mathrm{II}')$$

For the scalar field, the latter relations now imply

$$\mathrm{II}': \quad \begin{aligned} u(k,\sigma) &= u(k,\sigma)\, e^{-i\theta\sigma} \qquad \forall\, \theta \\ v(k,\sigma) &= v(k,\sigma)\, e^{i\theta\sigma} \end{aligned}$$

and so the scalar field describes massless particles with helicity $\sigma = 0$.

3.4 Dirac field

The results of the previous section can be used to construct field operators that create and annihilate particles with non-zero spin. As an example, we consider massive, charged spin-1/2 particles in the following (e.g. electrons or quarks).

In analogy to the complex scalar field, we first combine the creation and annihilation fields to

$$\psi_\alpha(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(u_\alpha(p,s)\, e^{-ipx}\, a(p,s) + v_\alpha(p,s)\, e^{ipx}\, b^+(p,s) \right)$$

where $a(p,s)$, $a^+(p,s)$ and $b(p,s)$, $b^+(p,s)$ are two independent sets of creation and annihilation operators, and the coefficients $u_\alpha(p,s)$ and $v_\alpha(p,s)$ have to obey the constraints (I) and (II) that we derived in the previous section (~ pages 173-174).

But which representation $\mathcal{D}(\Lambda)$ of the homogeneous Lorentz group do we use and which statistics do the creation and annihilation operators obey?

In the previous section we learned that the representations (a,b) contain representations of spin $j = |a-b|, \ldots, a+b$. The simplest representations that include $j = 1/2$ are thus the spinor representations $(1/2, 0)$ and $(0, 1/2)$. It turns out, however, that these two-component representations can only describe massive, neutral spin-1/2 particles (~ Majorana field) or massless spin-1/2 particles (~ Weyl field).

For massive, charged spin-1/2 particles, on the other hand, one instead has to start from the four-component direct sum representation

$$\mathcal{D}_{(0,1/2)\oplus(1/2,0)}(\Lambda) = \begin{pmatrix} \mathcal{D}_{(0,1/2)}(\Lambda) & 0 \\ 0 & \mathcal{D}_{(1/2,0)}(\Lambda) \end{pmatrix}$$

The field operator associated with this representation is called a <u>Dirac field</u>.

Let us now consider the constraints (I) and (II) for this representation explicitly. Due to its block-diagonal structure, we can discuss these constraints for the $(0, 1/2)$ and $(1/2, 0)$ representations separately.

We first consider rotations with

$$J^{ij} = \frac{1}{2} \varepsilon^{ijk} \sigma^k$$

for both the $(0, 1/2)$ and $(1/2, 0)$ representations. It follows

$$J^i = \frac{1}{2} \varepsilon^{ijk} J^{jk} = \frac{1}{4} \underbrace{\varepsilon^{ijk} \varepsilon^{jkl}}_{2\delta^{il}} \sigma^l = \frac{1}{2} \sigma^i$$

The generators of the rotations are thus given in the fundamental representation, which is in line with $j = 1/2$.

For the representation matrices we then obtain

$$D_{(1/2,0)}(R) = D_{(0,1/2)}(R) = e^{-\frac{i}{2} \omega_{ij} J^{ij}}$$

$$= e^{-\frac{i}{4} \omega_{ij} \varepsilon^{ijk} \sigma^k} \equiv e^{-\frac{i}{2} \theta \vec{n} \cdot \vec{\sigma}}$$

with $\theta n^k = \frac{1}{2} \omega_{ij} \varepsilon^{ijk}$. They thus turn out to be identical to the $j = 1/2$ Wigner function

$$D^{(1/2)}(R) = e^{-\frac{i}{2} \theta \vec{n} \cdot \vec{\sigma}}$$

$$= \cos \frac{\theta}{2} - i (\vec{n} \cdot \vec{\sigma}) \sin \frac{\theta}{2}$$

where θ is the rotation angle and $\vec{n}$ the rotation axis.

The relations (II) then become

$$\sum_{\alpha'} \left[\cos\frac{\theta}{2} - i(\vec{n}\cdot\vec{\sigma})\sin\frac{\theta}{2}\right]_{\alpha\alpha'} u_{\alpha'}(u,s) = \sum_{s'} u_\alpha(u,s')\left[\cos\frac{\theta}{2} - i(\vec{n}\cdot\vec{\sigma})\sin\frac{\theta}{2}\right]_{s's}$$

$$\sum_{\alpha'} \left[\cos\frac{\theta}{2} - i(\vec{n}\cdot\vec{\sigma})\sin\frac{\theta}{2}\right]_{\alpha\alpha'} v_{\alpha'}(u,s) = \sum_{s'} v_\alpha(u,s')\left[\cos\frac{\theta}{2} + i(\vec{n}\,\vec{\sigma}^{*})\sin\frac{\theta}{2}\right]_{s's}$$

which must hold for arbitrary rotations characterised by θ and $\vec{n}$.

We can write these expressions more compactly in a matrix notation with $U_{\alpha s} \equiv u_\alpha(u,s)$ and $V_{\alpha s} \equiv v_\alpha(u,s)$. The above relations are then satisfied if

$$\sigma^i U = U\sigma^i$$

$$\sigma^i V = V(-\sigma^i)^* = V\varepsilon^{-1}\sigma^i\varepsilon$$

$\varepsilon\,\sigma^{i*}\varepsilon^{-1} = -\sigma^i$
(~ page 167)

$$\Rightarrow \sigma^i(V\varepsilon^{-1}) = (V\varepsilon^{-1})\sigma^i$$

As the matrices U and $(V\varepsilon^{-1})$ commute with all Pauli matrices, they must be proportional to the identity matrix and hence

$$u_\alpha(u,s) = c_u\,\delta_{\alpha s}$$

$$v_\alpha(u,s) = c_v\,\varepsilon_{\alpha s}$$

where c_u and c_v are arbitrary constants. Explicitly

$$u(u,\tfrac{1}{2}) = \begin{pmatrix} c_u \\ 0 \end{pmatrix} \qquad v(u,\tfrac{1}{2}) = \begin{pmatrix} 0 \\ c_v \end{pmatrix}$$

$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$u(u,-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ c_u \end{pmatrix} \qquad v(u,-\tfrac{1}{2}) = \begin{pmatrix} -c_v \\ 0 \end{pmatrix}$$

We now turn to the boosts and rotations (I). In the following we will show that the standard boost $L(p)$ for massive particles is represented by

$$D_{(1/2,0)}(L(p)) = \frac{p^0+m+\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}}$$

$$D_{(0,1/2)}(L(p)) = \frac{p^0+m-\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}}$$

According to (I) the four-component coefficients of the Dirac field are then given by

$$u(p,s) = \begin{pmatrix} \frac{p^0+m-\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}} & 0 \\ 0 & \frac{p^0+m+\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}} \end{pmatrix} u(k,s)$$

and similarly for $v(p,s)$. In the rest frame of the massive particle, we further have

$$u(k,\tfrac{1}{2}) = \begin{pmatrix} c_u \\ 0 \\ c_u' \\ 0 \end{pmatrix} \qquad v(k,\tfrac{1}{2}) = \begin{pmatrix} 0 \\ c_v \\ 0 \\ c_v' \end{pmatrix}$$

$$u(k,-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ c_u \\ 0 \\ c_u' \end{pmatrix} \qquad v(k,-\tfrac{1}{2}) = \begin{pmatrix} -c_v \\ 0 \\ -c_v' \\ 0 \end{pmatrix}$$

with constants c_u, c_v, c_u' and c_v', which must be chosen such that the canonical commutation or anticommutation relations from page 125 are fulfilled. We anticipate here that this requires $c_u = c_v = c_u' = \sqrt{m}$ and $c_v' = -\sqrt{m}$.

In the rest frame the coefficients of the Dirac field are thus given by

$$u(k,\tfrac{1}{2}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad v(k,\tfrac{1}{2}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$$u(k,-\tfrac{1}{2}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \qquad v(k,-\tfrac{1}{2}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$$

which satisfy

$$\gamma^0\, u(k,s) = u(k,s)$$

$$\gamma^0\, v(k,s) = -\, v(k,s)$$

in the chiral representation of the Dirac matrices with $\gamma^0 = \begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}$.

We can use these relations to rewrite the coefficients $u(p,s)$ and $v(p,s)$ in a 4-vector notation

$\sigma^\mu p_\mu = p^0 - \vec{\sigma}\cdot\vec{p}$
$\bar{\sigma}^\mu p_\mu = p^0 + \vec{\sigma}\cdot\vec{p}$

$$u(p,s) = \frac{1}{\sqrt{2m(p^0+m)}}\left[\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix} + \begin{pmatrix}\sigma^\mu p_\mu & 0\\ 0 & \bar{\sigma}^\mu p_\mu\end{pmatrix}\underbrace{\gamma^0\gamma^0}_{=1\!\!1}\right] u(k,s)$$

$$= \frac{1}{\sqrt{2m(p^0+m)}}\begin{pmatrix}m & \sigma^\mu p_\mu\\ \bar{\sigma}^\mu p_\mu & m\end{pmatrix} u(k,s)$$

$$= \frac{\not{p}+m}{\sqrt{2m(p^0+m)}}\, u(k,s)$$

where we introduced the Feynman slash notation $\not{p} \equiv p_\mu\gamma^\mu$.

For the coefficients $v(p,s)$ one finds similarly

$$v(p,s) = - \frac{\not{p} - m}{\sqrt{2m(p^0+m)}} \, v(k,s)$$

Notice that

$$\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu$$

$$= \frac{1}{2} p_\mu p_\nu \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2g^{\mu\nu}} + \frac{1}{2} \underbrace{p_\mu p_\nu}_{\text{sym}} \underbrace{[\gamma^\mu, \gamma^\nu]}_{\text{antisym}} = p^2 = m^2$$

The coefficients $u(p,s)$ and $v(p,s)$ therefore satisfy the relations

$$(\not{p} - m)\, u(p,s) \sim (\not{p} - m)(\not{p} + m)\, u(k,s)$$

$$= (m^2 + m\not{p} - m\not{p} - m^2)\, u(k,s) = 0$$

$$(\not{p} + m)\, v(p,s) \sim (\not{p} + m)(\not{p} - m)\, v(k,s) = 0$$

We are now in the position to show that the Dirac field satisfies the <u>Dirac equation</u>

$$\boxed{(i\not{\partial} - m)\, \psi(x) = 0}$$

(in addition to the KG equation for each field component)

$$(i\partial\!\!\!/ - m)\,\psi(x)$$

$$= (i\partial\!\!\!/ - m) \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\left(u(p,s)\,e^{-ipx}\,a(p,s) + v(p,s)\,e^{ipx}\,b^+(p,s)\right)$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\left(\underbrace{(p\!\!\!/ - m)\,u(p,s)}_{=0}\,e^{-ipx}\,a(p,s) + \underbrace{(-p\!\!\!/ - m)\,v(p,s)}_{=0}\,e^{ipx}\,b^+(p,s)\right)$$

$$= 0$$

Can we find a Lagrangian that yields the Dirac equation as its Euler-Lagrange equation?

In analogy to the scalar field, we are looking for a free field Lagrangian that is quadratic in the fields and that transforms as a scalar under homogeneous LT

$$\mathcal{U}(\Lambda)\,\mathcal{L}(x)\,\mathcal{U}^{-1}(\Lambda) = \mathcal{L}(\Lambda x)$$

to yield a Lorentz-invariant action.

But since the representation matrices

$$\mathcal{D}_{(0,1/2)\oplus(1/2,0)}(\Lambda) = \begin{pmatrix} \mathcal{D}_{(0,1/2)}(\Lambda) & 0 \\ 0 & \mathcal{D}_{(1/2,0)}(\Lambda) \end{pmatrix} \equiv \mathcal{D}_D(\Lambda)$$

are not unitary, products like $\Psi^+(x)\,\Psi(x)$ do not have a simple transformation law

$$\mathcal{U}(\Lambda)\,\Psi^+(x)\,\Psi(x)\,\mathcal{U}^{-1}(\Lambda)$$

$$= \mathcal{U}(\Lambda)\,\Psi^+(x)\,\mathcal{U}^{-1}(\Lambda)\,\mathcal{U}(\Lambda)\,\Psi(x)\,\mathcal{U}^{-1}(\Lambda)$$

$$= \left[\mathcal{U}(\Lambda)\,\Psi(x)\,\mathcal{U}^{-1}(\Lambda)\right]^+ \left[\mathcal{U}(\Lambda)\,\Psi(x)\,\mathcal{U}^{-1}(\Lambda)\right]$$

$$= \left[\mathcal{D}_D(\Lambda^{-1})\,\Psi(\Lambda x)\right]^+ \left[\mathcal{D}_D(\Lambda^{-1})\,\Psi(\Lambda x)\right]$$

$$= \Psi^+(\Lambda x)\,\underbrace{\mathcal{D}_D^+(\Lambda^{-1})\,\mathcal{D}_D(\Lambda^{-1})}_{?}\,\Psi(\Lambda x)$$

The representation matrices satisfy however

$$\mathcal{D}_D^+(\Lambda)\,\gamma^0 = \gamma^0\,\mathcal{D}_D^{-1}(\Lambda)$$

which is a consequence of the relation (↝ page 166)

$$\left[\mathcal{D}_{(0,1/2)}(\Lambda)\right]^+ = \left[\mathcal{D}_{(1/2,0)}(\Lambda)\right]^{-1}$$

$$\mathcal{D}_D^+(\Lambda)\gamma^0 = \begin{pmatrix} \mathcal{D}_{(0,1/2)}(\Lambda)^+ & 0 \\ 0 & \mathcal{D}_{(1/2,0)}(\Lambda)^+ \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathcal{D}_{(1/2,0)}(\Lambda)^{-1} \\ \mathcal{D}_{(0,1/2)}(\Lambda)^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{D}_{(0,1/2)}(\Lambda)^{-1} & 0 \\ 0 & \mathcal{D}_{(1/2,0)}(\Lambda)^{-1} \end{pmatrix} = \gamma^0 \mathcal{D}_D^{-1}(\Lambda)$$

So if we define a new sort of adjoint

$$\bar{\psi}(x) \equiv \psi^+(x)\gamma^0$$

the product $\bar{\psi}(x)\psi(x)$ transforms as a scalar

$$\mathcal{U}(\Lambda)\bar{\psi}(x)\psi(x)\mathcal{U}^{-1}(\Lambda)$$

$$= \psi^+(\Lambda x)\underbrace{\mathcal{D}_D^+(\Lambda^{-1})\gamma^0}\mathcal{D}_D(\Lambda^{-1})\psi(\Lambda x)$$

$$= \psi^+(\Lambda x)\gamma^0\mathcal{D}_D^{-1}(\Lambda^{-1})\mathcal{D}_D(\Lambda^{-1})\psi(\Lambda x)$$

$$= \bar{\psi}(\Lambda x)\psi(\Lambda x)$$

The product $\bar{\Psi}(x)\gamma^\mu \Psi(x)$ then transforms as

$$\mathcal{U}(\Lambda)\, \bar{\Psi}(x)\gamma^\mu \Psi(x)\, \mathcal{U}^{-1}(\Lambda)$$

$$= \bar{\Psi}(\Lambda x)\, \mathcal{D}_D^{-1}(\Lambda^{-1})\, \gamma^\mu\, \mathcal{D}_D(\Lambda^{-1})\, \Psi(\Lambda x)$$

$$= (\Lambda^{-1})^\mu{}_\nu\, \bar{\Psi}(\Lambda x)\gamma^\nu \Psi(\Lambda x)$$

where we have used the transformation properties of the Dirac matrices (→ page 169)*

$$\mathcal{D}_D^{-1}(\Lambda)\, \gamma^\mu\, \mathcal{D}_D(\Lambda) = \Lambda^\mu{}_\nu\, \gamma^\nu$$

By contraction with

$$\partial_\mu^x = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial(\Lambda x)^\rho}\, \frac{\partial(\Lambda x)^\rho}{\partial x^\mu} = \partial_\rho^{\Lambda x}\, \Lambda^\rho{}_\mu$$

we thus see that

$$\mathcal{U}(\Lambda)\, \bar{\Psi}(x)\, \not{\partial}_x \Psi(x)\, \mathcal{U}^{-1}(\Lambda)$$

$$= \Lambda^\rho{}_\mu\, (\Lambda^{-1})^\mu{}_\nu\, \bar{\Psi}(\Lambda x)\gamma^\nu \partial_\rho^{\Lambda x} \Psi(\Lambda x)$$

$$= \bar{\Psi}(\Lambda x)\, \not{\partial}_{\Lambda x} \Psi(\Lambda x)$$

transforms as a scalar!

---

* This is the correct transformation law of a vector operator like e.g.

$$\mathcal{U}(\Lambda)\, P^\mu\, \mathcal{U}^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\, P^\nu \qquad \text{(→ tutorials)}$$

$$\mathcal{U}(\Lambda)\, A^\mu(x)\, \mathcal{U}^{-1}(\Lambda) = \left[\mathcal{D}_F(\Lambda^{-1})\, A(\Lambda x)\right]^\mu = (\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda x)$$

The procedure can easily be generalised to higher-rank tensors, which transform as

$$U(\Lambda)\, \bar{\psi}(x)\, \gamma^{\mu_1} \cdots \gamma^{\mu_n}\, \psi(x)\, U^{-1}(\Lambda)$$

$$= (\Lambda^{-1})^{\mu_1}{}_{\nu_1} \cdots (\Lambda^{-1})^{\mu_n}{}_{\nu_n}\, \bar{\psi}(\Lambda x)\, \gamma^{\nu_1} \cdots \gamma^{\nu_n}\, \psi(\Lambda x)$$

and so the Dirac bilinears are useful building blocks to construct Lorentz scalars.

In view of its simple transformation properties, it furthermore appears natural to consider $\bar{\psi}(x) = \psi^{+}(x)\,\gamma^{0}$ instead of $\psi^{+}(x)$ as an independent variable for the Dirac field.

We therefore consider the Lagrangian

$$\boxed{\mathcal{L}(\psi, \bar{\psi}, \partial_\mu \psi, \partial_\mu \bar{\psi}) = \bar{\psi}\,(i\not{\partial} - m)\,\psi}$$

which obviously is a Lorentz scalar.

The Lagrangian yields the desired Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 - (i\not{\partial} - m)\psi = 0$$

which is the Dirac equation, as well as

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu (\bar{\psi} i \gamma^\mu) + m \bar{\psi} = 0$$

$$\Rightarrow \quad \bar{\psi}(x) (i \overleftarrow{\not{\partial}} + m) = 0$$

which is the adjoint of the Dirac equation.

Notice that the above Lagrangian is, however, not hermitian since

$$(\bar{\psi}\psi)^+ = (\psi^+ \gamma^0 \psi)^+ = \psi^+ \gamma^{0+} \psi = \psi^+ \gamma^0 \psi = \bar{\psi}\psi \qquad \text{hermitian}$$

$$(i\bar{\psi}\not{\partial}\psi)^+ = (i\psi^+\gamma^0\gamma^\mu\partial_\mu\psi)^+$$

$$= -i(\partial_\mu\psi)^+ \underbrace{\gamma^0\gamma^0}_{=1} \gamma^{\mu+}\gamma^{0+}\psi$$

$$= -i(\partial_\mu\bar{\psi})\gamma^\mu\psi = -i\bar{\psi}\overleftarrow{\not{\partial}}\psi \qquad \underline{\text{not}} \text{ hermitian}$$

where we have used the relations

$$\gamma^0\gamma^{\mu+}\gamma^0 = \gamma^\mu \qquad\qquad (\gamma^0)^+ = \gamma^0$$

which we will prove in the tutorials.

One could, however, replace $i\bar{\psi}\partial\!\!\!/\psi$ in the Lagrangian by the hermitian operator

$$\frac{i}{2}(\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}\overleftarrow{\partial\!\!\!/}\psi)$$

but under the integral $\int d^4x$ this operator is related to $i\bar{\psi}\partial\!\!\!/\psi$ by a partial integration

$$\frac{i}{2}(\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}\overleftarrow{\partial\!\!\!/}\psi) = i\bar{\psi}\partial\!\!\!/\psi - \frac{i}{2}\partial_\nu(\bar{\psi}\gamma^\nu\psi)$$

As the surface term is irrelevant for the dynamics, one prefers to use the compact expression $i\bar{\psi}\partial\!\!\!/\psi$, which thus is – up to a surface term – hermitian.

In contrast to the scalar field, we observe that the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

contains only first-order derivatives, which turns out to be a characteristic feature of fermionic fields. But so far we have not clarified yet that spin-1/2 particles are fermions, so let us derive the conjugate fields and compute the canonical commutation / anticommutation relations.

We obtain

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi} i \gamma^0 = i \psi^+$$

as well as

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = 0$$

In contrast to the complex scalar field, where ϕ and ϕ^+ are independent variables with conjugate momenta $\pi = \partial^0 \phi^+$ and $\pi^+ = \partial^0 \phi$, the field ψ^+ (or $\bar{\psi}$) is essentially the conjugate field of ψ and there therefore exists only one set of canonical variables (ψ, π)

with

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(u(p,s)\, e^{-ipx}\, a(p,s) + v(p,s)\, e^{ipx}\, b^+(p,s) \right)$$

$$\pi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{i}{2p^0} \left(v^+(p,s)\, e^{-ipx}\, b(p,s) + u^+(p,s)\, e^{ipx}\, a^+(p,s) \right)$$

where we kept the spinor indices implicit. The fields obviously satisfy

$$[\psi(t,\vec{x}), \psi(t,\vec{y})]_{\mp} = [\pi(t,\vec{x}), \pi(t,\vec{y})]_{\mp} = 0$$

Let us now consider the non-trivial commutator / anticommutator at equal times

$$[\psi(t,\vec{x}), \pi(t,\vec{y})]_\mp$$

$$= \sum_s \sum_r \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{i}{2q^0}$$

$$\Big\{ u(p,s)\, e^{-ipx}\, u^+(q,r)\, e^{iqy} \underbrace{[a(p,s), a^+(q,r)]_\mp}_{(2\pi)^3 2q^0 \delta^{sr} \delta^{(3)}(\vec{p}-\vec{q})}$$

$$+ v(p,s)\, e^{ipx}\, v^+(q,r)\, e^{-iqy} \underbrace{[b^+(p,s), b(q,r)]_\mp}_{\mp (2\pi)^3 2q^0 \delta^{sr} \delta^{(3)}(\vec{p}-\vec{q})} \Big\}$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{i}{2p^0} \Big\{ u(p,s)\, u^+(p,s)\, e^{i\vec{p}(\vec{x}-\vec{y})} \mp v(p,s)\, v^+(p,s)\, e^{-i\vec{p}(\vec{x}-\vec{y})} \Big\} \underbrace{\gamma^0\gamma^0}_{=1}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2p^0} \Big\{ (\not{p}+m)\, e^{i\vec{p}(\vec{x}-\vec{y})} \mp (\not{p}-m)\, e^{-i\vec{p}(\vec{x}-\vec{y})} \Big\} \gamma^0$$

$\vec{p} \to -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2p^0}\, e^{i\vec{p}(\vec{x}-\vec{y})} \Big\{ p_0\gamma^0 - \vec{p}\vec{\gamma} + m \mp (p_0\gamma^0 + \vec{p}\vec{\gamma} - m) \Big\} \gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2p^0}\, e^{i\vec{p}(\vec{x}-\vec{y})} \begin{cases} 2(-\vec{p}\vec{\gamma}+m)\gamma^0 & \text{commutator} \\ 2p_0\gamma^0\gamma^0 & \text{anticommutator} \end{cases}$$

where in the third step we have used the spin sums

$$\sum_s u(p,s)\,\bar{u}(p,s) = \not{p}+m \qquad \sum_s v(p,s)\,\bar{v}(p,s) = \not{p}-m$$

with $\bar{u}(p,s) \equiv u^+(p,s)\gamma^0$, $\bar{v}(p,s) \equiv v^+(p,s)\gamma^0$, which we will prove in the tutorials.

We thus obtain the desired result for the anticommutator

$$\{\psi_\alpha(t,\vec{x}),\ \pi_\beta(t,\vec{y})\}$$

$$= i\,\delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}(\vec{x}-\vec{y})} = i\delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y})$$

The Dirac field thus describes spin-1/2 particles, which are found to obey Fermi statistics !*

We next compute the Hamiltonian density

$$\mathcal{H}(\psi,\pi,\vec{\nabla}\psi) = \pi\,(\partial_0\psi) - \mathcal{L}$$

$$= i\psi^+ \underbrace{\gamma^0\gamma^0}_{1} (\partial_0\psi) - \bar{\psi}\,(i\not{\partial}-m)\psi$$

$$= \bar{\psi}\,(i\gamma^0\partial_0 - i\gamma^0\partial_0 - i\vec{\gamma}\,\vec{\nabla} + m)\,\psi \qquad \partial_\mu = (\partial_0, +\vec{\nabla})\ !$$

$$= \bar{\psi}\,(-i\vec{\gamma}\vec{\nabla} + m)\,\psi = -i\pi\gamma^0\,(-i\vec{\gamma}\vec{\nabla} + m)\,\psi$$

---

* On page 181 we fixed the constants c_u, c_v, c'_u, c'_v such that this anticommutator comes out correctly, but there exists no choice for these constants to make the corresponding commutator canonical.

By expressing the field operators in terms of creation and annihilation operators, we will show in the tutorials that the Hamiltonian becomes

$$H = \int d^3x \ :\mathcal{H}:$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \ p^0 \left\{ a^+(p,s)\, a(p,s) + b^+(p,s)\, b(p,s) \right\}$$

similar to the complex scalar field*

One similarly shows that

$$\vec{P} = - \int d^3x \ : \pi \vec{\nabla} \psi :$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \ \vec{p} \left\{ a^+(p,s)\, a(p,s) + b^+(p,s)\, b(p,s) \right\}$$

---

* One can actually argue the other way round and show that commutation relations for the Dirac field would lead to a Hamiltonian with

$$\left\{ a^+(p,s)\, a(p,s) - b^+(p,s)\, b(p,s) \right\}$$

which would imply that one can lower the energy by producing more and more antiparticles. In order to have a stable vacuum state, one therefore comes to the conclusion that the Dirac field has to satisfy canonical anticommutation relations (cf. the discussion in Peskin / Schröder, chapter 3.5).

In analogy to the complex scalar field, the Lagrangian is furthermore invariant under phase transformations

$$\psi'(x) = e^{i\alpha}\,\psi(x) \qquad \alpha \in \mathbb{R}$$

and there exists a conserved Noether charge, which allows one to distinguish particle and antiparticle states (→ tutorials).

We next compute the anticommutators of the fields at different times $x^0 \neq y^0$. Apart from

$$\{\psi(x), \psi(y)\} = \{\pi(x), \pi(y)\} = 0$$

we obtain

$$\{\psi(x), \pi(y)\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2p^0} \left\{ (\not{p}+m)\, e^{-ip(x-y)} + (\not{p}-m)\, e^{ip(x-y)} \right\} \gamma^0$$

$$= i\,(i\not{\partial}_x + m)\,\gamma^0 \left\{ \Delta(x-y) - \Delta(y-x) \right\}$$

with $\Delta(z)$ from page 142. As the anticommutator is proportional to the same difference of amplitudes as the scalar commutator, it vanishes outside the light cone.

$$\Delta(z) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, e^{-ipz}$$

But since the uncertainty relation requires that commutators of observables vanish outside the light cone, it is not immediately clear how this anticommutator is related to causality in the fermionic case.

The point is that spinor fields are not observable, and that hermitian operators constructed out of spinor fields obey Bose statistics.

Consider e.g. the hermitian operator $\bar{\psi}\psi$ and the commutator

$$[\bar{\psi}_\alpha(x)\psi_\alpha(x), \bar{\psi}_\beta(y)\psi_\beta(y)]$$

$$= \bar{\psi}_\alpha(x)\underbrace{\psi_\alpha(x)\bar{\psi}_\beta(y)}\psi_\beta(y) - \bar{\psi}_\beta(y)\underbrace{\psi_\beta(y)\bar{\psi}_\alpha(x)}\psi_\alpha(x)$$

$$= -\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)\psi_\alpha(x)\psi_\beta(y) + \bar{\psi}_\alpha(x)\{\psi_\alpha(x),\bar{\psi}_\beta(y)\}\psi_\beta(y)$$

$\bar{\psi} = -i\pi\gamma^0$

$$+\underbrace{\bar{\psi}_\beta(y)\bar{\psi}_\alpha(x)\psi_\beta(y)\psi_\alpha(x)}_{(-1)^2\,\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)\psi_\alpha(x)\psi_\beta(y)} - \bar{\psi}_\beta(y)\{\psi_\beta(y),\bar{\psi}_\alpha(x)\}\psi_\alpha(x)$$

$$= -i\,\bar{\psi}_\alpha(x)\{\psi_\alpha(x),\pi_\gamma(y)\}\gamma^0_{\gamma\beta}\psi_\beta(y)$$

$$+i\,\bar{\psi}_\beta(y)\{\psi_\beta(y),\pi_\gamma(x)\}\gamma^0_{\gamma\alpha}\psi_\alpha(x).$$

which vanishes outside the light cone as required by causality. The structure of the anticommutator thus guarantees that observables that are built out of fermionic fields cannot influence each other if they are not causally connected.

Let us finally discuss the Green's function to the Dirac operator with

$$(i\partial\!\!\!/ - m)\, G(x-y) = +i\, \delta^{(4)}(x-y)$$

convention

In Fourier space, we now obtain

$$\int d^4x\, e^{ip(x-y)} (i\partial\!\!\!/_x - m)\, G(x-y)$$

p.I.

$$= \int d^4x\, (p\!\!\!/ - m)\, e^{ip(x-y)}\, G(x-y)$$

$$= \int d^4x\, e^{ip(x-y)} (+i)\, \delta^{(4)}(x-y) = +i$$

$$\Rightarrow \quad G(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p\!\!\!/ - m}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i\,(p\!\!\!/ + m)}{p^2 - m^2}$$

The integral has the same poles at $p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2}$ as the scalar propagator (~ page 147), and we define the Feynman propagator of a Dirac field by the same prescription

$$S_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i\,(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon}$$

What is the spacetime interpretation of this quantity?

To answer this question, we first consider

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle$$

⇝ create particle at y and annihilate at x

$$= \sum_s \sum_r \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}$$

$$u_\alpha(p,s)\, e^{-ipx}\, \bar{u}_\beta(q,r)\, e^{iqy}\, \underbrace{\langle 0|a(p,s)\, a^+(q,r)\,|0\rangle}_{(2\pi)^3\, 2q^0\, \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q})}$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\underbrace{\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s)}_{(\not{p}+m)_{\alpha\beta}}\, e^{-ip(x-y)}$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}(\not{p}+m)_{\alpha\beta}\, e^{-ip(x-y)}$$

and similarly

$$\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle$$

⇝ create antiparticle at x and annihilate at y

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}(\not{p}-m)_{\alpha\beta}\, e^{ip(x-y)}$$

which are actually the two expressions that enter the

anticommutator on page 195.

For the Feynman propagator we can then perform the p^0-integration with contour methods along the lines of the calculation on page 149, which yields

$$S_F(x-y)_{\alpha\beta}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left\{ \theta(y^0-x^0)\,(-\not p+m)_{\alpha\beta}\, e^{ip(x-y)} + \theta(x^0-y^0)\,(\not p+m)_{\alpha\beta}\, e^{-ip(x-y)} \right\}_{p^0=\sqrt{\vec p^2+m^2}}$$

$$= \theta(x^0-y^0)\,\langle 0 \,|\, \psi_\alpha(x)\,\bar\psi_\beta(y)\,|0\rangle - \theta(y^0-x^0)\,\langle 0 \,|\, \bar\psi_\beta(y)\,\psi_\alpha(x)\,|0\rangle$$

$$\equiv \langle 0 |\, T\, \psi_\alpha(x)\,\bar\psi_\beta(y)\, |0\rangle$$

where the appropriate definition of the time-ordering prescription for fermionic fields takes care of the fermion signs, e.g.

$$T\,\psi_\alpha(x)\,\bar\psi_\beta(y) = \theta(x^0-y^0)\,\psi_\alpha(x)\,\bar\psi_\beta(y) - \theta(y^0-x^0)\,\bar\psi_\beta(y)\,\psi_\alpha(x)$$

The Feynman propagator thus again describes propagation from earlier to later times, and the Feynman rule becomes in momentum space

$$\beta \xrightarrow{\;\to p\;} \alpha \quad = \quad \frac{i\,(\not p+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$

where the arrow indicates the direction of the particle flow.

($\bar\psi_\beta$ produces particle)

3.5 Vector field

As another example we construct a field operator that creates and annihilates neutral spin-1 particles (e.g. photons). It is, however, instructive to first consider the massive case, which is relevant e.g. for the theory of weak interactions (~ Z-bosons).

The simplest representations that contain $j=1$ are $(1,0)$, $(1/2,1/2)$ and $(0,1)$. As the $(1,0)$ and $(0,1)$ representations cannot be used individually to construct parity-invariant theories (like QED), we focus here on the 4-vector representation $(1/2,1/2)$ with

$$D_F(\Lambda) = \Lambda$$

We thus start from a real vector field

$$A^\mu(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\left(u^\mu(p,s)\, e^{-ipx}\, a(p,s) + v^\mu(p,s)\, e^{ipx}\, a^+(p,s)\right)$$

with coefficients $u^\mu(p,s)$ and $v^\mu(p,s)$ that have to satisfy the constraints (I) and (II) for the $(1/2,1/2)$ representation.

We start with the rotations and relations (II)

$$\Lambda(R)^\mu{}_\nu \, u^\nu(k,s) = \sum_{s'} u^\mu(k,s') \, D^{(j)}_{s's}(R)$$

$$\Lambda(R)^\mu{}_\nu \, v^\nu(k,s) = \sum_{s'} v^\mu(k,s') \, D^{(j)}_{s's}(R)^*$$

where $\Lambda(R)^\mu{}_\nu = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right)$ is a rotation embedded in the 4-vector rotation. For the $(1/2, 1/2)$ representation, we expect that these equations have solutions for $j=0$ and $j=1$.

We first consider $\underline{j=0}$ for which the above relations become

$$\Lambda(R)^\mu{}_\nu \, u^\nu(k) = u^\mu(k)$$

$$\Lambda(R)^\mu{}_\nu \, v^\nu(k) = v^\mu(k)$$

which must hold for arbitrary rotations R. The 4-vectors that are invariant under rotations are obviously those that only have a time component. We then fix their normalization to

$$u^\mu(k) = \begin{pmatrix} im \\ 0 \\ 0 \\ 0 \end{pmatrix} = ik^\mu \qquad v^\mu(k) = \begin{pmatrix} -mi \\ 0 \\ 0 \\ 0 \end{pmatrix} = -k^\mu i$$

such that the canonical commutation/anticommutation relations will be fulfilled (we include a factor i to make the field operator hermitian)

We next consider $j=1$. The generators of the rotation matrices R in the fundamental representation satisfy (see page 33)

$$(J^j)^i{}_k = i\,\varepsilon^{ijk} \qquad\qquad (J^j)^\mu{}_\nu = 0 \quad \text{if } \mu=0 \text{ or } \nu=0$$

The generators of the $j=1$ Wigner functions are, on the other hand, well known from quantum mechanics

$$(J^3)^{(j=1)} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$(J^+)^{(j=1)} = (J^x + i\,J^y)^{(j=1)} = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(J^-)^{(j=1)} = (J^x - i\,J^y)^{(j=1)} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

in the $s=+1, 0, -1$ basis.

The infinitesimal version of (II) for $j=1$ then implies

$$\mu=0 \qquad 0 = \sum_{s'} u^0(k,s')\,[J^m]^{(j=1)}_{s's}$$

$$\mu=i \qquad [J^m]^i{}_k\, u^k(k,s) = \sum_{s'} u^i(k,s')\,[J^m]^{(j=1)}_{s's}$$

which must hold for all rotations and hence for all $m = 1,2,3$. Similar relations with the complex conjugate generators $-[J^m]^{(j=1)\,*}_{s's}$ hold for the $v^\mu(k,s)$ coefficients.

page 49 $\quad T^a_{\bar{D}} = -T^{a\,*}_D$

The first relation implies

$$u^0(k,s) = 0$$

for $s = +1, 0, -1$. The solution to the second equation is straight-forward but cumbersome. One finds (see next page)

$$u^\mu(k,s) = \varepsilon^\mu(k,s)$$

$$v^\mu(k,s) = \varepsilon^\mu(k,s)^*$$

with

$$\varepsilon^\mu(k,+1) = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$$

$$\varepsilon^\mu(k,0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\varepsilon^\mu(k,-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}$$

We next turn to the boosts and rotations (I)

$$u^\mu(p,s) = L(p)^\mu{}_\nu \, u^\nu(k,s)$$

$$v^\mu(p,s) = L(p)^\mu{}_\nu \, u^\nu(k,s)$$

with the standard boost for massive particles $L(p)$ from page 82.

notation $u^i(k,s) \to u^i(s)$

	$n=3$	$n=+$	$n=-$
$[J^n]^i{}_k\, u^k(s)$	$\begin{pmatrix} -i u^2(s) \\ i u^1(s) \\ 0 \end{pmatrix}$	$\begin{pmatrix} -u^3(s) \\ -i u^3(s) \\ +u^1(s) + i u^2(s) \end{pmatrix}$	$\begin{pmatrix} +u^3(s) \\ -i u^3(s) \\ -u^1(s) + i u^2(s) \end{pmatrix}$
$\sum_{s'} u^i(s')\, [J^n]^{(j=1)}_{s's}$	$\begin{pmatrix} u^i(+) \\ 0 \\ -u^i(-) \end{pmatrix}$	$\begin{pmatrix} 0 \\ \sqrt{2}\, u^i(+) \\ \sqrt{2}\, u^i(0) \end{pmatrix}$	$\begin{pmatrix} \sqrt{2}\, u^i(0) \\ \sqrt{2}\, u^i(-) \\ 0 \end{pmatrix}$

$$\overset{i}{3} \times \overset{s}{3} \times \overset{n}{3} = 27 \text{ equations}$$

solution: $u^i(+) = -\frac{c_u}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ $\quad u^i(0) = c_u \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad u^i(-) = \frac{c_u}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$

with one overall constant c_u, which we fix such that the canonical commutation relations are fulfilled $\to$ $c_u = 1$

For $v^i(s)$ notice that $(J^\pm)^* = \left((J^x)^* \mp i (J^y)^* \right)$, i.e. $n=+$ and $n=-$ correspond to the opposite linear combinations

	$n=3$	$n=+$	$n=-$
$[J^n]^i{}_k\, v^k(s)$	$\begin{pmatrix} -i v^2(s) \\ i v^1(s) \\ 0 \end{pmatrix}$	$\begin{pmatrix} +v^3(s) \\ -i v^3(s) \\ -v^1(s) + i v^2(s) \end{pmatrix}$	$\begin{pmatrix} -v^3(s) \\ -i v^3(s) \\ +v^1(s) + i v^2(s) \end{pmatrix}$
$-\sum_{s'} v^i(s') \left([J^n]^{(j=1)}_{s's}\right)^*$	$\begin{pmatrix} -v^i(+) \\ 0 \\ +v^i(-) \end{pmatrix}$	$\begin{pmatrix} 0 \\ -\sqrt{2}\, v^i(+) \\ -\sqrt{2}\, v^i(0) \end{pmatrix}$	$\begin{pmatrix} -\sqrt{2}\, v^i(0) \\ -\sqrt{2}\, v^i(-) \\ 0 \end{pmatrix}$

$\Rightarrow$ $v^i(+) = -\frac{c_v}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$ $\quad v^i(0) = c_v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad v^i(-) = \frac{c_v}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$

where we now need to set $c_v = 1$.

For $\underline{j=0}$ we obtain

$$u^\mu(p) = L(p)^\mu{}_\nu\, u^\nu(k) = L(p)^\mu{}_\nu\, ik^\nu = ip^\mu$$

$$v^\mu(p) = L(p)^\mu{}_\nu\, v^\nu(k) = -ip^\mu$$

With these coefficients the vector field becomes

$$A^\mu(x) = -\partial^\mu \phi(x)$$

i.e. it is the 4-derivative of a scalar field ($\sim$ spin 0), which we already discussed above.

We therefore focus on the case $\underline{j=1}$ in the following with

$$u^\mu(p,s) = \varepsilon^\mu(p,s)$$

$$v^\mu(p,s) = \varepsilon^\mu(p,s)^* \qquad \text{(since } L(p) \text{ is real)}$$

where $\varepsilon^\mu(p,s) = L(p)^\mu{}_\nu\, \varepsilon^\nu(k,s)$ are usually called polarisation vectors, which fulfill

$$p_\mu\, \varepsilon^\mu(p,s) = 0$$

As this is a Lorentz-invariant equation, it can most easily be verified in the rest frame of the massive particle.

Apart from the KG equation

$$(\partial^2 + m^2)\, A^\mu(x) = 0$$

we thus find that the real vector field associated with spin-1 particles

$$A^\mu(x) = \sum_{s=0,\pm 1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(\varepsilon^\mu(p,s)\, e^{-ipx}\, a(p,s) + \varepsilon^\mu(p,s)^*\, e^{ipx}\, a^+(p,s) \right)$$

satisfies the equation

$$\partial_\mu A^\mu(x) = 0$$

Can we find a Lagrangian that yields this equation as its Euler-Lagrange equation?

As the vector field has simple transformation properties under homogeneous LT, it is much easier to construct a Lorentz-invariant action than for the Dirac field. In analogy to electrodynamics, it is convenient to introduce the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

which transforms as a second-rank tensor under LT.

The Lagrangian

$$\boxed{\mathcal{L}(A^\mu, \partial^\mu A^\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu}$$

is then quadratic in the fields and transforms as a Lorentz scalar.

The associated Euler-Lagrange equations are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu}$$

$$= \partial_\mu (-F^{\mu\nu}) - m^2 A^\nu = 0$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

which is sometimes called the Proca equation.

We may contract this equation with ∂_ν

$$\Rightarrow \cancel{\underbrace{\partial_\nu \partial_\mu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{antisym.}}}^{=0} + m^2 \partial_\nu A^\nu = 0$$

which is the desired equation. One then has

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = (\partial^2 + m^2) A^\nu - \partial^\nu \underbrace{\partial_\mu A^\mu}_{=0} = 0$$

and so the Proca equation also implies the KG equation.

We next derive the conjugate fields

$$\Pi_r = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^r)} = -F^0{}_r = F_{r0} = \partial_r A_0 - \partial_0 A_r$$

which implies $\Pi_0 = 0$, and the field A^0 is therefore not an independent variable. The full set of canonical variables thus includes $(\vec{A}, \vec{\pi})$ with

$$A^i(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(\varepsilon^i(p,s)\, e^{-ipx}\, a(p,s) + \varepsilon^i(p,s)^* e^{ipx}\, a^+(p,s) \right)$$

$$\pi^i(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big([+ip^i \varepsilon^0(p,s) - ip^0 \varepsilon^i(p,s)]\, e^{-ipx}\, a(p,s)$$
$$+ [-ip^i \varepsilon^0(p,s)^* + ip^0 \varepsilon^i(p,s)^*]\, e^{ipx}\, a^+(p,s) \Big)$$

For the following calculation we need the spin sum

real

$$\sum_s \varepsilon^\mu(p,s)\, \varepsilon^\nu(p,s)^* = L(p)^\mu{}_{\mu'}\, L(p)^\nu{}_{\nu'} \sum_s \varepsilon^{\mu'}(k,s)\, \varepsilon^{\nu'}(k,s)$$

$$= L(p)^\mu{}_{\mu'}\, L(p)^\nu{}_{\nu'} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu'\nu'}$$

$$= L(p)^\mu{}_{\mu'}\, L(p)^\nu{}_{\nu'} \left(-g^{\mu'\nu'} + \frac{k^{\mu'} k^{\nu'}}{m^2} \right)$$

$$= -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

since $g^{\mu\nu}$ is an invariant tensor.

The derivation of the equal-time commutation / anticommutation relations is by now standard, and one finds that the commutators satisfy (for details see next page)

$$[A^i(t,\vec{x}), A^j(t,\vec{y})] = [\pi^i(t,\vec{x}), \pi^j(t,\vec{y})] = 0$$

$$[A^i(t,\vec{x}), \pi^j(t,\vec{y})] = i\,\delta^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})$$

The vector field with $\partial_\nu A^\nu(x) = 0$ thus describes spin-1 particles, which are found to obey Bose statistics!

We next compute the Hamiltonian density

$$\mathcal{H}(A^i, \pi^i, \partial^i A^j, \partial^i \pi^j) = \pi_i\,(\partial^0 A^i) - \mathcal{L}$$

$$= F_{i0}\,(\partial^0 A^i) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\nu A^\nu$$

in which we need to express A^0 and $\partial^0 A^i$ in terms of the conjugate fields π^i.

To do so, we use the Proca equation for $\nu = 0$

$$\partial_\mu F^{\mu 0} + m^2 A^0 = \vec{\nabla}\cdot\vec{\pi} + m^2 A^0 = 0$$

$$\Rightarrow \quad A^0 = -\frac{1}{m^2}\,\vec{\nabla}\cdot\vec{\pi}$$

$$[A^i(t,\vec{x}), A^j(t,\vec{y})]_\mp$$

$$= \sum_s \sum_r \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0}$$

$$\Big\{ \varepsilon^i(p,s)\, e^{-ipx}\, \varepsilon^j(q,r)^*\, e^{iqy}\, \underbrace{[a(p,s), a^\dagger(q,r)]_\mp}_{(2\pi)^3\, 2q^0\, \delta^{sr}\, \delta^{(3)}(\vec{p}-\vec{q})}$$

$$+ \varepsilon^i(p,s)^*\, e^{ipx}\, \varepsilon^j(q,r)\, e^{-iqy}\, \underbrace{[a^\dagger(p,s), a(q,r)]_\mp}_{\mp(2\pi)^3\, 2q^0\, \delta^{sr}\, \delta^{(3)}(\vec{p}-\vec{q})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big\{ \sum_s \varepsilon^i(p,s)\, \varepsilon^j(p,s)^*\, e^{i\vec{p}(\vec{x}-\vec{y})} \mp \sum_s \varepsilon^i(p,s)^*\, \varepsilon^j(p,s)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(-g^{ij} + \frac{p^ip^j}{m^2}\right) \Big\{ e^{i\vec{p}(\vec{x}-\vec{y})} \mp \underbrace{e^{-i\vec{p}(\vec{x}-\vec{y})}}_{\vec{p}\to-\vec{p}} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(-g^{ij} + \frac{p^ip^j}{m^2}\right) \Big\{ e^{i\vec{p}(\vec{x}-\vec{y})} \mp e^{i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

which vanishes for the commutator.

$$[A^i(t,\vec{x}), \pi^j(t,\vec{y})]_\mp$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big\{ \sum_s \varepsilon^i(p,s) \left[-ip^j \varepsilon^0(p,s)^* + ip^0 \varepsilon^j(p,s)^*\right] e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$\mp \sum_s \varepsilon^i(p,s)^* \left[+ip^j \varepsilon^0(p,s) - ip^0 \varepsilon^j(p,s)\right] e^{-i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big\{ -ip^0 g^{ij}\, e^{i\vec{p}(\vec{x}-\vec{y})} \mp (+ip^0 g^{ij})\, \underbrace{e^{-i\vec{p}(\vec{x}-\vec{y})}}_{\vec{p}\to-\vec{p}} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2}\, e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ \delta^{ij} \pm \delta^{ij} \right\}$$

$$= \begin{cases} i\delta^{ij}\, \delta^{(3)}(\vec{x}-\vec{y}) \\ 0 \end{cases}$$

which has the right form for the commutator

$$\left[\pi^i(t,\vec{x}),\ \pi^j(t,\vec{y})\right]_\mp$$

$$= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2p^0} \Big\{ \sum_s \left[ip^i \varepsilon^0(p,s) - ip^0 \varepsilon^i(p,s)\right]\left[-ip^j \varepsilon^0(p,s)^* + ip^0 \varepsilon^j(p,s)^*\right] e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$\mp \sum_s \left[-ip^i \varepsilon^0(p,s)^* + ip^0 \varepsilon^i(p,s)^*\right]\left[ip^j \varepsilon^0(p,s) - ip^0 \varepsilon^j(p,s)\right] e^{-i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2p^0} \left(-g^{00} p^i p^j - g^{ij} p^0 p^0\right) \Big\{ e^{i\vec{p}(\vec{x}-\vec{y})} \mp \underbrace{e^{-i\vec{p}(\vec{x}-\vec{y})}}_{\vec{p} \to -\vec{p}} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2p^0} \left(-g^{00} p^i p^j - g^{ij} p^0 p^0\right) \Big\{ e^{i\vec{p}(\vec{x}-\vec{y})} \mp e^{i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

which vanishes again for the commutator

We further have

$$\pi^i = F^{i0} = \partial^i A^0 - \partial^0 A^i$$

$\partial^\mu = (\partial^0, -\vec{\partial})$

$$\Rightarrow \quad \partial^0 A^i = \partial^i A^0 - \pi^i$$

$$= + \frac{1}{m^2} \nabla^i \nabla^j \pi^j - \pi^i$$

and

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} \left(F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} \right)$$

$$= -\frac{1}{2} F^{0i} F^{0i} + \frac{1}{2} \left(\partial^i A^j \partial^i A^j - \partial^i A^j \partial^j A^i \right)$$

$$= -\frac{1}{2} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2$$

Since

$$(\vec{\nabla} \times \vec{A})^2 = \varepsilon^{kij} \partial^i A^j \varepsilon^{klm} \partial^l A^m$$

$$= (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}) \partial^i A^j \partial^l A^m$$

$$= \partial^i A^j \partial^i A^j - \partial^i A^j \partial^j A^i$$

and the Hamiltonian finally becomes

$$\mathcal{H} = -\frac{1}{m^2} \pi^i \overset{\text{P.I}}{\nabla^i} \nabla^j \pi^j + \vec{\pi}^2 - \frac{1}{2} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2$$

$$- \frac{1}{2} m^2 \left(-\frac{1}{m^2} \vec{\nabla} \cdot \vec{\pi} \right)^2 + \frac{1}{2} m^2 \vec{A}^2$$

$$= \frac{1}{2} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 + \frac{1}{2m^2} (\vec{\nabla} \cdot \vec{\pi})^2 + \frac{1}{2} m^2 \vec{A}^2$$

It is now a purely technical exercise to work out the Hamiltonian $H = \int d^3x \, :\mathcal{H}:$ and the 3-momentum operator $\vec{P}$. Also the interpretation of the commutator of the fields at different times and its implications for causality are similar to the one of the real scalar field.

So here we only consider the Green's function to the Proca equation explicitly and we derive the Feynman rule associated with the free propagation of neutral, massive spin-1 particles.

We thus start from

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = \left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\mu = 0$$

and hence

$$\left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] G_{\nu\beta}(x-y) = \underset{\text{convention}}{+i}\, \delta^{(4)}(x-y)\, \delta^\mu{}_\beta$$

which in Fourier space becomes

$$\int d^4x \, e^{ip(x-y)} \left[(\partial_x^2 + m^2) g^{\mu\nu} - \partial_x^\mu \partial_x^\nu \right] G_{\nu\beta}(x-y)$$

$$\overset{\text{P.I.}}{=} \int d^4x \left[(-p^2 + m^2) g^{\mu\nu} + p^\mu p^\nu \right] e^{ip(x-y)} G_{\nu\beta}(x-y)$$

$$= \int d^4x \, e^{ip(x-y)} \, i\, \delta^{(4)}(x-y)\, \delta^\mu{}_\beta = i\, \delta^\mu{}_\beta$$

We thus have to invert the tensor $[(-p^2+m^2)g^{\lambda\nu} + p^\lambda p^\nu]$.

To do so, we first note that the most general second-rank tensor that only depends on the 4-vector p^λ and the metric $g^{\lambda\nu}$ has the form

$$A(p^2)\, g^{\lambda\nu} + B(p^2)\, p^\lambda p^\nu$$

We therefore need to solve the equation

$$[(-p^2+m^2)g^{\lambda\nu} + p^\lambda p^\nu]\,[A(p^2)\, g_{\nu\rho} + B(p^2)\, p_\nu p_\rho] \overset{!}{=} \delta^\lambda{}_\rho$$

$$= A(p^2)(-p^2+m^2)\,\delta^\lambda{}_\rho + [B(p^2)(\cancel{-p^2}+m^2) + A(p^2) + \cancel{p^2 B(p^2)}]\, p^\lambda p_\rho$$

$$\Rightarrow \quad A(p^2) = \frac{-1}{p^2-m^2}$$

$$B(p^2) = -\frac{1}{m^2} A(p^2) = \frac{1}{m^2}\,\frac{1}{p^2-m^2}$$

The Green's function

$$G^{\lambda\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2-m^2}\left[-g^{\lambda\nu} + \frac{p^\lambda p^\nu}{m^2}\right]$$

then again has the same poles at $p^0 = \pm\sqrt{\vec{p}^{\,2}+m^2}$ as the scalar propagator (→ page 147).

We then define the corresponding <u>Feynman propagator</u> as

$$\Delta_F^{\mu\nu}(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \, e^{-ip(x-y)} \, \frac{i}{p^2-m^2+i\varepsilon} \left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right]$$

which again corresponds to a vacuum matrix element of a time-ordered product of fields.

To see this, we first consider

$$\langle 0 | A^\mu(x) A^\nu(y) | 0 \rangle$$

$$= \sum_s \sum_r \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0}$$

$$\varepsilon^\mu(p,s)\, e^{-ipx}\, \varepsilon^\nu(q,r)^*\, e^{iqy} \underbrace{\langle 0 | a(p,s)\, a^\dagger(q,r) | 0 \rangle}_{(2\pi)^3 2q^0 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_s \varepsilon^\mu(p,s)\, \varepsilon^\nu(p,s)^* \, e^{-ip(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) e^{-ip(x-y)}$$

where we used the spin sum from page 207.

For the Feynman propagator we then again follow along the lines of the calculation on page 149, which yields

$$\Delta_F^{\mu\nu}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right)$$

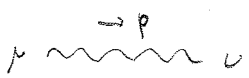
$$\left\{ \theta(y^0-x^0)\, e^{ip(x-y)} + \theta(x^0-y^0)\, e^{-ip(x-y)} \right\}_{p^0=\sqrt{\vec{p}^2+m^2}}$$

$$= \theta(y^0-x^0) \langle 0| A^\nu(y) A^\mu(x) |0\rangle + \theta(x^0-y^0) \langle 0| A^\mu(x) A^\nu(y) |0\rangle$$

$$= \langle 0| T A^\mu(x) A^\nu(y) |0\rangle$$

with the usual (bosonic) definition of the time-ordering prescription.

The Feynman rule in momentum space finally becomes



$$= \frac{i}{p^2-m^2+i\varepsilon} \left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right]$$

Let us now turn to the massless case, which is relevant for the description of photons. On the level of the Lagrangian, it seems that the limit $m \to 0$ is smooth and one obtains

$$\mathcal{L} = -\frac{1}{4} F_{\lambda\nu} F^{\lambda\nu}$$

which indeed yields the Maxwell equations for free fields

$$\partial_\lambda F^{\lambda\nu} = 0$$

On the level of the spin sum, however, the limit $m \to 0$ is not defined

$$\sum_s \varepsilon^\lambda(p,s)\, \varepsilon^\nu(p,s)^* = -g^{\lambda\nu} + \frac{p^\lambda p^\nu}{m^2} \to \infty$$

which is not surprising since massless spin-1 particles come in two helicity states (in parity-invariant theories), whereas massive spin-1 particles have three distinct spin configurations.

There is of course no point in taking the limit $m \to 0$ of the above equations, and one should instead go back to relations (I') and (II') from page 176 to construct the polarisation vectors of massless spin-1 particles. It turns out, however, that there exists no solution to these equations for the (1/2, 1/2) representation (for details see Weinberg, chapter 5.9). In other words, it is not possible to construct a Lorentz-covariant massless vector field for spin-1 particles, which transforms as

$$U(\Lambda)\, A^\mu(x)\, U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda x)$$

So how do we describe photons then?

One could start from the $(1,0) \oplus (0,1)$ representation, which would lead to a theory that is entirely formulated in terms of the field-strength tensor. Due to the derivatives in $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, the interactions in this theory would however be strongly suppressed at large distances, and the interactions that we observe in nature – like e.g. the electromagnetic interaction – simply do not belong to this class.

The more general case therefore consists in using a field operator

$$A^\mu(x) = \sum_{\sigma=\pm 1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(\varepsilon^\mu(p,\sigma)\, e^{-ipx}\, a(p,\sigma) + \varepsilon^\mu(p,\sigma)^* \, e^{ipx}\, a^+(p,\sigma) \right)$$

with two polarisation vectors that are constructed in analogy to the massive case from

$$\varepsilon^\mu(k,+1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \qquad \varepsilon^\mu(k,-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

but which now refer to a different reference frame with $k^\mu = (n,0,0,n)$.

We then obtain the polarisation vectors in an arbitrary frame as usual via relation (I')

$$\varepsilon^\mu(p,\sigma) = L(p)^\mu{}_\nu \, \varepsilon^\nu(k,\sigma)$$

where $L(p) = R\left(\frac{\vec{p}}{|\vec{p}|}\right) B\left(\frac{|\vec{p}|}{n}\right)$ is the standard boost for massless particles from page 89. As the boost $B\left(\frac{p}{n}\right)$ in z-direction leaves the x- and y-components invariant, we have

$$\varepsilon^\mu(p,\sigma) = R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\mu{}_\nu \, \varepsilon^\nu(k,\sigma)$$

where $R\left(\frac{\vec{p}}{|\vec{p}|}\right)$ is a rotation that carries the z-axis into the direction of $\vec{p}$.

The polarisation vectors thus fulfill

$$\varepsilon^0(p,\sigma) = 0$$

$$\vec{p}\cdot\vec{\varepsilon}\,(p,\sigma) = 0$$

since rotations leave the euclidean scalar product invariant and obviously $\vec{k}\cdot\vec{\varepsilon}\,(k,\sigma) = 0$. We further have

$$\sum_{\sigma=\pm1} \varepsilon^\mu(p,\sigma)\,\varepsilon^\nu(p,\sigma)^* = R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\mu{}_{\mu'} R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\nu{}_{\nu'} \sum_{\sigma=\pm1} \varepsilon^{\mu'}(k,\sigma)\,\varepsilon^{\nu'}(k,\sigma)$$

$$= R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\mu{}_{\mu'} R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\nu{}_{\nu'} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu'\nu'}$$

$$= R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\mu{}_{\mu'} R\left(\frac{\vec{p}}{|\vec{p}|}\right)^\nu{}_{\nu'} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} & \\ 0 & & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} & \\ 0 & & & \end{pmatrix}^{\mu\nu}$$

which explicitly shows that the construction is <u>not</u> covariant.

What is going on?

We argued that we cannot follow the systematic procedure from section 3.3 to construct a Lorentz-invariant action since the relations (I') and (II') have no solution in the massless case for the $(1/2, 1/2)$ representation.

Instead we suggested to use a field operator that satisfies

$$A^0(x) = 0 \qquad \rightsquigarrow \varepsilon^0(p,\sigma) = 0$$

$$\vec{\nabla}\cdot\vec{A}(x) = 0 \qquad \rightsquigarrow \vec{p}\cdot\vec{\varepsilon}(p,\sigma) = 0$$

i.e we are currently quantizing the electromagnetic field in Coulomb gauge. Because of this specific gauge choice, Lorentz-covariance is no longer manifest.

But the procedure in section 3.3 is to be understood as one particular method for constructing Lorentz-invariant actions. The field operator proposed here has a more complicated transformation law (for details see Weinberg, chapter 5.9)

$$U(\Lambda)\,A^{\prime}(x)\,U^{-1}(\Lambda) = (\Lambda^{-1})^{\prime}{}_{\nu}\,A^{\nu}(\Lambda x) + \underbrace{\partial^{\prime}_{x}\,\Omega(x,\Lambda)}_{\text{non-covariant term}}$$

with a scalar function $\Omega(x,\Lambda)$ whose explicit form is not needed here.

Can we construct a Lorentz-invariant action with this ingredient?

First of all we note that the non-covariant term drops out in the field-strength tensor, which then transforms as usual as a second-rank tensor

$$\mathcal{U}(\Lambda)\, F^{\mu\nu}(x)\, \mathcal{U}^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\mu'}\, (\Lambda^{-1})^{\nu}{}_{\nu'}\, F^{\mu'\nu'}(\Lambda x)$$

The non-covariant term is furthermore irrelevant if the vector field is coupled to a conserved current. For

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu$$

with $\partial^\mu j_\mu(x) = 0$, we have e.g.

$$\mathcal{U}(\Lambda)\, j_\mu(x)\, A^\mu(x)\, \mathcal{U}^{-1}(\Lambda)$$

$$= \mathcal{U}(\Lambda)\, j_\mu(x)\, \mathcal{U}^{-1}(\Lambda)\, \mathcal{U}(\Lambda)\, A^\mu(x)\, \mathcal{U}^{-1}(\Lambda)$$

$$= \left[(\Lambda^{-1})_\mu{}^{\rho}\, j_\rho(\Lambda x) \right] \left[(\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda x) + \partial^\mu_x\, \Omega(x,\Lambda) \right]$$

$\partial^\mu = \partial^\sigma_{\Lambda x}\, \Lambda_\sigma{}^\mu$ (page 187)

$$= j_\rho(\Lambda x)\, A^\rho(\Lambda x) + \underbrace{(\Lambda^{-1})_\mu{}^{\rho}\, \Lambda_\sigma{}^{\mu}}_{g_\sigma{}^\rho}\, j_\rho(\Lambda x)\, \partial^\sigma_{\Lambda x}\, \Omega(x,\Lambda)$$

$$\overset{\text{P.I.}}{=} j_\rho(\Lambda x)\, A^\rho(\Lambda x) - \underbrace{\left(\partial^\rho_{\Lambda x}\, j_\rho(\Lambda x) \right)}_{=0 \text{ on the physical states}} \Omega(x,\Lambda)$$

Our observations can be summarized as follows. A theory of massless spin-1 particles must be covariant under the transformations

$$A^\mu(x) \longrightarrow (\Lambda^{-1})^\mu{}_\nu \, A^\nu(\Lambda x) \qquad \rightsquigarrow \text{ Lorentz invariance}$$

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu \Omega(x) \qquad \rightsquigarrow \text{ gauge invariance}$$

Gauge symmetries, i.e. symmetries under local field transformations, thus arise naturally in a quantum theory of massless spin-1 particles!

We furthermore saw that massless spin-1 particles need to be coupled to conserved currents. Whenever a theory is invariant under a local symmetry transformation, it is of course also invariant under the corresponding global transformation, and according to Noether's theorem there then exists such a conserved current.

As it happens massless spin-1 particles play an important role in nature ($\rightsquigarrow$ photons, gluons), and we will therefore study gauge theories in more detail in later chapters.

But at this stage we want to derive the Feynman rule associated with the free propagation of massless spin-1 particles.

It turns out, however, that the canonical quantisation in Coulomb gauge is rather cumbersome (for details see Weinberg, chapters 8.3 - 8.5). One therefore often prefers to use a different gauge, in which Lorentz covariance is manifest. The Lorenz gauge $\partial_\mu A^\mu(x) = 0$ leads, however, to other complications. First of all, it turns out that the gauge condition cannot be implemented on the operator level, but it can only be realised on the level of the physical states $|\psi\rangle$

$$\langle\psi| \partial_\mu A^\mu(x) |\psi\rangle = 0$$

which is called the Gupta-Bleuler condition. The vector field in Lorenz gauge therefore describes four polarisations: two physical (transverse) polarisations and two unphysical (time-like and longitudinal) polarisations. The time-like polarisations furthermore have negative norm, which raises the question if the QFT is unitary (i.e. if the sum of probabilities is conserved). As we will illustrate in the tutorials, the Gupta-Bleuler condition ensures, however, that the unphysical polarisations do not contribute to physical observables. For more details about the Gupta-Bleuler formalism see e.g. appendix E of Schwabl II.

The most elegant method to derive the photon propagator is the path integral approach, which we will study in TPP II. There we will learn that the gauge-fixing procedure results in an additional term to the Lagrangian

$$\mathcal{L} = \underbrace{-\frac{1}{4} F_{\lambda\nu} F^{\lambda\nu}}_{\text{Lorentz and gauge invariant}} \underbrace{- \frac{1}{2\xi} (\partial_\lambda A^\lambda)^2}_{\text{invariant under } A^\lambda \to (\Lambda^{-1})^\lambda{}_\nu A^\nu \text{, but not gauge invariant since we have chose to work in Lorenz gauge}}$$

where ξ is an unphysical parameter that must drop out of all calculations for physical observables. The equations of motion then become

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu}$$

$$= \partial_\lambda \left(- F^{\lambda\nu} - \frac{1}{\xi} g^{\lambda\nu} (\partial_\beta A^\beta) \right) - 0 = 0$$

$$\Rightarrow \quad \partial_\lambda F^{\lambda\nu} + \frac{1}{\xi} \partial^\nu (\partial_\beta A^\beta)$$

$$= \left(\partial^2 g^{\lambda\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\lambda \partial^\nu \right) A_\lambda = 0$$

The corresponding Green's function thus satisfies

$$\left[\partial^2 g^{\mu\nu} - \left(1-\tfrac{1}{\xi}\right)\partial^\mu\partial^\nu\right] G_{\nu\rho}(x-y) = \underset{\text{convention}}{+i}\, \delta^{(4)}(x-y)\, \delta^\mu{}_\rho$$

We may then proceed in analogy to the massive case, which yields

$$\int d^4x\, e^{ip(x-y)} \left[\partial_x^2 g^{\mu\nu} - \left(1-\tfrac{1}{\xi}\right)\partial^\mu\partial^\nu\right] G_{\nu\rho}(x-y)$$

$$\downarrow \text{p.I.}$$

$$= \int d^4x\, e^{ip(x-y)} \left[-p^2 g^{\mu\nu} + \left(1-\tfrac{1}{\xi}\right)p^\mu p^\nu\right] e^{ip(x-y)}\, G_{\nu\rho}(x-y)$$

$$= \int d^4x\, e^{ip(x-y)}\, i\, \delta^{(4)}(x-y)\, \delta^\mu{}_\rho = i\, \delta^\mu{}_\rho$$

and hence (~ page 211)

$$\left[-p^2 g^{\mu\nu} + \left(1-\tfrac{1}{\xi}\right)p^\mu p^\nu\right]\left[A(p^2)\, g_{\nu\rho} + B(p^2)\, p_\nu p_\rho\right] \overset{!}{=} \delta^\mu{}_\rho$$

$$= -p^2 A(p^2)\, \delta^\mu{}_\rho + \left[\cancel{-p^2 B(p^2)} + \left(1-\tfrac{1}{\xi}\right)A(p^2) + \cancel{\left(1-\right.}\tfrac{1}{\xi}\Big)\, p^2 B(p^2)\right] p^\mu p_\rho$$

$$\Rightarrow \quad A(p^2) = \frac{-1}{p^2}$$

$$B(p^2) = -\frac{(1-\xi)}{p^2}\, A(p^2) = \frac{(1-\xi)}{p^4}$$

It follows

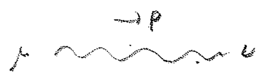
$$G^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

which has poles at $p^0 = \pm|\vec{p}|$.

The Feynman propagator in Lorenz gauge thus becomes

$$\Delta_F^{\mu\nu}(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2+i\varepsilon} \left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\varepsilon} \right]$$

which diverges in the limit $\xi \to \infty$, i.e. when we remove the gauge-fixing term from the Lagrangian. As physical observables are independent of the gauge parameter ξ, one is free to choose any numerical value for ξ, and one therefore often works with the momentum-space Feynman rule



$$= \frac{i}{p^2+i\varepsilon}\,(-g^{\mu\nu})$$

which corresponds to the <u>Feynman gauge</u> with $\xi = 1$.