2. Particle states

In this chapter we are going to study the implications of Lorentz invariance on the physical states of a quantum system. In particular, we will learn that one-particle states are associated with irreducible representations of the Poincaré group, and we will see how the concepts of spin and antiparticles arise in this context. We will then proceed and introduce many-particle states using the formalism of creation and annihilation operators. But before doing so, we start with a brief revision of the basic properties of Lorentz transformations and the associated Poincaré group.

2.1. Poincaré group

A relativistic quantum theory must incorporate Einstein's principle of relativity, which states that the physical laws of nature take the same form in all inertial reference frames, and that the speed of light is the same in each of these frames. The transformations that connect two different inertial frames are the Lorentz transformations (LT), which replace the Galilei transformations of Newtonian mechanics. As the LT mix time and spacial components, one combines them into a 4-vector

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$$

The 4-dimensional space-time with metric

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is called Minkowski space.

Due to the signs in the Minkowski metric, the notation distinguishes between upper and lower indices. Using Einstein's sum convention*, one associates with each <u>contravariant vector</u> x^μ a <u>covariant vector</u> $x_\mu \equiv (x_0, x_1, x_2, x_3)$ via

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3)$$

i.e $x_0 = x^0$, but $x_i = -x^i$ for $i = 1, 2, 3$.

Other important 4-vectors are

$$p^\mu = (p^0, \vec{p}) = (E, \vec{p})$$

with $E = \sqrt{m^2 + \vec{p}^{\,2}}$, and

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

which is a covariant vector and hence

$$\partial^\mu \equiv g^{\mu\nu} \partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

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* The notation implies implicit sums over equal indices of which one must be an upper and the other one a lower index.

One further introduces a scalar product in Minkowski space by "contracting" a contravariant and a covariant vector

$$x \cdot y \equiv x_\mu y^\mu = g_{\mu\nu} x^\nu y^\mu = x^0 y^0 - \vec{x} \cdot \vec{y}$$

In analogy to rotations that leave the euclidean scalar product invariant, the LT are those transformations which leave the Minkowskian scalar product invariant

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Rightarrow x' \cdot y' = g_{\mu\nu} x'^\mu y'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma y^\sigma$$

$$\stackrel{!}{=} x \cdot y = g_{\rho\sigma} x^\rho y^\sigma \qquad \forall x^\rho, y^\sigma$$

$$\Rightarrow g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

which becomes in matrix notation

$$\boxed{\Lambda^T g \Lambda = g}$$

which is the analog of $R^T \mathbb{1} R = \mathbb{1}$ for rotations in euclidean space. One can show that the invariance of the Minkowskian scalar product guarantees that the speed of light is the same in each inertial frame.

Similar to the rotations in euclidean space, the LT form a group. To see this, we first note that $\Lambda^T g \Lambda = g$ implies

$$\det(\Lambda^T g \Lambda) = (\det \Lambda)^2 \det g = \det g$$

$$\Rightarrow \det \Lambda = \pm 1$$

Moreover, one verifies that the group axioms are fulfilled

- closed since $\Lambda_1 \cdot \Lambda_2$ is a LT

$$(\Lambda_1 \Lambda_2)^T g (\Lambda_1 \Lambda_2) = \Lambda_2^T \underbrace{\Lambda_1^T g \Lambda_1}_{g} \Lambda_2 = \Lambda_2^T g \Lambda_2 = g \quad \checkmark$$

- associative ✓
- identity element $\mathbb{1}$ with $\mathbb{1}^T g \mathbb{1} = g$ ✓
- inverse to Λ exists since $\det \Lambda \neq 0$ and

$$\Lambda^T g \Lambda = g \quad \Rightarrow \quad \Lambda^{-1} = g \Lambda^T g$$

$$(\Lambda^{-1})^T g \Lambda^{-1} = g \Lambda \underbrace{g\, g}_{\mathbb{1}} \underbrace{g \Lambda^T g}_{\Lambda^{-1}} = g \Lambda \Lambda^{-1} = g \quad \checkmark$$

The condition $\Lambda^T g \Lambda$ yields 10 independent equations for 16 coefficients of a real 4×4 matrix. The dimension of the Lorentz group is therefore 6.

The LT are thus parametrised by 6 parameters of which 3 describe the usual rotations in 3-dimensional euclidean space,

$$\Lambda^\mu{}_\nu = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right) \qquad \text{with} \qquad R R^T = \mathbb{1}$$

The remaining parameters yield the velocity-dependent Lorentz boosts, which mix the time and the spacial components. A boost along the x-direction is e.g. given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\psi$, $v\gamma = \sinh\psi$ and ψ is called the rapidity. A boost along an arbitrary direction $\vec{v}$ is then characterised by three parameters v_x, v_y, v_z, and it can be obtained by combining the standard boost from above with appropriate rotations.

One finds

$$\Lambda^\mu{}_\nu = \left(\begin{array}{c|ccc} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ \hline -\gamma v_x & & & \\ -\gamma v_y & & \delta_{ij} + (\gamma-1)\frac{v_i v_j}{\vec{v}^2} & \\ -\gamma v_z & & & \end{array}\right) \qquad i,j = x,y,z$$

The Lorentz group is not compact since – roughly speaking – the rapidity $\psi \in \mathbb{R}$ is not bounded to a compact interval. It is furthermore not connected since there exist elements that are not continuously connected to the identity element. Apart from $\det \Lambda = \pm 1$, one classifies the elements of the Lorentz group according to their value of $\Lambda^0{}_0$. In general, one has

$$1 = g_{00} = g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2$$

$$\Rightarrow (\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \geq 1$$

and hence $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

The identity component thus consists of all elements with $\det \Lambda = +1$ and $\Lambda^0{}_0 \geq 1$, which is the subgroup of <u>proper, orthochronous LT</u> $L_+^\uparrow$.

$\uparrow$ – $\Lambda^0{}_0 \geq 1$

$+$ – $\det \Lambda = +1$

The remaining elements of the Lorentz group then follow by including parity and time reversal transformations, which by themselves are special LT

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad P^T g P = g \quad \checkmark$$

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad T^T g T = g \quad \checkmark$$

The Lorentz group thus has the following structure

$$\begin{tikzcd}
\begin{array}{c}\text{proper, orthochronous LT} \\ \det \Lambda = +1 \\ \Lambda^0{}_0 \geq 1\end{array} \arrow[r, "P"] \arrow[d, "T"'] & \begin{array}{c}\text{LT with} \\ \det \Lambda = -1 \\ \Lambda^0{}_0 \geq 1\end{array} \arrow[d, "T"] \\
\begin{array}{c}\text{LT with} \\ \det \Lambda = -1 \\ \Lambda^0{}_0 \leq -1\end{array} \arrow[r, "P"'] & \begin{array}{c}\text{LT with} \\ \det \Lambda = +1 . \\ \Lambda^0{}_0 \leq -1\end{array}
\end{tikzcd}$$

It turns out that Einstein's principle of relativity only applies to the subgroup of proper, orthochronous LT $L_+^\uparrow$. The weak interactions, in particular, are not invariant under parity and time reversal transformations, although the electromagnetic and strong interactions are so.

Apart from rotations and Lorentz boosts, two inertial frames may be connected by a translation in space-time. Including translations, one starts from

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

with $\Lambda^T g \Lambda = g$, which is called an inhomogeneous LT or a <u>Poincaré transformation</u>.

One again easily verifies that the group axioms are fulfilled

- two successive transformations give

$$x''^\mu = \Lambda_2{}^\mu{}_\nu x'^\nu + a_2{}^\mu$$

$$= \Lambda_2{}^\mu{}_\nu (\Lambda_1{}^\nu{}_\rho x^\rho + a_1{}^\nu) + a_2{}^\mu$$

$$= (\Lambda_2 \Lambda_1)^\mu{}_\rho x^\rho + (\Lambda_2 a_1 + a_2)^\mu \qquad (*)$$

which is again a Poincaré transformation since $\Lambda_2 \Lambda_1$ satisfies $\Lambda^T g \Lambda = g$ as shown above ✓

- In order to see that the transformation is associative, we first perform the combined transformation

$$x \xrightarrow{(\Lambda_1, a_1)} x' \xrightarrow{(\Lambda_2, a_2)} x'' \text{ followed by } x'' \xrightarrow{(\Lambda_3, a_3)} x''',$$

giving

$$x'''^{\mu} = \Lambda_3{}^{\mu}{}_{\nu}\, x''^{\nu} + a_3^{\mu}$$

$$= \Lambda_3{}^{\mu}{}_{\nu} \left((\Lambda_2 \Lambda_1)^{\nu}{}_{\beta}\, x^{\beta} + (\Lambda_2 a_1 + a_2)^{\nu} \right) + a_3^{\mu}$$

$$= (\Lambda_3 \Lambda_2 \Lambda_1)^{\mu}{}_{\beta}\, x^{\beta} + (\Lambda_3 \Lambda_2 a_1 + \Lambda_3 a_2 + a_3)^{\mu}$$

whereas the combined transformation $x' \xrightarrow{(\Lambda_2, a_2)} x'' \xrightarrow{(\Lambda_3, a_3)} x'''$

preceded by $x \xrightarrow{(\Lambda_1, a_1)} x'$ yields

$$x'''^{\mu} = (\Lambda_3 \Lambda_2)^{\mu}{}_{\nu}\, x'^{\nu} + (\Lambda_3 a_2 + a_3)^{\mu}$$

$$= (\Lambda_3 \Lambda_2)^{\mu}{}_{\nu} \left(\Lambda_1{}^{\nu}{}_{\beta}\, x^{\beta} + a_1^{\nu} \right) + (\Lambda_3 a_2 + a_3)^{\mu}$$

$$= (\Lambda_3 \Lambda_2 \Lambda_1)^{\mu}{}_{\beta}\, x^{\beta} + (\Lambda_3 \Lambda_2 a_1 + \Lambda_3 a_2 + a_3)^{\mu} \quad ✓$$

- the identity element is given by $(\Lambda, a) = (\mathbb{1}, 0)$ and it fulfills $\mathbb{1}^T g\, \mathbb{1} = g$ ✓
- from (*) we see that the inverse element to (Λ, a) is given by $(\Lambda^{-1}, -\Lambda^{-1} a)$, which again satisfies $\Lambda^T g\, \Lambda = g$ as shown above. ✓

The <u>Poincaré group</u> depends on 10 parameters, and it is again neither compact nor connected.

In a quantum theory one considers representations of the Poincaré group on the Hilbert space of the physical states $|\psi\rangle$. For a group element (Λ, a), there thus exists an operator $D(\Lambda, a)$ with

$$|\psi'\rangle = D(\Lambda, a)\, |\psi\rangle$$

As the results of all measurements must be the same in each inertial frame, one requires that the <u>probabilities</u> are equal for all states

$$|\langle \varphi' | \psi' \rangle|^2 = |\langle \varphi | \psi \rangle|^2$$

This implies that the operators $D(\Lambda, a)$ must fulfill the group multiplication law (*) up to an unobservable phase

$$D(\Lambda_2, a_2)\, D(\Lambda_1, a_1) = e^{i\alpha(\Lambda_1, a_1, \Lambda_2, a_2)}\, D(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

For $\alpha \neq 0$ this is called a projective representation (for details see Weinberg I, chapter 2.7). We have actually encountered such an example with a non-trivial phase at the end of the last chapter when we discussed the representations of the $SO(3)$ for half-integer values of j. There we saw that one can always remove the phases by considering the corresponding universal covering group, which is the $SU(2)$ in this specific case (see also the tutorials for more details). We therefore only consider the case with $\alpha = 0$ in the following.

The invariance of the probabilities implies that the operators $D(\Lambda, a)$ are <u>unitary</u>

$$\langle \varphi' | \psi' \rangle = \langle \varphi | D^\dagger(\Lambda, a) D(\Lambda, a) | \psi \rangle \overset{!}{=} \langle \varphi | \psi \rangle$$

$$\Rightarrow \quad D^\dagger(\Lambda, a) D(\Lambda, a) = \mathbb{1} \qquad \forall \; (\Lambda, a)$$

and we will therefore adopt the notation and write $U(\Lambda, a)$ for the representations of the Poincaré group in the following.*

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* Wigner's theorem states that the operators may also be anti-unitary. We will come back to this case when we discuss time-reversal transformations.

As usual it is convenient to consider the associated Lie algebra with

$$U(g) = e^{i\theta^a T^a_U} = 1\!\!1 + i\theta^a T^a_U + \dots$$

(T^a_U generators of the Lie algebra in the representation U)

A Poincaré transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \dots$$

$$a^\mu = \varepsilon^\mu + \dots$$

with infinitesimal $\omega^\mu{}_\nu$ and ε^μ satisfies

$$\begin{aligned} g_{\rho\sigma} &= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \\ &= g_{\mu\nu} (\delta^\mu{}_\rho + \omega^\mu{}_\rho + \dots)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma + \dots) \\ &= g_{\rho\sigma} + (\omega_{\sigma\rho} + \omega_{\rho\sigma}) + \dots \end{aligned}$$

$$\Rightarrow \quad \omega_{\rho\sigma} = -\omega_{\sigma\rho}$$

and so $\omega^\mu{}_\nu$ has only 6 independent components. Infinitesimal Poincaré transformations are thus described by 10 real parameters $\omega^\mu{}_\nu$ and ε^μ in agreement with the dimension of the Poincaré group.

Infinitesimal Poincaré transformations are represented on the Hilbert space by unitary operators

$$\mathcal{U}(\mathbb{1}+\omega, \varepsilon) = \mathbb{1} - \frac{i}{2}\,\omega_{\rho\nu}\,J^{\rho\nu}_{\mathcal{U}} + i\,\varepsilon_\rho\,P^\rho_{\mathcal{U}} + \dots$$

where $J^{\rho\nu}_{\mathcal{U}}$ and $P^\rho_{\mathcal{U}}$ are the generators of the <u>Poincaré algebra</u> in the representation $\mathcal{U}$ (we will suppress the subscript $\mathcal{U}$ in the following). The antisymmetry of $\omega_{\rho\nu}$ implies that the generators $J^{\rho\nu}$ are antisymmetric, too. We further have

$$\mathcal{U}^+(\mathbb{1}+\omega, \varepsilon) = \mathbb{1} + \frac{i}{2}\,\omega_{\rho\nu}\,(J^{\rho\nu})^+ - i\,\varepsilon_\rho\,(P^\rho)^+ + \dots$$

$$\Rightarrow \quad \mathbb{1} = \mathcal{U}^+(\mathbb{1}+\omega, \varepsilon)\,\mathcal{U}(\mathbb{1}+\omega, \varepsilon)$$

$$= \mathbb{1} - \frac{i}{2}\,\omega_{\rho\nu}\left(J^{\rho\nu} - (J^{\rho\nu})^+\right) + i\,\varepsilon_\rho\left(P^\rho - (P^\rho)^+\right) + \dots$$

$$\Rightarrow \quad J^{\rho\nu} = (J^{\rho\nu})^+$$

$$P^\rho = (P^\rho)^+$$

The generators of the Poincaré algebra are thus hermitian, and they may therefore correspond to physical observables.

By expanding the composition law

$$\mathcal{U}(\Lambda_2, a_2)\, \mathcal{U}(\Lambda_1, a_1) = \mathcal{U}(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

to first non-trivial order around the identity element $(\mathbb{1}, 0)$,

we will show in the tutorials that the generators of the Poincaré algebra satisfy the commutation relations

$$\boxed{\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= i\,(g^{\mu\sigma} J^{\nu\rho} + g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho}) \\ [P^\mu, J^{\rho\sigma}] &= i\,(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho) \\ [P^\mu, P^\nu] &= 0 \end{aligned}}$$

which is the analog of $[T^a, T^b] = i f^{abc} T^c$ and thus defines the structure constants of the Poincaré algebra.

We will see later that $P^0 = H$ is the Hamilton operator and $\vec{P}$ the momentum operator in a given frame. The commutator $[P^0, P^\nu] = 0$ for $\nu = 0,1,2,3$ therefore reflects energy-momentum conservation.

One can further group the generators $J^{\mu\nu}$ into two 3-vectors

$$\vec{J} = (J^1, J^2, J^3) \equiv (J^{23}, J^{31}, J^{12})$$

$$\vec{K} = (K^1, K^2, K^3) \equiv (J^{10}, J^{20}, J^{30})$$

such that $J^{ij} = \varepsilon^{ijk} J^k$ or $J^i = \frac{1}{2} \varepsilon^{ijk} J^{jk}$ and $K^i = J^{i0}$.

The commutators $[J^{\mu\nu}, J^{\rho\sigma}]$ then translate into

$$[J^i, J^j] = i\,\varepsilon^{ijk} J^k$$

$$[J^i, K^j] = i\,\varepsilon^{ijk} K^k$$

$$[K^i, K^j] = -i\,\varepsilon^{ijk} J^k$$

The first relation tells us that $\vec{J}$ is the angular momentum operator that generates rotations in 3-dimensional euclidean space, and so $\vec{K}$ must be the generator of the Lorentz boosts.

From $[P^\mu, J^{\rho\sigma}]$ we further derive

$$[P^0, J^i] = 0$$

which reflects angular momentum conservation, whereas

$$[P^0, K^i] = -i P^i$$

As the generators K^i are not conserved, we will not use the corresponding eigenvalues to label the physical states.

2.2. One-particle states

In a relativistic quantum theory the Hilbert space is thus equipped with unitary representations of the Poincaré group. It is then natural to define the one-particle states as those states that are associated with irreducible representations of the Poincaré group. As reducible representations contain invariant subspaces, which by themselves provide a relativistic invariant description of a part of the system, they are considered to reflect the many-particle states.*

In the group theory introduction we learned that irreducible representations can be classified according to the eigenvalues of the Casimir operators of the Lie algebra. As the eigenvalues of the Casimir operators do not change within an irreducible representation (recall Schur's lemma: $C \sim 1\!\!1$ in an irreducible representation), they provide the intrinsic properties that characterise a particle species.

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* Recall that unitary representations are always completely reducible.

Further properties of a particle are then described by the one-particle states, which are constructed as usual as the eigenstates of a complete set of commuting observables.

It is instructive to recall the example from pages 53-57:

* $\vec{J}^2$ is the only Casimir operator of the $su(2)$
* eigenvalues $\sim j$ characterise the irreducible representations
* $\vec{J}$ and J_z form a complete set of commuting observables

  $\rightarrow$ contain a common set of eigenstates $|j\,m\rangle$
* there exist $2j+1$ states within an irreducible representation with fixed j

So what are the characteristic properties of a particle species? In other words, which properties allow us to distinguish a particle species from another one?

In order to answer these questions we have to find the Casimir operators of the Poincaré algebra. It turns out that there exist two Casimir operators

- $P^2 = P^\mu P_\mu$
- $W^2 \equiv -W^\mu W_\mu$

where $W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$ is the <u>Pauli-Lubanski vector</u> and $\varepsilon_{\mu\nu\rho\sigma}$ is the four-dimensional totally antisymmetric Levi-Civita tensor (we use the convention $\varepsilon^{0123} = +1 = -\varepsilon_{0123}$). We will show in the tutorials that P^2 and W^2 commute with all generators of the Poincaré algebra.

We will see later that the eigenvalues of P^2 and W^2 are related with the <u>mass</u> and <u>spin</u> of a particle, which therefore provide a complete characterisation of a particle species from the perspective of Poincaré invariance. There may of course be additional "internal" symmetries realised in nature, which are not related to space time symmetries. The eigenvalues of the corresponding Casimir operators then provide further characteristics of a particle species (like electric charge or "colour").

Now that we understand how to characterise a particle species, we wonder what the corresponding one-particle states are and how they transform under Poincaré transformations.

As argued above one needs to find a complete set of commuting observables to characterise the one-particle states. One may choose e.g. P^μ and one component of W^μ, typically W^3. The one-particle states of a particle with mass m and spin j are then denoted by

$$|\underbrace{m, j}_{\substack{\text{eigenvalues of Casimirs}\\ \hat{=}\text{ intrinsic properties that}\\ \text{do not change}}}\,;\,\underbrace{p, j_3}_{\substack{\text{eigenvalues of } P^\mu \text{ and } W^3\\ \rightarrow \text{ 4-momentum and a}\\ \text{particular spin configuration}}}\,,\underbrace{\ldots}_{\text{further quantum numbers related to internal symmetries}}\rangle$$

It is instructive to adopt a short-hand notation for these states, in which the intrinsic properties mass and spin as well as the quantum numbers from the internal symmetries are suppressed. One furthermore typically writes s instead of j_3 for the spin configuration. A one-particle state will therefore be denoted by $|p, s\rangle$ in the following.

How do these states transform under Poincaré transformations?

$$U(\Lambda, a)\, |p, s\rangle = \ ?$$

Let us first consider translations with $U(\Lambda, a) = U(1\!\!1, a)$. As we have chosen the one-particle states to be eigenstates of $\underline{P}^\mu$,

$$\underset{\text{operator}}{\underline{P}^\mu}\, |p, s\rangle = \underset{\text{eigenvalue}}{p^\mu}\, |p, s\rangle$$

we simply obtain

$$U(1\!\!1, a)\, |p, s\rangle = e^{i a_\mu \underline{P}^\mu}\, |p, s\rangle = e^{i a_\mu p^\mu}\, |p, s\rangle$$

From non-relativistic quantum mechanics we further know that the Hamilton operator is the generator of time translations and the momentum operator is the generator of spacial translations, which justifies our interpretation in terms of $\underline{P}^0 = H$ and $\vec{\underline{P}}$ being the 3-momentum operator. Notice further that

$$\underline{P}^2 |p,s\rangle = p^2 |p,s\rangle$$

with $p^2 = E^2 - \vec{p}^{\,2} = m^2$. As anticipated, the eigenvalue of the Casimir operator $\underline{P}^2$ is thus related to the rest mass of a particle.

We next consider <u>homogeneous LT</u> with $U(\Lambda, a) = U(\Lambda, 0) \equiv U(\Lambda)$. It turns out that one has to distinguish the following cases:

(a) $m^2 > 0$: massive particles

(b) $m^2 = 0$: massless particles

(c) $m^2 < 0$: exotic particles (not realised in nature)

We will focus on (a) and (b) in the following. In particular, we will see that the transition from massive to massless particles in the limit $m \to 0$ is non-trivial, which is related to the fact that massless particles do not have a rest frame.

(a) $m^2 > 0$

In order to determine how massive one-particle states transform under homogeneous LT, we first write the momentum operator in the form *

$$\underline{P}^\mu = U(\Lambda)\,(\Lambda \underline{P})^\mu\, U^{-1}(\Lambda)$$

It follows

$$\underline{P}^\mu\, U(\Lambda)\,|p,s\rangle = U(\Lambda)\,(\Lambda\underline{P})^\mu \underbrace{U^{-1}(\Lambda)\,U(\Lambda)}_{1}\,|p,s\rangle$$

$$= (\Lambda p)^\mu\, U(\Lambda)\,|p,s\rangle$$

i.e. $U(\Lambda)\,|p,s\rangle$ is an eigenstate of $\underline{P}^\mu$ with eigenvalue $(\Lambda p)^\mu$.

We may therefore expand

$$U(\Lambda)\,|p,s\rangle = \sum_{s'} C_{s's}(\Lambda,p)\,|\Lambda p, s'\rangle \qquad (*)$$

where up to now we have neither used that the particle is massive nor that $U(\Lambda)$ is a homogeneous LT. The translations

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* This can be seen as follows

$$U(\Lambda)\,\underline{P}^\nu\, U^{-1}(\Lambda) = \Lambda_\sigma{}^\nu\, \underline{P}^\sigma \qquad (\sim \text{tutorials})$$

We further have $\Lambda^T g \Lambda = g \quad \rightarrow \quad \Lambda^T = g\Lambda^{-1} g$

$$\Rightarrow \Lambda^\mu{}_\nu\, U(\Lambda)\underline{P}^\nu U^{-1}(\Lambda) = \Lambda^\mu{}_\nu\, g^\nu{}_\sigma\, \underbrace{\Lambda_\sigma{}^\circ}_{\Lambda_\sigma{}^\nu}\, \underline{P}^\sigma$$

$$= (\Lambda g \Lambda^T)^\mu{}_\sigma\, \underline{P}^\sigma = \underbrace{(\Lambda g g \Lambda^{-1} g)^\mu{}_\sigma}_{g^\mu{}_\sigma}\, \underline{P}^\sigma = \underline{P}^\mu$$

from page 78 can therefore also be written in this form with

$\Lambda = 1\!\!1$ and $C_{s's}(1\!\!1, a, p) = e^{ia_\mu p^\mu} \delta_{ss'}$.

implicit in the above notation

In order to determine the coefficients $C_{s's}(\Lambda, p)$ for homogeneous LT, we consider the massive particle in its rest frame with $k^\mu = (m, \vec{0})$ and we denote the associated state by $|k, s\rangle$. We now have

$$W_\mu |k,s\rangle = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma |k,s\rangle = \frac{m}{2} \varepsilon_{\mu\nu\rho 0} J^{\nu\rho} |k,s\rangle$$

$$\Rightarrow \quad W_0 |k,s\rangle = 0$$

$$W_i |k,s\rangle = \frac{m}{2} \underbrace{\varepsilon_{ijk0}}_{+\varepsilon^{ijk}} J^{jk} |k,s\rangle = m J^i |k,s\rangle$$

$$\Rightarrow \quad W^2 |k,s\rangle = -W^\mu W_\mu |k,s\rangle$$

$$= -\frac{1}{2} \varepsilon^{i\nu\rho\sigma} J_{\nu\rho} P_\sigma \, m J^i |k,s\rangle$$

$$= -\frac{m}{2} \varepsilon^{i\nu\rho\sigma} J_{\nu\rho} J^i P_\sigma |k,s\rangle$$

commutator from below vanishes on $|k,s\rangle$

$$= -\frac{m^2}{2} \underbrace{\varepsilon^{ijk0}}_{-\varepsilon^{ijk}} J_{jk} J^i |k,s\rangle$$

$$= m^2 \vec{J}^2 |k,s\rangle$$

The eigenvalues of the Casimir operator W^2 are thus $m^2 j(j+1)$, i.e. they correspond to the <u>spin</u> j of a massive particle (note that the particle has no orbital angular momentum in its rest frame).

$$[P^\sigma, J^i] = \frac{1}{2} \varepsilon^{ijk} [P^\sigma, J^{jk}] = \frac{i}{2} \varepsilon^{ijk} (g^{\sigma j} P^k - g^{\sigma k} P^j)$$

As $W_3 \sim J^3$ we further identify the quantum number s with the corresponding $2s+1$ spin configurations.

We thus have seen what we have anticipated before. In a relativistic quantum theory the concepts of mass and spin arise as the eigenvalues of the Casimir operators of the Poincaré algebra (so far for $m^2 > 0$).

Now that we have clarified the interpretation of the states $|k,s\rangle$ in the rest frame, we wonder how we obtain the states $|p,s\rangle$ for arbitrary momenta $p^\mu = (p^0, \vec{p})$ from this configuration.

The two frames are connected by a Lorentz boost *

$$L(p)^\mu{}_\nu = \left(\begin{array}{c|c} \frac{p^0}{m} & \frac{p^i}{m} \\ \hline \frac{p^j}{m} & \delta_{ij} + \left(\frac{p^0}{m} - 1\right) \frac{p^i p^j}{\vec{p}^{\,2}} \end{array}\right)$$

with $p^\mu = L(p)^\mu{}_\nu k^\nu$.

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* Compare with the general formula for a Lorentz boost from page 63 with velocity $\vec{v} = -\frac{\vec{p}}{p^0}$ and $\gamma = \frac{1}{\sqrt{1-\vec{v}^2}} = \frac{p^0}{m}$.

We now <u>define</u>

$$|p,s\rangle \equiv U(L(p))\ |k,s\rangle$$

i.e the unitary operator that represents the Lorentz boost $L(p)$ on the Hilbert space transforms the states $|k,s\rangle$ into $|p,s\rangle$, without changing the spin configuration. This is in line with what we have learned in quantum mechanics, where we saw that spin configurations mix under <u>rotations</u> (as will also be discussed below).

The definition is moreover consistent with

$$\underline{P}^{\mu}|p,s\rangle = U(L(p))\ (L(p)\underline{P})^{\mu}\ \underbrace{U^{-1}(L(p))\ U(L(p))}_{1}\ |k,s\rangle$$

$$= (L(p)k)^{\mu}\ U(L(p))\ |k,s\rangle$$

$$= p^{\mu}\ |p,s\rangle \quad \checkmark$$

We then obtain

$$U(\Lambda)\ |p,s\rangle = U(\Lambda)\ U(L(p))\ |k,s\rangle$$

$$= U(\Lambda L(p))\ \ |k,s\rangle$$

$$= U(L(\Lambda p))\ \ U(\underbrace{L^{-1}(\Lambda p)\ \Lambda\ L(p)}_{\equiv R})\ |k,s\rangle$$

where R transforms the vector $k^\wedge$ into

$$k \xrightarrow{L(p)} p \xrightarrow{\Lambda} \Lambda p \xrightarrow{L^{-1}(\Lambda p)} k$$

i.e k' is left invariant! But the subgroup of LT that leave the vector $k^\wedge = (m, \vec{0})$ invariant are just the rotations (this is sometimes called the little group associated with $k^\wedge$).

We therefore have

$$\underbrace{U(R)}_{\text{operator}} \, |k, s\rangle = \sum_{s'} \underbrace{D^{(j)}_{s's}(R)}_{\text{matrix coefficients}} \, |k, s'\rangle$$

where $D^{(j)}_{s's}(R)$ is the <u>Wigner function</u> of a rotation R in the representation j.* For $j = 1/2$, one has e.g.

$$D^{(1/2)}_{s's}(R) = \cos\frac{\theta}{2}\, \delta_{s's} - i(\vec{n}\cdot\vec{\sigma})_{s's} \sin\frac{\theta}{2}$$

$\vec{\sigma}$: Pauli matrices

where $\vec{n}$ with $|\vec{n}|^2 = 1$ is the rotation axis and θ the rotation angle. (see e.g. tutorials)

---

* In quantum mechanics we typically write this as

insert complete set of states (for fixed j)

$$U(R)\, |j\, m\rangle = \sum_{m'} |j\, m'\rangle \langle j\, m'|\, U(R)\, |j\, m\rangle$$

$$\equiv \sum_{m'} |j\, m'\rangle\, D^{(j)}_{m'm}(R)$$

We finally obtain

$$U(\Lambda)\,|p,s\rangle = \sum_{s'} D^{(j)}_{s's}(R)\; U(L(\Lambda p))\,|k,s'\rangle$$

$$= \sum_{s'} D^{(j)}_{s's}(R)\;|\Lambda p, s'\rangle$$

and we have thus succeeded in determining the coefficients in (*)

for a homogeneous LT with

$$c_{s's}(\Lambda, p) = D^{(j)}_{s's}(R)$$

where $R = L^{-1}(\Lambda p)\,\Lambda\, L(p)$ is called the Wigner rotation

associated with the LT Λ and the momentum p (see also

tutorials for a specific example).

As eigenstates of hermitean operators, the one-particle states are

orthogonal. We choose a covariant normalisation

$$\langle p,s\,|\,p',s'\rangle \equiv (2\pi)^3\, 2p^0\, \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}\,')$$

with $p^0 = \sqrt{\vec{p}^{\,2}+m^2}$, which corresponds to a measure

$$\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0} = \int \frac{d^4p}{(2\pi)^4}\,\delta(p^2-m^2)\; 2\pi\;\theta(p^0)$$

and a completeness relation

$$\sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\,|p,s\rangle\langle p,s| = \mathbb{1}$$

(see also tutorials)

(b) $m^2 = 0$

The massless case is more complicated since one cannot build on the familiar properties of the angular momentum algebra in this case.

Starting from (*) on page 80, we again want to determine the coefficients $C_{s's}(\Lambda, p)$ and we also need to clarify the physical interpretation of the label s in the states $|p,s\rangle$.

As there does not exist a rest frame for massless particles, we now consider a reference frame in which the particle moves along the z-direction with $k^\mu = (n, 0, 0, n)$ and $k^2 = m^2 = 0$.

We denote the corresponding state by $|k, \sigma\rangle$ and obtain

$$W^\lambda |k,\sigma\rangle = \frac{1}{2} \varepsilon^{\lambda\nu\sigma\rho} J_{\nu\sigma} P_\rho |k,\sigma\rangle$$

$$= \frac{n}{2} \left(\varepsilon^{\lambda\nu\sigma 0} - \varepsilon^{\lambda\nu\sigma 3} \right) J_{\nu\sigma} |k,\sigma\rangle$$

$$\Rightarrow \quad W^0 |k,\sigma\rangle = -\frac{n}{2} \underbrace{\varepsilon^{0ij3}}_{= +\varepsilon^{3ij}} J_{ij} |k,\sigma\rangle = -n J^3 |k,\sigma\rangle$$

$$W^1 |k,\sigma\rangle = \frac{n}{2}\Big(\underbrace{\varepsilon^{1040}}_{-\varepsilon^{134}} J_{34} - \varepsilon^{1053} J_{53}\Big) |k,\sigma\rangle$$

$$= n\,(-J^1 + K^2)\, |k,\sigma\rangle$$

$$W^2 |k,\sigma\rangle = \frac{n}{2}\Big(\underbrace{\varepsilon^{2340}}_{-\varepsilon^{234}} J_{34} - \varepsilon^{2053} J_{53}\Big) |k,\sigma\rangle$$

$$= n\,(-J^2 - K^1)\, |k,\sigma\rangle$$

$$W^3 |k,\sigma\rangle = \frac{n}{2} \underbrace{\varepsilon^{3040}}_{-\varepsilon^{334}} J_{34} |k,\sigma\rangle = -n J^3 |k,\sigma\rangle$$

Instead of the angular momentum operator $\vec{J}$, we now obtain the generators $A \equiv K^1 + J^2$, $B \equiv K^2 - J^1$ and J^3 with

$$[J^3, A] = [J^3, K^1] + [J^3, J^2] = iK^2 - iJ^1 = iB$$

$$[J^3, B] = [J^3, K^2] - [J^3, J^1] = -iK^1 - iJ^2 = -iA$$

$$[A, B] = [K^1, K^2] - \underbrace{[K^1, J^1]}_{=0} + \underbrace{[J^2, K^2]}_{=0} - [J^2, J^1]$$

$$= -iJ^3 + iJ^3 = 0$$

The maximal set of commuting operators that is used to describe the one-particle states now consists of A and B (in the massive case this was just J^3, which implied that σ can be identified with the 3-component of the particle spin in its rest frame).

It turns out, however, that one would get a continuum of eigenstates if the eigenvalues of A and B were non-zero, which is not what is observed in nature (there are no particles with continuous internal degrees of freedom, for details see Weinberg I, chapter 2.5).

One therefore requires that the physical states are eigenstates of the operators A and B with eigenvalue zero. The physical states are then distinguished by the eigenvalue of the remaining generator J^3 with

$$J^3 |k,\sigma\rangle = \sigma\, |k,\sigma\rangle$$

We thus obtain

$$\begin{aligned} W^\mu |k,\sigma\rangle &= (-\kappa\sigma, 0, 0, -\kappa\sigma)\, |k,\sigma\rangle \\ &= -\sigma k^\mu\, |k,\sigma\rangle \end{aligned}$$

$$\Rightarrow\ W^2 |k,\sigma\rangle = -\sigma^2 k^2 |k,\sigma\rangle = 0$$

As the particle is moving into the 3-direction in the considered reference frame, σ corresponds to the <u>helicity</u> in this case. One further defines $|\sigma|$ to be the spin of a massless particle.

Starting from $|k,\sigma\rangle$ with $k^\mu = (\kappa, 0, 0, \kappa)$, we next determine the state $|p,\sigma\rangle$ associated with an arbitrary momentum $p^\mu = (p^0, \vec{p})$ and $p^0 = |\vec{p}|$. The two frames are connected by a LT

$p^\mu = L(p)^\mu{}_\nu k^\nu$ with

$$L(p) = R\left(\frac{\vec{p}}{|\vec{p}|}\right) B\left(\frac{|\vec{p}|}{\kappa}\right)$$

where

$$B(u) = \begin{pmatrix} \frac{1+u^2}{2u} & 0 & 0 & -\frac{1-u^2}{2u} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1-u^2}{2u} & 0 & 0 & \frac{1+u^2}{2u} \end{pmatrix}$$

is a boost into the 3-direction with $v = \frac{1-u^2}{1+u^2}$ and hence

$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1+u^2}{2u}$, which adjust the energy from $k^0 = \kappa$ to $p^0 = |\vec{p}|$, which is followed by a rotation that carries the 3-axis into the direction of $\vec{p}$.

We then proceed as in the massive case and <u>define</u>

$$|p,\sigma\rangle = U(L(p))\,|k,\sigma\rangle$$

which leads to

$$U(\Lambda)\,|p,\sigma\rangle = U(L(\Lambda p))\, U(\underbrace{L^{-1}(\Lambda p)\,\Lambda\, L(p)}_{\equiv W})\,|k,\sigma\rangle$$

where W now leaves the vector $k^\mu = (\kappa, 0, 0, \kappa)$ invariant.

The subgroup of homogeneous LT that leave k' invariant (i.e. the associated little group) are now those transformations that are generated by A, B and J^3. It turns out that W can always be written in the form*

$$W = L^{-1}(\Lambda p)\,\Lambda\, L(p) = S(\alpha,\beta)\,\bar{R}(\theta)$$

where

$$S(\alpha,\beta) = \begin{pmatrix} 1+\zeta & -\alpha & -\beta & -\zeta \\ -\alpha & 1 & 0 & \alpha \\ -\beta & 0 & 1 & \beta \\ \zeta & -\alpha & -\beta & 1-\zeta \end{pmatrix} \qquad \zeta = \frac{1}{2}(\alpha^2+\beta^2)$$

$$\bar{R}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For a given LT Λ and a momentum p, one then uses the above equation to determine the corresponding parameters α, β and θ.

---

* One easily verifies that W leaves $k' = (\kappa, 0, 0, \kappa)$ invariant.

The corresponding unitary operator that represents W on the Hilbert space is then given by

$$U(W) = e^{-i(\alpha A + \beta B)} \, e^{-i\theta J^3}$$

and hence

$$U(W)\,|k,\sigma\rangle = e^{-i\theta\sigma}\,|k,\sigma\rangle$$

since we argued above that the physical states are eigenstates of A and B with eigenvalue 0.

We finally obtain

$$U(\Lambda)\,|p\sigma\rangle = e^{-i\theta\sigma}\,U(L(\Lambda p))\,|k,\sigma\rangle$$

$$= e^{-i\theta\sigma}\,|\Lambda p,\sigma\rangle$$

and hence

$$C_{\sigma'\sigma}(\Lambda,p) = e^{-i\theta\sigma}\,\delta_{\sigma'\sigma}$$

where θ is determined for a given LT Λ and momentum p by the Wigner transformation W as outlined above.

Notice that the helicity of a massless particle is invariant under LT (whereas the spin configurations of a massive particle mix).

Remarks:

- The representations $U(\Lambda)$ are as usual projective with $U = \pm \mathbb{1}$ for a rotation with $\theta = 2\pi$

  $\Rightarrow$ σ is integer or half-integer!

- As helicity is a Lorentz-invariant concept, there may in principle exist a single helicity state for a massless particle as long as the theory is parity-invariant, however, we will see in the next section that the massless particle must have two helicity states corresponding to σ and $-\sigma$. This is the reason why the photon in QED has two physical polarisations.

Let us summarise the transformation properties of one-particle states

- $m^2 > 0$

massive particle with spin j and spin configuration $s = -j, -j+1, \dots, j$

$$\mathcal{U}(\Lambda, a)\,|p, s\rangle = \mathcal{U}(\mathbb{1}, a)\,\mathcal{U}(\Lambda, 0)\,|p, s\rangle$$

$$= e^{ia\Lambda p} \sum_{s'} D^{(j)}_{s's}(R)\,|\Lambda p, s'\rangle$$

Wigner function of rotation
$R = L^{-1}(\Lambda p)\,\Lambda\,L(p)$
in representation j
($L(p)$ from page 82)

spin configurations mix under LT

- $m^2 = 0$

massless particle with helicity σ

$$\mathcal{U}(\Lambda, a)\,|p, \sigma\rangle = \mathcal{U}(\mathbb{1}, a)\,\mathcal{U}(\Lambda, 0)\,|p, \sigma\rangle$$

$$= e^{ia\Lambda p}\, e^{-i\theta\sigma}\,|\Lambda p, \sigma\rangle$$

θ determined by
$W = L^{-1}(\Lambda p)\,\Lambda\,L(p)$
$= S(\alpha, \beta)\,\bar{R}(\theta)$
($L(p)$ from page 89)

helicity is invariant under LT

2.3 Parity and time reversal

The construction in the previous section was based on the Poincaré algebra, which reflects the identity component of the Poincaré group consisting of proper, orthochronous LT $\Lambda \in L_+^\uparrow$. Although we argued that Einstein's principle of relativity only applies to this subgroup, it turns out that the electromagnetic and the strong interactions are invariant under the full Poincaré group, and we therefore need to understand how the one-particle states transform under LT not captured by $L_+^\uparrow$.

We saw in section 2.1 that the disconnected components of the Poincaré group are related to $L_+^\uparrow$ by parity and time reversal transformations

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Beware of the notation:

P^μ : 4-momentum operator

$P^\mu{}_\nu$: $(\mu\nu)$-element of parity transformation matrix

P and T are special Lorentz transformations, which are realised on the Hilbert space by the operators

$$U(\Lambda, a) = U(P, 0) \equiv U_P$$

$$U(\Lambda, a) = U(T, 0) \equiv U_T$$

The general relations that we derived in the tutorials

$$U(\Lambda, a)\, i\, P^\mu\, U^{-1}(\Lambda, a) = i\, \Lambda_\rho{}^\mu\, P^\rho$$

$$U(\Lambda, a)\, i\, J^{\mu\nu}\, U^{-1}(\Lambda, a) = i\, \Lambda_\rho{}^\mu\, \Lambda_\sigma{}^\nu \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho \right)$$

imply

$$U_P\, i\, P^\mu\, U_P^{-1} = i\, P_\rho{}^\mu\, P^\rho \qquad (*)$$

$$U_P\, i\, J^{\mu\nu}\, U_P^{-1} = i\, P_\rho{}^\mu\, P_\sigma{}^\nu\, J^{\rho\sigma}$$

and similar relations hold for the time reversal transformation with $U_P \to U_T$ and $P_\mu{}^\nu \to T_\mu{}^\nu$. Notice that we did not cancel factors of i in these equations, since we have not yet clarified if the operators U_P and U_T are unitary or anti-unitary.

On page 69 we argued that the operators $\mathcal{U}(\Lambda, a)$ are linear

$$\mathcal{U}\left[c_1|\psi_1\rangle + c_2|\psi_2\rangle\right] = c_1\,\mathcal{U}|\psi_1\rangle + c_2\,\mathcal{U}|\psi_2\rangle$$

and unitary $\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger\mathcal{U} = \mathbb{1}$, since in this case we obtain

for $|\psi'\rangle = \mathcal{U}|\psi\rangle$

$$\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$$

According to Wigner's theorem, there exists however another possibility which leaves the probability $|\langle\varphi'|\psi'\rangle|^2$ invariant, namely the operator may be anti-linear

$$\mathcal{U}\left[c_1|\psi_1\rangle + c_2|\psi_2\rangle\right] = c_1^*\,\mathcal{U}|\psi_1\rangle + c_2^*\,\mathcal{U}|\psi_2\rangle$$

and anti-unitary with $\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger\mathcal{U} = \mathbb{1}$ and *

$$\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^* = \langle\psi|\varphi\rangle$$

Notice that for an anti-unitary operator, we have $\mathcal{U}i = -i\,\mathcal{U}$.

We will now show that <u>$\mathcal{U}_P$ is unitary</u> and

<u>$\mathcal{U}_T$ is anti-unitary</u>.

---

* The adjoint operator is now defined as $\langle\varphi|\mathcal{U}^\dagger\psi\rangle = \langle\mathcal{U}\varphi|\psi\rangle^*$

$$\Rightarrow \langle\varphi'|\psi'\rangle = \langle\mathcal{U}\varphi|\mathcal{U}\psi\rangle = \langle\varphi|\mathcal{U}^\dagger\mathcal{U}\psi\rangle^* = \langle\varphi|\psi\rangle^*$$

To this end, we consider the transformation relation (*) for the Hamiltonian $P^0 = H$

$$U_P \, i H U_P^{-1} = i H$$

Assuming that U_P was anti-unitary, we would have

$$U_P H = - H U_P$$

which for an eigenstate $|\psi\rangle$ with eigenvalue $E>0$ would imply

$$H U_P |\psi\rangle = - U_P H |\psi\rangle = - E \, U_P |\psi\rangle$$

i.e. there would exist a corresponding eigenstate $U_P|\psi\rangle$ with eigenvalue $-E<0$. In this case, however, the ground state of the theory with $E=0$ would be unstable. We therefore conclude that U_P must be a unitary operator.

It is easy to see that a unitary operator U_T would lead to the same problem since now

$$U_T \, i H U_T^{-1} = - i H$$

which for a unitary operator U_T would again lead to

$$U_T H = - H U_T$$

The generators of the Poincaré algebra thus satisfy the relations

$$\mathcal{U}_P H \mathcal{U}_P^{-1} = + H \qquad \mathcal{U}_T H \mathcal{U}_T^{-1} = + H$$

$$\mathcal{U}_P \vec{P} \mathcal{U}_P^{-1} = - \vec{P} \qquad \mathcal{U}_T \vec{P} \mathcal{U}_T^{-1} = - \vec{P}$$

$$\mathcal{U}_P \vec{J} \mathcal{U}_P^{-1} = + \vec{J} \qquad \mathcal{U}_T \vec{J} \mathcal{U}_T^{-1} = - \vec{J}$$

$$\mathcal{U}_P \vec{K} \mathcal{U}_P^{-1} = - \vec{K} \qquad \mathcal{U}_T \vec{K} \mathcal{U}_T^{-1} = + \vec{K}$$

But how do the one-particle states transform under parity and time reversal transformations?

We again have to distinguish the cases $m^2 > 0$ and $m^2 = 0$.

(a) $m^2 > 0$

For a massive particle we can again resort to its rest frame with $k^\mu = (m, \vec{0})$ and associated state $|m, s\rangle$. The above relations now imply

$$H \mathcal{U}_P |m,s\rangle = \mathcal{U}_P H |m,s\rangle = m \, \mathcal{U}_P |m,s\rangle$$

$$\vec{P} \mathcal{U}_P |m,s\rangle = - \mathcal{U}_P \vec{P} |m,s\rangle = 0$$

$$J^3 \mathcal{U}_P |m,s\rangle = \mathcal{U}_P J^3 |m,s\rangle = s \, \mathcal{U}_P |m,s\rangle$$

$\Rightarrow$ $\mathcal{U}_P |m,s\rangle$ is eigenstate of $H, \vec{P}$ and J^3 with the same eigenvalues as $|m,s\rangle$, i.e. $\mathcal{U}_P |m,s\rangle \sim |m,s\rangle$.

As $\mathcal{U}_P$ is unitary, we have

$$\mathcal{U}_P \, |k,s\rangle = \eta_P \, |k,s\rangle$$

where η_P with $|\eta_P| = 1$ is a pure phase, which in principle could depend on the spin configuration s. We will show in the tutorials that this is not the case.

We next boost to a general frame with $p^\mu = (p^0, \vec{p})$ via $p^\mu = L(p)^\mu{}_\nu \, k^\nu$ and $L(p)$ from page 82. We then obtain

$$\mathcal{U}_P \, |p,s\rangle = \mathcal{U}_P \, \mathcal{U}(L(p)) \, |k,s\rangle$$

$$= \mathcal{U}(P L(p) P^{-1}) \, \mathcal{U}_P \, |k,s\rangle$$

where $P L(p) P^{-1} = L(Pp)$ since

$$k \xrightarrow{P^{-1}} k \xrightarrow{L(p)} p \xrightarrow{P} Pp$$

$$\Rightarrow \quad \mathcal{U}_P |p,s\rangle = \eta_P \, \mathcal{U}(L(Pp)) \, |k,s\rangle$$

$$= \eta_P \, |Pp, s\rangle$$

where the <u>intrinsic parity</u> η_P only depends on the particle species on which the operator $\mathcal{U}_P$ acts (one typically has $\eta_P = +1$ or $\eta_P = -1$).

We will show in the tutorials that the corresponding relation for the time reversal transformation reads

$$U_T |p, s\rangle = \eta_T (-1)^{j-s} | \mathcal{P}p, -s\rangle \qquad \left\{ \mathcal{P}p = -\mathcal{T}p \right.$$

with a phase η_T that is however not observable since U_T is an anti-unitary operator.

(b) $m^2 = 0$

The massless case is again more complicated since the reference vector $k^\mu = (\kappa, 0, 0, \kappa)$ transforms non-trivially under P and T. We will not go into the details here and quote the results only (cf. Weinberg I, chapter 2.6)

$$U_P |p, \sigma\rangle = \eta_\sigma e^{\mp i\pi\sigma} | \mathcal{P}p, -\sigma\rangle$$

$$U_T |p, \sigma\rangle = \zeta_\sigma e^{\pm i\pi\sigma} | \mathcal{P}p, \sigma\rangle$$

where the upper / lower sign applies when the 2-component of $\vec{p}$ is positive or negative, respectively.

Remarks:

- The helicity σ changes its sign under a parity transformation, but not under a time reversal transformation. This is intuitively clear since

$$\begin{tikzcd} & \vec{J}\cdot(-\vec{p}) \\ \vec{J}\cdot\vec{p} \arrow[ur, "P"] \arrow[dr, "T"'] & \\ & (-\vec{J})\cdot(-\vec{p}) \end{tikzcd}$$

- As the parity transformation links the states with helicities σ and $-\sigma$, a parity-invariant theory of massless particles requires both of these states (as anticipated earlier).

- The prefactors $e^{\pm i\pi\sigma}$ are only relevant for half-integer values of σ and they arise because of projective representations.

2.4. Internal symmetries

We briefly mentioned in section 2.2 that a system may have additional "internal" symmetries, which are not related to Poincaré transformations. Assuming that the internal symmetry is associated with a Lie group G, the Coleman-Mandula theorem states that the combined symmetry group must be a direct product of the Poincaré and the internal group (given a few physically motivated assumptions). The elements $g \in G$ therefore commute with the Poincaré transformations, and our analysis from above can be trivially generalised in this case. In particular, the eigenvalues of the Casimir operators in G provide further characteristics of a particle and the set of commuting operators that describe the one-particle states has to be extended to include operators of G.

In the presence of internal symmetries, we denote the one-particle states by $|p, s; n\rangle$.

quantum numbers wrt G
(there may be various discrete labels)

These states then transform under the internal symmetry as

$$\mathcal{U}(g)\, |p, s; n\rangle = \sum_{n'} \mathcal{D}(g)_{n'n}\, |p, s; n'\rangle$$

operator

matrix representation of G
(unitary, irreducible)
$\hookrightarrow$ numerical coefficients

labels p, s are unaffected by internal symmetry transformation

We are now in the position to introduce the notion of antiparticles. For a particle of mass m and spin j (for $m > 0$) or helicity σ (for $m = 0$), which transforms with a representation $\mathcal{D}(g)$ under an internal symmetry, another particle is called the <u>antiparticle</u> if

* it has the same mass m
* it has the same spin j $(m > 0)$ or opposite helicity σ $(m = 0)$
* it transforms under <u>all</u> internal symmetries with the <u>complex conjugate representation</u> $\bar{\mathcal{D}}(g) = \mathcal{D}(g)^*$.

[If the representations $\mathcal{D}(g)$ are real, a particle can be considered to be its own antiparticle.]

We denote the antiparticle state associated with $|p,s;n\rangle$ by $|p,s;\bar{n}\rangle$, which according to the above definition, transforms as

$$U(g)\,|p,s;\bar{n}\rangle = \sum_{\bar{n}'} \underbrace{D(g)^*_{\bar{n}'\bar{n}}}_{\text{same } D(g) \text{ from above}} |p,s;\bar{n}'\rangle$$

As an example we consider phase transformations with $G = U(1)$. In this case there exists a single generator Q, which itself is a Casimir operator with

$$Q\,|q;p,s\rangle = q\,|q;p,s\rangle$$

where q is the $U(1)$ "charge" of the particle. Although we usually suppress the labels associated with the Casimir operators, we include it here to be able to distinguish the particle and antiparticle states in our notation. We now have

$$U(\theta)\,|q;p,s\rangle = e^{i\theta Q}\,|q;p,s\rangle = e^{i\theta q}\,|q;p,s\rangle$$

which for the antiparticle state $|\bar{q};p,s\rangle$ implies

$$U(\theta)\,|\bar{q};p,s\rangle = e^{i\theta Q}\,|\bar{q};p,s\rangle \overset{!}{=} e^{-i\theta q}\,|\bar{q};p,s\rangle$$

and hence

$$Q\,|\bar{q};p,s\rangle = \bar{q}\,|\bar{q};p,s\rangle = -q\,|\bar{q};p,s\rangle$$

$\Rightarrow$ the antiparticle has opposite $U(1)$ charge $\bar{q} = -q$!

It is furthermore convenient to introduce an operation that converts a particle state into its antiparticle state

$$U_c \, | p, s ; n \rangle = \eta_c \, | p, s ; \bar{n} \rangle$$

where η_c is a phase with $|\eta_c| = 1$. The operator U_c is called the <u>charge conjugation operator.</u>

2.5. Many-particle states

Starting from the one-particle states that we discussed in the previous sections, one can construct many-particle states by taking their direct products. Let us assume for the moment that there exists a single particle species, which is characterised by its mass, spin (or helicity) and possibly further quantum numbers related to internal symmetries. It will be convenient to introduce a short-hand notation in the following with

$$|p\rangle \equiv |p, s; n\rangle$$

We define an N-particle state by

$$|p_1 \ldots p_N\rangle = |p_1\rangle \otimes \ldots \otimes |p_N\rangle$$

which is an element of the direct product space

$$H = H_1 \otimes \ldots \otimes H_N$$

The representations of the one-particle states then induce a tensor product representation, which is completely reducible (since it is unitary) and can be decomposed into a direct sum of irreducible representations via the usual Clebsch-Gordon procedure.

The description of quantum systems with <u>identical particles</u> is fundamentally different from that of a classical system. Whereas classical particles can always be distinguished since one can follow their individual trajectories, a quantum mechanical measurement only reveals that a particle with a specific momentum and spin configuration has been measured, but it does not tell us <u>which</u> particle has been measured. Quantum states that are related by the exchange of identical particles therefore represent the same physical state. One has e.g.

$$|p_1 .. p_i .. p_j .. p_N\rangle = \alpha \; |p_1 .. p_j .. p_i .. p_N\rangle$$

where α must be a pure phase that does not depend on the specific configuration, but is rather a characteristic of the particle species.

The successive exchange of two identical particles implies

$$|p_1 \dots p_i \dots p_j \dots p_N\rangle = \alpha^2 \, |p_1 \dots p_i \dots p_j \dots p_N\rangle$$

and hence $\alpha = \pm 1$. For $\alpha = +1$ the states are symmetric under the exchange of two identical particles, which are called <u>bosons</u> in this case. Similarly, $\alpha = -1$ corresponds to antisymmetric states and the particles are called <u>fermions</u>. We will see in the next chapter that bosons / fermions have integer / half-integer spin.

As the states must be symmetric / antisymmetric under the exchange of any pair of identical particles, the bosonic / fermionic states must be totally symmetric / antisymmetric.

The many-particle states can then be constructed as follows:

- <u>vacuum</u>

$|0\rangle$ with $\langle 0|0\rangle = 1$

- <u>one particle</u>:

$$|p\rangle \quad \text{with} \quad \langle p|p'\rangle = \delta(p-p') \equiv (2\pi)^3\, 2p^0\, \delta_{ss'}\, \delta_{nn'}\, \delta^{(3)}(\vec{p}-\vec{p}\,')$$

- <u>two particles</u>:

$$|p_1\, p_2\rangle = \frac{1}{\sqrt{2}}\Big(|p_1\rangle|p_2\rangle \pm |p_2\rangle|p_1\rangle \Big) \qquad \begin{array}{l} + \text{ bosons} \\ - \text{ fermions} \end{array}$$

$$\Rightarrow \quad \langle p_1\, p_2 | p_1'\, p_2'\rangle = \delta(p_1-p_1')\,\delta(p_2-p_2') \pm \delta(p_1-p_2')\,\delta(p_2-p_1')$$

$\vdots$

- <u>N particles</u>:

number of transpositions of σ ($\to$ odd/even permutation)

$$|p_1 \dots p_N\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma} (\pm 1)^{n(\sigma)}\, P_\sigma \big(|p_1\rangle \dots |p_N\rangle \big)$$

permutation   permutation operator

$$\Rightarrow \quad \langle p_1 \dots p_M | p_1' \dots p_N'\rangle = \delta_{MN} \sum_{\sigma} (\pm 1)^{n(\sigma)} \prod_{i=1}^{N} \delta(p_i - p_{\sigma(i)}')$$

Notice that the antisymmetry of the fermionic states implies that two or more identical fermions cannot occupy the same one-particle state. This is known as <u>Pauli's exclusion principle.</u>

The set of symmetrized / antisymmetrized N-particle states forms a Hilbert space $H_{S/A}^{(N)}$, and their direct sum

$$F_{S/A} = \bigoplus_{N=0}^{\infty} H_{S/A}^{(N)}$$

is called Fock space.

The states that correspond to different numbers of particles are related by the creation operator $a^{+}(p)$, which adds a particle with configuration $|p\rangle = |p, s; n\rangle$ to the symmetrized / antisymmetrized states. We define

$$a^{+}(p)\,|p_1 \ldots p_N\rangle \equiv |p\, p_1 \ldots p_N\rangle$$

The N-particle states can thus be constructed by successively operating on the vacuum state with creation operators

$$|p_1 \ldots p_N\rangle = a^{+}(p_1) \ldots a^{+}(p_N)\,|0\rangle$$

The adjoint of the creation operator – the annihilation operator $a(p)$ – then removes a particle from the states.

We obtain

$$\langle p_1' .. p_M' \mid a(p) \mid p_1 .. p_N \rangle$$

↲ apply as $a^+(p)$

$$= \langle \underset{q_1}{p} \; \underset{q_2}{p_1'} \; .. \; \underset{q_{M+1}}{p_M'} \mid p_1 .. p_N \rangle$$

$$= \delta_{M+1,N} \sum_\sigma (\pm 1)^{n(\sigma)} \prod_{i=1}^{N} \delta(q_i - p_{\sigma(i)})$$

$$\overset{(*)}{=} \sum_{i=1}^{N} (\pm 1)^{i+1} \delta(p - p_i) \; \delta_{M+1,N} \sum_{\sigma'} (\pm 1)^{n(\sigma')} \prod_{j=1}^{M} \delta(p_j' - p_{\sigma'(j)})$$

$$= \sum_{i=1}^{N} (\pm 1)^{i+1} \delta(p - p_i) \; \langle p_1' \ldots p_M' \mid p_1 .. p_{i-1} \, p_{i+1} .. p_N \rangle$$

and since the state $|p_1' .. p_M'\rangle$ is arbitrary, we get

$$a(p) \, |p_1 .. p_N\rangle = \sum_{i=1}^{N} (\pm 1)^{i+1} \delta(p - p_i) \, |p_1 .. p_{i-1} \, p_{i+1} .. p_N\rangle$$

and in particular

$$a(p) \, |0\rangle = 0$$

which holds both in the bosonic and the fermionic case.

(*) Write permutations σ as a sum of terms in which p_i is associated with $q_1 = p$ and the remaining momenta $p_1 .. p_{i-1} \, p_{i+1} .. p_N$ form a permutation σ' with $p_1' .. p_M'$, e.g.

$$123 \pm 132 \pm 213 + 231 + 312 \pm 321$$

$$= 1\,[23 \pm 32] \pm 2\,(13 \pm 31) + 3\,(12 \pm 21)$$

The creation and annihilation operators satisfy important

<u>commutation relations</u>

$$[a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0$$

$$[a(p), a^\dagger(p')] = \delta(p-p')$$

in the bosonic case, and similar <u>anticommutation relations</u>

$$\{a(p), a(p')\} = \{a^\dagger(p), a^\dagger(p')\} = 0$$

$$\{a(p), a^\dagger(p')\} = \delta(p-p')$$

in the fermionic case.

This can be seen as follows

$$\left(a^\dagger(p)\, a^\dagger(p') \mp a^\dagger(p')\, a^\dagger(p)\right) |p_1 \dots p_N\rangle$$

$-$: bosons
$+$: fermions

$$= |p\, p'\, p_1 \dots p_N\rangle \mp \underbrace{|p'\, p\, p_1 \dots p_N\rangle}$$

symmetric / antisymmetric under the exchange of p and p'

$$= |p\, p'\, p_1 \dots p_N\rangle - |p\, p'\, p_1 \dots p_N\rangle$$

$$= 0$$

$$\Rightarrow \quad [a^\dagger(p), a^\dagger(p')] = 0 \qquad \text{bosons}$$

$$\{a^\dagger(p), a^\dagger(p')\} = 0 \qquad \text{fermions}$$

and by taking the adjoint of this relation, we get

$$[a(p), a(p')] = 0 \qquad \text{bosons}$$

$$\{a(p), a(p')\} = 0 \qquad \text{fermions}$$

We further have

$$\left(a(p)\, a^+(p') \mp a^+(p')\, a(p) \right) | p_1 \dots p_N \rangle$$

$$= a(p)\, | p'\, p_1 \dots p_N \rangle \mp a^+(p') \sum_{i=1}^{N} (\pm 1)^{i+1}\, \delta(p-p_i)\, | p_1 \dots p_{i-1}\, p_{i+1} \dots p_N \rangle$$

$$= \delta(p-p')\, | p_1 \dots p_N \rangle$$

since p_i is in the position $i+1$!

$$+ \sum_{i=1}^{N} (\pm 1)^{i+2}\, \delta(p-p_i)\, | p'\, p_1 \dots p_{i-1}\, p_{i+1} \dots p_N \rangle$$

$$\mp \sum_{i=1}^{N} (\pm 1)^{i+1}\, \delta(p-p_i)\, | p'\, p_1 \dots p_{i-1}\, p_{i+1} \dots p_N \rangle$$

$$= \delta(p-p')\, | p_1 \dots p_N \rangle$$

$$\Rightarrow \quad [\, a(p),\, a^+(p')\,] = \delta(p-p') \qquad \text{bosons}$$

$$\{\, a(p),\, a^+(p')\, \} = \delta(p-p') \qquad \text{fermions}$$

which is a short-hand notation of

$$[\, a(p,s;n),\, a^+(p',s';n')\,]$$

$$= (2\pi)^3\, 2p^0\, \delta_{ss'}\, \delta_{nn'}\, \delta^{(3)}(\vec{p}-\vec{p}\,')$$

and similarly for the anticommutator.

It should be stressed that the creation and annihilation operators introduced here have nothing to do with a harmonic approximation (notice that we did not even specify a Hamiltonian yet). We rather introduced these operators formally here as a means to connect states with different particle numbers in the Fock space. The operators then automatically take care of the required symmetrization / antisymmetrization of the states.

The importance of this formalism in QFT lies in the fact that <u>any</u> operator can be expressed in terms of creation and annihilation operators. We will encounter various examples below, the simplest one being the Hamiltonian of a spin-0 particle with

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \; p^0 \, a^+(p) \, a(p)$$

The creation and annihilation operators thus provide a universal "language" to discuss the action of Fock state operators on the physical states.

But there actually exists a more fundamental reason why QFT is formulated in terms of creation and annihilation operators. This is known as the <u>cluster decomposition principle</u>, which states that distant measurements must yield uncorrelated results. We will not go into the details here, but one can show that the representation of Fock state operators in terms of creation and annihilation operators automatically fulfills this requirement (see section 4 of Weinberg I).

We finally derive the transformation properties of the creation and annihilation operators under the various symmetries discussed in the previous sections. The transformation properties are actually fixed by the ones of the states, since

$$|p,s;n\rangle = a^{+}(p,s;n)\,|0\rangle$$

and $a(p,s;n)$ is the adjoint of $a^{+}(p,s;n)$. One further assumes that the vacuum state is invariant under all symmetry transformations (i.e. the symmetry is not spontaneously broken).

$m^2 > 0$:

$$\mathcal{U}(\Lambda,b)\, a^+(p,s;n)\, \mathcal{U}^{-1}(\Lambda,b) \underbrace{|0\rangle}_{= \mathcal{U}(\Lambda,b)|0\rangle}$$

$$= \mathcal{U}(\Lambda,b)\, a^+(p,s;n) \underbrace{\mathcal{U}^{-1}(\Lambda,b)\, \mathcal{U}(\Lambda,b)}_{= 1\!\!1} |0\rangle$$

$$= \mathcal{U}(\Lambda,b)\, |p,s;n\rangle$$

$$= e^{ib\Lambda p} \sum_{s'} D^{(j)}_{s's}(R)\, |\Lambda p, s'; n\rangle$$

$$= e^{ib\Lambda p} \sum_{s'} D^{(j)}_{s's}(R)\, a^+(\Lambda p, s'; n)\, |0\rangle$$

$$\Longrightarrow \quad \mathcal{U}(\Lambda,b)\, a^+(p,s;n)\, \mathcal{U}^{-1}(\Lambda,b) = e^{ib\Lambda p} \sum_{s'} D^{(j)}_{s's}(R)\, a^+(\Lambda p, s'; n)$$

$$\mathcal{U}(\Lambda,b)\, a(p,s;n)\, \mathcal{U}^{-1}(\Lambda,b) = e^{-ib\Lambda p} \sum_{s'} D^{(j)}_{s's}(R)^*\, a(\Lambda p, s'; n)$$

and similarly

$$\mathcal{U}_P\, a^+(p,s;n)\, \mathcal{U}_P^{-1} = \eta_P\, a^+(\mathcal{P}p, s; n)$$

$$\mathcal{U}_T\, a^+(p,s;n)\, \mathcal{U}_T^{-1} = \eta_T\, (-1)^{j-s}\, a^+(\mathcal{P}p, -s; n)$$

$$\mathcal{U}_C\, a^+(p,s;n)\, \mathcal{U}_C^{-1} = \eta_C\, a^+(p,s;\bar{n})$$

$$\mathcal{U}(g)\, a^+(p,s;n)\, \mathcal{U}^{-1}(g) = \sum_{n'} D(g)_{n'n}\, a^+(p,s;n')$$

$$\mathcal{U}(g)\, a^+(p,s;\bar{n})\, \mathcal{U}^{-1}(g) = \sum_{\bar{n}'} D(g)^*_{\bar{n}'\bar{n}}\, a^+(p,s;\bar{n}')$$

and the transformation of $a(p,s;n)$ again follows by taking the adjoint of the above relations.

$\underline{M^2 = 0:}$

$$\mathcal{U}(\Lambda,b)\, a^+(p,\sigma;n)\, \mathcal{U}^{-1}(\Lambda,b) = e^{ib\Lambda p}\, e^{-i\theta\sigma}\, a^+(\Lambda p,\sigma;n)$$

$$\mathcal{U}_P\, a^+(p,\sigma;n)\, \mathcal{U}_P^{-1} = \eta_\sigma\, e^{\mp i\pi\sigma}\, a^+(\mathcal{P}p,-\sigma;n)$$

$$\mathcal{U}_T\, a^+(p,\sigma;n)\, \mathcal{U}_T^{-1} = \mathcal{J}_\sigma\, e^{\pm i\pi\sigma}\, a^+(\mathcal{P}p,\sigma;n)$$

whereas the transformation properties under charge conjugation and the internal symmetries are the same as in the massive case.

Whenever a theory contains particles belonging to different species, it is convenient to use a convention in which the states are symmetric under the exchange of any two bosons or any boson with any fermion, but antisymmetric under the exchange of any two fermions. While this is not a fundamental requirement since distinct particles can clearly be distinguished in a quantum theory, this convention makes the implementation of approximate symmetries easier (like e.g. isospin symmetry in QCD).