0. Introduction

In this course we are going to study the fundamental laws of nature that govern the microscopic world at the smallest distances that mankind has probed so far ($\sim 10^{-18}$ m).

In the quantum mechanics course, we have learned that microscopic objects have characteristics of both particles and waves and that the time evolution of a quantum mechanical system is given by the Schrödinger equation.

The Schrödinger equation applies, however, only to systems that involve velocities that are small compared to the speed of light.

In order to describe scattering reactions of highly energetic particles, one has to reconcile the principles of quantum mechanics and special relativity. This endeavour turns out, however, to be notoriously difficult. Early attempts to write down a relativistic wave equation led to serious problems like negative probabilities and negative-energy states. In special relativity, we further learned that energy can be converted into mass and vice versa, but the creation and annihilation of particles is not captured by a single-particle wave equation.

The Schrödinger equation that we appreciated in the quantum mechanics course is incomplete in another respect. While it provides a consistent quantum mechanical description for non-relativistic electrons, the electromagnetic field the electrons interact with is not quantised at all.

As the quanta of the electromagnetic field - the photons - have zero rest mass, they travel at the speed of light and their quantum theory therefore necessarily needs to be relativistic.

Historically, the successful quantisation of the electromagnetic field can be viewed as the birth of quantum field theory (QFT). In a seminal work by Born, Heisenberg and Jordan from 1926, they considered the (free) electromagnetic field as an infinite set of harmonic oscillators and they applied the usual canonical quantisation procedure to these oscillators. As we will learn in the course of this lecture, this is the starting point of QFT, which is nothing but the quantum theory of systems with an infinite number of degrees of freedom.

Soon after the successful quantisation of the electromagnetic field, it was realised that the techniques can be applied to other particles as well, and that this procedure - known as second quantisation - circumvents the problems of a single-particle wave equation. In particular, it was found that a consistent quantisation of fermionic fields requires anticommutation relations, which is at the heart of the spin-statistics theorem.

We will not follow the historical development of QFT any further, but we will instead present QFT from the modern perspective in this course (for a historical account of the development of QFT, see chapter 1 of Weinberg, Vol I). To start with, we will examine the constraints from Lorentz invariance on the physical Hilbert space, and we will learn how the concepts of spin and antiparticles naturally arise in this context. We will then proceed and construct field operators for particles with integer and half-integer spin, and we will learn how to formulate interacting theories that are consistent with Lorentz invariance and causality. The goal of this course consists in developing the theoretical framework that is needed to compute scattering cross sections and decay rates within the Standard Model of particle physics, which is the QFT that reflects our current understanding of the microscopic world.

In this course I assume familiarity with the concepts of quantum mechanics, special relativity and electrodynamics. There is, however, another ingredient that usually falls short in the physics syllabus. At this stage, the students have probably already realised that symmetries play a central role in theoretical physics. The mathematical structures behind symmetries are groups, and we will therefore start with a brief introduction to group theory before we embark on our journey to construct the foundations of QFT.

Throughout this course we will use natural units with $c = \hbar = 1$.

1. Basic group theory

In the first course on theoretical physics we learned that continuous symmetries give rise to conserved quantities via Noether's theorem. In the quantum mechanics course, we could in particular appreciate that the use of symmetries can help to simplify a problem – just compare the elegant algebraic solution of the angular momentum algebra with the tedious way of solving the corresponding partial differential equations in the position space representation.

Our interest in QFT in symmetries and group theory is mainly twofold: On the one hand, we have to understand the implications of Lorentz invariance on the physical states and the field operators. In addition, the Standard Model (SM) is based on gauge symmetries, and we have to learn how to generalise the familiar gauge transformations from electrodynamics.

1.1. Definitions and examples

We start with the definition of a group.

A <u>group</u> $(G, \circ)$ is a set G with a group multiplication $\circ$, which associates any ordered pair of elements $a, b \in G$ a product $a \circ b \in G$, such that

i) $a \circ (b \circ c) = (a \circ b) \circ c \quad \forall\, a, b, c \in G$ (associativity)

ii) there is an element $e \in G$ with

$$e \circ a = a \circ e = a \qquad \forall\, a \in G \qquad \text{(identity element)}$$

iii) for each $a \in G$ there is an element $a^{-1} \in G$ with

$$a \circ a^{-1} = a^{-1} \circ a = e \qquad \text{(inverse element)}$$

<u>Further definitions</u>:

- A group is called <u>abelian</u> if the group multiplication is commutative, i.e

$$a \circ b = b \circ a \qquad \forall\, a, b \in G$$

Otherwise the group is called <u>non-abelian</u>.

- The number of elements of a group is called the <u>order</u> of the group (if it is finite).

- A subset H of G is called a <u>subgroup</u> of $(G, \circ)$ if $(H, \circ)$ itself forms a group under the same group multiplication as G.

- Two groups $(G, \circ)$ and $(G', \times)$ are said to be <u>isomorphic</u> if there exists a one-to-one correspondence between their elements, which preserves the law of group multiplication.

$$\begin{array}{ccc} G & & G' \\ a & \leftrightarrow & a' \\ b & \leftrightarrow & b' \\ c & \leftrightarrow & c' \end{array} \qquad \begin{array}{c} a \circ b = c \\ a' \times b' = c' \end{array}$$

We write $(G, \circ) \cong (G', \times)$
or simply $G \cong G'$

- Given two groups $(G, \circ)$ and $(G', \times)$ with elements $a, b, \ldots \in G$ and $a', b', \ldots \in G'$, one can define the <u>direct product group</u> $(G, \circ) \otimes (G', \times)$ via the group multiplication

$$\underbrace{(a, a') \bullet (b, b')}_{\text{elements of the direct product group}} = (a \circ b, a' \times b')$$

Let us illustrate these concepts with a few examples. For finite groups, one typically summarises the results of the group multiplication in a <u>multiplication table</u>.

<u>Examples</u>:

- Cyclic group C_2

	e	a
e	e	a
a	a	e

order 2

abelian

- Cyclic group C_3

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

order 3

abelian

For the cyclic groups C_n, it is instructive to denote the elements by $\{e, a, a^2, \ldots, a^{n-1}\}$ with $a^n = e$.

For the C_3 this amounts to renaming $b \equiv a^2$.

- Cyclic group C_6

	e	a	a^2	a^3	a^4	a^5
e	e	a	a^2	a^3	a^4	a^5
a	a	a^2	a^3	a^4	a^5	e
a^2	a^2	a^3	a^4	a^5	e	a
a^3	a^3	a^4	a^5	e	a	a^2
a^4	a^4	a^5	e	a	a^2	a^3
a^5	a^5	e	a	a^2	a^3	a^4

order 6

abelian

The cyclic group C_6 has two subgroups

$$H_1 = \{e, a^3\} \cong C_2$$

$$H_2 = \{e, a^2, a^4\} \cong C_3$$

and it is isomorphic to the direct product of C_2 and C_3, $\quad C_6 \cong C_2 \otimes C_3$

$H_1 \otimes H_2$		C_6
(e, e)	$\leftrightarrow$	e
(e, a^2)	$\leftrightarrow$	a^2
(e, a^4)	$\leftrightarrow$	a^4
(a^3, e)	$\leftrightarrow$	a^3
(a^3, a^2)	$\leftrightarrow$	a^5
(a^3, a^4)	$\leftrightarrow$	a

$$(e, a^2) \bullet (a^3, e) = (a^3, a^2)$$

$$a^2 \circ a^3 = a^5$$

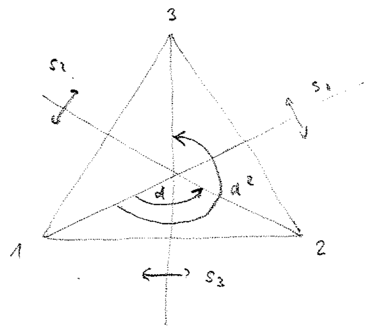
$$(a^3, a^2) \bullet (a^3, a^4) = (e, e)$$

$$a^5 \circ a = e$$

etc.

- Dihedral group D_3

This group can be generated from the symmetry transformations of an equilateral triangle. These are the identity e, rotations around the central point by $120°$ (d) or $240°$ (d^2) and reflections about the medians s_1, s_2, s_3.



The multiplication table is readily constructed

2 \ 1	e	d	d^2	s_1	s_2	s_3
e	e	d	d^2	s_1	s_2	s_3
d	d	d^2	e	s_3	s_1	s_2
d^2	d^2	e	d	s_2	s_3	s_1
s_1	s_1	s_2	s_3	e	d	d^2
s_2	s_2	s_3	s_1	d^2	e	d
s_3	s_3	s_1	s_2	d	d^2	e

order 6

non-abelian

The dihedral group has four subgroups

$$H_1 = \{e, s_1\} \cong C_2$$

$$H_2 = \{e, s_2\} \cong C_2$$

$$H_3 = \{e, s_3\} \cong C_2$$

$$H_4 = \{e, d, d^2\} \cong C_3$$

1.2. Lie groups

We now turn our attention to groups with an infinite number of elements, for which we cannot construct a multiplication table. To do so, we introduce another method for specifying the group structure.

We denote the elements of a finite group $(G, \circ)$ of order n in the form

$$G = \{g_1, \dots, g_n\}$$

The group structure then follows from specifying all products

$$g_i \circ g_j = g_k \qquad i,j,k = \{1, \dots, n\}$$

The group structure thus defines a <u>composition function</u>

$$\phi : \{1, \dots, n\} \times \{1, \dots, n\} \Longrightarrow \{1, \dots, n\}$$

$$\phi(i,j) = k$$

i.e. it gives the index of the group element which is given by the product of the ith and the jth group element.

This concept can be readily generalised to infinite groups, for which we label the group elements by a parameter instead of an index

$$g_i \longrightarrow g(a)$$

An infinite group can depend on several parameters and in this case we write $\vec{a} = (a_1, \dots, a_r)$. The group structure

$$g(\vec{a}) \circ g(\vec{b}) = g(\vec{c})$$

then defines r composition functions $\vec{\Phi}(\vec{a}, \vec{b}) = \vec{c}$. The number of real parameters r is called the <u>dimension</u> of the group.

We are in particular interested in <u>Lie groups</u> for which the composition functions $\vec{\Phi}(\vec{a}, \vec{b})$ are analytic functions of the parameters, i.e. they can be expanded at each point $(\vec{a}, \vec{b})$ into a convergent power series.

As an example consider the groups $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$.

Let us quickly verify if the group axioms are fulfilled:

- closed under addition ✓
- addition is associative ✓
- identity element $0 \in G$ ✓
- for each $x \in G$, there is an inverse $-x \in G$ ✓

The group are all abelian and infinite, but

- $(\mathbb{Z}, +)$ is not continous
- $(\mathbb{Q}, +)$ is continous but not a Lie group
- $(\mathbb{R}, +)$ is a Lie group

We introduce a few more definitions that are useful to characterise Lie groups

- A Lie group is <u>compact</u> if the range of the parameters $\vec{a} = (a_1, .., a_r)$ is a compact subset of $\mathbb{R}^r$.

(This is a bit sloppy, but ok for our purposes)

- Let $I \subseteq \mathbb{R}$ be an interval. A continous mapping

$$t \in I \rightarrow g(t) \in G$$

is called a <u>path</u> in G. It follows that

$$\{ g(\vec{a}(t)) \mid t \in I, \ \vec{a}(t) \text{ continous} \}$$

is a path in G.

- A group is called <u>connected</u> if every $g \in G$ can be connected to the identity element e via a path.
- A group is called <u>simply connected</u> if G is connected and every closed path in G can be contracted within G to a point.
- If the group is not connected, the subgroup G_0 of G, which contains all elements of G that are connected to the identity element e is called the <u>identity component</u>.

We illustrate these notions with a few examples.

Examples:

- $(\mathbb{R}^n, +)$

  This is the group of spatial translations in n-dimensional euclidean space

  $$\vec{x}' = \vec{x} + \vec{a} \qquad \vec{a} \in \mathbb{R}^n$$

  which similar to $(\mathbb{R}, +)$ from above is obviously a Lie group.

  The group is abelian, has dimension n, is not compact but is simply connected.

- $GL(n, \mathbb{C})$

  This is the first example of a matrix group, which is the set of $n \times n$ dimensional matrices with matrix multiplication as group multiplication. Matrix multiplication is associative but not commutative, and so the matrix groups are non-abelian for $n \geq 2$.

  The group $GL(n, \mathbb{C})$ is called the general linear group, which is the group of all complex, invertible $n \times n$ matrices.

  A matrix A is invertible if $\det A \neq 0$.

Let us verify the group axioms:

- closed, since $A \cdot B$ is a matrix with

$$\det(A \cdot B) = \det A \cdot \det B \neq 0 \quad ✓$$

- associative ✓
- identity element $\mathbb{1}$ with $\det \mathbb{1} = 1 \neq 0$ ✓
- inverse to A exists since $\det A \neq 0$ and

$$\det A^{-1} = \frac{1}{\det A} \neq 0$$

The group $GL(n,\mathbb{C})$ has dimension $2n^2$ and it is not compact.

It is furthermore connected, but not simply connected.

[which can be easily understood for $GL(1,\mathbb{C}) \cong \mathbb{C}\setminus\{0\}$]

- $\underline{SL(n,\mathbb{C})}$

A subgroup of $GL(n,\mathbb{C})$ is $SL(n,\mathbb{C})$, the <u>special linear group</u>, which consists of all complex $n \times n$ matrices with $\det A = 1$.

- closed, since $\det(A \cdot B) = \det A \cdot \det B = 1 \cdot 1 = 1$ ✓
- associative ✓
- identity element $\mathbb{1}$ has $\det \mathbb{1} = 1$ ✓
- inverse to A exists and $\det A^{-1} = \frac{1}{\det A} = 1$ ✓

The group $SL(n,\mathbb{C})$ has dimension $2n^2 - 2$ (since $\det A = 1$ gives two real constraints), it is not compact but simply connected.

[$SL(1,\mathbb{C})$ is special since this is a single point and hence it is finite and compact]

- $U(n)$

This is the group of complex, unitary $n \times n$ matrices known as the unitary group. Notice that $UU^\dagger = U^\dagger U = \mathbb{1}$ implies

$U^\dagger = (U^T)^*$

$$\det(UU^\dagger) = \det U \det U^\dagger = \det U \, (\det U)^*$$
$$= |\det U|^2 = \det(\mathbb{1}) = 1$$

$$\Rightarrow |\det U| = 1$$

- closed, since $U \cdot V$ is unitary

  $(UV)(UV)^\dagger = UVV^\dagger U^\dagger = \mathbb{1}$ and similar for $(UV)^\dagger UV = \mathbb{1}$ ✓

- associative ✓
- identity element $\mathbb{1}$ fulfils $\mathbb{1} \cdot \mathbb{1}^\dagger = \mathbb{1}^\dagger \mathbb{1} = \mathbb{1}$ ✓
- inverse to U exists since $\det U \neq 0$ and

  $UU^\dagger = \mathbb{1} \quad \rightarrow \quad U^{-1} = U^\dagger$ with

  $$(U^{-1})(U^{-1})^\dagger = U^\dagger (U^\dagger)^\dagger = U^\dagger U = \mathbb{1}$$
  $$(U^{-1})^\dagger (U^{-1}) = (U^\dagger)^\dagger U^\dagger = UU^\dagger = \mathbb{1}$$ ✓

To determine the dimension of $U(n)$, we note that the unitarity constraint gives:

- n real constraints for the diagonal elements
- $\frac{n(n-1)}{2}$ complex constraints for the offdiagonal elements

$$\Rightarrow \quad 2n^2 - n - \frac{n(n-1)}{2} \cdot 2 = n^2$$

The group $U(n)$ thus has dimension n^2, it is compact, connected, but not simply connected.

[which can again be easily understood for $U(1) \cong \{z \in \mathbb{C}, |z| = 1\}$, see also excercises]

- <u>$SU(n)$</u>

This is the subgroup of complex, unitary $n \times n$ matrices with $\det U = 1$, called the <u>special unitary group</u>. Like for $SL(n, \mathbb{C})$, it is easy to see that the constraint $\det U = 1$ does not spoil the group axioms.

The group has dimension $n^2 - 1$, since the constraint $\det U = 1$ just fixes the phase of $|\det U| = 1$, which holds for arbitrary unitary matrices. $SU(n)$ is furthermore compact and simply connected.

[$SU(1)$ is again special since it is a single point, see also excercises]

- <u>$O(n)$</u>

This is the group of real, orthogonal $n\times n$ matrices known as the <u>orthogonal group</u>. $RR^T = R^TR = \mathbb{1}$ now implies

$$\det(RR^T) = \det R \,\det R^T = (\det R)^2$$
$$= \det \mathbb{1} = 1$$

$$\Rightarrow \det R = \pm 1$$

- closed since $R\cdot S$ is orthogonal

$$(RS)(RS)^T = R S S^T R^T = \mathbb{1} \quad , \quad \text{similar for } (RS)^T(RS) = \mathbb{1} \quad ✓$$

- associative ✓
- identity element $\mathbb{1}$ with $\mathbb{1}\mathbb{1}^T = \mathbb{1}^T\mathbb{1} = \mathbb{1}$ ✓
- inverse to R exists since $\det R \neq 0$ and

$$RR^T = \mathbb{1} \quad \rightarrow \quad R^{-1} = R^T \quad \text{with}$$
$$(R^{-1})(R^{-1})^T = R^T(R^T)^T = R^TR = \mathbb{1}$$
$$(R^{-1})^T(R^{-1}) = (R^T)^T R^T = R R^T = \mathbb{1} \quad ✓$$

The orthogonality condition now yields

- n real constraints for the diagonal elements
- $\frac{n(n-1)}{2}$ real constraints for the off diagonal elements

$$\Rightarrow \quad n^2 - n - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$$

The group $O(n)$ thus has dimension $\frac{n(n-1)}{2}$, it is compact, but not connected since the elements with $\det R = -1$ are not connected to the identity component. The identity component of $O(n)$ is $SO(n)$.

One can formally define the group $O(n)$ as the set of transformations which leave the euclidean scalar product invariant

$$\vec{x}\,' = R\,\vec{x} \qquad\qquad \vec{x} \in \mathbb{R}^n$$

$$\Rightarrow \vec{y}\,'^T \cdot \vec{x}\,' = (R\vec{y})^T \cdot (R\vec{x}) = \vec{y}^{\,T} R^T R\,\vec{x} = \vec{y}^{\,T} \cdot \vec{x}$$

It consists of rotations with $\det R = +1$ and rotations that are combined with a reflection with $\det R = -1$.

- <u>$SO(n)$</u>

  This is the subgroup of real, orthogonal $n \times n$ matrices with $\det R = 1$, called the <u>special orthogonal group</u>.

  In view of our discussions on $SL(n, \mathbb{C})$ and $SU(n)$, the group axioms are again obviously fulfilled.

The group has dimension $\frac{n(n-1)}{2}$, since the condition $\det R = 1$ does not yield an independent constraint, but it rather rules out the component that is not connected to the identity element (which consists of rotations that are combined with a reflection). The group $SO(n)$ is also called the rotation group, and it is compact, connected, but not simply connected.

[$SO(1)$ is again a single point and $SO(2) \cong U(1)$]

1.3 Lie algebras

Instead of defining a Lie group as an infinite group with an analytic composition function, one can also define it as a differentiable manifold with a group structure. This starting point offers a new perspective on Lie groups that we are going to explore in this section.

For our purposes, it is sufficient to think about a manifold as a space M that locally looks like euclidean space, but on large scales it can be curved. There then exist a number of charts that map these locally flat regions to $\mathbb{R}^r$

$$f: M \to \mathbb{R}^r$$

Here r is the dimension of the manifold, which for a Lie group is equal to the dimension of the group.

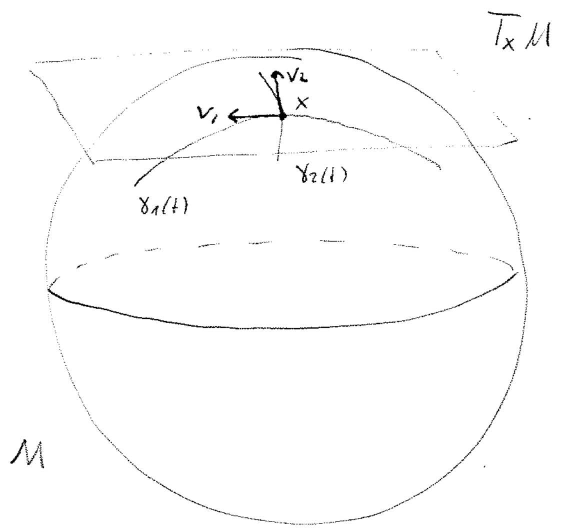
Let us now consider an arbitrary point x on the manifold.

If one collects all curves $\gamma : \mathbb{R} \to M$ on the manifold that go through that point, one can construct their respective tangential vectors v at $\gamma(0) = x$ via

$$v = \frac{d}{dt} \gamma(t) \Big|_{t=0}$$

The tangential vectors span a vector space attached to the point x with the same dimension as the manifold.

This space is called the <u>tangent space</u> $T_x M$.



In the context of Lie groups, we are in particular interested in the tangent space T_eG attached to the identity element e.

One then defines an <u>exponential map</u>

$$\begin{aligned} T_eG &\to G \\ v &\to \exp(v) \equiv \gamma_v(1) \end{aligned}$$

which allows one to reconstruct the group elements in the vicinity of the identity element from the elements in the tangent space.

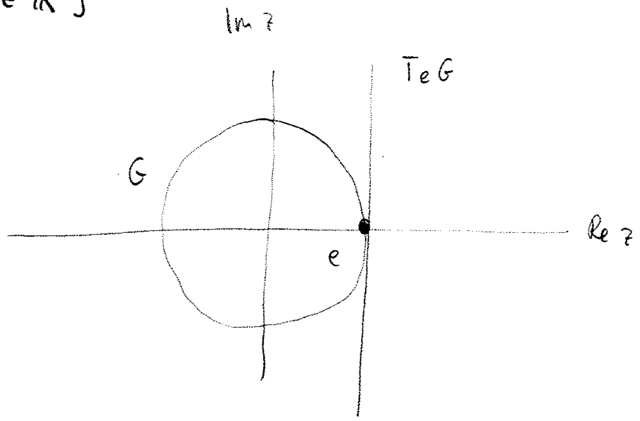
For the matrix groups that we mostly consider here, the exponential map coincide with the usual matrix exponential

$$\underbrace{\exp(V)}_{\in T_eG} = \sum_{k=0}^{\infty} \frac{V^k}{k!} = \mathbb{1} + V + \frac{1}{2} V^2 + \dots \quad \in G$$

Let us illustrate these notions with a simple example.

The group $U(1)$ consists of all unitary 1×1 matrices with norm 1. The group manifold thus corresponds to the unit circle centered at 0 in the complex plane, and the tangent space at e is given by the imaginary line

$\{ i\theta \mid \theta \in \mathbb{R} \}$



It turns out that the exponential map

$$i\theta \to g(\theta) = \exp(i\theta)$$

generates in this case all elements of $U(1)$, not only those which are in the vicinity of the identity element.

More generally, one can always represent an arbitrary group element in the vicinity of the identity element as

$$g(\vec{\theta}) = \exp\left(i \sum_{a=1}^{r} \theta^a T^a\right)$$

where r is the number of basis vectors T^a that span the vector space T_eG (which is equal to the dimension of the group) and $\vec{\theta} = (\theta^1, \dots, \theta^r)$ are arbitrary real coefficients.

In the following, we will adopt Einstein's summation convention to write this more concisely as

$$g(\vec{\theta}) = e^{i\theta^a T^a}$$

The key point is the observation that the group structure implies a certain structure on the tangent space T_eG.

To see this, we consider two group elements

$$g_1 = e^{i\varepsilon T^1} \qquad g_2 = e^{i\varepsilon T^2}$$

for $r \geq 2$. The inverse of these group elements is then simply given by

$$g_1^{-1} = e^{-i\varepsilon T^1} \qquad g_2^{-1} = e^{-i\varepsilon T^2}$$

since the considered group elements correspond to simple one-parameter subgroups.

Let us now consider the combination

$$g_1 \circ g_2 \cdot g_1^{-1} \cdot g_2^{-1} \equiv g'$$

which must give another group element $g' = e^{i\theta^a T^a}$ since the Lie group is closed.

Upon expanding the group elements around the identity element $\mathbb{1}$, we obtain

$$\left(\mathbb{1} + i\varepsilon T^1 - \frac{\varepsilon^2}{2} T^1 T^1 + \dots\right)\left(\mathbb{1} + i\varepsilon T^2 - \frac{\varepsilon^2}{2} T^2 T^2 + \dots\right)$$

$$\left(\mathbb{1} - i\varepsilon T^1 - \frac{\varepsilon^2}{2} T^1 T^1 + \dots\right)\left(\mathbb{1} - i\varepsilon T^2 - \frac{\varepsilon^2}{2} T^2 T^2 + \dots\right)$$

$$= \mathbb{1} - \varepsilon^2 \left(T^1 T^2 - T^2 T^1\right) + \dots$$

$$\stackrel{!}{=} \mathbb{1} + i\theta^a T^a + \dots$$

The group structure thus implies that the <u>commutator</u>

$$[T^a, T^b] \equiv T^a T^b - T^b T^a$$

yields another element of the tangent space!

As the tangent space is a vector space, we can expand each element in terms of the basis vectors. In general, we therefore write

$$\boxed{[T^a, T^b] = i f^{abc} T^c}$$

where the coefficients f^{abc} are called <u>structure constants</u>, which reflect the (local) structure of the underlying Lie group.

The tangent space T_eG together with the commutator

$$[\ldots,\ldots] : T_eG \times T_eG \rightarrow T_eG$$

has the structure of a Lie algebra.

In general, a <u>Lie algebra</u> $\mathcal{L}$ is a vector space V, which is closed under a bilinear operation $[\ldots,\ldots] : V \times V \rightarrow V$, with the properties

i) $[v,w] = -[w,v] \quad \forall\, v,w \in V$ (antisymmetry)

ii) $[v,[w,z]] + [w,[z,v]] + [z,[v,w]] = 0 \quad \forall\, v,w,z \in V$

(Jacobi identity)

This implies that the structure constants are antisymmetric in the first two indices, $f^{abc} = -f^{bac}$, and that they satisfy the relation

$$f^{abd} f^{cde} + f^{cad} f^{bde} + f^{bcd} f^{ade} = 0$$

A few remarks are in order

- First of all we note that a differentiable manifold is not automatically a Lie group. As we have seen above, the group structure was crucial to transform the tangent space $T_e G$ into a Lie algebra.

- The basis vectors T^a of the Lie algebra $(a = 1, \dots, r)$ are also called the <u>generators</u> of the Lie algebra.

- The derivation from above shows that the commutator vanishes if the underlying group is abelian

$$g_1 \circ \underbrace{g_2 \cdot g_1^{-1}}_{\text{commute}} \cdot g_2^{-1} = \underbrace{g_1 \circ g_1^{-1}}_{e} \cdot \underbrace{g_2 \cdot g_2^{-1}}_{e} = e$$

$$\Rightarrow \quad [T^a, T^b] = 0$$

This is in particular the case if the dimension of the group $r = 1$.

It is important to distinguish the concepts of a Lie group G and a Lie algebra $\mathcal{L}$. For the matrix groups, they are both represented by matrices, which have however completely different properties (see also the examples below). In particular

- The product of two elements of G yields another element of G since the group is closed. The product of two elements of $\mathcal{L}$ does not give, however, in general another element of $\mathcal{L}$ (only the commutator gives another element of $\mathcal{L}$!).
- As the Lie algebra is a vector space, one can construct linear combinations of the elements in $\mathcal{L}$. There does not exist a similar operation for the elements in G.

In order to distinguish the Lie algebra from the underlying Lie group, one typically uses lower case letters

$$G: \; SO(n), \; SU(n), \; SL(n,\mathbb{C}), \dots$$

$$\mathcal{L}: \; so(n), \; su(n), \; sl(n,\mathbb{C}), \dots$$

Whereas there exists a unique Lie algebra for each Lie group, the converse is not true. As the Lie algebra only specifies the Lie group in the vicinity of the identity element via the exponential map, two groups with the same algebra may differ by their global properties (like $O(n)$ and $SO(n)$). In general, one can show that the successive operation

$$e^{i\theta_1^a T^a} \, e^{i\theta_2^a T^a} \dots$$

generates only the elements of the identity component of the Lie group (which is the entire group if the group is connected).

Let us consider a few examples.

- $\underline{u(n)}$

Writing $U = e^{i\theta^a T^a} = 1\!\!1 + i\theta^a T^a + \dots$, where we recall that θ^a are real coefficients, we obtain

$$U^+ = 1\!\!1 - i\theta^a (T^a)^+ + \dots$$

$$\Rightarrow \quad UU^+ = 1\!\!1 + i\theta^a \left(T^a - (T^a)^+ \right) + \dots \overset{!}{=} 1\!\!1$$

$$\Rightarrow \quad T^a = (T^a)^+$$

The Lie algebra $u(n)$ is thus the space of all hermitian $n \times n$ matrices. Let us check if the dimension of the Lie algebra is the same as the one of the underlying Lie group

- diagonal elements are real
- off-diagonal elements are complex, but due to $a_{ij} = a_{ji}^*$, only half of them is independent

$$\Rightarrow \quad n + \frac{n(n-1)}{2} \cdot 2 = n^2 \quad \checkmark$$

- $su(n)$

Using $\det e^A = e^{\operatorname{tr} A}$ – which trivially holds for diagonal matrices, but can be shown to hold in the general case – the additional constraint $\det U = 1$ yields

$$\det e^{i\theta^a T^a} = e^{\operatorname{tr}(i\theta^a T^a)} = 1 + i\theta^a \operatorname{tr}(T^a) + \ldots \stackrel{!}{=} 1$$

$\Rightarrow \operatorname{tr}(T^a) = 0$

i.e. the Lie algebra $su(n)$ consists of all traceless, hermitian matrices. The dimension of $su(n)$ is $n^2 - 1$ ✓

The most prominent cases are $su(2)$ and $su(3)$:

- $su(2)$

dimension $2^2 - 1 = 3$

basis $T^a = \frac{\sigma^a}{2}$ with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

hermitian ✓
traceless ✓

$$\Rightarrow [T^a, T^b] = i\varepsilon^{abc} T^c$$

usual Levi-Civita tensor
(totally antisymmetric)

- su(3)

dimension $3^2 - 1 = 8$

basis $T^a = \frac{\lambda^a}{2}$ with Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

hermitian + traceless ✓

Structure constants are again totally antisymmetric and real with

abc	123	147	156	246	257	345	367	458	678
f^{abc}	1	1/2	−1/2	1/2	1/2	1/2	−1/2	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

The remaining structure constants are either determined by the antisymmetry or zero

$$[T^1, T^4] = \frac{1}{4}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i\,\frac{1}{2}\,T^7 \quad ✓$$

product $T^1 T^4$ is
not hermitian, only
the commutator is!

Jacobi identity: e.g. $a=1, b=2, c=4, e=5$

$$f^{123} f^{435} + f^{417} f^{275} + f^{246} f^{165} = 1 \cdot (-1/2) + (-1/2)(-1/2) + 1/2 \cdot 1/2 = 0 \quad ✓$$

- $O(n)$

Writing $R = e^{i\theta^a T^a} = 1\!\!1 + i\theta^a T^a + \ldots$

$$\Rightarrow \quad R^T = 1\!\!1 + i\theta^a (T^a)^T + \ldots$$

$$\Rightarrow \quad R R^T = 1\!\!1 + i\theta^a \left(T^a + (T^a)^T \right) + \ldots \overset{!}{=} 1\!\!1$$

$$\Rightarrow \quad T^a = -(T^a)^T$$

The Lie algebra $o(n)$ thus consists of all antisymmetric matrices. To determine the dimension of the algebra, we note that

* diagonal elements vanish
* off-diagonal elements are real, but only half of them is independent

$$\Rightarrow \quad \frac{n(n-1)}{2} \quad \checkmark$$

- $so(n)$

The constraint $\det R = 1$ implies again $\mathrm{tr}(T^a) = 0$, but antisymmetric matrices are anyway traceless and so this yields nothing new. In other words,

The Lie groups $O(n)$ and $SO(n)$ share the same Lie algebra.
(since they have the same identity component)

Let us consider the cases $so(2)$ and $so(3)$ in more detail.

- $so(2)$

  dimension $\frac{2(2-1)}{2} = 1 \quad \longrightarrow \quad so(2)$ is abelian

  only generator $T = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with $T^2 = \mathbb{1}$

  $$e^{i\theta T} = \mathbb{1} + i\theta T - \frac{\theta^2}{2}\mathbb{1} - \frac{i}{6}\theta^3 T + \dots$$

  $$= \cos\theta\,\mathbb{1} + iT\sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

  which is indeed the usual rotation matrix.

- $so(3)$

  dimension $\frac{3(3-1)}{2} = 3$

  basis $T^1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad T^2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad T^3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

  $$\Rightarrow \quad [T^a, T^b] = i\varepsilon^{abc}T^c$$

  We thus obtain the same structure functions as for the $su(2)$, i.e. $so(3) \cong su(2)$.

  (see exercises for a closer study of this connection)

1.4. Representations

In physics we typically do not work with abstract group elements, but we rather need to understand how a symmetry transformation is realized on the space of the physical states. So what we really use in practice are the representations of a group.

A <u>representation</u> $\mathcal{D}$ of a group G on a vector space V is a mapping

$$\mathcal{D} : G \longrightarrow \mathcal{O} \quad \text{(set of bijective linear operators on } V\text{)}$$

$$g \longrightarrow \mathcal{D}(g) : V \to V$$

which preserves the structure of the group (homomorphism)

$$\mathcal{D}(g_1 \circ g_2) = \mathcal{D}(g_1)\, \mathcal{D}(g_2) \qquad \forall\, g_1, g_2 \in G$$

<u>Remarks</u>:

- Although $\mathcal{D}(g)$ is bijective on V (one-to-one correspondence, in particular invertible), the mapping $g \to \mathcal{D}(g)$ is not necessarily bijective.

- A representation $\mathcal{D}$ is called <u>faithful</u> if the mapping $g \to \mathcal{D}(g)$ is injective, i.e.

$$\mathcal{D}(g) \neq \mathcal{D}(g') \qquad \forall g \neq g'$$

- The dimension of the underlying vector space V is called the <u>dimension</u> of the representation $\mathcal{D}$, $d(\mathcal{D}) = \dim V$.

If $d(\mathcal{D})$ is finite, one can think of the linear operators $\mathcal{D}(g)$ as matrices, i.e. $\mathcal{D}(g) \in GL(d(\mathcal{D}), V)$

set of invertible $d(\mathcal{D}) \times d(\mathcal{D})$ matrices defined on V

The simplest representation is given by

$$\mathcal{D}_1(g) = 1 \qquad \forall g \in G$$

which obviously preserves the group structure

$$\mathcal{D}_1(g_1)\, \mathcal{D}_1(g_2) = 1 \cdot 1 = 1 = \mathcal{D}_1(g_1 \circ g_2)$$

but is not faithful unless the group itself is the trivial group $G = \{e\}$. This representation is called the <u>trivial representation</u>.

For the matrix group $GL(n,\mathbb{C})$ or some of its subgroups that we discussed earlier, another representation comes to our mind

$$D_F(g) = g$$

which again fulfils the group structure

$$D_F(g_1)\, D_F(g_2) = g_1\, g_2 = D_F(g_1 \circ g_2)$$

This is called the defining or <u>fundamental representation</u>, which turns out to be the smallest dimensional representation that is faithful with $d(F) = n$. The elements of $SO(n)$ for instance are thus considered as linear operators acting on $\mathbb{R}^n$. But this is actually what we have been doing all the time! Instead of thinking of the elements of $SO(n)$ as abstract elements that leave the euclidean scalar product invariant and belong to the identity component (see page 16-17), we considered them as real $n \times n$ matrices with the properties $RR^T = \mathbb{1}$ and $\det R = 1$. The important point to note is that this is already a particular representation of the elements of $SO(n)$ (called the fundamental representation). But there are many others (actually an infinite number of them). Can we find e.g. a 4-dimensional representation of $SO(3)$?

A simple way to construct higher-dimensional representations consists in taking the direct sum of two representations.

Consider e.g. two representation $D_1(g)$ and $D_2(g)$, which are defined on the vector spaces V_1 and V_2 of dimension d_1 and d_2, respectively. A vector $\vec{x}_1 \in V_1$ thus transforms under $D_1(g): V_1 \to V_1$ as

$$\vec{x}_1' = D_1(g)\,\vec{x}_1$$

and similarly for $\vec{x}_2 \in V_2$

$$\vec{x}_2' = D_2(g)\,\vec{x}_2$$

The direct sum of the vector spaces V_1 and V_2 then consists of the elements

$$\vec{x} = \vec{x}_1 \oplus \vec{x}_2 = \begin{pmatrix} \vec{x}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{x}_2 \end{pmatrix} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} \begin{array}{l} \leftarrow d_1\text{-dimensional vector} \\ \leftarrow d_2\text{-dimensional vector} \end{array}$$

The vector space $V = V_1 \oplus V_2$ has dimension $d_1 + d_2$.

The obvious way to define the <u>direct sum</u> of two representations then consists in

$$
\mathcal{D}(g) = \mathcal{D}_1(g) \oplus \mathcal{D}_2(g) = \begin{array}{c} \begin{array}{cc} d_1\times d_1 & d_1\times d_2 \end{array} \\ \begin{pmatrix} \mathcal{D}_1(g) & 0 \\ 0 & \mathcal{D}_2(g) \end{pmatrix} \\ \begin{array}{cc} d_2\times d_1 & d_2\times d_2 \end{array} \end{array}
$$

since then

$$
\begin{aligned}
\mathcal{D}(g)\,\vec{x} &= (\mathcal{D}_1(g) \oplus \mathcal{D}_2(g))\,(\vec{x}_1 \oplus \vec{x}_2) \\
&= \begin{pmatrix} \mathcal{D}_1(g) & 0 \\ 0 & \mathcal{D}_2(g) \end{pmatrix} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_1(g)\,\vec{x}_1 \\ \mathcal{D}_2(g)\,\vec{x}_2 \end{pmatrix} \\
&= \mathcal{D}_1(g)\,\vec{x}_1 \oplus \mathcal{D}_2(g)\,\vec{x}_2
\end{aligned}
$$

One easily verifies that $\mathcal{D}(g)$ preserves the group structure

$$
\begin{aligned}
\mathcal{D}(g_1)\,\mathcal{D}(g_2) &= \begin{pmatrix} \mathcal{D}_1(g_1) & 0 \\ 0 & \mathcal{D}_2(g_1) \end{pmatrix} \begin{pmatrix} \mathcal{D}_1(g_2) & 0 \\ 0 & \mathcal{D}_2(g_2) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{D}_1(g_1)\,\mathcal{D}_1(g_2) & 0 \\ 0 & \mathcal{D}_2(g_1)\,\mathcal{D}_2(g_2) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{D}_1(g_1 \cdot g_2) & 0 \\ 0 & \mathcal{D}_2(g_1 \cdot g_2) \end{pmatrix} \qquad \text{$\mathcal{D}_1$ and $\mathcal{D}_2$ are representations} \\
&= \mathcal{D}(g_1 \cdot g_2)
\end{aligned}
$$

To come back to our question from above, a 4-dimensional representation of $SO(3)$ is thus given e.g. by the direct sum of the trivial and the fundamental representation

$$D(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \qquad R \in SO(3)$$

which is defined on the vector space $\mathbb{R}^4$.

It is obvious that one can construct representations of any dimension with this method, but the resulting representations are not particularly interesting. The point is that these representations are reducible.

A representation is called <u>reducible</u> if a non trivial subspace $U \subset V$ exists ($U \neq \{\}$ and $U \neq V$), which is invariant under the operation of D, i.e.

$$D(g)\,\vec{x} \in U \qquad \forall\ x \in U \ \text{and} \ \forall g \in G$$

If, on the other hand, no such invariant subspace of V exists, the representation is called <u>irreducible</u>.

One can show that a reducible representation $\mathcal{D}$ with an invariant subspace $\mathcal{U}$ can always be written in the form

$$\mathcal{D}(g) = \begin{pmatrix} \mathcal{D}_1(g) & a(g) \\ 0 & \mathcal{D}_2(g) \end{pmatrix} \qquad \forall g \in G$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are representations with

$\mathcal{D}_1(g): \mathcal{U} \to \mathcal{U}$ (invariant subspace)

$\mathcal{D}_2(g): V/\mathcal{U} \to V/\mathcal{U}$

$a(g): V/\mathcal{U} \to \mathcal{U}$

Here $V/\mathcal{U}$ is the quotient space which is obtained by collapsing $\mathcal{U}$ to zero*, and α is in general not a representation. If in addition $a(g) = 0 \ \forall g \in G$, the representation is said to be <u>completely reducible</u>. In this case $V/\mathcal{U}$ is also an invariant subspace and $\mathcal{D}$ is the direct sum $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$.

---

* Consider e.g. $V = \mathbb{R}^n$ with an invariant subspace $\mathcal{U} = \mathbb{R}^m$. One can then choose a basis such that the vectors in $\mathcal{U}$ take the form $(x_1, \dots, x_m, 0, \dots, 0)$. $V/\mathcal{U}$ then consists of the vectors $(0, \dots, 0, x_{m+1}, \dots, x_n)$ and it is isomorphic to $\mathbb{R}^{n-m}$. The formal definition of the quotient space makes use of equivalence classes.

It is, however, in general not obvious if a given representation is reducible, since the above statement only tells us that a reducible representation <u>can</u> be brought into block-triangular form.

Consider e.g. a basis transformation in the vector space V with

$$S\vec{x} = \vec{X}$$

The vector $\vec{x}\,' = D(g)\,\vec{x}$ then transforms as

$$S\vec{x}\,' = \vec{X}\,'$$

$$= S\,D(g)\,\vec{x} = \underbrace{S\,D(g)\,S^{-1}}_{=\tilde{D}(g)}\underbrace{S\,\vec{x}}_{\vec{X}} = \tilde{D}(g)\,\vec{X}$$

In a different basis of the vector space V, the representation D thus takes a different form with $\tilde{D}(g) = S\,D(g)\,S^{-1}$.

The above statement therefore only tells us that there exists a particular basis in V, in which a reducible representation takes a block-triangular form.

In general we say that two representation $\mathcal{D}_1$ and $\mathcal{D}_2$ are <u>equivalent</u> if an operator S exists with

$$\mathcal{D}_2(g) = S\, \mathcal{D}_1(g)\, S^{-1} \qquad \forall\, g \in G$$

This is called a similarity transformation.

Another method for constructing higher-dimensional representations consists in the tensor product. As an example consider two vector spaces V_1 and V_2 with dimensions $d_1 = 2$ and $d_2 = 3$, respectively. The direct product of two vectors $\vec{x} \in V_1$ and $\vec{y} \in V_2$ is then given by

$$\vec{x} \otimes \vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \end{pmatrix} \simeq \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_1 y_3 \\ x_2 y_1 \\ x_2 y_2 \\ x_2 y_3 \end{pmatrix}$$

The vector space $V = V_1 \otimes V_2$ thus has dimension $d_1 \cdot d_2$.

The <u>tensor product</u> of two representation $\underset{\hookrightarrow V_1}{D}$ and $\underset{\hookrightarrow V_2}{\tilde{D}}$ is then defined as

$$D \otimes \tilde{D} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} \\ \tilde{D}_{31} & \tilde{D}_{32} & \tilde{D}_{33} \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc} D_{11}\tilde{D}_{11} & D_{11}\tilde{D}_{12} & D_{11}\tilde{D}_{13} & D_{12}\tilde{D}_{11} & D_{12}\tilde{D}_{12} & D_{12}\tilde{D}_{13} \\ D_{11}\tilde{D}_{21} & D_{11}\tilde{D}_{22} & D_{11}\tilde{D}_{23} & D_{12}\tilde{D}_{21} & D_{12}\tilde{D}_{22} & D_{12}\tilde{D}_{23} \\ D_{11}\tilde{D}_{31} & D_{11}\tilde{D}_{32} & D_{11}\tilde{D}_{33} & D_{12}\tilde{D}_{31} & D_{12}\tilde{D}_{32} & D_{12}\tilde{D}_{33} \\ \hline D_{21}\tilde{D}_{11} & D_{21}\tilde{D}_{12} & D_{21}\tilde{D}_{13} & D_{22}\tilde{D}_{11} & D_{22}\tilde{D}_{12} & D_{22}\tilde{D}_{13} \\ D_{21}\tilde{D}_{21} & D_{21}\tilde{D}_{22} & D_{21}\tilde{D}_{23} & D_{22}\tilde{D}_{21} & D_{22}\tilde{D}_{22} & D_{22}\tilde{D}_{23} \\ D_{21}\tilde{D}_{31} & D_{21}\tilde{D}_{32} & D_{21}\tilde{D}_{33} & D_{22}\tilde{D}_{31} & D_{22}\tilde{D}_{32} & D_{22}\tilde{D}_{33} \end{array}\right)$$

So that

$$(D \otimes \tilde{D})(\vec{x} \otimes \vec{y}) = D\vec{x} \otimes \tilde{D}\vec{y}$$

which can be easily verified explicitly in the 2×3 dimensional case. The tensor product representation again preserves the group structure since

$$(D(g_1) \otimes \tilde{D}(g_1))(D(g_2) \otimes \tilde{D}(g_2))(\vec{x} \otimes \vec{y})$$

$$= (D(g_1) \otimes \tilde{D}(g_1))(D(g_2)\vec{x} \otimes \tilde{D}(g_2)\vec{y})$$

$$= D(g_1)D(g_2)\vec{x} \otimes \tilde{D}(g_1)\tilde{D}(g_2)\vec{y}$$

$$= D(g_1 \cdot g_2)\vec{x} \otimes \tilde{D}(g_1 \cdot g_2)\vec{y}$$

$$= (D(g_1 \cdot g_2) \otimes \tilde{D}(g_1 \cdot g_2))(\vec{x} \otimes \vec{y})$$

for arbitrary vectors $\vec{x} \in V_1$ and $\vec{y} \in V_2$.

Whereas we saw that the direct sum representation is trivially reducible, the decomposition of the tensor product representation in terms of irreducible representations is non-trivial and known as the <u>Clebsch-Gordan decomposition</u>. One writes

$$\underbrace{D_1 \otimes D_2}_{\substack{\text{tensor product} \\ \text{of } D_1 \text{ and } D_2}} = \underbrace{D_a \oplus D_b \oplus D_c \oplus \ldots}_{\substack{\text{direct sum of irreducible components} \\ \text{(if the rep. is completely reducible)}}}$$

Notice that even if D_1 and D_2 are irreducible representations on V_1 and V_2, the tensor product $D_1 \otimes D_2$ may not be irreducible on $V_1 \otimes V_2$.

Let us introduce two further concepts before we turn to the Lie algebra. First of all, for each representation $D: g \to D(g)$ there exists a <u>complex conjugate representation</u>

$$\bar{D} : g \to \bar{D}(g) \equiv (D(g))^*$$

which obviously preserves the group structure

$$\bar{D}(g_1)\,\bar{D}(g_2) = D(g_1)^*\, D(g_2)^* = (D(g_1)\, D(g_2))^*$$

$$= D(g_1 \cdot g_2)^* = \bar{D}(g_1 \cdot g_2)$$

As $\bar{D}$ acts on the same vector space V as D, we further have $\dim(\bar{D}) = \dim(D)$.

If the representation matrices are real, we have $\bar{D}(g) = D(g)\ \forall g \in G$ and the representation is said to be real. If, on the other hand, the representation matrices are complex with $\bar{D}(g) \neq D(g)$, this does not necessarily imply however that the representation is complex since D and $\bar{D}$ may be related by a similarity transformation with

$$\bar{D}(g) = S\, D(g)\, S^{-1} \qquad \forall g \in G$$

We will encounter an explicit example for such a representation in the tutorials. The complex conjugate representation plays an important role in particle physics since it is related to the notion of antiparticles.

Another important class of representations are <u>unitary representations</u>, which fulfill

$$D^\dagger(g)\, D(g) = \mathbb{1} \qquad \forall g \in G$$

Notice that this does not necessarily imply that the underlying group is unitary.

One can show that all finite-dimensional representations of <u>compact</u> Lie groups (as well as finite groups) are equivalent to a unitary representation. Unitary representations have moreover the remarkable property that they are completely reducible, i.e. there exists a particular basis of the vector space V in which they take a block-diagonal form.

Let us now turn to representations of Lie algebras.

A <u>representation</u> $\mathcal{D}$ of a Lie algebra $\mathcal{L}$ on a vector space V

is a mapping

$$\mathcal{D} : \mathcal{L} \to Q \qquad \text{(set of linear operators on } V\text{)}$$

$$A \to \mathcal{D}(A) : V \to V$$

which preserves the structure of the Lie algebra

$$\mathcal{D}([A,B]) = [\mathcal{D}(A), \mathcal{D}(B)] \qquad \forall\ A, B \in \mathcal{L}$$

The dimension of the representation $\mathcal{D}$ is then again given by the dimension of the underlying vector space V, $\dim(\mathcal{D}) = \dim V$.

Also, the notions faithful, equivalent, reducible and irreducible are defined in analogy to the group representations.

For a given representation $\mathcal{D}$, we write

$$\mathcal{D}(i\theta^a \underset{\in\mathcal{L}}{T^a}) \equiv i\theta^a T^a_{\mathcal{D}}$$

and the structure of the Lie algebra is preserved if the generators in the representation $\mathcal{D}$ satisfy

$$[T^a_{\mathcal{D}}, T^b_{\mathcal{D}}] = i f^{abc} T^c_{\mathcal{D}}$$

This can be seen as follows

$$\mathcal{D}([A,B]) = \mathcal{D}([\underbrace{i\theta^a T^a}_{A}, \underbrace{i\theta'^b T^b}_{B}]) = \mathcal{D}(i\theta^a i\theta'^b [T^a, T^b])$$

abstract elements of the Lie algebra

$$= \mathcal{D}(i\theta^a i\theta'^b i f^{abc} T^c) = i\theta^a i\theta'^b i f^{abc} T^c_{\mathcal{D}}$$

$$= i\theta^a i\theta'^b [T^a_{\mathcal{D}}, T^b_{\mathcal{D}}] = [i\theta^a T^a_{\mathcal{D}}, i\theta'^b T^b_{\mathcal{D}}]$$

$$= [\mathcal{D}(i\theta^a T^a), \mathcal{D}(i\theta'^b T^b)] = [\mathcal{D}(A), \mathcal{D}(B)]$$

Notice that the structure constants f^{abc} are independent of the representation since they reflect the structure of the underlying group!

In practise it is often easier to find representations of the Lie algebra than of the associated Lie group. But this is not a problem, since one can use the exponential map to reconstruct the corresponding group representation

$$\underset{\in G}{\mathcal{D}(g)} = \mathcal{D}(e^{i\theta^a T^a}) = e^{\underset{\in \mathcal{L}}{\mathcal{D}(i\theta^a T^a)}} = e^{i\theta^a T^a_{\mathcal{D}}}$$

On the level of the Lie algebra, the trivial representation implies

$$\mathcal{D}_1(g) = g \overset{!}{=} e^{\mathcal{D}_1(i\theta^a T^a)} = e^{i\theta^a T_1^a}$$

label of the trivial rep.

$$\Rightarrow T_1^a = 0 \qquad a = 1, \ldots, r$$

Similarly, the fundamental representation yields

$$\mathcal{D}_F(g) = g = e^{i\theta^a T^a} \overset{!}{=} e^{\mathcal{D}_F(i\theta^a T^a)} = e^{i\theta^a T_F^a}$$

$$\Rightarrow T_F^a = T^a$$

i.e. the generators in the fundamental representation are represented by themselves (similar to the group elements).

For the complex conjugate representation, we obtain

$$\bar{D}(g) = (D(g))^* = e^{\mathcal{D}(i\theta^a T^a)^*} \overset{!}{=} e^{\bar{\mathcal{D}}(i\theta^a T^a)}$$

$$\Rightarrow \bar{\mathcal{D}}(A) = \mathcal{D}(A)^* \qquad \forall A \in \mathcal{L}$$

which indeed preserves the structure of the Lie algebra

$$\bar{\mathcal{D}}([A,B]) = \mathcal{D}([A,B])^* = [\mathcal{D}(A), \mathcal{D}(B)]^*$$

$$= [\mathcal{D}(A)^*, \mathcal{D}(B)^*] = [\bar{\mathcal{D}}(A), \bar{\mathcal{D}}(B)]$$

The Lie algebra allows us to define another important representation, which is known as the <u>adjoint representation</u>. In this representation the generators are defined as

$$\left(T^b_{adj}\right)_{ac} \equiv i\, f^{abc} \qquad a,b,c = 1 \ldots r$$

where b denotes a specific generator and the tuple (a,c) refers to a particular entry in the $r \times r$ matrix that represents the generator T^b. The dimension of the adjoint representation ($\to a,c$) must therefore be equal to the dimension of the group ($\to b$),

$\dim(adj) = \dim G = r$.

Let us convince ourselves that the generators in the adjoint representation preserve the structure of the Lie algebra, i.e. that they satisfy

$$\left[T^a_{adj} , T^b_{adj} \right] = i\, f^{abc}\, T^c_{adj}$$

To this end, consider the ij element of the commutator

$$[T^a_{adj}, T^b_{adj}]_{ij}$$

$$= (T^a_{adj})_{ic} (T^b_{adj})_{cj} - (T^b_{adj})_{ic} (T^a_{adj})_{cj}$$

$$= i f^{iac} \, i f^{cbj} - i f^{ibc} \, i f^{caj}$$

$$= f^{\overset{ABD}{iac}} f^{\overset{CDE}{bcj}} + f^{\overset{CAD}{bic}} f^{\overset{BDE}{acj}} \qquad f^{abc} = - f^{bac}$$

$$= - f^{\overset{BCD}{abc}} f^{\overset{ADE}{icj}} \qquad \text{Jacobi identity}$$

$$= i f^{abc} (T^c_{adj})_{ij} \quad \checkmark$$

The structure constants thus induce a specific representation, which can be lifted to the level of the group via the exponential map

$$D_{adj}(g) = e^{\mathcal{D}_{adj}(i\theta^a T^a)} = e^{i\theta^a T^a_{adj}}$$

We finally introduce a concept, which is useful to classify the representations of a Lie algebra.

- For a Lie algebra $\mathcal{L}$ with basis T^a, a <u>Casimir operator</u> is a polynomial in the T^a that commutes with all elements of the Lie algebra.
- The maximum number of linear independent elements of $\mathcal{L}$, which commute with each other, is called the <u>rank</u> of the algebra.

This brings us to the <u>Lemma of Schur</u>:

Let $\mathcal{L}$ be an algebra and C a Casimir operator, i.e $[C, A] = 0 \quad \forall A \in \mathcal{L}$. The lemma states that C is then proportional to the identity operator in an irreducible representation. The constant of proportionality, i.e. the eigenvalues of the Casimir operator, can then be used to classify the irreducible representations of the Lie algebra.

As an example, we consider the representations of $su(2) \cong so(3)$.

In the preceding section we saw that the generators satisfy

$$[T^a, T^b] = i\,\varepsilon^{abc}\,T^c$$

which is of course nothing but the familiar angular momentum algebra from quantum mechanics. Let us therefore adopt the notation here and write $T^a \to J^i$ with $i = x, y, z$.

First of all, we note that there exists a Casimir operator $\vec{J}^2 = J^i J^i$, which indeed is a polynomial in the J^i that commutes with all generators and hence with all elements of the Lie algebra

$$
\begin{aligned}
[\vec{J}^2, J^j] &= J^i J^i J^j - J^j J^i J^i \\
&= J^i [J^i, J^j] - [J^j, J^i] J^i \\
&= i\varepsilon^{ijk} J^i J^k - \underbrace{i\varepsilon^{jik} J^k J^i}_{\text{rename } i \leftrightarrow k} \\
&= i\,(\varepsilon^{ijk} - \underbrace{\varepsilon^{jki}}_{=\varepsilon^{ijk}})\, J^i J^k = 0
\end{aligned}
$$

As $[J^i, J^j] \neq 0$ for $i \neq j$, the rank of $su(2)$ is 1.

We typically choose a basis in which J^2 is diagonal.

As $[\vec{J}^2, J^z] = 0$ the operators have a common set of eigenstates with

$$\vec{J}^2 |j\, m\rangle = j(j+1)\, |j\, m\rangle$$

$$J^z |j\, m\rangle = m\, |j\, m\rangle$$

One further introduces ladder operators $J^\pm = J^x \pm i J^y$ and finds

$$J^\pm |j\, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\; |j\, m\pm 1\rangle$$

The spectrum is furthermore bounded from above and below with

$$m = -j, -j+1, \dots, j-1, j$$

where $j = 0, 1/2, 1, \dots$

The irreducible representations of su(2) are thus characterised by a quantum number j, which is indeed related to the eigenvalues of the Casimir operator $\vec{J}^2$. For a given value of j, the corresponding irreducible representation has dimension $2j+1$.

Explicitly one finds that $j=0$ corresponds to the trivial representation with $J_1^i = 0$. $\quad \rightarrow \quad \vec{J}^2 = 0 \cdot \mathbb{1}$

$\uparrow$ trivial rep.

The representation with $j = 1/2$ is the fundamental representation of $su(2)$ with $J_F^i = \frac{\sigma^i}{2}$, which we discussed on page 30.

In particular, one verifies that $\vec{J}_F^2 = \frac{3}{4}\mathbb{1}$.

The representation with $j=1$ corresponds to the adjoint representation with $(J^j_{adj})_{ik} = i\,\varepsilon^{ijk}$, i.e

$$J^x_{adj} = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J^y_{adj} = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J^z_{adj} = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that this corresponds to the fundamental representation of $so(3)$, which we discussed on page 33. Moreover, $\vec{J}^2 = 2\,\mathbb{1}$.

This is, however, not yet the standard basis in which J^z is diagonal. Performing a similarity transformation with

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 & 0 \\ 0 & 0 & -\sqrt{2}\,i \\ -i & 1 & 0 \end{pmatrix} \qquad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 0 & i \\ 1 & 0 & 1 \\ 0 & i\sqrt{2} & 0 \end{pmatrix}$$

we obtain

$$\hat{J}^3 = S\, J^3_{adj}\, S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

as well as

$$\hat{J}^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{J}^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using the ladder operators one can construct all finite-dimensional irreducible representations of su(2).

The decomposition of tensor product representation in terms of irreducible representations is also familiar from quantum mechanics, e.g.

$$D_F \otimes D_F = D_1 \oplus D_{adj}$$

which is usually written in terms of the dimensions

$$2 \otimes 2 = 1 \oplus 3$$

The representations of the group $SU(2)$ then follow by $e^{i\theta^i J_i}$.

The situation is similar for the group $SO(3)$ except that half-integer values of j do not provide a representation in this case. One can show that

$$D(e) = \underbrace{D(R_{\vec{n}}(2\pi))}_{\substack{\text{rotation around axis } \vec{n} \\ \text{with angle } 2\pi}} = (-1)^{2j} \mathbb{1}$$

The identity element in the group is thus not represented by the identity element of the representation for half-integer j, violating

$$D(e)\, D(g) = -\mathbb{1} \cdot D(g) = -D(g) \neq D(e \cdot g) = D(g)$$

In this case one can, however, associate to each $g \in G$ two elements $\pm D(g)$ to restore the group multiplication law. This generalization can then be considered as a true representation of a larger group – the so-called universal covering group (see also exercises).