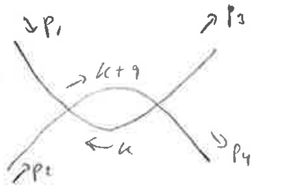
2. Renormalisation

So far we have calculated Green functions and scattering amplitudes in interacting theories only to leading order in the perturbative expansion. At higher orders, one encounters a new topological class of Feynman diagrams which involve integrations over loop momenta that are not constrained by momentum conservation. Consider e.g.

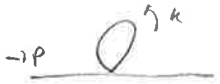


$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+q)^2-m^2]\,[k^2-m^2]}$$

where $q = p_1 + p_2 = p_3 + p_4$. Naive power counting suggests that the loop integral is logarithmically divergent in the limit $k^2 \to \infty$

$$\sim \int \frac{d^4k}{k^4} \sim \int^{\infty} \frac{dk}{k}$$

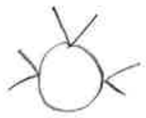
The situation is even worse for the two-point function



$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)} \sim \int \frac{d^4k}{k^2} \sim \int^{\infty} dk\, k$$

which appears to be quadratically divergent.

Higher n-point functions, on the other hand, involve more powers of propagators and therefore have a milder UV behaviour. The 6-point function e.g. yields

 $\sim \int \frac{d^4k}{(k^2)^3} \sim \int^{\infty} \frac{dk}{k^3}$

which is UV finite.

The appearance of UV divergences in loop diagrams not only limits the accuracy of our predictions (since we can only calculate leading-order cross sections that are free of these problems), but severely questions the applicability of the perturbative expansion.

The correct interpretation of the UV divergences is the subject of <u>renormalisation theory</u>.

2.1. The renormalisation program

The computation of loop integrals is involved and requires to develop quite a few calculational methods. Before entering these technicalities, we will outline first the basic strategy of the renormalisation program and we will focus on the interpretation of the results.

To be specific, we will consider the 4-point function in ϕ^4-theory to one-loop order

$i\mathcal{M}(s) =$  $+$  $+ \, \mathcal{O}(\lambda^3)$

where we have neglected the t-channel and u-channel diagrams, for simplicity (we postpone the rigorous calculation to a later section). We have argued above that the loop diagram is logarithmically divergent. The first step of the renormalisation program therefore consists in <u>regularising</u> the theory.

We are thus looking for a suitable regulator to parametrise the divergence. We will consider different types of regulators in detail below, and for the moment we will assume that the theory contains a parameter $\Lambda >> m, s$ which cuts off the UV frequencies of the loop integrals. The divergence is then schematically parametrised by

$$\sim \int^{\Lambda} \frac{dk}{k} \sim \ln \Lambda$$

Without going into the details here, the exact calculation of the amplitude in the cutoff-regularisation scheme gives

$$i\mathcal{M}(s) = -i\lambda + \frac{i\lambda^2}{32\pi^2}\left(\ln \frac{\Lambda^2}{s} + f(s,m) \right) + O(\lambda^3)$$

where $f(s,m)$ is a function that is independent of the UV-cutoff Λ. As the cross section for $\phi\phi \to \phi\phi$ scattering depends on $|\mathcal{M}(s)|^2$, our result seems to be meaningless since it depends on the cutoff Λ and diverges in the limit $\Lambda \to \infty$.

The key observation is that the parameter λ, which appears in the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4$$

is not a measurable quantity. After all λ describes the strength of the ϕ^4-interaction in the classical theory, but why shouldn't the quantum mechanical coupling strength that includes the full information on the quantum fluctuations be different? This even leads us to a more fundamental question: How do we actually define the coupling strength in a quantum field theory?

In an interacting theory, we cannot switch off the quantum fluctuations and therefore the coupling strength λ of the classical theory is not accessible experimentally.

The $\phi\phi \to \phi\phi$ scattering cross section should tell us something about the coupling strength of the ϕ^4-interaction. We could imagine e.g. to measure the cross section at some reference scale s_0, and we could use this measurement to define the quantum mechanical coupling strength

$$i\mathcal{M}(s_0) = -i\lambda + \frac{i\lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0, m)\right) + O(\lambda^3)$$

$$\equiv -i\lambda_R$$

This in turn implies a relation between the observable (renormalised) coupling λ_R that contains the full information on the quantum fluctuations and the unobservable (bare or unrenormalised) coupling λ

$$\lambda_R = \lambda - \frac{\lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0, m)\right) + O(\lambda^3)$$

We can invert this relation with the ansatz

$$\lambda = \lambda_R + c\,\lambda_R^2 + O(\lambda_R^3)$$

It follows

$$\lambda_R = \lambda_R + c\,\lambda_R^2 - \frac{\lambda_R^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0,m)\right) + O(\lambda_R^3)$$

$$\hookrightarrow \quad c = \frac{1}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0,m)\right)$$

and hence

$$\lambda = \lambda_R + \frac{\lambda_R^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0,m)\right) + O(\lambda_R^3)$$

We can then express the scattering amplitude in terms of the observable coupling λ_R, which gives

$$i\mathcal{M}(s) = -i\lambda_R - \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0,m)\right)$$
$$+ \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s} + f(s,m)\right) + O(\lambda_R^3)$$
$$= -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{s_0}{s} + f(s,m) - f(s_0,m)\right) + O(\lambda_R^3)$$

which is finite in the limit $\Lambda \to \infty$! We can therefore now safely remove the regulator by taking the limit $\Lambda \to \infty$, and we obtain a well-defined result for the physical cross section.

What has happened?

First of all, we notice that we have expressed an observable quantity (cross section at the scale s) in terms of another observable quantity (cross section at reference scale s_0) instead of an unobservable one (coupling strength of the classical theory).

The divergence thus gets absorbed by the relation between the bare and the renormalised couplings. While this seems to be a neat trick, it is not at all obvious that we can absorb all UV divergences into the parameters of the theory. In ϕ^4-theory there are e.g. only three such parameters: the coupling strength α, the mass m and the field normalisation Z. As we observed at the beginning of this chapter, there are however only a few amplitudes that are superficially divergent: the 2- and the 4-point functions.
Is it really possible to absorb the divergences into the parameters of the theory?

But what is the origin of the divergences?

The divergences are associated with configurations that have momenta which are much larger than all internal scales ($m, s, ..$) in the process. Relativistic invariance and causality, on the other hand, enforce us to formulate quantum field theories with <u>local</u> interactions. The locality then translates into momentum integrations that involve arbitrarily high energy scales. But do we really expect that our theory is valid to arbitrary high energies? Shouldn't at some point some new phenomena (like gravity, string theory, ..) kick in which provide a natural cutoff for our theory?

The uncertainty principle states, on the other hand, that we cannot resolve distances smaller than $\sim \frac{1}{E}$ in a scattering experiment with energy E. The physics at the scale E can therefore not depend on the UV physics since otherwise we could probe arbitrarily small distance scales with a finite energy resolution. This is exactly what we are seeing: the physics (cross section) is <u>independent</u> of the UV physics, and the divergences only show up in the relations between the bare and the renormalised parameters. The bare mass or coupling, on the other hand, are indeed dressed up with quantum fluctuations at arbitrarily small distances, but they are not experimentally accessible. Instead we can only measure renormalised parameters which always contain the effects of a certain "cloud" of virtual particles. The critical question therefore becomes how to define the renormalised parameters in a quantum field theory, i.e which of the virtual effects do we want to absorb into the definition?

To corroborate this point, let us repeat the above calculation with a different UV regulator. A particular elegant (but not very intuitive) way of regularising UV divergences consists in shifting the space-time dimension from $d=4$ to $d=4-2\varepsilon$. It is evident from our power-counting arguments that the one-loop integral for $\phi\phi \to \phi\phi$ scattering is then UV-finite as $k \to \infty$ as long as we choose $\varepsilon > 0$

 $$\sim \int \frac{d^{4-2\varepsilon}k}{k^4} \sim \int^{\infty} dk \; k^{-1-2\varepsilon}$$

In the limit $\varepsilon \to 0$, the UV divergence then shows up as a pole in ε. The amplitude now becomes

$$i\mathcal{M}(s) = -i\lambda + \frac{i\lambda^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s} + f(s,m) + 1\right) + O(\lambda^3)$$

where μ is a mass parameter that is introduced in dimensional regularisation on dimensional grounds. The naive result of the amplitude thus differs from the previous result – it is regulator-dependent – but the dependence on the internal scales encoded in $f(s,m)$ is the same.

Let us now impose the same renormalization condition as before

$$i\mathcal{M}(s_0) \equiv -i\lambda_R$$

which implies

$$\lambda_R = \lambda - \frac{\lambda^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s_0} + f(s_0,m) + 1\right) + O(\lambda^3)$$

$$\hookrightarrow \lambda = \lambda_R + \frac{\lambda_R^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s_0} + f(s_0,m) + 1\right) + O(\lambda_R^3)$$

and finally

$$i\mathcal{M}(s) = -i\lambda_R - \frac{i\lambda_R^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s_0} + f(s_0,m) + 1\right)$$

$$+ \frac{i\lambda_R^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s} + f(s,m) + 1\right) + O(\lambda_R^3)$$

$$= -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{s_0}{s} + f(s,m) - f(s_0,m)\right) + O(\lambda_R^3)$$

which is exactly the same result that we found before in the cutoff-regularisation scheme! The physical predictions are thus independent of how we treat or model the UV physics. The only difference between the two calculations shows up in the relation between the bare and the renormalised parameters, which we cannot probe (however) experimentally.

Let us now summarise what we have seen so far:

1) We first have to introduce a suitable regulator to parametrise the UV divergences.

2) We then have to define the renormalised parameters of the theory (these are called "renormalisation conditions").

3) We then rewrite the bare parameters in terms of the renormalised ones, and we express the observables in terms of renormalised parameters.

4) We can finally remove the regulator from the theory.

It is important to appreciate that the renormalisation program is not just a technical procedure to remove the disturbing divergences in the theory. In a finite quantum field theory, we would have to follow the same line of reasoning (without points 1+4) to give a physical meaning to the parameters of the theory.

Is there a more systematic way to organise the renormalisation program? Why can't we just get rid of the unphysical, bare parameters from the beginning and work with Feynman rules that depend on the physical, renormalised parameters only?

This is indeed possible. To do so, we rewrite the bare coupling as

$$\lambda = Z_\lambda \lambda_R$$

where Z_λ is a regulator-dependent object that is called a <u>renormalisation constant</u>. Notice that we have actually calculated this object before, we obtained eg.

$$Z_\lambda^\Lambda = 1 + \frac{\lambda_R}{32\pi^2}\left(\ln \frac{\Lambda^2}{s_0} + f(s_0, m) \right) + O(\lambda_R^2)$$

in the cutoff-regularisation scheme and

$$Z_\lambda^\varepsilon = 1 + \frac{\lambda_R}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{s_0} + f(s_0, m) + 1 \right) + O(\lambda_R^2)$$

in dimensional regularisation.

We then split the interaction term in the Lagrangian into two contributions

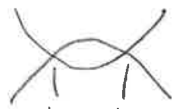
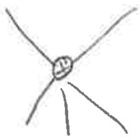
$$-\frac{\lambda}{4!}\phi^4 = -\frac{Z_\lambda \lambda_R}{4!}\phi^4$$

$$= -\frac{\lambda_R}{4!}\phi^4 - \frac{(Z_\lambda - 1)}{4!}\lambda_R \phi^4$$

gives the same Feynman rule as before, but in terms of the renormalized coupling

a new "counterterm" Feynman rule that we can tune to fulfill the renormalization conditions

Let us now repeat the above calculation in the cutoff scheme

$i\mathcal{M}(s) =$  $+$  $+$  $+ \mathcal{O}(\lambda_R^3)$

$$= -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s} + f(s,m)\right) - i(Z_\lambda - 1)\lambda_R + \mathcal{O}(\lambda_R^3)$$

The renormalization condition now implies

$$i\mathcal{M}(s_0) = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0,m)\right) - i(Z_\lambda - 1)\lambda_R + \mathcal{O}(\lambda_R^3)$$

$$\equiv -i\lambda_R$$

$$\Rightarrow \quad Z_\lambda = 1 + \frac{\lambda_R}{32\pi^2}\left(\ln\frac{\Lambda^2}{s_0} + f(s_0,m)\right) + \mathcal{O}(\lambda_R^2)$$

in agreement with what we have found before.

The scattering amplitude then becomes

$$i\mathcal{M}(s) = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{s_0}{s} + f(s,m) - f(s_0,m)\right) + O(\lambda_R^3)$$

as before.

We have found two different ways for organising the renormalisation program:

i) We can work with the usual Feynman rules in terms of the bare parameters and replace the bare parameters by the renormalised ones at the end of the calculation.

ii) We can work with the new Feynman rules in terms of the renormalised parameters, which involves additional counterterm contributions. This procedure is also called <u>renormalised perturbation theory</u> since we expand in the renormalised coupling constant.

The two methods are equivalent, but at higher orders renormalised perturbation theory often appears to be more transparent and more efficient.

Before we turn to the technical aspects of loop calculations, let us address another point. The renormalisation condition that we imposed above

$$i\mathcal{M}(s_0) \equiv -i\lambda_R$$

looks quite arbitrary. What happens if someone else fixes the renormalised parameter at a different scale s_1, imposing e.g.

$$i\mathcal{M}(s_1) \equiv -i\lambda_R'$$

The scattering amplitude then becomes

$$i\mathcal{M}(s) = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\left(\ln\frac{s_0}{s} + f(s,m) - f(s_0,m)\right) + O(\lambda_R^3)$$

$$= -i\lambda_R' + \frac{i\lambda_R'^2}{32\pi^2}\left(\ln\frac{s_1}{s} + f(s,m) - f(s_1,m)\right) + O(\lambda_R'^3)$$

which implies a relation between the two renormalised couplings, which we can again solve perturbatively.

We thus write

$$\lambda_R' = \lambda_R + c\,\lambda_R^2 + O(\lambda_R^3)$$

It follows

$$\lambda_R - \frac{\lambda_R^2}{32\pi^2}\left(\ln\frac{s_0}{s} + f(s,m) - f(s_0,m)\right) + O(\lambda_R^3)$$

$$= \lambda_R + c\,\lambda_R^2 - \frac{\lambda_R^2}{32\pi^2}\left(\ln\frac{s_1}{s} + f(s,m) - f(s_1,m)\right) + O(\lambda_R^3)$$

$$\Rightarrow \quad c = \frac{1}{32\pi^2}\left(\ln\frac{s_1}{s} + \cancel{f(s,m)} - f(s_1,m) - \ln\frac{s_0}{s} - \cancel{f(s,m)} + f(s_0,m)\right)$$

$$= \frac{1}{32\pi^2}\left(\ln\frac{s_1}{s_0} - f(s_1,m) + f(s_0,m)\right)$$

and hence

$$\lambda_R' = \lambda_R + \frac{\lambda_R^2}{32\pi^2}\left(\ln\frac{s_1}{s_0} - f(s_1,m) + f(s_0,m)\right) + O(\lambda_R^3)$$

(independent of s)

We thus learn that we can impose whatever renormalisation condition we want, since we can switch between different <u>renormalisation schemes</u> by translating the renormalised parameters from one scheme to the other. The renormalised couplings in different schemes then differ only by a <u>finite</u> renormalisation. The arbitrariness of the renormalisation conditions will be further explored when we discuss the <u>renormalisation group</u>.

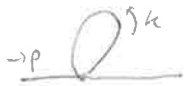
2.2 Loop integrals

We will now develop the technology that is needed to compute basic loop integrals. We start with the simplest one-loop integral

$$A(1,m^2) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-m^2+i\varepsilon]}$$

which appears in the calculation of the 2-point function

symmetry factor



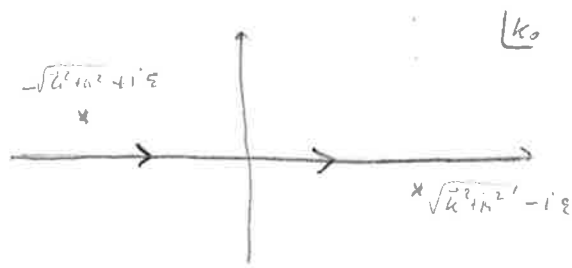
$$= \frac{1}{2}(-i\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} = \frac{\lambda}{2} A(1,m^2)$$

We argued above that the integral is quadratically divergent, and we thus have to introduce a suitable UV regulator. We will first consider the <u>cutoff regularisation scheme</u>.

We first recall that

$$\begin{aligned} k^2 - m^2 + i\varepsilon &= (k^0)^2 - \vec{k}^2 - m^2 + i\varepsilon \\ &= \left(k^0 - \sqrt{\vec{k}^2+m^2} + i\varepsilon\right)\left(k^0 + \sqrt{\vec{k}^2+m^2} - i\varepsilon\right) \end{aligned}$$

The integral therefore has two poles in the complex k^0-plane, which are located as follows:



As the integrand vanishes as $\frac{1}{(k^0)^2}$ for $k^0 \to \infty$, we can close the contour at infinity and compute the integral with the residue calculus. We are free, however, to deform the integration contour as long as the contour does not cross any poles. It is particularly convenient to perform a <u>Wick rotation</u>, i.e. we rotate the contour anticlockwise by $90°$

It follows

$$\int d^4k \, f(k^2) = \int_{-\infty}^{\infty} dk^0 \int d^3k \, f(k^2)$$

$$= \int_{-i\infty}^{+i\infty} dk^0 \int d^3k \, f(k^2)$$

We then substitute

$$k^0 = i\, k_E^0 \qquad\qquad \vec{k} = \vec{k}_E$$

such that

$$k^2 = (k^0)^2 - \vec{k}^2$$
$$= -(k_E^0)^2 - \vec{k}_E^2 \equiv - k_E^2$$

i.e. we have transformed the integral from Minkowskian to Euclidean space with $k_E^2 = (k_E^0)^2 + \vec{k}_E^2$. Moreover

$$\int d^4k\, f(k^2) = i \int_{-\infty}^{\infty} dk_E^0 \int d^3k_E\, f(-k_E^2) = i \int d^4k_E\, f(-k_E^2)$$

and our integral becomes

$$A(1, m^2) = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{-k_E^2 - m^2 + i\varepsilon}$$

$$= \frac{-i}{16\pi^4} \int d^4k_E \frac{1}{k_E^2 + m^2}$$

where we have dropped the $i\varepsilon$-prescription since the denominator does not vanish for any value of k_E^μ.

As the integral only depends on k_E^2, it is convenient to introduce 4-dimensional spherical coordinates with

$$d^4k_E = dk_E\, k_E^3\, d\Omega_4 = \frac{1}{2}\, dk_E^2\, k_E^2\, d\Omega_4$$

The angular integrations are trivial and give the surface of a 4-dimensional sphere

$$\int d\Omega_4 = 2\pi^2$$

(we will compute volume and surface of a d-dimensional sphere in the tutorials)

The "radial" k_E^2-integration, on the other hand, is divergent and we impose a hard cutoff with $k_E < \Lambda$. We thus obtain

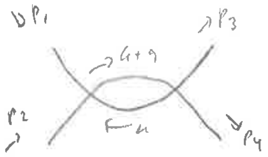
$$A(\Lambda, m^2) = \frac{-i}{16\pi^4} \frac{1}{2} 2\pi^2 \int_0^{\Lambda^2} dk_E^2 \frac{k_E^2}{k_E^2 + m^2}$$

$$= \frac{-i}{16\pi^2} \left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2 + m^2}{m^2} \right) \right]$$

$$= \frac{-i}{16\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} + O\left(\frac{m^4}{\Lambda^2} \right) \right]$$

where we expanded in $\frac{m^2}{\Lambda^2} \ll 1$, since we assume that the cutoff Λ is much larger than all internal scales in the problem. The integral is thus quadratically divergent as anticipated, and it contains an additional subleading logarithmic divergence.

We next consider the 4-point function, which involves the diagram



$$= \underbrace{\frac{1}{2}}_{\text{symmetry factor}} (-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{[(k+q)^2-m^2+i\varepsilon]\,[k^2-m^2+i\varepsilon]}$$

In contrast to the previous calculation, this integrand has a non-trivial angular dependence encoded in $k\cdot q = k^0q^0 - |\vec{k}|\,|\vec{q}|\cos\theta$. There is, however, a trick to circumvent this problem. Consider

$$\int_0^1 dx\, dy\; \delta(1-x-y)\, \frac{1}{[xA+yB]^2}$$

$$= \int_0^1 dx\, \frac{1}{[xA+(1-x)B]^2}$$

$$= -\frac{1}{A-B}\, \frac{1}{[xA+(1-x)B]}\Big|_0^1$$

$$= -\frac{1}{A-B}\left(\frac{1}{A}-\frac{1}{B}\right) = \frac{1}{AB}$$

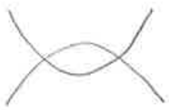
Reading this identity backwards, we see that we can combine two propagators at the expense of additional integrations over the "Feynman parameters" x and y.

Writing

$$x\,[(k+q)^2 - m^2 + i\varepsilon] + (1-x)\,[k^2 - m^2 + i\varepsilon]$$

$$= k^2 + 2x\,kq + x\,q^2 - m^2 + i\varepsilon$$

we obtain



$$= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \frac{1}{[k^2 + 2x\,kq + x q^2 - m^2 + i\varepsilon]^2}$$

We can now complete the square in the denominator, shifting

$$k = k' - x\,q$$

with trivial Jacobian $d^4k = d^4k'$. It follows

$$k^2 + 2x\,kq + xq^2 - m^2 + i\varepsilon$$

$$= k'^2 \cancel{- 2xk'q} + x^2q^2 \cancel{+ 2xk'q} - 2x^2q^2 + xq^2 - m^2 + i\varepsilon$$

$$= k'^2 - \Delta$$

where we defined $(\bar{x} = 1-x)$

$$\Delta = m^2 - x\bar{x}\,q^2 - i\varepsilon$$

The diagram finally becomes

$$= \frac{\lambda^2}{2} \int_0^1 dx \quad A(2,\Delta)$$

with

$$A(2,\Delta) = \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{[k^2 - \Delta]^2}$$

which is of a similar type as $A(1,m^2)$ from above.

For higher n-point functions, we need to generalize the Feynman trick to combine several propagators. In the tutorials we will derive the relation

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \frac{\Gamma(m_1 + \ldots + m_n)}{\Gamma(m_1) \cdots \Gamma(m_n)} \int_0^1 dx_1 \ldots dx_n \; \delta(1 - x_1 - \ldots - x_n)$$

$$\frac{x_1^{m_1 - 1} \cdots x_n^{m_n - 1}}{(x_1 A_1 + \ldots + x_n A_n)^{m_1 + \ldots + m_n}}$$

where

$$\Gamma(x) = \int_0^\infty dt \; t^{x-1} e^{-t}$$

is the Γ-function. Some important properties include

- $\Gamma(1) = \int_0^\infty dt \; e^{-t} = -e^{-t}\Big|_0^\infty = 1$
- $x\,\Gamma(x) = \int_0^\infty dt \; \underbrace{x\, t^{x-1}}_{\frac{d}{dt}(t^x)} e^{-t} = \int_0^\infty dt \; \frac{d}{dt}(t^x)\, e^{-t}$

$$= - \int_0^\infty dt \; t^x \frac{d}{dt}(e^{-t})$$

$$= \int_0^\infty dt \; t^x e^{-t} = \Gamma(1+x)$$

- for $n \in \mathbb{N}$, this implies $\Gamma(n) = (n-1)!$
- $\Gamma(x)$ has simple poles at $x = 0, -1, -2, \ldots$
- $\Gamma(1+\varepsilon) \simeq 1 - \gamma\varepsilon + \left(\frac{1}{2}\gamma^2 + \frac{\pi^2}{12}\right)\varepsilon^2 + O(\varepsilon^3)$

where $\gamma \simeq 0.577216$ is the Euler-Mascheroni constant.

For higher n-point functions, we can therefore follow the same strategy as before, and obtain an integral of the form

$$A(n,\Delta) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\Delta)^n}$$

where Δ is a function of the internal scales of the process and the Feynman parameters.

The integral can be calculated along the lines of the previous calculation

$$A(n,\Delta) = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(-k_E^2-\Delta)^n}$$

$$= (-1)^n \frac{i}{16\pi^2} \int_0^{\Lambda^2} dk_E^2 \frac{k_E^2}{(k_E^2+\Delta)^n}$$

For $n=2$ this yields

$$A(2,\Delta) = \frac{i}{16\pi^2} \left(\ln \frac{\Lambda^2+\Delta}{\Delta} - \frac{\Lambda^2}{\Lambda^2+\Delta} \right)$$

$$= \frac{i}{16\pi^2} \left(\ln \frac{\Lambda^2}{\Delta} - 1 + O\left(\frac{\Delta}{\Lambda^2}\right)\right)$$

which is indeed logarithmically divergent.

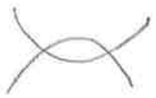
For $n=3$ we obtain

$$A(3,\Delta) = \frac{-i}{16\pi^2} \left(\frac{\Lambda^4}{2\Delta(\Lambda^2+\Delta)^2} \right)$$

$$= -\frac{i}{16\pi^2} \left(\frac{1}{2\Delta} + O\left(\frac{1}{\Lambda^2}\right) \right)$$

which is UV-finite.

Let us now come back to the 4-point function with



$$= \frac{\lambda^2}{2} \int_0^1 dx \; A(2,\Delta) \qquad\qquad \Delta = m^2 - x\bar{x}s - i\varepsilon$$

$$= \frac{\lambda^2}{2} \int_0^1 dx \; \frac{i}{16\pi^2} \left(\ln \frac{\Lambda^2}{\Delta} - 1 \right)$$

$$= \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left(\ln \frac{\Lambda^2}{s} + \ln \frac{s}{\Delta} - 1 \right)$$

$$\equiv \frac{i\lambda^2}{32\pi^2} \left(\ln \frac{\Lambda^2}{s} + f(s,m) \right)$$

where we can now read off the explicit form of $f(s,m)$, which we introduced on pg. 68

$$f(s,m) = \int_0^1 dx \; \ln \frac{s}{m^2 - x\bar{x}s - i\varepsilon} - 1$$

We will now repeat the calculation of the loop integrals in a different regularisation scheme. In <u>dimensional regularisation</u> (DR) we assume that the theory is formulated in arbitrary $d = 4-2\varepsilon$ dimensions. For a suitable value of $\varepsilon > 0$, the loop integrals are then UV-finite and can be computed as a function of ε. We finally analytically continue the result to $d=4$ by taking the limit $\varepsilon \to 0$.

Before computing loop integrals in DR, we have to address a subtle point, namely that the coupling constant has a non-trivial mass dimension in d dimensions. This can be seen as follows:

$$\text{path integral} \sim e^{iS} \Rightarrow \text{mass dimension } [S] = 0$$

$$S = \int d^d x \, \mathcal{L} \Rightarrow [\mathcal{L}] = d$$

$$\mathcal{L} \sim \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi \Rightarrow [\phi] = \frac{d-2}{2}$$

$$\mathcal{L} \sim \frac{m^2}{2} \phi^2 \Rightarrow [m] = \frac{d-(d-2)}{2} = 1 \checkmark$$

$$\mathcal{L} \sim \frac{\lambda}{4!} \phi^4 \Rightarrow [\lambda] = d - 2(d-2) = 4 - d = 2\varepsilon$$

But a perturbative expansion in a dimensionful parameter is not meaningful. We therefore rewrite $\lambda \to \lambda \mu^{2\varepsilon}$ with an arbitrary parameter μ that has mass dimension $[\mu] = 1$. But since this parameter is anyway arbitrary, we are free to rescale it by a constant, and the combination

$$\tilde{\mu} = \sqrt{\frac{e^{\gamma}}{4\pi}} \, \mu$$

with the Euler-Mascheroni constant $\gamma \simeq 0.577216$ turns out to be a particular convenient choice.

We thus have

$$\mathcal{L} \sim \frac{\lambda \tilde{\mu}^{2\varepsilon}}{4!} \phi^4$$

where for the moment λ is still the bare coupling with $[\lambda] = 0$, and the associated Feynman rule becomes

$$= (-i\lambda)\, \tilde{\mu}^{2\varepsilon}$$

The 2-point function in DR then reads

$$= \frac{1}{2}(-i\lambda)\, \tilde{\mu}^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\varepsilon} \equiv \frac{\lambda}{2} A(1, m^2)$$

where now

$$A(1, m^2) = \tilde{\mu}^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\varepsilon}$$

For the 4-point function, on the other hand, we can again introduce Feynman parameters and find

$$= \frac{\lambda^2}{2}\, \tilde{\mu}^{2\varepsilon} \int_0^1 dx\, A(2, \Delta)$$

as the tree-level diagram $\sim \tilde{\mu}^{2\varepsilon}$, we only absorb one factor of $\tilde{\mu}^{2\varepsilon}$ into the loop integral

with

$$A(2, \Delta) = \tilde{\mu}^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - \Delta]^2} \qquad \Delta = m^2 - x\bar{x}q^2 - i\varepsilon$$

In DR we thus obtain integrals of the form

$$A(n,\Delta) = \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2-\Delta]^n}$$

which we can calculate along the lines of the previous calculation.

$$d^dk_E = dk_E\, k_E^{d-1}\, d\Omega_d = \frac{1}{2} dk_E^2 (k_E^2)^{\frac{d-2}{2}} d\Omega_d \qquad \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\begin{aligned}
A(n,\Delta) &= i\mu^{2\varepsilon} \int \frac{d^dk_E}{(2\pi)^d} \frac{1}{[-k_E^2-\Delta]^n} \\
&= (-1)^n \frac{i}{(2\pi)^d} \mu^{2\varepsilon} \frac{1}{2} \int_0^\infty dk_E^2 \frac{(k_E^2)^{\frac{d-2}{2}}}{(k_E^2+\Delta)^n} \int d\Omega_d \\
&= (-1)^n \frac{i}{(4\pi)^{d/2}} \frac{\mu^{2\varepsilon}}{\Gamma(d/2)} \int_0^\infty dk_E^2 \frac{(k_E^2)^{\frac{d-2}{2}}}{(k_E^2+\Delta)^n}
\end{aligned}$$

We next substitute

$$k_E^2 = \frac{(1-x)}{x}\Delta \qquad dk_E^2 = -\frac{1}{x^2}\Delta\, dx \qquad x = \frac{\Delta}{k_E^2+\Delta}$$

to bring the "radial" integral into the form

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\begin{aligned}
\Rightarrow A(n,\Delta) &= (-1)^n \frac{i}{(4\pi)^{d/2}} \frac{\mu^{2\varepsilon}}{\Gamma(d/2)} \int_0^1 dx\, x^{n-\frac{d}{2}-1}(1-x)^{\frac{d}{2}-1}\Delta^{\frac{d}{2}-n} \\
&= (-1)^n \frac{i}{(4\pi)^{d/2}} \frac{\mu^{2\varepsilon}}{\cancel{\Gamma(d/2)}} \frac{\Gamma(n-d/2)\cancel{\Gamma(d/2)}}{\Gamma(n)} \Delta^{\frac{d}{2}-n} \\
&= \frac{i}{16\pi^2} \left(\mu^2 e^{\gamma}\right)^{\varepsilon} \frac{(-1)^n}{\Gamma(n)} \Gamma(n-d/2)\, \Delta^{\frac{d}{2}-n}
\end{aligned}$$

For $n=1$, this yields

$$A(1,m^2) = \frac{i}{16\pi^2} (\mu^2 e^{\gamma})^{\varepsilon} (-1) \underbrace{\Gamma(-1+\varepsilon)}_{\frac{\Gamma(\varepsilon)}{(-1+\varepsilon)} = \frac{\Gamma(1+\varepsilon)}{\varepsilon(-1+\varepsilon)} \simeq \frac{(1-\gamma\varepsilon)}{\varepsilon}(-1-\varepsilon) \simeq -\frac{1}{\varepsilon} - 1 + \gamma + O(\varepsilon)} (m^2)^{1-\varepsilon}$$

$$= \frac{i}{16\pi^2} m^2 \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{m^2} + 1 + O(\varepsilon) \right]$$

which is to be compared with the expression that we obtained in the cutoff regularisation scheme (cf. pg. 86)

$$A(1,m^2) = \frac{i}{16\pi^2} \left[-\Lambda^2 + m^2 \ln \frac{\Lambda^2}{m^2} + O\left(\frac{m^4}{\Lambda^2}\right) \right]$$

In contrast to the cutoff scheme, the integral is an analytic function of the regulator ε in DR. The logarithmic dependence on the physical scale m^2 is, moreover, the same in both regularisation schemes. We may actually be tempted to identify the logarithmic divergence in both schemes by

$$\ln \frac{\Lambda^2}{m^2} \longrightarrow \frac{1}{\varepsilon} + \ln \frac{\mu^2}{m^2}$$

but there is no analog of the quadratic divergence in DR.

For $n=2$ we get

$$A(2,\Delta) = \frac{i}{16\pi^2}\,(\mu^2 e^{\gamma})^{\varepsilon}\,\underbrace{\Gamma(\varepsilon)}_{\frac{\Gamma(1+\varepsilon)}{\varepsilon}\,\simeq\,\frac{1-\gamma\varepsilon}{\varepsilon}\,=\,\frac{1}{\varepsilon}-\gamma+O(\varepsilon)}\,\Delta^{-\varepsilon}$$

$$= \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{\Delta} + O(\varepsilon)\right)$$

whereas we obtained (cf. pg 90)

$$A(2,\Delta) = \frac{i}{16\pi^2}\left(\ln\frac{\Lambda^2}{\Delta} - 1 + O\left(\frac{\Delta}{\Lambda^2}\right)\right)$$

in the cutoff scheme. The logarithmic divergence is thus again captured by the above substitution rule.

We finally consider the case $n=3$

$$A(3,\Delta) = \frac{i}{16\pi^2}\,(\mu^2 e^{\gamma})^{\varepsilon}\,\frac{(-1)}{2}\,\underbrace{\Gamma(1+\varepsilon)}_{1+O(\varepsilon)}\,\Delta^{-1-\varepsilon}$$

$$= -\frac{i}{16\pi^2}\,\frac{1}{2\Delta} + O(\varepsilon)$$

which is UV-finite and indeed agrees with the result that we obtained in the cutoff regularisation scheme (cf. pg. 90).

For the 4-point function, we now obtain

$$= \frac{\lambda^2}{2}\,\tilde{\mu}^{2\varepsilon}\int_0^1 dx\; A(2,\Delta)$$

$$= \frac{i\lambda^2}{32\pi^2}\,\tilde{\mu}^{2\varepsilon}\int_0^1 dx\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{\Delta}\right)$$

$$= \frac{i\lambda^2}{32\pi^2}\,\tilde{\mu}^{2\varepsilon}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s} + f(s,m) + 1\right)$$

with the same function

$$f(s,m) = \int_0^1 dx\;\ln\frac{s}{m^2 - x\bar{x}s - i\varepsilon} - 1$$

as anticipated on pg. 75.

2.3 Mass and field renormalisation

Now that we have learned how to compute loop integrals, let us come back to the renormalisation program. We argued in section 2.1. that we can define a renormalised coupling strength by measuring a cross section in a particular kinematic configuration. But how can we define the renormalised mass of a particle?

First of all, we note that the situation is completely analogous to the one of the coupling constant. The bare mass parameter m in the Lagrangian corresponds to the mass of the particle in the classical theory, which is however not observable in an interacting QFT. We therefore have to define a renormalised mass parameter m_R by a suitable renormalisation condition. In physical terms, this implies that the mass definition always includes the effects of a certain "cloud" of virtual particles. But due to the freedom in choosing the renormalization condition, there then exist various definitions of what we call a particle mass in a QFT! Some mass definitions may be more intuitive or useful than others, but strictly speaking there does not exist a preferred mass definition. Towards the end of section 2.1, we argue now that we can switch between different schemes by a finite renormalization.

In this section we will introduce one of the most important mass definitions, which is sometimes called the physical mass scheme or the <u>on-shell renormalization scheme</u>.

Whereas we started from the $\phi\phi \to \phi\phi$ scattering amplitude to define the renormalised coupling constant, the natural object to define the renormalised particle mass is the two-point function

$$\langle \Omega | T\, \phi(x)\, \phi(y) | \Omega \rangle$$

vacuum of the interacting theory

Heisenberg operators

We will later impose a specific renormalisation condition on this object, but to do so we first have to better understand its general structure in an interacting QFT. The following analysis actually complements what we have seen in TPP1 (section 4.1) since it is not based on a perturbative expansion.

Our first step consists in inserting a complete set of momentum eigenstates in the form

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|$$

which includes the one-particle states of the interacting theory as well as all sorts of multi-particle states, which we distinguish in our notation by a label α. The momentum $\vec{p}$ corresponds to the total 3-momentum of the state $|\alpha_p\rangle$ with energy

$$E_p(\alpha) = \sqrt{\vec{p}^{\,2} + m_\alpha^2}$$

where m_α is the "mass" of the particle state, i.e. its energy in a frame with total momentum $\vec{p}=0$.

For the one-particle states, this is nothing but the familiar relativistic energy-momentum relation

$$E_p(\alpha) = \sqrt{\vec{p}^{\,2} + m_P^2}$$

which depends on the <u>physical mass parameter m_P</u>, which is defined as the eigenvalue of the Hamiltonian H in the particle rest frame

For a two-particle state with total momentum $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0$,

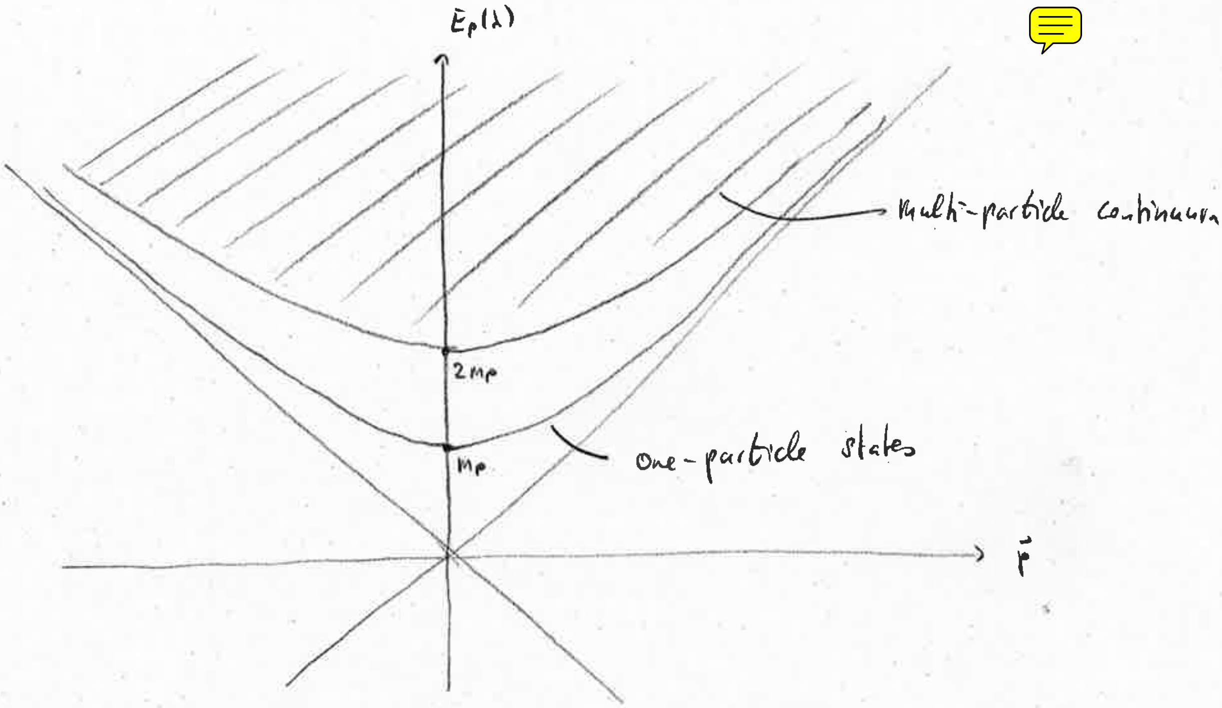
we further obtain for instance

$$m_\lambda^2 = \left(\sqrt{\vec{p}_1^{\,2} + m_p^2} + \sqrt{\vec{p}_2^{\,2} + m_p^2} \right)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

$$= 4 \left(m_p^2 + \vec{p}_1^{\,2} \right)$$

The eigenvalues $E_p(\lambda)$ of $P^0 = H$ and $\vec{p}$ of $\vec{P}$ can

thus be represented by a set of hyperboloids



and there may be additional bound-state contributions just

below the multi-particle continuum, which we disregard in

the following (see e.g. chapter 7.1 of Peskin / Schröder).

Similar to what we have seen in TPP1 (section 2.2), the states $|\lambda_p\rangle$ are related to the states $|\lambda_0\rangle$ in the CMS frame with $\vec{p}=0$ by a Lorentz transformation

$$|\lambda_p\rangle = U(L(p))\,|\lambda_0\rangle$$

where $L(p)$ is the standard boost for massive / massless particles.

Let us now come back to the two-point function and consider the specific time-ordering $x^0 > y^0$. After inserting the identity operator, we obtain

$$\langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle = \langle\Omega|\,\phi(x)\,|\Omega\rangle\langle\Omega|\,\phi(y)\,|\Omega\rangle$$

$$+ \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p(\lambda)}\,\langle\Omega|\,\phi(x)\,|\lambda_p\rangle\langle\lambda_p|\,\phi(y)\,|\Omega\rangle$$

It turns out that the first term is irrelevant for our analysis* and we therefore concentrate on the second term in the following.

---

* The first term vanishes for higher-spin fields due to Lorentz invariance, but it can also be set to zero in a scalar theory by a field redefinition.

The matrix element in the second term can be simplified as follows

$$\langle \Omega | \phi(x) | \lambda_p \rangle = \langle \Omega | e^{iPx} \phi(0) e^{-iPx} | \lambda_p \rangle \qquad P^\mu |\Omega\rangle = 0$$

↳ momentum eigenstate

$$= e^{-ipx} \langle \Omega | \phi(0) | \lambda_p \rangle \Big|_{p^0 = E_p(\lambda)}$$

$$= e^{-ipx} \langle \Omega | \phi(0)\, \mathcal{U}(L(p)) | \lambda_0 \rangle \Big|_{p^0 = E_p(\lambda)} \qquad \mathcal{U}(\Lambda)|\Omega\rangle = |\Omega\rangle$$

$$= e^{-ipx} \langle \Omega | \underbrace{\mathcal{U}(L(p)^{-1})\, \phi(0)\, \mathcal{U}(L(p)^{-1})^{-1}}_{= \phi(0) \quad \text{(TPP1, page 150)}} | \lambda_0 \rangle \Big|_{p^0 = E_p(\lambda)}$$

$$= e^{-ipx} \langle \Omega | \phi(0) | \lambda_0 \rangle \Big|_{p^0 = E_p(\lambda)}$$

which yields

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle$$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} e^{-ip(x-y)} \underbrace{|\langle \Omega | \phi(0) | \lambda_0 \rangle|^2}_{\text{independent of } \vec{p}} \Big|_{p^0 = E_p(\lambda)}$$

Following along the lines of the derivation of the scalar Feynman propagator (TPP1, page 143), the time-ordered product can then be written in the form

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$$

$$= \sum_\lambda \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 - m_\lambda^2 + i\varepsilon} |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2$$

where we recognise the familiar expression of the free Feynman propagator

$$\Delta_F(x-y, m_\lambda^2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 - m_\lambda^2 + i\varepsilon}$$

which depends on the mass m_λ of the propagating state $|\lambda_0\rangle$.

The two-point function is usually written in the form

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \int_0^\infty d\mu^2 \, \rho(\mu^2) \, \Delta_F(x-y, \mu^2)$$

which is known as the <u>Källén - Lehmann spectral representation</u>.

The <u>spectral density function</u> is then given by

$$\rho(\mu^2) = \sum_{\lambda} \delta(\mu^2 - m_\lambda^2) \; |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2$$

which is real and positive, and one can show that it is normalized to (see e.g. Weinberg I, chapter 10.7)

$$\int_0^\infty d\mu^2 \; \rho(\mu^2) = 1$$

To gain some intuition, let us evaluate the spectral density in a free theory with

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle =$$

In this case only one-particle intermediate states contribute and one has

$$\underbrace{\langle 0 | \phi(0) | k \rangle}_{\text{rest frame}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \langle 0 | a(p) + \cancelto{0}{a^\dagger(p)} | k \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} (2\pi)^3 \, 2p^0 \, \delta^{(3)}(\vec{p} - \vec{k}) \langle 0 | 0 \rangle$$

$$= 1$$

$$\Rightarrow \rho(\mu^2) = \delta(\mu^2 - m_p^2)$$

It follows

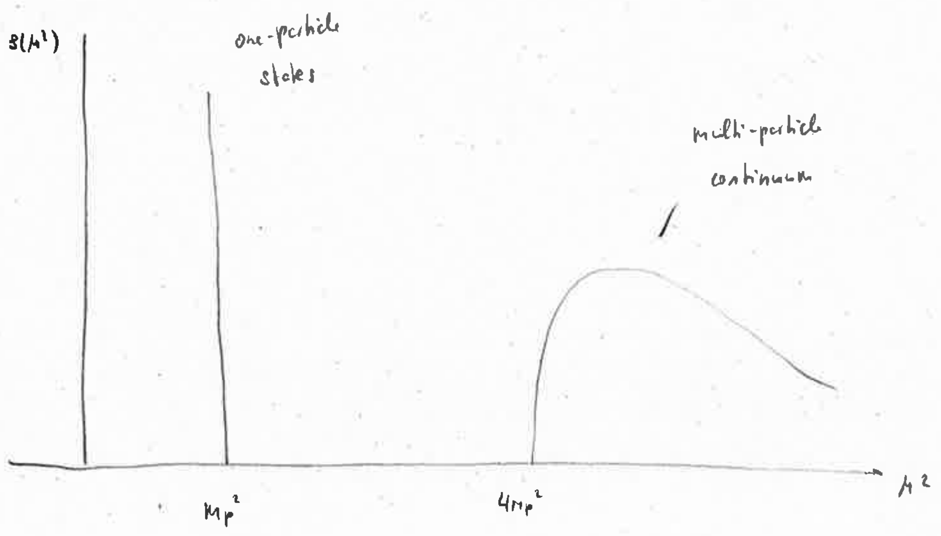
$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \Delta_F(x-y, m^2)$$

as expected since the physical (renormalised) mass m_p is of course equal to the bare mass m in a free theory.

In an interacting theory, on the other hand, all sorts of multi-particle intermediate states contribute to the two-point function. The general form of the spectral density is then given by

$$\rho(\mu^2) = \underbrace{Z\,\delta(\mu^2 - m_p^2)}_{\text{one-particle states}} + \underbrace{\theta(\mu^2 - 4m_p^2)\,\rho_{cont}(\mu^2)}_{\text{multi-particle continuum}}$$

where now $m_p \neq m$ in general and $Z = |\langle \Omega | \phi(0) | k \rangle|^2$. Due to the normalisation and the positivity of the spectral density, one has $0 \leq Z \leq 1$.



With this form of the spectral density function, the Fourier transform of the two-point function becomes

$$\boxed{\int d^4x \, e^{ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle = \frac{i Z}{p^2 - m_p^2 + i\varepsilon} + \int_{4m_p^2}^{\infty} d\mu^2 \, \varsigma_{\text{cont}}(\mu^2) \frac{i}{p^2 - \mu^2 + i\varepsilon}}$$

which is the central result of our analysis.

The formula states that the two-point function in an interacting QFT (the "full" or exact propagator) receives contributions from one-particle and multi-particle states, which can be distinguished by the strength of their analytic singularities in the complex p^2-plane:

- The one-particle states yield an isolated pole, which is located at the physical mass m_p of the particle (which is therefore also called the <u>pole mass</u> of the particle). The residue of this pole is given by the <u>on-shell field renormalization constant</u> Z, which describes the overlap of the field operator with a one-particle state.

- The multi-particle states contribute a branch-cut singularity, which opens up at the two-particle threshold, $p^2 \geq (2m_p)^2$.

The derivation of the Källén-Lehmann representation was based on general principles in QFT, and it can be directly generalized to higher-spin fields as well. We will not go into the details here, and instead only quote the result for a Dirac field

$$\int d^4x \, e^{ipx} \langle \Omega | T \psi_\alpha(x) \bar{\psi}_\beta(0) | \Omega \rangle$$

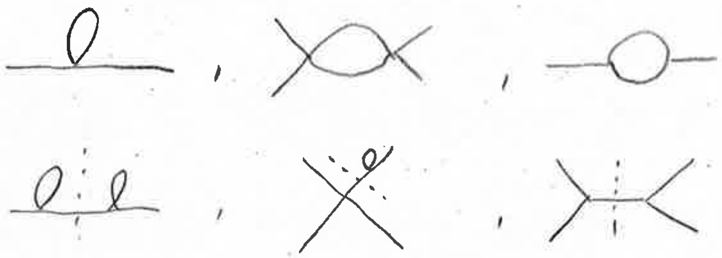
$$= \frac{iZ \, (\not{p}+m_p)_{\alpha\beta}}{p^2 - m_p^2 + i\varepsilon} + \int d\mu^2 \, S^{cont}_{\alpha\beta}(\mu^2) \, \frac{i}{p^2 - \mu^2 + i\varepsilon}$$

where the field renormalization constant is now defined as

$$\langle \Omega | \psi_\alpha(0) | k, s \rangle = \sqrt{Z} \, u_\alpha(k,s)$$

for a particle (rather than an antiparticle) state.

For a weakly-coupled theory, we may use the Källén-Lehmann representation to express the physical mass m_p and the on-shell field renormalization constant Z in terms of Feynman diagrams. To do so, we introduce the notion of <u>one-particle irreducible (1PI)</u> diagrams: a connected Feynman diagram is said to be 1PI if it cannot be disconnected by cutting a single internal line. A few examples



are 1PI

are not 1PI

We will denote the sum of all 1PI diagrams as

We now define the 1PI two-point function*

$-i\Pi(p^2) \equiv$ 

$=$  $+ O(\lambda^2)$

---

* Note that we are currently working with Feynman rules that are expressed in terms of bare parameters. There are thus no additional counterterm diagrams for the moment.

We can then write the exact propagator as

$$\int d^4x \, e^{ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle$$

$=$  (sum of all connected diagrams)

↳ we excluded the vacuum intermediate state

$=$ 

$$= \frac{i}{\underbrace{p^2 - m^2}_{\text{bare mass}} + i\varepsilon} + \frac{i}{p^2 - m^2 + i\varepsilon} \left(-i\Pi(p^2)\right) \frac{i}{p^2 - m^2 + i\varepsilon} + \dots$$

geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$= \frac{i}{p^2 - m^2 + i\varepsilon} \; \frac{1}{1 - \frac{\Pi(p^2)}{p^2 - m^2 + i\varepsilon}}$$

$$= \frac{i}{p^2 - m^2 - \Pi(p^2) + i\varepsilon}$$

The requirement that the full propagator has a pole at the physical mass m_p implies

$$m_p^2 - m^2 - \Pi(m_p^2) = 0$$

$$\Rightarrow \boxed{m_p^2 = m^2 + \underbrace{\Pi(m_p^2)}_{O(\lambda)}}$$

In order to extract the residue at the pole $p^2 = M_p^2$, we expand

$$\frac{i}{p^2 - m^2 - \pi(p^2)} = \frac{i}{\underbrace{M_p^2 - m^2 - \pi(M_p^2)}_{=0} + \left(1 - \frac{\partial \pi(p^2)}{\partial p^2}\Big|_{p^2=M_p^2}\right)(p^2 - M_p^2) + \dots}$$

$$= \frac{1}{1 - \frac{\partial\pi(p^2)}{\partial p^2}\Big|_{p^2=M_p^2}} \quad \frac{i}{p^2 - M_p^2} + \dots$$

and we identify the field renormalisation constant as

$$\boxed{Z = \frac{1}{1 - \frac{\partial \pi(p^2)}{\partial p^2}\Big|_{p^2 = M_p^2}}}$$

Our explicit result in dimensional regularisation (page 93-95)

$-i\,\pi(p^2) =$  $+ O(\lambda^2)$

$$= \frac{i\lambda}{32\pi^2}\, m^2 \left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m^2} + 1\right] + O(\lambda^2)$$

thus translates into

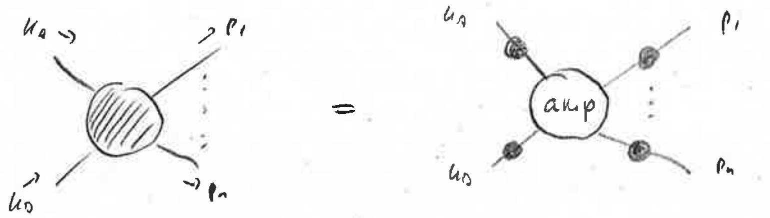
$$M_p^2 = \left(1 - \frac{\lambda}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m^2} + 1\right) + O(\lambda^2)\right) m^2$$

$$Z = 1 + O(\lambda^2)$$

since the tadpole diagram is independent of the external momentum.

The relation between the (observable) renormalised mass m_P and the (unobservable) bare mass m is thus again divergent, similar to what we have seen for the coupling constant (cf. page 76). We stress again that these results correspond to a particular renormalisation scheme (the on-shell scheme). Other mass definitions may be less intuitive (they in particular do not correspond to the eigenvalue of the Hamiltonian in the particle rest frame), but there is a priori no preferred renormalisation scheme. For the electron in QED, for instance, the physical or pole mass that we defined in this section seems to be a particular convenient choice, but for the quarks in QCD, which do not exist as free particles in nature, the pole mass is not necessarily a useful definition.

The results of this section can be generalised to higher correlation functions. We will not go into the details here (see e.g. Peskin/Schroeder, chapter 7.2), but the idea is to consider the Fourier transform of a $(n+2)$-point function



which defines the amputated piece of the Green's function as the one that does not contain any external leg corrections. The two-point function that we discussed above is thus related to the external leg corrections of higher n-point functions. As each of the two-point functions has a one-particle pole in the corresponding momentum variable, one finds

$$\prod_{i=1}^{2} \int d^4x_i \, e^{-ik_i x_i} \int \prod_{j=1}^{n} d^4 y_j \, e^{ip_j y_j} \langle \Omega | T \phi(y_1) \dots \phi(y_n) \phi(x_1) \phi(x_2) | \Omega \rangle$$

$$= \prod_{i=1}^{2} \frac{i\sqrt{Z}}{k_i^2 - m_p^2 + i\varepsilon} \prod_{j=1}^{n} \frac{i\sqrt{Z}}{p_j^2 - m_p^2 + i\varepsilon} \langle p_1 \dots p_n | S | k_A k_B \rangle + \underbrace{\dots}_{\text{multi-particle contributions}}$$

i.e the residue of the multiple one-particle poles is given by a product of on-shell field renormalization constants, which is multiplied by the corresponding $2 \to n$ S-matrix element! This formula thus relates S-matrix elements to correlation functions and it is known as the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula.

Our discussion clarifies the role of external leg corrections for the computation of scattering matrix elements, and it in particular justifies our adhoc prescription from TPP1 (chapter 4.7)

$$i\mathcal{M}(k_A k_B \to p_1 \dots p_n) = \left(\sqrt{Z}\right)^{n+2} \left(\begin{array}{l} \text{sum of all connected and amputated diagrams} \\ \text{with } k_A, k_B \text{ incoming and } p_1 \dots p_n \text{ outgoing} \end{array} \right)$$

where we now understand the role and the definition of the on-shell field renormalization factor Z.

2.4. Renormalised perturbation theory

We have seen before that there are two different ways to organise the renormalisation program. We can either work with the Feynman rules in terms of the bare parameters, or we apply the Feynman rules in terms of the renormalised quantities which include additional counterterm vertices. The second procedure is called renormalised perturbation theory, which we will now develop systematically to renormalise ϕ^4-theory at the one-loop level.

Note that from now on, we will change the notation and denote the bare parameters with an index "0", whereas we drop the index "R" for the renormalised parameters!

We thus start from

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{m_0^2}{2} \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4$$

and we introduce the renormalised parameters as follows

$$\phi_0 = \sqrt{Z} \, \phi$$

$$m_0^2 = Z_m \, m^2$$

$$\lambda_0 = \tilde{\mu}^{2\varepsilon} Z_\lambda \lambda$$

where we included a factor $\tilde{\mu}^{2\varepsilon}$ to keep the renormalised coupling λ dimensionless in DR.

We now split the Lagrangian into two terms

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$$

where $\mathcal{L}_r$ has the same functional form as $\mathcal{L}$, but expressed in terms of the renormalised parameters

$$\mathcal{L}_r = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda \tilde{\mu}^{2\varepsilon}}{4!} \phi^4$$

The counterterms, on the other hand, are described by

$$\mathcal{L}_{ct} = \frac{1}{2}(Z-1)\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}(Z Z_m - 1)\, m^2\phi^2$$

$$- (Z^2 Z_\lambda - 1)\frac{\lambda\mu^{-2\varepsilon}}{4!}\phi^4$$

and they will be considered as additional vertices since $Z_i - 1 = O(\lambda)$. (despite the fact that some terms are quadratic in the fields).

The Feynman rules in renormalised perturbation theory are thus

$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$= -i\lambda\mu^{-2\varepsilon}$$

$$= i\left[(Z-1)p^2 - (Z Z_m - 1)m^2\right]$$

$$= -i(Z^2 Z_\lambda - 1)\lambda\mu^{-2\varepsilon}$$

We recall that $\partial_\mu \to +ip_\mu$ for outgoing momenta in derivative interactions.

In renormalised perturbation theory the Green functions are naturally expressed in terms of the renormalised fields, which differ from the bare Green functions by factors of $\sqrt{Z}$

$$\langle\Omega| T\phi(x_1)\ldots\phi(x_n)|\Omega\rangle = (\sqrt{Z})^{-n}\langle\Omega|T\phi_0(x_1)\ldots\phi_0(x_n)|\Omega\rangle$$

This implies, in particular, that the one-particle contribution to the renormalised two-point function is now given by

$$\langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle = \frac{1}{Z}\langle\Omega|T\phi_0(x_1)\phi_0(x_2)|\Omega\rangle$$

$$\xrightarrow{p^2\sim m^2} \frac{1}{Z}\,\frac{i\,Z^{os}}{p^2-m^2+i\varepsilon} + \ldots$$

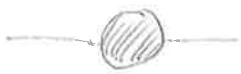
The residue $\frac{Z^{os}}{Z}$ is therefore trivial in the on-shell scheme, and finite in any other scheme.

Similarly, in the LSZ theorem we multiply the connected and amputated diagrams with a factor

$$\left(\sqrt{Z^{os}}\right)^{n+2} \longrightarrow \left(\sqrt{\frac{Z^{os}}{Z}}\right)^{n+2}$$ (for $2\to n$ scattering)

which again becomes trivial when we work with renormalised Green functions in the on-shell scheme.

We further have to specify <u>renormalisation conditions</u>, which define the renormalised parameters in the theory and thus also the renormalisation constants Z, Z_m and Z_λ. We may choose, for instance, to define Z and Z_m in the on-shell scheme. We thus require that the renormalised two-point function



$$= \frac{i}{p^2 - m_p^2 - \Pi(p^2) + i\varepsilon} = \frac{1}{Z}\underbrace{\frac{i}{p^2 - m_0^2 - \Pi_0(p^2) + i\varepsilon}}_{\text{old result from pg. 101 in terms of bare quantities}}$$

physical mass

1PI two-point function expressed in terms of ren. fields

has a pole at $p^2 = m_p^2$ with a trivial residue

$$\Rightarrow \boxed{\Pi(m_p^2) = 0}$$

$$\frac{1}{1 - \left.\frac{\partial \Pi(p^2)}{\partial p^2}\right|_{p^2 = m_p^2}} = 1 \quad \Rightarrow \quad \boxed{\left.\frac{\partial \Pi(p^2)}{\partial p^2}\right|_{p^2 = m_p^2} = 0}$$

Let us convince ourselves that this yields the same results that we obtained earlier. We now start from

$-i\Pi(p^2) =$  $+$  $+ O(\lambda^2)$

$$= \frac{i\lambda}{32\pi^2} m_p^2 \left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m_p^2} + 1 \right) + i\left[(Z-1)p^2 - (Z Z_m - 1) m_p^2 \right] + O(\lambda^2)$$

$$\Rightarrow \quad \frac{\partial \Pi(p^2)}{\partial p^2} = -(Z-1) + O(\lambda^2)$$

and hence in the on-shell scheme

$$\left.\frac{\partial \Pi(p^2)}{\partial p^2}\right|_{p^2=m_p^2} \overset{!}{=} 0 \quad \Rightarrow \quad Z = 1 + O(\lambda^2) \quad \checkmark$$

Moreover

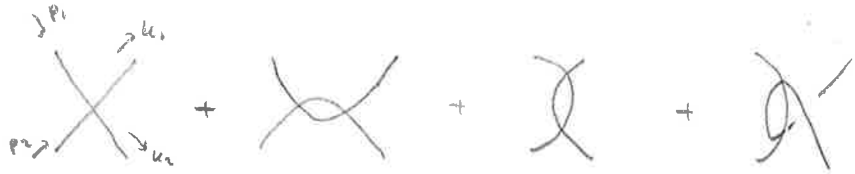
$$\Pi(m_p^2) = -\frac{\lambda}{32\pi^2} m_p^2 \left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m_p^2} + 1 \right] + (Z_m - 1) m_p^2 + O(\lambda^2)$$

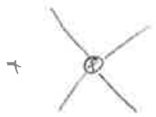
$$\overset{!}{=} 0$$

$$\Rightarrow \quad Z_m = 1 + \frac{\lambda}{32\pi^2} \left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m_p^2} + 1 \right] + O(\lambda^2)$$

which again agrees with the result on pg. 102.

We finally have to define the renormalized coupling. To this end, we consider the $\phi\phi \to \phi\phi$ scattering matrix element

$i\mathcal{M}(s,t,u) =$ 

 $+ \ O(\lambda^3)$

$$= -i\lambda\mu^{-2\varepsilon} + \frac{i\lambda^2}{32\pi^2}\mu^{-2\varepsilon}\Big(A(s) + A(t) + A(u)\Big)$$

$$-i(Z^2 Z_\lambda - 1)\lambda\mu^{-2\varepsilon} + O(\lambda^3)$$

where $s = (p_1+p_2)^2$, $t = (p_1-k_1)^2$ and $u = (p_1-k_2)^2$ are the Mandelstam variables and

$$A(s) = \frac{1}{\varepsilon} + \ln\frac{\mu^2}{s} + f(s,m_r) + 1$$

$$= \frac{1}{\varepsilon} + \ln\frac{\mu^2}{s} + \int_0^1 dx \, \ln\frac{s}{m_r^2 - x\bar{x}s - i\varepsilon} \qquad (\bar{x} = 1-x)$$

$$= \frac{1}{\varepsilon} + \ln\frac{\mu^2}{m_r^2} + \int_0^1 dx \, \ln\frac{m_r^2}{m_r^2 - x\bar{x}s - i\varepsilon}$$

We can define the renormalized coupling e.g. at threshold with $s = 4m_r^2$ and $t = u = 0$

$$i\mathcal{M}(4m_r^2, 0, 0) \equiv -i\lambda\mu^{-2\varepsilon}$$

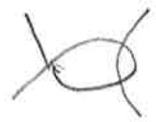
Using $Z = 1 + O(\lambda^2)$, we then obtain

$$Z_\lambda = 1 + \frac{\lambda}{32\pi^2}\left(A(4m_i^2) + 2A(0) \right) + O(\lambda^2)$$

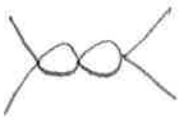
$$= 1 + \frac{\lambda}{32\pi^2}\left(\frac{3}{\varepsilon} + 3\ln\frac{\mu^2}{m_i^2} - \int_0^1 dx \, \ln(1 - 4x\bar{x} - i\varepsilon) \right) + O(\lambda^2)$$

This completes our discussion at the one-loop level. With these values of the renormalization constants, all Green functions are finite to one-loop order.

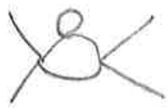
At $O(\lambda^2)$ we encounter diagrams of the form



genuine 2-loop



$(1\text{-loop})^2$



1-loop with self-energy insertion

1-loop counterterm insertions

$\sim \lambda^2 (Z_i - 1)$

the required $O(\lambda)$ is known

tree-level counterterm insertion

$\sim \lambda (Z_i - 1)$

the renormalization condition fixes the λ^2-term!

which suggests that the procedure is to be performed iteratively order by order in perturbation theory.

2.5. Renormalisation group

In renormalised perturbation theory, we introduce the renormalised parameters via

$$\phi_0 = \sqrt{Z}\ \phi$$

$$m_0^2 = Z_m\ m^2$$

$$\lambda_0 = \mu^{2\varepsilon} Z_\lambda\ \lambda \qquad \text{(in DR)}$$

and we in addition have to specify a set of renormalisation conditions that define the renormalised parameters. So far, we were working in a physically motivated renormalisation scheme, in which we imposed the conditions

- $\Pi(m_p^2) = 0$
- $\left.\dfrac{\partial\Pi(p^2)}{\partial p^2}\right|_{p^2=m_p^2} = 0$
- $i\mathcal{M}(4m_p^2,0,0) = -i\lambda\mu^{2\varepsilon}$

and the corresponding renormalised parameters were the physical mass m_p, the on-shell renormalisation constant Z and the coupling constant at threshold λ.

For the renormalization constants, we found at one-loop order

$$Z = 1 + O(\lambda^2)$$

$$Z_m = 1 + \frac{\lambda}{32\pi^2} \left[\frac{1}{\varepsilon} + \ln \frac{\mu^2}{m_p^2} + 1 \right] + O(\lambda^2)$$

$$Z_\lambda = 1 + \frac{\lambda}{32\pi^2} \left(A(4m_p^2) + 2A(0) \right) + O(\lambda^2)$$

$$= 1 + \frac{\lambda}{32\pi^2} \left(\frac{3}{\varepsilon} + 3 \ln \frac{\mu^2}{m_p^2} + \int_0^1 dx \, \ln \frac{m_p^2}{m_p^2 - x\bar{x}\, 4m_p^2 - i\varepsilon} \right) + O(\lambda^2)$$

For the $\phi\phi \to \phi\phi$ scattering amplitude, we then obtained

$$i\mathcal{M}(s,t,u) = -i\lambda \tilde{\mu}^{2\varepsilon} + \frac{i\lambda^2}{32\pi^2} \tilde{\mu}^{2\varepsilon} \Big(A(s) + A(t) + A(u) - A(4m_p^2) - 2A(0) \Big) + O(\lambda^3)$$

$$\overset{\varepsilon \to 0}{=} -i\lambda + \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left(\ln \frac{m_p^2}{m_p^2 - x\bar{x} s - i\varepsilon} + (s \to t) + (s \to u) - \ln \frac{m_p^2}{m_p^2 - x\bar{x}\, 4m_p^2 - i\varepsilon} \right) + O(\lambda^3)$$

which is indeed finite and fulfills $i\mathcal{M}(4m_p^2, 0, 0) = -i\lambda$.

We are, however, free to choose different renormalization conditions, which may not even be motivated on physical grounds. A particular convenient choice is the minimal subtraction scheme ($\overline{MS}$).

In the $\overline{MS}$ scheme we only subtract the pole terms in DR.

We thus simply have

$$\bar{Z} = 1 + O(\bar{\lambda}^2)$$

$$\bar{Z}_m = 1 + \frac{\bar{\lambda}}{32\pi^2}\,\frac{1}{\varepsilon} + O(\bar{\lambda}^2)$$

$$\bar{Z}_\lambda = 1 + \frac{\bar{\lambda}}{32\pi^2}\,\frac{3}{\varepsilon} + O(\bar{\lambda}^2)$$

and the associated renormalised parameters are called the $\overline{MS}$ mass $\bar{m}$, the $\overline{MS}$-renormalisation constant $\bar{Z}$ and the $\overline{MS}$ coupling $\bar{\lambda}$. In the $\overline{MS}$-scheme, the two-point function becomes

refers to the 2-point function in the $\overline{MS}$ scheme.

$$-i\,\overline{\Pi}(p^2 = m_p^2) = \frac{i\bar{\lambda}}{32\pi^2}\,\bar{m}^2\left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{\bar{m}^2} + 1\right] - i\bar{m}^2\,\frac{\bar{\lambda}}{32\pi^2}\,\frac{1}{\varepsilon} + O(\bar{\lambda}^2)$$

we need to evaluate the 2-point function at the physical mass

$$= \frac{i\bar{\lambda}}{32\pi^2}\,\bar{m}^2\left[\ln\frac{\mu^2}{\bar{m}^2} + 1\right] + O(\bar{\lambda}^2)$$

$$\neq 0$$

and therefore the $\overline{MS}$ mass differs from the physical (pole) mass.

The two renormalized masses can easily be connected using

$$m_0^2 = Z_m\, m_p^2 = \bar{Z}_m\, \bar{m}^2$$

$$\Rightarrow \quad \bar{m}^2 = \frac{Z_m}{\bar{Z}_m}\, m_p^2$$

$$= \frac{1 + \frac{\lambda}{32\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m_p^2} + 1\right) + O(\lambda^2)}{1 + \frac{\bar{\lambda}}{32\pi^2}\left(\frac{1}{\varepsilon}\right) + O(\bar{\lambda}^2)}\, m^2 \qquad \text{use } \bar{\lambda} = \lambda + O(\lambda^2)$$

$$= \left(1 + \frac{\lambda}{32\pi^2}\left(\ln\frac{\mu^2}{m_p^2} + 1\right) + O(\lambda^2)\right) m^2$$

The two masses thus differ by a finite renormalization, and we observe that the $\overline{MS}$ mass is explicitly renormalization scale dependent $\rightarrow \bar{m}(\mu)$!

We can proceed similarly for the coupling constant, writing

$$\lambda_0 = \mu^{2\varepsilon} Z_\lambda\, \lambda = \bar{\mu}^{2\varepsilon} \bar{Z}_\lambda\, \bar{\lambda}$$

$$\Rightarrow \quad \bar{\lambda} = \frac{Z_\lambda}{\bar{Z}_\lambda}\, \lambda$$

$$= \left(1 + \frac{\lambda}{32\pi^2}\left(3\ln\frac{\mu^2}{m_p^2} + \int_0^1 dx\, \ln\frac{m_p^2}{m_p^2 - x\bar{x}\, 4m_p^2 - i\varepsilon}\right) + O(\lambda^2)\right)\lambda$$

which is again scale-dependent $\rightarrow \bar{\lambda}(\mu)$!

In the $\overline{MS}$ scheme, the $\phi\phi \to \phi\phi$ scattering amplitude becomes

$$i\mathcal{M}(s,t,u) = -i\bar{\lambda}\,\tilde{\mu}^{-2\varepsilon} + \frac{i\bar{\lambda}^2}{32\pi^2}\,\tilde{\mu}^{2\varepsilon}\left(A(s) + A(t) + A(u) - \frac{3}{\varepsilon}\right) + O(\bar{\lambda}^3)$$

$$\overset{\varepsilon\to 0}{=} -i\bar{\lambda} + \frac{i\bar{\lambda}^2}{32\pi^2}\left(3\ln\frac{\mu^2}{m_p^2} + \int_0^1 dx\left[\ln\frac{m_p^2}{m_p^2 - x\bar{x}s - i\varepsilon} + (s\to t) + (s\to u)\right]\right) + O(\bar{\lambda}^3)$$

and we clearly have $i\mathcal{M}(4m_p^2,0,0) \neq -i\bar{\lambda}$. Once we convert this relation, however, to the old scheme, we obtain

$$i\mathcal{M}(s,t,u) = -i\lambda + \frac{i\lambda^2}{32\pi^2}\left(\cancel{3\ln\frac{\mu^2}{m_p^2}} + \int_0^1 dx\left[\ln\frac{m_p^2}{m_p^2 - x\bar{x}s - i\varepsilon} + (s\to t) + (s\to u)\right] - \cancel{3\ln\frac{\mu^2}{m_p^2}} - \int_0^1 dx\,\ln\frac{m_p^2}{m_p^2 - x\bar{x}4m_p^2 - i\varepsilon}\right) + O(\lambda^3)$$

which indeed agrees with our old result. We thus learn that by switching between different renormalization schemes, we only reshuffle the finite terms that are absorbed by the renormalization constants.

There is a priori no preferred renormalisation scheme. The on-shell scheme is often convenient since S-matrix elements are extracted from the poles of Green functions that are located at the physical masses (and the LSZ theorem involves trivial factors $\sqrt{\frac{Z^{os}}{Z}}$ in this case). In the $\overline{MS}$ scheme, on the other hand, the calculations often simplify considerably, and it is therefore a particular convenient choice for higher-order calculations.

There is, however, another important difference between <u>mass-dependent</u> renormalisation schemes (like the on-shell scheme) and <u>mass-independent</u> renormalisation schemes (like the $\overline{MS}$ scheme), which becomes apparent when we consider the high-energy behaviour of scattering processes.

In the first (mass-dependent) renormalization scheme, we obtain in the high-energy limit $s \sim -t \sim -u \gg m_r^2$

$$i\mathcal{M}(s,t,u) \simeq -i\lambda \left[1 + \frac{\lambda}{32\pi^2} \left(\ln \frac{-s-i\varepsilon}{m_r^2} + \ln \frac{-t-i\varepsilon}{m_r^2} + \ln \frac{-u-i\varepsilon}{m_r^2} + \dots \right) \right]$$

The logarithmic corrections can become very large, and they will eventually spoil the perturbative expansion when $\lambda \ln \frac{-s}{m_r^2} = O(1)$, even if $\lambda \ll 1$. The result may be surprising, since we would have expected that the mass is completely irrelevant in the high-energy limit. The problem is, however, that we have chosen renormalization conditions, for which the limit $m_r \to 0$ does not exist.

In the (mass-independent) $\overline{MS}$ scheme, on the other hand, we get

$$i\mathcal{M}(s,t,u) \simeq -i\bar{\lambda}(\mu) \left[1 + \frac{\bar{\lambda}(\mu)}{32\pi^2} \left(\ln \frac{-s-i\varepsilon}{\mu^2} + \ln \frac{-t-i\varepsilon}{\mu^2} + \ln \frac{-u-i\varepsilon}{\mu^2} + \dots \right) \right]$$

In the $\overline{MS}$ scheme, we can thus control the size of the logarithmic corrections, as long as we choose the arbitrary renormalisation scale $\mu \sim s \sim -t \sim -u$. The perturbative expansion is then valid as long as the running coupling $\bar{\lambda}(\mu) \ll 1$, and the limit $m_p \to 0$ exists.

But what do we know about the scale dependence of the running coupling $\bar{\lambda}(\mu)$?

First of all, we can easily relate the coupling at two different scales at fixed order

$$
\begin{aligned}
i\mathcal{M}(s,t,u) &= -i\,\bar{\lambda}(\mu_1) + \frac{i\,\bar{\lambda}(\mu_1)^2}{32\pi^2}\left(3\ln\frac{\mu_1^2}{m_p^2} + \dots\right) + O(\bar{\lambda}(\mu_1)^3) \\
&= -i\,\bar{\lambda}(\mu_2) + \frac{i\,\bar{\lambda}(\mu_2)^2}{32\pi^2}\left(3\ln\frac{\mu_2^2}{m_p^2} + \dots\right) + O(\bar{\lambda}(\mu_2)^3)
\end{aligned}
$$

where the dots refer to μ-independent terms. We thus find

$$
\bar{\lambda}(\mu_2) = \bar{\lambda}(\mu_1)\left[1 + \frac{3}{32\pi^2}\,\bar{\lambda}(\mu_1)\ln\frac{\mu_2^2}{\mu_1^2} + O(\bar{\lambda}(\mu_1)^2)\right]
$$

But we can learn even more by using the fact that the bare coupling constant is independent of the renormalisation scale.

$$\frac{d}{d\ln\mu}\lambda_0 = \mu\frac{d}{d\mu}\lambda_0 = 0$$

$\tilde{\mu} = \sqrt{\frac{e^{\gamma}}{4\pi}}\,\mu$

$$\mu\frac{d}{d\mu}\left(\tilde{\mu}^{2\varepsilon}\bar{Z}_\lambda\bar{\lambda}\right)$$

$$= 2\varepsilon\,\tilde{\mu}^{2\varepsilon}\bar{Z}_\lambda\bar{\lambda} + \tilde{\mu}^{2\varepsilon}\mu\frac{d\bar{Z}_\lambda}{d\mu}\bar{\lambda} + \tilde{\mu}^{2\varepsilon}\bar{Z}_\lambda\,\mu\frac{d\bar{\lambda}}{d\mu}$$

$$\Rightarrow \quad \mu\frac{d\bar{\lambda}}{d\mu} = -2\varepsilon\bar{\lambda} - \frac{1}{\bar{Z}_\lambda}\mu\frac{d\bar{Z}_\lambda}{d\mu}\bar{\lambda}$$

or equivalently

$$\boxed{\frac{d\bar{\lambda}}{d\ln\mu} = -2\varepsilon\bar{\lambda} + \beta(\bar{\lambda})}$$

which is called a renormalisation group (RG) equation. It specifies the scale-dependence of the running coupling $\bar{\lambda}(\mu)$, and it simply reflects the fact that physical observables are invariant under the way we organise our calculation (whether we express it in terms of $\bar{\lambda}(\mu_1)$ or $\bar{\lambda}(\mu_2)$ ...). The β-function

$$\beta(\bar{\lambda}) = -\frac{1}{\bar{Z}_\lambda}\frac{d\bar{Z}_\lambda}{d\ln\mu}\bar{\lambda}$$

$$\equiv 2\bar{\lambda}\left[\beta_0\frac{\bar{\lambda}}{4\pi} + \beta_1\left(\frac{\bar{\lambda}}{4\pi}\right)^2 + O(\bar{\lambda}^3)\right]$$

controls the scale dependence in $d=4$ dimensions.

At one-loop order, we found

$$\bar{Z}_\lambda = 1 + \frac{\bar{\lambda}}{32\pi^2}\frac{3}{\varepsilon} + O(\bar{\lambda}^2)$$

$$\rightarrow \frac{d\bar{Z}_\lambda}{d\ln_r} = \underbrace{\frac{d\bar{\lambda}}{d\ln_r}}_{-2\varepsilon\bar{\lambda}+O(\bar{\lambda}^2)}\frac{3}{32\pi^2}\frac{1}{\varepsilon} + O(\bar{\lambda}^2)$$

$$= -\frac{3}{16\pi^2}\bar{\lambda} + O(\bar{\lambda}^2)$$

and therefore

$$\beta(\bar{\lambda}) = -\underbrace{\frac{1}{\bar{Z}_\lambda}}_{1+O(\bar{\lambda})}\frac{d\bar{Z}_\lambda}{d\ln_r}\bar{\lambda}$$

$$= \frac{3}{16\pi^2}\bar{\lambda}^2 + O(\bar{\lambda}^3)$$

which implies $\beta_0 = \frac{3}{8\pi} > 0$.

The solution of the RG equation in $d=4$ dimensions becomes at one-loop order

$$\frac{d\bar{\lambda}}{d\ln_r} = 2\frac{\beta_0}{4\pi}\bar{\lambda}^2$$

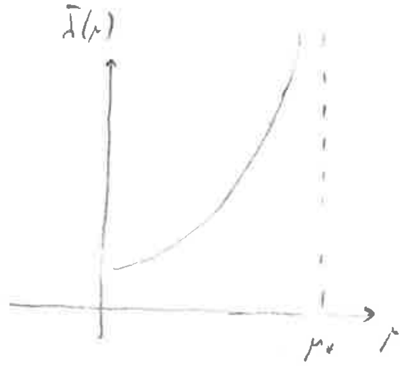
$$\rightarrow \int_{\bar{\lambda}(\mu_1)}^{\bar{\lambda}(\mu_2)}\frac{d\bar{\lambda}}{\bar{\lambda}^2} = 2\frac{\beta_0}{4\pi}\int_{\mu_1}^{\mu_2} d\ln_r$$

$$\rightarrow \frac{1}{\bar{\lambda}(\mu_1)} - \frac{1}{\bar{\lambda}(\mu_2)} = \frac{\beta_0}{4\pi}\ln\frac{\mu_2^2}{\mu_1^2}$$

$$\rightarrow \boxed{\bar{\lambda}(\mu_2) = \frac{\bar{\lambda}(\mu_1)}{1-\frac{\beta_0}{4\pi}\bar{\lambda}(\mu_1)\ln\frac{\mu_2^2}{\mu_1^2}}}$$

Given an initial condition for $\bar{\alpha}(\mu_1)$, we may thus calculate the running coupling at any scale μ_2. Notice that since $\beta_0 > 0$, the coupling $\bar{\alpha}(\mu)$ increases with increasing μ.

It actually formally diverges for



$$1 - \frac{\beta_0}{4\pi} \bar{\alpha}(\mu_1) \ln \frac{\mu_*^2}{\mu_1^2} = 0$$

$$\Rightarrow \quad \mu_*^2 = \exp\left(\frac{4\pi}{\beta_0 \bar{\alpha}(\mu_1)}\right) \mu_1^2$$

which is called the Landau pole. We should, however, keep in mind that our one-loop analysis (based on β_0) will break down when $\bar{\alpha}(\mu)$ becomes large. The exact behavior of $\bar{\alpha}(\mu)$ for large values of the renormalization scale μ is therefore not accessible to perturbation theory.

We may expand the above result in terms of $\bar{\alpha}(\mu_1)$

$$\bar{\alpha}(\mu_2) = \bar{\alpha}(\mu_1) \left[1 + \frac{\beta_0}{4\pi} \bar{\alpha}(\mu_1) \ln \frac{\mu_2^2}{\mu_1^2} + \left(\frac{\beta_0}{4\pi}\right)^2 \bar{\alpha}(\mu_1)^2 \ln^2 \frac{\mu_2^2}{\mu_1^2} + \dots \right]$$

which indeed reproduces our fixed-order result from pg. 119 for $\beta_0 = \frac{3}{8\pi}$. By solving the RG equation, we have however in addition resummed an infinite tower of higher-order corrections!

2.6. Renormalisability

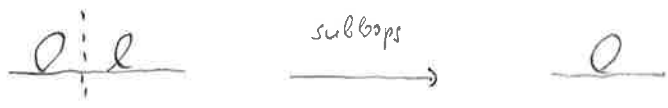
We have seen that the UV divergences in ϕ^4-theory can be absorbed into the renormalised parameters at the one-loop level. But does this procedure work to all orders in perturbation theory and make all Green functions finite? If so the theory is said to be renormalisable, but how do we know if a theory is renormalisable? And what happens if a theory turns out to be non-renormalisable?

To answer these questions, it is instructive to study a general scalar field theory with a ϕ^n-interaction term

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{n!}\phi^n$$

in d spacetime dimensions. For the analysis of UV divergences, it is furthermore sufficient to consider 1PI Feynman diagrams, since they contain the full information on the loop structure of the theory.

e.g.



is not 1PI   is 1PI

We now consider a general 1PI diagram in ϕ^n-theory with

- L loops
- E external lines
- P propagators
- V vertices

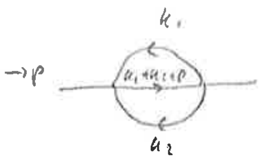
Based on our power counting arguments, we can then easily read off the scaling of the diagram in the limit in which all loop momenta $k_i \to \infty$ simultaneously

$$D = d\,L - 2\,P$$

loop measures $d^d k_i$ propagators $\frac{1}{k_i^2}$

D is called the <u>superficial degree of divergence</u>. A diagram with $D \geq 0$ is UV divergent.

Example:



$$\overset{k_i \to \infty}{\sim} \int d^d k_1\, d^d k_2 \, \frac{1}{k_1^2\, k_2^2\, (k_1+k_2)^2}$$

$$D = 2d - 6$$

- $d = 4 \to D = 2$, quadratic divergence
- $d = 3 \to D = 0$, logarithmic divergence
- $d = 2 \to D = -2$, UV finite

A connected Feynman diagram fulfils the topological relation

$$L = P - V + 1$$

which can easily be understood in momentum space, in which we can assign P arbitrary momenta to the propagators. Each vertex then gives a constraint which reflects momentum conservation of which one is the overall momentum conservation that does not involve any loop momenta.

(in the example above, we have $P = 3, V = 2 \rightarrow L = 2$ ✓)

We further know that

$$2P + E = nV$$

since propagators have two ends and external lines one end that connects to the vertices. Each vertex joins n ends in ϕ^n - theory.

$$\Rightarrow \quad D = d\,(P - V + 1) - 2P$$

$$= \left(\tfrac{d}{2} - 1\right)(nV - E) - dV + d$$

$$= d - \left(\tfrac{d}{2} - 1\right)E - \left(d - n\left(\tfrac{d}{2} - 1\right)\right)V$$

For $d > 2$, we thus see that diagrams with an increasing number of external legs are less (superficially) divergent.

In $d = 4$ dimensions, we obtain

$$D = 4 - E - (4-n)\,V$$

Let us consider a few examples:

- <u>ϕ^3-theory</u>

  We now have $D = 4 - E - V$

  The UV behaviour therefore becomes milder at higher orders in perturbation theory, and there are only a small number of 1PI diagrams that are superficially divergent with

  $$\begin{aligned} E &= 0,\ 1,\ 2,\ 3 \\ V &\leq 4,\ 3,\ 2,\ 1 \end{aligned}$$

  If we disregard the vacuum diagrams, these are

  Their divergences can be absorbed by the counterterms

  We thus learn that we have to introduce a linear term $c \cdot \phi$ in the Lagrangian, which generates the tadpole counterterm.

  ϕ^3-theory in 4 dimensions is said to be <u>superrenormalisable</u>.

- ϕ^4- theory

We have $\quad D = 4 - E$

There are only a few 1PI Green functions that are superficially divergent with $E = 0, 1, 2, 3, 4$, but their divergences now arise in every order in perturbation theory. If we disregard the vacuum diagrams, the divergences are now contained in

and their divergences can again be removed by counterterms

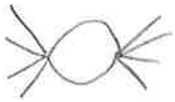
ϕ^4- theory in 4 dimensions is said to be <u>renormalisable</u>.

- ϕ^6- theory

We have $\quad D = 4 - E + 2V$

The UV behaviour now gets worse and worse at each order in perturbation theory, and therefore all 1PI Green functions will start to become divergent at some order with $V \geq \frac{E-4}{2}$ vertices.

The diagram

 $\sim \lambda^2 \ln \Lambda$

is e.g. logarithmically divergent. We therefore need to introduce a ϕ^8-term, which generates the corresponding counterterm

$\mathcal{L} \sim \frac{\lambda'}{8!} \phi^8 \longrightarrow$ 

This, however, then introduces a logarithmic divergence in

 $\sim \lambda \lambda' \ln \Lambda$

and we need to add a ϕ^{10} term to cancel this divergence, and so on. Our attempt to absorb the UV divergence into renormalised parameters now introduces an <u>infinite</u> number of parameters, and it therefore becomes impossible to determine them by experimental measurements. The theory thus loses its predictive power.

ϕ^6-theory in 4 dimensions is said to be <u>nonrenormalisable</u>.

Let us now come back to the general expression for the superficial degree of divergence

$$\mathcal{D} = d - \left(\frac{d}{2}-1\right)E - \left(d - n\left(\frac{d}{2}-1\right)\right)V$$

We have just learned that the quantity

$$\Delta = d - n\left(\frac{d}{2}-1\right)$$

tells us if a theory is renormalisable:

$\Delta > 0$ superrenormalisable

$\Delta = 0$ renormalisable

$\Delta < 0$ non renormalisable

To better understand what this implies, we will perform a simple dimensional analysis of ϕ^n-theory in d dimensions.
(cf. page 92)

$$[\mathcal{L}] = d$$

$$\mathcal{L} \sim \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \quad \rightarrow \quad [\phi] = \frac{d-2}{2}$$

$$\mathcal{L} \sim \frac{\lambda}{n!}\phi^n \quad \rightarrow \quad [\lambda] = d - n\left(\frac{d}{2}-1\right) = \Delta$$

$\Rightarrow$ We thus see that

- Couplings with positive mass dimension $\rightarrow$ superrenormalisable interactions
- dimensionless couplings $\rightarrow$ renormalisable interactions
- Couplings with negative mass dimension $\rightarrow$ non renormalisable interactions

One can actually show that this distinction also holds in theories with higher-spin fields and with derivative interactions.

We thus have learned that we can check very easily if a theory is renormalizable; we only need to read off the mass dimensions of the coupling constants.

This observation also sheds light onto the role of non renormalizable theories. To this end, let us from now on take the idea of an UV cutoff seriously, i.e. we will assume that our theory is only a low-energy approximation that loses its validity when the relevant scattering energies are of the order of the UV cutoff.

As an example, consider e.g. the Fermi theory of weak interactions with

$$\mathcal{L}_{Fermi} = \frac{G_F}{\sqrt{2}} \, \bar{\nu}_\mu \gamma^\alpha (1-\gamma_5) \mu \; \bar{e} \gamma_\alpha (1-\gamma_5) \nu_e$$

which successfully describes the muon decay $\mu \to e^- \bar{\nu}_e \nu_\mu$ at low energies $E \sim m_\mu \ll M_W$. At high energies, On the other hand, one starts to resolve the W-boson that mediates the weak interaction

$$\frac{1}{q^2-M_W^2} \approx -\frac{1}{M_W^2}$$

and the Fermi theory breaks down. The theory therefore has an intrinsic <u>physical</u> cutoff $\Lambda \sim M_W$.

We can easily verify that the Fermi theory is nonrenormalisable

$$\mathcal{L} \sim \bar{\psi} i \not{\partial} \psi \quad \longrightarrow \quad [\psi] = \frac{4-1}{2}$$

$$\mathcal{L} \sim G_F \bar{\psi}\psi\bar{\psi}\psi \quad \longrightarrow \quad [G_F] = 4 - 4\,\frac{3}{2} = -2$$

But does this imply that the Fermi theory is useless? Of course not, since it is a very successful description of the weak interactions at low energies. As we know what the underlying "UV completion" is, we further learn that $G_F \sim \frac{g^2}{M_W^2}$.

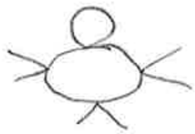
This suggests that in a non-renormalisable theory, the coupling constant with negative mass dimension Δ is of the form

$$g \sim \frac{c}{\Lambda^{|\Delta|}}$$

where c is a dimensionless number of order 1 and Λ is the UV cutoff. Whenever we consider scattering processes at low energies $E << \Lambda$, the contributions of the non-renormalisable interactions are therefore suppressed by powers of $\left(\frac{E}{\Lambda}\right)^{|\Delta|}$.

For a given precision, it is therefore sufficient to consider a finite number of higher-dimensional operators. Their couplings can then indeed be determined by experimental measurements, and the non-renormalisable theory turns out to be predictive as long as it is viewed as a low-energy approximation. This is the concept of effective field theories.

Our analysis thus far has a serious flaw. We assumed that 1PI diagrams with $D < 0$ are UV finite, but this is not always the case. Consider e.g. the diagram



in ϕ^4-theory in 4 dimensions. The diagram has $D = -2$, but the superficial degree of divergence only accounts for the scaling in which all loop momenta $k_i \to \infty$ simultaneously. The diagram has, however, a divergent subdiagram with $D = 2$

i.e the diagram diverges quadratically in the limit $k_1 \to \infty$, when k_2 is kept fixed. The divergent subdiagram is, however, just the familiar two-point function and there exists a counterterm that absorbs this divergence. In the sum of the diagrams

the UV divergence therefore drops out.

The proof that ϕ^4-theory is renormalisable in 4 dimensions is complicated precisely because of these nested (and overlapping) divergences. This leads to the BPHZ theorem (Bogoliubov, Parasiuk, Hepp, Zimmermann), which states that in a renormalisable theory all divergences can be absorbed by counterterms that correspond to superficially divergent 1PI diagrams.