4. Non-abelian gauge theories

We have seen that the concept of gauge symmetry leads to a consistent quantum field theory for massless spin-1 particles. So far, we have only considered the simplest example of a gauge theory, which is based on the abelian gauge group $U(1)$. The question arises which Lie groups lead to a sensible gauge theory.

Without going into details here (cf. e.g. Weinberg II), it turns out that the most general group is a direct product of compact and simple Lie groups and $U(1)$ factors. Here "compact" refers to a compact group manifold and "simple" implies that the group has no invariant subgroup.

Interestingly, there are only few Lie groups which fall into this class:

- $SU(N)$: special unitary group
  unitary $N \times N$ matrices with determinant 1
  $\dim G = N^2 - 1$
- $SO(N)$: special orthogonal group
  orthogonal $N \times N$ matrices with determinant 1
  $\dim G = \frac{N(N-1)}{2}$
- $Sp(N)$: symplectic group
  unitary $N \times N$ matrices that preserve the antisymmetric inner product
  $\dim G = \frac{N(N+1)}{2}$
- Exceptional groups G_2, F_4, E_6, E_7, E_8

In the following we will restrict our attention to $SU(N)$ gauge theories, which happen to play an important role in particle physics.

4.1 Non-abelian gauge invariance (chapter 5.2 of TPP 1)

In the non-abelian case, the infinitesimal transformation law of the fermion field becomes

$$\psi'_a(x) = \psi_a(x) + i\,\varepsilon^A(x)\, T^A_{ab}\, \psi_b(x)$$

real-valued functions

where $A = 1, \dots, \dim G = N^2-1$. The generators T^A are hermitian and traceless, and they depend on the representation R of the symmetry group with $a,b = 1,\dots,\dim R$.

The generators fulfil the Lie algebra

$$\boxed{[T^A, T^B] = i f^{ABC}\, T^C}$$

The structure constants f^{ABC} are real and totally antisymmetric, and they are independent of the representation R. They fulfil the Jacobi identity

$$f^{ADE} f^{BCD} + f^{BDE} f^{CAD} + f^{CDE} f^{ABD} = 0$$

Finite transformations can be obtained as usual by exponentiating

$$\psi'_a(x) = U_{ab}(x)\, \psi_b(x)$$

where $U(x) = e^{i\varepsilon^A(x) T^A}$ is a unitary operator.

The generators are normalised according to

$$\mathrm{Tr}\,[T^A_R\, T^B_R] = T_R\, \delta^{AB}$$

and T_R is called the Dynkin index of the representation R.

As $T^A T^A$ commutes with all generators, its matrix representation is proportional to the unit matrix

$$T^A_R\, T^A_R = C_R\, \mathbb{1}_R$$

and C_R is called the quadratic Casimir operator.

Writing

$$\mathrm{Tr}\,[T^A_R T^B_R]\,\delta^{AB} = T_R\,\delta^{AB}\delta^{AB} = T_R \dim G$$

$$= \mathrm{Tr}\,[T^A_R T^A_R] = C_R\,\mathrm{Tr}\,[\mathbb{1}_R] = C_R \dim R$$

we obtain

$$C_R = \frac{\dim G}{\dim R}\,T_R.$$

The simplest representation is the <u>fundamental representation</u>, in which the group element is represented by itself, i.e. $\dim F = N$. For $N > 2$ this is a complex representation and there exists an inequivalent conjugate representation $\bar{F}$. By convention we fix $T_F = \frac{1}{2}$, which implies

$$C_F = \frac{N^2-1}{2N}$$

The simplest examples are

$$SU(2): \qquad T^A_F = \frac{\sigma^A}{2} \quad \text{— Pauli matrices}$$

$$SU(3) \qquad T^A_F = \frac{\lambda^A}{2} \quad \text{— Gell-Mann matrices}$$

For a transformation $\phi' = \phi + i\epsilon^A T^A_R \phi$

the complex conjugate is $\phi'^* = \phi^* - i\epsilon^A T^{A*}_R \phi^*$

↳ the conjugate representation is $T^A_{\bar{R}} = -T^{A*}_R$

Another important representation is the <u>adjoint representation</u> with

$$(T_{ad}^B)_{AC} = i f^{ABC}$$

The commutation relation then gives the Jacobi identity.

The adjoint representation is a real representation (since $i\varepsilon^A T^A_{ad}$ is real) and $\dim \mathrm{ad} = \dim G = N^2 - 1$.

We can further decompose

$$\underbrace{N}_{F} \otimes \underbrace{\bar{N}}_{\bar{F}} = 1 \oplus \underbrace{(N^2-1)}_{\text{adjoint}}$$

with

$$T^A_{N\times\bar{N}} = T^A_N \otimes 1\!\!1_{\bar{N}} + 1\!\!1_N \otimes T^A_{\bar{N}}$$

no Casimir operator in singlet representation

$$\rightarrow \mathrm{Tr}\,[T^A_{N\times\bar{N}}\, T^A_{N\times\bar{N}}] = 0 + C_{ad}\, \mathrm{Tr}\,[1\!\!1_{ad}] = C_{ad}\,(N^2-1)$$

$$= \mathrm{Tr}\,[\,T^A_N T^A_N \otimes 1\!\!1_{\bar{N}} + 2\, \underbrace{T^A_N}_{\to 0} \otimes \underbrace{T^A_{\bar{N}}}_{\to 0} + 1\!\!1_N \otimes T^A_{\bar{N}} T^A_{\bar{N}}\,]$$

$$= (C_F + C_{\bar{F}})\, \mathrm{Tr}\,(1\!\!1_N \otimes 1\!\!1_{\bar{N}}) = 2 C_F N^2$$

$$\rightarrow \quad C_{ad} = \frac{2 C_F N^2}{N^2-1} = N \qquad \text{(typically we write } C_{ad} = C_A\text{)}$$

$$T_{ad} = \frac{\dim \mathrm{ad}}{\dim G}\, C_{ad} = N$$

We aim at constructing a Lagrangian that is invariant under <u>local</u> SU(N) transformations. As in the abelian case, we introduce the covariant derivative

$$D_{\mu,ab} = \partial_\mu \delta_{ab} - ig\, A_\mu^A(x)\, T^A_{ab}$$

which introduces N^2-1 massless vector fields. We impose the transformation law $D_\mu'\psi' = \mathcal{U}\, D_\mu \psi$, i.e. $D_\mu' = \mathcal{U}\, D_\mu\, \mathcal{U}^+$. Writing $A_\mu(x) \equiv A_\mu^A(x)\, T^A$, this implies

$$\begin{aligned} D_\mu'\psi' &= (\partial_\mu - ig A_\mu')\, \mathcal{U}\psi \\ &= \mathcal{U}\partial_\mu\psi + (\partial_\mu \mathcal{U})\psi - ig A_\mu' \mathcal{U}\psi \\ &= \mathcal{U} D_\mu \psi + \underbrace{ig\, \mathcal{U} A_\mu \psi - ig A_\mu' \mathcal{U}\psi + (\partial_\mu \mathcal{U})\psi}_{=0} \end{aligned}$$

$$\Rightarrow \quad A_\mu' = \mathcal{U} A_\mu \mathcal{U}^+ - \frac{i}{g} (\partial_\mu \mathcal{U})\, \mathcal{U}^+$$

For infinitesimal transformations this implies

$$
\begin{aligned}
A_\mu' &= A_\mu'^{A}\, T^A \\
&= A_\mu^A T^A + i\varepsilon^B T^B A_\mu^A T^A - i A_\mu^A T^A \varepsilon^B T^B - \frac{i}{g}\, i\,\partial_\mu \varepsilon^A T^A \\
&= A_\mu^A T^A + i\,\varepsilon^{\overset{C}{B}} A_\mu^{\overset{B}{A}}\; i f^{\overset{CBA}{BAC}}\, T^{\overset{A}{C}} + \frac{1}{g}\,\partial_\mu \varepsilon^A T^A \\
&= \left(A_\mu^A + \frac{1}{g}\,\partial_\mu \varepsilon^A + f^{ABC} A_\mu^B \varepsilon^C\right) T^A
\end{aligned}
$$

and therefore

$$
A_\mu'^{A} = A_\mu^A + \frac{1}{g}\,\partial_\mu \varepsilon^A + f^{ABC} A_\mu^B \varepsilon^C
$$

Notice that the last term even contributes under <u>global</u> SU(N) transformations, and it therefore gives an additional contribution to the Noether current.

The above relations reduce to the abelian transformation law, $A_\mu' = A_\mu + \partial_\mu \omega$, for $U = e^{ie\omega}$, $g = e$, $f^{ABC} = 0$ and $\varepsilon^A = e\omega$.

We next construct the non-abelian field-strength tensor.

We first note that in QED

$$\frac{i}{e}[D_\mu, D_\nu]\psi$$

$$= \frac{i}{e}\Big(\underbrace{[\partial_\mu,\partial_\nu]}_{=0} - ie[A_\mu,\partial_\nu] - ie[\partial_\mu, A_\nu] - e^2\underbrace{[A_\mu, A_\nu]}_{=0}\Big)\psi$$

$$= A_\mu \partial_\nu \psi - \partial_\nu (A_\mu \psi) + \partial_\mu (A_\nu \psi) - A_\nu \partial_\mu \psi$$

$$= (-\partial_\nu A_\mu + \partial_\mu A_\nu)\psi = F_{\mu\nu}\psi$$

$$\Rightarrow \quad F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu]$$

In the non-abelian case, we define in analogy

$$G_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

The non-abelian field-strength tensor is <u>not</u> gauge invariant but transforms as

$$G'_{\mu\nu} = \frac{i}{g}[D'_\mu, D'_\nu]$$

$$= \frac{i}{g}[U D_\mu \underbrace{U^\dagger U}_{1} D_\nu U^\dagger - U D_\nu \underbrace{U^\dagger U}_{1} D_\mu U^\dagger]$$

$$= U G_{\mu\nu} U^\dagger$$

The combination $Tr[G_{\mu\nu} G^{\mu\nu}]$ is, however, gauge invariant.

Writing $G_{\mu\nu}(x) = G_{\mu\nu}^A(x)\, T^A$ we further have

$$G_{\mu\nu} = (\partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f^{ABC} A_\mu^B A_\nu^C)\, T^A$$

$$\Rightarrow \quad G_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f^{ABC} A_\mu^B A_\nu^C$$

which transforms as

$$\begin{aligned} G'_{\mu\nu} &= U G_{\mu\nu} U^\dagger \\ &= G_{\mu\nu}^A T^A + i\varepsilon^B T^B G_{\mu\nu}^A T^A - i G_{\mu\nu}^A T^A \varepsilon^B T^B \\ &= (G_{\mu\nu}^A + f^{ABC} G_{\mu\nu}^B \varepsilon^C)\, T^A \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad G'^A_{\mu\nu} &= G_{\mu\nu}^A + f^{ABC} G_{\mu\nu}^B \varepsilon^C \\ &= G_{\mu\nu}^A + i\varepsilon^C \underbrace{(T^C_{\text{ad}})_{AB}}_{i f^{ACB}} G_{\mu\nu}^B \end{aligned}$$

i.e. the field-strength tensor transforms in the adjoint representation (which is not true for the gauge field because of the inhomogeneous term $\frac{1}{g}\partial_\mu \varepsilon^A$ in the transformation law).

Note also that

$$\mathrm{Tr}\,(G_{\mu\nu} G^{\mu\nu}) = G_{\mu\nu}^A G^{B,\mu\nu}\, \mathrm{Tr}\,(T^A T^B) = \underbrace{T_R}_{\text{representation of the fermions}} G_{\mu\nu}^A G^{A,\mu\nu}$$

We can now write down a gauge-invariant Lagrangian in full analogy to QED (*)

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^{A} G^{A,\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu^A \partial^\mu A^{A,\nu} - \partial_\mu A_\nu^A \partial^\nu A^{A,\mu}\right) + \bar{\psi}(i\not{\partial} - m)\psi$$

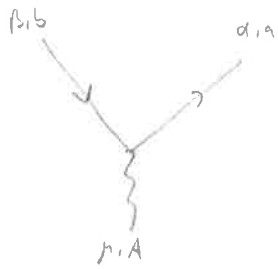
$$+ g\bar{\psi}\gamma^\mu T^A \psi A^A_\mu - g f^{ABC} (\partial_\mu A_\nu^A) A^{B,\mu} A^{C,\nu}$$

$$- \frac{g^2}{4} f^{ABC} f^{ADE} A_\mu^B A_\nu^C A^{D,\mu} A^{E,\nu}$$

In contrast to QED, we obtain cubic and quartic self-interactions of the gauge bosons, since they are themselves charged under the non-abelian gauge group. The strength of the three interaction terms is furthermore constrained by gauge invariance.

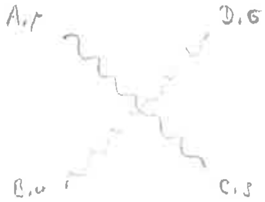
(*) There exists another term that respects the symmetries and is renormalizable, $\theta\, \epsilon_{\mu\nu\rho\sigma} G^{A,\mu\nu} G^{A,\rho\sigma}$, which can be removed if one is willing to impose parity conservation. The term is actually a total derivative and can as such be neglected in QED. This is, however, not possible in a non-abelian gauge theory due to its non-trivial topological structure of the vacuum. We disregard this term in the following.

The Lagrangian gives rise to the following Feynman rules



$$= i g \gamma^\mu_{\alpha\beta} T^A_{ab}$$

$$= g f^{ABC} \left[g^{\mu\nu} (k-p)^\rho + g^{\nu\rho} (p-q)^\mu + g^{\rho\mu} (q-k)^\nu \right]$$



$$= -i g^2 \Big[f^{ABE} f^{CDE} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$
$$+ f^{ACE} f^{BDE} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$
$$+ f^{ADE} f^{BCE} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) \Big]$$

The fermion propagator is diagonal in group space with

$$= \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon} \, \delta_{ab}$$

The gauge boson propagator is also diagonal in group space, and one is tempted to read off the Feynman rule "in analogy to QED" (in generalized Lorentz gauge)

$$= \frac{i}{q^2+i\varepsilon} \left[-g^{\mu\nu} + (1-\xi) \frac{q^\mu q^\nu}{q^2+i\varepsilon} \right] \delta^{AB}$$

We will see in the next section that this is the correct expression, but that the gauge-fixing procedure is more complicated in non-abelian gauge theories and gives rise to additional Feynman rules.

Note also that in QED one has the freedom to rescale the generator $Q \to e_f Q$ since $[Q, Q] = 0$. Each fermion may therefore have a different electromagnetic charge, e.g. $e_e = -1$, $e_u = +\frac{2}{3}$, $e_d = -\frac{1}{3}$ etc. There is no such freedom in non-abelian gauge theories since

$$[T^a, T^b] = i f^{abc} T^c.$$

We saw on page 189 that the non-abelian gauge field transforms non-trivially under global transformations and gives an additional contribution to the Noether current.

Specifically, we obtain (before gauge fixing)

$$j^{A,\mu} = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}}_{i\bar{\psi}\gamma^\mu} \underbrace{\delta^A \psi}_{iT^A\psi} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^B)}}_{-G^{B,\mu\nu}} \underbrace{\delta^A A_\nu^B}_{f^{ABC} A_\nu^C}$$

$$= - f^{ABC} G^{B,\mu\nu} A_\nu^C - \bar{\psi}\gamma^\mu T^A \psi$$

which is conserved, $\partial_\mu j^{A,\mu} = 0$. Notice that the Noether current is <u>not</u> gauge invariant, and that the matter current $j_M^{A,\mu} \equiv - \bar{\psi}\gamma^\mu T^A \psi$ is <u>not</u> conserved, due to the bosonic contribution to the Noether current.

We next compute the equations of motion for the gauge field

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^A)} \right) = - \partial_\mu G^{A,\mu\nu}$$

$$= \frac{\partial \mathcal{L}}{\partial A_\nu^A} = \frac{\partial}{\partial A_\nu^A}\left(-\frac{1}{4} G_{\mu\nu}^B G^{B,\mu\nu} + \bar{\psi}(i\not{D}-m)\psi\right)$$

$$= \frac{\partial}{\partial G_{\rho\sigma}^C}\left(-\frac{1}{4} G_{\mu\nu}^B G^{B,\mu\nu}\right)\frac{\partial G_{\rho\sigma}^C}{\partial A_\nu^A} + g\bar{\psi}\gamma^\nu T^A \psi$$

$$= -\frac{1}{2} G^{C,\rho\sigma}\, g f^{CDE}\left(\delta_\rho^\nu \delta^{DA} A_\sigma^E + A_\rho^D \delta_\sigma^\nu \delta^{EA}\right) + g\bar{\psi}\gamma^\nu T^A\psi$$

$$= g f^{ABC} G^{B,\nu\rho} A_\rho^C + g\bar{\psi}\gamma^\nu T^A\psi$$

$$= -g j^{A,\nu}$$

We thus obtain

$$\partial_\mu G^{A,\mu\nu} = g j^{A,\nu}$$

which reduces to the Maxwell equation $\partial_\mu F^{\mu\nu} = -e\bar{\psi}\gamma^\nu\psi$ in the abelian case. The objects $G^{A,\mu\nu}$ and $j^{A,\nu}$ have no simple transformation laws, but the equation of motion can be written in covariant form

$$\partial_\mu G^{A,\mu\nu} = -g f^{ABC} G^{B,\nu\rho} A_\rho^C + g j_\mu^{A,\nu}$$

$$\Rightarrow \quad D_\mu^{AB} G^{B,\mu\nu} = g j_\mu^{A,\nu}$$

where $D_\mu^{AB} = \partial_\mu \delta^{AB} - ig A_\mu^C \underbrace{(T_{ad}^C)_{AB}}_{if^{ACB}}$ is the covariant derivative in the adjoint representation.

We further have

$$D_\mu^{AB}\, j_\mu^{B,\mu} = \frac{1}{g}\, D_\mu^{AB}\, D_\nu^{BC}\, G^{C,\nu\mu}$$

$$= \frac{1}{2g}\, [D_\mu, D_\nu]^{AC}\, G^{C,\nu\mu}$$ since $G^{C,\nu\mu}$ is antisymmetric in (μ,ν)

$$= -\frac{i}{2}\, G_{\mu\nu}^{B}\, (T^{B}_{\mathrm{ad}})_{AC}\, G^{C,\nu\mu}$$ $\frac{i}{g}[D_\mu, D_\nu]^{AC} = G_{\mu\nu}^{B}(T^{B}_{\mathrm{ad}})_{AC}$

$$= \frac{1}{2}\, f^{ABC}\, G_{\mu\nu}^{B}\, G^{C,\nu\mu}$$

$$= -\frac{1}{2}\, f^{ABC}\, \underbrace{G_{\mu\nu}^{B}\, G^{C,\mu\nu}}_{\text{symm. in } B\leftrightarrow C}$$

$$= 0$$

and we say that the matter current is "covariantly conserved".

4.2 Faddeev - Popov ghosts

As in the abelian case, the functional integral over the non-abelian gauge fields

$$Z[J] = N \int \mathcal{D}A_\mu^A \, e^{i\int d^4x \left(-\frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} - J_\mu^A A^{A,\mu}\right)}$$

is ill-defined, and one has to single out the contribution from physically inequivalent configurations. In the non-abelian case, this procedure is more involved and we have to carefully reconsider the steps of the Faddeev - Popov method.

We first introduce a gauge-fixing condition $G[A]=0$ in the form

$$1 = \int \mathcal{D}\varepsilon \quad \delta(G[A^\varepsilon]) \det\left(\frac{\delta G[A^\varepsilon]}{\delta \varepsilon}\right)$$

with the gauge-transformed field

$$A_\mu^{A,\varepsilon} = A_\mu^A + \frac{1}{g}\partial_\mu \varepsilon^A + f^{ABC} A_\mu^B \varepsilon^C$$

new term

$$= A_\mu^A + \frac{1}{g} D_\mu^{AB} \varepsilon^B$$

As long as the gauge condition is linear in the gauge field, the determinant again does not depend on ε.

But in contrast to the abelian case, it now depends on A_μ!

We thus arrive at

$$Z[J] = N \int \mathcal{D}\varepsilon \int \mathcal{D}A_\mu^A \; e^{i\int d^4x \left(-\frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} - J_\mu^A A^{A,\mu}\right)}$$

$$\delta(G(A^\varepsilon)) \; \det\left(\frac{\delta G(A^\varepsilon)}{\delta\varepsilon}\right)$$

We next shift the integration variable $A_\mu^A \to A_\mu^{A,\varepsilon}$.

The Jacobian of this transformation gives (in the abelian case this was a constant shift)

$\det e^A = e^{\operatorname{Tr} A}$

$$\left|\det \frac{\delta A_\mu^{A,\varepsilon}(x)}{\delta A_\nu^B(y)}\right| = \exp \operatorname{Tr} \ln\left(\delta^{AB} g_\mu{}^\nu \delta^{(4)}(x-y) + f^{ACD}\delta^{BC} g_\mu{}^\nu \varepsilon^D \delta^{(4)}(x-y)\right)$$

$\ln(1+x) = x + O(x^2)$

$$\simeq \exp \operatorname{Tr}\left(f^{ABC} g_\mu{}^\nu \varepsilon^C \delta^{(4)}(x-y)\right)$$

$$= 1 \qquad \text{since the trace sets } A=B,\ \mu=\nu,\ x=y \to f^{AAC}=0$$

$$\Rightarrow \quad \mathcal{D}A_\mu^A = \mathcal{D}A_\mu^{A,\varepsilon}$$

The kinetic term is obviously gauge invariant

$$-\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} = -\frac{1}{4} G^{A,\varepsilon}_{\mu\nu} G^{A,\varepsilon,\mu\nu}$$

and the source term gives

$$\int d^4x \; J^A_\mu A^{A,\mu}$$

$$= \int d^4x \left(J^A_\mu A^{A,\varepsilon,\mu} - \frac{1}{g} J^A_\mu \partial^\mu \varepsilon^A - f^{ABC} J^A_\mu A^{B,\mu} \varepsilon^C \right)$$

$$\Big/ \text{p.I.}$$

$$= \int d^4x \left(J^A_\mu A^{A,\varepsilon,\mu} + \frac{1}{g} (\partial^\mu J^A_\mu) \varepsilon^A + f^{ACB} J^B_\mu A^{C,\mu} \varepsilon^A \right)$$

$$= \int d^4x \left(J^A_\mu A^{A,\varepsilon,\mu} + \frac{1}{g} \underbrace{D^{AB}_\mu J^{B,\mu}}_{=0 \text{ covariantly conserved !}} \varepsilon^A \right)$$

We finally have to transform the determinant, which itself is a functional of A^μ. To this end, we rewrite

$$1 = \det[A_\mu] \int \mathcal{D}\varepsilon' \, \delta(G[A^{\varepsilon'}])$$

short-hand notation of FP determinant, which is independent of ε'

$$\Rightarrow \det[A^\varepsilon_\mu]^{-1} = \int \mathcal{D}\varepsilon' \, \delta(G[A^{\varepsilon\varepsilon'}])$$

$\varepsilon'' = \varepsilon\varepsilon'$
$\mathcal{D}\varepsilon'' = \mathcal{D}\varepsilon'$
(group measure is invariant)

$$= \int \mathcal{D}\varepsilon'' \, \delta(G[A^{\varepsilon''}])$$

← read backwards

$$= \det[A_\mu]^{-1}$$

i.e. the determinant is also invariant under gauge transformations.

Renaming again A_μ^ε by A_μ, we obtain

$$Z[J] = N \int \mathcal{D}\varepsilon \quad \text{— irrelevant divergent prefactor} \qquad \text{new contribution!}$$

$$\underbrace{\int \mathcal{D}A_\mu^A \, e^{i\int d^4x \left(-\frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} - J_\mu^A A^{A,\mu}\right)} \delta(G(A)) \det\left(\frac{\delta G(A^\varepsilon)}{\delta\varepsilon}\right)[A]}_{\text{integration over physically inequivalent configurations}}$$

In order to bring the generating functional into a form that is suited for perturbative calculations, we specify to the class of generalised Lorenz gauges

$$G[A] = \partial^\mu A_\mu^A(x) - \alpha^A(x)$$

$$\hookrightarrow \quad \frac{\delta G[A_\mu^{A,\varepsilon}](x)}{\delta \varepsilon^B(y)} = \frac{1}{g}\,\partial^\mu D_\mu^{AB}\,\delta^{(4)}(x-y)$$

and integrate over $\alpha^A(x)$ with a Gaussian weight factor. This yields (→ page 146)

$$Z[J] = N\, N(\xi) \int \mathcal{D}\varepsilon \int \mathcal{D}A_\mu^A \, e^{i\int d^4x \left(-\frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} - J_\mu^A A^{A,\mu}\right)}$$

$$e^{-i\int d^4x \frac{(\partial^\mu A_\mu^A)^2}{2\xi}} \det\left(\frac{1}{g}\,\partial^\mu D_\mu^{AB}\,\delta^{(4)}(x-y)\right)$$

We will see that the determinant does not modify the quadratic terms in A_μ^A, and we therefore obtain the same gauge boson propagator as in the abelian case, as anticipated on page 193.

The determinant represents a new contribution, which we would like to write as an additional contribution to the Lagrangian. To this end, we recall that a Gaussian integral over Grassmann fields gives the determinant of the coefficient of the quadratic term, cf. page 59. Faddeev and Popov therefore proposed to write

$$\det\left(\frac{1}{g}\partial^\mu D_\mu^{AB}\delta^{(4)}(x-y)\right)$$

$$= \int \mathcal{D}\bar{c}^A \int \mathcal{D}c^A \, e^{-\int d^4x\, d^4y\, \bar{c}^A(x)\frac{1}{g}\partial^\mu D_\mu^{AB}\delta^{(4)}(x-y)c^B(y)}$$

Several comments are in order:

* c^A and $\bar{c}^A$ are independent, real Grassmann fields, i.e. $c^{A*} = c^A$ and $\bar{c}^{A*} = \bar{c}^A$. There is <u>no</u> relation between c^A and $\bar{c}^A$.
* c^A and $\bar{c}^A$ are scalars under Lorentz transformations.

  → they violate the spin-statistics theorem!

  But remember that they do not describe physical fields (and we are not going to study Green functions that involve c^A and $\bar{c}^A$), but they are rather a technical construct to rewrite the determinant. The fields are called <u>Faddeev - Popov ghosts</u>, which are needed to cancel the effects from the non-physical polarisation states of the non-abelian gauge fields (as we will show in the next section).

* in order to obtain the canonical normalisation of the kinetic term, we rescale the ghost field $c^A \to i g c^A$, while $\bar{c}^A$ remains unchanged. The field c^A then becomes antihermitian, $c^{A\dagger} = - c^A$, which is what is needed for a hermitian kinetic term

$$(\partial_\mu \bar{c}^A \partial^\mu c^A)^\dagger = - \partial_\mu c^A \partial^\mu \bar{c}^A = \partial_\mu \bar{c}^A \partial^\mu c^A$$

In summary, the Faddeev - Popov determinant gives an additional contribution to the Lagrangian of the form

$$\mathcal{L}_{FP} = - \bar{c}^A \partial^\mu D_\mu^{AB} c^B$$

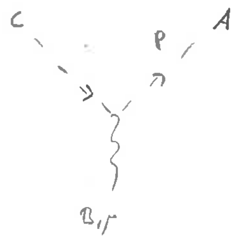
$$= - \bar{c}^A (\partial^2 \delta^{AB} - g f^{ABC} \partial^\mu A_\mu^C) c^B$$

$$\overset{P.I.}{=} \partial^\mu \bar{c}^A \partial_\mu c^A - g f^{ABC} (\partial^\mu \bar{c}^A) c^B A_\mu^C$$

which gives rise to the Feynman rules

$$A \; \text{- - -}\overset{k}{\rightarrow}\text{- - -} \; B = \frac{i}{k^2 + i\varepsilon} \delta^{AB}$$

[convention: arrow points towards c^A]



$$= - g f^{ABC} p^\mu$$

We see that the strength of the gauge boson - ghost interaction is constrained by gauge invariance. As the ghost field is Grassmann - valued, we also obtain factors of (-1) for closed ghost loops.

Notice that in QED with $f^{ABC} = 0$, the ghost fields are irrelevant since they do not couple to the physical degrees of freedom. In a rather exotic non-linear gauge, there are however non-trivial ghost fields even in QED ($\rightarrow$ problem sheet).

The ghost fields are an artefact of the quantisation. In particular, there are specific gauge choices which are ghost-free. Consider e.g. an axial gauge, $n \cdot A^A = 0$, with

$$\frac{\delta G[A_\mu^{A,\varepsilon}](x)}{\delta \varepsilon^B(y)} = \frac{1}{g}\, n\cdot\partial\, \delta^{AB}\, \delta^{(4)}(x-y) + f^{ACB}\, \underbrace{n\cdot A^C}_{=0}\, \delta^{(4)}(x-y)$$

$$= \frac{1}{g}\, n\cdot\partial\, \delta^{AB}\, \delta^{(4)}(x-y)$$

which does <u>not</u> depend on A_μ^A! In an axial gauge, the ghost fields therefore do not couple to the physical degrees of freedom and can be neglected.

To sum up, the Lagrangian for a non-abelian $SU(N)$ gauge theory after gauge-fixing becomes

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} + \bar{\psi}(i \not{D} - m)\psi$$

$$-\frac{1}{2\xi}(\partial^\mu A_\mu^A)^2 + \partial^\mu \bar{c}^A \partial_\mu c^A - g f^{ABC} (\partial^\mu \bar{c}^A) c^B A_\mu^C$$

↳ additional contributions to Noether current and EOM...

4.3 BRST symmetry

In QED we found that the unphysical polarisation states of the photon do not contribute to S-matrix elements due to the conservation of the electromagnetic current (or more generally the Ward-Takahashi identity).
In a non-abelian gauge theory, on the other hand, the scalar and longitudinal polarisations of the gauge field do not cancel, but the cancellation is restored when the contributions of the ghost fields are included. An elegant formalism that shows this cancellation has been introduced by Becchi, Rouet and Stora and, independently, by Tyutin. It is based on the observation that the gauge-fixed Lagrangian is invariant under a global, fermionic symmetry transformation - the BRST symmetry.

The symmetry is most easily identified if one introduces an additional (bosonic) scalar field B^A such that the gauge-fixed Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$$+ \frac{3}{2}(B^A)^2 + B^A \partial^\mu A^A_\mu - \bar{c}^A \partial^\mu D^{AB}_\mu c^B$$

Note that the new field does not have a kinetic term, it is just an <u>auxiliary field</u>. It can therefore easily be eliminated using the equations of motion (or equivalently by performing the functional integral)

$$\frac{\partial \mathcal{L}}{\partial B^A} = 3 B^A + \partial^\mu A^A_\mu = 0$$

$$\rightarrow \quad B^A = -\frac{1}{3} \partial^\mu A^A_\mu$$

which brings us back to the original version of the gauge-fixed Lagrangian.

Let us now consider the following infinitesimal transformation

gauge transformation

$$\delta\psi = i g \theta c^A T^A \psi \qquad i\varepsilon^A T^A \psi$$

$$\delta A_\mu^A = \theta D_\mu^{AB} c^B \qquad \frac{1}{g} D_\mu^{AB} \varepsilon^B$$

$$\delta c^A = -\frac{1}{2} g \theta f^{ABC} c^B c^C$$

$$\delta \bar{c}^A = \theta B^A$$

$$\delta B^A = 0$$

where θ is an anticommuting parameter that does not depend on x ($\to$ global, fermionic symmetry). Notice that each bosonic / fermionic field transforms with an even / odd number of Grassmann-valued objects.

We will now verify that the gauge-fixed Lagrangian is invariant under this transformation:

* the BRST transformation acts on the fermions and the gauge bosons as a gauge transformation with

$$\varepsilon^A(x) = g\,\theta\, c^A(x)$$

$\to \; -\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$ is BRST invariant

* $\frac{3}{2}(B^A)^2$ is trivially invariant since $\delta B^A = 0$

* The last two terms give

$$\delta(B^A \partial^\mu A_\mu^A - \bar c^A \partial^\mu D_\mu^{AB} c^B)$$

$$= B^A \partial^\mu \theta D_\mu^{AB} c^B - \theta B^A \partial^\mu D_\mu^{AB} c^B - \bar c^A \partial^\mu \delta(D_\mu^{AB} c^B)$$

$$= -\bar c^A \partial^\mu \delta(D_\mu^{AB} c^B)$$

We thus need to evaluate $\quad D_\mu^{AB} = \partial_\mu \delta^{AB} - g f^{ABC} A_\mu^C$

$$\delta(D_\mu^{AB} c^B) = -g f^{ABC} \underbrace{(\theta D_\mu^{CD} c^D)}_{\delta A_\mu^C} c^B + D_\mu^{AB} \underbrace{(-\tfrac12 g\theta f^{BCD} c^C c^D)}_{\delta c^B}$$

$$= -g\theta f^{ABC}(\partial_\mu c^C - g f^{CDE} A_\mu^E c^D) c^B$$

$$- \tfrac12 g\theta f^{BCD}(\delta^{AB}(\partial_\mu c^C c^D) - g f^{ABE} A_\mu^E c^C c^D)$$

$$= -g\theta f^{ABC}(\partial_\mu c^C) c^B + g^2\theta f^{ABC} f^{CDE} A_\mu^E c^D c^B$$

$$- \tfrac12 g\theta f^{ACD}(\partial_\mu c^C) c^D - \tfrac12 g\theta f^{ACD} c^C(\partial_\mu c^D) + \tfrac12 g^2 \theta f^{BCD} f^{ABE} A_\mu^E c^C c^D$$

$$= \underbrace{g\theta f^{ACB}(\partial_\mu c^C) c^B - \tfrac12 g\theta f^{ACB}(\partial_\mu c^C) c^B - \tfrac12 g\theta f^{ACB}(\partial_\mu c^C) c^B}_{=0}$$

$$+ \tfrac12 g^2\theta A_\mu^E (f^{ABC} f^{CDE} c^D c^B - f^{ADC} f^{CBE} c^D c^B + f^{CDB} f^{ACE} c^D c^B)$$

$$= \tfrac12 g^2\theta A_\mu^E (f^{ABC} f^{CDE} - f^{ADC} f^{CBE} + f^{CDB} f^{ACE}) c^D c^B$$

$$= \tfrac12 g^2\theta A_\mu^E c^D c^B \underbrace{(f^{DCE} f^{BAC} + f^{BCE} f^{ADC} + f^{ACE} f^{DBC})}_{\text{Jacobi identity} \to 0}$$

$$= 0$$

$\Rightarrow$ the BRST transformation is a global symmetry of the gauge-fixed Lagrangian

We can formally consider the BRST transformation as the action of an operator

$$\delta \phi = \theta s \phi$$

e.g. $s A_\mu^A = D_\mu^{AB} c^B$. The BRST operator s has an interesting property, it is <u>nilpotent</u>

$$s^2 O(x) = 0 \qquad \forall O(x)$$

This can easily be verified for the fundamental fields, e.g.

$$s^2 \bar{c}^A = s B^A = 0$$

and we will show in the tutorials that this extends to arbitrary operators.

As for any continous symmetry, there exists a conserved Noether current and a conserved charge associated with the BRST symmetry. We do not need their explicit expressions here, but we make use the general result that the Noether charge generates the symmetry transformation.

$$\delta \phi = i [\theta Q, \phi]$$

It follows

$$\delta\phi = \theta s\,\phi = i\,\theta Q\,\phi - i\,\phi\,\theta Q = \theta\, i\,[Q,\phi]_\mp$$

for a bosonic / fermionic field ϕ and

$$\begin{aligned} s^2\phi &= s\, i\,[Q,\phi]_\mp \\ &= -\,[Q,[Q,\phi]_\mp]_\pm \qquad s\phi \text{ has opposite statistics than } \phi\,! \\ &= -\big(Q\,[Q,\phi]_\mp \pm [Q,\phi]_\mp Q\big) \\ &= -\,Q^2\phi \pm Q\phi Q \mp (Q\phi Q \mp \phi Q^2) \\ &= \phi Q^2 - Q^2\phi \qquad \forall\,\phi \end{aligned}$$

and hence $Q^2 = 0$ since s is nilpotent ($Q^2 \sim \mathbb{1}$ is not possible, since the explicit expression for Q shows that it carries ghost number). As the BRST charge is conserved, it also commutes with the Hamiltonian, $[Q,H]=0$.

The BRST charge allows us to identify the <u>physical Hilbert space</u>. In general, a nilpotent operator that commutes with the Hamiltonian divides the eigenstates of H into three subspaces

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2$$

Here

* $\mathcal{H}_1$ contains the states that are not annihilated by Q

$$Q|\psi_1\rangle \neq 0 \qquad \forall \quad |\psi_1\rangle \in \mathcal{H}_1$$

* $\mathcal{H}_2$ contains the states of the form

$$|\psi_2\rangle = Q|\psi_1\rangle \qquad \text{with} \quad |\psi_1\rangle \in \mathcal{H}_1$$

which are annihilated by Q

$$Q|\psi_2\rangle = Q^2|\psi_1\rangle = 0$$

* $\mathcal{H}_0$ contains the remaining states that are annihilated by Q with

$$Q|\psi_0\rangle = 0 \qquad \text{and} \qquad |\psi_0\rangle \neq Q|\psi_1\rangle$$

Notice that the states of $\mathcal{H}_2$ have zero norm

$$\langle\psi_2|\psi_2\rangle = \langle\psi_1|Q^2|\psi_1\rangle = 0$$

and that the scalar product between elements of $\mathcal{H}_0$ and $\mathcal{H}_2$ vanish

$$\langle\psi_0|\psi_2\rangle = \underbrace{\langle\psi_0|Q}_{=0}|\psi_1\rangle = 0$$

We next classify the asymptotic states ($\rightarrow$ localised wave packets of free particles in the far past / future) according to the three subspaces $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$. To this end, we examine the implications of a BRST transformation with $g = 0$

$$\delta A_\mu^A = \theta \partial_\mu c^A$$

$$\delta \bar{c}^A = \theta B^A$$

$$\delta \psi = \delta c^A = \delta B^A = 0$$

In order to identify the polarisation states of the non-abelian gauge field, we write

$\varepsilon_\mu^{\perp}(k)$: two transverse polarisations

$\varepsilon_\mu^{\pm}(k) = \frac{1}{\sqrt{2}}\left(1, \mp \frac{\vec{k}}{|\vec{k}|}\right)$: linear combinations of scalar and longitudinal polarisation

with
$$\varepsilon_\mu^{\perp}(k)\, \varepsilon^{\mu\pm}(k) = 0$$
$$\left[\varepsilon_\mu^{\perp}(k)\right]^2 = 0$$
$$\varepsilon_\mu^{+}(k)\, \varepsilon^{\mu -}(k) = 1$$

and
$$\varepsilon_\mu^{\perp}(k)\, k^\mu = 0 \qquad (k = \sqrt{2}\,|\vec{k}|\,\varepsilon^{+})$$
$$\varepsilon_\mu^{+}(k)\, k^\mu = 0$$
$$\varepsilon_\mu^{-}(k)\, k^\mu = \sqrt{2}\,|\vec{k}|$$

In momentum space the BRST transformation of the gauge field implies

$$\delta(\varepsilon^\mu(k) A_\mu(k)) \sim \varepsilon^\mu(k) k_\mu c^A(k)$$

which is non-zero only for $\varepsilon_\mu^-(k)$. On the left-hand side, this projects out the $\varepsilon_\mu^+(k)$-component of $A_\mu(k)$ and hence

$$|\varepsilon^+(k)\rangle \in \mathcal{H}_1$$
$$|c(k)\rangle \in \mathcal{H}_2$$

We proceed similarly for the BRST transformation of the anti-ghost field

$$\delta\bar{c}^A \sim B^A \sim k^\mu A_\mu(k)$$

$$\Rightarrow \quad |\bar{c}(k)\rangle \in \mathcal{H}_1$$
$$|\varepsilon^-(k)\rangle \in \mathcal{H}_2$$

since k^μ projects out the $\varepsilon_\mu^-(k)$-component of $A_\mu(k)$.

The remaining fields with $\delta\phi = 0$ are annihilated by a BRST transformation and cannot be written as a BRST variation of another field

$$\Rightarrow \quad |\varepsilon^\perp(k)\rangle \in \mathcal{H}_0$$
$$|\psi(k)\rangle \in \mathcal{H}_0$$

This suggests that $\mathcal{H}_0$ is the physical Hilbert space.

Without going into the details here (cf. e.g. Weinberg II), we note that the proof relies on gauge invariance of the S-matrix, which implies

$$Q \, | \, p, s \, ; \, {}^{in}/_{out} \rangle = 0$$

for the asymptotic states of the physical Hilbert space.

After eliminating the zero-norm modes, one then confirms that $\mathcal{H}_0$ is the physical Hilbert space.

(note also that the vacuum state with $Q|\Omega\rangle = 0$ and $\langle\Omega|\Omega\rangle = 1$ also belongs to $\mathcal{H}_0$)

Having identified the asymptotic states, we still need to show that they cannot evolve into an unphysical state in a scattering process, i.e. we have to show that the S-matrix is unitary on the physical Hilbert space.

To do so, we first use that the S-matrix is unitary on the full Hilbert space (since H is hermitean)

$$\langle \psi_0' | S^+ S | \psi_0 \rangle = \langle \psi_0' | \psi_0 \rangle \qquad |\psi_0\rangle \in \mathcal{H}_0$$

As $[Q,H]=0$, Q also commutes with S and

$$Q S |\psi_0\rangle = S Q |\psi_0\rangle = 0$$

$$\rightarrow \quad S|\psi_0\rangle \in \mathcal{H}_0 + \mathcal{H}_2$$

We obtain

$$\langle \psi_0' | S^+ S | \psi_0 \rangle = \sum_{\phi \in \mathcal{H}_0 + \mathcal{H}_2} \langle \psi_0' | S^+ | \phi \rangle \langle \phi | S | \psi_0 \rangle$$

$$= \sum_{\phi \in \mathcal{H}_0} \langle \psi_0' | S^+ | \phi \rangle \langle \phi | S | \psi_0 \rangle \qquad \text{since the states in } \mathcal{H}_2 \text{ fulfill } \langle \psi_0 | \psi_2 \rangle = 0$$

$$= \langle \psi_0' | S^+ S |_{\mathcal{H}_0} | \psi_0 \rangle .$$

$$\Rightarrow \quad S^+ S |_{\mathcal{H}_0} = \mathbb{1} .$$

The BRST symmetry also provides a means to derive the Ward identities associated with non-abelian gauge invariance, the so-called Slavnov-Taylor identities. Details can be found e.g. in Ryder (chapter 7.6) or Pokorski (chapter 8.1).

4.4. Renormalisation

In renormalised perturbation theory, we introduce renormalised parameters as

$$\boxed{\begin{aligned} \psi_0 &= \sqrt{Z_\psi}\,\psi & \qquad A_0^{A,\mu} &= \sqrt{Z_A}\,A^{A,\mu} \\ m_0 &= Z_m\, m & \xi_0 &= Z_3\,\xi \\ g_0 &= \mu^{\varepsilon} Z_g\, g & c_0^A &= \sqrt{Z_c}\, c^A \end{aligned}}$$

and we are free to choose $\bar{c}_0^A = \sqrt{Z_c}\,\bar{c}^A$ since the Lagrangian only involves combinations of the $\bar{c}^A c^B$.

Writing $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$, we obtain

$$\begin{aligned} \mathcal{L}_r = & \left.\begin{aligned} & -\tfrac{1}{4}\left(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A_\mu^A\right)^2 \\ & + \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi + (\partial^\mu \bar{c}^A)\,\partial_\mu c^A \end{aligned}\right\} \mathcal{L}_0 \\ & \left.\begin{aligned} & + g\mu^{\varepsilon}\,\bar{\psi}\gamma^\mu T^A \psi A_\mu^A - g\mu^{\varepsilon} f^{ABC}\left(\partial_\mu A_\nu^A\right) A^{B,\mu} A^{C,\nu} \\ & - \frac{g^2}{4}\mu^{2\varepsilon} f^{ABC} f^{ADE} A_\mu^B A_\nu^C A^{D,\mu} A^{E,\nu} \\ & - g\mu^{\varepsilon} f^{ABC}\left(\partial^\mu \bar{c}^A\right) c^B A_\mu^C \end{aligned}\right\} \mathcal{L}_{int} \end{aligned}$$

and

$$\mathcal{L}_{ct} = -\frac{1}{4}(Z_A-1)(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A)^2 - \left(\frac{Z_A}{Z_\xi}-1\right)\frac{1}{\xi}(\partial^\mu A_\mu^A)^2$$

$$+ (Z_\psi-1)\,\bar\psi i\partial\!\!\!/\psi - (Z_\psi Z_m-1)\, m\,\bar\psi\psi + (Z_c-1)(\partial^\mu \bar c^A)\partial_\mu c^A$$

$$+ (Z_g Z_\psi \sqrt{Z_A}-1)\, g\mu^{\varepsilon}\,\bar\psi\gamma^\mu T^A\psi A_\mu^A$$

$$- (Z_g Z_A^{3/2}-1)\, g\mu^{\varepsilon} f^{ABC}(\partial_\mu A_\nu^A)A^{B,\mu}A^{C,\nu}$$

$$- (Z_g^2 Z_A^2-1)\,\frac{g^2}{4}\mu^{2\varepsilon} f^{ABC}f^{ADE}A_\mu^B A_\nu^C A^{D,\mu}A^{E,\nu}$$

$$- (Z_g Z_c\sqrt{Z_A}-1)\, g\mu^{\varepsilon} f^{ABC}(\partial^\mu\bar c^A)c^B A_\mu^C$$

$\mathcal{L}_{int}$

The counterterms give rise to the Feynman rules

$$= (-i)\left[(Z_A-1)(q^2 g^{\mu\nu} - q^\mu q^\nu) + \left(\frac{Z_A}{Z_\xi}-1\right)\frac{1}{\xi}q^\mu q^\nu\right]\delta^{AB}$$

$$= i\left[(Z_\psi-1)\not p - (Z_\psi Z_m-1)m\right]_{\alpha\beta}\delta_{ab}$$

$$= i(Z_c-1)k^2\delta^{AB}$$

and the vertices

follow from the expressions on page 193 and 204

with the corresponding combination of Z-factors.

(note that those expressions contain factors of μ^ε in DR)

In QED we found that gauge invariance implies exact relations between Green functions – the Ward – Takahashi identities – and we explicitly showed that

$$Z_e Z_\psi \sqrt{Z_A} = Z_\psi \qquad\qquad Z_3 = Z_A$$

in the on-shell scheme (the relations actually also hold in the $\overline{MS}$ scheme). The first relation allowed us to write

$$(Z_\psi - 1)\, \bar\psi\, i \partial\!\!\!/ \psi + (Z_e Z_\psi \sqrt{Z_A} - 1)\, e \hat{\mu}^\varepsilon \bar\psi \gamma^\mu \psi A_\mu$$

$$= (Z_\psi - 1)\, \bar\psi\, i D\!\!\!\!/\, \psi$$

in a gauge-invariant combination. We also noted that each fermion can have a different charge e_ψ in QED, and that the relation $Z_e = \frac{1}{\sqrt{Z_A}}$ ensures that the renormalisation of the electromagnetic charge is independent of the fermion species.

In a non-abelian gauge theory, the Slavnov-Taylor identities again imply

$$Z_3 = Z_A$$

and they also provide exact relations between the renormalisation factors of the $\bar{\psi}\psi A$, AAA, $AAAA$ and $\bar{c}cA$ counterterms (note that there are only 6 renormalisation constants in a non-abelian theory but 9 counterterms — the ST identities provide the three missing relations). In other words, a universal value of Z_g simultaneously makes the four interaction vertices finite to all orders in perturbation theory. There is, on the other hand, no relation between Z_g and Z_1 in a non-abelian theory, and one finds

$$Z_g Z_4 \sqrt{Z_1} \neq Z_4$$

and the counterterms are therefore no longer individually gauge-invariant. The fermions, moreover, always couple with the same strength g to the gauge field in a non-abelian theory.

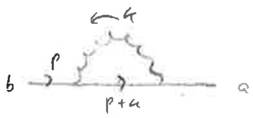
The most important quantum effect in non-abelian gauge theories is <u>asymptotic freedom</u>, i.e the coupling strength becomes weaker at high energies and the theory asymptotically resembles a free theory. In technical terms, the β-function that governs the running of the coupling constant is negative. Since this is such a central result, we will sketch the computation of the 1-loop β-function here without entering all the technical details. We will adopt the $\overline{MS}$ scheme in the following for convenience.

In QED the relation $Z_e = \frac{1}{\sqrt{Z_A}}$ implies that the β-function can be calculated directly from the vacuum polarisation. From our result on page 181, we read off

$$\bar{Z}_e = 1 + \frac{\bar{\alpha}}{6\pi\varepsilon} + O(\bar{\alpha}^2)$$

In a non-abelian theory this is not possible, and we instead have to compute the loop corrections to any of the four interaction vertices. Here, we will choose the $\bar\psi\psi A$ vertex. We thus first have to compute the renormalization constants Z_ψ and Z_A from the fermion and gauge boson self energies, respectively.

We start with the fermion self energy (in Feynman gauge)

$-i\,\Sigma(p) =$  $+$  $+\ O(\bar\alpha^2)$

$$= (i\bar g\mu^{\varepsilon})^2\,[T^a T^a]_{ab} \int \frac{d^dk}{(2\pi)^d}\,\frac{\gamma^\mu\, i(\not p+\not k+m)\,\gamma_\mu}{((p+k)^2-m^2)}\,\frac{(-i)}{k^2}$$

$$+\,i\left[(\bar Z_\psi-1)\not p - (\bar Z_\psi \bar Z_m - 1)\,m\right]\delta_{ab} \overset{!}{=} \text{UV-finite}$$

We have

$$[T^a T^a]_{ab} = C_F\,\delta_{ab}$$

and the evaluation of the loop integral gives

$$\boxed{\bar Z_\psi = 1 - \frac{\bar\alpha C_F}{4\pi}\,\frac{1}{\varepsilon}} \qquad \alpha = \frac{g^2}{4\pi}$$

We next consider the gauge boson self energy

$i\,\Pi^{AB}_{\mu\nu}(q^2) =$

$+ \; \mathcal{O}(\bar{\alpha}^2)$

where

$$= (-i)\,(\bar{Z}_A - 1)\,(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\delta^{AB}$$

since $\bar{Z}_3 = \bar{Z}_A$. We already encountered the first diagram in QED (→ page 177), which in the non-abelian case gets multiplied with

$$\mathrm{Tr}\,[T^A T^B] = T_F\,\delta^{AB}$$

Assuming that there are n_f different fermion species in the fundamental representation, we thus obtain

$$= \underbrace{(-i)\,\frac{\bar{\alpha}}{3\pi\varepsilon}\,(q^2 g^{\mu\nu} - q^\mu q^\nu)}_{\text{QED result}}\; n_f\,T_F\,\delta^{AB} + \ldots$$

As to the non-abelian diagrams, we have to account for a symmetry factor 1/2 for both diagrams with an internal gauge boson loop, and a Grassmann factor (−1) for the diagram with the ghost loop.

All of the non-abelian diagrams involve the group factor

$$f^{ACD} f^{BCD} = (T^A_{ad})_{CD} \, (T^B_{ad})_{DC} \qquad (T^B_{ad})_{AC} = i f^{ABC}$$

$$= \mathrm{Tr}\,[\, T^A_{ad} \, T^B_{ad} \,]$$

$$= T_{ad}\, \delta^{AB} = N \, \delta^{AB}$$

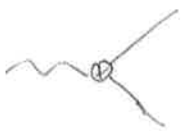
As in QED, the gauge boson self energy is transverse due to gauge invariance. It is interesting to note, however, that the non-abelian diagrams are not individually transverse, but their sum is. This is another example that illustrates that gauge invariance is restored only when the ghost contributions are included.

The evaluation of the diagrams gives (in Feynman gauge)

$$\boxed{\bar{Z}_A = 1 + \frac{\bar{\alpha}}{4\pi} \left(\frac{5}{3} N - \frac{4}{3} n_f T_F \right) \frac{1}{\varepsilon}}$$

We finally consider the corrections to the $\bar\psi\psi A$ vertex

where



$$= (\bar Z_g \bar Z_\psi \sqrt{\bar Z_A} - 1)\, i \bar g \tilde\mu^\varepsilon \gamma^\mu T^A$$

The first diagram involves the group factor

$$T^B T^A T^B = T^B (T^B T^A + i f^{ABC} T^C)$$

$$= C_F T^A + \frac{i}{2} f^{ABC} \left([T^B, T^C] + \underbrace{\cancel{\{T^B, T^C\}}}_{\to 0} \right)$$

$$= C_F T^A + \frac{i}{2} f^{ABC}\, i f^{BCD}\, T^D$$

$$= \left(C_F - \frac{N}{2}\right) T^A$$

and for the second diagram we obtain

$$i f^{ABC} T^B T^C = -\frac{N}{2} T^A$$

The evaluation of the loop diagrams yields

$$\bar Z_g \bar Z_\psi \sqrt{\bar Z_A} = 1 - \frac{\bar\alpha}{4\pi} (C_F + N) \frac{1}{\varepsilon} + \bar Z_\psi$$

and finally

$$\boxed{\bar Z_g = 1 - \frac{\bar\alpha}{4\pi} \left(\frac{11}{6} N - \frac{2}{3} n_f T_F \right) \frac{1}{\varepsilon}}$$

which reduces to the QED result from page 221

for $N = 0$ and $T_F = 1$.

The running of the $\overline{MS}$-coupling constant is governed by the RG equation

$$\frac{d\bar{\alpha}}{d\ln\mu} = -2\varepsilon\bar{\alpha} + \beta(\bar{\alpha})$$

with β-function (→ page 120)

$$\beta(\bar{\alpha}) = -\frac{1}{\bar{Z}_\alpha}\frac{d\bar{Z}_\alpha}{d\ln\mu}\,\bar{\alpha}$$

$$\equiv 2\bar{\alpha}\left[\beta_0\frac{\bar{\alpha}}{4\pi} + \beta_1\left(\frac{\bar{\alpha}}{4\pi}\right)^2 + O(\bar{\alpha}^3)\right]$$

and

$$\bar{Z}_\alpha = \bar{Z}_g^{\,2} = 1 - \frac{\bar{\alpha}}{4\pi}\left(\frac{11}{3}N - \frac{4}{3}n_f T_F\right)\frac{1}{\varepsilon} + O(\bar{\alpha}^2)$$

$$\Rightarrow \beta(\bar{\alpha}) = -1\left(-\frac{1}{4\pi}\right)\underbrace{\frac{d\bar{\alpha}}{d\ln\mu}}_{-2\varepsilon\bar{\alpha}}\left(\frac{11}{3}N - \frac{4}{3}n_f T_F\right)\frac{1}{\varepsilon}\,\bar{\alpha}$$

$$= 2\bar{\alpha}\,\frac{\bar{\alpha}}{4\pi}\left(-\frac{11}{3}N + \frac{4}{3}n_f T_F\right)$$

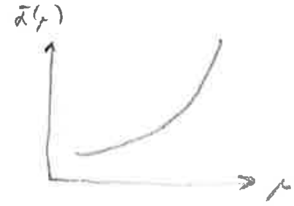
and hence $\beta_0 = -\frac{11}{3}N + \frac{4}{3}n_f T_F$. The one-loop solution to the RG equation is given by

$$\boxed{\bar{\alpha}(\mu) = \frac{\bar{\alpha}(\mu_0)}{1 - \frac{\bar{\alpha}(\mu_0)}{4\pi}\beta_0 \ln\frac{\mu^2}{\mu_0^2}}}$$

For the two most important examples, we obtain

- QED : $N=0,\ T_F = 1 \ \rightarrow\ \beta_0 = \frac{4}{3} n_f$

  $\bar{\alpha}(m_e) \simeq \frac{1}{137}\ ,\ \bar{\alpha}(M_Z) \simeq \frac{1}{128}$



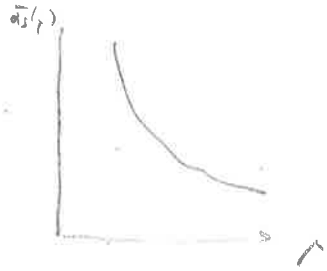
- QCD : $N=3,\ T_F = \frac{1}{2} \ \rightarrow\ \beta_0 = -11 + \frac{2}{3} n_f$

  6 quarks $\rightarrow\ \beta_0 = -7 \ \underline{\underline{< 0}}$

  $\bar{\alpha}_s(m_b \simeq 5\ \text{GeV}) \simeq 0.2$

  $\bar{\alpha}_s(M_Z \simeq 90\ \text{GeV}) \simeq 0.12$

  $\bar{\alpha}_s(\mu \rightarrow \infty) \rightarrow 0 \quad \Rightarrow$ <u>asymptotic freedom</u>



In QED we argued that the running of the coupling constant can be understood as a polarisation effect of the vacuum due to the virtual e^+e^- pairs that screen the electromagnetic charge. In a non-abelian theory we observe the same effect for the fermions, but we get an additional contribution since the gauge bosons themselves are charged under the gauge group. It turns out that the contribution of the gauge bosons is opposite in sign and they therefore lead to an <u>antiscreening</u> of the bare charge. For a heuristic explanation, cf. Peskin / Schröder, chapter 16.7.

In the opposite limit the coupling constant becomes large and the one-loop solution formally diverges at $\mu = \Lambda_{QCD}$ with

$$1 - \frac{\bar{\alpha}_s(\mu)}{4\pi} \beta_0 \ln \frac{\Lambda_{QCD}^2}{\mu^2} = 0$$

$$\Rightarrow \quad \bar{\alpha}_s(\mu) = - \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda_{QCD}^2}}$$

Although we started with a dimensionless coupling, there exists an intrinsic reference scale

$$\Lambda_{QCD} \simeq 200 \text{ MeV}$$

at which the strong interactions become non-perturbative (this is sometimes called dimensional transmutation).

This does not explain the <u>confinement</u> of quarks and gluons into colour-neutral hadrons, but it at least makes it plausible that the strong interactions become very strong at distances $r \geq \frac{1}{200 \text{ MeV}} \sim 10^{-15}$ m and generates bound states. Typical hadron masses are indeed of this order of magnitude

$$m_p \simeq 940 \text{ MeV} \qquad m_\rho \simeq 780 \text{ MeV}$$

[We will see that the role of pions and kaons is special in this context, since they are pseudo Goldstone bosons of chiral symmetry breaking]