Particle Physics Phenomenology
3. Evolution equations and final-state showers

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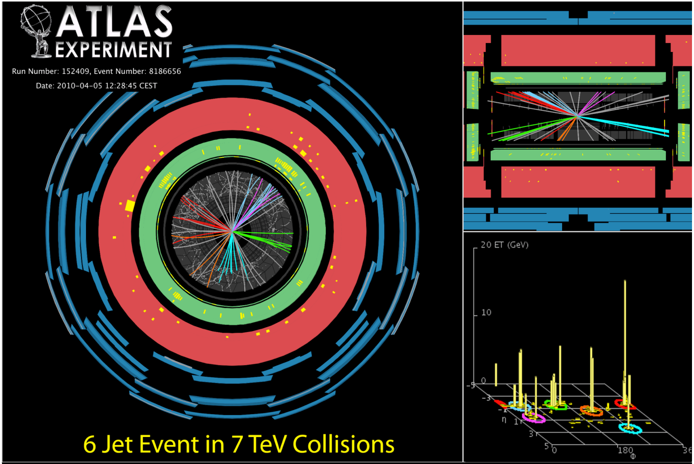


An event with 6 jets taken on April 4th, 2010. The jets have calibrated transverse momenta between 30 GeV and 70 GeV and are well separated in the detector. 12

QED bremsstrahlung – 1

Accelerated electric charges radiate photons,

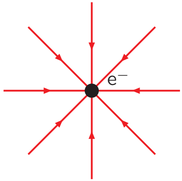
see e.g. J.D. Jackson, *Classical Electrodynamics*.
A charge ze that changes its velocity vector from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$ radiates a spectrum of photons that depends on its trajectory. In the long-wavelength limit it reduces to

$$\lim_{\omega \to 0} \frac{\mathrm{d}^2 I}{\mathrm{d}\omega \mathrm{d}\Omega} = \frac{z^2 e^2}{4\pi^2} \left| \boldsymbol{\epsilon}^* \left(\frac{\boldsymbol{\beta}'}{1 - \mathbf{n}\boldsymbol{\beta}'} - \frac{\boldsymbol{\beta}}{1 - \mathbf{n}\boldsymbol{\beta}} \right) \right|^2$$

where $\mathbf{n}$ is a vector on the unit sphere Ω, ω is the energy of the radiated photon, and $\boldsymbol{\epsilon}$ its polarization.

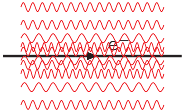
1. For fast particles radiation collinear with the $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ directions is strongly enhanced.
2. $\mathrm{d}N/\mathrm{d}\omega = (1/\omega)\mathrm{d}I/\mathrm{d}\omega \propto 1/\omega$ so infinitely many infinitely soft photons are emitted, but the net energy taken away is finite.

An electrical charge, say an electron, is surrounded by a field:



For a rapidly moving charge this field can be expressed in terms of an equivalent flux of photons:

$$\mathrm{dn}_\gamma \approx \frac{2\alpha_{\mathrm{em}}}{\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$



Equivalent Photon Approximation, or method of virtual quanta (e.g. Jackson) (Bohr; Fermi; Weiszäcker, Williams ∼1934)

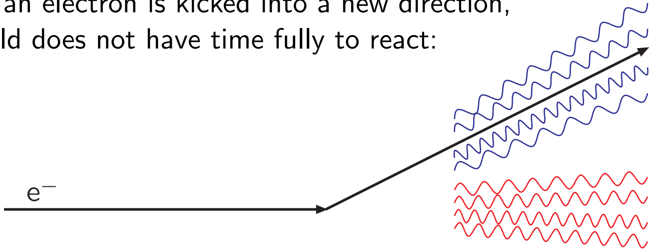
θ: collinear divergence, saved by $m_{\mathrm{e}} > 0$ in full expression.

ω: true divergence, $n_\gamma \propto \int \mathrm{d}\omega/\omega = \infty$, but $E_\gamma \propto \int \omega\,\mathrm{d}\omega/\omega$ finite.

These are virtual photons: continuously emitted and reabsorbed.

QED bremsstrahlung – 3

When an electron is kicked into a new direction,
the field does not have time fully to react:



- Initial State Radiation (ISR):
  part of it continues $\sim$ in original direction of e
- Final State Radiation (FSR):
  the field needs to be regenerated around outgoing e,
  and transients are emitted $\sim$ around outgoing e direction

Emission rate provided by equivalent photon flux in both cases.
Approximate cutoffs related to timescale of process:
the more violent the hard collision, the more radiation!

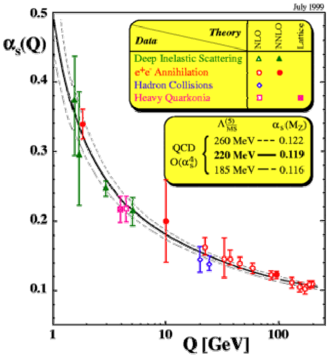
Divergences

Emission rate $\mathrm{q} \to \mathrm{qg}$ diverges when

- collinear: opening angle $\theta_{\mathrm{qg}} \to 0$
- soft: gluon energy $E_{\mathrm{g}} \to 0$

Almost identical to $\mathrm{e} \to \mathrm{e}\gamma$
but QCD is non-Abelian so additionally

- $\mathrm{g} \to \mathrm{gg}$ similarly divergent
- $\alpha_s(Q^2)$ diverges for $Q^2 \to 0$
  (actually for $Q^2 \to \Lambda^2_{\mathrm{QCD}}$)

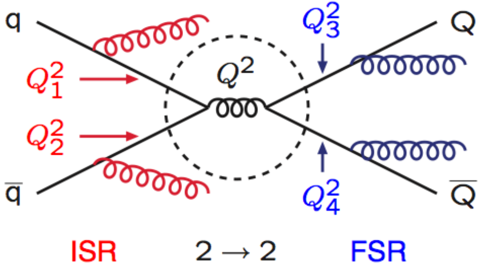


Big probability for one emission $\Longrightarrow$ also big for several
$\Longrightarrow$ with ME's need to calculate to high order **and** with many loops $\Longrightarrow$ extremely demanding technically (not solved!), and involving big cancellations between positive and negative contributions.

Alternative approach: **parton showers**

The Parton-Shower Approach

$$2 \to n \quad = \quad (2 \to 2) \oplus \text{ISR} \oplus \text{FSR}$$

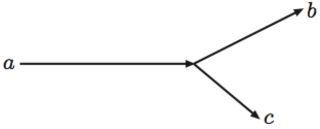


FSR = Final-State Radiation = timelike shower
$Q_i^2 \sim m^2 > 0$ decreasing
ISR = Initial-State Radiation = spacelike showers
$Q_i^2 \sim -m^2 > 0$ increasing

Why "time" like and "space" like?

Consider four-momentum conservation in a branching $a \to b\,c$

$$\mathbf{p}_{\perp a} = 0 \quad \Rightarrow \quad \mathbf{p}_{\perp c} = -\mathbf{p}_{\perp b}$$
$$p_{+} = E + p_{\mathrm{L}} \quad \Rightarrow \quad p_{+a} = p_{+b} + p_{+c}$$
$$p_{-} = E - p_{\mathrm{L}} \quad \Rightarrow \quad p_{-a} = p_{-b} + p_{-c}$$



Define $p_{+b} = z\,p_{+a}, \quad p_{+c} = (1-z)\,p_{+a}$

Use $p_{+}p_{-} = E^2 - p_{\mathrm{L}}^2 = m^2 + p_{\perp}^2$

$$\frac{m_a^2 + p_{\perp a}^2}{p_{+a}} = \frac{m_b^2 + p_{\perp b}^2}{z\,p_{+a}} + \frac{m_c^2 + p_{\perp c}^2}{(1-z)\,p_{+a}}$$

$$\Rightarrow \quad m_a^2 = \frac{m_b^2 + p_{\perp}^2}{z} + \frac{m_c^2 + p_{\perp}^2}{1-z} = \frac{m_b^2}{z} + \frac{m_c^2}{1-z} + \frac{p_{\perp}^2}{z(1-z)}$$

Final-state shower: $m_b = m_c = 0 \Rightarrow m_a^2 = \frac{p_{\perp}^2}{z(1-z)} > 0 \Rightarrow$ timelike

Initial-state shower: $m_a = m_c = 0 \Rightarrow m_b^2 = -\frac{p_{\perp}^2}{1-z} < 0 \Rightarrow$ spacelike

Showers and cross sections

Shower evolution is viewed as a probabilistic process, which occurs with unit total probability:
the cross section is not directly affected

However, more complicated than so

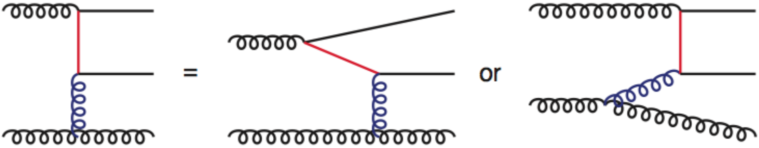
- PDF evolution $\approx$ showers $\Rightarrow$ enters in convoluted cross section, e.g. for $2 \to 2$ processes

$$\sigma = \iiint \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}\hat{t} \; f_i(x_1, Q^2) \, f_j(x_2, Q^2) \, \frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}\hat{t}}$$

- Shower affects event shape
  E.g. start from 2-jet event with $\mathrm{p}_{\perp 1} = \mathrm{p}_{\perp 2} = 100$ GeV.
  ISR gives third jet, plus recoil to existing two, so
  $\mathrm{p}_{\perp 1} = 110$ GeV, $\mathrm{p}_{\perp 2} = 90$ GeV, $\mathrm{p}_{\perp 1} = 20$ GeV:
  - hardest $\mathrm{p}_{\perp \mathrm{jet}}$ spectrum goes up
  - two-jets with both jets above some $p_{\perp \mathrm{min}}$ comes down
  - three-jet rate goes up
  - inclusive $\mathrm{p}_{\perp \mathrm{jet}}$ spectrum goes up (steeply falling slope!)

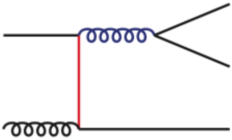
Doublecounting

A $2 \to n$ graph can be “simplified” to $2 \to 2$ in different ways:



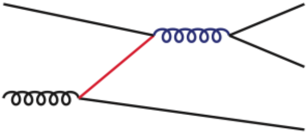
$g \to q\bar{q} \oplus qg \to qg$ $\quad$ $g \to gg \oplus gg \to q\bar{q}$

or deform



FSR

to



ISR

Do not doublecount: $2 \to 2$ = most virtual = shortest distance

Conflict: theory derivations assume virtualities strongly ordered; interesting physics often in regions where this is not true!

Timelike shower: the prototype

For the rest of this lecture, consider specifically

Final State Radiation

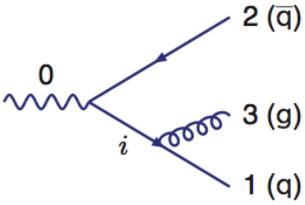
Timelike shower: the prototype

For the rest of this lecture, consider specifically

Final State Radiation

Standard process: $\boxed{\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}\overline{\mathrm{q}}\mathrm{g}}$ (or $\mathrm{q}\overline{\mathrm{q}} \to \gamma^*/\mathrm{Z}^0/\mathrm{W}^\pm \to \mathrm{q}\overline{\mathrm{q}}\mathrm{g}$).

Contributions from two Feynman diagrams:



For simplicity neglect quark masses.
Flavour dependence and mix of γ^* and Z^0 propagators divides out in ratio $\sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}\overline{\mathrm{q}}\mathrm{g})/\sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}\overline{\mathrm{q}})$.

The phase space – 1

Define $p_0 = p_1 + p_2 + p_3$, and

$$x_i = \frac{2 p_0 p_i}{p_0^2} = \frac{2 E_{\mathrm{cm}} E_i}{E_{\mathrm{cm}}^2} = \frac{2 E_i}{E_{\mathrm{cm}}}$$

so that $x_1 + x_2 + x_3 = 2$. Also note that

$$y_{jk} = \frac{m_{jk}^2}{E_{\mathrm{cm}}^2} = \frac{(p_j + p_k)^2}{p_0^2} = \frac{(p_0 - p_i)^2}{p_0^2} = \frac{p_0^2 - 2 p_0 p_i + p_i^2}{p_0^2} = 1 - x_i$$

with i, j, k permutation of $1, 2, 3$.
Hence $y_{23} + y_{13} + y_{12} = 3 - (x_1 + x_2 + x_3) = 1$.

Limit $x_1, x_2 \to 1 \Leftrightarrow y_{23}, y_{13} \to 0 \Leftrightarrow m_{\overline{\mathrm{q}}\mathrm{g}}^2, m_{\mathrm{qg}}^2 \to 0$,
i.e. propagator for $\overline{\mathrm{q}} \to \overline{\mathrm{q}}\mathrm{g}, \mathrm{q} \to \mathrm{qg}$ goes on shell.

The phase space – 2

Since $0 \leq y_{jk} \leq 1$ and $x_i = 1 - y_{jk}$ it follows that $0 \leq x_i \leq 1$.
Also obvious from momentum conservation:

$$\mathbf{p}_i + \mathbf{p}_j + \mathbf{p}_k = \mathbf{0}$$
$$\Rightarrow |\mathbf{p}_i| = |-(\mathbf{p}_j + \mathbf{p}_k)| \leq |\mathbf{p}_j| + |\mathbf{p}_k|$$
$$\Leftrightarrow E_i \leq E_j + E_k \Leftrightarrow x_i \leq x_j + x_k$$
$$\Rightarrow x_i \leq (x_i + x_j + x_k)/2 = 1.$$

The phase space – 2

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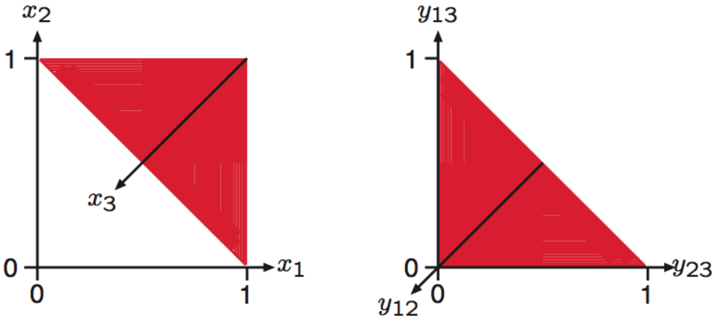
$\Leftrightarrow E_i \le E_j + E_k \Leftrightarrow x_i \le x_j + x_k$
$\Rightarrow x_i \le (x_i + x_j + x_k)/2 = 1.$

Allowed triangular region of phase space:



The matrix elements – 1

$$
\begin{aligned}
\sigma_0 &= \sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}\overline{\mathrm{q}}) = \frac{4\pi\alpha_{\mathrm{em}}^2}{3} N_c \left\{ \frac{q_f^2}{s} \right. \\
&- \left. \frac{2q_f v_f}{4\sin^2\theta_W} \frac{s - m_{\mathrm{Z}}^2}{(s - m_{\mathrm{Z}}^2)^2 + m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2} + \frac{v_f^2 + a_f^2}{(4\sin^2\theta_W)^2} \frac{s}{(s - m_{\mathrm{Z}}^2)^2 + m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2} \right\}
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}\sigma_{\mathrm{ME}}}{\sigma_0} = \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{4}{3} \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} \mathrm{d}x_1 \, \mathrm{d}x_2}
$$

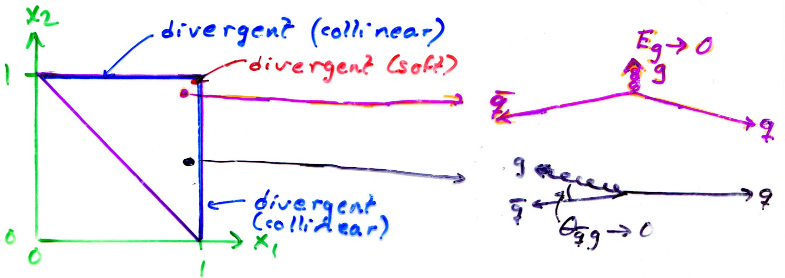
Note on Phase Space:

$$
\begin{aligned}
\mathrm{dPS}_2 &\propto \mathrm{d}\cos\theta_{\mathrm{q}} \, \mathrm{d}\phi_{\mathrm{q}} \\
\mathrm{dPS}_3 &\propto \mathrm{d}\cos\theta_{\mathrm{q}} \, \mathrm{d}\phi_{\mathrm{q}} \, \mathrm{d}\chi_{\mathrm{qg}} \, \mathrm{d}x_1 \, \mathrm{d}x_2
\end{aligned}
$$

Since $x_1, x_2 \to 1$ dominates, $x_1^2 + x_2^2 \approx 2$, and

$$
\frac{\mathrm{d}\sigma_{\mathrm{ME}}}{\sigma_0} \approx \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{4}{3} \frac{2 \, \mathrm{d}x_1 \, \mathrm{d}x_2}{(1-x_1)(1-x_2)} = \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{8}{3} \frac{\mathrm{d}y_{23} \, \mathrm{d}y_{13}}{y_{23} y_{13}} = \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{8}{3} \frac{\mathrm{d}m_{23}^2 \, \mathrm{d}m_{13}^2}{m_{23}^2 m_{13}^2}
$$

The matrix elements – 2



Two “kinds” of singularities, collinear and soft, closely interlinked.

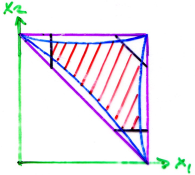
Convenient (but arbitrary) subdivision:

$$\frac{1}{(1-x_1)(1-x_2)} \frac{(1-x_1)+(1-x_2)}{x_3} = \frac{1}{(1-x_2)x_3} + \frac{1}{(1-x_1)x_3}$$

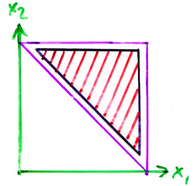
so split into radiation “from” q and “from” $\overline{\mathrm{q}}$

$$\frac{\mathrm{d}\sigma_{\mathrm{ME}}}{\sigma_0} \approx \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{8}{3} \left(\frac{\mathrm{d}m_{13}^2}{m_{13}^2} \frac{\mathrm{d}x_3}{x_3} + \frac{\mathrm{d}m_{23}^2}{m_{23}^2} \frac{\mathrm{d}x_3}{x_3} \right)$$

Divergences necessitates introduction of cuts for physical rate:



(ϵ, δ) cuts (Sterman–Weinberg)
$x_i < 1 - \epsilon$, $\theta_{ij} > \delta$
"all but a fraction ϵ of the energy is within cones of size δ"



y cut: $y_{ij} > y_{\mathrm{min}}$
Also called Thrust cut, where
$T = \max(x_1, x_2, x_3) = 1 - \min(y_{12}, y_{13}, y_{23})$
for three-jets.

$$\int_{y_{\mathrm{min}}} \frac{\mathrm{d}\sigma_{\mathrm{ME}}}{\sigma_0} \approx \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{8}{3} \int_{y_{\mathrm{min}}} \frac{\mathrm{d}y_{23}\,\mathrm{d}y_{13}}{y_{23}\,y_{13}} \approx \frac{4\alpha_{\mathrm{s}}}{3\pi} \ln^2 y_{\mathrm{min}}$$

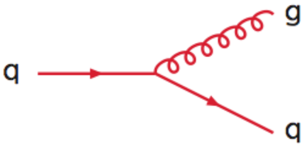
i.e. double logarithmic divergence.

From matrix elements to parton showers

Rewrite for $x_2 \to 1$, i.e. q–g collinear limit:

$$1 - x_2 = \frac{m_{13}^2}{E_{\mathrm{cm}}^2} = \frac{Q^2}{E_{\mathrm{cm}}^2} \Rightarrow \mathrm{d}x_2 = \frac{\mathrm{d}Q^2}{E_{\mathrm{cm}}^2}$$

define z as fraction q retains in branching $\mathrm{q} \to \mathrm{qg}$



$$\begin{aligned} x_1 &\approx z \Rightarrow \mathrm{d}x_1 \approx \mathrm{d}z \\ x_3 &\approx 1 - z \end{aligned}$$

$$\Rightarrow \mathrm{d}\mathcal{P} = \frac{\mathrm{d}\sigma}{\sigma_0} = \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{\mathrm{d}x_2}{(1 - x_2)} \frac{4}{3} \frac{x_2^2 + x_1^2}{(1 - x_1)} \mathrm{d}x_1 \approx \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{\mathrm{d}Q^2}{Q^2} \frac{4}{3} \frac{1 + z^2}{1 - z} \mathrm{d}z$$

In limit $x_1 \to 1$ same result, but for $\overline{\mathrm{q}} \to \overline{\mathrm{q}}\mathrm{g}$.

$\mathrm{d}Q^2/Q^2 = \mathrm{d}m^2/m^2$: "mass (or collinear) singularity"

$\mathrm{d}z/(1 - z) = \mathrm{d}\omega/\omega$ "soft singularity"

The DGLAP equations

Generalizes to

DGLAP (Dokshitzer–Gribov–Lipatov–Altarelli–Parisi)

$$
\begin{aligned}
\mathrm{d}\mathcal{P}_{a\to bc} &= \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z \\
P_{\mathrm{q}\to\mathrm{qg}} &= \frac{4}{3}\,\frac{1+z^2}{1-z} \\
P_{\mathrm{g}\to\mathrm{gg}} &= 3\,\frac{(1-z(1-z))^2}{z(1-z)} \\
P_{\mathrm{g}\to\mathrm{q\overline{q}}} &= \frac{n_f}{2}\,(z^2+(1-z)^2) \quad (n_f = \text{no. of quark flavours})
\end{aligned}
$$

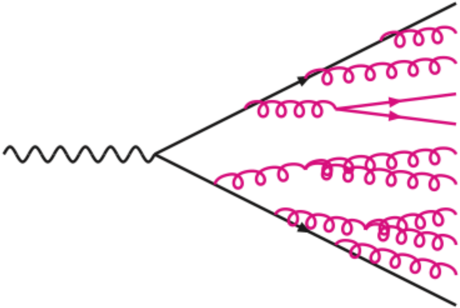
Universality: any matrix element reduces to DGLAP in collinear limit.

$$
\text{e.g.}\quad \frac{\mathrm{d}\sigma(\mathrm{H}^0\to\mathrm{q\overline{q}g})}{\mathrm{d}\sigma(\mathrm{H}^0\to\mathrm{q\overline{q}})} = \frac{\mathrm{d}\sigma(\mathrm{Z}^0\to\mathrm{q\overline{q}g})}{\mathrm{d}\sigma(\mathrm{Z}^0\to\mathrm{q\overline{q}})} \quad \text{in collinear limit}
$$

The iterative structure

Generalizes to many consecutive emissions if strongly ordered,
$Q_1^2 \gg Q_2^2 \gg Q_3^2 \ldots$ ($\approx$ time-ordered).
To cover "all" of phase space use DGLAP in whole region
$Q_1^2 > Q_2^2 > Q_3^2 \ldots$.

Iteration gives final-state parton showers:



Need soft/collinear cuts to stay away from nonperturbative physics.
Details model-dependent, but around 1 GeV scale.

The ordering variable

In the evolution with

$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z$$

- Q^2 is a "temporal" variable: ordering of emissions (memory)
- z is a "spatial" variable: no ordering (no memory)

If $Q^2 = m^2$ is one possible evolution variable
then $Q'^2 = f(z)Q^2$ is also allowed, since

$$\left|\frac{\mathrm{d}(Q'^2, z)}{\mathrm{d}(Q^2, z)}\right| = \left|\begin{array}{cc} \frac{\partial Q'^2}{\partial Q^2} & \frac{\partial Q'^2}{\partial z} \\ \frac{\partial z}{\partial Q^2} & \frac{\partial z}{\partial z} \end{array}\right| = \left|\begin{array}{cc} f(z) & f'(z)Q^2 \\ 0 & 1 \end{array}\right| = f(z)$$

$$\Rightarrow \mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{f(z)\mathrm{d}Q^2}{f(z)Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{\mathrm{d}Q'^2}{Q'^2}\,P_{a\to bc}(z)\,\mathrm{d}z$$

- $Q'^2 = E_a^2\theta_{a\to bc}^2 \approx m^2/(z(1-z))$; angular-ordered shower
- $Q'^2 = p_\perp^2 \approx m^2 z(1-z)$; transverse-momentum-ordered

The Sudakov form factor – 1

(Back to radioactive decay:) Conservation of total probability:
$\mathcal{P}(\text{no decay}) = 1 - \mathcal{P}(\text{decay})$
"multiplicativeness" in "time" evolution:
$\mathcal{P}_{\text{no}}(0 < t \leq T) = \mathcal{P}_{\text{no}}(0 < t \leq T_1)\, \mathcal{P}_{\text{no}}(T_1 < t \leq T)$

Subdivide further, with $T_i = (i/n)T$, $0 \leq i \leq n$:

$$
\begin{aligned}
\mathcal{P}_{\text{no}}(0 < t \leq T) &= \lim_{n \to \infty} \prod_{i=0}^{n-1} \mathcal{P}_{\text{no}}(T_i < t \leq T_{i+1}) \\
&= \lim_{n \to \infty} \prod_{i=0}^{n-1} (1 - \mathcal{P}_{\text{decay}}(T_i < t \leq T_{i+1})) \\
&= \exp\left(- \lim_{n \to \infty} \sum_{i=0}^{n-1} \mathcal{P}_{\text{decay}}(T_i < t \leq T_{i+1}) \right) \\
&= \exp\left(- \int_0^T \frac{\mathrm{d}\mathcal{P}_{\text{decay}}(t)}{\mathrm{d}t} \mathrm{d}t \right) \\
\Longrightarrow \quad \mathrm{d}\mathcal{P}_{\text{first}}(T) &= \mathrm{d}\mathcal{P}_{\text{decay}}(T) \exp\left(- \int_0^T \frac{\mathrm{d}\mathcal{P}_{\text{decay}}(t)}{\mathrm{d}t} \mathrm{d}t \right)
\end{aligned}
$$

The Sudakov form factor – 2

Correspondingly, with $Q \sim 1/t$ (Heisenberg)

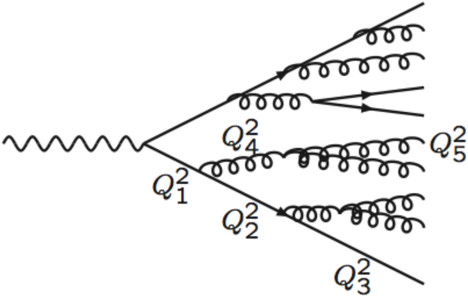
$$\begin{aligned} \mathrm{d}\mathcal{P}_{a\to bc} &= \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z \\ & \quad \exp\left(-\sum_{b,c}\int_{Q^2}^{Q^2_{\mathrm{max}}}\frac{\mathrm{d}Q'^2}{Q'^2}\int\frac{\alpha_{\mathrm{s}}}{2\pi}\,P_{a\to bc}(z')\,\mathrm{d}z'\right) \end{aligned}$$

where the exponent is (one definition of) the Sudakov form factor

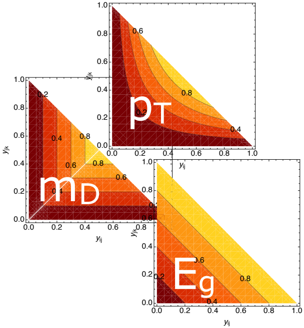
A given parton can only branch once, i.e. if it did not already do so

Note that $\sum_{b,c}\int\int\ \mathrm{d}\mathcal{P}_{a\to bc} \equiv 1 \Rightarrow$ *convenient for Monte Carlo*
($\equiv 1$ if extended over whole phase space, else possibly nothing happens before you reach $Q_0 \approx 1$ GeV).

More common/strict definition of Sudakov form factor based on $\int_{Q_0^2}^{Q^2}$ instead of $\int_{Q^2}^{Q^2_{\mathrm{max}}}$.

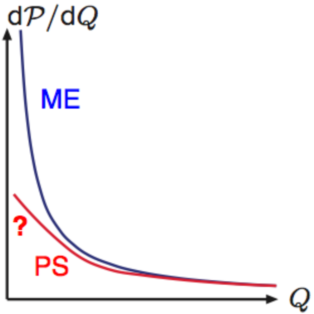


Sudakov form factor provides
“time” ordering of shower:
lower $Q^2 \Longleftrightarrow$ longer times
$Q_1^2 > Q_2^2 > Q_3^2$
$Q_1^2 > Q_4^2 > Q_5^2$
etc.



The Sudakov form factor – 4

Sudakov regulates singularity for *first* emission ...



... but in limit of *repeated soft* emissions $\mathrm{q} \to \mathrm{qg}$ (but no $\mathrm{g} \to \mathrm{gg}$) one obtains the same inclusive Q emission spectrum as for ME,
i.e. divergent ME spectrum
$\Longleftrightarrow$ infinite number of PS emissions
Proof: as for veto algorithm (what is probability to have an emission at Q after 0, 1, 2, 3, ... previous ones?)

More complicated in reality:

- energy-momentum conservation effects big since α_{s} big, so hard emissions frequent
- $\mathrm{g} \to \mathrm{gg}$ branchings leads to accelerated multiplication of partons

Matrix elements and the Sudakov

Recall $\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}\overline{\mathrm{q}}\mathrm{g}$, with phase space slicing at y:

$$\begin{aligned}
\frac{\sigma_{\mathrm{R}}(y)}{\sigma_0} &\approx \frac{4\alpha_{\mathrm{s}}}{3\pi} \ln^2 y \\
\frac{\sigma_{\mathrm{V}}(y)}{\sigma_0} &\approx \frac{\alpha_{\mathrm{s}}}{\pi} - \frac{4\alpha_{\mathrm{s}}}{3\pi} \ln^2 y \\
\sigma_0 + \sigma_{\mathrm{R}}(y) + \sigma_{\mathrm{V}}(y) &= \left(1 + \frac{\alpha_{\mathrm{s}}}{\pi}\right) \sigma_0
\end{aligned}$$

If small finite term α_{s}/π neglected then $\boxed{\sigma_{\mathrm{V}}(y) = -\sigma_{\mathrm{R}}(y)}$

If shower emissions ordered in y then

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}y} &= \frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_{\mathrm{R}}}{\mathrm{d}y} \exp\left(-\int_y^1 \frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_{\mathrm{R}}}{\mathrm{d}y'}\,\mathrm{d}y'\right) \\
&= \frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_{\mathrm{R}}}{\mathrm{d}y} \exp\left(\frac{\sigma_{\mathrm{V}}(y)}{\sigma_0}\right) \\
&= \frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_{\mathrm{R}}}{\mathrm{d}y} \left(1 + \frac{\sigma_{\mathrm{V}}(y)}{\sigma_0} + \cdots\right)
\end{aligned}$$

Veto algorithm crucial for practical implementation

- Allowed Q^2-dependent range $z_{\min}(Q^2) \leq z \leq z_{\max}(Q^2)$ overestimated by maximum range $z_{\min}(Q_0^2) \leq z \leq z_{\max}(Q_0^2)$.
- Splitting functions overestimated to simplify, e.g. $1 + z^2 \leq 2$.
- For now assume α_{s} fixed, so combine to upper estimate $I_z \simeq (\alpha_{\mathrm{s}}/2\pi) \int P(z)\,\mathrm{d}z$
- Splitting probability overestimated by

$$I_z \frac{\mathrm{d}Q^2}{Q^2} \exp\left(- \int_{Q^2}^{Q^2_{\max}} I_z \frac{\mathrm{d}Q'^2}{Q'^2} \right)$$

- "Radioactive decays" formula gives solution

$$\exp\left(-I_z \ln \frac{Q^2_{\max}}{Q^2} \right) = \left(\frac{Q^2}{Q^2_{\max}} \right)^{I_z} = R \quad \Rightarrow \quad Q^2 = Q^2_{\max} R^{1/I_z}$$

Shower evolution generation – 2

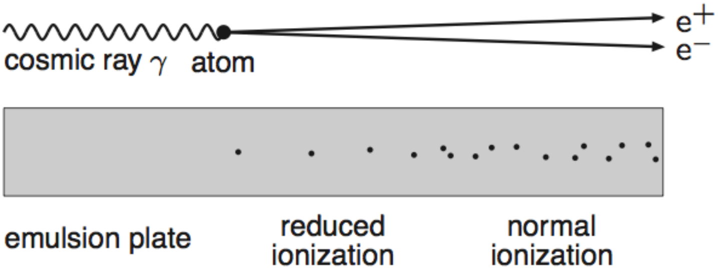
- If $Q < Q_0$ then stop evolution
- Pick z in overestimated range with overestimated P, e.g.

$$\int_{z_{\min}(Q_0^2)}^{z} \frac{2\,\mathrm{d}z'}{1-z'} = R \int_{z_{\min}(Q_0^2)}^{z_{\max}(Q_0^2)} \frac{2\,\mathrm{d}z'}{1-z'}$$

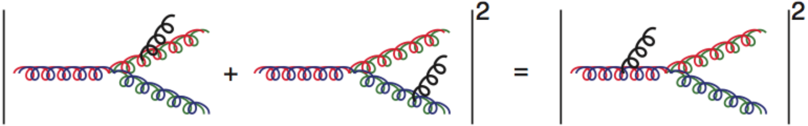
- If z outside allowed range for chosen Q^2 or if $P_{\mathrm{true}}(z)/P_{\mathrm{overestimation}}(z) < R$ then put $Q^2_{\max} = Q^2$ and continue downwards evolution (veto algorithm)
- If two possible channels, like $\mathrm{g} \to \mathrm{gg}$ and $\mathrm{g} \to \mathrm{q\overline{q}}$, then combined overestimation or use "the winner take it all", i.e. pick one Q^2 scale for each channel and retain the largest
- If $\alpha_{\mathrm{s}} = \alpha_{\mathrm{s}}(Q^2) \propto 1/\ln(Q^2/\Lambda^2)$ then use that

$$\int_{Q^2}^{Q^2_{\max}} \frac{1}{\ln \frac{Q'^2}{\Lambda^2}} \frac{\mathrm{d}Q'^2}{Q'^2} = \int_{Q^2}^{Q^2_{\max}} \frac{\mathrm{d}\ln \frac{Q'^2}{\Lambda^2}}{\ln \frac{Q'^2}{\Lambda^2}} = \ln\ln \frac{Q^2_{\max}}{\Lambda^2} - \ln\ln \frac{Q^2}{\Lambda^2}$$

QED: Chudakov effect (mid-fifties)



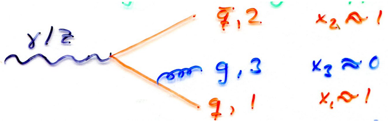
QCD: colour coherence for **soft** gluon emission



solved by • requiring emission angles to be decreasing
or • requiring transverse momenta to be decreasing

Coherence – 2

Simple illustration of principles: consider emission of soft gluon off fast moving colour singlet $\mathrm{q}\overline{\mathrm{q}}$ system:



$$\text{recall} \quad \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2} \propto \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \sim \frac{2}{(1-x_1)(1-x_2)}$$

$$1-x_1 = y_{23} = \frac{(p_2+p_3)^2}{(p_1+p_2+p_3)^2} \approx \frac{2p_2p_3}{2p_1p_2} = \frac{E_2E_3(1-\cos\theta_{23})}{E_1E_2(1-\cos\theta_{12})} = \frac{E_3a_{23}}{E_1a_{12}}$$

with $a_{ij} = 1 - \cos\theta_{ij}$.

Recasting matrix element in gluon momentum variables:

$$\frac{\mathrm{d}\sigma}{\sigma} \propto \frac{\mathrm{d}E_3}{E_3}\,\mathrm{d}\Omega_3\,\frac{a_{12}}{a_{13}a_{23}}$$

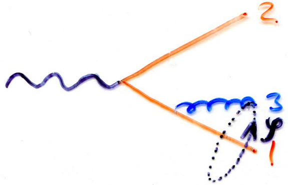
which is singular both for $a_{13} \to 0$ and $a_{23} \to 0$.

Coherence – 3

Rewrite angular dependence

$$\frac{2a_{12}}{a_{13}a_{23}} = \frac{2a_{12} - a_{13} - a_{23}}{a_{13}a_{23}} + \frac{1}{a_{13}} + \frac{1}{a_{23}}$$
$$= \frac{1}{a_{13}}\left(1 + \frac{a_{12} - a_{13}}{a_{23}}\right) + \frac{1}{a_{23}}\left(1 + \frac{a_{12} - a_{23}}{a_{13}}\right)$$

where $(a_{12} - a_{13})/a_{23}$ is not singular:
$a_{23} \to 0 \Rightarrow$ 2 and 3 become parallel $\Rightarrow a_{12} - a_{13} \to 0$.

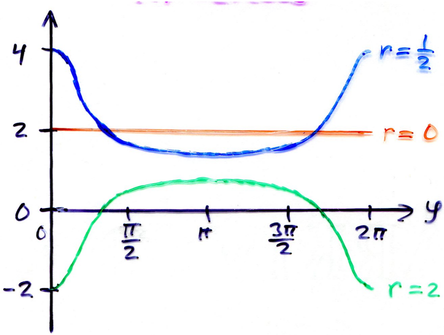


$$\cos\theta_{23} = \sin\theta_{12}\,\sin\theta_{13}\,\cos\varphi + \cos\theta_{12}\,\cos\theta_{13}$$

Also $\theta_{ij} \to 0 \Rightarrow a_{ij} = 1 - \cos\theta_{ij} \approx \theta_{ij}^2/2$.

$$1+\frac{a_{12}-a_{13}}{a_{23}} \approx 1+\frac{\theta_{12}^2-\theta_{13}^2}{\theta_{12}^2+\theta_{13}^2-\theta_{12}\theta_{13}\cos\varphi} = 1+\frac{1-r^2}{1+r^2-2r\cos\varphi}$$

with $r=\theta_{13}/\theta_{12}$.

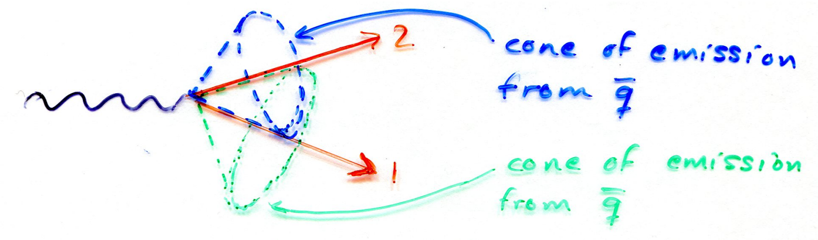


$$\int_0^{2\pi}\left(1+\frac{1-r^2}{1+r^2-2r\cos\varphi}\right)\mathrm{d}\varphi = 2\pi\left(1+\mathrm{sign}(1-r^2)\right) = 4\pi\Theta(1-r)$$

i.e. $= 4\pi$ for $r<1$ but $=0$ for $r>1$.

Thus, averaged over φ

$$\frac{\mathrm{d}\sigma}{\sigma} \propto \frac{\mathrm{d}E_3}{E_3}\,\mathrm{d}\Omega_3 \left\{ \frac{2}{\theta_{13}^2}\Theta(\theta_{12}-\theta_{13}) + \frac{2}{\theta_{23}^2}\Theta(\theta_{12}-\theta_{23}) \right\}$$



Reasoning above generalizes also when more than two partons may radiate.

Key consequences of angular ordering:

- slower multiplicity growth of partons (and thus of hadrons),
- depletion of particle production at small $x = E/E_{\mathrm{jet}}$.

Derivation of soft radiation pattern for generic topology:
Soft gluons = long wavelength.
Thus internal lines of hard processes are not resolved,
i.e. only external lines contribute,
and these may be treated as classical currents:

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \, \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega} \, \left| \sum_{i=1}^{n} g_s \, \mathbf{T}_i \, \frac{p_i}{p_i k} \right|^2$$

where $\mathbf{T}_i$ are colour charge matrices, $\omega = |\mathbf{k}|$ and

$$\frac{p_i}{p_i k} = \frac{E_i \, (1, \boldsymbol{\beta}_i)}{E_i \, (1, \boldsymbol{\beta}_i) \, \omega \, (1, \mathbf{n})} = \frac{1}{\omega} \, \frac{(1, \boldsymbol{\beta}_i)}{1 - \mathbf{n} \boldsymbol{\beta}_i}$$

is $\propto (\phi, \mathbf{A})$ of a moving charge (cf. Jackson).

The eikonal approximation – 2

Expand square

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \, \frac{\mathrm{d}^3 k}{2\pi\omega} \, \frac{\alpha_\mathrm{s}}{2\pi} \sum_{i,j} (-\mathbf{T}_i \mathbf{T}_j) \, \frac{p_i p_j}{(p_i k)(p_j k)}$$

Note that for $p_i^2 = m_i^2 = 0$ the $i = j$ terms vanish,
the radiation is not by a single charge but by a dipole.
E.g.

$$\begin{aligned}
\frac{\mathrm{d}\sigma_3}{\sigma_2} &= \frac{\mathrm{d}^3 p_3}{2\pi E_3} \, \frac{\alpha_\mathrm{s}}{2\pi} \, \frac{4}{3} \, \frac{p_1 p_2}{(p_1 p_3)(p_2 p_3)} \, 2 \\
&= \frac{\alpha_\mathrm{s}}{2\pi} \, \frac{4}{3} \, \frac{s(1-x_3)/2}{s^2(1-x_2)(1-x_1)/4} \, 2 \frac{\mathrm{d}^3 p_3}{2\pi E_3} \\
&\approx \frac{\alpha_\mathrm{s}}{2\pi} \, \frac{4}{3} \, \frac{2}{(1-x_1)(1-x_2)} \, \mathrm{d}x_1 \, \mathrm{d}x_2
\end{aligned}$$

The eikonal approximation – 3

Generalizes to higher multiplicities,
e.g. $\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}(1)\,\overline{\mathrm{q}}(2)\,\mathrm{g}(3)\,\mathrm{g}(k)$:

$$\begin{aligned}\frac{\mathrm{d}\sigma_4}{\sigma_3} &\propto \frac{\alpha_\mathrm{s}}{2\pi}\left\{(-\mathbf{T}_1\mathbf{T}_3)\,\widehat{13} + (-\mathbf{T}_2\mathbf{T}_3)\,\widehat{23} + (-\mathbf{T}_1\mathbf{T}_2)\,\widehat{12}\right\}\frac{\mathrm{d}^3k}{2\pi\omega} \\ &\propto \frac{\alpha_\mathrm{s}}{2\pi}\left\{\widehat{13} + \widehat{23} - \frac{1}{N_C^2}\,\widehat{12}\right\}\frac{\mathrm{d}^3k}{2\pi\omega}\end{aligned}$$

where

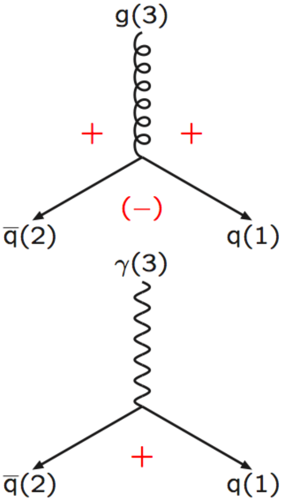
$$\widehat{ij} = \frac{p_i p_j}{(p_i k)(p_j k)}$$

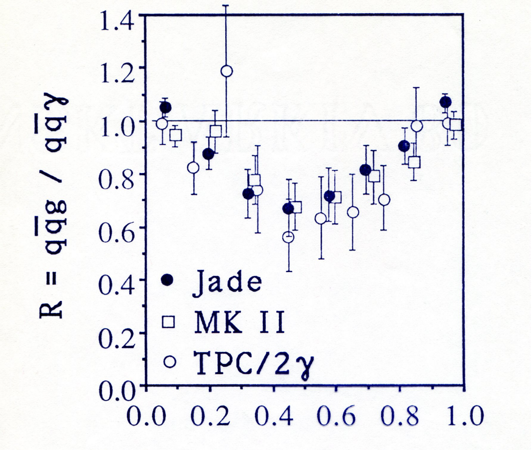
encode radiating dipoles.

Neglecting $1/N_c^2$-suppressed terms, radiation can be viewed as coming from two "independent" dipoles, qg and $\overline{\mathrm{q}}\mathrm{g}$.

The eikonal approximation – 4

Compare with $\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}(1)\,\overline{\mathrm{q}}(2)\,\gamma(3)\,\mathrm{g}(k)$:
the γ does not carry colour, so here only $\mathrm{q}\overline{\mathrm{q}}$ dipole.





x axis: angular range between q and $\overline{\mathrm{q}}$.

Scale choice

Branching rate of $a \to bc$ is proportional to $\alpha_{\mathrm{s}}(Q^2)$.
Naive guess $Q^2 = m_a^2$.
Then next-to-leading branching kernel in limit $z \to 1$ (soft g)

$$
\begin{aligned}
\frac{\alpha_{\mathrm{s}}}{2\pi} P_{\mathrm{q}\to\mathrm{qg}}(z) &\to \frac{\alpha_{\mathrm{s}}}{2\pi} \left\{ P_{\mathrm{q}\to\mathrm{qg}}(z) + \frac{\alpha_{\mathrm{s}}}{2\pi} P'_{\mathrm{q}\to\mathrm{qg}}(z) \right\} \\
&\approx \frac{\alpha_{\mathrm{s}}(m_a^2)}{2\pi} \left\{ 1 - \frac{\ln(1-z)}{\ln(m_a^2/\Lambda^2)} \right\} P_{\mathrm{q}\to\mathrm{qg}}(z) \\
&\approx \frac{\alpha_{\mathrm{s}}((1-z)m_a^2)}{2\pi} P_{\mathrm{q}\to\mathrm{qg}}(z) \approx \frac{\alpha_{\mathrm{s}}(p_\perp^2)}{2\pi} P_{\mathrm{q}\to\mathrm{qg}}(z)
\end{aligned}
$$

Corresponding results hold for $\mathrm{g} \to \mathrm{gg}$, $z \to 0$ and $z \to 1$,
so natural to pick generically $\boxed{Q^2 = p_\perp^2 = z(1-z)m_a^2}$.
Does not absorb finite corrections for non-singular z,
so not complete NLO, but part of the way.

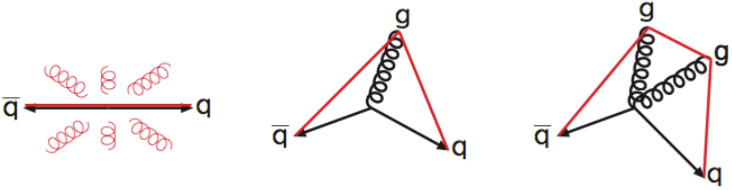
The Common Showering Algorithms (LEP era)

Three main approaches to showering in common use:

Two are based on the standard shower language
of $a \to bc$ successive branchings:

HERWIG: $Q^2 \approx E^2(1-\cos\theta) \approx E^2\theta^2/2$
PYTHIA: $Q^2 = m^2$ (timelike) or $= -m^2$ (spacelike)

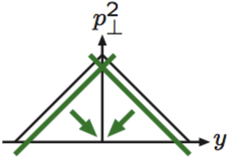
One is based on a picture of dipole emission $ab \to cde$:



ARIADNE: $Q^2 = p_\perp^2$; FSR mainly, ISR is primitive

Ordering variables in final-state radiation (LEP era)

PYTHIA: $Q^2 = m^2$



large mass first

$\Rightarrow$ "hardness" ordered

coherence brute force

covers phase space

ME merging simple

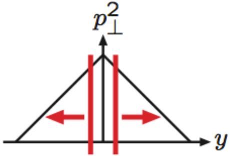
$\mathrm{g} \to \mathrm{q}\overline{\mathrm{q}}$ simple

not Lorentz invariant

no stop/restart

ISR: $m^2 \to -m^2$

HERWIG: $Q^2 \sim E^2\theta^2$



large angle first

$\Rightarrow$ **hardness not ordered**

coherence inherent

gaps in coverage

ME merging messy

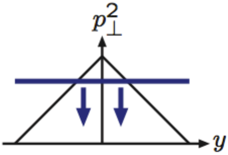
$\mathrm{g} \to \mathrm{q}\overline{\mathrm{q}}$ simple

not Lorentz invariant

no stop/restart

ISR: $\theta \to \theta$

ARIADNE: $Q^2 = p_\perp^2$



large $p_\perp$ first

$\Rightarrow$ "hardness" ordered

coherence inherent

covers phase space

ME merging simple

$\mathrm{g} \to \mathrm{q}\overline{\mathrm{q}}$ **messy**

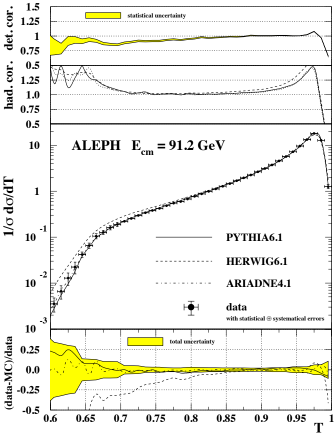
Lorentz invariant

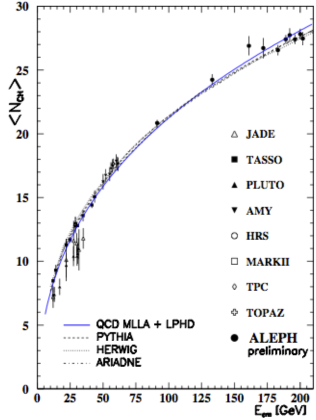
can stop/restart

ISR: more messy

Data comparisons (LEP)

All three algorithms do a reasonable job of describing LEP data, but typically ARIADNE ($p_\perp^2$) > PYTHIA (m^2) > HERWIG (θ)





...and programs evolve to do even better ...

The HERWIG algorithm

Basic ideas, to which much has been added over the years:

1. Evolution in $Q_a^2 = E_a^2 \xi_a$ with $\xi_a \approx 1 - \cos\theta_a$, i.e.

$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_\mathrm{s}}{2\pi} \frac{\mathrm{d}(E_a^2\xi_a)}{E_a^2\xi_a} P_{a\to bc}(z)\,\mathrm{d}z = \frac{\alpha_\mathrm{s}}{2\pi} \frac{\mathrm{d}\xi_a}{\xi_a} P_{a\to bc}(z)\,\mathrm{d}z$$

   Require ordering of consecutive ξ values, i.e. $(\xi_b)_\mathrm{max} < \xi_a$ and $(\xi_c)_\mathrm{max} < \xi_a$.

2. Reconstruct masses backwards in algorithm
   $m_a^2 = m_b^2 + m_c^2 + 2E_bE_c\xi_a$
   Note: $\xi_a = 1 - \cos\theta_a$ only holds for $m_b = m_c = 0$.

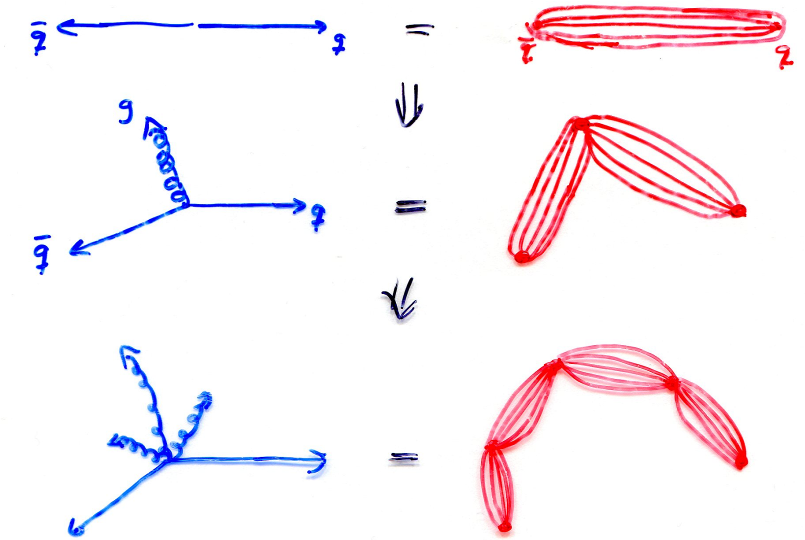
3. Reconstruct complete kinematics of shower (forward again).

\+ angular ordering built in from start

− total jet/system mass not known beforehand ($\Rightarrow$ boosts)

− some wide-angle regions never populated, "dead zones" partonic picture

equivalent dipole picture



The ARIADNE algorithm – 2

- Dual description of topology:
  a dipole connects two partons
  a gluon connects two dipoles
- Dual description of particle production
  shower: parton $\to$ parton $+$ parton
  dipole: dipole $\to$ dipole $+$ dipole
- Use effective 3-jet matrix elements
  in rest frame of each new dipole
- Evolve downwards in $p_\perp$
  $\Rightarrow$ angular ordering for free
- No need to assign virtualities to partons;
  $(E, \mathbf{p})$ conserved inside dipole
- $(\mathrm{qg}) \to (\mathrm{qg}) + (\mathrm{gg})$ and $(\mathrm{gg}) \to (\mathrm{gg}) + (\mathrm{gg})$ standard fare;
  generalization of $\mathrm{g} \to \mathrm{q}\overline{\mathrm{q}}$ nontrivial

The ARIADNE algorithm – 3

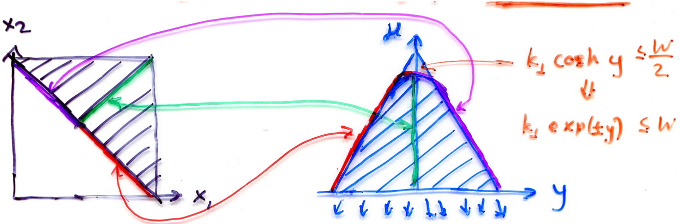
Transform 3-jet variables from (x_1, x_2) to (κ, y) where

$$e^{\kappa} = p_{\perp}^2 = E_3^2 = \frac{(E_1 E_3)(E_2 E_3)}{E_1 E_2} \approx \frac{(p_1 p_3)(p_2 p_3)}{s/4} = s(1-x_1)(1-x_2)$$

(in frame with q and $\overline{\mathrm{q}}$ opposite and g at 90°)

$$y \approx \frac{1}{2} \ln \frac{(E+p_z)_{\mathrm{g}}}{(E-p_z)_{\mathrm{g}}} \approx \frac{1}{2} \ln \frac{(1-x_1)(E+p_z)_{\mathrm{q}}}{(1-x_2)(E-p_z)_{\overline{\mathrm{q}}}} = \frac{1}{2} \ln \frac{1-x_1}{1-x_2}$$

$$\frac{\mathrm{d}\sigma}{\sigma} \approx \frac{8}{3} \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(1-x_1)(1-x_2)} = \frac{8}{3} \frac{\alpha_{\mathrm{s}}(p_{\perp}^2)}{2\pi} \frac{\mathrm{d}p_{\perp}^2}{p_{\perp}^2}\,\mathrm{d}y = \frac{4\alpha_{\mathrm{s}}(p_{\perp}^2)}{3\pi}\,\mathrm{d}\kappa\,\mathrm{d}y$$



Comments on dipole showers – 1 (Frank Krauss)

Features of dipole showers

- Quantum coherence on similar grounds for angular and k_T-ordering, typical ordering in dipole showers by $k_\perp$.
- Many new shower formulations in past few years, many (nearly all) based on dipoles in one way or the other.
- Seemingly closer link to NLO calculations: Use subtraction kernels like antennae or Catani-Seymour kernels.
- Typically: First emission fully accounted for.

Comments on dipole showers – 2 (Frank Krauss)

Survey of existing showering tools

Tools	evolution	AO/Coherence
Ariadne	$k_\perp$-ordered	by construction
Herwig	angular ordering	by construction
Herwig++	improved angular ordering	by construction
Pythia	old: virtuality ordered	by hand
	new: $k_\perp$-ordered	by construction
Sherpa	virtuality ordered (like old Pythia)	by hand
	new: $k_\perp$-ordering	by construction
Vincia	$k_\perp$-ordered	by construction

Nowadays also Herwig++/Herwig7 has a dipole shower as an option.

Dipoles and recoils

Consider dipole emission $\mathrm{q\overline{q}} \to \mathrm{q\overline{q}g}$.
Given internal topology, e.g. (x_1, x_2), how is event oriented, i.e. how is recoil of emission shared?

- ARIADNE: minimize $p_{\perp\mathrm{q}}^2 + p_{\perp\overline{\mathrm{q}}}^2$ of new q and $\overline{\mathrm{q}}$ with respect to original $\mathrm{q\overline{q}}$ axis.
- PYTHIA and others: keep either q or $\overline{\mathrm{q}}$ direction fix, e.g. with $\mathcal{P} = m_{\mathrm{qg}}^2/(m_{\mathrm{qg}}^2 + m_{\overline{\mathrm{q}}\mathrm{g}}^2)$ for q to keep its direction.

(In addition isotropic φ angle around original $\mathrm{q\overline{q}}$ axis.)
Such differences carry over to subsequent emissions, with further ambiguities, and can affect final results.
Exemplifies "subleading effects".

Prompt photon production in showers

In shower evolution:

$$
\begin{aligned}
P_{\mathrm{q}\to\mathrm{qg}} &= \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{4}{3}\,\frac{1+z^2}{1-z} \\
P_{\mathrm{q}\to\mathrm{q}\gamma} &= \frac{\alpha_{\mathrm{em}}}{2\pi}\,\mathrm{e}_{\mathrm{q}}^2\,\frac{1+z^2}{1-z} \\
\frac{P_{\mathrm{q}\to\mathrm{q}\gamma}}{P_{\mathrm{q}\to\mathrm{qg}}} &= \frac{\alpha_{\mathrm{em}}\,\mathrm{e}_{\mathrm{q}}^2}{\alpha_{\mathrm{s}}\,\frac{4}{3}} \approx \frac{1}{200}
\end{aligned}
$$

but

- no direct emission from $\mathrm{g} \Rightarrow$ dilution
- γ uncharged $\Rightarrow$ no coherence $\Rightarrow$ more, but negligibly
- at small $p_\perp$ overwhelming background from $\pi^0 \to \gamma\gamma$
- competition with g emission reduces by factor 2–3 relative to if only γ emission from quark
- reasonably well tested/understood at LEP

Leading Log and beyond

Neglecting Sudakovs, rate of one emission is:

$$\mathcal{P}_{\mathrm{q}\to\mathrm{qg}} \approx \int_{Q^2_{\mathrm{max}}}^{Q^2_{\mathrm{max}}} \frac{\mathrm{d}Q^2}{Q^2} \int_{z_{\mathrm{min}}}^{z_{\mathrm{max}}} \mathrm{d}z \, \frac{\alpha_{\mathrm{s}}}{2\pi} \, \frac{4}{3} \, \frac{1+z^2}{1-z} \sim \alpha_{\mathrm{s}} \ln^2$$

Rate for n emissions is of form:

$$\mathcal{P}_{\mathrm{q}\to\mathrm{q}n\mathrm{g}} \sim (\mathcal{P}_{\mathrm{q}\to\mathrm{qg}})^n \sim \alpha_{\mathrm{s}}^n \ln^{2n}$$

Next-to-leading log (NLL): include *all* corrections of type $\alpha_{\mathrm{s}}^n \ln^{2n-1}$

No existing generator completely NLL (?), but

- energy-momentum conservation (and “recoil” effects)
- coherence
- $2/(1-z) \to (1+z^2)/(1-z)$
- scale choice $\alpha_{\mathrm{s}}(p_{\perp}^2)$ absorbs singular terms $\propto \ln z, \ln(1-z)$ in $\mathcal{O}(\alpha_{\mathrm{s}}^2)$ splitting kernels $P_{\mathrm{q}\to\mathrm{qg}}$ and $P_{\mathrm{g}\to\mathrm{gg}}$
- ...

$\Rightarrow$ far better than naive, analytical LL

On the road to NLL showers

Key ingredient to NLL is NLO splitting kernels:

- $1 \to 3$, i.e $\mathrm{q} \to \mathrm{qgg}$, $\mathrm{q} \to \mathrm{qq'\overline{q}'}$, $\mathrm{g} \to \mathrm{ggg}$, $\mathrm{g} \to \mathrm{q\overline{q}g}$;
- virtual corrections to $1 \to 2$ (not positive definite).

Are known in terms of energy sharing, but less well defined in $p_\perp$.

- NLLJET (Kato, Munehisa, $\sim$ 1990): first with NLO splitting kernels for $\mathrm{e^+e^-}$, but other problems.
- VINCIA (Skands et al., 2017): combine $\mathcal{O}(\alpha_\mathrm{s}^2)$-corrected iterated $2 \to 3$ kernels for ordered emissions with tree-level $2 \to 4$ kernels for unordered emissions, but only for $\mathrm{e^+e^-}$.
- DIRE (Höche, Prestel, 2017): dipole shower with NLO splitting kernels, including handling of negative weights, also for hadron–hadron.

Further efforts under way, e.g. for Herwig7.

Also colour effects relevant, including $1/N_C^2$-suppressed terms, studied by Plätzer, Sjödahl and Thorén.