Particle Physics Phenomenology
2. Phase space and matrix elements

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Four-vectors

$$\begin{aligned}
\text{four}-\text{vector}: \quad & p = (E; \mathbf{p}) = (E; p_x, p_y, p_z) \\
\text{vector sum}: \quad & p_1 + p_2 = (E_1 + E_2; \mathbf{p}_1 + \mathbf{p}_2) \\
\text{vector product}: \quad & p_1 p_2 = E_1 E_2 - \mathbf{p}_1 \mathbf{p}_2 \\
& = E_1 E_2 - p_{x1} p_{x2} - p_{y1} p_{y2} - p_{z1} p_{z2} \\
& = E_1 E_2 - |\mathbf{p}_1|\,|\mathbf{p}_2| \cos\theta_{12} \\
\text{square}: \quad & p^2 = E^2 - \mathbf{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 \\
\text{transverse mom.}: \quad & p_\perp = \sqrt{p_x^2 + p_y^2} \\
\text{transverse mass}: \quad & m_\perp = \sqrt{m^2 + p_x^2 + p_y^2} = \sqrt{m^2 + p_\perp^2} \\
& E^2 = m^2 + \mathbf{p}^2 = m^2 + p_\perp^2 + p_z^2 = m_\perp^2 + p_z^2
\end{aligned}$$

Warning: No standard to distinguish $p = (E; p_x, p_y, p_z)$ and $p = |\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$, but usually clear from context.
When we remember, we will try to use $\underline{p} = |\mathbf{p}|$, since $\overline{p} = \mathbf{p}$.

Simple scattering theory and phase space – 1

Transition rate $W_{fi} \propto |\langle \psi_f | U(x) | \psi_i \rangle|^2$

Plane wave ansatz $\psi(\mathbf{x}) = N e^{i(\mathbf{px} - Et)}$ with $\hbar = c = 1$.

Nonrelativistic phase space: box with volume $V = L^3$ suggests normalization $N = 1/\sqrt{V}$ such that $\int_V |\psi(\mathbf{x})|^2 \, \mathrm{d}^3x = 1$.

Periodic boundary conditions $\psi(x + L, y, z) = \psi(x, y, z)$ gives quantization

$$L p_x = 2\pi n_x \quad \Rightarrow \quad \mathrm{d}n_x = \frac{L \mathrm{d}p_x}{2\pi} \quad \Rightarrow \quad \mathrm{d}^3 n = V \, \frac{\mathrm{d}^3 p}{(2\pi)^3}$$

Dependence on V cancels between normalization and phase space for outgoing particles, and similarly for influx normalization, so put $V = 1$ and drop.

But $\mathrm{d}^3 p$ shrinks under Lorentz boosts, i.e. not Lorentz invariant.

Simple scattering theory and phase space – 2

$$\text{Solution}: \psi(x) \to \sqrt{2E}\,\exp(-ipx) \text{ and } \mathrm{d}^3 n \to \frac{1}{(2\pi)^3}\,\frac{\mathrm{d}^3 p}{2E}\ .$$

Now Lorentz invariant, e.g. boost along z axis, so $\mathbf{p}_\perp$ unchanged:

$$\begin{aligned}
p'_z &= \gamma(p_z + \beta E) \\
E' &= \gamma(E + \beta p_z) \\
\frac{\mathrm{d}^3 p'}{E'} &= \mathrm{d}^2 p_\perp\,\frac{\gamma(\mathrm{d}p_z + \beta\mathrm{d}E)}{\gamma(E + \beta p_z)} \\
&= \mathrm{d}^2 p_\perp\,\frac{\gamma(\mathrm{d}p_z + \beta(p_z/E)\mathrm{d}p_z)}{\gamma(E + \beta p_z)} = \mathrm{d}^2 p_\perp\,\frac{\mathrm{d}p_z}{E} = \frac{\mathrm{d}^3 p}{E}\ ,
\end{aligned}$$

using that $E^2 = m_\perp^2 + p_z^2 \Rightarrow E\,\mathrm{d}E = p_z\,\mathrm{d}p_z$.

Also $(\rho, \mathbf{j}) = 2(E, \mathbf{p})|\psi|^2$ is a natural four-vector.

Application: Lorentz invariant production cross sections $E\,\mathrm{d}\sigma/\mathrm{d}^3 p$

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Phase space – 1

n bodies in final state gives one factor per particle:

$$\prod_{j=1}^{n} \frac{1}{(2\pi)^3} \frac{\mathrm{d}^3 p_j}{2E_j}$$

Also energy–momentum conservation, e.g. for $U(x)$ constant

$$\begin{aligned} V_{fi} &\propto \int \langle \psi_f | U(x) | \psi_i \rangle \, \mathrm{d}^4 x \\ &\propto \int e^{i(p_f - p_i)x} \, \mathrm{d}^4 x = (2\pi)^4 \delta^{(4)}(p_f - p_i) \end{aligned}$$

where each 2π follows from the rules of a Fourier transform, but can also be viewed as getting back 2π for each degree of freedom that is killed.

Can show that $\mathrm{d}^4 p$ and thus $\delta^{(4)}(p)$ are Lorentz invariant, e.g. with lightcone coordinates (see later).

Phase space – 2

Putting it all together, with changed labels, one obtains :

n-body phase space:

$$\mathrm{d}\Phi_n = (2\pi)^4 \delta^{(4)}(P - \sum_{i=1}^{n} p_i) \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}$$

Alternative explicit Lorentz covariant form:

$$\mathrm{d}^4 p_i \, \delta(p_i^2 - m_i^2) \, \theta(E_i) = \mathrm{d}^4 p_i \, \delta(E_i^2 - (\mathbf{p}_i^2 + m_i^2)) \, \theta(E_i) = \frac{\mathrm{d}^3 p_i}{2E_i}$$

with $E_i = \sqrt{\mathbf{p}_i^2 + m_i^2}$ and using

$$\delta(f(x)) = \sum_{x_j, f(x_j)=0} \frac{1}{|f'(x_j)|} \, \delta(x - x_j)$$

Cross sections

For collision cross section need to normalize per incoming flux, which for $a + b$ in the rest frame of b is

$$2E_a\,2E_b\,v_a = 2\underline{p}_a\,2m_b$$

which can be written in Lorentz invariant form noting that

$$\begin{aligned} p_a p_b &= E_a E_b - \mathbf{p}_a \mathbf{p}_b = E_a m_b \\ (p_a p_b)^2 - m_a^2 m_b^2 &= E_a^2 m_b^2 - m_a^2 m_b^2 = (\mathbf{p}_a^2 + m_a^2) m_b^2 - m_a^2 m_b^2 = \underline{p}_a^2 m_b^2 \end{aligned}$$

Collision process cross section, $2 \to n$:

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}\,\mathrm{d}\Phi_n \approx \frac{|\mathcal{M}|^2}{2s}\,\mathrm{d}\Phi_n$$

Integrated it gives collision rate

$$N = \sigma \int \mathcal{L}(t)\,\mathrm{d}t \quad \text{with} \quad \mathcal{L} \approx f\,\frac{n_1 n_2}{A}$$

in a theorist's approximation of the luminosity $\mathcal{L}$ for a collider.

Decay widths

For decay of particle at rest normalization per particle is $2M$, so decay width for $1 \to n$ is

$$\mathrm{d}\Gamma = \frac{|\mathcal{M}|^2}{2M} \, \mathrm{d}\Phi_n \; .$$

Integrated it gives an exponential decay rate

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \Gamma e^{-\Gamma t} \;\; \text{and} \;\; \langle \tau \rangle = 1/\Gamma$$

and Breit–Wigner shapes (to come).

If not at rest then time dilatation:

$$\frac{1}{2E} = \frac{1}{2M} \, \frac{M}{E} = \frac{1}{2M} \, \frac{1}{\gamma} \Rightarrow \langle t \rangle = \gamma \langle \tau \rangle = \frac{\gamma}{\Gamma} \; .$$

Two-body phase space

Evaluate in rest frame, i.e. $P = (E_{\mathrm{cm}}, \mathbf{0})$.

$$
\begin{aligned}
\mathrm{d}\Phi_2 &= (2\pi)^4 \delta^{(4)}(P - p_1 - p_2) \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \\
&= \frac{1}{16\pi^2} \delta(E_{\mathrm{cm}} - E_1 - E_2) \frac{\mathrm{d}^3 p_1}{E_1 E_2} \\
&= \frac{1}{16\pi^2} \delta(\sqrt{m_1^2 + \underline{p}^2} + \sqrt{m_2^2 + \underline{p}^2} - E_{\mathrm{cm}}) \frac{\underline{p}^2 \, \mathrm{d}\underline{p} \, \mathrm{d}\Omega}{E_1 E_2} \\
&= \frac{1}{16\pi^2} \frac{\delta(\underline{p} - \underline{p}^*)}{|\frac{\underline{p}}{E_1} + \frac{\underline{p}}{E_2}|} \frac{\underline{p}^2 \, \mathrm{d}\underline{p} \, \mathrm{d}\Omega}{E_1 E_2} \\
&= \frac{1}{16\pi^2} \frac{E_1 E_2}{E_1 + E_2} \frac{\underline{p} \, \mathrm{d}\Omega}{E_1 E_2} \\
&= \frac{\underline{p} \, \mathrm{d}\Omega}{16\pi^2 \, E_{\mathrm{cm}}}
\end{aligned}
$$

The Källén function – 1

$$\sqrt{m_1^2 + \underline{p}^2} + \sqrt{m_2^2 + \underline{p}^2} = E_{\mathrm{cm}}$$

gives solution

$$\begin{aligned}
E_1 &= \frac{E_{\mathrm{cm}}^2 + m_1^2 - m_2^2}{2E_{\mathrm{cm}}} \\
E_2 &= \frac{E_{\mathrm{cm}}^2 + m_2^2 - m_1^2}{2E_{\mathrm{cm}}} \\
\underline{p} &= \frac{1}{2E_{\mathrm{cm}}}\sqrt{(E_{\mathrm{cm}}^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2} = \frac{1}{2E_{\mathrm{cm}}}\sqrt{\lambda(E_{\mathrm{cm}}^2, m_1^2, m_2^2)}
\end{aligned}$$

where Källén λ function is

$$\begin{aligned}
\lambda(a^2, b^2, c^2) &= (a^2 - b^2 - c^2)^2 - 4b^2c^2 \\
&= a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \\
&= (a^2 - (b+c)^2)(a^2 - (b-c)^2) \\
&= (a+b+c)(a-b-c)(a-b+c)(a+b-c)
\end{aligned}$$

The Källén function – 2

Hides everywhere in kinematics, e.g.

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}\,\mathrm{d}\Phi_n$$

has

$$\begin{aligned}
4((p_1 p_2)^2 - m_1^2 m_2^2) &= (p_1^2 + 2p_1 p_2 + p_2^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \\
&= ((p_1 + p_2)^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \\
&= \lambda(E_{\mathrm{cm}}^2, m_1^2, m_2^2)
\end{aligned}$$

so

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(E_{\mathrm{cm}}^2, m_1^2, m_2^2)}}\,\mathrm{d}\Phi_n$$

Three-body phase space

Three-body final states has $3 \cdot 3 - 4 = 5$ degrees of freedom.
In massless case straightforward to show that, in CM frame,

$$\begin{aligned} \mathrm{d}\Phi_3 &= (2\pi)^4 \delta^{(4)}(P - p_1 - p_2 - p_3) \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \\ &= \frac{1}{8(2\pi)^5} \mathrm{d}E_1 \, \mathrm{d}E_2 \, \mathrm{d}\cos\theta_1 \, \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_{21} \end{aligned}$$

with θ_1, φ_1 polar coordinates of 1 and
φ_{21} azimuthal angle of 2 around 1 axis (Euler angles).
Phase space limits $0 \leq E_{1,2} \leq E_{\mathrm{cm}}/2$ and
$E_1 + E_2 = E_{\mathrm{cm}} - E_3 > E_{\mathrm{cm}}/2$.

Same simple phase space expression holds in massive case,
but phase space limits much more complicated!

Higher multiplicities increasingly difficult to understand.
One solution: recursion!

Factorized three-body phase space

Drop factors of 2π, and don't write implicit integral signs.
Introduce intermediate "particle" $12 = 1 + 2$.

$$
\begin{aligned}
& \mathrm{d}\Phi_3(P; p_1, p_2, p_3) \\
\sim\ & \delta^{(4)}(P - p_1 - p_2 - p_3)\, \frac{\mathrm{d}^3 p_1}{2E_1}\, \frac{\mathrm{d}^3 p_2}{2E_2}\, \frac{\mathrm{d}^3 p_3}{2E_3}\, \delta^{(4)}(p_{12} - p_1 - p_2)\, \mathrm{d}^4 p_{12} \\
=\ & \delta^{(4)}(P - p_{12} - p_3)\, \mathrm{d}^4 p_{12}\, \frac{\mathrm{d}^3 p_3}{2E_3} \left[\delta^{(4)}(p_{12} - p_1 - p_2)\, \frac{\mathrm{d}^3 p_1}{2E_1}\, \frac{\mathrm{d}^3 p_2}{2E_2} \right] \\
=\ & \delta^{(4)}(P - p_{12} - p_3)\, \mathrm{d}^4 p_{12}\, \delta(p_{12}^2 - m_{12}^2)\, \mathrm{d}m_{12}^2\, \frac{\mathrm{d}^3 p_3}{2E_3} \mathrm{d}\Phi_2(p_{12}; p_1, p_2) \\
=\ & \mathrm{d}m_{12}^2 \left[\delta^{(4)}(P - p_{12} - p_3)\, \frac{\mathrm{d}^3 p_{12}}{2E_{12}}\, \frac{\mathrm{d}^3 p_3}{2E_3} \right] \mathrm{d}\Phi_2(p_{12}; p_1, p_2) \\
=\ & \mathrm{d}m_{12}^2\, \mathrm{d}\Phi_2(P; p_{12}, p_3)\, \mathrm{d}\Phi_2(p_{12}; p_1, p_2)
\end{aligned}
$$

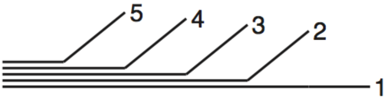
Note: here 4 angles + 1 mass^2; last slide 3 angles + 2 energies.

Recursive phase space – 1

Generalizes to

$$\begin{aligned} \mathrm{d}\Phi_n(P; p_1, \ldots, p_n) &= \mathrm{d}m^2_{12\ldots(n-1)}\, \mathrm{d}\Phi_2(P; p_{12\ldots(n-1)}, p_n) \\ &\times \mathrm{d}\Phi_{n-1}(P; p_1, \ldots, p_{(n-1)}) \end{aligned}$$

Can be viewed as a sequential decay chain, with undetermined intermediate masses.



$$\text{Recall } \mathrm{d}\Phi_2(P; p_1, p_2) \propto \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{M^2}\, \mathrm{d}\Omega_{12}$$

where $\mathrm{d}\Omega_{12}$ is the unit sphere *in the 1+2 rest frame*.
Now can write down e.g. 4-body phase space:

The M-generator

$$\mathrm{d}\Phi_4(P; p_1, p_2, p_3, p_4) \propto \frac{\sqrt{\lambda(M^2; m_4^2, m_{123}^2)}}{M^2}\, m_{123}\, \mathrm{d}m_{123}\, \mathrm{d}\Omega_{1234}$$

$$\times \frac{\sqrt{\lambda(m_{123}^2; m_3^2, m_{12}^2)}}{m_{123}^2}\, m_{12}\, \mathrm{d}m_{12}\, \mathrm{d}\Omega_{123}\, \frac{\sqrt{\lambda(m_{12}^2; m_1^2, m_2^2)}}{m_{12}^2}\, \mathrm{d}\Omega_{12}$$

Mass limits coupled, but can be decoupled: pick two random numbers $0 < R_{1,2} < 1$ and order them $R_1 < R_2$. Then

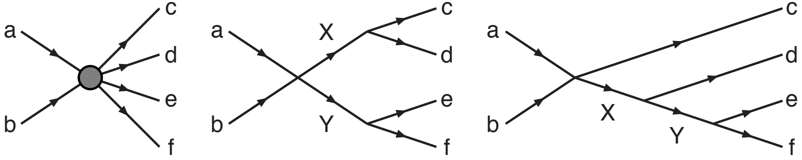
$$\begin{aligned} \Delta &= M - (m_1 + m_2 + m_3 + m_4) \\ m_{12} &= m_1 + m_2 + R_1 \Delta \\ m_{123} &= m_1 + m_2 + m_3 + R_2 \Delta \end{aligned}$$

uniformly covers $\mathrm{d}m_{12}\, \mathrm{d}m_{123}$ space with weight

$$\frac{\sqrt{\lambda(M^2; m_4^2, m_{123}^2)}}{M}\, \frac{\sqrt{\lambda(m_{123}^2; m_3^2, m_{12}^2)}}{m_{123}}\, \frac{\sqrt{\lambda(m_{12}^2; m_1^2, m_2^2)}}{m_{12}}$$

Recursive phase space – 2

Alternative splits possible, e.g. for $a + b \rightarrow c + d + e + f$:



$$
\begin{aligned}
\text{either } \mathrm{d}\Phi_4 &\propto \iint \mathrm{d}M_X^2 \, \mathrm{d}M_Y^2 \, \mathrm{d}\Phi_2(a + b \rightarrow X + Y) \\
&\times \mathrm{d}\Phi_2(X \rightarrow c + d) \, \mathrm{d}\Phi_2(Y \rightarrow e + f) \\
\text{or } \mathrm{d}\Phi_4 &\propto \iint \mathrm{d}M_X^2 \, \mathrm{d}M_Y^2 \, \mathrm{d}\Phi_2(a + b \rightarrow c + X) \\
&\times \mathrm{d}\Phi_2(X \rightarrow d + Y) \, \mathrm{d}\Phi_2(Y \rightarrow e + f)
\end{aligned}
$$

- Either recipe covers whole phase space.
- X, Y not resonances but *if* enhancement someplace *then* make use of it.

RAMBO

For massless case a smart solution is RAMBO (RAndom Momenta and BOosts), which is 100% efficient:

RAMBO

1. Pick n massless 4-vectors p_i according to

$$E_i e^{-E_i}\, \mathrm{d}\Omega_i$$

2. boost all of them by a common boost vector that brings them to their overall rest frame
3. rescale them by a common factor that brings them to the desired mass M

Can be modified for massive cases, but then no longer 100% efficiency; gets worse the bigger $\sum m_i/M$ is.
MAMBO: workaround for high multiplicities

Efficiency troubles

Even if you can pick phase space points uniformly, $|\mathcal{M}|^2$ is not!
A n-body process receives contributions from a large number of Feynman graphs, plus interferences.
Can lead to extremely low Monte Carlo efficiency.
Intermediate resonances $\Rightarrow$ narrow spikes when $(p_i + p_j)^2 \approx M_{\mathrm{res}}^2$.
t-channel graphs $\Rightarrow$ peaked at small $p_{\perp}$.

Multichannel techniques:

$$|\mathcal{M}|^2 = \frac{|\sum_i \mathcal{M}_i|^2}{\sum_i |\mathcal{M}_i|^2} \sum_i |\mathcal{M}_i|^2$$

so pick optimized for either $|\mathcal{M}_i|^2$ according to their relative integral, and use ratio as weight.
Still major challenge in real life!

Spherical symmetry

Spherical coordinates:

$$\frac{\mathrm{d}^3 p}{E} = \frac{\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z}{E} = \frac{\underline{p}^2\,\mathrm{d}\underline{p}\,\mathrm{d}\Omega}{E} = \frac{\underline{p}\,E\mathrm{d}E\,\mathrm{d}\Omega}{E} = \underline{p}\,\mathrm{d}E\,\mathrm{d}\Omega$$

where Ω is the unit sphere,

$$\begin{aligned} \mathrm{d}\Omega &= \mathrm{d}(\cos\theta)\,\mathrm{d}\phi = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ p_x &= \underline{p}\,\sin\theta\,\cos\varphi \\ p_y &= \underline{p}\,\sin\theta\,\sin\varphi \\ p_z &= \underline{p}\,\cos\theta \end{aligned}$$

and $E^2 = \underline{p}^2 + m^2 \Rightarrow E\,\mathrm{d}E = \underline{p}\,\mathrm{d}\underline{p}$.

Convenient for use e.g. in resonance decays,
but not for standard QCD physics in pp collisions.
Instead:

Cylindrical symmetry and rapidity

Cylindrical coordinates:

$$\frac{\mathrm{d}^3 p}{E} = \frac{\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z}{E} = \frac{\mathrm{d}^2 p_\perp\,\mathrm{d}p_z}{E} = \mathrm{d}^2 p_\perp\,\mathrm{d}y$$

with rapidity y given by

$$\begin{aligned} y &= \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \frac{1}{2}\ln\frac{(E+p_z)^2}{(E+p_z)(E-p_z)} = \frac{1}{2}\ln\frac{(E+p_z)^2}{m^2+p_\perp^2} \\ &= \ln\frac{E+p_z}{m_\perp} = \ln\frac{m_\perp}{E-p_z} \end{aligned}$$

The relation $\mathrm{d}y = \mathrm{dp}_z/E$ can be shown by

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}p_z} &= \frac{\mathrm{d}}{\mathrm{d}p_z}\left(\ln\frac{E+p_z}{m_\perp}\right) = \frac{\mathrm{d}}{\mathrm{d}p_z}\left(\ln(\sqrt{m_\perp^2+p_z^2}+p_z) - \ln m_\perp\right) \\ &= \frac{\frac{1}{2}\frac{2p_z}{\sqrt{m_\perp^2+p_z^2}}+1}{\sqrt{m_\perp^2+p_z^2}+p_z} = \frac{\frac{p_z+E}{E}}{E+p_z} = \frac{1}{E} \end{aligned}$$

Lightcone kinematics and boosts

Introduce (lightcone) $p^+ = E + p_z$ and $p^- = E - p_z$.
Note that $p^+ p^- = E^2 - p_z^2 = m_\perp^2$.
Consider boost along z axis with velocity β, and $\gamma = 1/\sqrt{1-\beta^2}$.

$$
\begin{aligned}
p'_{x,y} &= p_{x,y} \\
p'_z &= \gamma(p_z + \beta\, E) \\
E' &= \gamma(E + \beta\, p_z) \\
p'^+ &= \gamma(1+\beta)p^+ = \sqrt{\frac{1+\beta}{1-\beta}}\, p^+ = k\, p^+ \\
p'^- &= \gamma(1-\beta)p^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, p^- = \frac{p^-}{k} \\
y' &= \frac{1}{2}\ln\frac{p'^+}{p'^-} = \frac{1}{2}\ln\frac{k\, p^+}{p'^-/k} = y + \ln k \\
y'_2 - y'_1 &= (y_2 + \ln k) - (y_1 + \ln k) = y_2 - y_1
\end{aligned}
$$

Pseudorapidity

If experimentalists cannot measure m they may assume $m = 0$.
Instead of rapidity y they then measure pseudorapidity η:

$$y = \frac{1}{2} \ln \frac{\sqrt{m^2 + \mathbf{p}^2} + p_z}{\sqrt{m^2 + \mathbf{p}^2} - p_z} \quad \Rightarrow \quad \eta = \frac{1}{2} \ln \frac{|\mathbf{p}| + p_z}{|\mathbf{p}| - p_z} = \ln \frac{|\mathbf{p}| + p_z}{p_\perp}$$

or

$$\begin{aligned} \eta &= \frac{1}{2} \ln \frac{\underline{p} + \underline{p}\, \cos\theta}{\underline{p} - \underline{p}\, \cos\theta} = \frac{1}{2} \ln \frac{1 + \cos\theta}{1 - \cos\theta} \\ &= \frac{1}{2} \ln \frac{2\cos^2\theta/2}{2\sin^2\theta/2} = \ln \frac{\cos\theta/2}{\sin\theta/2} = -\ln\tan\frac{\theta}{2} \end{aligned}$$

which thus only depends on polar angle.
η is **not** simple under boosts: $\eta'_2 - \eta'_1 \neq \eta_2 - \eta_1$.
You may even flip sign!
Assume $m = m_\pi$ for all charged $\Rightarrow y_\pi$; intermediate to y and η.

The pseudorapidity dip

By analogy with $\mathrm{d}y/\mathrm{d}p_z = 1/E$ it follows that $\mathrm{d}\eta/\mathrm{d}p_z = 1/\underline{p}$.

Thus

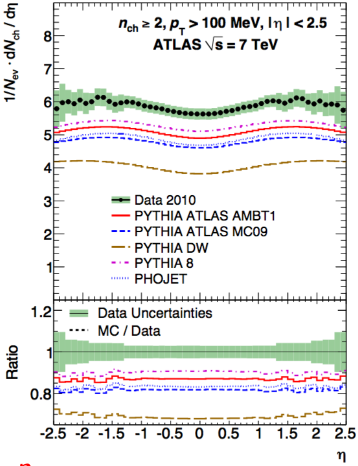
$$\frac{\mathrm{d}\eta}{\mathrm{d}y} = \frac{\mathrm{d}\eta/\mathrm{d}p_z}{\mathrm{d}y/\mathrm{d}p_z} = \frac{E}{\underline{p}} > 1$$

with limits

$$\frac{\mathrm{d}\eta}{\mathrm{d}y} \rightarrow \frac{m_\perp}{p_\perp} \text{ for } p_z \rightarrow 0$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}y} \rightarrow 1 \text{ for } p_z \rightarrow \pm\infty$$

so if $\mathrm{d}n/\mathrm{d}y$ is flat for $y \approx 0$
then $\mathrm{d}n/\mathrm{d}\eta$ has a dip there.



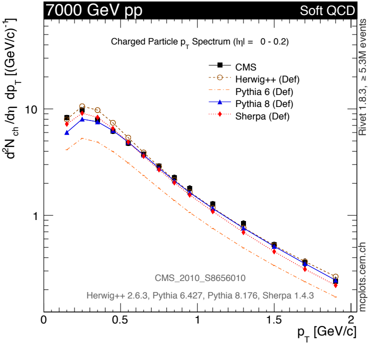
$$\eta - y = \ln\frac{\underline{p} + p_z}{p_\perp} - \ln\frac{E + p_z}{m_\perp} = \ln\frac{\underline{p} + p_z}{E + p_z}\frac{m_\perp}{p_\perp} \rightarrow \ln\frac{m_\perp}{p_\perp} \text{ when } p_z \gg m_\perp$$

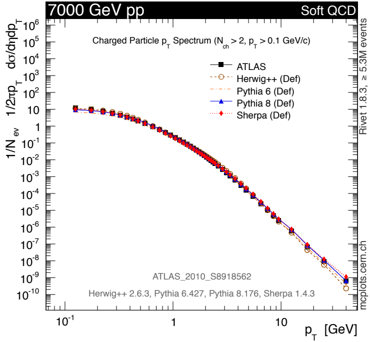
Single-particle cross sections

Normalization to d^3p/E separates kinematics from dynamics.
E.g. single-particle cross sections

$$E\frac{\mathrm{d}^3\sigma}{\mathrm{d}p^3} = \frac{\mathrm{d}^3\sigma}{\mathrm{d}y\,\mathrm{d}^2p_\perp} = \frac{\mathrm{d}^3\sigma}{\mathrm{d}y\,p_\perp\,\mathrm{d}p_\perp\,\mathrm{d}\varphi} \to \frac{1}{\pi}\,\frac{\mathrm{d}^2\sigma}{\mathrm{d}y\,\mathrm{d}p_\perp^2}$$

Note that $\mathrm{d}^2p_\perp \propto p_\perp\,\mathrm{d}p_\perp$



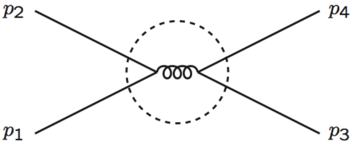


so left-plot dip for $p_\perp \to 0$ kinematics, not physics.

Mandelstam variables

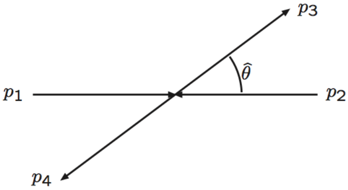
For process $1 + 2 \to 3 + 4$

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2
\end{aligned}
$$



In rest frame, massless limit: $m_1 = m_2 = m_3 = m_4 = 0$,

$$
\begin{aligned}
p_{1,2} &= \frac{E_{\mathrm{cm}}}{2}(1; 0, 0, \pm 1) \\
p_{3,4} &= \frac{E_{\mathrm{cm}}}{2}(1; \pm \sin\hat{\theta}, 0, \pm\cos\hat{\theta}) \\
s &= E_{\mathrm{cm}}^2 \\
t &= -2p_1p_3 = -\frac{s}{2}(1 - \cos\hat{\theta}) \\
u &= -2p_1p_4 = -\frac{s}{2}(1 + \cos\hat{\theta})
\end{aligned}
$$



$$s + t + u = 0$$

Mandelstam variables with masses

$$
\begin{aligned}
\beta_{34} &= \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{s} \\
p_{3,4} &= \frac{\sqrt{s}}{2} \left(1 \pm \frac{m_3^2 - m_4^2}{s} ; \pm\beta_{34} \sin\hat{\theta}, 0, \pm\beta_{34} \cos\hat{\theta} \right) \\
t &= m_1^2 + m_3^2 - \frac{s}{2} \left(1 + \frac{m_1^2 - m_2^2}{s} \right) \left(1 + \frac{m_3^2 - m_4^2}{s} \right) \\
&\quad + \frac{s}{2} \, \beta_{12} \, \beta_{34} \cos\hat{\theta} \\
\mathrm{d}\sigma &= \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(s, m_1^2, m_2^2)}} \, \frac{p_{34}}{\sqrt{s}} \, \frac{\mathrm{d}\cos\hat{\theta} \, \mathrm{d}\varphi}{16\pi^2} = \frac{|\mathcal{M}|^2}{2s\beta_{12}} \, \frac{\beta_{34}}{2} \, \frac{\mathrm{d}\cos\hat{\theta}}{8\pi}
\end{aligned}
$$

assuming no polarization $\Rightarrow$ no φ dependence

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}cos\hat{\theta}} \, \frac{\mathrm{d}cos\hat{\theta}}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{16\pi s^2 \beta_{12}^2}
$$

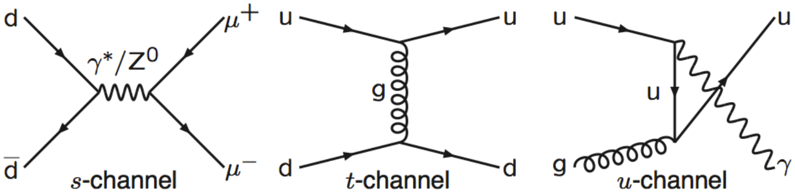
Mandelstam variables with final-state masses

Usually $m_{1,2} \approx 0$, while often $m_{3,4}$ non-negligible

$$\begin{aligned}
t, u &= -\frac{1}{2}\left[s - m_3^2 - m_4^2 \mp s\beta_{34}\cos\hat{\theta}\right] \\
\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{|\mathcal{M}|^2}{16\pi s^2} \\
s + t + u &= m_3^2 + m_4^2 \\
tu &= \frac{1}{4}\left[(s - m_3^2 - m_4^2)^2 - s^2\beta_{34}^2\cos^2\hat{\theta}\right] \\
&= \frac{1}{4}\left[s^2\beta_{34}^2 + 4m_3^2 m_4^2 - s^2\beta_{34}^2\cos^2\hat{\theta}\right] \\
&= \frac{1}{4}s^2\beta_{34}^2\sin^2\hat{\theta} + m_3^2 m_4^2 = sp_\perp^2 + m_3^2 m_4^2 \\
p_\perp^2 &= \frac{tu - m_3^2 m_4^2}{s}
\end{aligned}$$

s-, t- and u-channel processes

Classify $2 \to 2$ diagrams by character of propagator, e.g.



Singularities reflect channel character, e.g. pure t-channel:

$$\frac{\mathrm{d}\sigma(\mathrm{qq}' \to \mathrm{qq}')}{\mathrm{d}t} = \frac{\pi}{s^2}\frac{4}{9}\alpha_\mathrm{s}^2\frac{s^2+u^2}{t^2}$$

peaked at $t \to 0 \Rightarrow u \approx -s$, so

$$\frac{\mathrm{d}\sigma(\mathrm{qq}' \to \mathrm{qq}')}{\mathrm{d}t} \approx \frac{8\pi\alpha_\mathrm{s}^2}{9t^2} = \frac{32\pi\alpha_\mathrm{s}^2}{9s^2(1-\cos\hat{\theta})^2} = \frac{8\pi\alpha_\mathrm{s}^2}{9s^2\sin^4\hat{\theta}/2} \approx \frac{8\pi\alpha_\mathrm{s}^2}{9p_\perp^4}$$

i.e. Rutherford scattering!

Order-of-magnitude cross sections

With masses neglected:

$$
\begin{aligned}
s\mathrm{-channel} &: \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim \frac{\pi}{s^2} \\
t\mathrm{-channel,~spin}~1 &: \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim \frac{\pi}{t^2} \\
t\mathrm{-channel,~spin}~\frac{1}{2} &: \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim \frac{\pi}{-st} \\
u\mathrm{-channel} &: \quad \mathrm{same~with}~t \to u
\end{aligned}
$$

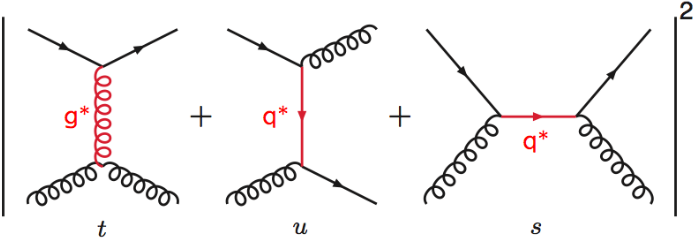
Add couplings at vertices:

$$
\begin{aligned}
\mathrm{qqg} &: \quad C_F \alpha_{\mathrm{s}} \qquad (\alpha_{\mathrm{s}} = g_s^2/4\pi) \\
\mathrm{ggg} &: \quad N_c \alpha_{\mathrm{s}} \\
\mathrm{f\,f}\gamma &: \quad e_{\mathrm{f}}^2 \alpha_{\mathrm{em}} \qquad (\alpha_{\mathrm{em}} = e^2/4\pi) \\
\mathrm{f\,f'W} &: \quad |V_{\mathrm{ff'}}|^2 \frac{\alpha_{\mathrm{em}}}{4 \sin^2\theta_W} \\
\mathrm{f\,f'Z} &: \quad (v_{\mathrm{f}}^2 + a_{\mathrm{f}}^2) \frac{\alpha_{\mathrm{em}}}{16 \sin^2\theta_W \cos^2\theta_W}
\end{aligned}
$$

Closeup: $\mathrm{qg} \to \mathrm{qg}$

Consider $\mathrm{q}(1)\,\mathrm{g}(2) \to \mathrm{q}(3)\,\mathrm{g}(4)$:

$|\mathcal{M}|^2 =$



$t : p_{\mathrm{g}^*} = p_1 - p_3 \Rightarrow m_{\mathrm{g}^*}^2 = (p_1 - p_3)^2 = t \Rightarrow \mathrm{d}\sigma/\mathrm{d}t \sim 1/t^2$

$u : p_{\mathrm{q}^*} = p_1 - p_4 \Rightarrow m_{\mathrm{q}^*}^2 = (p_1 - p_4)^2 = u \Rightarrow \mathrm{d}\sigma/\mathrm{d}t \sim -1/su$

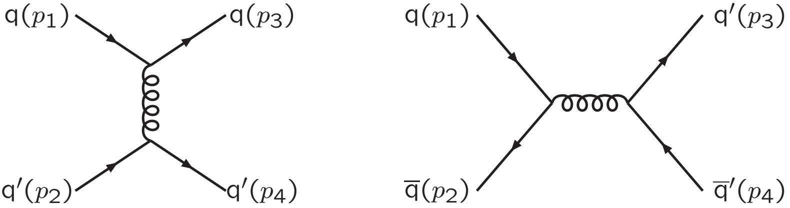
$s : p_{\mathrm{q}^*} = p_1 + p_2 \Rightarrow m_{\mathrm{q}^*}^2 = (p_1 + p_2)^2 = s \Rightarrow \mathrm{d}\sigma/\mathrm{d}t \sim 1/s^2$

Contribution of each sub-graph is gauge-dependent, only sum is well-defined:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_{\mathrm{s}}^2}{s^2}\left[\frac{s^2+u^2}{t^2} + \frac{4}{9}\frac{s}{(-u)} + \frac{4}{9}\frac{(-u)}{s}\right]$$

Crossing

Matrix elements do not distinguish incoming particle and outgoing antiparticle with opposite four-momentum, so have crossing symmetries:



$$
\begin{aligned}
t = (p_1 - p_3)^2 &\quad\Rightarrow\quad (p_1 + p_2)^2 = s \\
s = (p_1 + p_2)^2 &\quad\Rightarrow\quad (p_1 - p_4)^2 = u \\
u = (p_1 - p_4)^2 &\quad\Rightarrow\quad (p_1 - p_3)^2 = t \\
|\mathcal{M}|^2 \propto \frac{s^2 + u^2}{t^2} &\quad\Rightarrow\quad \frac{u^2 + t^2}{s^2}
\end{aligned}
$$

Colour and spin summing/averaging may give different prefactors.

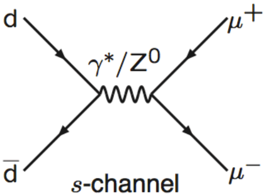
Scale choice

What Q^2 scale to use for $\alpha_{\mathrm{s}} = \alpha_{\mathrm{s}}(Q^2)$?
Should be characteristic virtuality scale of process!
But e.g. for $\mathrm{q}\,\mathrm{g} \to \mathrm{q}\,\mathrm{g}$: both s-, t- and u-channel + interference.
At small t the t-channel graph dominates $\Rightarrow Q^2 \sim |t|$,
at small u the u-channel graph dominates $\Rightarrow Q^2 \sim |u|$,
in between all graphs comparably important $\Rightarrow Q^2 \sim s \sim |t| \sim |u|$.
Suitable interpolation:

$$Q^2 = p_\perp^2 = \frac{tu}{s} \quad \begin{array}{ll} \to & -t \text{ for } t \to 0 \\ \to & -u \text{ for } u \to 0 \\ \to & \dfrac{s}{4} \text{ for } t = u = -\dfrac{s}{2} \end{array}$$

but could equally well be multiple of $p_\perp^2$, or more complicated
$\Rightarrow$ one limitation of LO calculations.

s-channel processes



A pure s-channel process $2 \to 1 \to 2$ can be calculated just like any other $2 \to 2$ one, but can also be factorized into production and decay *if* massive particle with small width.

For exchanged massive particle (notably vector bosons W/Z):

$$\mathcal{M} \propto \frac{1}{s} \to \frac{1}{s - m^2}, \quad \mathcal{M} \propto \frac{1}{t} \to \frac{1}{t - m^2}, \quad \mathcal{M} \propto \frac{1}{u} \to \frac{1}{u - m^2}$$

For s-channel need correction from finite width/lifetime:

$$|\psi|^2 \propto e^{-\Gamma t} \quad \Rightarrow \quad \psi \propto e^{-imt} e^{-\Gamma t/2} = e^{-i(m - i\Gamma/2)t}$$

$$|\mathcal{M}|^2 \propto \left| \frac{1}{s - (m - i\Gamma/2)^2} \right|^2 \approx \left| \frac{1}{s - m^2 + im\Gamma} \right|^2 = \frac{1}{(s - m^2)^2 + m^2\Gamma^2}$$

Resonances

Resonance shape given by Breit-Wigner

$$1 \;\mapsto\; \rho(s) = \frac{1}{\pi}\frac{m\Gamma}{(s-m^2)^2 + m^2\Gamma^2}$$

$$\mapsto\; \frac{1}{\pi}\frac{s\Gamma(m)/m}{(s-m^2)^2 + s^2\Gamma^2(m)/m^2}$$

where $m \mapsto \sqrt{s}$ in phase space and $\Gamma(s) \mapsto \Gamma(m)\sqrt{s}/m$
for gauge bosons, neglecting thresholds.
Latter shape suppressed below and enhanced above peak; tilted.
For $s \to 0$ $\rho(s)$ goes to constant or like s.
PDF's tend to be peaked at small x: convolution enhances small s.
Can give secondary mass-spectrum "peak" in $s \to 0$ region.
But note that

$$|\mathcal{M}|^2 = |\mathcal{M}_{\mathrm{signal}} + \mathcal{M}_{\mathrm{background}}|^2$$

so in many cases Breit-Wigner cannot be trusted except in the neighbourhood of the peak, where signal should dominate.

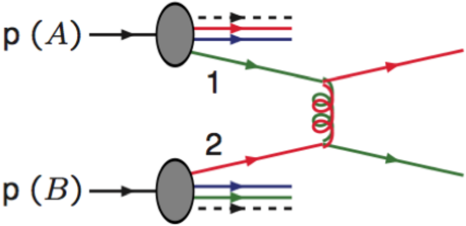
Composite beams

In reality all beams are composite:

$\mathrm{p} : \mathrm{q}, \mathrm{g}, \overline{\mathrm{q}}, \ldots$

$e^- : \mathrm{e}^-, \gamma, \mathrm{e}^+, \ldots$

$\gamma : \mathrm{e}^\pm, \mathrm{q}, \overline{\mathrm{q}}, \mathrm{g}$



Factorization

$$\sigma^{AB} = \sum_{i,j} \iint \mathrm{d}x_1\, \mathrm{d}x_2\, f_i^{(A)}(x_1, Q^2)\, f_j^{(B)}(x_2, Q^2)\, \hat{\sigma}_{ij}$$

x: momentum fraction, e.g. $p_i = x_1 p_A$; $p_j = x_2 p_B$

Q^2: factorization scale, "typical momentum transfer scale"

Factorization only proven for a few cases, like γ^*/Z^0 prodution, and strictly speaking not correct e.g. for jet production,

but good first approximation and unsurpassed physics insight.

Subprocess kinematics – 1

If $p_A + p_B = (E_{\mathrm{cm}}; \mathbf{0})$, A, B along $\pm z$ axis, and $1, 2$ collinear with A, B then conviniently put them massless:

$$\begin{aligned} p_1 &= x_1 \, (E_{\mathrm{cm}}/2)(1; 0, 0, 1) \\ p_2 &= x_2 \, (E_{\mathrm{cm}}/2)(1; 0, 0, -1) \end{aligned}$$

such that $\hat{s} = (p_1 + p_2)^2 = x_1 \, x_2 \, s = \tau \, s$. Velocity of subsystem is

$$\beta_z = \frac{p_z}{E} = \frac{x_1 - x_2}{x_1 + x_2}$$

and its rapidity

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{x_1}{x_2}$$

$\mathrm{d}x_1 \, \mathrm{d}x_2 = \mathrm{d}\tau \, \mathrm{d}y$ convenient for Monte Carlo.
Historically $x_F = 2p_z/E_{\mathrm{cm}} = x_1 - x_2$.
Subprocess $2 \to 2$ kinematics for $\hat{\sigma}$: $\hat{s}, \hat{t}, \hat{u}$.

Subprocess kinematics – 2

$f_i(x, Q^2)$ peaked at small $x \Rightarrow f_i(x, Q^2)\mathrm{d}x \rightarrow xf_i(x, Q^2)\,(\mathrm{d}x/x)$.
$2 \rightarrow 2$ process $\Rightarrow \hat{\sigma} \rightarrow (\mathrm{d}\hat{\sigma}/\mathrm{d}\hat{t})\,\mathrm{d}\hat{t}$.
Thus integration phase space conveniently of form

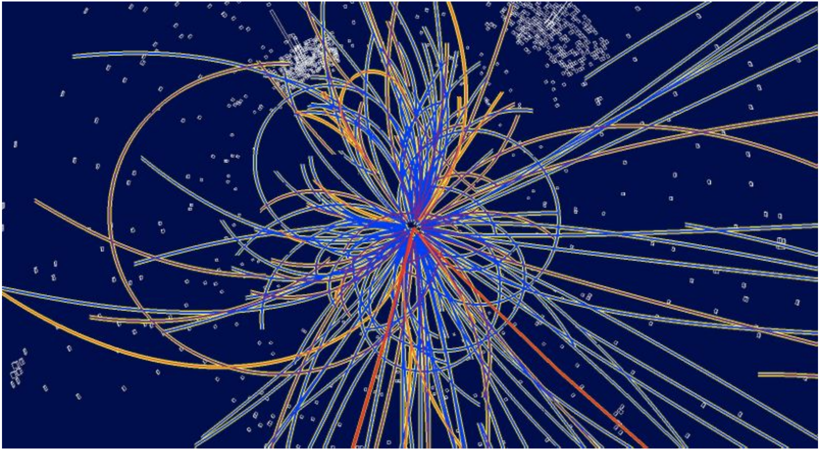
$$\frac{\mathrm{d}x_1}{x_1}\,\frac{\mathrm{d}x_2}{x_2}\,\mathrm{d}\hat{t} = \frac{\mathrm{d}\tau\,\mathrm{d}y}{\tau}\,\mathrm{d}\hat{t} = \frac{\mathrm{d}\hat{s}}{\hat{s}}\,\mathrm{d}y\,\mathrm{d}\hat{t}$$

If all partons massless then

$$\begin{aligned}
y_{3,4} &= y \pm \frac{\Delta}{2} \Rightarrow \mathrm{d}y_3\,\mathrm{d}y_4 = \mathrm{d}y\,\mathrm{d}\Delta \\
\Delta &= 2\frac{1}{2}\ln\left(\frac{1-\cos\hat{\theta}}{1+\cos\hat{\theta}}\right) = \ln\left(\frac{\hat{t}}{\hat{u}}\right) = \ln\left(\frac{-\hat{t}}{\hat{s}+\hat{t}}\right) \\
p_\perp^2 &= \frac{\hat{t}\,\hat{u}}{\hat{s}} = \frac{\hat{t}(-\hat{s}-\hat{t})}{\hat{s}} = -\hat{t} - \frac{\hat{t}^2}{\hat{s}} \Rightarrow \frac{\mathrm{d}(\Delta, p_\perp^2)}{\mathrm{d}(\hat{s}, \hat{t})} = \frac{1}{\hat{s}} \\
\frac{\mathrm{d}\hat{s}}{\hat{s}}\,\mathrm{d}y\,\mathrm{d}\hat{t} &= \mathrm{d}y\,\mathrm{d}\Delta\,\mathrm{d}p_\perp^2 = \mathrm{d}y_3\,\mathrm{d}y_4\,\mathrm{d}p_\perp^2
\end{aligned}$$

Matrix Elements and Their Usage

$\mathcal{L} \Rightarrow$ Feynman rules $\Rightarrow$ Matrix Elements $\Rightarrow$ Cross Sections
$+$ Kinematics $\Rightarrow$ Processes $\Rightarrow \ldots \Rightarrow$



(Higgs simulation in CMS)

QCD at Fixed Order

Distribution of observable: O

In production of X + anything

Fixed Order (all orders)

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{O}}\right|_{\mathrm{ME}} = \sum_{k=0} \int \mathrm{d}\Phi_{X+k} \left| \sum_{\ell=0} M_{X+k}^{(\ell)} \right|^2 \delta(\mathcal{O} - \mathcal{O}(\{p\}_{X+k}))$$

Cross Section differentially in O

Sum over "anything" ≈ legs

Phase Space

Sum over identical amplitudes, then square

Matrix Elements for X+k at (l) loops

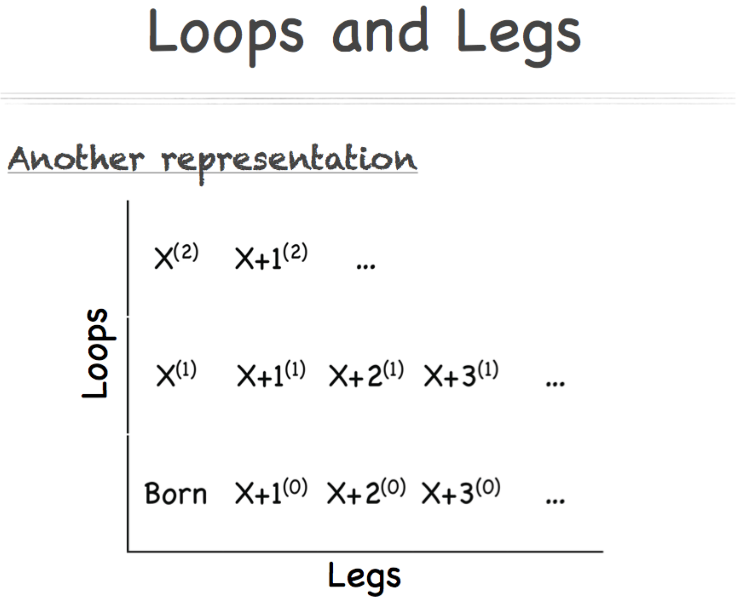
Momentum configuration

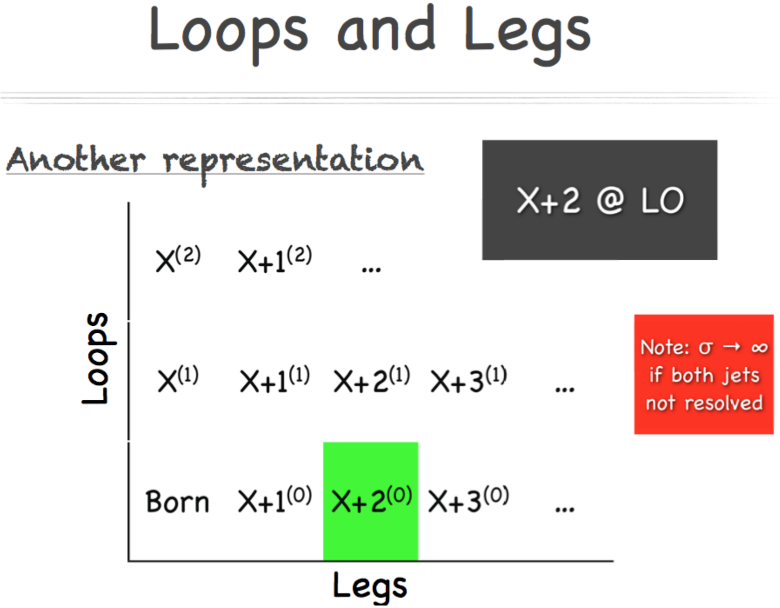
Evaluate observable → differential in O

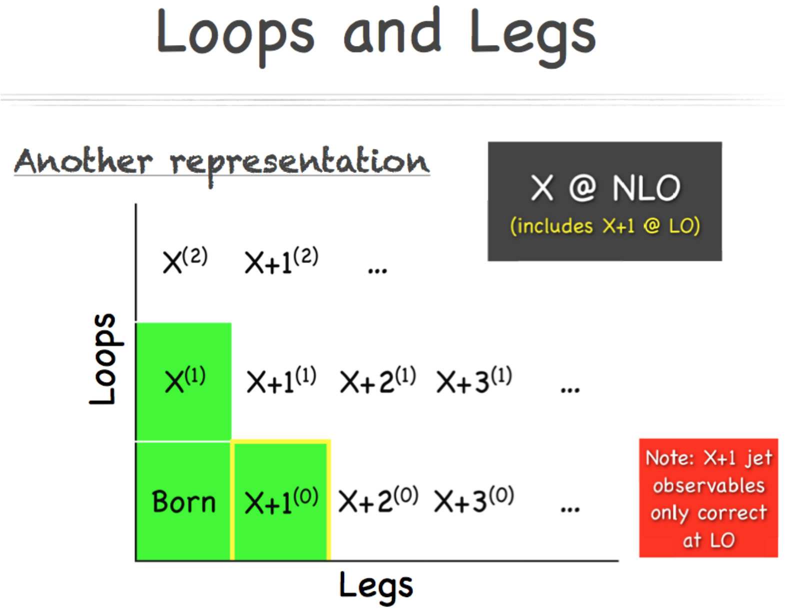
Truncate at k=n, l=0

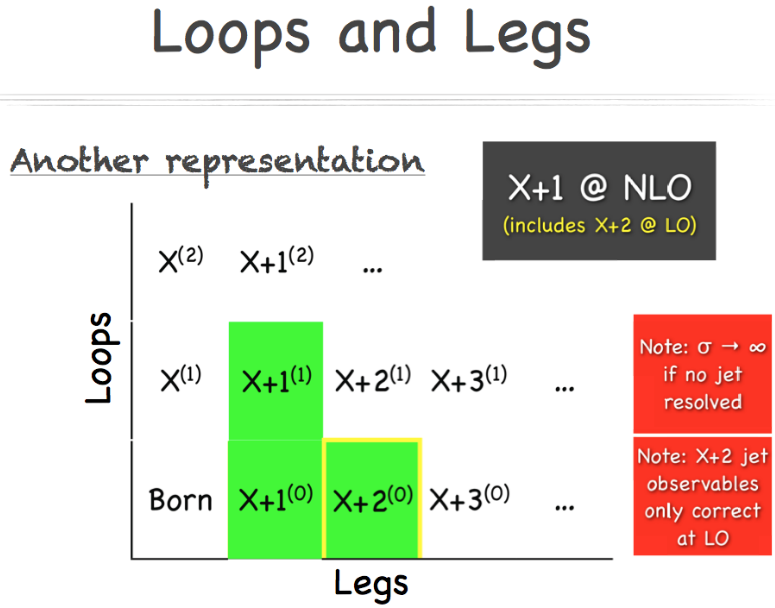
→ **Leading Order** for X + n

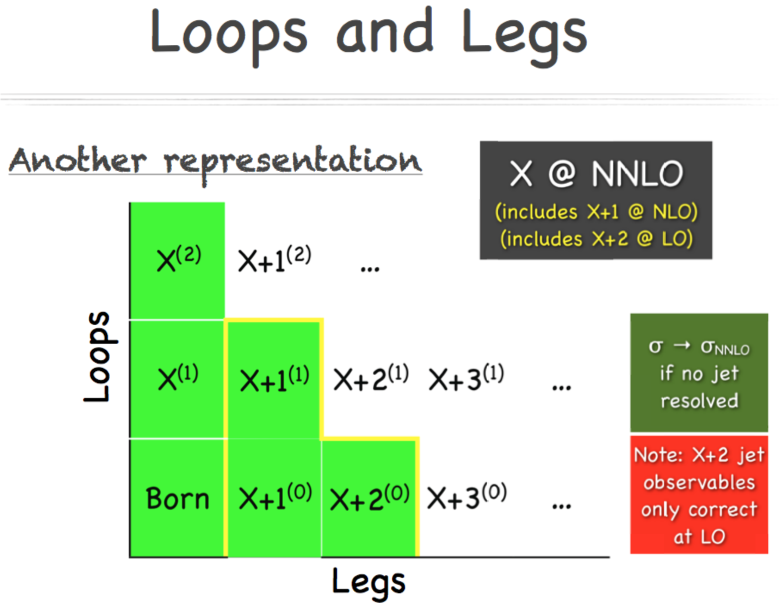
Lowest order at which X + n happens


Loops and Legs
Another representation
Loops
$X^{(2)}$ $X+1^{(2)}$ ...
$X^{(1)}$ $X+1^{(1)}$ $X+2^{(1)}$ $X+3^{(1)}$ ...
Born $X+1^{(0)}$ $X+2^{(0)}$ $X+3^{(0)}$ ...
Legs


Loops and Legs
Another representation
X+2 @ LO
Note: σ → ∞
if both jets
not resolved
Loops
X(2) X+1(2) ...
X(1) X+1(1) X+2(1) X+3(1) ...
Born X+1(0) X+2(0) X+3(0) ...
Legs


Loops and Legs
Another representation
X @ NLO
(includes X+1 @ LO)
$X^{(2)}$
$X+1^{(2)}$
...
Loops
$X^{(1)}$
$X+1^{(1)}$
$X+2^{(1)}$
$X+3^{(1)}$
...
Born
$X+1^{(0)}$
$X+2^{(0)}$
$X+3^{(0)}$
...
Note: X+1 jet observables only correct at LO
Legs


Loops and Legs
Another representation
X+1 @ NLO
(includes X+2 @ LO)
Loops
Legs
X(2)
X+1(2)
...
X(1)
X+1(1)
X+2(1)
X+3(1)
...
Born
X+1(0)
X+2(0)
X+3(0)
...
Note: σ → ∞ if no jet resolved
Note: X+2 jet observables only correct at LO


Loops and Legs
Another representation
X @ NNLO
(includes X+1 @ NLO)
(includes X+2 @ LO)
X(2)
X+1(2)
...
X(1)
X+1(1)
X+2(1)
X+3(1)
...
Born
X+1(0)
X+2(0)
X+3(0)
...
Loops
Legs
σ → σNNLO
if no jet
resolved
Note: X+2 jet
observables
only correct
at LO

Born level calculations – 1 (Frank Krauss)

Stating the problem(s)

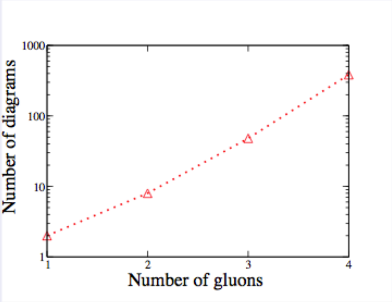
- Multi-particle final states for signals & backgrounds.
- Need to evaluate $\mathrm{d}\sigma_N$:

$$\int_{\mathrm{cuts}} \left[\prod_{i=1}^{N} \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i}\right] \delta^4 \left(p_1 + p_2 - \sum_i q_i\right) \left|\mathcal{M}_{p_1 p_2 \to N}\right|^2 .$$

- Problem 1: Factorial growth of number of amplitudes.
- Problem 2: Complicated phase-space structure.
- Solutions: Numerical methods.

Example for factorial growth: $e^+e^- \to q\bar{q} + ng$

n	$\#_{\text{diags}}$
0	1
1	2
2	8
3	48
4	384



Remember: to be squared for number of squared MEs.

Basic ideas of efficient ME calculation

Need to evaluate $|\mathcal{M}|^2 = \left|\sum_i \mathcal{M}_i\right|^2$

- Obvious: Traditional textbook methods (squaring, completeness relations, traces) fail
  $\Longrightarrow$ result in proliferation of terms ($\mathcal{M}_i \mathcal{M}_j^*$)
  $\Longrightarrow$ Better: Amplitudes are complex numbers,
  $\Longrightarrow$ add them before squaring!
- Remember: spinors, gamma matrices have explicit form could be evaluated numerically (brute force)
  But: Rough method, lack of elegance, CPU-expensive

Helicity method

- Introduce basic helicity spinors (needs to "gauge"-vectors)
- Write everything as spinor products, e.g.

$$\bar{u}(p_1, h_1)u(p_2, h_2) = \text{complex numbers.}$$

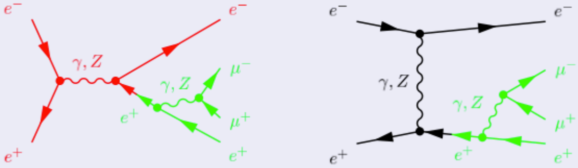
Completeness rel'n: $(\not{p} + m) \implies \frac{1}{2}\sum_h \left[\left(1 + \frac{m^2}{p^2}\right)\bar{u}(p, h)u(p, h) + \left(1 - \frac{m^2}{p^2}\right)\bar{v}(p, h)v(p, h)\right]$

- There are other genuine expressions ...
- Translate Feynman diagrams into "helicity amplitudes": complex-valued functions of momenta & helicities.
- Spin-correlations etc. nearly come for free.

Taming the factorial growth

- In the helicity method
  - Reusing pieces: Calculate only once!
  - Factoring out: Reduce number of multiplications!

  Implemented as a-posteriori manipulations of amplitudes.



- Better method: Recursion relations (recycling built in).
  Best candidate so far: Off-shell recursions

(Dyson-Schwinger, Berends-Giele etc.)

Efficient phase space integration

("Amateurs study strategy, professionals study logistics")

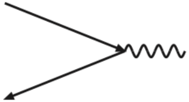
- Democratic, process-blind integration methods:
  - Rambo/Mambo: Flat & isotropic

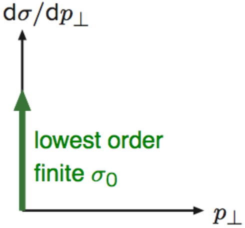
    R.Kleiss, W.J.Stirling & S.D.Ellis, Comput. Phys. Commun. **40** (1986) 359;
  - HAAG/Sarge: Follows QCD antenna pattern

    A.van Hameren & C.G.Papadopoulos, Eur. Phys. J. C **25** (2002) 563.
- Multi-channeling: Each Feynman diagram related to a phase space mapping (= "channel"), optimise their relative weights. R.Kleiss & R.Pittau, Comput. Phys. Commun. **83** (1994) 141.
- Main problem: practical only up to $\mathcal{O}(10\mathrm{k})$ channels.
- Some improvement by building phase space mappings recursively: More channels feasible, efficiency drops a bit.

Next-to-leading order (NLO) graphs

I. Lowest order,
$\mathcal{O}(\alpha_{\mathrm{em}})$:
$\mathrm{q}\overline{\mathrm{q}} \to \mathrm{Z}^0$





Next-to-leading order (NLO) graphs

I. Lowest order,
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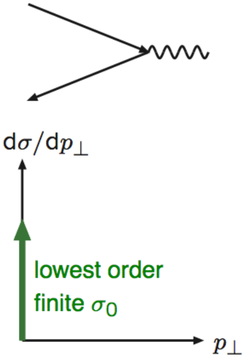
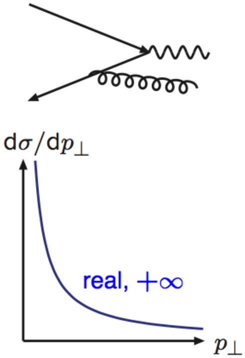
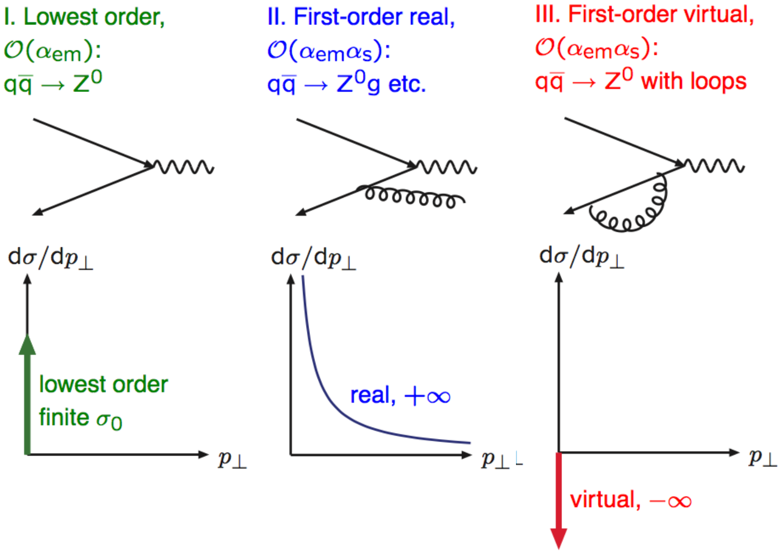
$\mathrm{q\bar{q}} \to \mathrm{Z}^0$



II. First-order real,
$\mathcal{O}(\alpha_{\mathrm{em}}\alpha_{\mathrm{s}})$:
$\mathrm{q\bar{q}} \to \mathrm{Z}^0\mathrm{g}$ etc.



Next-to-leading order (NLO) graphs



NLO calculations – 1

$$\sigma_{\mathrm{NLO}} = \int_n \mathrm{d}\sigma_{\mathrm{LO}} + \int_{n+1} \mathrm{d}\sigma_{\mathrm{Real}} + \int_n \mathrm{d}\sigma_{\mathrm{Virt}}$$

Simple one-dimensional example: $x \sim p_{\perp}/p_{\perp\mathrm{max}}$, $0 \leq x \leq 1$
Divergences regularized by $d = 4 - 2\epsilon$ dimensions, $\epsilon < 0$

$$\sigma_{\mathrm{R+V}} = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0$$

KLN cancellation theorem: $M(0) = M_0$

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Phase Space Slicing:
Introduce arbitrary *finite* cutoff $\delta \ll 1$ (so $\delta \gg |\epsilon|$)

$$\begin{aligned}
\sigma_{\mathrm{R+V}} &= \int_\delta^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) + \int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0 \\
&\approx \int_\delta^1 \frac{\mathrm{d}x}{x} M(x) + \int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \\
&= \int_\delta^1 \frac{\mathrm{d}x}{x} M(x) + \frac{1}{\epsilon} \left(1 - \delta^{-\epsilon}\right) M_0 \approx \int_\delta^1 \frac{\mathrm{d}x}{x} M(x) + \ln\delta\, M_0
\end{aligned}$$

NLO calculations – 2

Alternatively **Subtraction**:

$$
\begin{aligned}
\sigma_{\mathrm{R+V}} &= \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) - \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \\
&= \int_0^1 \frac{M(x) - M_0}{x^{1+\epsilon}} \mathrm{d}x + \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon} \right) M_0 \\
&\approx \int_0^1 \frac{M(x) - M_0}{x} \mathrm{d}x + \mathcal{O}(1) M_0
\end{aligned}
$$

NLO calculations – 2

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$$
\begin{aligned}
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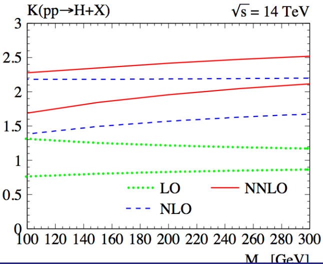
&\approx \int_0^1 \frac{M(x) - M_0}{x} \mathrm{d}x + \mathcal{O}(1) M_0
\end{aligned}
$$

NLO provides a more accurate answer for an integrated cross section:



Warning!

Neither approach operates with positive definite quantities.

No obvious event-generator implementation.

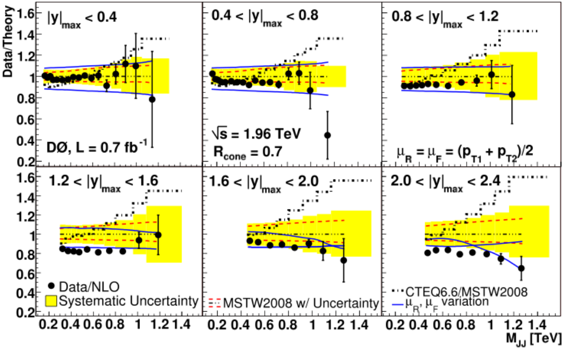
No trivial connection to physical events

Cross section depends on factorization scale μ_F
and renormalization scale μ_R:

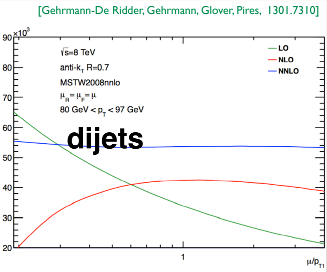
$$\sigma^{AB} = \sum_{i,j} \iint \mathrm{d}x_1\,\mathrm{d}x_2\, f_i^{(A)}(x_1,\mu_F)\, f_j^{(B)}(x_2,\mu_F)\, \hat{\sigma}_{ij}(\alpha_\mathrm{s}(\mu_R),\mu_F,\mu_R)$$

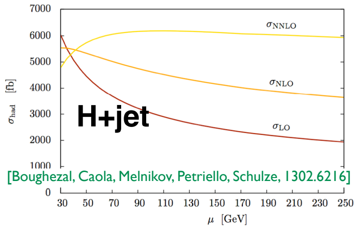
Historically common to put $Q = \mu_F = \mu_R$ but nowadays varied independently to gauge undertainty of cross section prediction.

Typical variation factor $2^{\pm 1}$ around "natural value", but beware



How reliable is scale variation?





There's no shortage of cases where (sometimes partial) NNLO is at or beyond edge of NLO scale variation

[Czakon, Fiedler & Mitov 1303.6254]

$t\bar{t}$ @ LHC8

LO:	145^{+49}_{-34} pb
NLO:	213^{+25}_{-27} pb
NNLO:	239^{+9}_{-15} pb

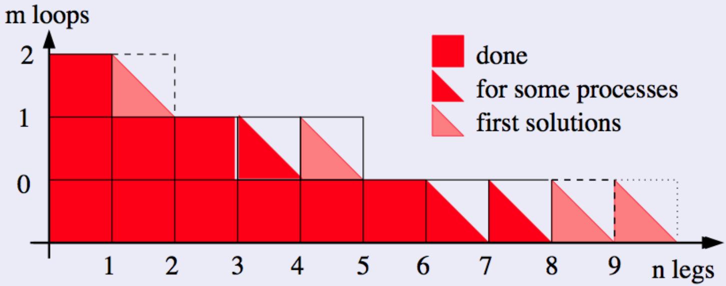
top++, MSTW2008NNLO, μ = m_t

1

Current status (N)(N)LO (Frank Krauss)

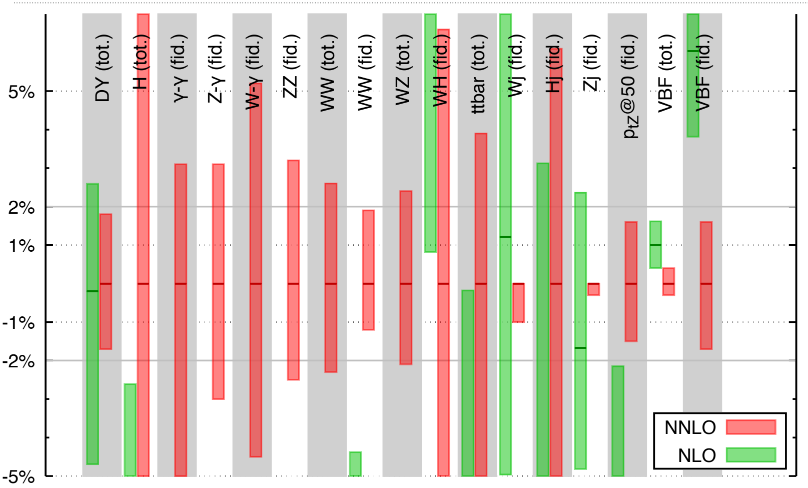
Availability of exact calculations (hadron colliders)

- Fixed order matrix elements ("parton level") are exact to a given perturbative order. (and often quite a pain!)
- Important to understand limitations:
  Only tree-level fully automated, 1-loop-level ongoing.



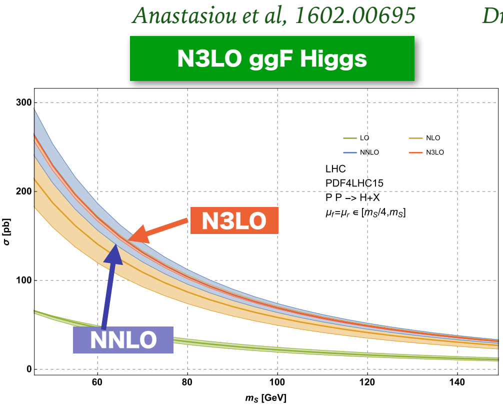
MadGraph5_aMC@NLO: automated NLO now available!

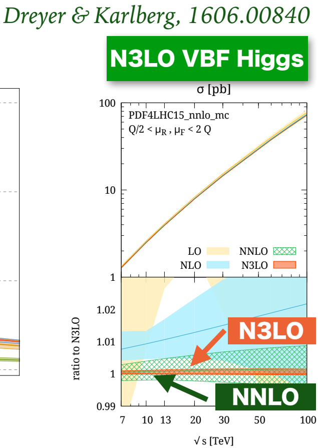
What precision at NNLO? (Gavin Salam)



For many processes NNLO scale band is $\sim \pm 2\%$.
But only in 3/17 cases is NNLO central value within NLO band.

N3LO convergence? (Gavin Salam)

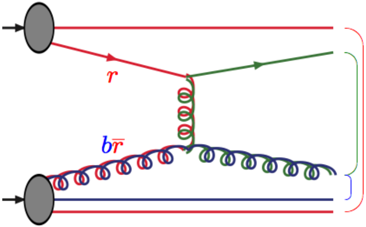


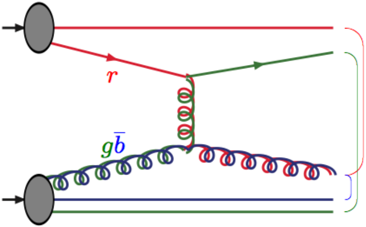


VBF converges much faster than ggF.
But in both calculations NNLO fell outside NLO scale band, while N^{3}LO (with good central scale choice) is very close to NNLO.

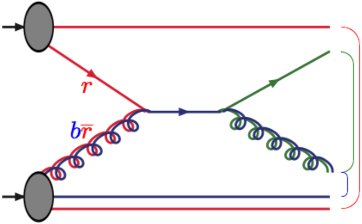
Colour flow in hard processes – 1

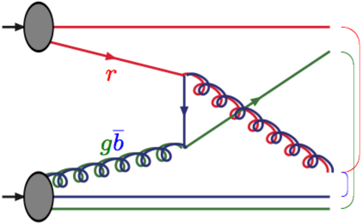
One Feynman graph can correspond to several possible colour flows, e.g. for $\mathrm{qg} \to \mathrm{qg}$:





while other $\mathrm{qg} \to \mathrm{qg}$ graphs only admit one colour flow:





Colour flow in hard processes – 2

so nontrivial mix of kinematics variables $(\hat{s}, \hat{t})$
and colour flow topologies I, II:

$$\begin{aligned} |\mathcal{A}(\hat{s},\hat{t})|^2 &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t}) + \mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2 \\ &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})|^2 + |\mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2 + 2\,\mathcal{R}e\left(\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})\mathcal{A}^*_{\mathrm{II}}(\hat{s},\hat{t})\right) \end{aligned}$$

with $\mathcal{R}e\left(\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})\mathcal{A}^*_{\mathrm{II}}(\hat{s},\hat{t})\right) \neq 0$
$\Rightarrow$ indeterminate colour flow, while
- showers *should* know it (coherence),
- hadronization *must* know it (hadrons singlets).

Normal solution:

$$\frac{\text{interference}}{\text{total}} \propto \frac{1}{N_{\mathrm{C}}^2 - 1}$$

so split $\mathrm{I} : \mathrm{II}$ according to proportions in the $N_{\mathrm{C}} \to \infty$ limit, i.e.

$$\begin{aligned} |\mathcal{A}(\hat{s},\hat{t})|^2 &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})|^2_{\mathrm{mod}} + |\mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2_{\mathrm{mod}} \\ |\mathcal{A}_{\mathrm{I(II)}}(\hat{s},\hat{t})|^2_{\mathrm{mod}} &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t}) + \mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2 \left(\frac{|\mathcal{A}_{\mathrm{I(II)}}(\hat{s},\hat{t})|^2}{|\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})|^2 + |\mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2} \right)_{N_{\mathrm{C}} \to \infty} \end{aligned}$$