4. Interacting QFTs

So far we have only discussed QFTs of free particles. The dynamics of these theories is encoded in Lagrangians that are quadratic in the fields and that transform as scalars under homogeneous LT.

Explicitly we found

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2}\phi^2$$ massive, neutral, spin-0

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$ massive, charged, spin-1/2

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$$ massive, neutral, spin-1

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 ,$$ massless, neutral, spin-1 in Lorenz gauge

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The resulting Euler-Lagrange equations represent linear differential equations for the field operators, which are solved by plane-wave solutions

$$\psi_\alpha(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\left(u_\alpha(p,s)\, e^{-ipx}\, a(p,s) + v_\alpha(p,s)\, e^{ipx}\, b^+(p,s)\right)$$

with coefficients $u_\alpha(p,s)$ and $v_\alpha(p,s)$ that reflect the transformation properties of the field operators under homogeneous LT.

How can we construct a QFT of interacting particles?

The idea is to add higher-order terms to the free Lagrangians that respect the constraints from Lorentz invariance and possibly also from further internal symmetries of the theory (like gauge invariance). The interaction terms should furthermore again be local to avoid any conflicts with causality. But even in the simplest case of a real scalar field, there seems to exist an infinite number of possible interaction terms since there does not exist any internal symmetry for such a theory and terms like $\phi^n(x)$ with $n \geq 3$ are all allowed by Lorentz invariance*.

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* The free theory Lagrangian actually corresponds to the most general case with $n \leq 2$, since a constant term with $n=0$ is irrelevant for the dynamics and a term with $n=1$ can always be removed by a field redefinition (which corresponds to a different choice of generalised coordinates).

There exists, however, a very stringent constraint that excludes almost all of the interaction terms, namely a QFT of interacting particles should be renormalisable. We will discuss renormalisation in TPP II when we analyse the structure of Feynman diagrams beyond the leading order in perturbation theory. At higher orders it turns out that the momenta of the virtual particles are no longer fixed by the kinematics of the considered reaction, and one instead has to integrate over all momentum configuration of the virtual particles. These loop integrals often turn out, however, to be formally divergent, but there exists a systematic procedure to get rid of these divergences as long as the theory is renormalisable.

Fortunately, one can tell very easily if a theory is renormalisable since all coupling constants in renormalisable QFTs must have non-negative mass dimension.

How can we determine the mass dimension of a field operator?

In natural units with $c = \hbar = 1$ the action is dimensionless, and one writes $[S] = 0$ to indicate that S has mass dimension m^0.

$E = mc^2$
$\vec{p} = \gamma m \vec{v}$
e^{ipx}

One obviously has $[m] = 1$, $[p^\mu] = 1$ and $[x^\mu] = -1$. From $S = \int d^4x \, \mathcal{L}(x)$, we then obtain $[\mathcal{L}] = 4$ and from the free theory Lagrangians we read off

$[\phi] = 1$ scalar field

$[\psi] = 3/2$ Dirac field

$[A^\mu] = 1$ vector field

A renormalizable QFT of interacting neutral spin-0 particles therefore is of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{d_3}{3!} \phi^3 - \frac{d_4}{4!} \phi^4$$

where d_3 and d_4 are called coupling constants and $[d_3] = 1$ and $[d_4] = 0$. Higher-order terms of the form $\frac{d_n}{n!} \phi^n$ are, on the other hand, excluded since the corresponding coupling constant would have mass dimension $[d_n] = 4 - n < 0$ for $n \geq 5$.

For $\lambda_4 = 0$ the ϕ^3-interaction is actually problematic for a different reason since the corresponding vacuum state would be unstable*.

The only acceptable self-interactions of spin-0 particles are therefore ϕ^3- and ϕ^4-interactions. Spin-1 particles, on the other hand, can have self-interactions of the form $A_\mu A^\mu A_\nu A^\nu$ or $A_\mu A^\mu (\partial_\nu A^\nu)$, which are however not allowed for photons because of gauge invariance. In the QFT of the strong interactions these terms do however exist ($\rightsquigarrow$ gluons have self-interactions), and they are responsible for the qualitative difference between so-called abelian and non-abelian gauge theories. Finally, spin-1/2 particles do not have any self-interactions at all in a renormalizable QFT since any Lorentz-invariant product of the form $(\bar\psi \ldots \psi)(\bar\psi \ldots \psi)$ would already have dimension 6.

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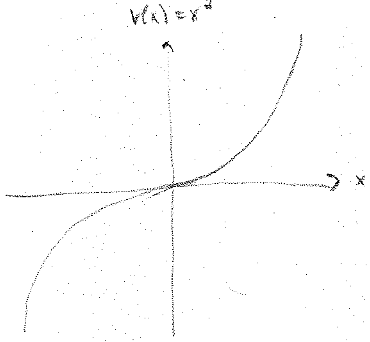
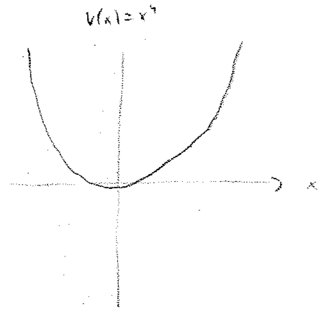
* This is similar to a one-dimensional system with $L = T - V$ and



no stable ground state



stable ground state at $x = 0$

In a theory with different particle species, there exist on the other hand a variety of interaction terms like e.g.

- $\bar{\psi}\gamma^\mu\psi A_\mu$ — electron-photon interaction
- $\bar{\psi}\psi\phi$ — top-Higgs interaction
- $A_\mu(\partial^\mu\phi)\phi$

but the structure of the allowed interactions is again highly constrained in renormalisable QFTs.

Throughout this section we find it convenient to illustrate the new aspects of interacting QFTs with an explicit example, which is ϕ^4-theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

The Euler-Lagrange equations of this theory read

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu(\partial^\mu\phi) - \left(-m^2\phi - \frac{\lambda}{3!}\phi^3\right) = 0$$

$$\Rightarrow\quad (\partial^2 + m^2)\,\phi(x) = -\frac{\lambda}{3!}\phi^3(x)$$

which is a <u>non-linear</u> differential equation that we do not know to solve exactly.

One therefore has to resort to approximate solutions, and the two most popular approaches are:

- time-dependent perturbation theory

  Whenever a theory is weakly coupled, i.e. $\alpha \ll 1$, the interacting theory resembles the free theory and one can find a solution as a perturbative expansion in the coupling constant that can be systematically improved.

- simulations on a spacetime lattice

  For strongly-interacting theories with $\alpha = O(1)$, the most popular method starts from (euclidean) path integrals, which are evaluated numerically on a discretized spacetime lattice.

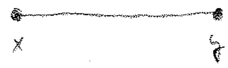
In this lecture we will focus on the perturbative approach, which will lead to a nice pictorial representation of scattering amplitudes in terms of Feynman diagrams.

The outline of this chapter is as follows. We will first learn how to compute (time-ordered) correlation functions in perturbation theory, which are the basic ingredients of relativistic QFTs. In particular, we will see that the mathematical expressions of the terms in the perturbative expansion follow a specific pattern and that they can be systematically deduced from a handful of so-called Feynman rules. We will then introduce the S-matrix and learn how to compute scattering cross sections and particle decay rates in QFT. Whereas most of the new concepts will be introduced for theories with scalar fields for simplicity, we will discuss the specifics of fermionic theories towards the end of this chapter.

4.1 Time-dependent perturbation theory

In the previous chapter we introduced the Green's function or two-point function

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \Delta_F (x-y) =$$



which represents the amplitude for the propagation of a particle from x to y (if $y^0 > x^0$) and from y to x (for $x^0 > y^0$) in a free theory with Hamiltonian H_0. Here $\phi(x)$ is a time-dependent Heisenberg operator

$$\phi(t, \vec{x}) = e^{iH_0 (t-t_0)} \underbrace{\phi(t_0, \vec{x})}_{\text{time-independent operator in Schrödinger picture}} e^{-iH_0(t-t_0)}$$

which has an expansion in terms of plane-wave solutions and creation and annihilation operators

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(e^{-ipx} a(p) + e^{ipx} a^\dagger(p) \right)$$

The particle states are on the other hand time-independent, and we have chosen them to be eigenstates of the Hamiltonian H_0. In particular, the vacuum state $|0\rangle$ is annihilated by all $a(p)$ and it is therefore the state of lowest energy $E = 0$.

We are now going to evaluate the same object

$$\langle \Omega | T\, \phi(x)\, \phi(y) | \Omega \rangle$$

in an interacting theory with $H = H_0 + H_{int}$, where H_0 is e.g. the free Hamiltonian of a real scalar field and $H_{int} = \int d^3x \, \frac{1}{4!} \phi^4(x)$ in ϕ^4-theory. Due to the complicated non-linear Euler-Lagrange equations in interacting theories, the Heisenberg operator $\phi(x)$ then no longer has a simple expansion in plane-wave solutions, and the vacuum state of the interacting theory $|\Omega\rangle$ becomes a non-trivial eigenstate of the full Hamiltonian H.

The time evolution of the field operator is furthermore now governed by the full Hamiltonian H with

$$\phi(t,\vec{x}) = e^{iH(t-t_0)} \underbrace{\phi(t_0,\vec{x})}_{\text{time-independent operator in Schrödinger picture}} e^{-iH(t-t_0)}$$

We then define an <u>interaction-picture field</u> $\phi_I(x)$, which absorbs the time evolution of the free Hamiltonian H_0

$$\phi_I(t,\vec{x}) \equiv e^{iH_0(t-t_0)} \underbrace{\phi(t_0,\vec{x})}_{\text{Schrödinger picture reference time } t_0} e^{-iH_0(t-t_0)}$$

The interaction-picture field therefore has the usual expansion

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left(e^{-ipx} a(p) + e^{ipx} a^\dagger(p) \right)$$

in terms of creation and annihilation operators, which act on the particle state of the free theory, i.e. e.g. $a(p)|0\rangle = 0$ but $a(p)|\Omega\rangle \neq 0$.

Our goal consists in expressing the Heisenberg operator $\phi(x)$ and the vacuum state $|\Omega\rangle$ in terms of $\phi_I(x)$ and $|0\rangle$ to relate calculations in interacting theories to the familiar Fock space manipulations of a free theory (in a perturbative expansion).

We start with the field operator $\phi(x)$. By eliminating the Schrödinger picture field, we get

$$\phi(x) = e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(x) e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$\equiv \mathcal{U}^\dagger(t,t_0)\, \phi_I(x)\, \mathcal{U}(t,t_0)$$

where

$$\mathcal{U}(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

is a unitary operator known as the <u>time-evolution operator</u>.

We still need to express the time-evolution operator in terms of $\phi_I(x)$.

To do so, we note that U obeys the boundary condition $U(t_0,t_0) = 1$,

and so it can be determined from a first-order differential equation

$$i \frac{\partial}{\partial t} U(t,t_0) = e^{iH_0(t-t_0)} (H-H_0) e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$= H_I(t)\, U(t,t_0)$$

where

$$H_I(t) = e^{iH_0(t-t_0)} \underset{\text{(reference time } t_0}{H_{int}} e^{-iH_0(t-t_0)}$$

is the interaction Hamiltonian in the interaction picture (since its time evolution is governed by H_0). Whenever H_{int} is a polynomial in the field $\phi(x)$, $H_I(t)$ has the <u>same</u> functional form as H_{int} but in terms of the interaction-picture field $\phi_I(x)$, e.g.

$$H_I(t) = \int d^3x \, \frac{\lambda}{4!} \phi_I^4(x)$$

for ϕ^4-theory*.

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* We do not consider theories with derivative interactions here, which are more easily discussed in the path-integral formalism.

We can formally integrate the above differential equation to

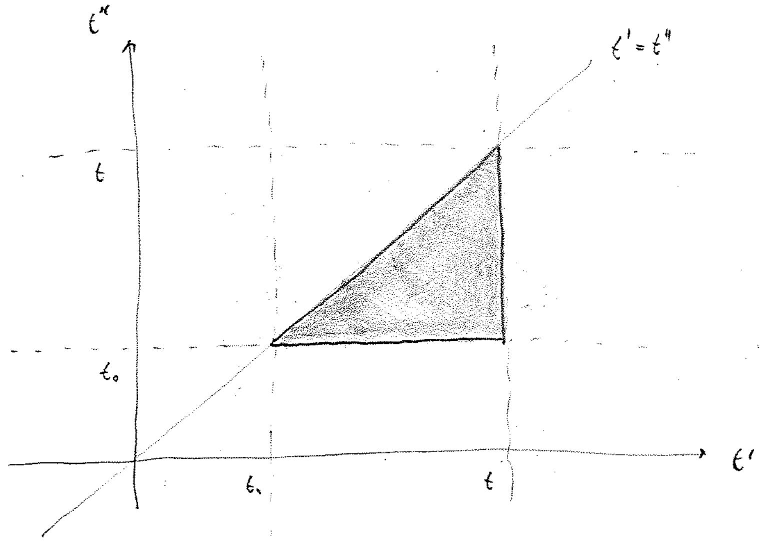
$$U(t,t_0) = 1 - i \int_{t_0}^{t} dt' \, H_I(t') \, U(t',t_0)$$

which reflects the boundary condition $U(t_0,t_0) = 1$. We may then construct an iterative solution as

$$U(t,t_0) = 1 - i \int_{t_0}^{t} dt' \, H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, H_I(t') H_I(t'') + \dots$$

which looks like we are building up an exponential except that the operators $H_I(t)$ do not necessarily commute (recall that $[\phi(x),\phi(y)] \neq 0$ inside the light cone).

We therefore rewrite the second term in the expansion in a symmetric form. To do so, let us illustrate the integration domain (for $t > t_0$)



It follows

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, H_I(t') \, H_I(t'')$$

$$= \int_{t_0}^{t} dt'' \int_{t''}^{t} dt' \, H_I(t') \, H_I(t'') \qquad \text{(exchange order of integrations)}$$

$$= \int_{t_0}^{t} dt' \int_{t'}^{t} dt'' \, H_I(t'') \, H_I(t') \qquad \text{(rename } t' \leftrightarrow t'')$$

We may therefore rewrite the expansion in $U(t,t_0)$ as an average

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, H_I(t') \, H_I(t'')$$

$$= \frac{1}{2} \int_{t_0}^{t} dt' \left(\int_{t_0}^{t'} dt'' \, H_I(t') \, H_I(t'') + \int_{t'}^{t} dt'' \, H_I(t'') \, H_I(t') \right)$$

$$= \frac{1}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \left(\theta(t'-t'') \, H_I(t') \, H_I(t'') + \theta(t''-t') \, H_I(t'') \, H_I(t') \right)$$

$$= \frac{1}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \; T \, H_I(t') \, H_I(t'')$$

where T is the usual time-ordering prescription. One proceeds similarly for the higher-order terms and obtains

$$U(t,t_0) = T \exp\left(-i \int_{t_0}^{t} dt' \, H_I(t') \right) \qquad (t \geq t_0)$$

where the time-ordering of an exponential is defined as the usual Taylor series with each term time-ordered

$$U(t,t_0) = 1 - i \int_{t_0}^{t} dt' \, H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \; T \, H_I(t') \, H_I(t'') + \ldots$$

which is known as the Dyson series.

Remarks:

- The definition of the time-evolution operator on page 235 refers to the time t_0, at which the operators in the Schrödinger-, Heisenberg- and interaction picture coincide. For arbitrary times t', one defines

$$U(t,t') = e^{iH_0(t-t_0)} \, e^{-iH(t-t')} \, e^{-iH_0(t'-t_0)}$$

and one easily verifies that this operator is unitary and obeys the same differential equation with boundary condition $U(t',t') = 1$. We therefore have

$$U(t,t') = T\exp\left(-i \int_{t'}^{t} dt'' \, H_I(t'')\right) \qquad (t \geq t')$$

- The time-evolution operator satisfies the composition rules

$$U(t_1,t_2)\, U(t_2,t_3) = U(t_1,t_3)$$

$$U(t_1,t_3)\, U^{-1}(t_2,t_3) = U(t_1,t_2)$$

Let us now turn to the vacuum state of the interacting theory $|\Omega\rangle$.

How can we relate $|\Omega\rangle$ to the vacuum state of the free theory $|0\rangle$?

We start from the expression

$$e^{-iH(T+t_0)}|0\rangle = \sum_n |n\rangle\langle n| e^{-iH(T+t_0)}|0\rangle$$

where we have inserted a complete set of eigenstates of the full Hamiltonian H with

$$H|n\rangle = E_n|n\rangle$$

of which $|\Omega\rangle$ is the eigenstate of lowest energy $E_\Omega < E_n \;\forall n \neq \Omega$.

It follows

$$e^{-iH(T+t_0)}|0\rangle = e^{-iE_\Omega(T+t_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{-iE_n(T+t_0)}|n\rangle\langle n|0\rangle$$

As $E_n > E_\Omega$ the terms in the sum vanish faster than the vacuum state in the limit $T\to\infty(1-i\varepsilon)$. We may thus write

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_\Omega(T+t_0)}\langle\Omega|0\rangle}$$

as long as $\langle\Omega|0\rangle \neq 0$, which is a reasonable assumption in a perturbative expansion.

We want to rewrite this expression in terms of the time-evolution operator U. To do so, we use

$$|0\rangle = e^{-iH_0(-T-t_0)}|0\rangle$$

which holds since $H_0|0\rangle = 0$. We then obtain

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(t_0-(-T))}\, e^{-iH_0(-T-t_0)}|0\rangle}{e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle}$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{U(t_0,-T)|0\rangle}{e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle}$$

which tells us that we can get $|\Omega\rangle$ by evolving $|0\rangle$ from the infinite past to t_0 with U.

Starting from $\langle 0| e^{-iH(T-t_0)}$, one derives similarly

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U(T,t_0)}{e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle}$$

which is <u>not</u> the hermitian conjugate of the above relation since this would lead to a representation of $\langle\Omega|$ in a different limit $T\to\infty(1+i\varepsilon)$.

We now have assembled all ingredients to express the correlation function $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$ in terms of $\phi_I(x)$ and $|0\rangle$. We first consider the time ordering $x^0 > y^0$

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | \mathcal{U}(T, t_0)\, \mathcal{U}^+(x^0, t_0)\, \phi_I(x)\, \mathcal{U}(x^0, t_0)\, \mathcal{U}^+(y^0, t_0)\, \phi_I(y)\, \mathcal{U}(y^0, t_0)\, \mathcal{U}(t_0, -T) | 0 \rangle}{e^{-iE_\Omega (T - t_0)} \langle 0 | \Omega \rangle\, e^{-iE_\Omega (t_0 - (-T))} \langle \Omega | 0 \rangle}$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | \mathcal{U}(T, x^0)\, \phi_I(x)\, \mathcal{U}(x^0, y^0)\, \phi_I(y)\, \mathcal{U}(y^0, -T) | 0 \rangle}{e^{-2iE_\Omega T} \langle 0 | \Omega \rangle \langle \Omega | 0 \rangle}$$

which is independent of the reference time t_0.

The vacuum state in the interacting theory should furthermore be normalised to 1, and hence

$$\langle \Omega | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | \mathcal{U}(T, -T) | 0 \rangle}{e^{-2iE_\Omega T} \langle 0 | \Omega \rangle \langle \Omega | 0 \rangle} \stackrel{!}{=} 1$$

For $x^0 > y^0$ our expression can thus be rewritten as

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | \mathcal{U}(T, x^0) \, \phi_I(x) \, \mathcal{U}(x^0, y^0) \, \phi_I(y) \, \mathcal{U}(y^0, -T) \, | 0 \rangle}{\langle 0 | \mathcal{U}(T, -T) | 0 \rangle}$$

and we recall that $\mathcal{U}(t, t') = T \exp\left(-i \int_{t'}^{t} dt'' \, H_I(t'')\right)$ for $t \geq t'$.

The fields in the above expression are thus fully time-ordered and the same would be true if we considered $\langle \Omega | \phi(y) \phi(x) | \Omega \rangle$ for $y^0 > x^0$. This allows us to put everything into one time-ordering operator such that our final result for the two-point function in the interacting theory becomes

$$\boxed{\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T \, \phi_I(x) \, \phi_I(y) \, \exp\left(-i \int_{-T}^{T} dt' \, H_I(t')\right) | 0 \rangle}{\langle 0 | T \exp\left(-i \int_{-T}^{T} dt' \, H_I(t')\right) | 0 \rangle}}$$

Remarks:

- The result is ideally suited for a perturbative approach in which one truncates the Taylor expansion of the exponentials at a certain order.
- We will see later that the $i\varepsilon$-prescription (which was needed to project out the vacuum state $|\Omega\rangle$) is related to the Feynman prescription of the propagators. If we keep this prescription implicit, the exponentials can be written in a more compact form

$$\lim_{T\to\infty} \exp\left(-i\int_{-T}^{T} dt\; H_I(t)\right) = \exp\left(-i\int d^4x\; \mathcal{H}_I(x)\right)$$

where $\mathcal{H}_I(x)$ is the interaction Hamiltonian density in the interaction picture.

- The generalisation to n-point functions is obvious: for each Heisenberg operator $\phi(x_i)$ on the left-hand side of the equation just add a corresponding interaction-picture field $\phi_I(x_i)$ on the right-hand side.

4.2 Feynman rules

The calculation of time-ordered correlation functions in perturbation theory is thus reduced to evaluating expressions of the form

$$\langle 0 | T\, \phi_I(x_1) \ldots \phi_I(x_n) | 0 \rangle$$

As the interaction-picture field $\phi_I(x)$ has the same decomposition in terms of creation and annihilation operators as a free field (which act on the free particle states), the calculations are similar to the ones we performed in chapter 3 (see e.g. page 143). The goal of this section consists in developing an efficient method for performing these calculations.

To do so, we decompose the interaction-picture field into annihilation and creation fields, $\phi_I(x) = \phi_I^{(+)}(x) + \phi_I^{(-)}(x)$, with

$$\phi_I^{(+)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx} a(p)$$

$$\phi_I^{(-)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{ipx} a^+(p)$$

In order to keep the notation transparent, we write

$$\phi_I^{(+)}(x) \to \phi^+(x) \qquad \phi_I^{(-)}(x) \to \phi^-(x)$$

in the following, but is should be clear that we consider fields in the interaction picture in this section.

As we are interested in evaluating vacuum matrix elements, we bring the operators into normal order (see page 133), i.e. we commute all ϕ^- fields to the left of all ϕ^+ fields. E.g.

$$\phi(x)\,\phi(y) = \phi^+(x)\,\phi^+(y) + \phi^+(x)\,\phi^-(y) + \phi^-(x)\,\phi^+(y) + \phi^-(x)\,\phi^-(y)$$

$$= \phi^+(x)\,\phi^+(y) + \phi^-(y)\,\phi^+(x) + \phi^-(x)\,\phi^+(y) + \phi^-(x)\,\phi^-(y)$$

$$+ [\phi^+(x), \phi^-(y)]$$

$$= :\phi(x)\,\phi(y): + [\phi^+(x), \phi^-(y)]$$

and similarly

$$\phi(y)\,\phi(x) = :\phi(x)\,\phi(y): + [\phi^+(y), \phi^-(x)]$$

since the ordering of the fields within the normal-ordering symbol does not matter.

For the time-ordered product we thus obtain

$$T\,\phi(x)\,\phi(y) = \,:\phi(x)\,\phi(y):\, + \overline{\phi(x)\,\phi(y)}$$

where we introduced the Wick contraction of two fields as

$$\overline{\phi(x)\,\phi(y)} \equiv \theta(x^0-y^0)\,[\phi^+(x), \phi^-(y)] + \theta(y^0-x^0)\,[\phi^+(y), \phi^-(x)]$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0} \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{2q^0}$$

$$\Big\{ \theta(x^0-y^0)\; e^{-ipx}\; e^{iqy}\; \overbrace{[a(p), a^+(q)]}^{(2\pi)^3 2q^0 \delta^{(3)}(\vec p-\vec q)}$$

$$+\,\theta(y^0-x^0)\; e^{ipx}\; e^{-iqy}\; \underbrace{[a(q), a^+(p)]}_{(2\pi)^3 2q^0 \delta^{(3)}(\vec p-\vec q)} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\;\Big\{ \theta(x^0-y^0)\; e^{-ip(x-y)}$$

$$+\,\theta(y^0-x^0)\; e^{ip(x-y)} \Big\}$$

$$= \Delta_F(x-y)$$

which is just the Feynman propagator as can be seen by comparison with the expressions on page 149.

When we take the vacuum matrix element, the contribution from the normal-ordered field operators drops out since

$$\phi^+(x)\,|0\rangle = 0 \quad \text{and} \quad \langle 0|\,\phi^-(x) = 0.$$

We thus obtain the well-known result

$$\langle 0|\, T\phi(x)\phi(y)\,|0\rangle = \langle 0|\, \contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y)\,|0\rangle = \Delta_F(x-y)$$

Our strategy therefore consists in rewriting time-ordered products in terms of normal-ordered products. This can be achieved with the help of <u>Wick's theorem</u>, which states that

$$\boxed{T\phi(x_1)\ldots\phi(x_n) = :\phi(x_1)\ldots\phi(x_n) + \text{all possible contractions}:}$$

The theorem obviously holds for $n=2$, and we will prove it by induction in the tutorials.

As an example consider the 4-point function (writing $\phi(x_i)\equiv\phi_i$)

$$\begin{aligned}
T\phi_1\phi_2\phi_3\phi_4 = :\ & \phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1\phi_2}{\phi}\phi_1\phi_2\phi_3\phi_4 \\
& + \contraction{}{\phi}{_1\phi_2\phi_3}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{\phi_1}{\phi}{_2}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{\phi_1}{\phi}{_2\phi_3}{\phi}\phi_1\phi_2\phi_3\phi_4 \\
& + \contraction{\phi_1\phi_2}{\phi}{_3}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1}{\phi}\contraction{\phi_1\phi_2}{\phi}{_3}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1\phi_2}{\phi}\bcontraction{\phi_1}{\phi}{_2\phi_3}{\phi}\phi_1\phi_2\phi_3\phi_4 \\
& + \contraction[2ex]{}{\phi}{_1\phi_2\phi_3}{\phi}\contraction{\phi_1}{\phi}{_2}{\phi}\phi_1\phi_2\phi_3\phi_4\ :
\end{aligned}$$

When we take the vacuum matrix element of this expression, all terms with uncontracted fields vanish (since vacuum matrix elements of normal-ordered products vanish), and so

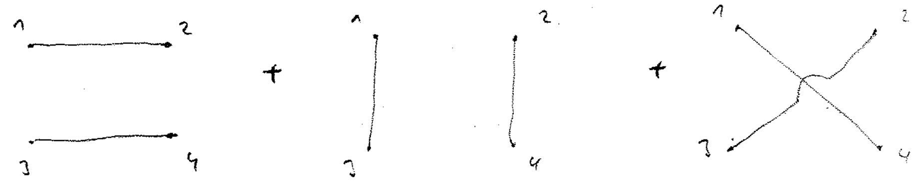
$$\langle 0|\, T\phi_1\phi_2\phi_3\phi_4\, |0\rangle$$

$$= \langle 0|\, \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \overline{\phi_1\overline{\phi_2\phi_3}\phi_4}\, |0\rangle$$

$$= \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4)$$

$$+ \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)$$

=



which is the coherent sum of all amplitudes that describe the free propagation of two particles between two points (from earlier to later times).

Let us now come back to the two-point function in interacting theories from page 243. Keeping the $i\varepsilon$-prescription implicit for the moment, the numerator of this expression becomes for $\mathcal{H}_I(x) = \frac{\lambda}{4!}\phi^4(x)$ (recall that fields without index are interaction-picture fields in this section)

$$\langle 0 | T\, \phi(x)\phi(y) \exp\left(-i \int d^4z \frac{\lambda}{4!}\phi^4(z)\right) |0\rangle$$

$$= \langle 0 | T\, \phi(x)\phi(y) |0\rangle$$

$$- \frac{i\lambda}{4!} \int d^4z \, \langle 0 | T\, \phi(x)\phi(y)\phi^4(z) |0\rangle + \mathcal{O}(\lambda^2)$$

The first term is just the free propagator and the higher-order terms can be evaluated with Wick's theorem. But since the interaction $\mathcal{H}_I(x)$ involves fields at the same point, many contractions lead to the same Feynman diagrams.

Let us consider the first-order term explicitly. First of all, we recall that only full contractions have non-vanishing vacuum matrix elements, and for six fields there are $5 \cdot 3 \cdot 1 = 15$ terms with full contractions. These terms furthermore fall into two classes depending on whether or not the fields $\phi(x)$ and $\phi(y)$ are contracted with each other

- $\overbrace{\phi(x)\,\phi(y)}^{1}\;\overbrace{\phi(z)\,\phi(z)}^{3}\;\overbrace{\phi(z)\,\phi(z)}^{1}$

  since there are 3 possibilities to contract the fields at the point z with each other

- $\phi(x)\,\phi(y)\,\phi(z)\,\phi(z)\,\overbrace{\phi(z)\,\phi(z)}^{1}$ (4, 3)

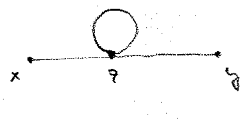
  and so there are 12 possibilities to contract x with z and y with z and to contract the remaining fields at the point z with each other

We thus obtain for the first-order correction to the numerator

$$-\frac{i\lambda}{4!}\int d^4z\,\langle 0|\,T\,\phi(x)\,\phi(y)\,\phi^4(z)\,|0\rangle$$

$$= -\frac{i\lambda}{4!}\,3\int d^4z\;\Delta_F(x-y)\;\Delta_F(z-z)\;\Delta_F(z-z)$$

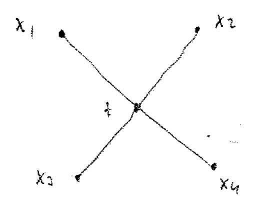
$$-\frac{i\lambda}{4!}\,12\int d^4z\;\Delta_F(x-z)\;\Delta_F(y-z)\;\Delta_F(z-z)$$

=  + 

where we again represented a propagator $\Delta_F(x_1-x_2)$ by a line between the points x_1 and x_2.

Apart from the external points x and y, we thus see that the interaction $\mathcal{H}_I(z)$ introduces additional (internal) points, where four lines meet (in ϕ^4-theory). In accordance with the superposition principle of quantum mechanics, we then have to sum over the internal points and we therefore associate the expression $(-i\lambda)\int d^4z$ with each vertex. But what about the factors $\frac{3}{4!} = \frac{1}{8}$ and $\frac{12}{4!} = \frac{1}{2}$?

These factors indicate that the diagrams we considered above have certain (topological) symmetries. For an unsymmetric diagram like



$$\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\,\phi(z)\,\phi(z)\,\phi(z)\,\phi(z)$$

the factor is just $\frac{4\cdot 3\cdot 2\cdot 1}{4!} = 1$, which explains why we have written the interaction term as $\frac{\lambda}{4!}\phi^4(z)$. In contrast, the diagrams from above have lines that start and end at the same point, which yields a non-trivial <u>symmetry factor</u>.

In realistic theories like QED there are no non-trivial symmetry factors since the fields at the corresponding vertex $\bar{\Psi}\gamma^\mu\Psi A_\mu$ are all different. In ϕ^4-theory, however, the symmetry factors are more complicated and they can be evaluated according to the following rules:

(i) factor of 2 for each line that starts and ends at the same vertex

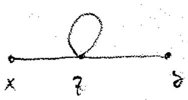
(ii) factor of $n!$ for n lines that connect the same vertices

(iii) factor of $n!$ for n equivalent vertices

In addition one gets a factor of $n!$ from the interchange of n vertices, which is however cancelled by the factor $\frac{1}{n!}$ from the Taylor expansion of $\exp\left(-i\int d^4z\, \mathcal{H}_I(z)\right)$.

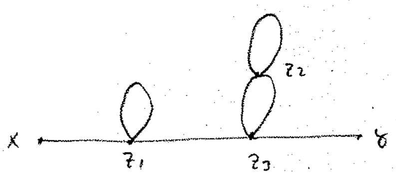
Let us apply these rules to the above diagrams

(i) $2 \cdot 2$
(ii) 2 $\Rightarrow S = 8$ ✓



(i) 2 $\Rightarrow S = 2$ ✓

Let us consider two more examples



(i) $2\cdot 2$ (z_1, z_2)

(ii) 2 $(z_2 - z_3)$ $\Rightarrow$ $S = 8$

<u>Check:</u>

4 3 6 2 1

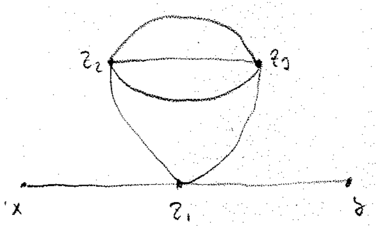
$xy \quad z_1z_1z_1z_1 \quad z_2z_2z_2z_2 \quad z_3z_3z_3z_3$

3 4

$$\frac{1}{3!}\; 3!\; \frac{4\cdot 3\cdot 6\cdot 2\cdot 1\cdot 3\cdot 4}{4!\;4!\;4!} = \frac{1}{8} \quad ✓$$

Taylor expansion ($\frac{1}{3!}$)

rename z_1, z_2, z_3 ($3!$)



(ii) $3!$ $(z_2 - z_3)$

(iii) 2 (z_2, z_3) $\Rightarrow$ $S = 12$

<u>Check:</u>

4

$xy \quad z_1z_1z_1z_1 \quad z_2z_2z_2z_2 \quad z_3z_3z_3z_3$

3

$z_1z_1z_1z_1$: call these points 1, 2

$z_2z_2z_2z_2$: A, B, C, D

$z_3z_3z_3z_3$: W, X, Y, Z

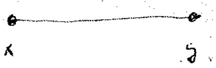
$\Rightarrow$ 1A2W 1B2W
1A2X ⋮
1A2Y
1A2Z

$\Rightarrow$ $4\cdot 4$ possibilities with $3!$ contractions for contracting the remaining fields in each case

$$\frac{1}{3!}\; 3!\; \frac{4\cdot 3\cdot 4\cdot 4\cdot 3!}{4!\;4!\;4!} = \frac{1}{12} \quad ✓$$

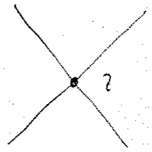
We are now in the position to formulate the <u>Feynman rules</u> of ϕ^4-theory for the computation of correlation functions (<u>in position space</u>)

1) For each propagator



$$= \Delta_F(x-y)$$

2) For each vertex



$$= -i\lambda \int d^4z$$

3) Divide by the symmetry factor of the diagram (which can be determined according to the rules (i) - (iii) from above)

For the numerator of the two-point function in ϕ^4-theory (~ page 243) we thus draw all diagrams with two external points and k vertices in k-th order in perturbation theory. With the help of the Feynman rules, it then becomes an easy task to translate the Feynman diagrams into the corresponding mathematical expressions. The numerator is finally given by the sum of all diagrams that contribute to the considered order in perturbation theory.

As the Feynman propagator takes a simple form in Fourier space, one often prefers to work with Feynman rules that are directly formulated in momentum space. So let us reconsider the first-order correction from page 251 and insert the representation

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \tilde{\Delta}_F(p)$$

with $\tilde{\Delta}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$ for each propagator.

$$\Rightarrow -\frac{i\lambda}{4!}\int d^4z\, \langle 0|\, T\, \phi(x)\,\phi(y)\,\phi^4(z)\,|0\rangle$$

$$= -\frac{i\lambda}{4!}\, 3 \cancel{\int d^4z} \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \tilde{\Delta}_F(p) \int \frac{d^4q}{(2\pi)^4}\, \cancel{e^{-iq(z-z)}}\, \tilde{\Delta}_F(q)$$

$$\int \frac{d^4k}{(2\pi)^4}\, \cancel{e^{-ik(z-z)}}\, \tilde{\Delta}_F(k) \qquad (2\pi)^4\,\delta^{(4)}(q-q+k-k)$$

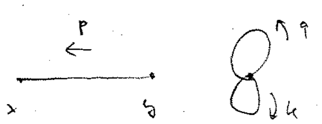
$$-\frac{i\lambda}{4!}\, 12 \cancel{\int d^4z} \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-\cancel{z})}\, \tilde{\Delta}_F(p) \int \frac{d^4q}{(2\pi)^4}\, e^{-iq(\cancel{z}-y)}\, \tilde{\Delta}_F(q)$$

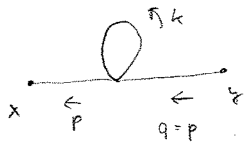
$$\int \frac{d^4k}{(2\pi)^4}\, \cancel{e^{-ik(z-z)}}\, \tilde{\Delta}_F(k) \qquad (2\pi)^4\,\delta^{(4)}(p-q+k-k)$$

↳ $q=p$

$$= -\frac{i\lambda}{4!}\, 3 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4}\, e^{-ipx}\, e^{ipy}\, \tilde{\Delta}_F(p)\, \tilde{\Delta}_F(q)\, \tilde{\Delta}_F(k)\, (2\pi)^4\, \delta^{(4)}(0)$$

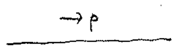
$$-\frac{i\lambda}{4!}\, 12 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4}\, e^{-ipx}\, e^{ipy}\, \tilde{\Delta}_F(p)\, \tilde{\Delta}_F(p)\, \tilde{\Delta}_F(k)$$

= 

+ 

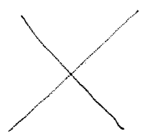
We thus see that the integral $\int d^4z$ turns into a δ-function, which reflects 4-momentum conservation at the vertex. The <u>momentum-space Feynman rules</u> for the computation of correlation functions in ϕ^4-theory then become:

1) For each propagator



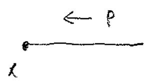
$$= \tilde{\Delta}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

2) For each vertex



$$= -i\lambda$$

3) For each external point



$$= e^{-ipx}$$

4) Impose momentum conservation at each vertex and integrate over all undetermined momenta $\int \frac{d^4p}{(2\pi)^4}$

5) Divide by the symmetry factor of the diagram

With these Feynman rules we reproduce the second diagram in the above expression. But what about the (divergent) factor $(2\pi)^4\,\delta^{(4)}(0)$ in the first diagram?

From the technical point of view, the factor $(2\pi)^4 \delta^{(4)}(0)$ arises because the internal point z is disconnected from the external points x and y in the first diagram. Diagrams with this topological structure are called <u>vacuum bubbles</u>.

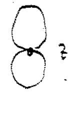
To better understand the role of vacuum bubbles, let us now turn to the denominator of the two-point function in ϕ^4-theory from page 243. In analogy to the numerator, we obtain

$$\langle 0 | T \exp\left(-i \int d^4z \, \frac{\lambda}{4!} \phi^4(z)\right) | 0 \rangle$$

$$= \underbrace{\langle 0 | 0 \rangle}_{=1} - \frac{i\lambda}{4!} \int d^4z \, \langle 0 | T \phi^4(z) | 0 \rangle + O(\lambda^2)$$

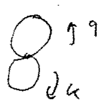
and the first-order correction to the denominator becomes

$$-\frac{i\lambda}{4!} \int d^4z \, \langle 0 | T \phi^4(z) | 0 \rangle$$

$$= -\frac{i\lambda}{4!} \, 3 \int d^4z \, \Delta_F(z-z) \, \Delta_F(z-z) =$$ 

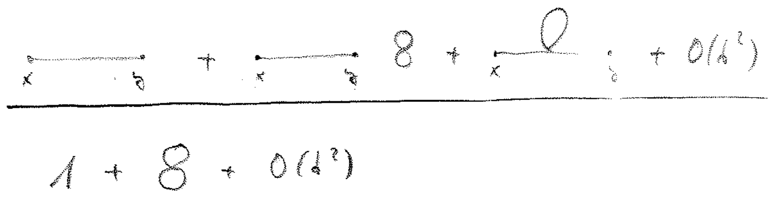
$$= -\frac{i\lambda}{4!} \, 3 \int d^4z \int \frac{d^4q}{(2\pi)^4} e^{-iq(z-z)} \tilde{\Delta}_F(q) \int \frac{d^4k}{(2\pi)^4} e^{-ik(z-z)} \tilde{\Delta}_F(k)$$

$$= -\frac{i\lambda}{4!} \, 3 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \tilde{\Delta}_F(q) \, \tilde{\Delta}_F(k) \, (2\pi)^4 \delta^{(4)}(0)$$

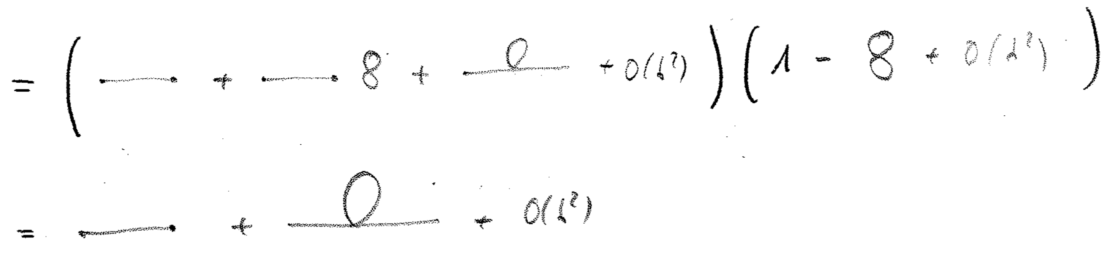
$=$ 

We thus see that in the ratio

$$\langle \Omega | T \phi_H(x) \phi_H(y) | \Omega \rangle =$$



Heisenberg operators



the vacuum bubbles drop out. It turns out that the same is true to all orders in perturbation theory since the vacuum bubbles are by definition disconnected from the external points and so they simply factor out.

Our formula for the computation of n-point functions in interacting theories from page 243 can therefore finally be written very concisely as

$$\boxed{\langle \Omega | T \phi_H(x_1) \dots \phi_H(x_n) | \Omega \rangle = \left(\begin{array}{l} \text{sum of all diagrams with } n \text{ external} \\ \text{points without vacuum bubbles} \end{array} \right)}$$

and the momentum-space Feynman rules from page 257 tell us how to translate those diagrams into the corresponding mathematical expressions in ϕ^4-theory.

Let us now come back to the limit $T \to \infty(1-i\varepsilon)$, which we have completely disregarded in this section. In the above expressions, we should therefore replace

$$\int d^4 z \longrightarrow \lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \int d^3 z$$

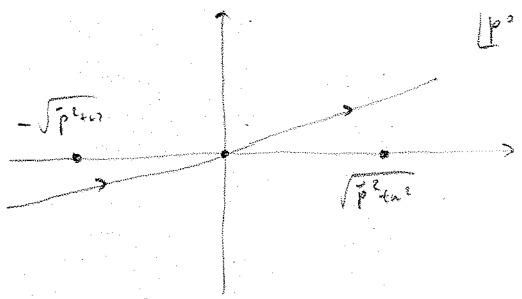
which is relevant for the Fourier transformations like

$$\lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \; e^{i(p^0-q^0)z^0} \qquad z^0 = \tilde{z}^0(1-i\varepsilon)$$

$$= \int_{-\infty}^{\infty} d\tilde{z}^0 \; e^{i(p^0-q^0)\tilde{z}^0(1-i\varepsilon)} \qquad \begin{aligned} p^0 &= \tilde{p}^0(1+i\varepsilon) \\ q^0 &= \tilde{q}^0(1+i\varepsilon) \end{aligned}$$

$$= \int_{-\infty}^{\infty} d\tilde{z}^0 \; e^{i(\tilde{p}^0-\tilde{q}^0)\tilde{z}^0} = (2\pi)\,\delta(\tilde{p}^0-\tilde{q}^0)$$

which is the same result as before except that the contour have been rotated slightly in the complex plane according to



But this rotation precisely corresponds to the Feynman prescription of the propagators, i.e. our results from above are consistent with the $\lim_{T\to\infty(1-i\varepsilon)}$ prescription!

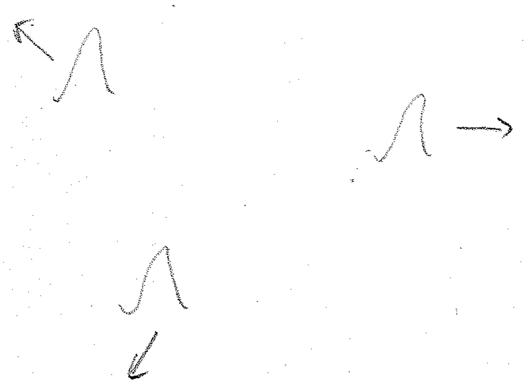
4.3 S-matrix

So far we have learned how to compute vacuum correlation functions in perturbation theory. In this section we will extend the formalism to S-matrix elements, which are the objects that are needed to describe scattering reactions.

The main idea of scattering theory is that interactions "turn off" at large negative and positive times. One assumes that one can prepare wavepackets of <u>free</u> particles that are sufficiently localised in the remote past

$(t \to -\infty)$

which in the course of the scattering process overlap and interact with each other, and produce final-state particles that can again be considered as wavepackets of free particles in the distant future



$(t \to +\infty)$

In practise one is not interested in the complicated time evolution of a scattering process, but one rather wants to determine the transition probability between the initial and the final state.

In addition to what we have seen before, we thus need to understand how to describe the external states of a scattering process in QFT. To this end, one assumes that the particle detector measures the momenta of the scattered particles, i.e. it projects onto the eigenstates of momentum. Due to the finite detector resolution, one is moreover usually not sensitive to the momentum spread of the initial-state wavepackets. One therefore often approximates the initial-state wavepackets by plane waves with fixed momentum, although this is strictly speaking in conflict with the assumption that the wavepackets are localised (for a more careful treatment of wavepackets see e.g. Peskin / Schroeder, chapter 4.5, and references therein).

Our goal thus consists in computing transition amplitudes of the form

$$ {}_{out}\langle p_1 p_2 \dots | k_A k_B \rangle_{in} $$

where the <u>in- and out-states</u> are eigenstates of the full Hamiltonian $H = H_0 + H_{int}$, which approach free-particle states in the limit $t \to \pm\infty$.

In other words, the corresponding (time-dependent) Schrödinger-picture states satisfy

$$ |k_A k_B ; t\rangle_{in} = e^{-iHt} |k_A k_B ; t=0\rangle_{in} \xrightarrow{t\to-\infty} e^{-iH_0 t} |k_A k_B ; t=0\rangle $$

$$ {}_{out}\langle p_1 p_2 \dots ; t| = {}_{out}\langle p_1 p_2 \dots ; t=0| e^{iHt} \xrightarrow{t\to+\infty} \langle p_1 p_2 \dots ; t=0| e^{iH_0 t} $$

where the states on the right-hand side are free-particle states.

As the Schrödinger and Heisenberg states coincide at the reference time $t=0$, the corresponding relations between the in- and out-states and the free-particle states in the Heisenberg picture are

$$ |k_A k_B\rangle_{in} = \lim_{t\to-\infty} e^{iHt} e^{-iH_0 t} |k_A k_B\rangle $$

$$ {}_{out}\langle p_1 p_2 \dots | = \lim_{t\to+\infty} \langle p_1 p_2 \dots | e^{iH_0 t} e^{-iHt} $$

time-independent Heisenberg state
eigenstate of H

time-independent Heisenberg state
eigenstate of H_0

We thus obtain

$$
\begin{aligned}
{}_{out}\langle p_1 p_2 \ldots | k_A k_B \rangle_{in}
&= \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle p_1 p_2 \ldots | e^{iH_0 t_f} e^{-iH(t_f - t_i)} e^{-iH_0 t_i} | k_A k_B \rangle \\
&= \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle p_1 p_2 \ldots | U(t_f, t_i) | k_A k_B \rangle \\
&= \langle p_1 p_2 \ldots | S | k_A k_B \rangle
\end{aligned}
$$

where U is the time-evolution operator from page 239 and

$$
S \equiv \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} U(t_f, t_i) = T \exp\left(-i \int_{-\infty}^{\infty} dt \, H_I(t)\right)
$$

is the scattering operator or <u>S-matrix</u>.

<u>Remarks</u>:

- The S-matrix is a unitary operator, $SS^\dagger = S^\dagger S = \mathbb{1}$, which guarantees that probabilities are conserved in the scattering process*.

---

* This is actually quite subtle since the limit of a unitary operator need not be unitary, unless its range is the full Hilbert space. One therefore assumes that the in- and out states form a complete basis of the Hilbert space.

• The S-matrix can be written in the form

$$S = \mathbb{1} + i T$$

where the first term reflects the fact that the particles may not interact in the scattering process at all. The interesting part of the S-matrix is thus encoded in the transition operator or <u>T-matrix</u>.

As the 4-momentum is conserved in a scattering reaction, one writes

$$\langle p_1 p_2 \ldots | i T | k_A k_B \rangle$$

$$\equiv (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_i p_i) \; i\mathcal{M}(k_A k_B \to p_1 p_2 \ldots)$$

We will see below that the Lorentz-invariant transition matrix element $i\mathcal{M}(k_A k_B \to p_1 p_2 \ldots)$ enters the formulae for the computation of scattering cross sections and particle decay rates.

Let us now turn to the evaluation of S-matrix elements in perturbation theory. For concreteness we consider a $2 \to 2$ scattering process in ϕ^4-theory with

$$\langle p_1 p_2 | S | k_A k_B \rangle = \langle p_1 p_2 | T \exp\left(-i \int d^4x \, \frac{\lambda}{4!} \phi^4(x)\right) | k_A k_B \rangle$$

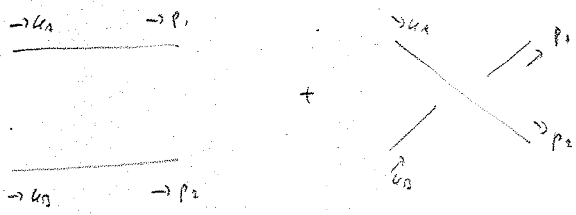
where $\phi(x)$ is an interaction-picture field and $|k_A k_B\rangle$ and $|p_1 p_2\rangle$ are free-particle states.

To lowest order in perturbation theory we have

$$\langle p_1 p_2 | k_A k_B \rangle$$

$$= \langle 0 | a(p_2)\, a(p_1)\, a^\dagger(k_A)\, a^\dagger(k_B) | 0 \rangle$$

$$= (2\pi)^6 \, 2k_A^0 \, 2k_B^0 \left\{ \delta^{(3)}(\vec{k}_A - \vec{p}_1)\, \delta^{(3)}(\vec{k}_B - \vec{p}_2) + \delta^{(3)}(\vec{k}_A - \vec{p}_2)\, \delta^{(3)}(\vec{k}_B - \vec{p}_1) \right\}$$

$=$ 

which gives a contribution to the $\mathbb{1}$-operator in S.

We next turn to the first-order correction

$$-i\frac{\lambda}{4!}\int d^4z \quad \langle p_1\, p_2 |T\phi^4(z)| k_A\, k_B\rangle$$

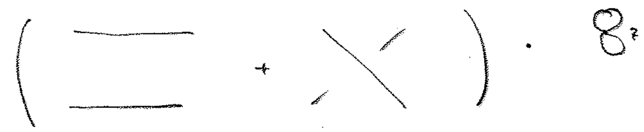
for which we can still use Wick's theorem to bring the operators into normal order

$$T\phi(z)\phi(z)\phi(z)\phi(z) = :\phi(z)\phi(z)\phi(z)\phi(z) + 6\,\phi(z)\phi(z)\overline{\phi(z)\phi(z)}$$
$$+\, 3\,\overline{\phi(z)\phi(z)}\;\overline{\phi(z)\phi(z)}:$$

but since the external states are not vacuum states, all terms now give a non-zero contribution.

Let us start with the term with two contractions

$$-\frac{i\lambda}{4!}\, 3\int d^4z \quad \Delta_F(z-z)\,\Delta_F(z-z) \quad \langle p_1\, p_2 | k_A\, k_B\rangle$$

$$=$$ 

which corresponds to the lowest-order term multiplied by a vacuum bubble, which again gives a contribution to the $\mathbb{1}$-operator in S.

In order to evaluate the remaining terms, we first note that

$$[\phi^+(x), a^+(p)] = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} e^{-iqx} \underbrace{[a(q), a^+(p)]}_{(2\pi)^3 2q^0 \delta^{(3)}(\vec{p}-\vec{q})} = e^{-ipx}$$

The term with one contraction involves the matrix element

$$\langle p_1 p_2 | : \phi(z) \phi(z) : | k_A k_B \rangle$$

$$= \langle 0 | a(p_2) a(p_1) [\cancel{\phi^+(z) \phi^+(z)}^{\,0} + 2 \phi^-(z) \phi^+(z) + \cancel{\phi^-(z) \phi^-(z)}^{\,0}] a^+(k_A) a^+(k_B) | 0 \rangle$$

$$= 2 \{ \langle 0 | a(p_2) a(p_1) \phi^-(z) a^+(k_A) \phi^+(z) a^+(k_B) | 0 \rangle$$

$$+ e^{-ik_A z} \langle 0 | a(p_2) a(p_1) \phi^-(z) a^+(k_B) | 0 \rangle \}$$

$$= 2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} e^{iqz}$$

$$\{ e^{-ik_B z} \langle p_1 p_2 | q k_A \rangle + e^{-ik_A z} \langle p_1 p_2 | q k_B \rangle \}$$

$$= 2 \{ e^{-ik_B z} e^{ip_2 z} (2\pi)^3 2k_A^0 \delta^{(3)}(\vec{k}_A - \vec{p}_1)$$

$$+ e^{-ik_B z} e^{ip_1 z} (2\pi)^3 2k_A^0 \delta^{(3)}(\vec{k}_A - \vec{p}_2)$$

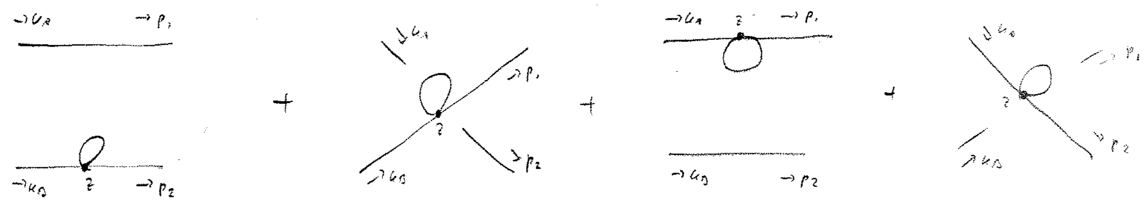
$$+ e^{-ik_A z} e^{ip_1 z} (2\pi)^3 2k_B^0 \delta^{(3)}(\vec{k}_B - \vec{p}_2)$$

$$+ e^{-ik_A z} e^{ip_2 z} (2\pi)^3 2k_B^0 \delta^{(3)}(\vec{k}_B - \vec{p}_1) \}$$

In terms of Feynman diagrams, we get

$$-i\frac{\lambda}{4!}\,6\int d^4z\;\Delta_F(z-z)\;\langle p_1 p_2|:\phi(z)\phi(z):|k_A k_B\rangle$$

$$=$$



which corresponds to the leading-order term with λ-corrections on the propagators. These diagrams thus again contribute to the $\mathbb{1}$-operator in S.

So let us finally consider the term without any contractions, which involves the matrix element

$$\langle p_1 p_2|:\phi(z)\phi(z)\phi(z)\phi(z):|k_A k_B\rangle$$

$$= \langle 0|a(p_2)a(p_1)\left[6\,\phi^-(z)\phi^-(z)\underline{\phi^+(z)\phi^+(z)\right]a^\dagger(k_A)}\,a^\dagger(k_B)|0\rangle$$

$$= 6\left\{\langle 0|a(p_2)a(p_1)\phi^-(z)\phi^-(z)a^\dagger(k_A)\phi^+(z)\underline{\phi^+(z)a^\dagger(k_B)}|0\rangle \longrightarrow 0\right.$$

$$\left.+\,2e^{-ik_Az}\langle 0|a(p_2)a(p_1)\phi^-(z)\phi^-(z)\underline{\phi^+(z)a^\dagger(k_B)}|0\rangle\right\}$$

$$= 12\,e^{-ik_Az}e^{-ik_Bz}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}e^{iqz}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}e^{ikz}\langle p_1p_2|q\,k\rangle$$

$$= 24\,e^{-ik_Az}e^{-ik_Bz}e^{ip_1z}e^{ip_2z}$$

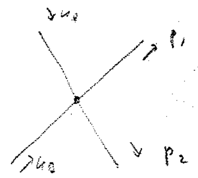
We thus obtain

$$-\frac{i\lambda}{4!}\int d^4z \;\langle p_1 p_2 |\, :\phi(z)\,\phi(z)\,\phi(z)\,\phi(z):\, | k_A k_B\rangle$$

$$= -i\frac{\lambda}{4!}\int d^4z \;24\; e^{-ik_A z}\, e^{-ik_B z}\, e^{ip_1 z}\, e^{ip_2 z}$$

$$= -i\lambda\, (2\pi)^4\, \delta^{(4)}(k_A + k_B - p_1 - p_2)$$

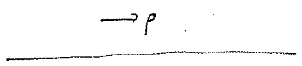
which contributes to the T-matrix and is indeed of the form anticipated on page 265. The lowest-order contribution to the transition matrix element is thus given by

$$i\mathcal{M}(k_A k_B \to p_1 p_2) = -i\lambda =$$ 

If we compare this expression with the momentum-space Feynman rules that we derived on page 257, we see that the external lines do not yield Feynman propagators here and there are obviously no factors like e^{-ipx} since the diagram has no external points.

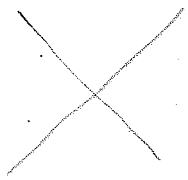
The <u>momentum-space Feynman rules</u> for the computation of scattering matrix elements can thus be summarised as follows

1) For each internal line



$$= \tilde{\Delta}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

2) For each vertex



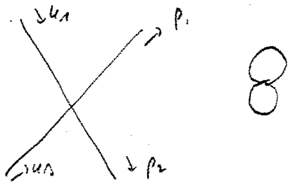
$$= -i\lambda$$

3) Impose momentum conservation at each vertex and integrate over all undetermined momenta $\int \frac{d^4p}{(2\pi)^4}$

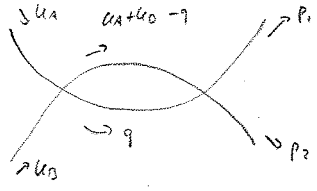
4) Divide by the symmetry factor of the diagram.

But which Feynman diagrams contribute to the scattering matrix element at higher orders?

First of all, it is clear that only diagrams in which all external lines are connected with each other describe a scattering process and give a contribution to the T-matrix. At order λ^2 there are three classes of diagrams of this type:

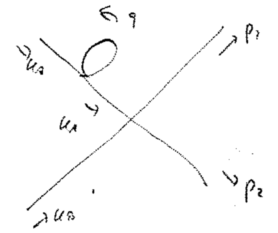
(i) 

These diagrams describe the leading-order scattering process in a non-trivial background. As they do not yield any new information on the scattering process itself, one should normalise the S-matrix in a way such that diagrams with vacuum bubbles drop out (similar to correlation functions). With such a normalisation, only fully connected diagrams contribute to the T-matrix.

(ii) 

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{(k_A+k_B-q)^2-m^2+i\varepsilon} \frac{i}{q^2-m^2+i\varepsilon}$$

These diagrams indeed give a non-trivial λ^2-correction to the scattering matrix element. We will learn how to evaluate such 1-loop diagrams in TPP II.

(iii) 

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{k_A^2-m^2+i\varepsilon} \frac{i}{q^2-m^2+i\varepsilon}$$

These diagrams turn out to be problematic since the propagator

$$\frac{i}{k_A^2-m^2+i\varepsilon} = \frac{i}{i\varepsilon} \to \infty$$ is ill-defined in the limit $\varepsilon \to 0$.

This happens in all diagrams with "external leg corrections".

The problem with external leg corrections is a deep one. As these corrections are obviously independent of the "proper" scattering process, one may wonder if one can absorb them into the external states of the scattering process. The problem indeed reveals that our assumption that interactions turn off at large negative and positive times was too naive, since even if the wavepackets in the initial state (or final state) are widely separated and do not feel each other, the particles still have self interactions which cannot be turned off. In other words, the free particle states in interacting theories are different from the particle states that we constructed for free theories since they must include the effects from the self interactions.

We will see later when we discuss renormalisation in TPP II that the self interactions have two important implications: they contribute to the particle mass (which is defined as the eigenvalue of P^2) and they modify the normalisation of the particle states. To put it differently, the decomposition $H = H_0' + H_{int}$ in interacting theories contains a term H_0' that is quadratic in the fields, but which is <u>different</u> from the free Hamiltonian H_0 since it must account for the shifted particle mass and the modified normalisation of the particle states.

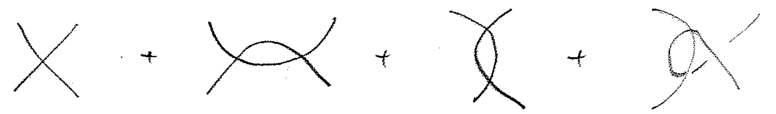
The upshot of this discussion is that the transition matrix element for a $2 \to n$ scattering process can be calculated according to

$$\boxed{\begin{aligned} &i\mathcal{M}(k_A k_B \to p_1 p_2 \dots p_n) \\ &= \left(\sqrt{Z}\right)^{n+2} \begin{pmatrix} \text{sum of all connected and amputated diagrams} \\ \text{with } k_A, k_B \text{ incoming and } p_1 \dots p_n \text{ outgoing} \end{pmatrix} \end{aligned}}$$

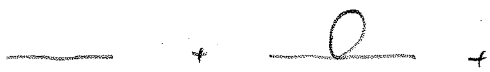
where "amputated" means that external leg corrections are to be discarded and the factor $\sqrt{Z}$ accounts for the modified normalisation of the particle states due to the self interactions.

Including first-order corrections, the $2 \to 2$ scattering matrix element in ϕ^4-theory thus becomes

$$i\mathcal{M}(k_1 k_2 \to p_1 p_2)$$

$$= Z^2 \Big($$  $$+ O(\lambda^2)\Big)$$

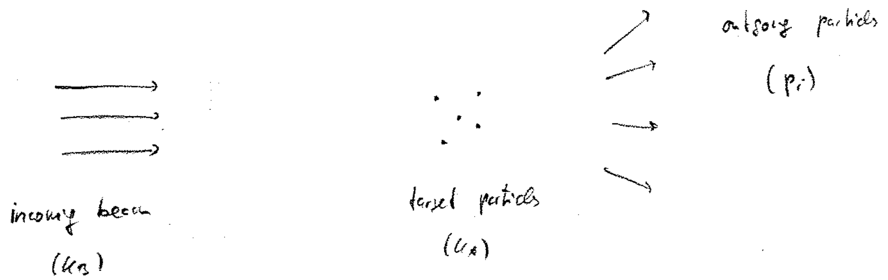
with a Z-factor that is to be determined from

 $$+ O(\lambda^2)$$

4.4 Cross sections and decay rates

Now that we have learned how to compute S-matrix elements, we are going to work out their relation to physical observables.

We start by considering a $2 \to n$ scattering process in the rest frame of one of the particles



The observable of interest is the <u>differential cross section</u>

$$d\sigma = \frac{\text{differential transition probability per unit time and target particles}}{\text{flux of incoming particles}}$$

which is differential in some kinematic variables like e.g. the angles or energies of the final-state particles.

First of all, we have to address a technical point which arises because we work with incoming and outgoing plane waves instead of wavepackets. We therefore have to regularise the norm of the particle states

$$\langle p|p\rangle = (2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p}-\vec{p}) = 2p^0 \int d^3x = 2p^0\, V$$

by a finite volume V, and we only take the limit $V\to\infty$ at the end of the calculation.

As we consider the scattering of single particles rather than bunches of identical particles, the initial state $|u_A\, u_B\rangle$ corresponds to

- number of target particles : 1
- flux of incoming particles : $\phi = \underbrace{\frac{1}{V}}_{\text{density of particles}}\ \underbrace{|\vec{v}_B|}_{\text{velocity}}$

In a different setup, in which the particles come in from both sides, the flux is determined by the difference between the particle velocities, $\phi = \dfrac{|\vec{v}_A-\vec{v}_B|}{V}$. For two back-to-back beams of massless particles, one has e.g. $|\vec{v}_A-\vec{v}_B| = 2$.

The differential transition probability is proportional to the square of the S-matrix element. The normalised differential probability is

$$dP(k_A k_B \to p_1 \dots p_n) = \frac{|\langle p_1 \dots p_n | S | k_A k_B \rangle|^2}{\langle k_A | k_A \rangle \langle k_B | k_B \rangle \langle p_1 | p_1 \rangle \cdots \langle p_n | p_n \rangle} \, d\Pi_n$$

where $d\Pi_n$ is the region of final-state momenta we are interested in. It is normalised to $\int d\Pi_n = 1$ and so

$$d\Pi_n = \left(V \frac{d^3 p_1}{(2\pi)^3} \right) \dots \left(V \frac{d^3 p_n}{(2\pi)^3} \right)$$

As we do not consider processes in which the initial and final states are identical, only the T-matrix contributes and

$$|\langle p_1 \dots p_n | S | k_A k_B \rangle|^2$$

$$= (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_i p_i) \, (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_i p_i)$$

$$|\mathcal{M}(k_A k_B \to p_1 \dots p_n)|^2$$

The square of the δ-function seems problematic, but as long as we work with a finite volume V and a finite time interval T, it is replaced by

$$\delta^{(4)}(k_A + k_B - \sum_i p_i)\ \delta^{(4)}(k_A + k_B - \sum_i p_i)$$

$$= \int \frac{d^4x}{(2\pi)^4}\, e^{i(k_A + k_B - \sum_i p_i)x}\ \delta^{(4)}(k_A + k_B - \sum_i p_i)$$

$$= \frac{1}{(2\pi)^4} \int d^4x\ \delta^{(4)}(k_A + k_B - \sum_i p_i) = \frac{VT}{(2\pi)^4}\ \delta^{(4)}(k_A + k_B - \sum_i p_i)$$

On the level of the cross section, which depends on the differential transition rate $\frac{dP}{T}$, we see that the various factors of V and T cancel and the limits $V \to \infty$ and $T \to \infty$ are well-defined

$$d\sigma = \frac{dP/T \cdot 1}{|\vec{v}_A - \vec{v}_B| / V}$$

$$= \left(V \frac{d^3p_1}{(2\pi)^3} \right) \cdots \left(V \frac{d^3p_n}{(2\pi)^3} \right) \frac{V}{|\vec{v}_A - \vec{v}_B|} \frac{1}{T}$$

$$\frac{VT\ (2\pi)^4\ \delta^{(4)}(k_A + k_B - \sum_i p_i)\ |\mathcal{M}(k_A k_B \to p_1 \ldots p_n)|^2}{(2k_A^0 V)(2k_B^0 V)(2p_1^0 V) \cdots (2p_n^0 V)}$$

We thus obtain the following master formula for the computation of scattering cross sections

$$\boxed{\begin{aligned} &d\sigma(k_A k_B \to p_1 \dots p_n) \\ &= \frac{1}{2k_A^0\, 2k_B^0\, |\vec{v}_A - \vec{v}_B|} \quad \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0} \cdots \frac{d^3p_n}{(2\pi)^3}\frac{1}{2p_n^0} \\ &\quad (2\pi)^4\, \delta^{(4)}\Big(k_A + k_B - \sum_{i=1}^{n} p_i\Big) \; |\mathcal{M}(k_A k_B \to p_1 \dots p_n)|^2 \end{aligned}}$$

The formula consists of three factors:

(i) The square of the Lorentz-invariant transition matrix element $|\mathcal{M}(k_A k_B \to p_1 \dots p_n)|^2$, which encodes the information about the dynamics of the scattering process and can be calculated with the help of Feynman diagrams (see page 274).

(ii) A flux factor

$$\frac{1}{2k_A^0\, 2k_B^0\, |\vec{v}_A - \vec{v}_B|} = \frac{1}{2k_A^0\, 2k_B^0 \left| \frac{\vec{k}_A}{k_A^0} - \frac{\vec{k}_B}{k_B^0} \right|} = \frac{1}{4\, |\vec{k}_A k_B^0 - \vec{k}_B k_A^0|}$$

which is invariant under boosts along the beam axis.

Consider beams in z-direction

$$k_A^\mu = (k_A^0, 0, 0, |\vec{k}_A|) \qquad (k_A^0)^2 - \vec{k}_A^2 = m_A^2$$

$$k_B^\mu = (k_B^0, 0, 0, -|\vec{k}_B|) \qquad (k_B^0)^2 - \vec{k}_B^2 = m_B^2$$

Perform boost along z-direction

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$\Rightarrow k_A'^\mu = \left(\gamma(k_A^0 + v|\vec{k}_A|), 0, 0, \gamma(|\vec{k}_A| + v k_A^0) \right)$$

$$k_B'^\mu = \left(\gamma(k_B^0 - v|\vec{k}_B|), 0, 0, \gamma(-|\vec{k}_B| + v k_B^0) \right)$$

$$\Rightarrow |\vec{k}_A' k_B'^0 - \vec{k}_B' k_A'^0|$$

$$= |\gamma^2(|\vec{k}_A| + v k_A^0)(k_B^0 - v|\vec{k}_B|) - \gamma^2(-|\vec{k}_B| + v k_B^0)(k_A^0 + v|\vec{k}_A|)|$$

$$= |\gamma^2 \left[|\vec{k}_A| k_B^0 - \cancel{v|\vec{k}_A||\vec{k}_B|} - v^2 k_A^0 |\vec{k}_B| + \cancel{v k_A^0 k_B^0} \right]$$

$$- \gamma^2 \left[v^2 |\vec{k}_A| k_B^0 - \cancel{v|\vec{k}_A||\vec{k}_B|} - k_A^0 |\vec{k}_B| + \cancel{v k_A^0 k_B^0} \right] |$$

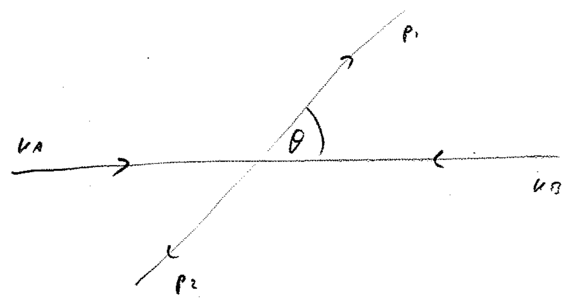
$$= |\gamma^2 (1 - v^2)(|\vec{k}_A| k_B^0 + |\vec{k}_B| k_A^0)|$$

$$= |\vec{k}_A k_B^0 - \vec{k}_B k_A^0| \quad \checkmark$$

(iii) The Lorentz-invariant n-body phase space

$$dPS_n \equiv \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0}\cdots\frac{d^3p_n}{(2\pi)^3}\frac{1}{2p_n^0}(2\pi)^4\,\delta^{(4)}(u_A+u_B-\sum_i p_i)$$

As an example consider a $2\to 2$ scattering reaction, for which 4 out of 6 integrals can be performed with the δ-function and the result is usually quoted as $\frac{d\sigma}{d\Omega}$ in the center-of-mass frame. For reactions that are symmetric about the beam axis, one can trivially integrate over the azimuthal angle and obtains $\frac{d\sigma}{d\theta}$ as a function of the scattering angle θ (see also tutorials).



Further remarks:

- The above formula for the differential cross section holds for both distinct and identical particles in the final state. When integrating this formula to obtain the total cross section, one has however to be careful not to double count the same configurations. For n identical particles in the final state, the total cross section is therefore obtained as

$$\sigma = \frac{1}{n!}\int d\sigma$$

- For particles with spin, one usually does not measure the spin configuration of the final-state particles, and one therefore has to sum over all configurations. As one typically collides unpolarised beams, one furthermore has to average over the initial-state spin configurations. The spin-averaged squared transition matrix element is then given by

$$\underbrace{\frac{1}{n_A n_B}}_{\text{number of spin configurations of initial-state particles}} \sum_{s_A, s_B} \sum_{s_1, \dots, s_n} |M|^2$$

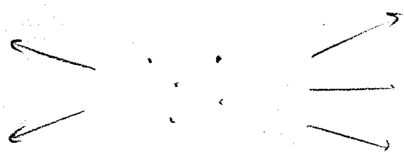
- Collider experiments essentially count events with specific signatures and one needs to know their luminosity to translate the number of observed events into a cross section. The precise relation is

$$\underbrace{\frac{dN}{dt}}_{\text{number of events per unit time}} = \underbrace{L}_{\text{luminosity}} \, d\sigma$$

Cross sections are measured in units of area; in particle physics one uses barn with $1b = 10^{-24} cm^2$. Typical cross sections can vary by many orders of magnitude in the range μb, nb or pb. Typical luminosities of today's collider experiments are in the range

$10^{32} - 10^{34} \frac{1}{cm^2 s}$.

Another class of observables that are measured at collider experiments are decay rates of unstable particles



One defines the <u>differential decay rate</u> as

$d\Gamma$ = differential decay probability per unit time and particle

The process can be considered as a $1 \to n$ scattering reaction, and one can therefore apply the same formalism as above to derive a similar master formula for the computation of decay rates

$$\boxed{\begin{aligned} d\Gamma(k_A \to p_1 \ldots p_n) &= \frac{1}{2k_A^0} \frac{d^3p_1}{(2\pi)^3} \frac{1}{2p_1^0} \cdots \frac{d^3p_n}{(2\pi)^3} \frac{1}{2p_n^0} \\ &\quad (2\pi)^4 \delta^{(4)}(k_A - \sum_{i=1}^{n} p_i) \; |\mathcal{M}(k_A \to p_1 \ldots p_n)|^2 \end{aligned}}$$

and the same considerations about the spin average and identical particles apply in this case.

The decay rate is usually quoted in the rest frame of the decaying particle. As the transition matrix element and the n-body phase space are Lorentz-invariant, the decay rate transforms as

$$d\Gamma = \frac{m_A}{k_A^0}\, d\Gamma_{rest} = \frac{1}{\gamma}\, d\Gamma_{rest}$$

under Lorentz transformations. The decay rate is thus smaller for fast-moving particles than for particles at rest, which reflects the usual effect from time dilation.

If one sums over all final states i, one obtains the total decay rate

$$\Gamma_{tot} = \sum_i \Gamma_i$$

which is related to the lifetime τ of the unstable particle by

$$\tau = \frac{1}{\Gamma_{tot}}$$

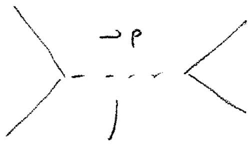
One further introduces the branching ratio for a decay into the final state i

$$Br_i = \frac{\Gamma_i}{\Gamma_{tot}}$$

such that $\sum_i Br_i = 1$.

The treatment of unstable particles within the S-matrix formalism seems, however, odd since an unstable particle cannot be represented by an asymptotic state for $t \to -\infty$. It turns out that the above formula is nevertheless correct as long as the unstable particle is long-lived, i.e. $\Gamma_{tot} \ll m_A$.

A more realistic treatment of unstable particles must consider both the production and decay of the unstable particle



unstable particle

$p^\mu = (E, \vec{0})$ in cms frame

If the energy E of the production process is close to the mass m of the unstable particle, the propagator of the unstable particle can be approximated by

$$\frac{1}{p^2 - m^2 + i m \Gamma} \overset{E \sim m}{\approx} \frac{1}{2E\left(E - m + \frac{i\Gamma}{2}\right)}$$

assumes $\Gamma \sim (E-m) \ll m$

For $\Gamma \to 0$ this is just the free propagator of a scalar particle.
We will learn later in TPP II that one has to sum up infinitely many Feynman diagrams to generate a finite width Γ.

On the level of the cross section, the unstable particle shows up as a resonance structure around the mass of the unstable particle in form of a Breit-Wigner distribution

$$\sigma \overset{E \sim m}{\underset{\Gamma \ll m}{\approx}} \frac{1}{(E-m)^2 + \Gamma^2/4}$$

and the width of the resonance peak is equal to the decay rate of the unstable particle.

4.5. Feynman rules for fermions

We now have assembled all ingredients to compute physical observables in interacting QFTs of spin-0 particles. The formalism immediately carries over to theories with higher-spin bosons, but it is not obvious if the same is true for theories with fermionic field operators. We have seen e.g. at the end of section 3.4 that the time-ordering prescription for fermionic fields is defined with a minus sign

$$T\,\psi_\alpha(x)\,\bar\psi_\beta(y) = \theta(x^0-y^0)\,\psi_\alpha(x)\,\bar\psi_\beta(y) - \theta(y^0-x^0)\,\bar\psi_\beta(y)\,\psi_\alpha(x)$$

and we will therefore have to reconsider Wick's theorem in the presence of fermionic fields.

As in the bosonic case, we first consider the two-point function

$$\langle\Omega|\,T\,\underbrace{\psi_\alpha(x)\,\bar\psi_\beta(y)}_{\text{Heisenberg operators}}\,\underbrace{|\Omega\rangle}_{\text{vacuum of interacting theory}}$$

in an interacting theory of Dirac fermions. As spin-½ particles do not couple to themselves in renormalizable QFTs (see page 229), one needs to couple them to another field. For completeness, we consider the simplest example in this section, which is the <u>Yukawa theory</u> with interaction Hamiltonian

$$H_{int} = \int d^3x \; g \; \bar{\psi}(x) \, \psi(x) \, \phi(x)$$

where $\phi(x)$ is a real scalar field and g a dimensionless coupling constant. This interaction is realized e.g. within the Standard Model by the coupling of (spin-0) Higgs bosons to (spin-1/2) quarks or leptons.

Lorentz invariance requires in general that the interaction Hamiltonian contains a product $\bar{\psi}(x) \psi(x)$, and it can therefore be considered as a bosonic quantity (see also the discussion on page 196). The definition of the time-evolution operator from page 239

$$U(t,t') = T \exp\left(-i \int_{t'}^{t} dt'' \, H_I(t'')\right) \qquad (t \geq t')$$

therefore immediately carries over to fermionic theories with the usual <u>bosonic</u> definition of the time-ordering prescription.

For the considered two-point function, one then obtains

interaction-picture

vacuum of free theory

$$\langle \Omega | T \psi_\alpha(x) \bar{\psi}_\beta(y) | \Omega \rangle = \frac{\langle 0 | T \psi_\alpha^I(x) \bar{\psi}_\beta^I(y) \exp\left(-i \int d^4 z \, \mathcal{H}_I(z)\right) | 0 \rangle}{\langle 0 | T \exp\left(-i \int d^4 z \, \mathcal{H}_I(z)\right) | 0 \rangle}$$

where the time-ordering symbol accounts for a minus sign if the order of $\psi_\alpha^I(x)$ and $\bar{\psi}_\beta^I(y)$ is interchanged, irrespective of the insertions of the interaction Hamiltonian.

For higher n-point functions, one needs to generalise the time-ordering prescription. The definition is obvious: for each interchange of fermionic fields, one picks up a minus sign, e.g.

$$T \, \psi(x_1) \psi(x_2) \psi(x_3) \psi(x_4) = (-1)^3 \, \psi(x_3) \psi(x_1) \psi(x_4) \psi(x_2)$$

in the region with $x_3^0 > x_1^0 > x_4^0 > x_2^0$. Notice that this definition mirrors the one of the normal-ordering prescription, which we introduced in section 3.2 (see page 139).

We next decompose the Dirac field operator into annihilation and creation fields, $\psi_\alpha(x) = \psi_\alpha^+(x) + \psi_\alpha^-(x)$, with

$$\psi_\alpha^+(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \, u_\alpha(p,s) \, e^{-ipx} \, a(p,s)$$

$$\psi_\alpha^-(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \, v_\alpha(p,s) \, e^{ipx} \, b^+(p,s)$$

where we have again suppressed the index I of the interaction-picture field for convenience. As $\bar{\psi} = -i\pi\gamma^0$, we have (see page 131)

$$\bar{\psi}_\alpha^+(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \, \bar{v}_\alpha(p,s) \, e^{-ipx} \, b(p,s)$$

$$\bar{\psi}_\alpha^-(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \, \bar{u}_\alpha(p,s) \, e^{ipx} \, a^+(p,s)$$

The non-trivial anticommutators therefore involve $\{\psi^+, \bar{\psi}^-\}$ and $\{\psi^-, \bar{\psi}^+\}$.

Proceeding in analogy to section 4.2, we then obtain

$$\begin{aligned}
\psi_\alpha(x)\,\bar{\psi}_\beta(y) &= \psi_\alpha^+(x)\,\bar{\psi}_\beta^+(y) + \psi_\alpha^+(x)\,\bar{\psi}_\beta^-(y) + \psi_\alpha^-(x)\,\bar{\psi}_\beta^+(y) + \psi_\alpha^-(x)\,\bar{\psi}_\beta^-(y) \\
&= \psi_\alpha^+(x)\,\bar{\psi}_\beta^+(y) - \bar{\psi}_\beta^-(y)\,\psi_\alpha^+(x) + \psi_\alpha^-(x)\,\bar{\psi}_\beta^+(y) + \psi_\alpha^-(x)\,\bar{\psi}_\beta^-(y) \\
&\quad + \{\psi_\alpha^+(x), \bar{\psi}_\beta^-(y)\} \\
&= :\psi_\alpha(x)\,\bar{\psi}_\beta(y): + \{\psi_\alpha^+(x), \bar{\psi}_\beta^-(x)\}
\end{aligned}$$

whereas

$$\bar{\psi}_\beta(y)\,\psi_\alpha(x) = \,:\bar{\psi}_\beta(y)\,\psi_\alpha(x): + \{\bar{\psi}_\beta^+(y), \psi_\alpha^-(x)\}$$

$$= -:\psi_\alpha(x)\,\bar{\psi}_\beta(y): + \{\bar{\psi}_\beta^+(y), \psi_\alpha^-(x)\}$$

since the ordering of the fields within the normal-ordering prescription now yields another minus sign.

For the time-ordered product, we thus find

$$T\,\psi_\alpha(x)\,\bar{\psi}_\beta(y)$$

$$= \theta(x^0-y^0)\,\psi_\alpha(x)\,\bar{\psi}_\beta(y) - \theta(y^0-x^0)\,\bar{\psi}_\beta(y)\,\psi_\alpha(x)$$

$$= \,:\psi_\alpha(x)\,\bar{\psi}_\beta(y): + \overline{\psi_\alpha(x)\,\bar{\psi}_\beta(y)}$$

where the Wick contraction of fermionic fields is defined as

$$\overline{\psi_\alpha(x)\,\bar{\psi}_\beta(y)}$$

$$\equiv \theta(x^0-y^0)\,\{\psi_\alpha^+(x), \bar{\psi}_\beta^-(y)\} - \theta(y^0-x^0)\,\{\bar{\psi}_\beta^+(y), \psi_\alpha^-(x)\}$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\sum_r \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}$$

$$\Big\{\theta(x^0-y^0)\,u_\alpha(p,s)\,e^{-ipx}\,\bar{u}_\beta(q,r)\,e^{iqy}\,\underbrace{\{a(p,s), a^+(q,r)\}}_{(2\pi)^3 2q^0\,\delta_{rs}\,\delta^{(3)}(\vec{p}-\vec{q})}$$

$$-\theta(y^0-x^0)\,\bar{v}_\beta(q,r)\,e^{-iqy}\,v_\alpha(p,s)\,e^{ipx}\,\{b(q,r), b^+(p,s)\}\Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\Big\{\theta(x^0-y^0)\underbrace{\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s)}_{(\not{p}+m)_{\alpha\beta}}e^{-ip(x-y)}$$

$$-\theta(y^0-x^0)\underbrace{\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s)}_{(\not{p}-m)_{\alpha\beta}}e^{ip(x-y)}\Big\}$$

$$= S_F(x-y)_{\alpha\beta}$$

as can be seen by comparison with the expressions on page 199.

As the corresponding anticommutators vanish, it is obvious that

$$\overbrace{\psi_\alpha(x)\,\psi_\beta(y)} = \overbrace{\bar\psi_\alpha(x)\,\bar\psi_\beta(y)} = 0$$

and since vacuum matrix elements of normal-ordered products vanish, we reproduce the familiar results

$$\langle 0 |\, T\,\psi_\alpha(x)\,\psi_\beta(y)\,| 0\rangle = \langle 0 |\, T\,\bar\psi_\alpha(x)\,\bar\psi_\beta(y)\,| 0\rangle = 0$$

$$\langle 0 |\, T\,\psi_\alpha(x)\,\bar\psi_\beta(y)\,| 0\rangle = S_F(x-y)_{\alpha\beta}$$

With the above definitions of the time-ordering and normal-ordering prescriptions, one can show that Wick's theorem takes the same form as before

$$\boxed{T\,\psi(x_1)\ldots\psi(x_n) = :\psi(x_1)\ldots\psi(x_n) + \text{all possible contractions}:}$$

can involve ψ and $\bar\psi$ fields

since the additional minus signs for fermionic fields are consistently taken into account on both sides of the equation.

Notice that for contractions under the normal-ordering symbol, one now has e.g.

$$:\overbrace{\psi_\alpha(x_1)\,\psi_\beta(x_2)\,\bar\psi_\gamma}(x_3)\,\bar\psi_\delta(x_4):$$

$$= -\,:\overbrace{\psi_\alpha(x_1)\,\bar\psi_\gamma(x_3)}\,\psi_\beta(x_2)\,\bar\psi_\delta(x_4):$$

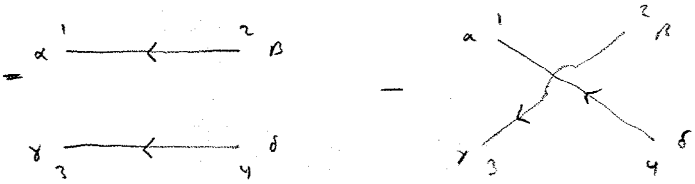
$$= -\,S_F(x_1-x_3)_{\alpha\gamma}\,:\psi_\beta(x_2)\,\bar\psi_\delta(x_4):$$

For the 4-point function, one then obtains

$$\langle 0|\, T\, \psi_\alpha(x_1)\, \bar\psi_\beta(x_2)\, \psi_\gamma(x_3)\, \bar\psi_\delta(x_4)\, |0\rangle$$

$$= \langle 0|\, \overbrace{\psi_\alpha(x_1)\, \bar\psi_\beta(x_2)}\, \overbrace{\psi_\gamma(x_3)\, \bar\psi_\delta(x_4)} + \overbrace{\psi_\alpha(x_1)\, \overbrace{\bar\psi_\beta(x_2)\, \psi_\gamma(x_3)}\, \bar\psi_\delta(x_4)}\, |0\rangle$$

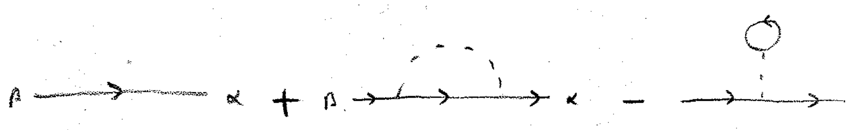
$$= S_F(x_1-x_2)_{\alpha\beta}\, S_F(x_3-x_4)_{\gamma\delta} - S_F(x_1-x_4)_{\alpha\delta}\, S_F(x_3-x_2)_{\gamma\beta}$$

$=$ 

where the arrows again indicate the direction of the particle flow. Due to the underlying Wick contractions, the two diagrams thus contribute with opposite signs!

In Yukawa theory the first correction to the free fermion propagator arises at $O(g^2)$, since one needs to contract an even number of bosonic fields. As the vacuum bubbles again drop out in the ratio, one obtains

$$\langle \Omega|\, T\, \psi_\alpha(x)\, \bar\psi_\beta(y)\, |\Omega\rangle$$

$=$  $+\, O(g^4)$

where the dashed line represents the propagator of the scalar field.

The signs of the diagrams can again be reconstructed from the underlying Wick contractions

$$\psi(x)\,\bar\psi(y)\quad \bar\psi(z_1)\,\psi(z_1)\,\phi(z_1)\quad \bar\psi(z_2)\,\psi(z_2)\,\phi(z_2) \longrightarrow (-1)^2 = +1$$

$$\psi(x)\,\bar\psi(y)\quad \bar\psi(z_1)\,\psi(z_1)\,\phi(z_1)\quad \bar\psi(z_2)\,\psi(z_2)\,\phi(z_2) \longrightarrow (-1)^3 = -1$$

for the second and third diagram, respectively.

We could easily proceed along the lines of section 4.2 to derive the corresponding Feynman rules for the computation of correlation functions, but we will instead turn to the evaluation of S-matrix elements here. Notice however that one often prefers to adopt a convention in which the correlation functions are given by the <u>sum</u> of all Feynman diagrams, and that the corresponding fermion signs are reproduced by additional Feynman rules. The sign in the last diagram from above is actually a reflection of the fact that a closed fermion loop always yields a factor of (-1) as we will see below.

Let us now turn to fermion scattering $ff \to ff$ in Yukawa theory.

As in the bosonic case, we start from

$$\langle p_1 p_2 | S | k_A k_B \rangle = \langle p_1 p_2 | T \exp\left(-i \int d^4 z \; g \underbrace{\bar{\psi}(z) \psi(z) \phi(z)}_{\text{interaction picture}}\right) | k_A k_B \rangle$$

free-particle states

where the states refer to fermions f (rather than antifermions $\bar{f}$ or bosons ϕ), which are created and annihilated by $a^+(p)$ and $a(p)$.

Notice that we keep the corresponding spin configurations s_A, s_B, s_1 and s_2 implicit in our notation for brevity!

Whereas the time-ordering symbol in the S-matrix refers to the bosonic definition, the non-trivial fermion statistics is now encoded in the particle states with

$$|k_A k_B\rangle = a^+(k_A)\, a^+(k_B)\, |0\rangle$$

$$= -a^+(k_B)\, a^+(k_A)\, |0\rangle = -|k_B k_A\rangle$$

and

$$\langle p_1 p_2 | = \left(|p_1 p_2\rangle \right)^+ = \left(a^+(p_1)\, a^+(p_2)\, |0\rangle \right)^+$$

$$= \langle 0 | a(p_2)\, a(p_1)$$

$$= - \langle 0 | a(p_1)\, a(p_2) = - \langle p_2 p_1 |$$

The leading contribution to the T-matrix then arises at $O(g^2)$ from the contraction

$$T \; \bar{\Psi}(z_1)\Psi(z_1)\phi(z_1) \quad \bar{\Psi}(z_2)\Psi(z_2)\phi(z_2)$$

$$\longrightarrow \; : \bar{\Psi}(z_1)\Psi(z_1)\phi(z_1) \quad \bar{\Psi}(z_2)\Psi(z_2)\phi(z_2) :$$

Proceeding along the lines of section 4.3, we first evaluate the expression

$$\{\psi_\alpha^+(x), a^+(p,s)\}$$

$$= \sum_r \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\, u_\alpha(q,r)\, e^{-iqx}\, \underbrace{\{a(q,r), a^+(p,s)\}}_{(2\pi)^3\, 2q^0\, \delta_{rs}\, \delta^{(3)}(\vec{p}-\vec{q})}$$

$$= u_\alpha(p,s)\, e^{-ipx}$$

We will furthermore make use of the relation*

$$[\psi_\alpha^+(x)\,\psi_\beta^+(y), a^+(p,s)]$$

$$= \psi_\alpha^+(x)\,\{\psi_\beta^+(y), a^+(p,s)\} - \{\psi_\alpha^+(x), a^+(p,s)\}\,\psi_\beta^+(y)$$

$$= u_\beta(p,s)\, e^{-ipy}\, \psi_\alpha^+(x) - u_\alpha(p,s)\, e^{-ipx}\, \psi_\beta^+(y)$$

---

* $[AB, C] = ABC + ACB - ACB - CAB$

$= A\{B,C\} - \{A,C\}B$

$\psi^+ \sim a$
$\bar\psi^- \sim a^+$

We thus obtain

$$\langle p_1 p_2 | : \bar\psi_\alpha(z_1)\psi_\alpha(z_1)\phi(z_1)\;\bar\psi_\beta(z_2)\psi_\beta(z_2)\phi(z_2) : | k_A k_B\rangle$$

$$= \Delta_F(z_1-z_2)\,\langle 0|a(p_2)a(p_1)\left[-\bar\psi_\alpha^-(z_1)\bar\psi_\beta^-(z_2)\underbrace{\psi_\alpha^+(z_1)\psi_\beta^+(z_2)\right]a^+(k_A)}\,a^+(k_B)|0\rangle$$

$$= -\Delta_F(z_1-z_2)$$

$$\Big\{\langle 0|a(p_2)a(p_1)\bar\psi_\alpha^-(z_1)\bar\psi_\beta^-(z_2)a^+(k_A)\psi_\alpha^+(z_1)\overset{\to 0}{\cancel{\psi_\beta^+(z_2)\,a^+(k_B)|0\rangle}}$$

$$+\,u_\beta(k_A)\,e^{-ik_A z_2}\,\langle 0|a(p_2)a(p_1)\bar\psi_\alpha^-(z_1)\bar\psi_\beta^-(z_2)\underbrace{\psi_\alpha^+(z_1)a^+(k_B)}|0\rangle$$

$$-\,u_\alpha(k_A)\,e^{-ik_A z_1}\,\langle 0|a(p_2)a(p_1)\bar\psi_\alpha^-(z_1)\bar\psi_\beta^-(z_2)\underbrace{\psi_\beta^+(z_2)a^+(k_B)}|0\rangle\Big\}$$

$$= -\Delta_F(z_1-z_2)$$

$$\Big\{u_\beta(k_A)e^{-ik_A z_2}\,u_\alpha(k_B)e^{-ik_B z_1} - u_\alpha(k_A)e^{-ik_A z_1}\,u_\beta(k_B)e^{-ik_B z_2}\Big\}$$

$$\langle 0|a(p_2)a(p_1)\bar\psi_\alpha^-(z_1)\bar\psi_\beta^-(z_2)|0\rangle$$

indices $s_A \leftrightarrow s_B$

$$= -\Delta_F(z_1-z_2)$$

$$\Big\{u_\beta(k_A)u_\alpha(k_B)\,e^{-ik_A z_2}e^{-ik_B z_1} - (k_A\leftrightarrow k_B)\Big\}$$

$$\sum_r\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\,\bar u_\alpha(q,r)\,e^{iqz_1}\;\sum_t\int\frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}\,\bar u_\beta(k,t)\,e^{ikz_2}$$

$$\underbrace{\langle 0|a(p_2)a(p_1)a^+(q)a^+(k)|0\rangle}$$

$$= \langle p_1p_2|qk\rangle$$

page 109 $\to$
$$= (2\pi)^6\,2q^0\,2k^0\,\Big\{\delta_{rs_1}\delta_{ts_2}\,\delta^{(3)}(\vec q-\vec p_1)\,\delta^{(3)}(\vec k-\vec p_2)$$
$$-\,\delta_{rs_2}\delta_{ts_1}\,\delta^{(3)}(\vec q-\vec p_2)\,\delta^{(3)}(\vec k-\vec p_1)\Big\}$$

$$= -\Delta_F(z_1-z_2)$$

$$\Big\{u_\beta(k_A)u_\alpha(k_B)\,e^{-ik_A z_2}e^{-ik_B z_1} - (k_A\leftrightarrow k_B)\Big\}$$

$$\Big\{\bar u_\alpha(p_1)\bar u_\beta(p_2)\,e^{ip_1z_1}e^{ip_2z_2} - (p_1\leftrightarrow p_2)\Big\}$$

$$= \Delta_F(z_1 - z_2)$$

$$\{ \bar u(p_1)\, u(k_A)\; \bar u(p_2)\, u(k_B)\; e^{-i(k_B-p_2)z_1}\; e^{-i(k_A-p_1)z_2}$$

$$- \bar u(p_1)\, u(k_B)\; \bar u(p_2)\, u(k_A)\; e^{-i(k_B-p_1)z_1}\; e^{-i(k_A-p_2)z_2}$$

$$+ (z_1 \leftrightarrow z_2) \}$$

which leads to a contribution to the S-matrix of the form

$$\frac{1}{2!}(-ig)^2 \int d^4z_1 \int d^4z_2 \;\langle p_1 p_2 | : \bar\psi(z_1)\psi(z_1)\phi(z_1)\,\bar\psi(z_2)\psi(z_2)\phi(z_2) : | k_A k_B\rangle$$

$$= \frac{1}{2!}(-ig)^2 \int d^4z_1 \int d^4z_2 \;\; 2\int \frac{d^4q}{(2\pi)^4}\, e^{-iq(z_1-z_2)}\, \tilde\Delta_F(q)$$

$$\{ \bar u(p_1)\, u(k_A)\, \bar u(p_2)\, u(k_B)\; e^{-i(k_B-p_2)z_1}\; e^{-i(k_A-p_1)z_2}$$

$$- (p_1 \leftrightarrow p_2) \}$$

$z_1 \to (2\pi)^4\, \delta^{(4)}(q + k_B - p_2)$
$z_2 \to (2\pi)^4\, \delta^{(4)}(-q + k_A - p_1)$

$$= (-ig)^2 \{ \tilde\Delta_F(k_A - p_1)\; \bar u(p_1)\, u(k_A)\;\; \bar u(p_2)\, u(k_B) \; - \; (p_1 \leftrightarrow p_2) \}$$

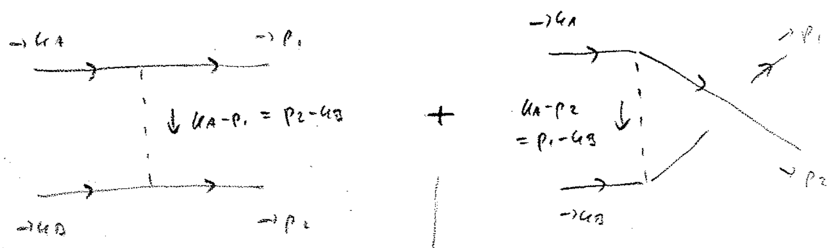
$$(2\pi)^4\, \delta^{(4)}(k_A + k_B - p_1 - p_2)$$

The lowest-order contribution to the transition matrix element is thus given by

$$i\mathcal{M}(k_A k_B \to p_1 p_2)$$

$$= (-ig)^2 \{ \tilde\Delta_F(k_A - p_1)\; \bar u(p_1)\, u(k_A)\; \bar u(p_2)\, u(k_B) \; - \; (p_1 \leftrightarrow p_2) \}$$

$\equiv$

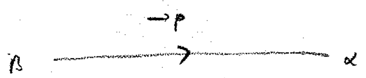


Fermion sign will be reproduced by Feynman rule

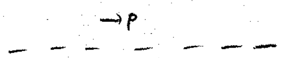
In comparison with ϕ^4-theory, we thus obtain spinors $u(k_A)$, $\bar{u}(p_1)$ etc associated with external lines as well as non-trivial fermion signs.

The <u>momentum-space Feynman rules</u> for the computation of scattering matrix elements in Yukawa theory can then be summarized as follows ($\rightsquigarrow$ fermion mass m, boson mass M).

1) For each internal line

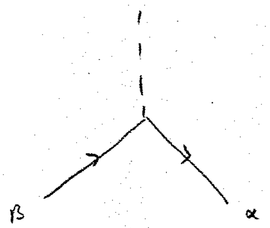


$$= \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$



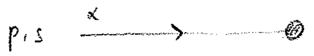
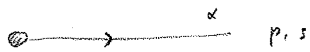
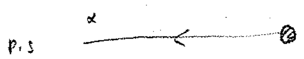
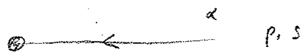
$$= \frac{i}{p^2-M^2+i\varepsilon}$$

2) For each vertex



$$= -ig\,\delta_{\alpha\beta}$$

3) For each external line*

	$= u_\alpha(p,s)$	incoming fermion
	$= \bar{u}_\alpha(p,s)$	outgoing fermion
	$= \bar{v}_\alpha(p,s)$	incoming antifermion
	$= v_\alpha(p,s)$	outgoing antifermion
	$= 1$	incoming boson
	$= 1$	outgoing boson

4) Impose momentum conservation at each vertex and integrate over all undetermined momenta $\int \frac{d^4 q}{(2\pi)^4}$

5) Multiply with the fermion sign of the diagram.

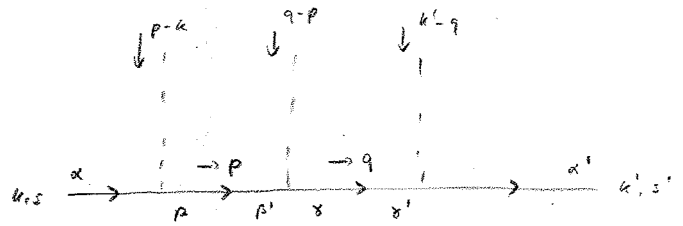
---

* In this lecture we always draw incoming/outgoing particles on the left/right of the Feynman diagram (the conventions differ in the literature).

Remarks:

- As the vertex $\bar{\psi}\psi\phi$ contains three different fields, there are no symmetry factors in Yukawa theory. The $\frac{1}{n!}$ from the Taylor expansion of $\exp(-i\int d^4z\, \mathcal{H}_I(z))$ furthermore always gets cancelled by identical contributions that arise from the interchange of n vertices, as we have seen explicitly above.

- Each fermion line generates an independent string of Dirac matrices, which are to be multiplied in the <u>opposite</u> direction of the particle flow ($\sim$ arrow)



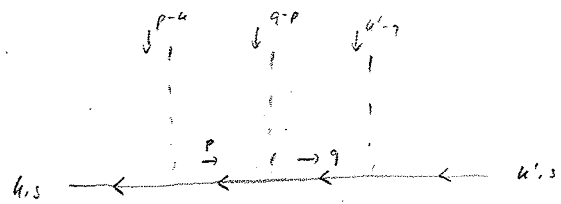
$$= u_\alpha(k,s)\;(-ig\,\delta_{\rho\alpha})\;\frac{i(\not p+m)_{\rho'\rho}}{p^2-m^2+i\varepsilon}\;(-ig\,\delta_{\gamma\rho'})\;\frac{i(\not q+m)_{\gamma'\gamma}}{q^2-m^2+i\varepsilon}\;(-ig\,\delta_{\alpha'\gamma'})\;\bar u_{\alpha'}(k',s')$$

$$= (-ig)^3\;\frac{\bar u(k',s')\; i(\not q+m)\; i(\not p+m)\; u(k,s)}{(q^2-m^2+i\varepsilon)\;(p^2-m^2+i\varepsilon)}$$

i.e. one can suppress the spinor indices as long as the Dirac strings are written down starting from their end!

For an antifermion line, one obtains similarly

momenta pointing into direction of particle flow!



$$= (-ig)^3 \, \frac{\bar{v}(k,s)\, i(-\not{p}+m)\, i(-\not{q}+m)\, v(k',s')}{(p^2-m^2+i\varepsilon)(q^2-m^2+i\varepsilon)}$$

but one has to be careful with the direction of the particle momenta in the Dirac propagators.

- For the correlation functions we could determine the overall sign of a Feynman diagram from the underlying Wick contractions (see page 294). For S-matrix elements one in addition needs contractions with external states, which are introduced as

$a \sim \psi^+$
$a^\dagger \sim \bar{\psi}^-$

$$\langle p_1 p_2 | (\bar\psi\psi\phi)(\bar\psi\psi\phi) | k_A k_B \rangle$$

$$= \langle 0 | a(p_2)\, a(p_1)\, (\bar\psi\psi\phi)(\bar\psi\psi\phi)\, a^\dagger(k_A)\, a^\dagger(k_B) | 0 \rangle \longrightarrow (-1)^2 = +1$$

$$\langle p_1 p_2 | (\bar\psi\psi\phi)(\bar\psi\psi\phi) | k_A k_B \rangle$$

$$= \langle 0 | a(p_2)\, a(p_1)\, (\bar\psi\psi\phi)(\bar\psi\psi\phi)\, a^\dagger(k_A)\, a^\dagger(k_B) | 0 \rangle \longrightarrow (-1)^3 = -1$$

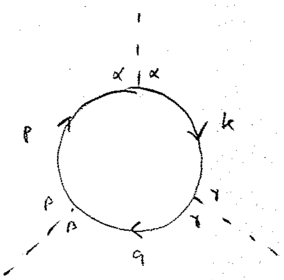
for the first and second diagram on page 298, respectively.

In general there are two different sources of fermion signs for the computation of S-matrix elements:

(i) We get a factor of (-1) for each interchange of external identical fermions, which is a reflection of Fermi statistics.

(ii) We get a factor of (-1) for each closed fermion loop.

For the second point consider e.g. a contribution of the form



$$\sim (\bar{\psi}_\alpha \psi_\alpha \phi) \; (\bar{\psi}_\beta \psi_\beta \phi) \; (\bar{\psi}_\gamma \psi_\gamma \phi)$$

$$\rightarrow (-1) \; (\not{p}+m)_{\alpha\beta} \; (\not{q}+m)_{\beta\gamma} \; (\not{k}+m)_{\gamma\alpha}$$

$$= (-1) \; \mathrm{Tr}\left[(\not{p}+m) (\not{q}+m) (\not{k}+m) \right]$$

i.e. a closed fermion loop always gives a factor of (-1) as well as a trace over the corresponding Dirac string!

To complete this section, we will compute the $ff \to ff$ scattering cross section in Yukawa theory explicitly here. In order to streamline the calculation somewhat, we will set the fermion mass to zero, i.e. we consider the high-energy limit of a theory of fermions, whose interaction is mediated by a heavy scalar particle.

The kinematics of a $2 \to 2$ scattering process is most conveniently described in terms of Lorentz-invariant <u>Mandelstam variables</u>. Denoting the incoming momenta by k_A, k_B and the outgoing momenta by p_1, p_2, one defines

$$s = (k_A + k_B)^2 = (p_1 + p_2)^2$$

$$t = (k_A - p_1)^2 = (p_2 - k_B)^2$$

$$u = (k_A - p_2)^2 = (p_1 - k_B)^2$$

and we will show in the tutorials that the Mandelstam variables fulfill the relation

$$s + t + u = m_A^2 + m_B^2 + m_1^2 + m_2^2$$

The transition matrix element for $ff \to ff$ scattering can then be written in the form (making the spin configurations explicit again)

$$i\mathcal{M}(k_A k_B \to p_1 p_2) = -i g^2$$

$$\left\{ \frac{\bar{u}(p_1,s_1)\, u(k_A,s_A)\, \bar{u}(p_2,s_2)\, u(k_B,s_B)}{t-\mu^2} - \frac{\bar{u}(p_2,s_2)\, u(k_A,s_A)\, \bar{u}(p_1,s_1)\, u(k_B,s_B)}{u-\mu^2} \right\}$$

In order to determine the complex conjugate expression, we note that

$$[\bar{u}(p,s)\, u(k,r)]^* = [u^+(p,s)\, \gamma^0\, u(k,r)]^*$$

$$= u^+(k,r) \underbrace{\gamma^{0+}}_{=\gamma^0} u(p,s) = \bar{u}(k,r)\, u(p,s)$$

It follows

$$-i\mathcal{M}^*(k_A k_B \to p_1 p_2) = i g^2$$

$$\left\{ \frac{\bar{u}(k_A,s_A)\, u(p_1,s_1)\, \bar{u}(k_B,s_B)\, u(p_2,s_2)}{t-\mu^2} - \frac{\bar{u}(k_A,s_A)\, u(p_2,s_2)\, \bar{u}(k_B,s_B)\, u(p_1,s_1)}{u-\mu^2} \right\}$$

As mentioned on page 282, we are furthermore not interested in specific spin configurations of the initial and final-state fermions. The spin-averaged squared transition matrix element is then given by

$$\frac{1}{2\cdot 2} \sum_{s_A,s_B} \sum_{s_1,s_2} |\mathcal{M}|^2 \equiv \overline{|\mathcal{M}|}^2$$

We obtain

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4} \sum_{s_A, s_B} \sum_{s_1, s_2} g^4$$

$$\left\{ \frac{\bar{u}_\alpha(p_1,s_1)\, u_\alpha(k_A,s_A)\, \bar{u}_\beta(k_A,s_A)\, u_\beta(p_1,s_1)\, \bar{u}_\gamma(p_2,s_2)\, u_\gamma(k_B,s_B)\, \bar{u}_\delta(k_B,s_B)\, u_\delta(p_2,s_2)}{(t-\mu^2)^2} \right.$$

relative sign of diagrams matters in interference term!

$$- \frac{\bar{u}_\alpha(p_1,s_1)\, u_\alpha(k_A,s_A)\, \bar{u}_\beta(k_B,s_A)\, u_\beta(p_2,s_2)\, \bar{u}_\gamma(p_2,s_2)\, u_\gamma(k_B,s_B)\, \bar{u}_\delta(k_B,s_B)\, u_\delta(p_1,s_1)}{(t-\mu^2)(u-\mu^2)}$$

$$- \frac{\bar{u}_\alpha(p_2,s_2)\, u_\alpha(k_A,s_A)\, \bar{u}_\beta(k_A,s_A)\, u_\beta(p_1,s_1)\, \bar{u}_\gamma(p_1,s_1)\, u_\gamma(k_B,s_B)\, \bar{u}_\delta(k_B,s_B)\, u_\delta(p_2,s_2)}{(t-\mu^2)(u-\mu^2)}$$

$$\left. + \frac{\bar{u}_\alpha(p_2,s_2)\, u_\alpha(k_A,s_A)\, \bar{u}_\beta(k_A,s_A)\, u_\beta(p_2,s_2)\, \bar{u}_\gamma(p_1,s_1)\, u_\gamma(k_B,s_B)\, \bar{u}_\delta(k_B,s_B)\, u_\delta(p_1,s_1)}{(u-\mu^2)^2} \right\}$$

$\sum_s u(p,s)\, \bar{u}(p,s) = \not{p}$

$$= \frac{g^4}{4} \left\{ \frac{(\not{k}_A)_{\alpha\beta}(\not{p}_1)_{\beta\alpha}(\not{k}_B)_{\gamma\delta}(\not{p}_2)_{\delta\gamma}}{(t-\mu^2)^2} - \frac{(\not{k}_A)_{\alpha\beta}(\not{p}_2)_{\beta\gamma}(\not{k}_B)_{\gamma\delta}(\not{p}_1)_{\delta\alpha}}{(t-\mu^2)(u-\mu^2)} \right.$$

$$\left. - \frac{(\not{k}_A)_{\alpha\beta}(\not{p}_1)_{\beta\gamma}(\not{k}_B)_{\gamma\delta}(\not{p}_2)_{\delta\alpha}}{(t-\mu^2)(u-\mu^2)} + \frac{(\not{k}_A)_{\alpha\beta}(\not{p}_2)_{\beta\alpha}(\not{k}_B)_{\gamma\delta}(\not{p}_1)_{\delta\gamma}}{(u-\mu^2)^2} \right\}$$

$$= \frac{g^4}{4} \left\{ \frac{\mathrm{tr}(\not{k}_A \not{p}_1)\,\mathrm{tr}(\not{k}_B \not{p}_2)}{(t-\mu^2)^2} - \frac{\mathrm{tr}(\not{k}_A \not{p}_2 \not{k}_B \not{p}_1)}{(t-\mu^2)(u-\mu^2)} \right.$$

$$\left. - \frac{\mathrm{tr}(\not{k}_A \not{p}_1 \not{k}_B \not{p}_2)}{(t-\mu^2)(u-\mu^2)} + \frac{\mathrm{tr}(\not{k}_A \not{p}_2)\,\mathrm{tr}(\not{k}_B \not{p}_1)}{(u-\mu^2)^2} \right\}$$

from which we see that spin-averaged squared transition matrix elements lead to traces over strings of Dirac matrices.

The Dirac traces can be evaluated according to (→ tutorials)

$$\mathrm{Tr}\,[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$$

$$\mathrm{Tr}\,[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

and the resulting scalar products can be turned into Mandelstam variables via

$$s = (k_A + k_B)^2 = 2k_A k_B = (p_1+p_2)^2 = 2p_1p_2$$

$$t = (k_A - p_1)^2 = -2k_A p_1 = (p_2 - k_B)^2 = -2k_B p_2$$

$$u = (k_A - p_2)^2 = -2k_A p_2 = (p_1 - k_B)^2 = -2k_B p_1$$

since for massless fermions $k_A^2 = k_B^2 = p_1^2 = p_2^2 = 0$.

It follows

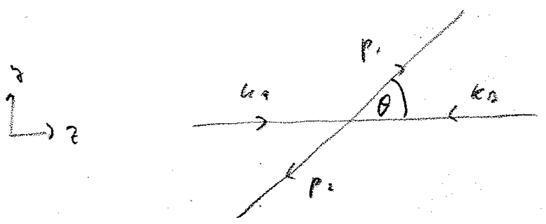
$$|\overline{\mathcal{M}}|^2 = \frac{g^4}{4}\left\{\frac{(-2t)(-2t)}{(t-\mu^2)^2} - \frac{u^2 - s^2 + t^2}{(t-\mu^2)(u-\mu^2)} - \frac{t^2 - s^2 + u^2}{(t-\mu^2)(u-\mu^2)} + \frac{(-2u)(-2u)}{(u-\mu^2)^2}\right\}$$

$s+t+u=0$

$$= g^4\left\{\frac{t^2}{(t-\mu^2)^2} + \frac{ut}{(t-\mu^2)(u-\mu^2)} + \frac{u^2}{(u-\mu^2)^2}\right\}$$

which is a Lorentz-invariant expression which depends on two independent kinematic variables.

In the center-of-mass frame those kinematic variables are the center-of-mass energy E_{CM} and the scattering angle θ, which is defined as



$$k_A^\mu = \frac{E_{CM}}{2}(1,0,0,1)$$
$$k_B^\mu = \frac{E_{CM}}{2}(1,0,0,-1)$$
$$p_1^\mu = \frac{E_{CM}}{2}(1,0,\sin\theta,\cos\theta)$$
$$p_2^\mu = \frac{E_{CM}}{2}(1,0,-\sin\theta,-\cos\theta)$$

$$\Rightarrow \quad s = 2k_A\cdot k_B = E_{CM}^2$$
$$t = -2k_A\cdot p_1 = -E_{CM}^2\,\frac{1-\cos\theta}{2}$$
$$u = -2k_A\cdot p_2 = -E_{CM}^2\,\frac{1+\cos\theta}{2}$$

We will furthermore show in the tutorials that our master formula for the computation of scattering cross sections from page 280 can be written for a $2\to 2$ scattering process in the center-of-mass frame as

$$\frac{d\sigma}{d\cos\theta} = \frac{\lambda(s,m_1^2,m_2^2)}{32\pi s\,\lambda(s,m_A^2,m_B^2)}\,|\mathcal{M}|^2$$

where $\lambda(x,y,z) = \sqrt{x^2+y^2+z^2-2xy-2xz-2yz}$ is the Källén function.

For massless fermions, we then obtain

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s}\, |\overline{\mathcal{M}}|^2$$

$$= \frac{g^4}{128\pi}\, E_{cm}^2 \left\{ \frac{(1-\cos\theta)^2}{(\mu^2 + \frac{E_{cm}^2}{2}(1-\cos\theta))^2} + \frac{(1+\cos\theta)^2}{[\mu^2 + \frac{E_{cm}^2}{2}(1+\cos\theta)]^2} + \frac{(1-\cos\theta)(1+\cos\theta)}{[\mu^2 + \frac{E_{cm}^2}{2}(1-\cos\theta)][\mu^2 + \frac{E_{cm}^2}{2}(1+\cos\theta)]} \right\}$$

A measurement of the differential cross section could be used e.g. to determine the mass of the spin-0 particle. For $m^2 \ll \mu^2 \ll E_{cm}^2$, one obtains e.g. a flat distribution in $\cos\theta$

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{3g^4}{32\pi E_{cm}^2} + \mathcal{O}\left(\frac{\mu^2}{E_{cm}^2}\right)$$

whereas in the opposite limit $m^2 \ll E_{cm}^2 \ll \mu^2$

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{g^4 E_{cm}^2}{128\pi\, \mu^4}\,(3+\cos^2\theta) + \mathcal{O}\left(\frac{E_{cm}^2}{\mu^2}\right)$$

the distribution is peaked in the forward direction. Notice that since the particles cannot be distinguished, only the distribution in the interval $\cos\theta \in [0,1]$ can be measured.

The total cross section, on the other hand, is mostly sensitive to the coupling constant g

$$\sigma = \frac{1}{2} \int_{-1}^{1} d\cos\theta \; \frac{d\sigma}{d\cos\theta}$$

identical particles !

$$= \frac{g^4}{32\pi E_{cm}^2} (3+5x) \left\{ \frac{1}{1+x} - \frac{2x}{(1+2x)} \ln \frac{1+x}{x} \right\}$$

where $x = \frac{\mu^2}{E_{cm}^2}$.