Why is π constant?

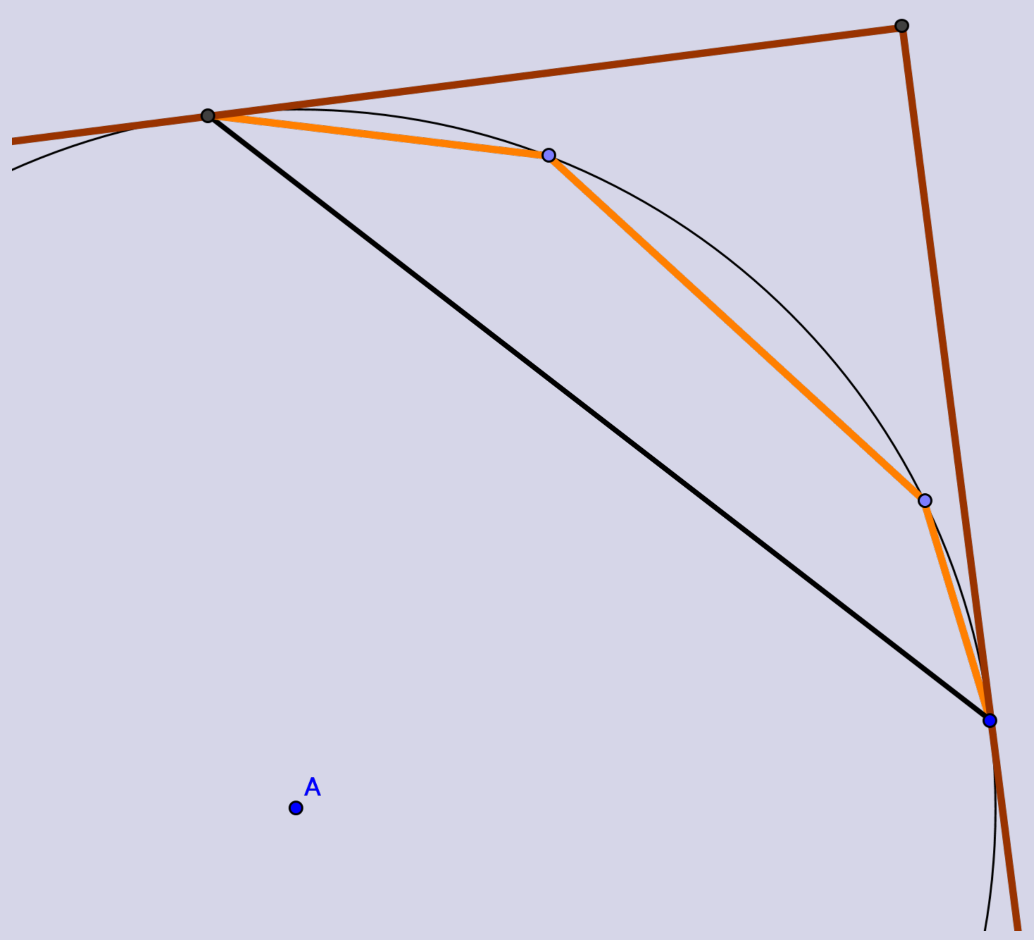
What is π?

By now you know that we define π as the ratio C/D for *any* (Euclidean) circle's circumference C and diameter D. For this to make sense, we need to know two things:

- For a given circle, C and D make sense and are finite.
- For any two circles, the ratio C/D is the same.

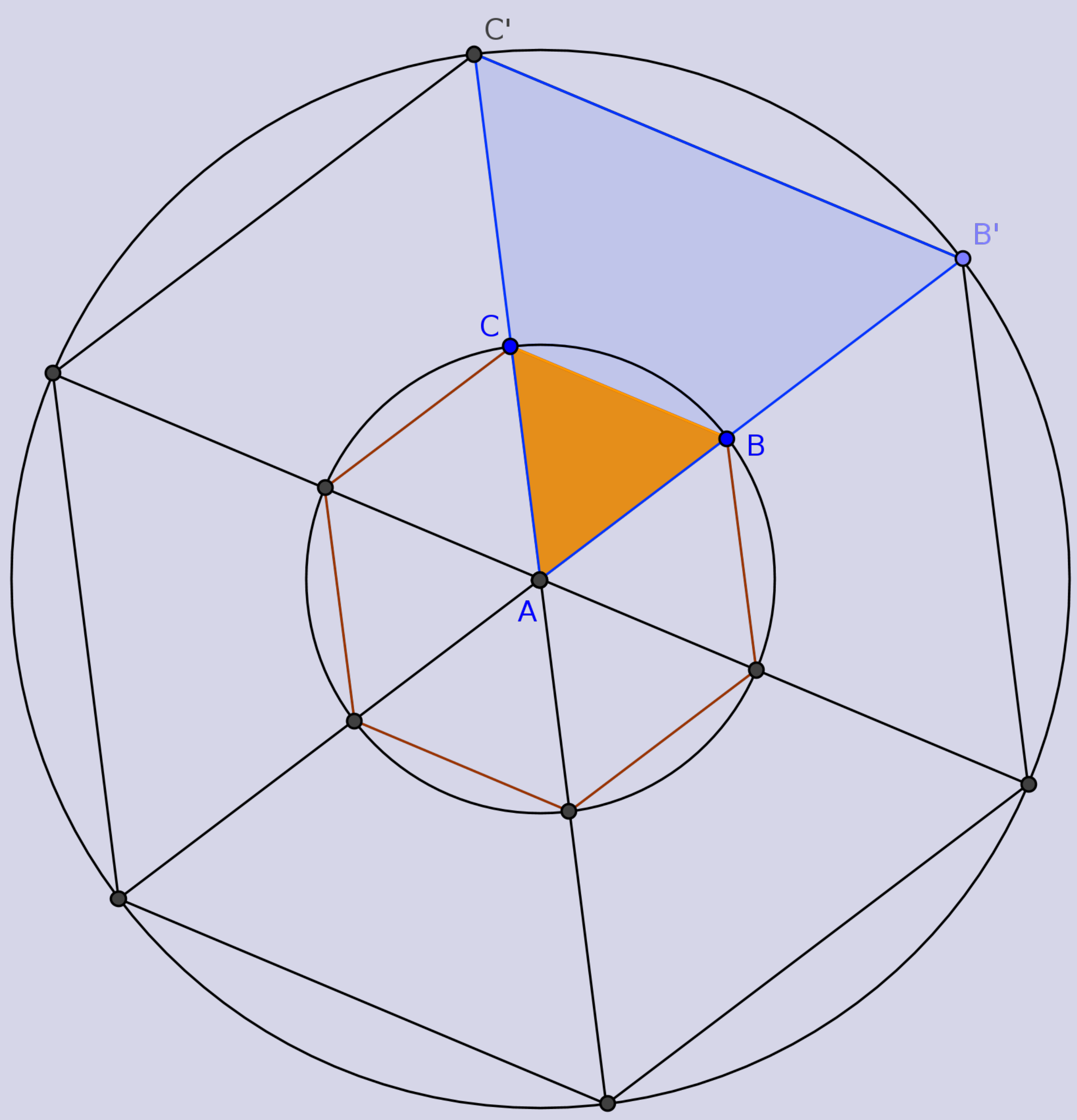
What is "circumference"?

Since a circle is curvy, measuring its circumference is not straightforward (pun intended). We define the length of a curve to be the limit of the lengths of piecewise-linear paths; for the circle, this is the inscribed polygons you've already seen!
But why does this limit exist? One can show that as the number of sides increases (as the "mesh is refined"), the perimeter increases; and the length is bounded above by the perimeter of a circumscribed square.



Different circles

Consider two different circles. Translate one so that they are concentric; this doesn't change the circumference and diameter of the moved circle. Now compare inscribed polygons of the two circles. Divide the polygons into triangles with a vertex at the center of the circles.



The resulting triangles are isoceles with the same apex angle, hence are similar, with ratio r_2/r_1. This implies that the perimeters of the polygons have the same ratio r_2/r_1.
This is true for every fixed number of edges in the inscribed polygons, and so is true also for the limit. That is,

$$C_2/C_1 = r_2/r_1 = D_2/D_1,$$

and so

$$C_2/D_2 = C_1/D_1.$$