Math 241, Sections BL1 and BL2

Quiz # 7   Solutions

December 6, 2012

Solve both exercises. Show work to get credit.

1) [5pts.] Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Find ∇r.

Solution: We have that $r = \sqrt{x^2 + y^2 + z^2}$, so

$$\begin{aligned}
\nabla r &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \sqrt{x^2 + y^2 + z^2} \\
&= \left\langle \frac{\partial}{\partial x}(\sqrt{x^2 + y^2 + z^2}), \frac{\partial}{\partial y}(\sqrt{x^2 + y^2 + z^2}), \frac{\partial}{\partial z}(\sqrt{x^2 + y^2 + z^2}) \right\rangle \\
&= \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle \\
&= \frac{1}{r}\vec{r}.
\end{aligned}$$

2) [5pts.] Compute the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of the vector field $\vec{F}(x,y,z) = \langle x^4, -x^3z^2, 4xy^2z\rangle$ across the boundary S of the solid E bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $z=x+7$, and oriented inward.

Solution: By the Divergence Theorem, the flux is equal to

$$-\iiint_E \operatorname{div}(\vec{F})\, dV,$$

where we use the negative since the desired normal vectors point inward instead of outward. We compute

$$\begin{aligned}
\operatorname{div}(\vec{F}) &= \nabla \cdot \vec{F} \\
&= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle x^4, -x^3z^2, 4xy^2z\rangle \\
&= \frac{\partial}{\partial x}(x^4) + \frac{\partial}{\partial y}(-x^3z^2) + \frac{\partial}{\partial z}(4xy^2z) \\
&= 4x^3 + 0 + 4xy^2.
\end{aligned}$$

So the flux is given by

$$-4\iiint_E x(x^2+y^2)\, dx\, dy\, dz.$$

We need to parameterize the region E; this is easiest in cylindrical coordinates:

$$\begin{aligned}
-4\iiint_E x(x^2+y^2)\, dx\, dy\, dz &= 4\int_0^{2\pi}\int_0^2\int_0^{r\cos\theta+7} (r\cos\theta)(r^2)\,(r\, dz\, dr\, d\theta) \\
&= -4\int_0^{2\pi}\int_0^2 \left(r^5\cos^2\theta + 7r^4\cos\theta\right)\, dr\, d\theta \\
&= -4\int_0^{2\pi} \left(\frac{2^6}{6}\cos^2\theta + \frac{2^5 7}{5}\cos\theta\right)\, d\theta \\
&= -4\left[\frac{2^6}{6}\frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta)) + \frac{2^5 7}{5}\sin\theta\right]_0^{2\pi} \\
&= -4\left(\frac{2^5}{6}(2\pi) + 0 + 0\right) \\
&= -\frac{2^7\pi}{3} \\
&= -\frac{128\pi}{3}.
\end{aligned}$$

(In the process of antidifferentiating, we use the identity $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$.)