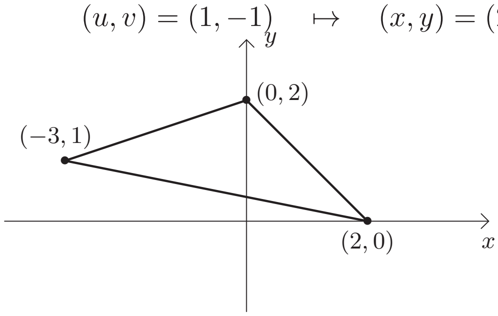
Solutions to the 10/02 Worksheet

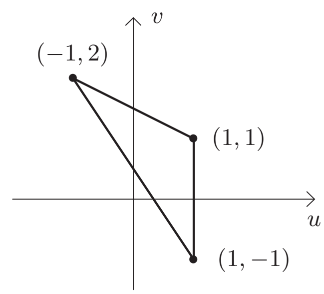
1) $f(x,y) = 2\cos x - y^2 + e^{xy}$.
(a) $f_x = -2\sin x + ye^{xy}, \quad f_y = -2y + xe^{xy}$.
$f_x(0,0) = 0, \quad f_y(0,0) = 0 \quad \Longrightarrow \quad (0,0)$ critical point for f.
(b) $f_{xx} = -2\cos x + y^2 e^{xy}, \quad f_{yy} = -2 + x^2 e^{xy}, \quad f_{xy} = f_{yx} = e^{xy} + xye^{xy} = (1+xy)e^{xy}$.
$f_{xx}(0,0) = -2, \quad f_{yy}(0,0) = -2, \quad f_{xy}(0,0) = f_{yx}(0,0) = 1$.
(c) $h(0) = f(0,0) = 2 - 0 + 1 = 3$.

$$\begin{aligned}
h'(t) &= \frac{\partial f}{\partial x}(tx,ty)x + \frac{\partial f}{\partial y}(tx,ty)y = f_x(tx,ty)x + f_y(tx,ty)y \\
h''(t) &= \left(\frac{\partial f_x}{\partial x}(tx,ty)x + \frac{\partial f_x}{\partial y}(tx,ty)y\right)x + \left(\frac{\partial f_y}{\partial x}(tx,ty)x + \frac{\partial f_y}{\partial y}(tx,ty)y\right)y \\
&= f_{xx}(tx,ty)x^2 + 2f_{xy}(tx,ty)xy + f_{yy}(tx,ty)y^2 \\
h'(0) &= f_x(0,0)x + f_y(0,0)y = 0 \\
h''(0) &= \underbrace{f_{xx}(0,0)}_{-2}x^2 + \underbrace{2f_{xy}(0,0)}_{2}xy + \underbrace{f_{yy}(0,0)}_{-2}y^2 = -2x^2 + 2xy - 2y^2 \\
h(1) &= f(x,y) \approx h(0) + \underbrace{h'(0)}_{0} + \frac{h''(0)}{2} = 3 - x^2 + xy - y^2 = g(x,y).
\end{aligned}$$

2) $g_x = -2x + y, \quad g_y = x - 2y$
$g_x(0,0) = g_y(0,0) = 0 \quad \Longrightarrow \quad (0,0)$ critical point for g.
$g_{xx} = -2, \quad g_{yy} = -1, \quad g_{xy} = g_{yx} = 1$.
$D_g(x,y) = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 = D_g(0,0) > 0$ and $g_{xx}(0,0) < 0 \quad \Longrightarrow \quad (0,0)$ local MAX for g.
$D_f(0,0) = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} > 0$ and $f_{xx}(0,0) = -2 < 0 \quad \Longrightarrow \quad (0,0)$ local MAX for f.
Remark that $D_f(0,0) = D_g(0,0)$ and $f_{xx}(0,0) = g_{xx}(0,0)$, thus g captures all the relevant information concerning the second derivative test for f at the critical point $(0,0)$.

3) $\begin{cases} x = u - v \\ y = u + x \end{cases}$ $\quad (0,0)$ is mapped into $(0,0)$.
(a) $(u,v) = (-1,2) \quad \mapsto \quad (x,y) = (-3,1)$
$(u,v) = (1,1) \quad \mapsto \quad (x,y) = (0,2)$
$(u,v) = (1,-1) \quad \mapsto \quad (x,y) = (2,0)$.





(b) $g(x,y) = 3 - x^2 + xy - y^2 = 3 - (u-v)^2 + (u-v)(u+v) - (u+v)^2 = 3 - u^2 - 3v^2 = G(u,v)$.

(c) Because $G = G(u, v)$ has a local MAX at $(0, 0)$ with $G(0, 0) = g(0, 0) = 3$.
(d)

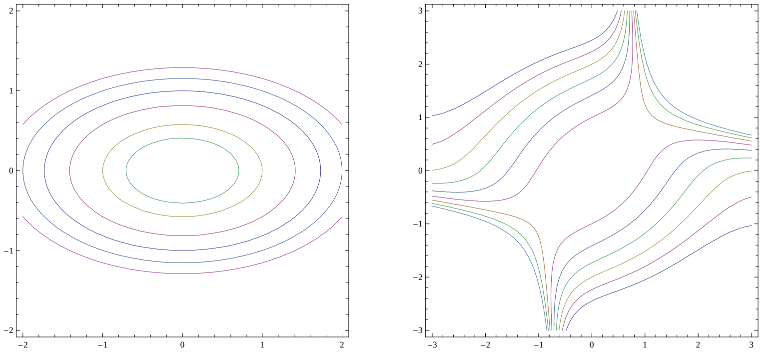


FIGURE 1. Level curves for $G = G(u, v)$ and respectively $g = g(x, y)$

Note that the change of variables $(u, v) \mapsto (x, y) = (u - v, u + v)$ amounts to a counterclockwise rotation by $\pi/4$ and a stretching by a factor of $\sqrt{2}$. This can be seen by writing $u = r\cos\theta$, $v = r\sin\theta$, which give:

$$x = u - v = r(\cos\theta - \sin\theta) = r\sqrt{2}\Big(\cos\theta\cos(\pi/4) - \sin\theta\sin(\pi/4)\Big) = r\sqrt{2}\cos(\theta + \pi/4),$$

$$y = u + v = r(\cos\theta + \sin\theta) = r\sqrt{2}\Big(\cos\theta\cos(\pi/4) + \sin\theta\sin(\pi/4)\Big) = r\sqrt{2}\sin(\theta + \pi/4).$$

Try to reproduce this calculation to see how does the inverse transformation

$$(x, y) \mapsto \left(u = \frac{x + y}{2}, v = \frac{y - x}{2}\right)$$

act geometrically.

4) $f(x, y) = 3xe^y - x^3 - e^{3y}$.
(a) $f_x = 3e^y - 3x^2$, $\quad f_y = 3xe^y - 3e^{3y}$.

Critical points: solve the system $\begin{cases} 3e^y - 3x^2 = 0 \\ 3xe^y - 3e^{3y} = 0 \end{cases}$

The first equation yields $e^y = x^2$. Inserting this into the second equation we find $e^{3y} = x^6 = xe^y = x^3$, which gives $x = 0$ or $x = 1$. But $x = 0$ implies $e^y = 0$ which is not possible, so it remains that $(x, y) = (1, 0)$ is the only critical point for f.

$f_{xx} = -6x$, $\quad f_{yy} = 3xe^y - 9e^{3y}$, $\quad f_{xy} = f_{yx} = 3e^y$.

$D = D(1, 0) = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = (-6)(-6) - 9 > 0$ and $f_{xx}(1, 0) = -6 < 0$, showing that $(1, 0)$ is a local MAX for f.

(b) The domain $\mathbb{R}^2$ of f is not bounded. The function f has no absolute MAX on $\mathbb{R}^2$. This can be seen for instance by noticing that $\lim_{x\to+\infty} f(x, x) = \lim_{x\to+\infty} \left(3xe^x - x^3 - e^{3x}\right) = -\infty$ because the exponential e^{3x} grows much faster than any polynomial.