Workshop 15 October 20, 2011

1. For each of the following, either give an example or justify that no examples exist.
   (a) A convergent series $\sum a_n$ for which $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$.
   (b) A divergent series $\sum a_n$ for which $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$.
   (c) An absolutely convergent series that diverges.
   (d) A conditionally convergent series that diverges.
   (e) A conditionally convergent geometric series.
2. Try to use the ratio test for $\sum \frac{1}{n^p}$. What happens? What if you try the ratio test on any series whose terms are rational functions of n? What other test(s) will work, if the ratio test doesn't?
3. Let's prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$.
   (a) First, prove that the series converges without computing its value. (After you're done with the rest of this problem, you won't have needed to do this, but it's good practice.)
   (b) Let $h_N = \sum_{n=1}^{N} \frac{1}{n}$, the partial sum of the harmonic series. Define the sequence $t_N = h_N - \ln(N)$. Prove that t_N is a bounded decreasing sequence (use an appropriate picture!). What does this say about $\lim_{N\to\infty} t_N$? (This is related to the *Euler-Mascheroni constant*, but we won't go there today.)
   (c) Let $s_N = \sum_{n=1}^{N} \frac{(-1)^n}{n}$. Use a bit of algebra to show that $s_{2N} = h_N - h_{2N}$; it may be helpful to consider $h_N = 2(\frac{1}{2}h_N)$.
   (d) Why do $h_N - \ln N$ and $h_{2N} - \ln 2N$ converge to the same limit at $N \to \infty$?
   (e) Use the previous two parts to show that $\lim\limits_{N\to\infty} s_N = \ln 2$, as desired.
4. *Commutativity is Weird in Infinite Sums*

   Actually, I should say that commutativity is weird in conditionally convergent series. Consider the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots + \frac{1}{2k+1} - \frac{1}{4k+2} - \frac{1}{4k+4} + \cdots.$$

Verify that this series is a rearrangement of the alternating harmonic series (i.e., check that all the terms show up here eventually). Now, simplifying the first two terms from each chunk of three terms, find the value of the series.

(If you're being careful, you might object to the simplifications you made; after all, it's a bit like the associativity we demonstrated didn't work a few worksheets ago. If you're worried that we've lied above, consider the partial sums of the above series and compare them to the partial sums of the simplified series. Why does this justify the simplification?)

(It turns out that you can make the above series converge to whatever your favorite number is (if it's a real number or $\pm\infty$) with an appropriate rearrangement!)

5. Often the ratio and root tests are interchangeable. (Extra question to squeeze in here: can these two tests ever tell you that a series converges conditionally?) Here's an example where that doesn't happen. Try to use both tests on the series

$$\sum_{n=0}^{\infty} 2^{-n+(-1)^n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \cdots .$$

(After you're done with that, look back at the expanded form of the series. If you can commute the terms, what series is it equivalent to? It turns out that, if a series converges *absolutely*, then you can commute the terms, so you actually know the value of this series!)

6. So far we've dealt with series whose terms were just plain old real numbers. Let's jazz it up a bit with variables! For each value of x we can form the series $\sum_{n=0}^{\infty} x^n$, and this will either diverge or converge to some real number. So it makes sense to create the *function* $f(x) = \sum_{n=0}^{\infty} x^n$.

   (a) What is $f(0)$? (Note: it makes sense to call $0^0 = 1$, and we will always do so in this class.)

   (b) What is $f(-1/2)$?

   (c) Why can't I ask what $f(1)$ is?

   (d) What's the domain of f?

   (e) What's a simpler name for f?

   (f) Discuss why it might be helpful to think of f in this series form instead of your answer to the previous part. (Hint: computers love addition, like multiplication, and hate division.)