Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:50am to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$
$$\sin(2t) = 2\sin(t)\cos(t)$$

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points: | 13 | 16 | 18 | 20 | 18 | 15 | 100 |
| Score: | | | | | | | |

1. (13 points) Let $R$ be the region bounded by the curves $x-2y=2$, $x-2y=4$, $2x+y=1$, and $2x+y=3$. Transform $\iint_R (3x-y)\,dx\,dy$ into an integral over a rectangle.

Let $u = x-2y$ so $2 \le u \le 4$
$v = 2x+y$ $\quad 1 \le v \le 3$

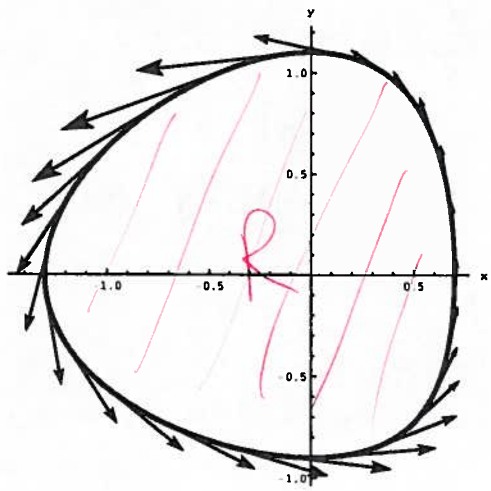
Then $u+2v = 5x+0y$ & $v-2u = 0x+5y$
$\frac{1}{5}(u+2v) = x$ $\quad \frac{1}{5}(v-2u) = y$

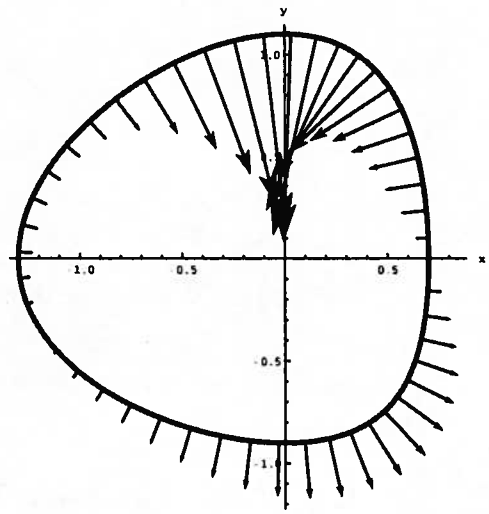
So $A = \begin{vmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{25}(1+4) = \frac{1}{5}$.

Note that $3x-y = u+v$.

$$\boxed{\int_1^3 \int_2^4 (u+v)\cdot\frac{1}{5}\,du\,dv}$$

2. (16 points) Below, a curve $C$ is shown together with the tangential then normal components of a vector field $\mathbf{F}(x, y) = \langle m(x, y), n(x, y)\rangle$ on that curve. Each picture helps you understand some flow. For each, (i) write down a path integral to measure the appropriate net flow, (ii) write down a double integral to measure the appropriate net flow, (iii) say whether the double integral will be positive or negative based on the picture, and (iv) give the interpretation of your answer to (iii) in terms of flow. (Assume $C$ is parametrized counterclockwise wherever necessary. Write integrals in terms of $\mathbf{F}, x, y$; no $t$ necessary.)




(a) path integral:

$\int_C F \cdot \langle dx, dy \rangle$

(b) double integral:

$\iint_R \text{curl}\, F \; dx\, dy$

(c) numeric value of integral (b) is
positive/zero/negative (circle one)

(d) interpretation as net flow:

Net flow along C is counterclockwise.

(e) path integral:

$\int_C F \cdot \langle dy, -dx \rangle$

(f) double integral:

$\iint_R \text{div}\, F \; dx\, dy$

(g) numeric value of integral (f) is
positive/zero/negative (circle one)

(h) interpretation as net flow:

Net flow across C is inward.

3. Let $\mathbf{F}(x,y) = \langle \ln y + 1, \frac{x}{y} + 3 \rangle$ (defined only for $y > 0$).

(a) (8 points) Find a potential function for $\mathbf{F}$.

$\partial_x f = \ln y + 1$ $\qquad$ $\partial_y f = \frac{x}{y} + 3$

$\Rightarrow f(x,y) = x \ln y + x + g(y) \longrightarrow \frac{x}{y} + 0 + g'(y) = \frac{x}{y} + 3$

$\Rightarrow g'(y) = 3$

$\Rightarrow g(y) = 3y \quad + C$

$$\boxed{f(x,y) = x \ln y + x + 3y}$$

(b) (10 points) Compute the flow of $\mathbf{F}$ along the part of the curve $y = 5 + \sin(\pi x)$ going from $(0,5)$ to $(7,5)$.

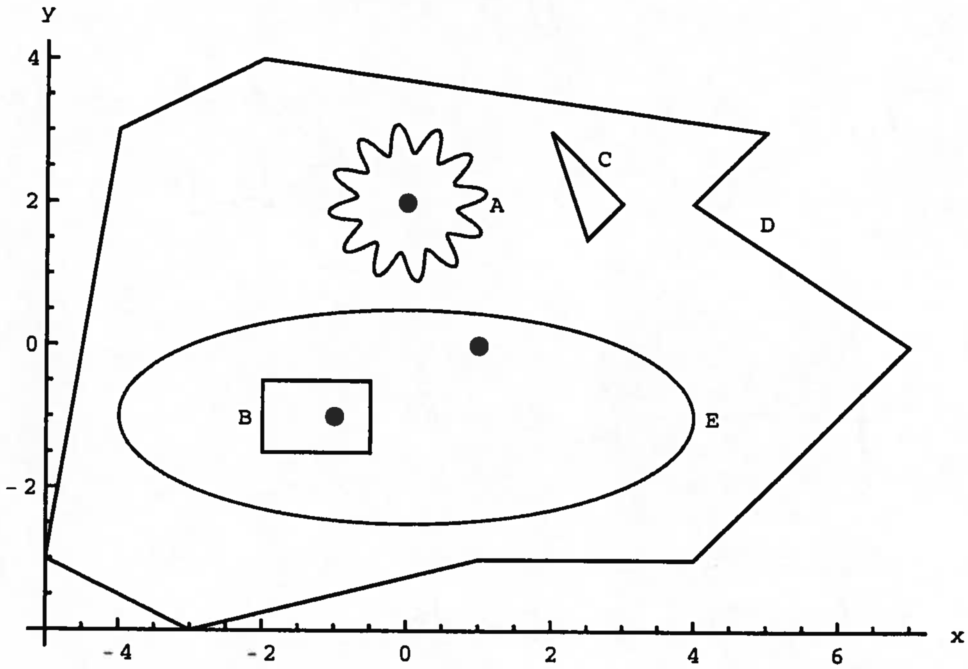
By the Fundamental Thm of Path Integrals,

$= f(7,5) - f(0,5)$

$= (7 \ln 5 + 7 + 15) - (0 + 0 + 15)$

$= \boxed{7 \ln 5 + 7}$

4. A vector field $\mathbf{F}(x,y) = \langle m(x,y), n(x,y)\rangle$ has singularities at $(1,0)$, $(0,2)$, and $(-1,-1)$; other than at these points, the divergence of this field is zero. Below are pictured several curves together with these singularities.




You are given the values of the following integrals.

$$\int_D m\,dy - n\,dx = 7 \qquad \int_E m\,dy - n\,dx = 0$$

For each of the following, either compute the value (with brief justification) or say that it cannot be determined from the given information.

(a) (5 points) $\int_A m\,dy - n\,dx$ $= \int_D \cdots - \int_E \cdots = 7 - 0 = 7$

no singularities between A & E

(b) (5 points) $\int_B m\,dy - n\,dx$ Cannot be determined

(c) (5 points) $\int_C m\,dy - n\,dx$ $= 0$ ; div $F = 0$ everywhere inside $C$.

(d) (5 points) Say as much as you can about which points are sources/sinks/neither.

From (a), $(0,2)$ is a source.

Since $\int_E \cdots = 0$, the points $(1,0)$ & $(-1,-1)$ are either neither sources nor sinks, or one is a source & the other a sink (of equal "magnitude").

No other points are sources or sinks (div $= 0$).

5. (18 points) Let $Q$ be the part of the unit disk that lies in the third quadrant (so is described by $x^2+y^2 \leq 1$, $x \leq 0$, $y \leq 0$). Use an appropriate transformation to compute

$$\iint_Q \frac{2xy}{x^2+y^2}\,dx\,dy.$$

Polar: $x = r\cos\theta$, $y = r\sin\theta$

$3^{rd}$ quadrant: $\theta \in [\pi, \frac{3\pi}{2}]$

unit disk: $r \in [0, 1]$

Area conversion

$$\int_{\pi}^{3\pi/2}\int_0^1 \frac{2r^2\sin\theta\cos\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_{\pi}^{3\pi/2}\int_0^1 2r\sin\theta\cos\theta\,dr\,d\theta$$

$$= \int_{\pi}^{3\pi/2} \sin\theta\cos\theta\,\left[r^2\right]_0^1\,d\theta$$

$$= \int_{\pi}^{3\pi/2} \sin\theta\cos\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \int_0^{-1} u\,du$$

$$= \frac{1}{2}\left[u^2\right]_0^{-1}$$

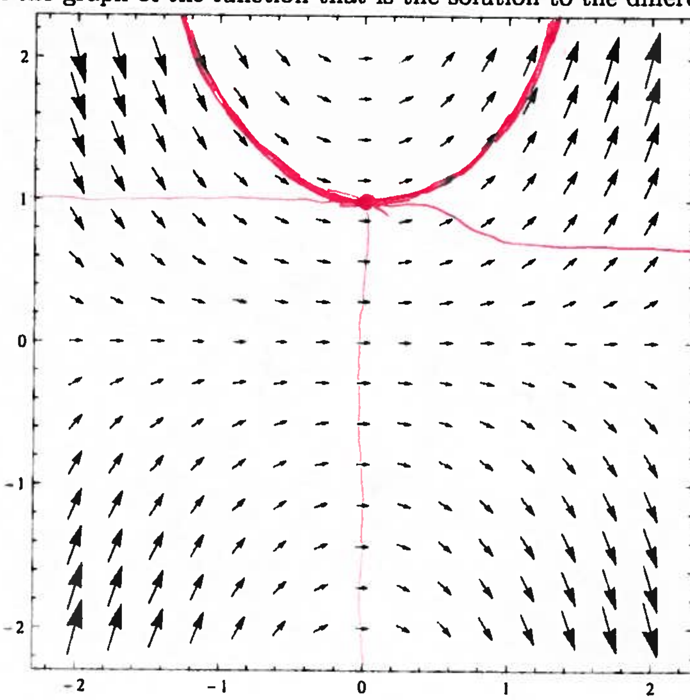
$$= \boxed{\frac{1}{2}}$$

6. Consider the differential equation $y' = xy$, $y(0) = 1$.

   (a) (5 points) Define a vector field $\mathbf{F}(x, y)$ that would help to analyze solutions of this differential equation.

   $F(x,y) = \langle 1, xy \rangle$

   (b) (10 points) Below is the plot of a correct answer to the previous part (with some scaling applied). On this plot, sketch the graph of the function that is the solution to the differential equation.




$y(0) = 1$

**Scratch Paper - Do Not Remove**