# Circles in $L^p$ space

## What is $L^p$?

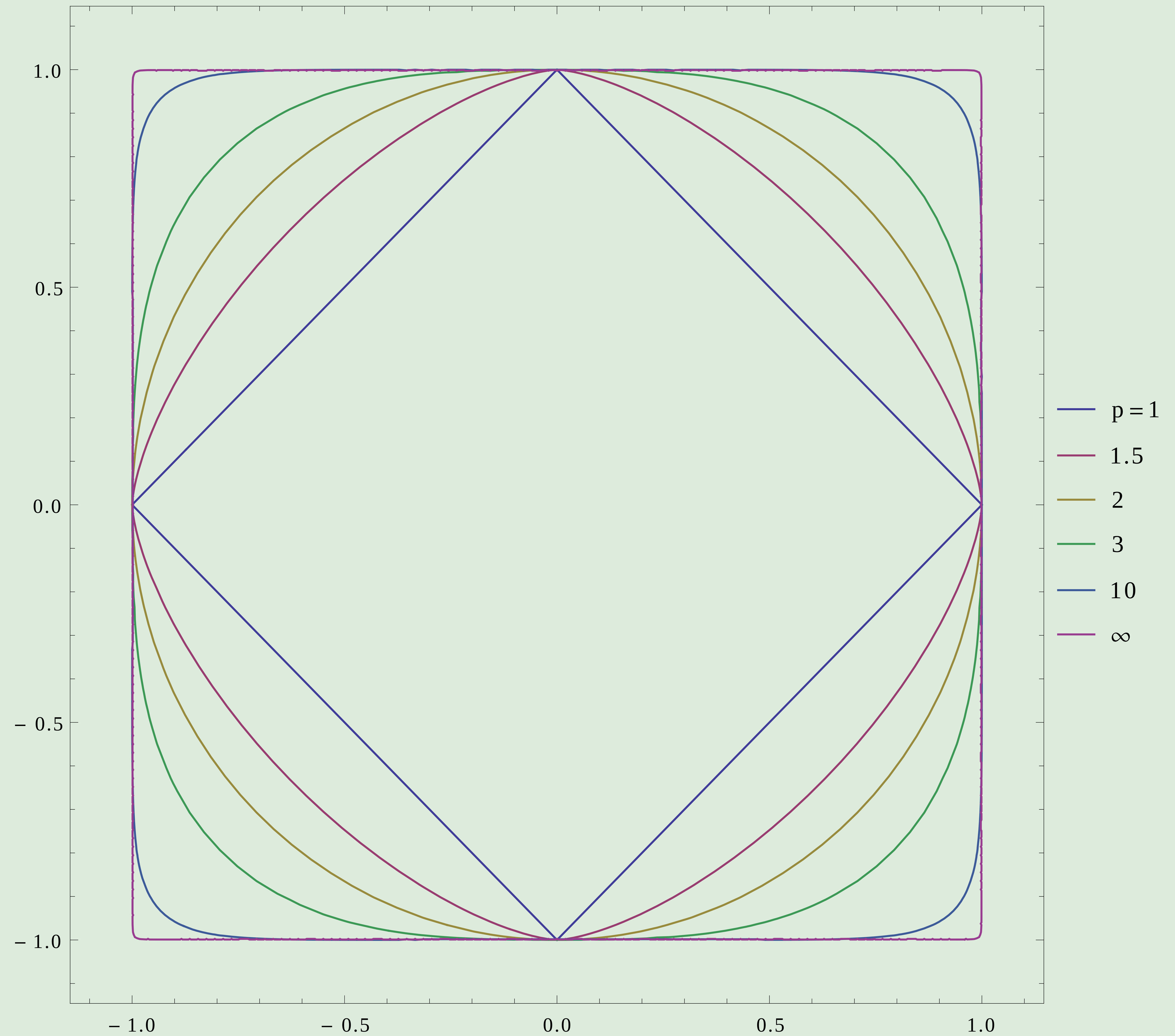
For a real number $p \geq 1$ (or $\infty$), $L^p$ is a different metric on the plane. Taxicab geometry is $L^1$ and ordinary Euclidean geometry is $L^2$.
The distance between the points $(x, y)$ and $(x', y')$ in $L^p$ is defined as

$$(|x - x'|^p + |y - y'|^p)^{1/p}.$$

The circle centered at the origin with radius $r$ is given by

$$|x|^p + |y|^p = r^p.$$

Here are the circles centered at the origin of radius 1 for several values of $p$:




## What is "circumference"?

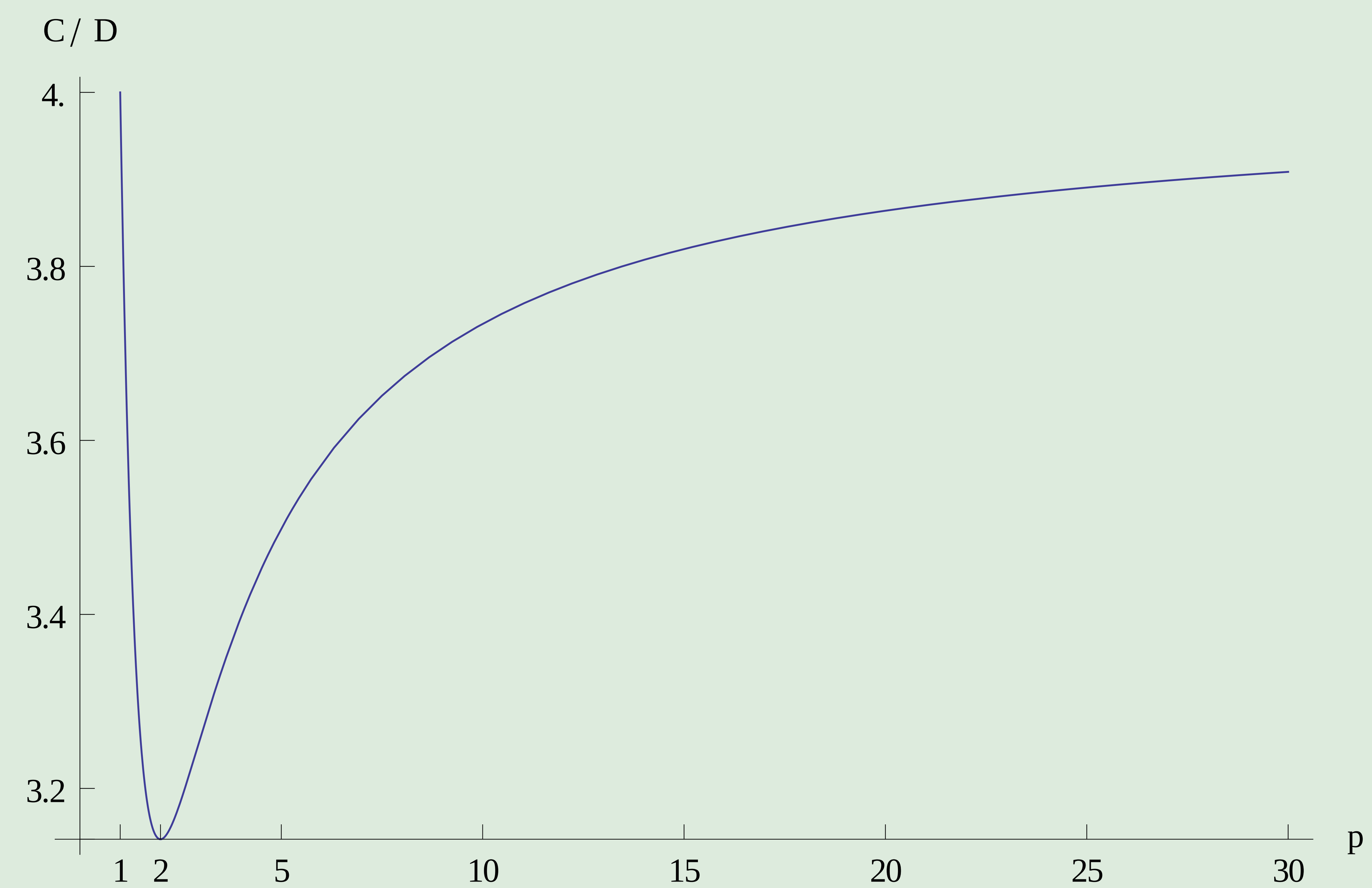
The length of a curve in $L^p$ is defined using calculus:

$$\int_C ds = \int_C (|dx|^p + |dy|^p)^{1/p} = \int_C \left(1 + \left|\frac{dy}{dx}\right|^p\right)^{1/p} |dx|.$$

The circles centered at the origin are symmetric about both axes and the line $y = x$, so we can compute the circumference as 8 times the piece with $0 \leq x \leq y$. There we have $\frac{dy}{dx} = \frac{-x^{p-1}}{(r^p - x^p)^{1-1/p}}$. So the ratio of circumference to diameter is

$$\frac{8}{2r} \int_{x=0}^{(r^p/2)^{1/p}} \left(1 + \frac{x^{p(p-1)}}{(r^p - x^p)^{p-1}}\right)^{1/p} dx.$$

(The transformation $x = rt$ shows that this is independent of $r$.)
This can be computed exactly when $p = 1, 2, \infty$; the values are $4, \pi, 4$ respectively. For other $p$ we can compute the integral numerically. The results appear below.




Notice that the value of $C/D$ has a minimum at $p = 2$, the classic value $\pi$. For every number between $\pi$ and 4 there are two values, $p$ and $q$, with that value of $C/D$. These numbers satisfy $\frac{1}{p} + \frac{1}{q} = 1$.