Discussion 4 – Thursday, September 2nd

1. Find the distance between the point $P(2,1,3)$ and the line $\mathbf{l}(t) = (2,3,-2)+t(-1,1,-2)$.
Solution: See #4 from Discussion 3 (Tuesday August 31) worksheet.

2. Calculate the area of the parallelogram having vertices $(1,-2)$, $(4,-3)$, $(3,-6)$ and $(0,-5)$.
Solution: The two (distinct) vector forms of edges are $(4,-3)-(1,-2) = (3,-1)$ and $(4,-3)-(3,-6) = (1,3)$, so the volume is given by

$$\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = (3)(3) - (-1)(1) = 10.$$

3. Consider the triangle where two of the sides are given by the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$. Compute the area of the triangle.
Solution: The area is half of the area of the parallelogram spanned by these vectors, so is

$$\frac{1}{2}\begin{vmatrix} 1 & -3 \\ 4 & 3 \end{vmatrix} = \frac{1}{2}((1)(3) - (-3)(4)) = \frac{15}{2}.$$

4.
(a) Find the volume of the parallelopiped formed by the vectors $\mathbf{a} = (-2,0,5)$, $\mathbf{b} = (4,-1,0)$ and $\mathbf{c} = (3,1,1)$.
Solution: Again we take a determinant,

$$\begin{aligned} \begin{vmatrix} -2 & 0 & 5 \\ 4 & -1 & 0 \\ 3 & 1 & 1 \end{vmatrix} &= (-2)\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + (5)\begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} \\ &= (-2)(-1) + 0 + (5)(7) \\ &= 37. \end{aligned}$$

(b) Find the volume of the parallelopiped formed by the vectors $\mathbf{a} = (-2,0,5)$, $\mathbf{b} = (4,-1,0)$ and $\mathbf{d} = (8,-1,-10)$. Interpret your answer geometrically. Specifically, what can you say about these three vectors?
Solution:

$$\begin{aligned} \begin{vmatrix} -2 & 0 & 5 \\ 4 & -1 & 0 \\ 8 & -1 & -10 \end{vmatrix} &= (-2)\begin{vmatrix} -1 & 0 \\ -1 & -10 \end{vmatrix} - (0)\begin{vmatrix} 4 & 0 \\ 8 & -10 \end{vmatrix} + (5)\begin{vmatrix} 4 & -1 \\ 8 & -1 \end{vmatrix} \\ &= (-2)(10) + 0 + (5)(4) \\ &= 0. \end{aligned}$$

If the volume of the parallelopiped is zero, it must be “flat”; indeed, the three vectors given here all lie in the same plane $\mathbf{p}(t) = t\mathbf{a} + s\mathbf{b}$, for all $s, t \in \mathbb{R}$. Notice that $\mathbf{d} = -2\mathbf{a} + \mathbf{b}$.

5.
(a) Calculate $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = (3,-2,3)$ and $\mathbf{b} = (2,-4,1)$. Calculate $\mathbf{b} \times \mathbf{a}$. How do these two

cross products compare?
Solution:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 3 \\ 2 & -4 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ -4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 2 & -4 \end{vmatrix}\mathbf{k}$$
$$= 10\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}.$$
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 1 \\ 3 & -2 & 3 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ -2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -4 \\ 3 & -2 \end{vmatrix}\mathbf{k}$$
$$= -10\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}.$$

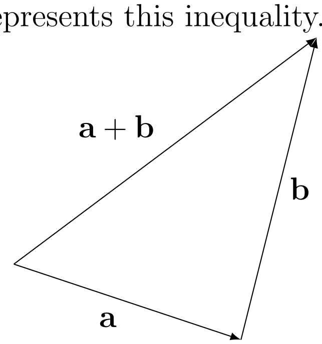
Notice that these are *not* equal! $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

The Right Hand Rule: If the fingers of your right hand curl in the direction of a rotation from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.
(b) Without doing any calculation, use the right hand rule to find a vector parallel to $\mathbf{i} \times \mathbf{j}$.
Answer: $\mathbf{k}$
(c) Find a vector parallel to $\mathbf{j} \times \mathbf{k}$.
Answer: $\mathbf{i}$
(d) Find a vector parallel to $-\mathbf{k} \times \mathbf{i}$
Answer: $-\mathbf{j}$
(e) Find a vector parallel to $(\mathbf{i} + \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$.
Answer: $-\mathbf{j}$
(f) Calculate $(\mathbf{i}+\mathbf{k}) \times (\mathbf{k}-\mathbf{i})$. Are your answers to (e) and (f) the same? What is different?
Solution:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}\mathbf{k}$$
$$= -2\mathbf{j}.$$

The vector from (e) has the same direction as this vector, but has different magnitude.

6. The Triangle Inequality.
Let $\mathbf{a}$ and $\mathbf{b}$ be any vectors in $\mathbb{R}^n$. The triangle inequality states that $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$.
(a) Give a geometric interpretation of the triangle inequality. (i.e. Draw a picture in $\mathbb{R}^2$ or $\mathbb{R}^3$ that represents this inequality.)



Solution: The triangle inequality says that the sum of the lengths of two sides of a triangle is at least as large as the length of the third side.

(b) Use the Cauchy-Schwarz inequality to prove the triangle inequality. (Hint: $||\mathbf{a}+\mathbf{b}||^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$)

$$\begin{aligned}
||\mathbf{a}+\mathbf{b}||^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
&= ||\mathbf{a}||^2 + 2\mathbf{a}\cdot\mathbf{b} + ||\mathbf{b}||^2 \\
&\leq ||\mathbf{a}||^2 + 2|\mathbf{a}\cdot\mathbf{b}| + ||\mathbf{b}||^2 \\
&\leq ||\mathbf{a}||^2 + 2||\mathbf{a}||\ ||\mathbf{b}|| + ||\mathbf{b}||^2 \\
&= (||\mathbf{a}|| + ||\mathbf{b}||)^2,
\end{aligned}$$

and now taking square roots on each side gives us the triangle inequality.