Name: Solutions

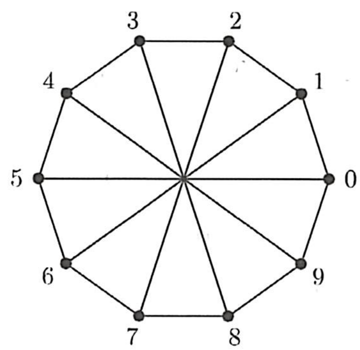
- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have seventy-five (75) minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.

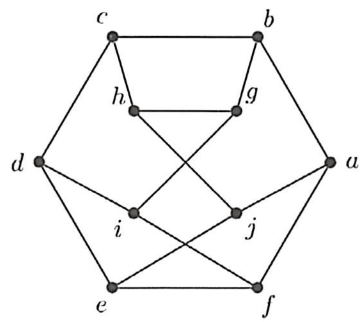
- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.

Question:	1	2	3	4	5	6	7	Total
Points:	12	12	12	12	12	20	40	120
Score:								

Short answer

1. (12 points) Determine whether the following two graphs are isomorphic. (Give an isomorphism or a short argument why they are not isomorphic.)



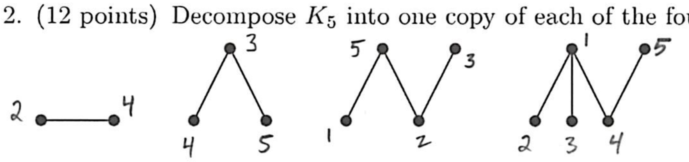


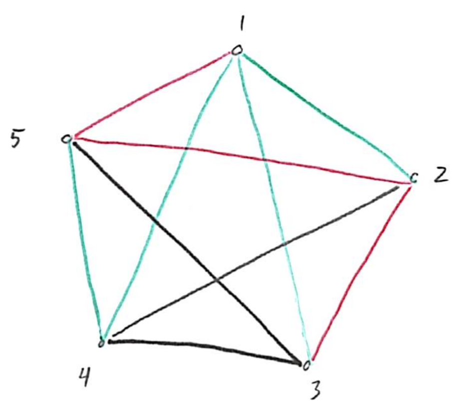
They are not isomorphic.

The first graph is bipartite — the even vts & the odd vts form indep. sets.

The second graph is not — bcdig is a 5-cycle.

2. (12 points) Decompose K_5 into one copy of each of the four trees below.





Algorithms

Give brief justifications for your answers (but not necessarily full proofs; a computation with each step clearly written may suffice).

3. (12 points) Determine whether the following sequences are graphic (the degree sequence of a simple graph).

   (a) 6 5 5 5 2 2 2 1 1

   odd # of odd degree vts

   $\Rightarrow$ not graphic

   (b) ~~6~~ 5 5 5 2 2 2 1 not graphic

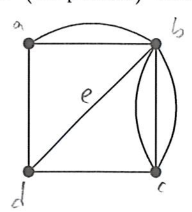
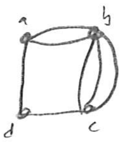
   ~~4~~ 4 4 1 1 1 1

   3 3 0 0 1 1 (Havel-Hakimi)

   ~~3~~ 3 1 1 0 0

   2 0 0 0 0 not graphic

4. (12 points) How many spanning trees does the following graph have?

$\tau($  $) = \tau($  $) + \tau($  $)$

$= 1\cdot1\cdot3 + 2\cdot1\cdot1 + 3\cdot2\cdot1 + 1\cdot3\cdot2 \quad + \quad 3\cdot4$

$= 17 + 12 = 29$

OR

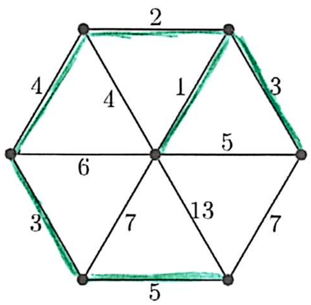
$$Q = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 3 & -2 & 0 & -1 \\ b & -2 & 6 & -3 & -1 \\ c & 0 & -3 & 4 & -1 \\ d & -1 & -1 & -1 & 3 \end{array} \qquad Q^* = \begin{array}{c|ccc} & a & c & d \\ \hline a & 3 & 0 & -1 \\ c & 0 & 4 & -1 \\ d & -1 & -1 & 3 \end{array}$$

$$\det Q^* = 3\cdot\det\begin{bmatrix}4 & -1\\ -1 & 3\end{bmatrix} - 0\cdot\det[\quad] - 1\det\begin{bmatrix}0 & 4\\ -1 & -1\end{bmatrix}$$

$$= 3\cdot 11 + 0 - 4 = 29$$

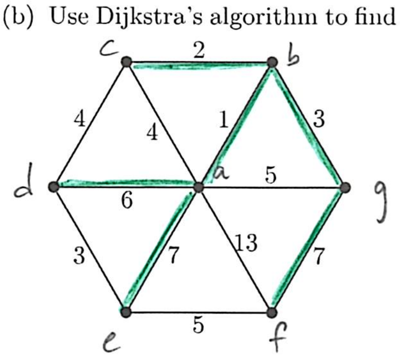
5. (12 points) *Weighted graph algorithms*

(a) Use either Kruskal's or Prim's algorithm to find a minimum spanning tree in the following weighted graph.



Kruskal:
The edges w/ wt 1,2 are added first,
then both edges w/ wt 3 (in either order).
Now only one edge w/ wt 4 is allowed,
then only one edge w/ wt 5 is allowed,
and we are done
(w/o considering the edges of wt 6, 7, 13).

(b) Use Dijkstra's algorithm to find the minimum distances from the central vertex to each other vertex.



	a	b	c	d	e	f	g
	$\boxed{0}$	∞	∞	∞	∞	∞	∞
a:		$\boxed{1}$	4	$\boxed{6}$	$\boxed{7}$	13	5
b:			$\boxed{3}$	X	X	X	$\boxed{4}$
c:				X	X	X	X
g:				X	X	$\boxed{11}$	
d:					X	X	
e:						X	
f							

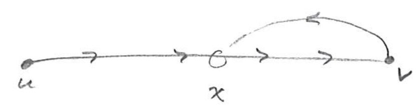
Proofs

6. (20 points) *Digraphs*

(a) Prove that if D is a digraph with $\delta^+(D) \geq 1$, then D has a (directed) cycle.

Consider a maximal path P in D, say from u to v.

$d^+(v) \geq \delta^+(D) \geq 1$, but all of the outnbrs of v are on P (else P is not max'l).



So $\exists$ $x \in V(P)$ w/ $v \to x$ in D.

The part of P from x to v together w/ this edge forms a cycle.

(b) Use the statement in part (a) to prove that if D is a digraph with $d^+(v) = d^-(v)$ at every vertex v, then D decomposes into (directed) cycles.

By induction on $m = |E(D)|$. If $m = 0$, this is trivial.

If $m > 0$, let H be a nontrivial weak component of D.

Every vtx v of H has $d^+(v) = d^-(v) \geq 1$, so (a) $\Rightarrow$ H has a cycle C. Note C is also a cycle of D.

Now $D - E(C)$ is a digraph w/ $d^+(v) = d^-(v)$ $\forall v \in V(D)$ [because each vtx of C loses one in-edge & one out-edge & each other vtx loses no edges],

so by the I.H. it decomposes into cycles $C^1, C^2, \dots, C^k$,

and now D decomposes into $C, C^1, C^2, \dots, C^k$.

7. (40 points) Give complete careful proofs of **2 of the following 3** statements.

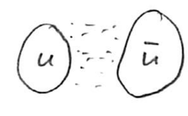
(i) If G is disconnected, then $\overline{G}$ is connected.

(ii) A graph is a tree if and only if it is loopless and has exactly one spanning tree.

(iii) If G is even (i.e., every vertex degree is even), then G has no cut edge.

(i) Let G be disconnected. Then G has a component $H \subsetneq G$; let $U = V(H)$ & $\overline{U} = V(G) - V(H)$. $(U, \overline{U} \neq \emptyset)$

Then G has no edges joining vts of U to those of $\overline{U}$.



Consider two vertices $x, y \in V(\overline{G})$:

- if $x \in U$ & $y \in \overline{U}$ (or vice-versa), then $xy \notin E(G)$ so $xy \in E(\overline{G})$.
- if $x, y \in U$, then $\exists\ z \in \overline{U}$, and $xz, zy \notin E(G)$ so $xz, zy \in E(\overline{G})$
- if $x, y \in \overline{U}$, - - - - U - - - - - -

In any case, we have found an xy-path (of length 1 or 2) in $\overline{G}$, so $\overline{G}$ is connected.

(ii) "only if": a tree is acyclic, so loopless,
& the only spanning tree of a tree is the whole tree:
any proper spanning subgraph has too few edges to be a tree.

"if": Say G is acyclic & has only $T \leq G$ as a sp. tree.

If $G \neq T$, then $\exists\ e \in E(G) - E(T)$.

But then $T + e$ has a cycle (of length ≥ 2 since e is not a loop), say w/ an edge e'. Then $T + e - e'$ is another sp. tree, $\lightning$.

(The statements are repeated here for your convenience.)

(i) If G is disconnected, then $\overline{G}$ is connected.

(ii) A graph is a tree if and only if it is loopless and has exactly one spanning tree.

(iii) If G is even (i.e., every vertex degree is even), then G has no cut edge.

(iii) Proof 1: Every cpnt of G has an Eulerian circuit. Deleting $e \in E(G)$ leaves an Eulerian trail, (in the cpnt containing e) so $G-e$ is still connected.

Proof 2: G decomposes into cycles. Cut-edges are precisely those not in any cycles.

Proof 3: Say $e \in E(G)$, & let H be the cpnt of G containing e. If e is a cut-edge, then $H-e$ has two cpnts A & B. But every vtx of A is even, except for the endpoint of e which now has odd degree. But graphs (here A) cannot have an odd number (here 1) of odd-degree vts. $\lightning$

Scratch Paper - Do Not Remove