SECTION 1.5 (Part II)

Polynomials - Introduction to Polynomials

Objective: Evaluate, add, and subtract polynomials.

Many applications in mathematics have to do with what are called polynomials. Polynomials are made up of terms. **Terms** are a product of numbers and/or variables. For example, $5x$, $2y^2$, -5, ab^3c, and x are all terms. Terms are connected to each other by addition or subtraction. Expressions are often named based on the number of terms in them. A **monomial** has one term, such as $3x^2$. A **binomial** has two terms, such as $a^2 - b^2$. A Trinomial has three terms, such as $ax^2 + bx + c$. The term **polynomial** means many terms. Monomials, binomials, trinomials, and expressions with more terms all fall under the umbrella of "polynomials".

If we know what the variable in a polynomial represents we can replace the variable with the number and evaluate the polynomial as shown in the following example.

Example

$2x^2 - 4x + 6$ when $x = -4$	Replace variable x with -4
$2(-4)^2 - 4(-4) + 6$	Exponents first
$2(16) - 4(-4) + 6$	Multiplication (we can do all terms at once)
$32 + 16 + 6$	Add
54	Our Solution

It is important to be careful with negative variables and exponents. Remember the exponent only effects the number it is physically attached to. This means $-3^2 = -9$ because the exponent is only attached to the 3. Also, $(-3)^2 = 9$ because the exponent is attached to the parenthesis and effects everything inside. When we replace a variable with parenthesis like in the previous example, the substituted value is in parenthesis. So the $(-4)^2 = 16$ in the example. However, consider the next example.

Example

$-x^2 + 2x + 6$ when $x = 3$	Replace variable x with 3
$-(3)^2 + 2(3) + 6$	Exponent only on the 3, not negative
$-9 + 2(3) + 6$	Multiply
$-9 + 6 + 6$	Add
3	Our Solution

World View Note: Ada Lovelace in 1842 described a Difference Engine that would be used to caluclate values of polynomials. Her work became the foundation for what would become the modern computer (the programming language Ada was named in her honor), more than 100 years after her death from cancer.

Generally when working with polynomials we do not know the value of the variable, so we will try and simplify instead. The simplest operation with polynomials is addition. When adding polynomials we are mearly combining like terms. Consider the following example

Example

$$(4x^3 - 2x + 8) + (3x^3 - 9x^2 - 11) \quad \text{Combine like terms } 4x^3 + 3x^3 \text{ and } 8 - 11$$
$$7x^3 - 9x^2 - 2x - 3 \quad \text{Our Solution}$$

Generally final answers for polynomials are written so the exponent on the variable counts down. Example 3 demonstrates this with the exponent counting down 3, 2, 1, 0 (recall $x^0 = 1$). Subtracting polynomials is almost as fast. One extra step comes from the minus in front of the parenthesis. When we have a negative in front of parenthesis we distribute it through, changing the signs of everything inside. The same is done for the subtraction sign.

Example .

$$(5x^2 - 2x + 7) - (3x^2 + 6x - 4) \quad \text{Distribute negative through second part}$$
$$5x^2 - 2x + 7 - 3x^2 - 6x + 4 \quad \text{Combine like terms } 5x^2 - 3x^3, -2x - 6x, \text{ and } 7 + 4$$
$$2x^2 - 8x + 11 \quad \text{Our Solution}$$

Addition and subtraction can also be combined into the same problem as shown in this final example.

Example

$$(2x^2 - 4x + 3) + (5x^2 - 6x + 1) - (x^2 - 9x + 8) \quad \text{Distribute negative through}$$
$$2x^2 - 4x + 3 + 5x^2 - 6x + 1 - x^2 + 9x - 8 \quad \text{Combine like terms}$$
$$6x^2 - x - 4 \quad \text{Our Solution}$$

Practice - Introduction to Polynomials

Simplify each expression.

1) $-a^3 - a^2 + 6a - 21$ when $a = -4$

2) $n^2 + 3n - 11$ when $n = -6$

3) $n^3 - 7n^2 + 15n - 20$ when $n = 2$

4) $n^3 - 9n^2 + 23n - 21$ when $n = 5$

5) $-5n^4 - 11n^3 - 9n^2 - n - 5$ when $n = -1$

6) $x^4 - 5x^3 - x + 13$ when $x = 5$

7) $x^2 + 9x + 23$ when $x = -3$

8) $-6x^3 + 41x^2 - 32x + 11$ when $x = 6$

9) $x^4 - 6x^3 + x^2 - 24$ when $x = 6$

10) $m^4 + 8m^3 + 14m^2 + 13m + 5$ when $m = -6$

11) $(5p - 5p^4) - (8p - 8p^4)$

12) $(7m^2 + 5m^3) - (6m^3 - 5m^2)$

13) $(3n^2 + n^3) - (2n^3 - 7n^2)$

14) $(x^2 + 5x^3) + (7x^2 + 3x^3)$

15) $(8n + n^4) - (3n - 4n^4)$

16) $(3v^4 + 1) + (5 - v^4)$

17) $(1 + 5p^3) - (1 - 8p^3)$

18) $(6x^3 + 5x) - (8x + 6x^3)$

19) $(5n^4 + 6n^3) + (8 - 3n^3 - 5n^4)$

20) $(8x^2 + 1) - (6 - x^2 - x^4)$

21) $(3+b^4)+(7+2b+b^4)$

22) $(1+6r^2)+(6r^2-2-3r^4)$

23) $(8x^3+1)-(5x^4-6x^3+2)$

24) $(4n^4+6)-(4n-1-n^4)$

25) $(2a+2a^4)-(3a^2-5a^4+4a)$

26) $(6v+8v^3)+(3+4v^3-3v)$

27) $(4p^2-3-2p)-(3p^2-6p+3)$

28) $(7+4m+8m^4)-(5m^4+1+6m)$

29) $(4b^3+7b^2-3)+(8+5b^2+b^3)$

30) $(7n+1-8n^4)-(3n+7n^4+7)$

31) $(3+2n^2+4n^4)+(n^3-7n^2-4n^4)$

32) $(7x^2+2x^4+7x^3)+(6x^3-8x^4-7x^2)$

33) $(n-5n^4+7)+(n^2-7n^4-n)$

34) $(8x^2+2x^4+7x^3)+(7x^4-7x^3+2x^2)$

35) $(8r^4-5r^3+5r^2)+(2r^2+2r^3-7r^4+1)$

36) $(4x^3+x-7x^2)+(x^2-8+2x+6x^3)$

37) $(2n^2+7n^4-2)+(2+2n^3+4n^2+2n^4)$

38) $(7b^3-4b+4b^4)-(8b^3-4b^2+2b^4-8b)$

39) $(8-b+7b^3)-(3b^4+7b-8+7b^2)+(3-3b+6b^3)$

40) $(1-3n^4-8n^3)+(7n^4+2-6n^2+3n^3)+(4n^3+8n^4+7)$

41) $(8x^4+2x^3+2x)+(2x+2-2x^3-x^4)-(x^3+5x^4+8x)$

42) $(6x-5x^4-4x^2)-(2x-7x^2-4x^4-8)-(8-6x^2-4x^4)$

Polynomials - Multiplying Polynomials

Objective: Multiply polynomials.

Multiplying polynomials can take several different forms based on what we are multiplying. We will first look at multiplying monomials, then monomials by polynomials and finish with polynomials by polynomials.

Multiplying monomials is done by multiplying the numbers or coefficients and then adding the exponents on like factors. This is shown in the next example.

Example

$(4x^3y^4z)(2x^2y^6z^3)$	Multiply numbers and add exponents for x, y, and z
$8x^5y^{10}z^4$	Our Solution

In the previous example it is important to remember that the z has an exponent of 1 when no exponent is written. Thus for our answer the z has an exponent of $1+3=4$. Be very careful with exponents in polynomials. If we are adding or subtracting the exponnets will stay the same, but when we multiply (or divide) the exponents will be changing.

Next we consider multiplying a monomial by a polynomial. We have seen this operation before with distributing through parenthesis. Here we will see the exact same process.

Example

$4x^3(5x^2-2x+5)$	Distribute the $4x^3$, multiplying numbers, adding exponents
$20x^5-8x^4+20x^3$	Our Solution

Following is another example with more variables. When distributing the exponents on a are added and the exponents on b are added.

Example

$2a^3b(3\mathrm{ab}^2-4a)$	Distribute, multiplying numbers and adding exponents
$6a^4b^3-8a^4b$	Our Solution

There are several different methods for multiplying polynomials. All of which work, often students prefer the method they are first taught. Here three methods will be discussed. All three methods will be used to solve the same two multiplication problems.

Multiply by Distributing

Just as we distribute a monomial through parenthesis we can distribute an entire polynomial. As we do this we take each term of the second polynomial and put it in front of the first polynomial.

Example

$$
\begin{array}{rl}
(4x+7y)(3x-2y) & \text{Distribute } (4x+7y) \text{ through parenthesis} \\
3x(\mathbf{4x+7y})-2y(\mathbf{4x+7y}) & \text{Distribute the } 3x \text{ and } -2y \\
12x^2+21xy-8xy-14y^2 & \text{Combine like terms } 21xy-8xy \\
12x^2+13xy-14y^2 & \text{Our Solution}
\end{array}
$$

This example illustrates an important point, the negative/subtraction sign stays with the $2y$. Which means on the second step the negative is also distributed through the last set of parenthesis.

Multiplying by distributing can easily be extended to problems with more terms. First distribute the front parenthesis onto each term, then distribute again!

Example

$$
\begin{array}{rl}
(2x-5)(4x^2-7x+3) & \text{Distribute } (2x-5) \text{ through parenthesis} \\
4x^2(\mathbf{2x-5})-7x(\mathbf{2x-5})+3(\mathbf{2x-5}) & \text{Distribute again through each parenthesis} \\
8x^3-20x^2-14x^2+35x+6x-15 & \text{Combine like terms} \\
8x^3-34x^2+41x-15 & \text{Our Solution}
\end{array}
$$

This process of multiplying by distributing can easily be reversed to do an important procedure known as factoring. Factoring will be addressed in a future lesson.

Multiply by FOIL

Another form of multiplying is known as FOIL. Using the FOIL method we multiply each term in the first binomial by each term in the second binomial. The letters of FOIL help us remember every combination. F stands for First, we multiply the first term of each binomial. O stand for Outside, we multiply the outside two terms. I stands for Inside, we multiply the inside two terms. L stands for Last, we multiply the last term of each binomial. This is shown in the next example:

Example

$$
\begin{array}{rl}
(4x+7y)(3x-2y) & \text{Use FOIL to multiply} \\
(4x)(3x)=12x^2 & F-\text{First terms } (4x)(3x) \\
(4x)(-2y)=-8xy & O-\text{Outside terms } (4x)(-2y) \\
(7y)(3x)=21xy & I-\text{Inside terms } (7y)(3x) \\
(7y)(-2y)=-14y^2 & L-\text{Last terms } (7y)(-2y) \\
12x^2-8xy+21xy-14y^2 & \text{Combine like terms } -8xy+21xy \\
12x^2+13xy-14y^2 & \text{Our Solution}
\end{array}
$$

Some students like to think of the FOIL method as distributing the first term $4x$ through the $(3x-2y)$ and distributing the second term $7y$ through the $(3x-2y)$. Thinking about FOIL in this way makes it possible to extend this method to problems with more terms.

Example

$$\begin{array}{rl}
(2x-5)(4x^2-7x+3) & \text{Distribute } 2x \text{ and } -5 \\
(2x)(4x^2)+(2x)(-7x)+(2x)(3)-5(4x^2)-5(-7x)-5(3) & \text{Multiply out each term} \\
8x^3-14x^2+6x-20x^2+35x-15 & \text{Combine like terms} \\
8x^3-34x^2+41x-15 & \text{Our Solution}
\end{array}$$

The second step of the FOIL method is often not written, for example, consider the previous example, a student will often go from the problem $(4x+7y)(3x-2y)$ and do the multiplication mentally to come up with $12x^2-8xy+21xy-14y^2$ and then combine like terms to come up with the final solution.

Multiplying in rows

A third method for multiplying polynomials looks very similar to multiplying numbers. Consider the problem:

$$\begin{array}{rl}
35 & \\
\underline{\times 27} & \\
245 & \text{Multiply 7 by 5 then 3} \\
\underline{700} & \text{Use 0 for placeholder, multiply 2 by 5 then 3} \\
945 & \text{Add to get Our Solution}
\end{array}$$

World View Note: The first known system that used place values comes from Chinese mathematics, dating back to 190 AD or earlier.

The same process can be done with polynomials. Multiply each term on the bottom with each term on the top.

Example

$$\begin{array}{rl}
(4x+7y)(3x-2y) & \text{Rewrite as vertical problem} \\
4x+7y & \\
\underline{\times 3x-2y} & \\
-8xy-14y^2 & \text{Multiply } -2y \text{ by } 7y \text{ then } 4x \\
\underline{12x^2+21xy\qquad\quad} & \text{Multiply } 3x \text{ by } 7y \text{ then } 4x\text{. Line up like terms} \\
12x^2+13xy-14y^2 & \text{Add like terms to get Our Solution}
\end{array}$$

This same process is easily expanded to a problem with more terms.

Example

$(2x-5)(4x^2-7x+3)$	Rewrite as vertical problem
$4x^3-7x+3$	Put polynomial with most terms on top
$\times\, 2x-5$	
$-20x^2+35x-15$	Multiply -5 by each term
$8x^3-14x^2+6x$	Multiply $2x$ by each term. Line up like terms
$8x^3-34x^2+41x-15$	Add like terms to get our solution

This method of multiplying in rows also works with multiplying a monomial by a polynomial!

Any of the three described methods work to multiply polynomials. It is suggested that you are very comfortable with at least one of these methods as you work through the practice problems. All three methods are shown side by side in the example.

Example

$$(2x-y)(4x-5y)$$

Distribute	FOIL	Rows
$4x(2x-y)-5y(2x-y)$	$2x(4x)+2x(-5y)-y(4x)-y(-5y)$	$2x-y$
$8x^2-4xy-10xy-5y^2$	$8x^2-10xy-4xy+5y^2$	$\times\, 4x-5y$
$8x^2-14xy-5y^2$	$8x^2-14xy+5y^2$	$-10xy+5y^2$
		$8x^2-4xy$
		$8x^2-14xy+5y^2$

When we are multiplying a monomial by a polynomial by a polynomial we can solve by first multiplying the polynomials then distributing the coefficient last. This is shown in the last example.

Example

$3(2x-4)(x+5)$	Multiply the binomials, we will use FOIL
$3(2x^2+10x-4x-20)$	Combine like terms
$3(2x^2+6x-20)$	Distribute the 3
$6x^2+18x-60$	Our Solution

A common error students do is distribute the three at the start into both parenthesis. While we can distribute the 3 into the $(2x-4)$ factor, distributing into both would be wrong. Be careful of this error. This is why it is suggested to multiply the binomials first, then distribute the coeffienct last.

Practice - Multiply Polynomials

Find each product.

1) $6(p-7)$

2) $4k(8k+4)$

3) $2(6x+3)$

4) $3n^2(6n+7)$

5) $5m^4(4m+4)$

6) $3(4r-7)$

7) $(4n+6)(8n+8)$

8) $(2x+1)(x-4)$

9) $(8b+3)(7b-5)$

10) $(r+8)(4r+8)$

11) $(4x+5)(2x+3)$

12) $(7n-6)(n+7)$

13) $(3v-4)(5v-2)$

14) $(6a+4)(a-8)$

15) $(6x-7)(4x+1)$

16) $(5x-6)(4x-1)$

17) $(5x+y)(6x-4y)$

18) $(2u+3v)(8u-7v)$

19) $(x+3y)(3x+4y)$

20) $(8u+6v)(5u-8v)$

21) $(7x+5y)(8x+3y)$

22) $(5a+8b)(a-3b)$

23) $(r-7)(6r^2-r+5)$

24) $(4x+8)(4x^2+3x+5)$

25) $(6n-4)(2n^2-2n+5)$

26) $(2b-3)(4b^2+4b+4)$

27) $(6x+3y)(6x^2-7xy+4y^2)$

28) $(3m-2n)(7m^2+6mn+4n^2)$

29) $(8n^2+4n+6)(6n^2-5n+6)$

30) $(2a^2+6a+3)(7a^2-6a+1)$

31) $(5k^2+3k+3)(3k^2+3k+6)$

32) $(7u^2+8uv-6v^2)(6u^2+4uv+3v^2)$

33) $3(3x-4)(2x+1)$

34) $5(x-4)(2x-3)$

35) $3(2x+1)(4x-5)$

36) $2(4x+1)(2x-6)$

37) $7(x-5)(x-2)$

38) $5(2x-1)(4x+1)$

39) $6(4x-1)(4x+1)$

40) $3(2x+3)(6x+9)$

Polynomials - Multiply Special Products

Objective: Recognize and use special product rules of a sum and difference and perfect squares to multiply polynomials.

There are a few shortcuts that we can take when multiplying polynomials. If we can recognize them the shortcuts can help us arrive at the solution much quicker. These shortcuts will also be useful to us as our study of algebra continues.

The first shortcut is often called a **sum and a difference**. A sum and a difference is easily recognized as the numbers and variables are exactly the same, but the sign in the middle is different (one sum, one difference). To illustrate the shortcut consider the following example, multiplied by the distributing method.

Example

$$\begin{array}{rl} (a+b)(a-b) & \text{Distribute } (a+b) \\ a(a+b)-b(a+b) & \text{Distribute } a \text{ and } -b \\ a^2+ab-ab-b^2 & \text{Combine like terms } ab-ab \\ a^2-b^2 & \text{Our Solution} \end{array}$$

The important part of this example is the middle terms subtracted to zero. Rather than going through all this work, when we have a sum and a difference we will jump right to our solution by squaring the first term and squaring the last term, putting a subtraction between them. This is illustrated in the following example

Example

$$\begin{array}{rl} (x-5)(x+5) & \text{Recognize sum and difference} \\ x^2-25 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

This is much quicker than going through the work of multiplying and combining like terms. Often students ask if they can just multiply out using another method and not learn the shortcut. These shortcuts are going to be very useful when we get to factoring polynomials, or reversing the multiplication process. For this reason it is very important to be able to recognize these shortcuts. More examples are shown here.

Example

$$\begin{array}{rl}(3x+7)(3x-7) & \text{Recognize sum and difference}\\ 9x^2-49 & \text{Square both, put subtraction between. Our Solution}\end{array}$$

Example

$$\begin{array}{rl}(2x-6y)(2x+6y) & \text{Recognize sum and difference}\\ 4x^2-36y^2 & \text{Square both, put subtraction between. Our Solution}\end{array}$$

It is interesting to note that while we can multiply and get an answer like a^2-b^2 (with subtraction), it is impossible to multiply real numbers and end up with a product such as a^2+b^2 (with addition).

Another shortcut used to multiply is known as a **perfect square**. These are easy to recognize as we will have a binomial with a 2 in the exponent. The following example illustrates multiplying a perfect square

Example

$$\begin{array}{rl}(a+b)^2 & \text{Squared is same as multiplying by itself}\\ (a+b)(a+b) & \text{Distribute } (a+b)\\ a(a+b)+b(a+b) & \text{Distribute again through final parenthesis}\\ a^2+ab+ab+b^2 & \text{Combine like terms } ab+ab\\ a^2+2\text{ab}+b^2 & \text{Our Solution}\end{array}$$

This problem also helps us find our shortcut for multiplying. The first term in the answer is the square of the first term in the problem. The middle term is 2 times the first term times the second term. The last term is the square of the last term. This can be shortened to square the first, twice the product, square the last. If we can remember this shortcut we can square any binomial. This is illustrated in the following example

Example

$$\begin{array}{rl}(x-5)^2 & \text{Recognize perfect square}\\ x^2 & \text{Square the first}\\ 2(x)(-5)=-10x & \text{Twice the product}\\ (-5)^2=25 & \text{Square the last}\\ x^2-10x+25 & \text{Our Solution}\end{array}$$

Be very careful when we are squaring a binomial to **NOT** distribute the square through the parenthesis. A common error is to do the following: $(x-5)^2 = x^2 - 25$ (or $x^2 + 25$). Notice both of these are missing the middle term, $-10x$. This is why it is important to use the shortcut to help us find the correct solution. Another important observation is that the middle term in the solution always has the same sign as the middle term in the problem. This is illustrated in the next examples.

Example

$(2x+5)^2$	Recognize perfect square
$(2x)^2 = 4x^2$	Square the first
$2(2x)(5) = 20x$	Twice the product
$5^2 = 25$	Square the last
$4x^2 + 20x + 25$	Our Solution

Example

$(3x-7y)^2$	Recognize perfect square
$9x^2 - 42\text{xy} + 49y^2$	Square the first, twice the product, square the last. Our Solution

Example

$(5a+9b)^2$	Recognize perfect square
$25a^2 + 90\text{ab} + 81b^2$	Square the first, twice the product, square the last. Our Solution

These two formulas will be important to commit to memory. The more familiar we are with them, the easier factoring, or multiplying in reverse, will be. The final example covers both types of problems (two perfect squares, one positive, one negative), be sure to notice the difference between the examples and how each formula is used

Example .

$(4x-7)(4x+7)$	$(4x+7)^2$	$(4x-7)^2$
$16x^2 - 49$	$16x^2 + 56x + 49$	$16x^2 - 56x + 49$

World View Note: There are also formulas for higher powers of binomials as well, such as $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. While French mathematician Blaise Pascal often gets credit for working with these expansions of binomials in the 17th century, Chinese mathematicians had been working with them almost 400 years earlier!

Practice - Multiply Special Products

Find each product.

1) $(x+8)(x-8)$

2) $(a-4)(a+4)$

3) $(1+3p)(1-3p)$

4) $(x-3)(x+3)$

5) $(1-7n)(1+7n)$

6) $(8m+5)(8m-5)$

7) $(5n-8)(5n+8)$

8) $(2r+3)(2r-3)$

9) $(4x+8)(4x-8)$

10) $(b-7)(b+7)$

11) $(4y-x)(4y+x)$

12) $(7a+7b)(7a-7b)$

13) $(4m-8n)(4m+8n)$

14) $(3y-3x)(3y+3x)$

15) $(6x-2y)(6x+2y)$

16) $(1+5n)^2$

17) $(a+5)^2$

18) $(v+4)^2$

19) $(x-8)^2$

20) $(1-6n)^2$

21) $(p+7)^2$

22) $(7k-7)^2$

23) $(7-5n)^2$

24) $(4x-5)^2$

25) $(5m-8)^2$

26) $(3a+3b)^2$

27) $(5x+7y)^2$

28) $(4m-n)^2$

29) $(2x+2y)^2$

30) $(8x+5y)^2$

31) $(5+2r)^2$

32) $(m-7)^2$

33) $(2+5x)^2$

34) $(8n+7)(8n-7)$

35) $(4v-7)(4v+7)$

36) $(b+4)(b-4)$

37) $(n-5)(n+5)$

38) $(7x+7)^2$

39) $(4k+2)^2$

40) $(3a-8)(3a+8)$

5.7

Polynomials - Divide Polynomials

Objective: Divide polynomials using long division.

Dividing polynomials is a process very similar to long division of whole numbers. But before we look at that, we will first want to be able to master dividing a polynomial by a monomial. The way we do this is very similar to distributing, but the operation we distribute is the division, dividing each term by the monomial and reducing the resulting expression. This is shown in the following examples

Example

$$\frac{9x^5+6x^4-18x^3-24x^2}{3x^2} \quad \text{Divide each term in the numerator by } 3x^2$$

$$\frac{9x^5}{3x^2}+\frac{6x^4}{3x^2}-\frac{18x^3}{3x^2}-\frac{24x^2}{3x^2} \quad \text{Reduce each fraction, subtracting exponents}$$

$$3x^3+2x^2-6x-8 \quad \text{Our Solution}$$

Example

$$\frac{8x^3+4x^2-2x+6}{4x^2} \quad \text{Divide each term in the numerator by } 4x^2$$

$$\frac{8x^3}{4x^2}+\frac{4x^2}{4x^2}-\frac{2x}{4x^2}+\frac{6}{4x^2} \quad \text{Reduce each fraction, subtracting exponents}$$

Remember negative exponents are moved to denominator

$$2x+1-\frac{1}{2x}+\frac{3}{2x^2} \quad \text{Our Solution}$$

The previous example illustrates that sometimes we will have fractions in our solution, as long as they are reduced this will be correct for our solution. Also interesting in this problem is the second term $\frac{4x^2}{4x^2}$ divided out completely. Remember that this means the reduced answer is 1 not 0.

Long division is required when we divide by more than just a monomial. Long division with polynomials works very similar to long division with whole numbers.

An example is given to review the (general) steps that are used with whole numbers that we will also use with polynomials

Example

$4|\overline{631}$ Divide front numbers: $\frac{6}{4}=1\ldots$

$$\begin{array}{rl} \mathbf{1} & \\ 4|\overline{631} & \text{Multiply this number by divisor: } 1\cdot 4=4 \\ \underline{\mathbf{-4}} & \text{Change the sign of this number (make it subtract) and combine} \\ \mathbf{23} & \text{Bring down next number} \end{array}$$

$$\begin{array}{rl} \mathbf{15} & \text{Repeat, divide front numbers: } \frac{23}{4}=5\ldots \\ 4|\overline{631} & \\ \underline{-4} & \\ 23 & \text{Multiply this number by divisor: } 5\cdot 4=20 \\ \underline{\mathbf{-20}} & \text{Change the sign of this number (make it subtract) and combine} \\ \mathbf{31} & \text{Bring down next number} \end{array}$$

$$\begin{array}{rl} \mathbf{157} & \text{Repeat, divide front numbers: } \frac{31}{4}=7\ldots \\ 4|\overline{631} & \\ \underline{-4} & \\ 23 & \\ \underline{-20} & \\ 31 & \text{Multiply this number by divisor: } 7\cdot 4=28 \\ \underline{\mathbf{-28}} & \text{Change the sign of this number (make it subtract) and combine} \\ \mathbf{3} & \text{We will write our remainder as } a \text{ fraction, over the divisor, added to the end} \end{array}$$

$157\frac{3}{4}$ Our Solution

This same process will be used to multiply polynomials. The only difference is we will replace the word "number" with the word "term"

Dividing Polynomials

1. Divide front terms

2. Multiply this term by the divisor

3. Change the sign of the terms and combine

4. Bring down the next term

5. Repeat

Step number 3 tends to be the one that students skip, not changing the signs of the terms would be equivalent to adding instead of subtracting on long division with whole numbers. Be sure not to miss this step! This process is illustrated in the following two examples.

Example

$\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4}$ Rewrite problem as long division

$x - 4\overline{)3x^3 - 5x^2 - 32x + 7}$ Divide front terms: $\dfrac{3x^3}{x} = 3x^2$

$$\begin{array}{rll} & \mathbf{3x^2} & \\ x - 4 & \overline{)3x^3 - 5x^2 - 32x + 7} & \text{Multiply this term by divisor: } 3x^2(x-4) = 3x^3 - 12x^2 \\ & \underline{\mathbf{-3x^3 + 12x^2}} & \text{Change the signs and combine} \\ & \mathbf{7x^2 - 32x} & \text{Bring down the next term} \end{array}$$

$$\begin{array}{rll} & 3x^2 + \mathbf{7x} & \text{Repeat, divide front terms: } \dfrac{7x^2}{x} = 7x \\ x - 4 & \overline{)3x^3 - 5x^2 - 32x + 7} & \\ & \underline{-3x^3 + 12x^2} & \\ & 7x^2 - 32x & \text{Multiply this term by divisor: } 7x(x-4) = 7x^2 - 28x \\ & \underline{\mathbf{-7x^2 + 28x}} & \text{Change the signs and combine} \\ & \mathbf{-4x + 7} & \text{Bring down the next term} \end{array}$$

$$\begin{array}{rll} & 3x^2 + 7x \mathbf{- 4} & \text{Repeat, divide front terms: } \dfrac{-4x}{x} = -4 \\ x - 4 & \overline{)3x^3 - 5x^2 - 32x + 7} & \\ & \underline{-3x^3 + 12x^2} & \\ & 7x^2 - 32x & \\ & \underline{-7x^2 + 28x} & \\ & -4x + 7 & \text{Multiply this term by divisor: } -4(x-4) = -4x + 16 \\ & \underline{\mathbf{+4x - 16}} & \text{Change the signs and combine} \\ & \mathbf{-9} & \text{Remainder put over divisor and subtracted (due to negative)} \end{array}$$

$$3x^2+7x-4-\frac{9}{x-4} \quad \text{Our Solution}$$

Example

$$\frac{6x^3-8x^2+10x+103}{2x+4} \quad \text{Rewrite problem as long division}$$

$$2x+4\,\overline{)6x^3-8x^2+10x+103} \quad \text{Divide front terms: } \frac{6x^3}{2x}=3x^2$$

$$\begin{array}{rll}
 & \mathbf{3x^2} & \\
2x+4 & \overline{)6x^3-8x^2+10x+103} & \text{Multiply term by divisor: } 3x^2(2x+4)=6x^3+12x^2 \\
 & \underline{\mathbf{-6x^3-12x^2}} & \text{Change the signs and combine} \\
 & \mathbf{-20x^2+10x} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rll}
 & 3x^2\mathbf{-10x} & \\
2x+4 & \overline{)6x^3-8x^2+10x+103} & \text{Repeat, divide front terms: } \frac{-20x^2}{2x}=-10x \\
 & \underline{-6x^3-12x^2} & \text{Multiply this term by divisor:} \\
 & -20x^2+10x & -10x(2x+4)=-20x^2-40x \\
 & \underline{\mathbf{+20x^2+40x}} & \text{Change the signs and combine} \\
 & \mathbf{50x+103} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rll}
 & 3x^2-10x\mathbf{+25} & \\
2x+4 & \overline{)6x^3-8x^2+10x+103} & \text{Repeat, divide front terms: } \frac{50x}{2x}=25 \\
 & \underline{-6x^3-12x^2} & \\
 & -20x^2+10x & \\
 & \underline{+20x^2+40x} & \\
 & 50x+103 & \text{Multiply this term by divisor: } 25(2x+4)=50x+100 \\
 & \underline{\mathbf{-50x-100}} & \text{Change the signs and combine} \\
 & 3 & \text{Remainder is put over divsor and added (due to positive)}
\end{array}$$

$$3x^2-10x+25+\frac{3}{2x+4} \quad \text{Our Solution}$$

In both of the previous example the dividends had the exponents on our variable counting down, no exponent skipped, third power, second power, first power, zero power (remember $x^0 = 1$ so there is no variable on zero power). This is very important in long division, the variables must count down and no exponent can be skipped. If they don't count down we must put them in order. If an exponent is skipped we will have to add a term to the problem, with zero for its coefficient. This is demonstrated in the following example.

Example

$$\frac{2x^3+42-4x}{x+3}$$ Reorder dividend, need x^2 term, add $0x^2$ for this

$x+3\overline{)2x^3+\mathbf{0x^2}-4x+42}$ Divide front terms: $\frac{2x^3}{x}=2x^2$

$$\begin{array}{rl}
\mathbf{2x^2} & \\
x+3\overline{)2x^3+0x^2-4x+42} & \text{Multiply this term by divisor: } 2x^2(x+3)=2x^3+6x^2 \\
\underline{\mathbf{-2x^3-6x^2}} & \text{Change the signs and combine} \\
\mathbf{-6x^2-4x} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
2x^2\mathbf{-6x} & \\
x+3\overline{)2x^3+0x^2-4x+42} & \text{Repeat, divide front terms: } \frac{-6x^2}{x}=-6x \\
\underline{-2x^3-6x^2} & \\
-6x^2-4x & \text{Multiply this term by divisor: } -6x(x+3)=-6x^2-18x \\
\underline{\mathbf{+6x^2+18x}} & \text{Change the signs and combine} \\
\mathbf{14x+42} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
2x^2-6x\mathbf{+14} & \\
x+3\overline{)2x^3+0x^2-4x+42} & \text{Repeat, divide front terms: } \frac{14x}{x}=14 \\
\underline{-2x^3-6x^2} & \\
-6x^2-4x & \\
\underline{+6x^2+18x} & \\
14x+42 & \text{Multiply this term by divisor: } 14(x+3)=14x+42 \\
\underline{\mathbf{-14x-42}} & \text{Change the signs and combine} \\
0 & \text{No remainder}
\end{array}$$

$2x^2-6x+14$ Our Solution

It is important to take a moment to check each problem to verify that the exponents count down and no exponent is skipped. If so we will have to adjust the problem. Also, this final example illustrates, just as in regular long division, sometimes we have no remainder in a problem.

World View Note: Paolo Ruffini was an Italian Mathematician of the early 19th century. In 1809 he was the first to describe a process called synthetic division which could also be used to divide polynomials.

Practice - Divide Polynomials

Divide.

1) $\frac{20x^4+x^3+2x^2}{4x^3}$

2) $\frac{5x^4+45x^3+4x^2}{9x}$

3) $\frac{20n^4+n^3+40n^2}{10n}$

4) $\frac{3k^3+4k^2+2k}{8k}$

5) $\frac{12x^4+24x^3+3x^2}{6x}$

6) $\frac{5p^4+16p^3+16p^2}{4p}$

7) $\frac{10n^4+50n^3+2n^2}{10n^2}$

8) $\frac{3m^4+18m^3+27m^2}{9m^2}$

9) $\frac{x^2-2x-71}{x+8}$

10) $\frac{r^2-3r-53}{r-9}$

11) $\frac{n^2+13n+32}{n+5}$

12) $\frac{b^2-10b+16}{b-7}$

13) $\frac{v^2-2v-89}{v-10}$

14) $\frac{x^2+4x-26}{x+7}$

15) $\frac{a^2-4a-38}{a-8}$

16) $\frac{x^2-10x+22}{x-4}$

17) $\frac{45p^2+56p+19}{9p+4}$

18) $\frac{48k^2-70k+16}{6k-2}$

19) $\frac{10x^2-32x+9}{10x-2}$

20) $\frac{n^2+7n+15}{n+4}$

21) $\frac{4r^2-r-1}{4r+3}$

22) $\frac{3m^2+9m-9}{3m-3}$

23) $\frac{n^2-4}{n-2}$

24) $\frac{2x^2-5x-8}{2x+3}$

25) $\frac{27b^2+87b+35}{3b+8}$

26) $\frac{3v^2-32}{3v-9}$

27) $\frac{4x^2-33x+28}{4x-5}$

28) $\frac{4n^2-23n-38}{4n+5}$

29) $\frac{a^3+15a^2+49a-55}{a+7}$

30) $\frac{8k^3-66k^2+12k+37}{k-8}$

31) $\frac{x^3-26x-41}{x+4}$

32) $\frac{x^3-16x^2+71x-56}{x-8}$

33) $\frac{3n^3+9n^2-64n-68}{n+6}$

34) $\frac{k^3-4k^2-6k+4}{k-1}$

35) $\frac{x^3-46x+22}{x+7}$

36) $\frac{2n^3+21n^2+25n}{2n+3}$

37) $\frac{9p^3+45p^2+27p-5}{9p+9}$

38) $\frac{8m^3-57m^2+42}{8m+7}$

39) $\frac{r^3-r^2-16r+8}{r-4}$

40) $\frac{2x^3+12x^2+4x-37}{2x+6}$

41) $\frac{12n^3+12n^2-15n-4}{2n+3}$

42) $\frac{24b^3-38b^2+29b-60}{4b-7}$

43) $\frac{4v^3-21v^2+6v+19}{4v+3}$