Solutions to the 10/04 Worksheet

1) (a) C is the curve $x^3 + y^3 = 16$ with $(x, y) \in \mathbb{R}^2$.

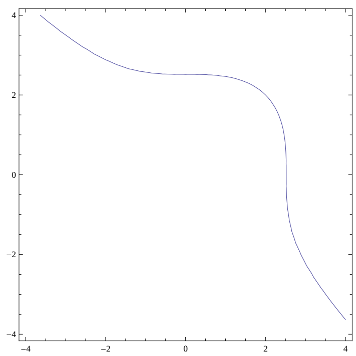

FIGURE 1. Sketch of the curve C (using Mathematica)

(b) C is not bounded because there is no restriction on x or y and when, say, x approaches $+\infty$, $y = y(x) = \sqrt[3]{16 - x^3}$ approaches $-\infty$.

(c) C is closed because the boundary of C coincides with C, thus it is included into C.

2) $f(x, y) = e^{xy}$.

(a) f is continuous, being the composition $f = g \circ h$ of the continuous functions $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x, y) = xy$, and $g : \mathbb{R} \to \mathbb{R}$, $g(u) = e^u$. The Extreme Value Theorem says nothing about the existence of global min or max on C because the set C is not bounded.

(b) We have to find the points of local extremum for the function $f(x, y) = e^{xy}$ subject to the constraint $g(x, y) = 16$, where $g(x, y) = x^3 + y^3$. The Lagrange multipliers method says that such a point (x, y) must satisfy $\begin{cases} (\nabla f)(x, y) = \lambda(\nabla g)(x, y) \\ x^3 + y^3 = 16. \end{cases}$ for some $\lambda \neq 0$, or equivalently

$$\begin{cases} ye^{xy} = 3\lambda x^2 \\ xe^{xy} = 3\lambda y^2 \\ x^3 + y^3 = 16. \end{cases}$$

When $x \neq 0$ we divide the first two equations and get $y^3 = x^3$ hence $y = x$. Taking into account the third equation we find $x = y = 2$. The case $x = 0$ cannot occur because on one hand it would give $y^3 = 16$ and on the other hand $y = 0$, which contradict each other.

The point $(2, 2)$ is the only local max value for f on C and when (x, y) moves towards the end (at infinity) of C the product xy tends to $-\infty$ so e^{xy} tends to 0. Therefore the max value of f on C is $f(2, 2) = e^4$ and there is no min value of f on C.

3) (a) S is the surface given by $z^2 = x^2 + y^2$.

(b) We wish to minimize the function $f(x, y, z) := (x - 4)^2 + (y - 2)^2 + z^2$ that represents the square of the distance between the points (x, y, z) and $(4, 2, 0)$, subject to constraint $g(x, y, z) :=$

$x^2 + y^2 - z^2 = 0$. Plainly $\nabla f = \langle 2(x-4), 2(y-2), 2z\rangle$, $\nabla g = \langle 2x, 2y, -2z\rangle$, and by Lagrange multipliers method we have to solve the system of equations

$$\begin{cases} (\nabla f)(x,y,z) = \lambda(\nabla g)(x,y,z) \\ x^2+y^2 = z^2 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} 2(x-4) = 2\lambda x \\ 2(y-2) = \lambda y \\ 2z = -2\lambda z \\ x^2+y^2 = z^2. \end{cases}$$

If $z = 0$ then the last equality gives $x = y = 0$, contradicting the first and the second equation. So $z \neq 0$ and the third equation gives $\lambda = -1$ and then $x - 4 = -x$ so $x = 2$, $y - 2 = -y$ so $y = 1$, and $z^2 = 2^2 + 1^2$, so $z = \pm\sqrt{5}$.

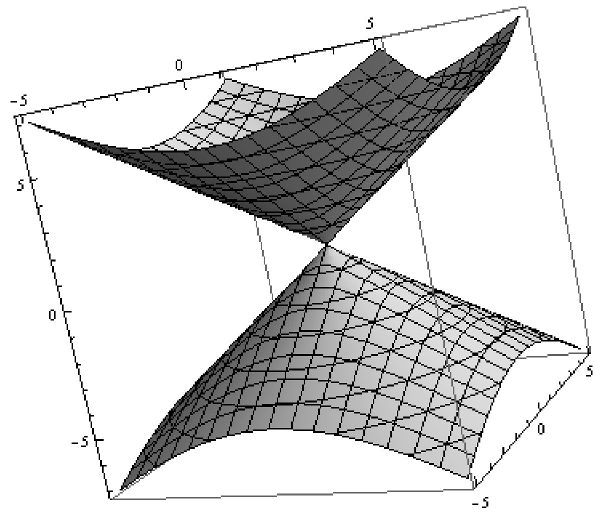

FIGURE 2. Sketch of the surface S (using Mathematica)

Thus the points on S closest to $(4,2,0)$ are $(2,1,-\sqrt{5})$ and $(2,1,\sqrt{5}$ and the shortest distance is $\sqrt{f(2,1,\pm\sqrt{5})} = \sqrt{(-1)^2+(-1)^2+5} = \sqrt{10}$.

4) An inspection of the level sets map reveals that the graph of the surface $z = f(x,y)$ bottoms at $(-1,1)$ and $(-1,-1)$, and peaks at $(1,0)$, hence $(-1,1)$ and $(-1,-1)$ are candidates for local minima while $(1,0)$ is a candidate for local max.

Furthermore, near $(-1,0)$ the function $x \mapsto f(x,0)$ is seen to have a local min at $x = -1$ and $y \mapsto f(-1,y)$ a local max at $y = 0$, suggesting that $(-1,0)$ is a saddle point. Near $(1,1)$ the function $x \mapsto f(x,1)$ has a local max at $y = 1$ and the function $y \mapsto f(1,y)$ has a local min at $y = 1$, suggesting that $(1,1)$ is a saddle point. Near $(1,-1)$ the function $x \mapsto f(x,-1)$ has a local max at $x = 1$ and the function $y \mapsto f(1,y)$ has a local min at $y = -1$, suggesting that $(1,-1)$ is a saddle point.

Now we check these guesses using the explicit formula $f(x,y) = 3x - x^3 - 2y^2 + y^4$. The critical points of f are obtained by solving the system

$$\begin{cases} f_x = 3 - 3x^2 = 0 \\ f_y = -4y + 4y^3 = 0 \end{cases}$$

whose solutions are given exactly by $x \in \{-1, 1\}$ and $y \in \{-1, 0, 1\}$, or

$$(x,y) \in \{(-1,-1), (-1,0), (-1,1), (1,-1), (1,0), (1,1)\}.$$

Next we compute the second order partial derivatives $f_{xx} = -6x$, $f_{yy} = -4y + 4y^3$, $f_{xy} = f_{yx} = 0$, and the Hessian

$$D(x,y) = \begin{vmatrix} -6x & 0 \\ 0 & -4 + 12y^2 \end{vmatrix} = 24x(1 - 3y^2).$$

Since $D(-1,1) > 0$, $f_{xx}(-1,1) > 0$, $D(-1,-1) > 0$, $f_{xx}(-1,-1) > 0$, $D(1,0) > 0$, $f_{xx}(1,0) < 0$, and $D(-1,0) < 0$, $D(1,1) < 0$, $D(1,-1) < 0$, the information gathered from the contour map turns out to be correct.

5) We have to maximize the function $V(x,y,z) = xyz$ under the constraint $g(x,y,z) := x^2 + y^2 + z^2 = L^2$ and $x,y,z > 0$. Using Lagrange multipliers we have to solve the system

$$\begin{cases} (\nabla V)(x,y,z) = \lambda(\nabla g)(x,y,z) \\ x^2 + y^2 + z^2 = L^2 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} yz = 2\lambda x \\ xz = 2\lambda y \\ xy = 2\lambda z \\ x^2 + y^2 + z^2 = L^2 \end{cases}$$

Dividing the first two equations we find $y^2 = x^2$ so $y = x$ (because $x, y > 0$). Similarly from the second and the third equation we find $z = y$, so $x = y = z$ and the fourth equation now yields $x = y = z = \sqrt{\frac{L^2}{3}} = \frac{L}{\sqrt{3}}$. It follows that $V_{\max} = V\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right) = \frac{L^3}{3\sqrt{3}}$.