Worksheet April 18, 2014

1. Warmup: find the eigenvalues and corresponding eigenspaces of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

2. Diagonalize the matrix from (1).

3. What needs to be true about the eigenspaces of A for you to be able to diagonalize A by the method from class?

4. It turns out that a matrix is diagonalizable if and only if our method from class works.

   (a) Why is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ not diagonalizable?

   (b) Why is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ not diagonalizable?

5. An application of diagonalization: For various dynamical systems including simple predator-prey models, almost everything is encoded in a *transition matrix* A. For example, there may be a population of r rabbits, f foxes, and s abominable snowmen. Let $\mathbf{x} = \begin{bmatrix} r \\ f \\ s \end{bmatrix}$. Under suitable conditions, the population after one year may be given by $A\mathbf{x}$.

   (a) If you can write $A = PDP^{-1}$, then find a simple formula for A^n for any (positive) integer n.

   (b) Apply this with the matrix A from (1) to give an easily-computed formula for the number of rabbits, foxes, and snowmen after n years from an arbitrary starting population. (The matrix is a terrible one for modelling these populations. Just pretend.)

6. Another application: Fibonacci numbers.

   (a) You probably know the Fibonacci numbers: $f_1 = 1$, $f_2 = 1$, and for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Consider the vector $v_n = \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$. Find a matrix A so that $v_n = Av_{n-1}$ for all $n \geq 3$.

(b) Diagonalize A.

(c) Find a formula for A^n.

(d) Use A^n and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to find a formula for v_n. Hence give a closed form for f_n.