WORKSHOP 6: §2.2-2.3 JANUARY 31, 2017

Name: ______________________________

A few definitions:

- An integer m *divides* another integer n if $n = km$ for some integer k. We also say that m is a *factor* of n, that n is a *multiple* of m, and that n is *divisible* by m.
- An integer greater than 1 is *prime* if it has no factors other than 1 and itself. A positive integer is *composite* if it has some factor other than 1 or itself. (In this class, we won't refer to negative numbers when discussing prime/composite. The integers 0 and 1 are special and are neither prime nor composite.)
- A *rational number* is a number that can be expressed as m/n for some integers m, n with $n \neq 0$. A real number that is not rational is *irrational*.

Last time, I showed how a "direct proof" of a statement of the form $\forall x\ (P(x) \to Q(x))$ can be proven using formal logic, then how that translates into a less-formal written proof. A "proof by contrapositive" is just a direct proof of the contrapositive $\forall x\ (\neg Q(x) \to \neg P(x))$. Thus, in general, a proof by contrapositive goes as follows:

> "We prove the contrapositive. *[You might want to write down the contrapositive here.]*
> Let x be an arbitrary $\langle$object in the domain$\rangle$ such that $\langle \neg Q(x) \rangle$.
> $\langle$using the above, make some argument that finishes with the following$\rangle$
> Therefore $\langle \neg P(x) \rangle$. $\square$ "

(1) What is wrong with the following "proof" that the sum of any two rational numbers is rational?

> Let x and y be arbitrary rational numbers. If $x+y$ is rational, then $x+y = m/n$ for some integers m and n with $n \neq 0$. Since x and y are rational, $x = a/b$ and $y = c/d$ for some integers a, b, c, d with $b, d \neq 0$. So we have
> $$x + y = \frac{a}{b} + \frac{c}{d} = \frac{m}{n},$$
> which is a ratio of two integers with nonzero denominator, i.e. it is a rational number. $\square$

(2) Prove or disprove the following statements. Be sure to mention the proof technique you are using.

- (a) The sum of any two rational numbers is rational.
- (b) For every integer n, if n is even, then n^2 is divisible by 4.
- (c) For every integer n, if n^2 is divisible by 4, then n is divisible by 4.
- (d) For every integer n, if n^2 is divisible by 4, then n is even.
- (e) For every positive integer n, $n^2 \geq n$.
- (f) For every integer n, if $n^3 + 5$ is odd, then n is even.
- (g) The quotient of any two rational numbers is rational.

(3) Prove that for every integer $p \geq 2$, if $2^p - 1$ is prime, then so is p. *(Hint: the last step is easiest if you know something about geometric sums.)*

(4) Let n be an integer greater than 1. Prove that n is composite if and only if there are two integers a, b with $n = ab$ and $a, b < n$. *(Recall that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$, so you are actually proving two things: the "if" part and, separately, the "only if" part.)*