Math 241 C8

Quiz # 1 Solutions

January 30, 2013

No electronic devices or interpersonal communication allowed. Show work to get credit.

1) [5pts.] Parametrize the curve $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$.

Solution: As usual, the ellipse is parametrized by $x = 2\cos(t)$, $y = 5\sin(t)$, for $t \in [0, 2\pi]$.

2) [5pts.] Find all points on the curve $(x, y) = (t + e^{4t}, 2e^{2t})$ where the slope of the tangent line is 1. (Hint: toward the end, think about $u = e^{2t}$.)

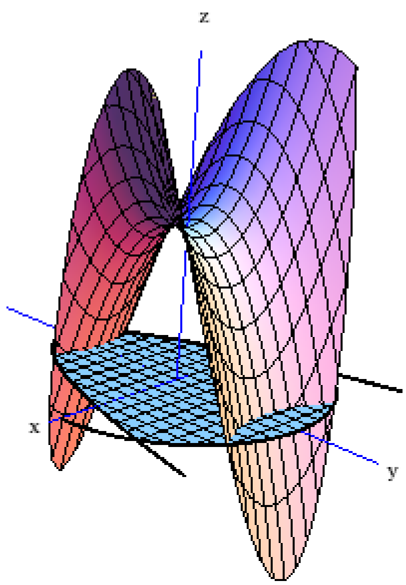
Solution: We know that slopes of tangent lines to parametric curves are given by $y'(t)/x'(t)$. So we wish to solve

$$\begin{aligned} \frac{4e^{2t}}{1+4e^{4t}} &= 1 \\ 4e^{2t} &= 1 + 4e^{4t} \\ 4e^{4t} - 4e^{2t} + 1 &= 0 \\ (2e^{2t} - 1)^2 &= 0. \end{aligned}$$

Hence $e^{2t} = 1/2$, and $t = \ln(1/2)/2$. We wanted the point on the curve, so plug this t into the parametric equations to obtain

$$\left(\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{4}, 1\right).$$

3) Consider the region R in the xy-plane bounded by the curves $y = \sqrt{4-x^2}$ and $y = x^2 - 2$. Let $f(x,y) = x^2 - y^2 + 2$. Here is a picture of the graph of f above the region R:



a) [5 pts.] Without performing any computation, is $\iint_R f(x,y)\,dx\,dy$ positive or negative? Explain.

Solution: The integral is positive, because there is a lot of space under the surface and above R (which counts positive), and only a small amount of space below R and above the surface (which counts negative).

b) [5 pts.] Start to compute $\iint_R f(x,y)\,dx\,dy$. Stop when you have a single integral (there should only be one variable, but you don't need to simplify anything).

Solution: The easiest way to compute this integral is by slicing in the y-direction first. Thus the bounds on the inner integral will be $x^2 - 2$ and $\sqrt{4-x^2}$. The bounds on the outer integral are the extreme x-values, which occur at the intersection of the two curves. So we solve $x^2 - 2 = \sqrt{4-x^2}$:

$$\begin{aligned}
(x^2-2)^2 &= 4 - x^2 \\
x^4 - 4x^2 + 4 &= 4 - x^2 \\
x^4 - 3x^2 &= 0 \\
x^2(x^2-3) &= 0.
\end{aligned}$$

$x = 0$ is not a solution (obvious from the picture or the original equation), so we're left with $x = \pm\sqrt{3}$. Hence the desired integral is

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2-2}^{\sqrt{4-x^2}} x^2 - y^2 + 2\,dy\,dx.$$

To get down to a single integral, we compute the inner one above:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \left[x^2 y - \frac{1}{3}y^3 + 2y\right]_{x^2-2}^{\sqrt{4-x^2}} dx$$
$$= \int_{-\sqrt{3}}^{\sqrt{3}} \Big(x^2\sqrt{4-x^2} - \frac{1}{3}(\sqrt{4-x^2})^3 + 2\sqrt{4-x^2}$$
$$-x^2(x^2-2) + \frac{1}{3}(x^2-2)^3 - 2(x^2-2) \Big)\, dx.$$

Big and terrible, but a single integral.

c) [bonus] Start to compute $\iint_S f(x,y)\,dx\,dy$, where S is the unit disk in the xy-plane. Again, stop when you have a single integral.

Solution: This time it makes sense to use Gauss-Green. The boundary is parametrized by $x = \cos(t)$, $y = \sin(t)$, for $t \in [0, 2\pi]$. Take $m = 0$ and $n = \frac{1}{3}x^3 - xy^2 + 2x$, so

$$\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \iint_R (\partial_x n - \partial_y m)\;dx\,dy \\ &= \int_0^{2\pi} (mx' + ny')\,dt \\ &= \int_0^{2\pi} \left(\frac{1}{3}\cos^3(t) - \cos(t)\sin^2(t) + 2\cos(t)\right)(\cos(t))\,dt.\end{aligned}$$

Single integral. Now the calc 2 students can take over.