HW 6

1) Let f, g be injective.

Suppose $a, b \in X$ are such that $g\circ f(a) = g\circ f(b)$. [Want to prove that $a = b$.]

By the def. of composition, $g(f(a)) = g(f(b))$.

Since g is injective, $f(a) = f(b)$.

Since f is injective, $a = b$. $\square$

2) No, it is not possible.

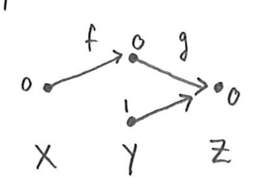
<u>Theorem</u>: If f is not injective, then $g\circ f$ is not injective.

<u>Proof</u> (Direct proof) Suppose f is not injective. That is, there are $a, b \in X$, $a \neq b$, such that $f(a) = f(b)$.

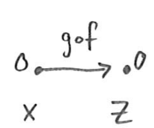
Then $g(f(a)) = g(f(b))$,

i.e. $g\circ f(a) = g\circ f(b)$, so $g\circ f$ is not injective. $\square$

3) Yes, it is possible. For example:



gives



4) If f is not injective, then some element $y \in Y$ has more than one $x \in X$ with $(y, x) \in f^{-1}$.

If f is not surjective, then some element $y \in Y$ has no $x \in X$ with $(y, x) \in f^{-1}$.

5) Input: $a_1, \cdots, a_n$ a sequence of real #s, $n \geq 2$
n, the length

Output: The second-smallest entry (counting a repeated smallest element as also the second-smallest)

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If a_1 ≤ a_2
    min := a_1
    next := a_2
End-if
If a_1 > a_2
    min := a_2
    next := a_1

For i = 3 to n
    If a_i ≤ min
        next := min
        min := a_i
    End-if
    If min < a_i < next
        next := a_i
    End-if
End-for
Return next
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6) Direct proof: Let $f = O(h)$ and $g = O(h)$.

Then there are $c > 0$ and n_0 such that if $n \geq n_0$ then $f(n) \leq c\,h(n)$, and there are $d > 0$ and m_0 such that if $n \geq m_0$ then $g(n) \leq d\,h(n)$.

Let $l_0 = \max(n_0, m_0)$ and $b = c + d$. (Note that $l_0, b > 0$.)

Then if $n \geq l_0$, we have both $n \geq n_0$ and $n \geq m_0$, so

$$f(n) \leq c\,h(n) \text{ and } g(n) \leq d\,h(n),$$

and so $(f+g)(n) = f(n) + g(n) \leq c\,h(n) + d\,h(n) = (c+d)\,h(n) = b\,h(n)$.

Thus $f+g = O(h)$. $\square$