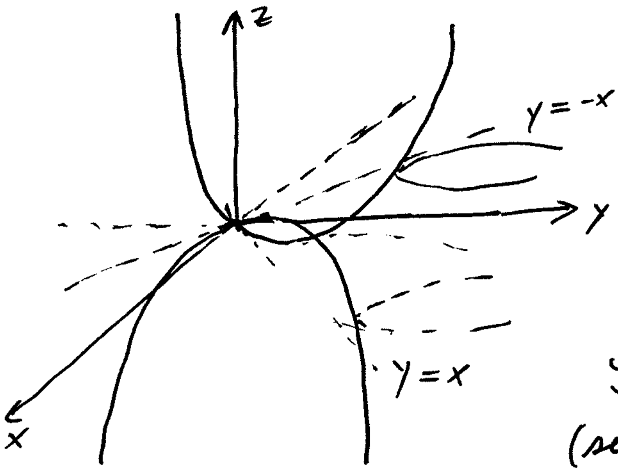
SOLUTIONS TO THE 09/13/12 WORKSHEET

1) (a) $f(x,y) = xy$

Level sets: $xy = K$ (constant).

Two lines $x=0$ and $y=0$ if $K=0$

Two hyperbola branches if $K \neq 0$



The graph is challenging to draw by hand. To get some idea about its shape consider also its traces in the planes $y=x$ ($\Rightarrow z = x^2$ in this plane) and $y=-x$ ($\Rightarrow z = -x^2$).

Such a surface is a <u>hyperbolic paraboloid</u> (see Section 12.6). Since the family of hyperbolas $xy=K$ is transformed into the family of hyperbolas $x^2-y^2=K$ after rotation by $45°$ about the z-axis, the shape of the surfaces $z=xy$ and $z=x^2-y^2$ are similar.

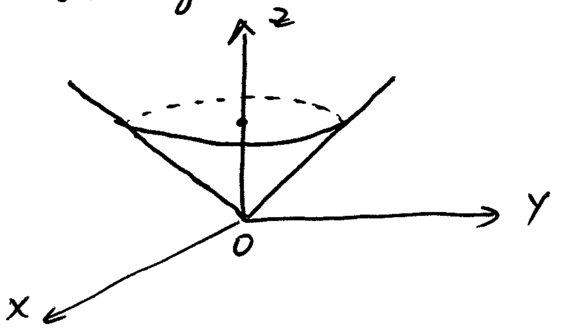
(b) $f(\underline{x}) = |\underline{x}|$ $\left(\Longleftrightarrow f(x,y)=\sqrt{x^2+y^2}\right)$.

Level sets $f(x,y)=K \Longleftrightarrow \sqrt{x^2+y^2}=K$
- $\to K<0$ $\quad \phi$ no solution
- $\to K=0$ $\quad (x,y)=0$
- $\to K>0$ $\quad x^2+y^2=K$ a circle

Since $f(x,y)$ only depends on $r=\sqrt{x^2+y^2}$, the surface must stay invariant when rotated about the z-axis.

Finally the trace in the plane yz ($\Longleftrightarrow x=0$) is $z=\sqrt{y^2}=|y|$.

The surface is a <u>cone</u>.



2) $f(x,y) = \dfrac{2x^3 y}{x^6+y^2}$, $(x,y) \neq (0,0)$.

(a) On the x-axis $f(x,0)=0$.
On the y-axis $f(0,y)=0$.
This shows that IF the limit $L = \lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists, then $L=0$.

(b) Any such line (except for x or y-axis) is given by $y = Cx$ (C constant) $C \neq 0$
Along this line $f(x,y) = \dfrac{2x^4 \cdot Cx}{x^6 + C^2x^2} = \dfrac{2Cx^3}{x^4+C^2} \xrightarrow[x\to 0]{} 0$,
showing that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along each line through the origin.

(c) Consider the curves C_1 and C_2, defined as the graphs of the functions $g(x) = x^3$ and respectively $h(x) = -x^3$.

$$\left.\begin{aligned} f(x,x^3) &= \frac{2x^3 \cdot x^3}{x^6+x^6} = \frac{2x^6}{2x^6} \underset{x\neq 0}{=} 1 \xrightarrow[x\to 0]{} 1 \\ f(x,-x^3) &= \frac{2x^3(-x^3)}{x^6+(-x^3)^2} = \frac{-2x^6}{2x^6} = -1 \xrightarrow[x\to 0]{} -1 \end{aligned}\right\} \Longrightarrow \lim_{(x,y)\to(0,0)} f(x,y) \text{ DNE.}$$

3) $f(x,y) = \dfrac{xy^2}{\sqrt{x^2+y^2}}$ for $(x,y) \neq (0,0)$

(a) $\underline{h} = (x,y)$

$$\left.\begin{aligned} |f(\underline{h})| = \left|\frac{xy^2}{\sqrt{x^2+y^2}}\right| = \frac{|x|}{\sqrt{x^2+y^2}}\, y^2 \\ |x| \leq \sqrt{x^2+y^2} \end{aligned}\right\} \Longrightarrow |f(\underline{h})| \leq y^2 \qquad (1)$$

(since, for instance, $|x| = |r\cos\theta| = r|\cos\theta| \leq r = \sqrt{x^2+y^2}$)
$y^2 < \frac{1}{2}$ would ensure $|f(\underline{h})| < \frac{1}{2}$, so taking $\delta = \frac{1}{\sqrt{2}}$ we have
$0 < |\underline{h}| = \sqrt{x^2+y^2} < \frac{1}{\sqrt{2}} \Longrightarrow y^2 \leq x^2+y^2 < \frac{1}{2} \overset{(1)}{\Longrightarrow} |f(\underline{h})| < \frac{1}{2}$.

(b)+(c) Notice that for given $\varepsilon > 0$, the choice $\delta = \delta(\varepsilon) = \sqrt{\varepsilon}$ would give $0 < |\underline{h}| = \sqrt{x^2+y^2} < \delta = \sqrt{\varepsilon} \Longrightarrow y^2 \leq x^2+y^2 < \varepsilon \overset{(1)}{\Longrightarrow} |f(\underline{h})| < \varepsilon$.
This way we proved $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

(d) $g(x,y) = xy^2 \longrightarrow 0$ as $(x,y) \to (0,0)$, $h(x,y) = \sqrt{x^2+y^2} \longrightarrow 0$ as $(x,y)\to(0,0)$,
but this does not say anything about $f(x,y) = \dfrac{g(x,y)}{h(x,y)}$ as $(x,y) \to (0,0)$

because $\frac{0}{0}$ is an indeterminate form and a version of l'Hospital's rule is not available for functions of two variables.