Math 241 X8 **Name(s):** Solutions

Homework 3 supplement

This is a written homework supplement to the homework for Unit 3: Perpendicularity. It also covers some material from Unit 2: Vectors.

In 2D, there is only one "flat" object: lines. In 3D, there are both lines and planes. Two lines are *parallel* if they have the same direction. Two planes are *parallel* if they have the same normal direction.

(1) Find an equation for the line passing through (4,5,6) and intersecting the plane $3x + 2y - z = 1$ at a right angle.

Intersecting $3x+2y-z=1$ at a right angle

$\Rightarrow$ direction vector is normal to the plane,

e.g. $\langle 3, 2, -1 \rangle$.

$\therefore \quad \ell(t) = \langle 4,5,6 \rangle + t\langle 3,2,-1 \rangle$

is such an equation.

(2) In each of the following, two lines are given. If they are parallel or if they intersect, find an equation of the plane containing both of them. Otherwise, find the equations of the two planes, one containing each line, such that the planes are parallel to each other.

(a) $\ell_1(t) = (3,1,4) + t(6,1,8)$, $\ell_2(t) = (13,4,26) + t(2,1,7)$

(b) $\ell_3(t) = (2,-1,8) + t(3,2,3)$, $\ell_4(t) = (-1,1,0) + t(4,1,-1)$

(c) $\ell_5(t) = (5,4,2) + t(3,-1,5)$, $\ell_6(t) = (2,1,1) + t(-6,2,-10)$

(d) $\ell_7(t) = (6,15,-17) + t(2,4,-8)$, $\ell_8(t) = (0,3,7) + t(-1,-2,4)$ (hint: this is a trick question)

(a) Not parallel: $\langle 6,1,8\rangle \neq c\langle 2,1,7\rangle$ for any c.

Intersect? $(3,1,4) + t(6,1,8) \overset{?}{=} (13,4,26) + s(2,1,7)$

$\Leftrightarrow \quad t(6,1,8) = (10,3,22) + s(2,1,7)$

$$\Leftrightarrow \begin{cases} 6t = 10 + 2s \\ t = 3 + s \\ 8t = 22 + 7s \end{cases} \Rightarrow 6(3+s) = 10 + 2s \Rightarrow s = -2 \Rightarrow t = 1 \quad \checkmark$$

Intersect @ $(9,2,12)$.

Normal vector $= \langle 6,1,8\rangle \times \langle 2,1,7\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 1 & 8 \\ 2 & 1 & 7 \end{vmatrix} = (1\cdot 7 - 1\cdot 8)\hat{\imath} - (6\cdot 7 - 2\cdot 8)\hat{\jmath} + (6\cdot 1 + 2\cdot 1)\hat{k}$

$= -\hat{\imath} - 26\hat{\jmath} + 8\hat{k}$

$= \langle -1, -26, 8\rangle$

So plane is $-1(x-9) - 26(y-2) + 8(z-12) = 0$.

(2) In each of the following, two lines are given. If they are parallel or if they intersect, find an equation of the plane containing both of them. Otherwise, find the equations of the two planes, one containing each line, such that the planes are parallel to each other.

(a) $\ell_1(t) = (3,1,4) + t(6,1,8)$, $\ell_2(t) = (13,4,26) + t(2,1,7)$

(b) $\ell_3(t) = (2,-1,8) + t(3,2,3)$, $\ell_4(t) = (-1,1,0) + t(4,1,-1)$

(c) $\ell_5(t) = (5,4,2) + t(3,-1,5)$, $\ell_6(t) = (2,1,1) + t(-6,2,-10)$

(d) $\ell_7(t) = (6,15,-17) + t(2,4,-8)$, $\ell_8(t) = (0,3,7) + t(-1,-2,4)$ (hint: this is a trick question)

(b) Not parallel: $\langle 3,2,3\rangle \neq c\langle 4,1,-1\rangle$ for any c.

Intersect? $\Leftrightarrow \begin{cases} 2+3t = -1+4s \\ -1+2t = 1+s \\ 8+3t = -s \end{cases}$ $\Rightarrow 7+5t = 1 \Rightarrow t = -\frac{6}{5}$ $\Rightarrow s = -\frac{22}{5}$ $\Rightarrow -\frac{8}{5} = -\frac{93}{5}$ ✗

so no intersection.

The desired planes are parallel to both lines, so

normal vector $= \langle 3,2,3\rangle \times \langle 4,1,-1\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 3 \\ 4 & 1 & -1 \end{vmatrix} = (2(-1)-1\cdot 3)\hat{i} - (3(-1)-4\cdot 3)\hat{j} + (3\cdot 1 - 4\cdot 2)\hat{k}$$

$$= -5\hat{i} + 15\hat{j} - 5\hat{k} = \langle -5, 15, -5\rangle.$$

[Can take $\langle 1,-3,1\rangle$ (since $\langle -5,15,-5\rangle$ is parallel to $\langle 1,-3,1\rangle$).

So the planes are $1(x-2) - 3(y+1) + 1(z-8) = 0$

& $1(x+1) - 3(y-1) + 1(z-0) = 0$.

(2) In each of the following, two lines are given. If they are parallel or if they intersect, find an equation of the plane containing both of them. Otherwise, find the equations of the two planes, one containing each line, such that the planes are parallel to each other.

(a) $\ell_1(t) = (3,1,4) + t(6,1,8)$, $\quad \ell_2(t) = (13,4,26) + t(2,1,7)$

(b) $\ell_3(t) = (2,-1,8) + t(3,2,3)$, $\quad \ell_4(t) = (-1,1,0) + t(4,1,-1)$

(c) $\ell_5(t) = (5,4,2) + t(3,-1,5)$, $\quad \ell_6(t) = (2,1,1) + t(-6,2,-10)$

(d) $\ell_7(t) = (6,15,-17) + t(2,4,-8)$, $\quad \ell_8(t) = (0,3,7) + t(-1,-2,4)$ (hint: this is a trick question)

(c) $\langle -6, 2, -10\rangle = -2 \cdot \langle 3, -1, 5\rangle$, so these lines are parallel.

We need another vector parallel to the plane to get its equation. A vector from one line to the other will work:

$$\vec{v} = (5,4,2) - (2,1,1)$$
$$= \langle 3, 3, 1\rangle$$

$$\vec{n} = \langle 3,-1,5\rangle \times \langle 3,3,1\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 5 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (-1\cdot 1 - 3\cdot 5)\hat{\imath} - (3\cdot 1 - 3\cdot 5)\hat{\jmath} + (3\cdot 3 - (-1)\cdot 3)\hat{k}$$

$$= -16\hat{\imath} + 12\hat{\jmath} + 12\hat{k} = \langle -16, 12, 12\rangle.$$

So plane is $\quad -16(x-5) + 12(y-4) + 12(z-2) = 0.$

(d) $\langle 2,4,-8\rangle = -2\langle -1,-2,4\rangle$, so parallel. Trying the same approach from (c), we get $\vec{v} = (6,15,-17) - (0,3,7) = \langle 6, 12, -24\rangle$

$$= 3\langle 2, 4, -8\rangle \;!$$

is actually

These are the same line!