Worksheet 23 April 18, 2011

1. Consider the region in the plane between the x-axis, $y = 1/x$, $x = 1$, and $x = M$, where $M > 1$.
   (a) Find the area of this region.
   (b) What happens to the area as $M \to \infty$?
   (c) Now suppose the region is rotated about the x-axis. Compute the resulting solid's volume.
   (d) What happens to the volume as $M \to \infty$?

2. Consider the region under one hump of the graph of $f(t) = \cos t$, say $t \in [-\pi/2, \pi/2]$. Recall that its area is 2. Now rotate this region about the y-axis and compute its volume. You'll need to calculate the derivative of $t \sin t + \cos t$ to complete this problem.

3. Consider the same region, but now rotate about the x-axis; what is the resulting solid's volume? You'll need a trig identity for this one.

4. We'll now prove the formula for the volume of a ball. Start by writing down an equation for a half-circle with radius r. Then rotate this curve about an appropriate axis to obtain a sphere, and use the disk method to compute the resulting volume.

5. Suppose the velocity of your car at the beginning of a trip is given by $v(t) = 1 + \dfrac{3t}{(t^2+4)^2}$. Find the distance traveled during the interval $t \in [0, 2]$, and simplify your answer. If you wanted to travel the same distance in the same time but with constant speed, what speed should you travel at? How does this relate to the formula you've learned for the average value of a function?

6. Suppose the average velocity of your car during the trip is 40 mph. Prove that it must be true that at some point you were traveling at exactly 40 mph. What assumptions about your position and/or velocity function are you making?

7. Consider the region bounded by $y = x^2$, $y = 1 + x^3$, $x = 0$, and $x = 1$.
   (a) The region is rotated about the y-axis. Compute the resulting solid's volume.
   (b) The region is rotated about the line $y = -2$. Compute the resulting solid's volume.
   (c) The region is the base of a solid. The cross sections of the solid perpendicular to the x-axis are squares. What is its volume?
   (d) The region is the base of a solid. The cross sections of the solid perpendicular to the y-axis are triangles with height equal to the square of their (common) y-coordinate. What is its volume? (Notice here that I haven't given you enough information to know what the solid really looks like, but you can nonetheless compute its volume.)

8. Calculus is a fundamental part of probability when dealing with situations that have a continuum (or really just infinitely many) possible outcomes. For instance, age or height of people, lifetime of a machine, or time between passing cars all have infinitely many possible outcomes. We compute probabilities of these events by integrating a *probability density function*.
   (a) For example, suppose the amount of time you spend in a line at a local bakery has a density function $f(t) = 0.1e^{-0.1t}$, $t \geq 0$. To find the probability that you wait between 1 minute and 5 minutes, you need to calculate $\int_1^5 f(t)\, dt$. Do so.

(b) What should be the probability that you wait at least 0 minutes? Compute the probability that you wait between 0 and M minutes, then take the limit as $M \to \infty$ to verify that this is true for the given density function.

(c) Calculus also gives us a way to compute the *expected* waiting time. To do this we just compute $\int t \cdot f(t)\, dt$. Do so, integrating from 0 to M, then let $M \to \infty$ to get the expected waiting time. In order to complete this, you will need to first compute the derivative of $-(t+10)e^{-0.1t}$.