Tuesday, September 18 $*$ **Solutions** $*$ *Partial derivatives and differentiability.*

1.
$$f_x = \frac{\partial f}{\partial x} = 2x^2y^2e^{x^2} + y^2e^{x^2}; \qquad f_y = \frac{\partial f}{\partial y} = 2xye^{x^2}$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = 4x^2ye^{x^2} + 2ye^{x^2}$$
$$f_{xy}\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = 4x^2ye^{x^2} + 2ye^{x^2}.$$

We notice that $f_{xy} = f_{yx}$, as predicted by Clairaut's Theorem (Here, as required by Clairaut, the functions f_{xy} and f_{yx} exist and are continuous.)

2. Consider $P(a,b)$.

(a) Keep $y = b$ constant and allow x to vary around a. Then the function $g_1(x) = f(x,b)$ is seen to increase, showing that $g_1'(a) = f_x(a,b) > 0$.

(b) Keep $x = a$ constant and allow y to vary around b. The function $g_2(y) = f(a,y)$ is seen to satisfy $g_2(b+h) - g_2(b) = f(a,b+h) - f(a,b) \geq 0$ as $h \to 0^+$, showing that $g_2'(b) = f_y(a,b) \geq 0$. On the other hand $g_2(b-h) - g_2(b) \geq 0$ as $h \to 0^+$, showing that $\lim_{h \to 0^+} \frac{g_2(b-h)-g_2(b)}{-h} = g_2'(b) = f_y(a,b) \leq 0$. We conclude that $f_y(a,b) = 0$. (Note that we assumed that $f_y(a,b) = g_2'(b)$ does exist.)

(c) Fixing $y = b$ and allowing x to vary near a, we see that to the right of P the level curves are closer together that to the left of P, suggesting that f_x increases in the positive direction with respect to x, and thus $f_{xx} = (f_x)_x$ is positive at P.

(d) An analog argument as in (c) shows that f_y is increasing faster with respect to y in the positive direction above P than below P, so $f_{yy} = (f_y)_y$ is positive at P.

(e) $f_{xy} = \frac{\partial}{\partial y}(f_x)$, so f_{xy} represents the rate of change of f_x as y increases. The level curves are closer together in the x-direction at points above P than below P. This shows that f increases faster with respect to x for y-values above P than below P. As a result f_x increases while moving through P in the positive y-direction, and so $f_{xy} = f_{yx}$ is positive at P.

3. (a) The quantity
$$f_T(-20,40) = \lim_{h \to 0} \frac{f(-20+h,40) - f(-20,40)}{h}$$
can be approximated ($h = 5$ respectively $h = -5$) as:
$$f_T(-20,40) \approx \frac{f(-15,40) - f(-20,40)}{5} = \frac{-27-(-34)}{5} = \frac{7}{5},$$
$$f_T(-20,40) \approx \frac{f(-25,40) - f(-20,40)}{-5} = \frac{-41-(-34)}{-5} = \frac{7}{5}.$$

Averaging these values, we obtain $f_T(-20,40) \approx \frac{7}{5}$.

Similarly,

$$f_v(-20,40) = \lim_{h\to 0} \frac{f(-20,40+h) - f(-20,40)}{h}$$

can be approximated (considering $h = 10$ respectively $h = -10$) as:

$$f_v(-20,40) \approx \frac{f(-20,50) - f(-20,40)}{10} = \frac{-35-(-34)}{10} = -\frac{1}{10},$$
$$f_v(-20,40) \approx \frac{f(-20,30) - f(-20,40)}{-10} = \frac{-33-(-34)}{-10} = -\frac{1}{10}.$$

Averaging these values, we find $f_v(-20,40) \approx -\frac{1}{10}$.

(b) The linear approximation of f at $(-20,40)$ is

$$f(T,v) \approx L(T,v) = f(-20,40) + f_T(-20,40)\big(T-(-20)\big) + f_v(-20,40)(v-40)$$
$$\approx -34 + \frac{7}{5}(T+20) - \frac{1}{10}(v-40).$$

(c) $f(-22,45) \approx L(-22,45) \approx -34 + \frac{7}{5}(-22+20) - \frac{1}{10}(45-40) = -37.3.$

4. (a) The domain of f is defined by $1 - x^2 - y^2 \geq 0$, or equivalently

$$D = \{(x,y) : x^2 + y^2 \leq 1\},$$

which represents the closed disc of center $(0,0)$ and radius 1.

(b) The equality $z = \sqrt{1-x^2-y^2}$ is equivalent to

$$\begin{cases} z \geq 0 \\ x^2+y^2+z^2 = 1, \end{cases}$$

which represents the upper hemisphere of the sphere of center $(0,0,0)$ and radius 1.

(c) Differentiating $z = f(x,y)$ with respect to x and y we find

$$\frac{\partial z}{\partial x} = -\frac{x}{1-\sqrt{1-x^2-y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$
$$\frac{\partial z}{\partial x}\left(\frac{1}{2},\frac{1}{2}\right) = -\frac{1/2}{\sqrt{1-1/4-1/2}} = -\frac{1}{\sqrt{2}} = \frac{\partial z}{\partial y}\left(\frac{1}{2},\frac{1}{2}\right).$$

So an equation of the tangent plane to the surface $z = \sqrt{1-x^2-y^2}$ at $\left(\frac{1}{2},\frac{1}{2}\right)$ is

$$z - f\left(\frac{1}{2},\frac{1}{2}\right) = f_x\left(\frac{1}{2},\frac{1}{2}\right)\left(x-\frac{1}{2}\right) + f_y\left(\frac{1}{2},\frac{1}{2}\right)\left(y-\frac{1}{2}\right), \qquad \text{or equivalently}$$
$$z - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(x-\frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(y-\frac{1}{2}\right),$$

giving

$$x + y + z\sqrt{2} = 2. \tag{1}$$

(d) Equation (1) shows that a normal vector to the tangent plane is $\vec{n} = \langle 1, 1, \sqrt{2}\rangle$. The angle $\theta \in [0, \pi]$ between $\vec{v}$ and $\vec{n}$ is characterized by

$$\cos\theta = \frac{\vec{v}\cdot\vec{n}}{|\vec{v}|\,|\vec{n}|} = \frac{\langle 1/2, 1/2, 1/\sqrt{2}\rangle \cdot \langle 1, 1, \sqrt{2}\rangle}{1\sqrt{1+1+2}} = \frac{2}{2} = 1,$$

showing that $\theta = 0$. (We could have directly noticed that the vectors $\vec{n}$ and $\vec{v}$ are proportional.)

5. (a) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt[3]{h}}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1.$

   (b) As in part (a) we find $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = 1.$

   Suppose that f is differentiable at $(0,0)$.Then

$$\begin{aligned}\Delta z = f(\Delta x, \Delta y) - f(0,0) = \sqrt[3]{(\Delta x)^3 + (\Delta y)^3} &= f_x(0,0)\Delta x + f_y(0,0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ &= (1+\varepsilon_1)\Delta x + (1+\varepsilon_2)\Delta y,\end{aligned}$$

   with $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta_2 \to 0$. The particular choice $y = x$ gives $\Delta x = \Delta y$ and so

$$\sqrt[3]{2}\,\Delta x = (2 + \varepsilon_1 + \varepsilon_2)\Delta x \qquad \text{with } \varepsilon_2, \varepsilon_2 \to 0 \text{ as } \Delta x \to 0.$$

   This yields $\sqrt[3]{2} = 1$, which is impossible.

   It remains that f cannot be differentiable at $(0,0)$.