Math 241, Sections BL1 and BL2

Quiz # 6 Solutions

November 15, 2012

Solve both exercises. Show work to get credit.

1) [5pts.] Find the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 36$.

Solution: By the symmetry of the solid, we have $\bar{x} = \bar{y} = 0$. Notice that we can describe the region E in spherical coordinates by $0 \leq \rho \leq 6$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi/4$. For $\bar{z}$, we need to compute

$$\begin{aligned}
\text{Volume of } E = \iiint_E 1\, dV &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^6 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi \\
&= \frac{6^3}{3} \int_0^{\pi/4} \int_0^{2\pi} \sin\varphi \, d\theta \, d\varphi \\
&= \frac{2 \cdot 6^3 \pi}{3} \int_0^{\pi/4} \sin\varphi \, d\theta \\
&= -\frac{2 \cdot 6^3 \pi}{3} \left[\cos\varphi\right]_0^{\pi/4} \\
&= \frac{2 \cdot 6^3 \pi}{3} \left(1 - \frac{\sqrt{2}}{2}\right),
\end{aligned}$$

and also

$$\begin{aligned}
\iiint_E z\, dV &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^6 (\rho\cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi \\
&= \frac{6^4}{4} \int_0^{\pi/4} \int_0^{2\pi} \sin\varphi \cos\varphi \, d\theta \, d\varphi \\
&= \frac{2 \cdot 6^4 \pi}{4} \int_0^{\pi/4} \sin\varphi \cos\varphi \, d\varphi \\
&= \frac{2 \cdot 6^4 \pi}{4} \int_0^{1/\sqrt{2}} u \, du \\
&= \frac{2 \cdot 6^4 \pi}{4} \left(\frac{1/2}{2} - 0\right) \\
&= \frac{2 \cdot 6^4 \pi}{16}
\end{aligned}$$

(where we've used the substitution $u = \sin\varphi$ in the last integral). Hence we have

$$\bar{z} = \frac{\iiint_E z\, dV}{\iiint_E 1\, dV} = \frac{9/8}{1 - \frac{\sqrt{2}}{2}}.$$

2) [5pts.] Evaluate $I = \int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x,y) = \langle e^{3x} + x^2y, e^{3y} - xy^2 \rangle,$$

and C is the circle $x^2 + y^2 = 9$ oriented clockwise.

Solution: We'd like to use Green's Theorem. Note that the orientation of C is reverse of what we would like: we'll negate the integral to compensate (recall that reversing directions of the path in a vector field line integral reverses the sign of the result). Let D be the disc bounded by C.

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= -\int_{-C} \cdot d\vec{r} \\
&= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= -\iint_D \left(\frac{\partial}{\partial x}(e^{3y} - xy^2) - \frac{\partial}{\partial y}(e^{3x} + x^2y) \right) dA \\
&= -\iint_D \left(-y^2 - x^2 \right) dA
\end{aligned}$$

Now to compute this integral we use polar coordinates.

$$\begin{aligned}
&= -\int_0^{2\pi} \int_0^3 (-r^2) r \, dr \, d\theta \\
&= (2\pi) \left(\frac{3^4}{4} - 0 \right) \\
&= \frac{81\pi}{2}.
\end{aligned}$$