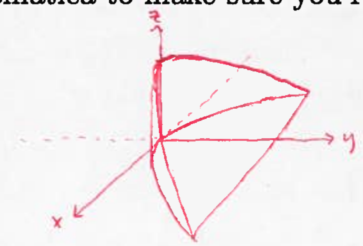
Math 241 X8 **Name(s):** Solutions

Homework 10 supplement

This is a written homework supplement to the homework for Units 10 and 11: 3D Transformations and Spherical Coordinates.

(1) For each of the 6 ways to slice up the region (6 orders of integration), write a triple integral that represents the volume of the region bounded by the surfaces $y = x^2$, $z = 0$, and $y + 2z = 4$. (At least try to sketch the region by hand first; you may use Mathematica to make sure you're thinking of the correct region.)

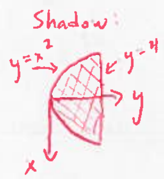


Either method will give all 6.

Sticks

Slices

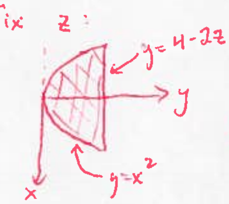
z-direction:



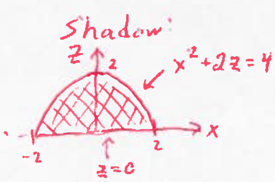
$$\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-\frac{y}{2}} dz\,dy\,dx$$

$$\int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{2-\frac{y}{2}} dz\,dx\,dy$$

fix z:



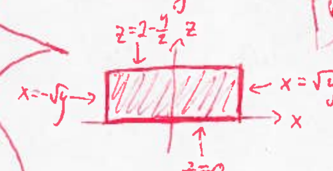
y-direction:



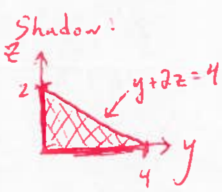
$$\int_{-2}^{2}\int_{0}^{2-\frac{x^2}{2}}\int_{x^2}^{4-2z} dy\,dz\,dx$$

$$\int_{0}^{2}\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} dy\,dx\,dz$$

fix y:



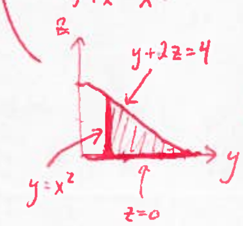
x-direction:



$$\int_{0}^{2}\int_{0}^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$$

$$\int_{0}^{4}\int_{0}^{2-\frac{y}{2}}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$$

fix x:



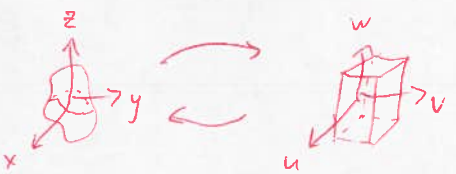
(2) Sometimes you can get away with less work than the "standard approach" to transformations. Consider the 3D region that is bounded by the planes $x+y+2z=-1$, $x+y+2z=5$, $x-y-2z=-3$, $x-y-2z=6$, and the surfaces $\sinh(x)-y+2z=-1$ and $\sinh(x)-y+2z=2$. You should agree that this begs for the transformation

$$u = x+y+2z, \qquad v = x-y-2z, \qquad w = \sinh(x)-y+2z.$$

Now the usual next step is to solve for x,y,z in terms of u,v,w, but *we don't have to do this here.* Instead, compute the volume conversion factor $V_{uvw}(x,y,z)$. Then with a little bit of common sense, conclude what $V_{xyz}(u,v,w)$ is. With that in hand, you're ready to complete the computation: do it. (The derivative of $\sinh(x)$ is $\cosh(x)$.)

$$V_{uvw}(x,y,z) = \begin{array}{c|ccc|} & u & v & w \\ \partial_x & 1 & 1 & \cosh x \\ \partial_y & 1 & -1 & -1 \\ \partial_z & 2 & -2 & 2 \end{array} = 1\begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + \cosh x \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}$$

$$= -4 - 4 + 0 = -8.$$



So uvw-volume is 8 times as large as corresponding xyz-volume. Hence xyz-volume is $\frac{1}{8}$ that of uvw-volume, i.e. $|V_{xyz}(u,v,w)| = \frac{1}{8}$.

So $\text{Volume(region)} = \iiint_{\text{region}} 1\,dx\,dy\,dz = \int_{-1}^{2}\int_{-3}^{6}\int_{-1}^{5} 1\cdot\frac{1}{8}\,du\,dv\,dw$

$$= \frac{1}{8}\cdot 3\cdot 9\cdot 6 = \frac{81}{4}.$$

(3) Quickie: compute $\iiint_R z\,dV$ where R is the portion of the unit ball in the first octant.

Spherical: $z = \rho\cos\varphi$, $J = \rho^2\sin\varphi$, region R is $0 \le \rho \le 1$, $0 \le \varphi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$.

So $\iiint_R z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\cos\varphi)(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{4}\cos\varphi\sin\varphi\,d\varphi\,d\theta = \int_0^{\pi/2}\frac{1}{8}\,d\theta = \frac{\pi}{16}.$$

$u = \sin\varphi$
$du = \cos\varphi\,d\varphi$

$\frac{1}{4}\left.\frac{u^2}{2}\right|_0^1$