Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:50am to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

1. Consider the parametric equations $x = t^2$, $y = \sin(\pi t)$.

   (a) Find the slope(s) of the tangent line(s) to the curve at the point(s) where $x = 9$.

$x = t^2 = 9 \iff t = \pm 3$

$\text{slope} = \frac{y'}{x'} = \frac{\pi \cos(\pi t)}{2t}$ $\quad$ @ $t=3$: $-\frac{\pi}{6}$

@ $t=-3$: $\frac{\pi}{6}$

   (b) What are the minimum and maximum values taken by x and y in this curve?

$x' = 0 \iff t = 0$

$x \geq 0$, and $\lim_{t\to\pm\infty} x = +\infty$, so $\min x = 0$

$\max x$ DNE

$y' = 0 \iff t = \frac{\pi}{2} + \pi n$ for integer n;

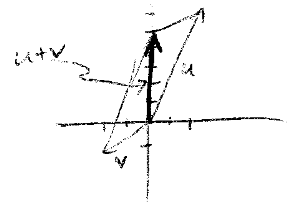
$y = \sin(\pi t)$ $\Rightarrow$ $\min y = -1$

$(t \in \mathbb{R})$ $\quad$ $\max y = +1$

2. Let $\vec{u} = \langle 2, 6\rangle$ and $\vec{v} = \langle -2, -1\rangle$. Compute and plot the following together with u and v.

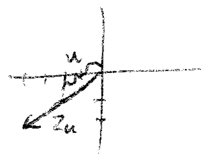
(a) $\vec{u} + \vec{v}$

$= \langle 0, 5\rangle$



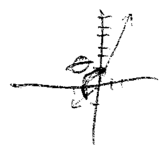
(b) $2\vec{v}$

$= \langle -4, -2\rangle$



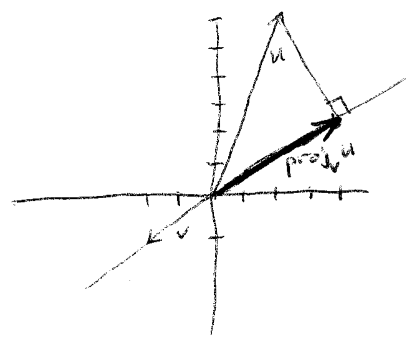
(c) the angle between $\vec{u}$ and $\vec{v}$

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-4-6}{\sqrt{4+36}\cdot\sqrt{4+1}} = \frac{-10}{2\sqrt{10}\,\sqrt{5}} = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3}{4}\pi$$



(d) the push/projection of $\vec{u}$ in the direction of $\vec{v}$

$$= \frac{u \cdot v}{v \cdot v} v = \frac{-10}{5}\langle -2, -1\rangle = \langle 4, 2\rangle$$



(e) If $\vec{u}$ and $\vec{v}$ live in the plane of this paper, and you consider the paper in the 3D classroom, then which direction does $\vec{u} \times \vec{v}$ point?

By right hand rule, it points up out of the page (toward us).

Also, $\langle 2,6,0\rangle \times \langle -2,-1,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 0 \\ -2 & -1 & 0 \end{vmatrix} = \langle 0, 0, -2+12\rangle$
$= \langle 0, 0, 10\rangle$ ✓

3. Consider the lines $\ell_1(t) = (0,1,3)+t(-2,-2,-10)$ and $\ell_2(t) = (-5,2,3)+t(-9,9,30)$. Are they parallel, perpendicular, or neither? Do they intersect or not? If they intersect or are parallel, find an xyz-equation of the plane containing them. ~~Find the distance between them.~~

~~Parallel:~~ not parallel: if $\langle -2,-2,-10\rangle = c\langle -9,9,30\rangle$

then $c>0$ in x-coord, $c<0$ in y- & z-coord ↯

not $\perp$: $\langle -2,-2,-10\rangle \cdot \langle -9,9,30\rangle = 18-18-300 \neq 0$

intersect? $\begin{cases} 0-2t = -5-9s \\ 1-2t = 2+9s \\ 3-10t = 3+30s \Rightarrow t=-3s \end{cases}$ $\Rightarrow 1+6s = 2+9s \Rightarrow s=-\frac{1}{3} \Rightarrow t=1$ ✓

yes, @ $(-2,-1,-7)$.

$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & -10 \\ -9 & 9 & 30 \end{vmatrix} = 6 \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 5 \\ -3 & 3 & 10 \end{vmatrix} = 6\langle 10-15, -(10+15), 3+3\rangle = c\langle -5,-25,6\rangle$

plane: $-5(x+2)-25(y+1)+6(z+7)=0$.

4. Consider the lines $\ell_1(t) = (0,1,3)+t(2,1,5)$ from above and $\ell_3(t) = (3,0,1)+t(-2,-1,1)$. Are they parallel, perpendicular, or neither? Do they intersect or not? If they intersect or are parallel, find an xyz-equation of the plane containing them. Find the distance between them.

Perpendicular: $\langle 2,1,5\rangle \cdot \langle -2,-1,1\rangle = -4-1+5=0$

intersect? $\begin{cases} 0+2t = 3-2s \\ 1+t = 0-s \\ 3+5t = 1+s \end{cases}$ $\Rightarrow 4+6t = 1 \Rightarrow t=-\frac{1}{2} \Rightarrow s=-\frac{1}{2} \Rightarrow -1=4$ ↯

no

Distance is minimized along a line segment $\perp$ to both lines.

$\vec{n}_0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 5 \\ -2 & -1 & 1 \end{vmatrix} = \langle 1+5, -(2+10), -2+2\rangle = \langle 6,-12,0\rangle = c\cdot\langle 1,-2,0\rangle$.

$\vec{n} = \langle 1,-2,0\rangle$

$\vec{v} = (3,0,1)-(0,1,3) = \langle 3,-1,-2\rangle$

distance $= |\mathrm{proj}_{\vec{n}}\vec{v}| = \left|\frac{v\cdot n}{n\cdot n}n\right| = \left|\frac{3+2+0}{1+4+0}\langle 1,-2,0\rangle\right| = |\langle 1,-2,0\rangle| = \sqrt{5}$.

5. Consider the two planes given by equations

$$3x - y + z = 4$$
$$2x + y - 2z = 6.$$

Find an equation of the line that is the intersection of these planes.

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \langle 2-1, -(3+2), 3+2 \rangle = \langle 1, -5, 5 \rangle$$

point: try $z=0$: $\begin{cases} 3x - y = 4 \\ 2x + y = 6 \end{cases}$

$\oplus \quad 5x = 10$

$x = 2$

$y = 2$ $\qquad p = \langle 2, 2, 0 \rangle$

$$\ell(t) = \langle 2, 2, 0 \rangle + t \langle 1, -5, 5 \rangle.$$

6. Consider the two planes given by equations

$$3x + 12y - 3z = 1 \rightarrow P_1 = (\tfrac{1}{3}, 0, 0)$$
$$2x + 8y - 2z = 7. \rightarrow P_2 = (\tfrac{7}{2}, 0, 0)$$

Find the distance between them.

$$\vec{n} = \langle 1, 4, -1 \rangle$$

$$\vec{v} = P_2 - P_1 = \langle \tfrac{19}{6}, 0, 0 \rangle$$

$$\text{distance} = \left| \operatorname{proj}_n v \right| = \left| \frac{v \cdot n}{n \cdot n} n \right|$$

$$= \left| \frac{19/6 + 0 + 0}{1 + 16 + 1} \langle 1, 4, -1 \rangle \right|$$

$$= \cancel{\frac{19}{108}} = \frac{19}{6} \frac{\sqrt{18}}{18} = \frac{19}{18\sqrt{2}}.$$

7. Suppose a particle moves in the plane, with position at time t given by (t^2, t^3) at time t.

$p(t) = (t^2, t^3)$

(a) Find the velocity at time $t = 9$.

$v(t) = p'(t) = \langle 2t, 3t^2 \rangle$

$v(9) = p'(9) = 9\langle 2, 27 \rangle$

(b) Find the acceleration at time $t = 9$.

$a(t) = v'(t) = \langle 2, 6t \rangle$

$a(9) = v'(9) = 2\langle 1, 27 \rangle$

(c) Find the tangential component of acceleration (i.e. the push of acceleration in the direction tangential to motion) at time $t = 9$.

$$= \mathrm{proj}_{v(9)} a(9) = \frac{a(9) \cdot v(9)}{v(9) \cdot v(9)} v(9) = \frac{18(2 + 27^2)}{81(4 + 27^2)} 9\langle 2, 27 \rangle$$

$$= 2\frac{2 + 27^2}{4 + 27^2}\langle 2, 27 \rangle .$$

(d) Find the normal component of acceleration (i.e. the push of acceleration in the direction perpendicular to motion.

$$= a(9) - \mathrm{proj}_{v(9)} a(9) = 2\langle 1, 27 \rangle - 2\frac{2 + 27^2}{4 + 27^2}\langle 2, 27 \rangle .$$

(e) What do you know about how the speed of the particle is changing at $t = 9$?

Speed is increasing because $a(9) \cdot v(9) > 0$.

8. Give parametric equations for the unit circle in the plane $3x - y + z = 4$ centered at $(1, 0, 1)$.

Need $\perp$ unit vectors that span the plane.

$$v_1 = (\overset{P_1}{1,1,2}) - (\overset{P_2}{1,0,1}) = \langle 0, 1, 1 \rangle$$

$$v_2 = v_1 \times n = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} = \langle 1+1, -(0-3), 0-3 \rangle$$

$|v_1| = \sqrt{0+1+1}$ $\quad |v_2| = \sqrt{4+9+9}$ $\quad = \langle 2, 3, -3 \rangle$

$$\hat{v}_1 = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$$

$$\hat{v}_2 = \frac{1}{\sqrt{22}} \langle 2, 3, -3 \rangle$$

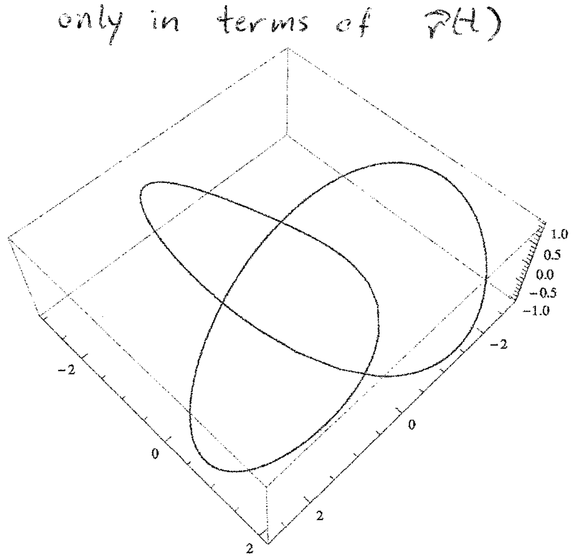
$$p(t) = (1, 0, 1) + \cos t \, \hat{v}_1 + \sin t \, \hat{v}_2, \quad t \in [0, 2\pi).$$

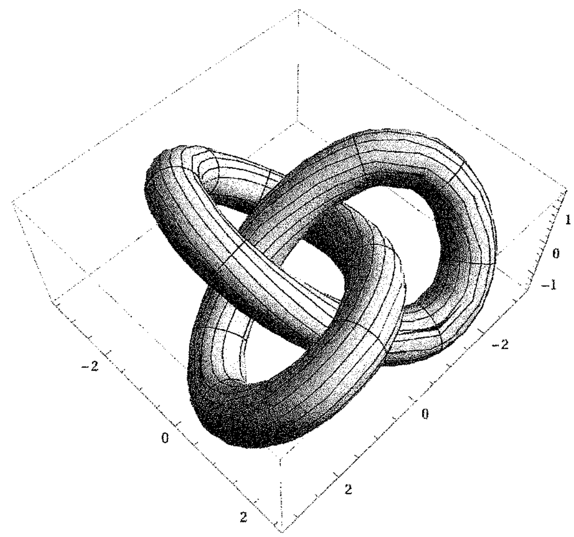
9. Below is shown a curve (the trefoil knot) and a "fattening" of it into a closed tube. The curve can be parametrized by

$$\cancel{\vec{r}(t) = \langle \sin t + 2\sin 2t\,,\ \cos t - 2\cos 2t\,,\ -\sin 3t \rangle, \qquad t \in [0, 2\pi).}$$

Give parametric equations for the tube. It consists of circles of radius 0.5 centered on the curve that lie in planes that cut the curve perpendicularly.

only in terms of $\vec{r}(t)$





$$r'(t) = \langle \cos t + 4\cos 2t\,,\ -\sin t + 4\sin 2t\,,\ -3\cos 3t \rangle$$

$$\text{unittan}(t) = \frac{r'(t)}{|r'(t)|} \qquad \text{unitnormal}(t) = \frac{\text{unittan}'(t)}{|\text{unittan}'(t)|}$$

$$\text{unitbinormal}(t) = \text{unittan}(t) \times \text{unitnormal}(t)$$

$$\text{tube}(s,t) = r(t) + 0.5\cos s\ \text{unitnormal}(t) + 0.5\sin s\ \text{unitbinormal}(t), \qquad t, s \in [0, 2\pi).$$

10. Find the maximum and minimum values of $f(x,y) = xy$ on the disk $\underbrace{x^2 + y^2}_{g(x,y)} \leq 4$.

Interior: $\nabla f = \langle y, x \rangle = \langle 0, 0 \rangle \iff x=0$ & $y=0$

$$\boxed{f(0,0) = 0.}$$

Boundary: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \iff \begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2+y^2 = 4 \end{cases} \Rightarrow x = 2\lambda(2\lambda x)$

$x = 4\lambda^2 x$

$0 = (4\lambda^2 - 1)x$

$\Rightarrow \lambda = \pm\frac{1}{2}$ or $x = 0$

$y = \pm x$

$x^2 + y^2 = 4$

$2x^2 = 4$

$x = \pm\sqrt{2}$

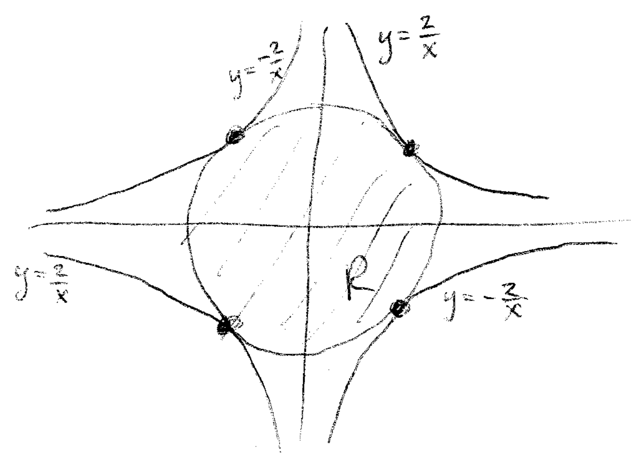
$y = \pm\sqrt{2}$

$y = 0$

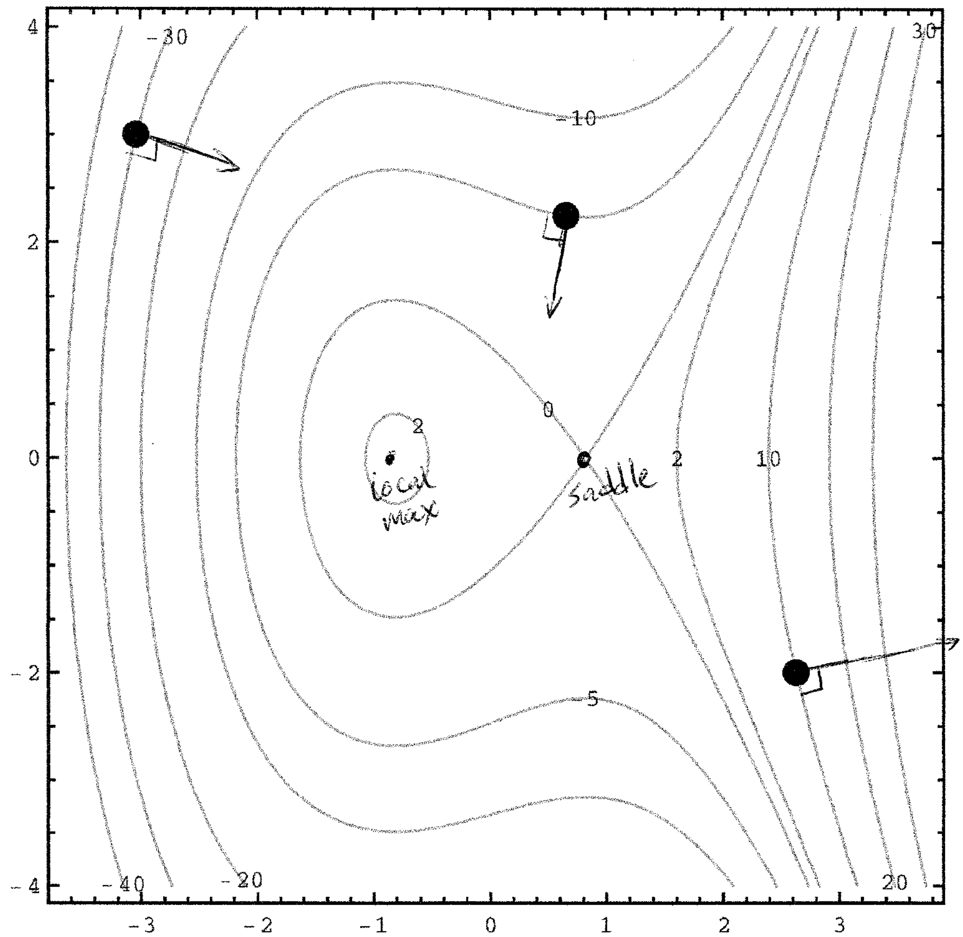
$x^2+y^2 = 0 \neq 4$ ↯

$$\boxed{\begin{array}{l} f(-\sqrt{2},-\sqrt{2}) = f(\sqrt{2},\sqrt{2}) = 2 \\ f(-\sqrt{2},+\sqrt{2}) = f(+\sqrt{2},-\sqrt{2}) = -2 \end{array}}$$

So max $= 2$ min $= -2$

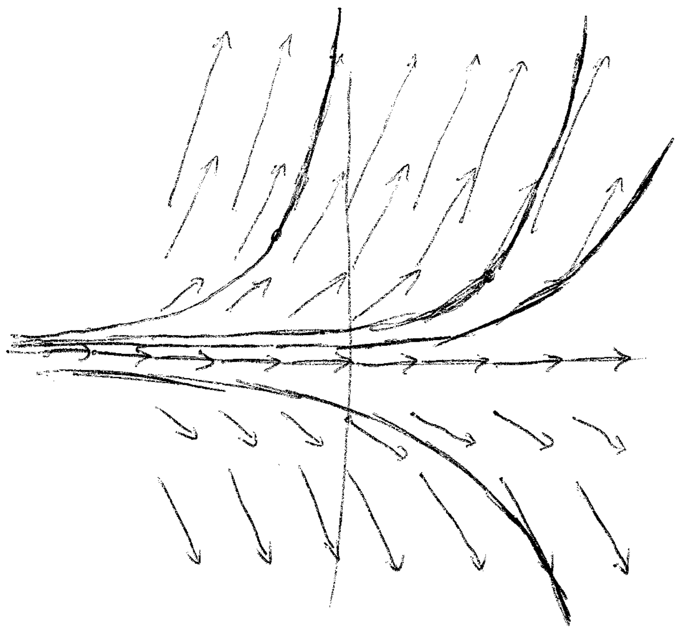


11. Below is a plot of several level curves of a function $f(x, y)$. At the indicated points, sketch in the gradient vectors. Find the (approximate) locations of the critical points of f, then classify them.



12. Consider $F(x, y) = \langle 1, y \rangle$.

(a) Draw enough vectors from F to get the feel for what it looks like.



(b) Add to the plot a few trajectories.

(c) Give the differential equations that define the trajectories of F.

$x' = 1$
$y' = y$

(d) Solve the system of differential equations to find the trajectory that passes through $(2, 1)$.

say at $t=0$

$x' = 1 \Rightarrow x = t + c_1$

$y' = y \Rightarrow y = e^{t + c_2}$

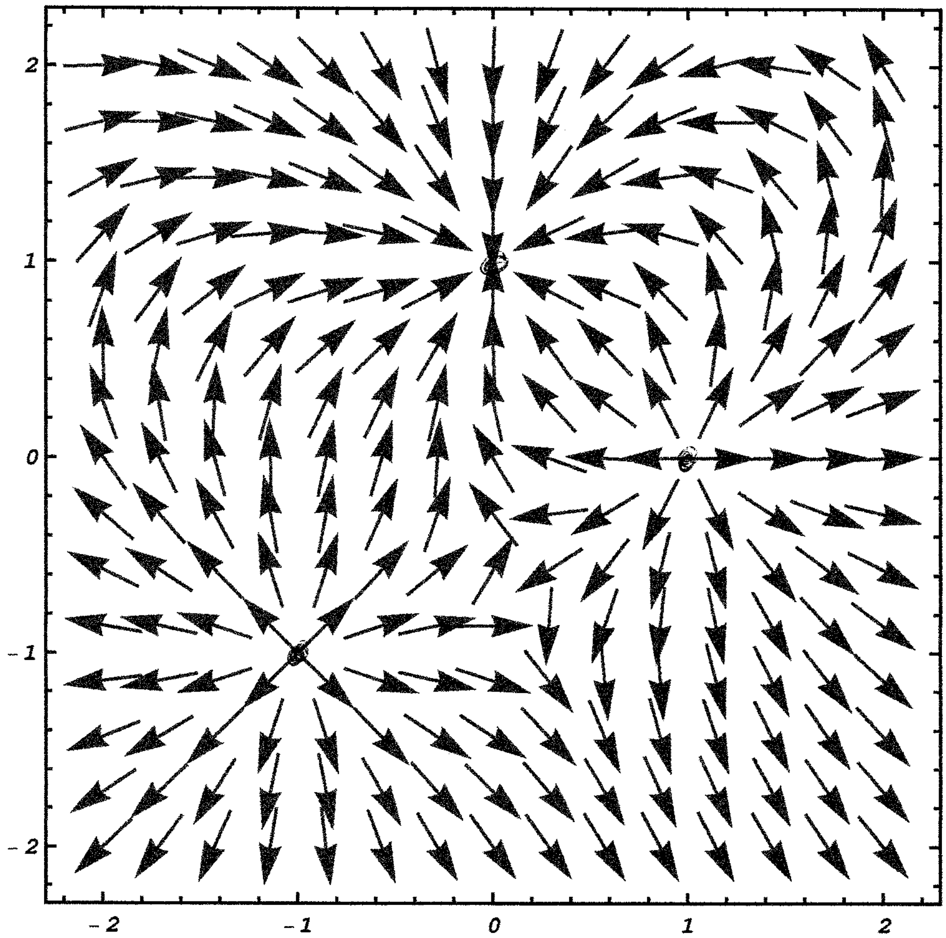
$\Rightarrow 2 = 0 + c_1$
$1 = e^{0 + c_2}$

$\Rightarrow c_1 = 2$
$c_2 = 0$

$x = t + 2$
$y = e^t$

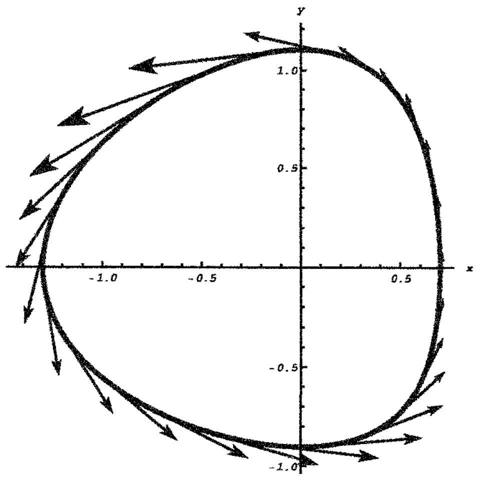
(eliminating t, this is $y = e^{x-2}$, which fits the trajectory above)

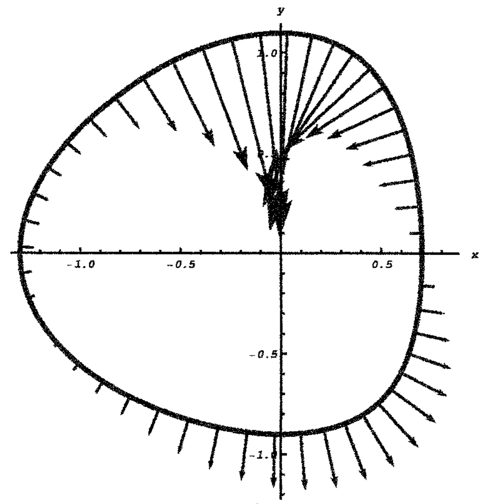
13. Here's a plot of the gradient field $F = \nabla f$ for some function f. What do the trajectories in the field tell you about f?



Since ∇f points in direction of greatest (initial) increase, the points $(-1,-1)$ and $(1,0)$ are local min's for f, and $(0,1)$ is a local max.

14. Here are the tangential and normal components of some F on a curve C. What do they suggest about the net flow of F along and across C?





Net flow is counterclockwise

Net flow is inward (bigger arrows in than out)