# $\pi$ in different bases

## What does that mean?

You know that any real number can be written in *decimal*: as a sum of powers of 10 with coefficients (*digits*) from 0-9. The same is true for powers of any integer bigger than 1. To the right you see $\pi$ written in several different bases.

For example, $\pi$ in base 6 is $3.12418\ldots$; this means

$$\pi = 3 \cdot 6^0 + 1 \cdot 6^{-1} + 2 \cdot 6^{-2} + 4 \cdot 6^{-3} + 1 \cdot 6^{-4} + 8 \cdot 6^{-5} + \cdots .$$

When the base is larger than 10, we need new “digits”; it is common to use $a = 10$, $b = 11$, and so on.

For base 26, we have chosen to use instead of the usual digits, $a = 0, b = 1, \ldots, z = 25$.

There are also two unusual expressions given. Using the golden ratio $\phi \approx 1.618$ as a base and “digits” 0 and 1, forbidding consecutive 1’s gives a unique representation of any real number. Second, we use *balanced ternary*, which uses powers of 3 but “digits” $-1, 0, 1$. We’ve used the symbol $T$ for the digit $-1$.

## Irrational, Normal

A number is **rational** if it can be written as a fraction with integer numerator and denominator. It is **irrational** otherwise. In any integer base, being rational is the same as having a terminating or eventually repeating representation. Since $\pi$ is irrational, none of the representations above can be eventually repeating.

A number is **normal in base $b$** if every string of base-$b$ digits occurs as often as it would in a random infinite string of digits. That is, every single digit occurs as often as any other, any two-digit string occurs as often as any other, etc. A number is **absolutely normal** if it is normal in base $b$ for every integer $b \geq 2$. We don’t know whether $\pi$ is normal, but it is widely believed to be. The representations above are short, but you can see that the digits seem to appear roughly equally often.

## Representations of $\pi$

Base 2: 11.001001000011111101101010100010001000010110100011000010001101...
Base 3: 10.010211012222010211002111110221222220111201212121200121100 1...
Base 4: 3.021003331222202020112203002031030103012120222023200031300130 31...
Base 5: 3.032322143033432411241224041402314211143020310022003444132211...
Base 6: 3.050330051415124105234414053125321102301214442004115252553314 2...
Base 7: 3.066365143203613411026340224465222664352065024015544321542643 1...
Base 8: 3.110375524210264302151423063050560067016321122011160210514763...
Base 9: 3.124188124074427886451777617310358285165453534626523011263214 5...
Base 10: 3.141592653589793238462643383279502884197169399375105820974944...
Base 11: 3.16150702865a485235215259777529418386688488531 63a1a54213004658...
Base 12: 3.184809493b918664573a6211bb151551a05729290a7809a492742140a60a5...
Base 16: 3.243f6a8885a308d313198a2e03707344a4093822299f31d0082efa98ec4e6...

Base 26*: d.drsqlolyrtrodnlhnqtgkudqgtuirxneqbckbszivqqvgdmelmu
exroiqiyalvuzvebmijpqqxlkplrncfwjpbymggohjmmqisms...

Base $\varphi$: 100.0100101010010001010101000001010 0...
Balanced ternary: 10.011T111T000T011T1101T111111...

## Fun with base 26

In the modified base 26 representation, we can look for words (or sentences!) in the digits of $\pi$.

Already in the part of the representation given above, we see the word “lo” at the 6th place, and in fact the text “lol” is there. Shortly after, we find the word “trod”.

If $\pi$ is indeed normal, then every possible word will occur infinitely often in this expansion of $\pi$. The first $n$-letter words to appear are “o”, “lo,” “rod,” “trod,” “steel,” “oxygen,” “subplot,” and “armagnac.” There are websites that will allow you to search the first several million or billion digits of $\pi$ in various bases for your favorite string; you might find your name, birthday, phone number, ...

You could instead code up $\pi$ using base 27 and include a space as a digit, or some larger base with punctuation as some digits. If $\pi$ really is normal, then you can find an entire Shakespeare play somewhere in $\pi$’s digits–and even find it infinitely often!!!