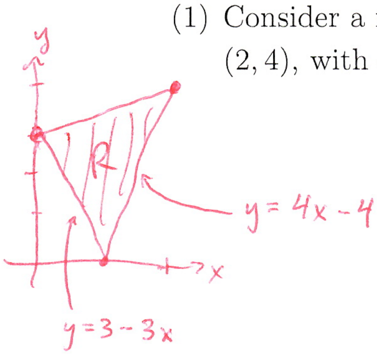
Math 241 X8 **Name(s):** Solutions

Homework 9 supplement

This is a written homework supplement to the homework for Unit 9: 2D Transformations.

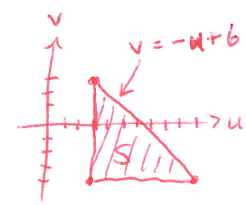
(1) Consider a metal plate in the shape of the triangle with vertices $(1,0)$, $(0,3)$, and $(2,4)$, with density at each point (x,y) given by y kg/m^2. Compute its mass.



Take $u = 3x + y$ (other possibilities)
$v = y - 4x$.

$\Rightarrow x = \frac{1}{7}(u - v)$
$y = \frac{1}{7}(4u + 3v)$

(x,y)	(u,v)
$(1,0)$	$(3,-4)$
$(0,3)$	$(3,3)$
$(2,4)$	$(10,-4)$



$$\text{mass} = \iint_R \underset{\text{density}}{y}\, dx\,dy = \iint_S \left(\tfrac{1}{7}(4u+3v)\right) |J|\, du\,dv$$

$$J = \begin{vmatrix} 1/7 & -1/7 \\ 4/7 & 3/7 \end{vmatrix} = \frac{1}{49}(3+4) = \frac{1}{7}$$

$$= \int_3^{10}\int_{-4}^{6-u} \frac{1}{49}(4u+3v)\, dv\,du \quad \text{or} \quad \int_{-4}^{3}\int_{3}^{6-v} \frac{1}{49}(4u+3v)\, du\,dv$$

$$= \frac{1}{49}\int_3^{10} 4u(10-u) + \frac{3}{2}\left((6-u)^2 - (-4)^2\right) du$$

$$= \frac{1}{49}\int_3^{10} \left(-\frac{5}{2}u^2 + 22u + 30\right) du$$

$$= \frac{1}{49}\left[-\frac{5}{6}u^3 + 11u^2 + 30u\right]_3^{10}$$

$$= \frac{1}{49}\left(-\frac{5000}{6} + 1100 + 300 + \frac{45}{2} - 99 - 90\right).$$

$$\left(= \frac{49}{6}\right)$$

(2) Compute the integral $\iint_R x^2 \, dA$, where R is the region bounded by the ellipse $4x^2 + y^2 = 1$, by the following method[1]:

(a) Transform the integral into one over the unit disk by an appropriate linear transformation.

Take $u = 2x$, $v = y$.

Then $4x^2 + y^2 \leq 1 \iff u^2 + v^2 \leq 1$,

and $J = \begin{vmatrix} 1/2 & 0 \\ 0 & 1 \end{vmatrix} = 1/2$, so

$$\iint_R x^2 \, dA = \iint_{\substack{\text{unit} \\ \text{disk} \\ \text{in } uv\text{-plane}}} \left(\frac{u^2}{4}\right) \cdot \tfrac{1}{2} \, du \, dv.$$

(b) Compute the new integral using polar coordinates.

$u = r\cos\theta$

$v = r\sin\theta$

$$\int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{8} \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{\cos^2\theta}{8} \left.\frac{r^4}{4}\right|_0^1 d\theta = \frac{1}{32}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{64}\int_0^{2\pi} (1 + \cos(2\theta)) \, d\theta$$

$$= \frac{1}{64} \cdot 2\pi = \frac{\pi}{32}.$$

[1] In Mathematica this is easy to do in one transformation; by hand it's easier to do in two steps, since you already have memorized the Jacobian for the polar transform.