SECTION 1.7 RATIONAL EXPRESSIONS

Simplifying rational expressions

The fundamental principle of rational expressions

One of the most important procedures we can use when we work with rational expressions is the simplification of the rational expression. To do this, we use a principle called the **fundamental principle of rational expressions.**

Fundamental principle of rational expressions

If P is any polynomial and Q and R are nonzero polynomials, then

$$\frac{PR}{QR} = \frac{P}{Q} \text{ and } \frac{P}{Q} = \frac{PR}{QR}$$

Concept

To change the appearance of a rational expression without changing its value, we may multiply or divide both the numerator and the denominator by the same nonzero polynomial.

This property is based on 1 being the identity element for multiplication. That is,

$$\frac{PR}{QR} = \frac{P}{Q} \cdot \frac{R}{R} = \frac{P}{Q} \cdot 1 = \frac{P}{Q}$$

This property permits us to **reduce** rational expressions to *lowest terms*. A rational expression is *completely reduced* if the greatest factor common to both the numerator and the denominator is 1 or -1. We can see that the key to reducing rational expressions is *finding* and dividing out *factors* that are common to both the numerator and the denominator.

To reduce a rational expression

1. Write the numerator and the denominator in factored form.
2. Divide the numerator and the denominator by all common factors.

EXAMPLES

Simplify the following rational expressions by reducing to lowest terms. Assume that all denominators are nonzero.

1. $\dfrac{45}{60} = \dfrac{3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 5}$ Factor numerator and denominator to prime factors

$= \dfrac{3 \cdot (3 \cdot 5)}{2 \cdot 2 \cdot (3 \cdot 5)}$ Group common factors $(3 \cdot 5)$

$= \dfrac{3}{2 \cdot 2}$ Divide numerator and denominator by $(3 \cdot 5)$

$= \dfrac{3}{4}$ Multiply remaining factors

2. $\dfrac{14x^2}{10x^3} = \dfrac{7 \cdot 2 \cdot x \cdot x}{5 \cdot 2 \cdot x \cdot x \cdot x}$ Factor numerator and denominator

$= \dfrac{7 \cdot (2 \cdot x \cdot x)}{5 \cdot x \cdot (2 \cdot x \cdot x)}$ Group common factors $(2 \cdot x \cdot x)$

$= \dfrac{7}{5x}$ Divide numerator and denominator by common factor $(2 \cdot x \cdot x)$

3. $\dfrac{5a - 15}{4a - 12} = \dfrac{5(a - 3)}{4(a - 3)}$ Factor numerator and denominator

$= \dfrac{5}{4}$ Divide numerator and denominator by common factor $(a - 3)$

4. $\dfrac{y - 7}{y^2 - 49} = \dfrac{y - 7}{(y + 7)(y - 7)}$ Factor denominator

$= \dfrac{1}{y + 7}$ Divide numerator and denominator by common factor $(y - 7)$

▶ ***Quick check*** Reduce $\dfrac{25z^4}{15z^5}$ and $\dfrac{a^2 - 36}{a^2 - a - 30}$ to lowest terms.

Note This reducing process is often called "cancelling" the common factors. It is important to remember that we are *dividing* both the numerator and the denominator by the same common factor; we are *not* just "crossing out" quantities. This leads us to our next topic.

Common errors when reducing

The fundamental property allows us to divide by common *factors only.* A common error in example 4 is to divide the numerator and the denominator by y and 7. These are *terms* and this cannot be done. For example,

$$\frac{9}{11} = \frac{8+1}{8+3} \neq \frac{\cancel{8}+1}{\cancel{8}+3} = \frac{1}{3}$$

This error can be avoided by always remembering that the *fundamental principle of rational expressions* allows us to *divide* the numerator and the denominator by common factors. y and 7 are *terms* and *not factors.*

Reducing $\frac{a-b}{b-a}$

Consider the rational expression $\frac{x-5}{5-x}$, which does not appear to be reducible by a common factor. However,

$$\begin{aligned} 5 - x &= -1(-5 + x) && \text{Factor out } -1 \\ &= -1(x - 5) && \text{Commute the terms and use the definition of subtraction} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{x-5}{5-x} &= \frac{x-5}{-1(x-5)} \\ &= \frac{1}{-1} && \text{Reduce by common factor } (x-5) \\ &= -1 \end{aligned}$$

In general, for all real numbers a and b, $a \neq b$,

$$\frac{a-b}{b-a} = -1$$

EXAMPLES

Simplify the following rational expressions by reducing to lowest terms. Assume that no denominator equals zero.

1. $\dfrac{4-x}{x^2-16}$

$= \dfrac{4-x}{(x-4)(x+4)}$ Completely factor the denominator

$= \dfrac{4-x}{x-4} \cdot \dfrac{1}{x+4}$ Factor opposites $4-x$ and $x-4$

$= -1 \cdot \dfrac{1}{x+4}$ $\dfrac{4-x}{x-4} = -1$

$= \dfrac{-1}{x+4}$ Multiply numerator by -1

2. $\dfrac{1-x^2}{2x^2+x-3}$

$= \dfrac{(1-x)(1+x)}{(x-1)(2x+3)}$ Completely factor numerator and denominator

$= \dfrac{1-x}{x-1} \cdot \dfrac{1+x}{2x+3}$ Factor opposites $1-x$ and $x-1$

$= -1 \cdot \dfrac{1+x}{2x+3}$ $\dfrac{1-x}{x-1} = -1$

$= \dfrac{-1(1+x)}{2x+3}$ Multiply numerator by -1

$= \dfrac{-1-x}{2x+3}$ or $\dfrac{-x-1}{2x+3}$ Alternative forms of answer

▶ ***Quick check*** Reduce $\dfrac{16-y^2}{3y^2-11y-4}$ to lowest terms.

Mastery points

Can you

- Reduce a rational expression to lowest terms using the fundamental principle of rational expressions?
- Recognize factors $a-b$ and $b-a$ and use $\dfrac{a-b}{b-a} = -1$?

EXERCISES

Simplify the following rational expressions by reducing to lowest terms. Assume that no denominator equals zero.

1. $\frac{54}{72}$
2. $\frac{75}{145}$
3. $\frac{6x}{15}$
4. $\frac{8a}{10}$
5. $\frac{16x^2}{12x}$
6. $\frac{15b^3}{20b}$
7. $\frac{-8x^2}{6x^4}$
8. $\frac{3a^6}{-9a^3}$
9. $\frac{16a^2b}{20ab^2}$
10. $\frac{15a^2x^3}{35ax^2}$
11. $\frac{20ab^2c^3}{-4ab^2c^3}$
12. $\frac{-72x^4y^3z^2}{9x^4y^3z^2}$
13. $\frac{10(x+5)}{8(x+5)}$
14. $\frac{24(x-3)}{15(x-3)}$
15. $\frac{6(x-2)}{(x+3)(x-2)}$
16. $\frac{-8(x+1)}{4(x+1)(x-6)}$

17. $\frac{a+b}{a^2-b^2}$
18. $\frac{x^2-y^2}{x-y}$
19. $\frac{3m-6}{5m-10}$
20. $\frac{8b+12}{10b+15}$
21. $\frac{3x-3}{6x+6}$
22. $\frac{6y-6}{8y^2-8}$
23. $\frac{x^2-9}{x^2+6x+9}$
24. $\frac{a^2-10a+25}{a^2-25}$
25. $\frac{x^2-3x-10}{x^2-x-6}$
26. $\frac{y^2-y-42}{y^2+12y+36}$
27. $\frac{2y^2-3y-9}{4y^2-13y+3}$
28. $\frac{4m^2-15m-4}{8m^2-18m-5}$
29. $\frac{x-3}{x^3-27}$
30. $\frac{x+2}{x^3+8}$
31. $\frac{a^2-b^2}{a^3+b^3}$
32. $\frac{x^3+y^3}{x^2-y^2}$

Simplify by reducing to lowest terms. Assume that no denominator is equal to zero. See example 5–

Example $\frac{16-y^2}{3y^2-11y-4}$

Solution

$= \frac{(4-y)(4+y)}{(y-4)(3y+1)}$ Factor numerator and denominator

$= \frac{4-y}{y-4}\cdot\frac{4+y}{3y+1}$

$= -1\cdot\frac{4+y}{3y+1}$ $\frac{4-y}{y-4} = -1$

$= \frac{-1(4+y)}{3y+1}$ Multiply the numerator by -1

$= \frac{-4-y}{3y+1}$ or $\frac{-y-4}{3y+1}$ Alternative forms of answer

33. $\frac{4x-4y}{y-x}$
34. $\frac{8b-8a}{a-b}$
35. $\frac{2x-8}{12-3x}$
36. $\frac{12a-8b}{10b-15a}$
37. $\frac{2y^2-2x^2}{x-y}$
38. $\frac{3p-3q}{6q^2-6p^2}$
39. $\frac{(x-y)^2}{y^2-x^2}$
40. $\frac{a-b}{b^2-a^2}$
41. $\frac{n^2-m^2}{(m+n)^2}$
42. $\frac{p^2-q^2}{q^2-p^2}$
43. $\frac{4x-4y}{y^2-x^2}$
44. $\frac{4-y}{2y^2-7y-4}$
45. $\frac{x-3}{12-x-x^2}$

Multiplication and division of rational expressions

Multiplication of rational expressions

Recall that to multiply two real number fractions we multiply the numerators and multiply the denominators.

Multiplication property of fractions

If a, b, c, and d are real numbers, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \; (b, d \neq 0)$$

Note Any possible reduction is performed *before* the multiplication takes place.

EXAMPLE

Multiply the fractions $\frac{3}{7}$ and $\frac{14}{27}$ and simplify the product.

$$\frac{3}{7} \cdot \frac{14}{27} = \frac{3 \cdot 14}{7 \cdot 27}$$ Multiply the numerators
Multiply the denominators

$$= \frac{3 \cdot 2 \cdot 7}{7 \cdot 3 \cdot 3 \cdot 3}$$ Factor the numerator and the denominator

$$= \frac{2 \cdot (3 \cdot 7)}{3 \cdot 3 \cdot (3 \cdot 7)}$$ Group the common factors $(3 \cdot 7)$

$$= \frac{2}{3 \cdot 3}$$ Divide numerator and denominator by $(3 \cdot 7)$

$$= \frac{2}{9}$$ Multiply the remaining factors ■

This same procedure is followed when we multiply two rational expressions.

Multiplication property of rational expressions

Given rational expressions $\frac{P}{Q}$ and $\frac{R}{S}$, then

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S} \; (Q, S \neq 0)$$

Since we want the resulting product to be stated in lowest terms, we apply the *fundamental principle of rational expressions* and divide both the numerator and the denominator by their common factors. That is, we *reduce* by dividing out the common factors.

Multiplication of rational expressions

1. State the numerators and denominators as indicated products. (Do not multiply.)
2. Factor the numerator and the denominator.
3. Divide the numerators and the denominators by the factors that are common.
4. Multiply the remaining factors in the numerator and place this product over the product of the remaining factors in the denominator.

EXAMPLES

Perform the indicated multiplication and simplify your answer. Assume that no denominator equals zero.

1. $\frac{3x}{4} \cdot \frac{5}{2y}$

$= \frac{3x \cdot 5}{4 \cdot 2y}$ Multiply numerators and denominators

$= \frac{15x}{8y}$ Will not reduce

2. $\frac{x+1}{x-3} \cdot \frac{4}{x+2}$

$= \frac{(x+1) \cdot 4}{(x-3)(x+2)}$ Multiply numerators and denominators

$= \frac{4x+4}{x^2-x-6}$ Will not reduce

3. $\dfrac{4}{9x} \cdot \dfrac{3x^2}{2}$

$$= \frac{4 \cdot 3x^2}{9x \cdot 2}$$ Multiply numerators / Multiply denominators

$$= \frac{2 \cdot 2 \cdot 3 \cdot x \cdot x}{3 \cdot 3 \cdot 2 \cdot x}$$ Factor numerator and denominator

$$= \frac{2 \cdot x \cdot (2 \cdot 3 \cdot x)}{3 \cdot (2 \cdot 3 \cdot x)}$$ Identify common factors

$$= \frac{2x}{3}$$ Divide numerator and denominator by $(2 \cdot 3 \cdot x)$

4. $\dfrac{x+1}{3-x} \cdot \dfrac{(x-3)^2}{x-2}$

$$= \frac{(x+1) \cdot (x-3)^2}{(3-x) \cdot (x-2)}$$ Multiply numerators / Multiply denominators

$$= \frac{x-3}{3-x} \cdot \frac{(x+1)(x-3)}{x-2}$$ Factor opposites

$$= \frac{-1 \cdot (x+1)(x-3)}{x-2}$$ $\dfrac{x-3}{3-x} = -1$

$$= \frac{-1(x^2-2x-3)}{x-2}$$ Multiply as indicated

$$= \frac{-x^2+2x+3}{x-2}$$

5. $\dfrac{x^2-8x+16}{x^2+3x-10} \cdot \dfrac{x^2-4}{x^2-5x+4}$

$$= \frac{(x^2-8x+16)(x^2-4)}{(x^2+3x-10)(x^2-5x+4)}$$ Multiply numerators and denominators

$$= \frac{(x-4)(x-4)}{(x-4)(x-2)} \frac{(x-2)(x+2)}{(x+5)(x-1)}$$ Factor numerator and denominator

$$= \frac{(x-4)(x+2)}{(x+5)(x-1)}$$ Reduce by common factors $(x-4)$ and $(x-2)$

$$= \frac{x^2-2x-8}{x^2+4x-5}$$ Multiply remaining factors

► ***Quick check*** Multiply $\dfrac{12}{5y} \cdot \dfrac{15y^2}{4}$ ■

Division of rational expressions

Recall that to divide two fractions $\frac{a}{b}$ and $\frac{c}{d}$, we multiply $\frac{a}{b}$ by the *reciprocal* of $\frac{c}{d}$, which is $\frac{d}{c}$.

Division property of fractions

If a, b, c, and d are real numbers, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} \quad (b, c, d \neq 0)$$

EXAMPLE

Find the indicated quotient. Reduce your answer to lowest terms.

$$\frac{18}{25} \div \frac{9}{5}$$

$= \frac{18}{25} \cdot \frac{5}{9}$ Multiply by the reciprocal of $\frac{9}{5}$

$= \frac{2 \cdot 3 \cdot 3 \cdot 5}{5 \cdot 5 \cdot 3 \cdot 3}$ Factor in numerator and denominator

$= \frac{2 \cdot (3 \cdot 3 \cdot 5)}{5 \cdot (3 \cdot 3 \cdot 5)}$ Group the common factors $(3 \cdot 3 \cdot 5)$

$= \frac{2}{5}$ Reduce by the common factors $(3 \cdot 3 \cdot 5)$

Division of rational expressions is done in the same way.

Division property of rational expressions

If $\frac{P}{Q}$ and $\frac{R}{S}$ are rational expressions, then

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{P \cdot S}{Q \cdot R} \quad (Q, R, S \neq 0)$$

Notice that once the operation of division has been changed to multiplication, we proceed exactly as we did with the multiplication of rational expressions.

Division of rational expressions

1. Multiply the first rational expression by the reciprocal of the second.
2. Proceed as in the multiplication of rational expressions.

EXAMPLES

Find the indicated quotients. Express the answer in reduced form.

1. $\dfrac{3ab}{5} \div \dfrac{9abc}{10}$

$= \dfrac{3ab}{5} \cdot \dfrac{10}{9abc}$ — Multiply by the reciprocal of $\dfrac{9abc}{10}$

$= \dfrac{3ab \cdot 2 \cdot 5}{5 \cdot 3 \cdot 3 \cdot abc}$ — Factor numerator and denominator

$= \dfrac{2 \cdot (3 \cdot 5 \cdot ab)}{3 \cdot c \cdot (3 \cdot 5 \cdot ab)}$ — Group common factors $(3 \cdot 5 \cdot ab)$

$= \dfrac{2}{3c}$ — Divide numerator and denominator by common factors $(3 \cdot 5 \cdot ab)$

2. $\dfrac{x^2 - 4}{5} \div \dfrac{x - 2}{15}$

$= \dfrac{x^2 - 4}{5} \cdot \dfrac{15}{x - 2}$ — Multiply by the reciprocal of $\dfrac{x-2}{15}$

$= \dfrac{(x - 2)(x + 2) \cdot 3 \cdot 5}{5 \cdot (x - 2)}$ — Factor numerator and denominator

$= \dfrac{3 \cdot (x + 2) \cdot 5(x - 2)}{5(x - 2)}$ — Locate common factors $5(x - 2)$

$= \dfrac{3(x + 2)}{1} = 3x + 6$ — Reduce to lowest terms by dividing numerator and denominator by $5(x - 2)$

3. $\dfrac{4x+2}{x-1} \div \dfrac{2x+1}{4-4x}$

$$= \frac{4x+2}{x-1} \cdot \frac{4-4x}{2x+1}$$

Multiply by the reciprocal of $\dfrac{2x+1}{4-4x}$

$$= \frac{2(2x+1)(-4)(x-1)}{(x-1)(2x+1)}$$

Factor $4 - 4x = -4(x-1)$ and $4x + 2 = 2(2x+1)$

$$= \frac{2(-4)(2x+1)(x-1)}{(2x+1)(x-1)}$$

Reduce by $(x-1)(2x+1)$

$$= 2(-4)$$
$$= -8$$

4. $\dfrac{x^2-9}{2x+1} \div \dfrac{3-x}{2x^2+7x+3}$

$$= \frac{x^2-9}{2x+1} \cdot \frac{2x^2+7x+3}{3-x}$$

Multiply by the reciprocal of $\dfrac{3-x}{2x^2+7x+3}$

$$= \frac{(x-3)(x+3) \cdot (2x+1)(x+3)}{(2x+1)(3-x)}$$

Factor numerators

$$= \frac{x-3}{3-x} \cdot \frac{(2x+1)(x+3)(x+3)}{2x+1}$$

Factor opposites

$$= -1 \cdot (x+3)(x+3)$$

$\dfrac{x-3}{3-x} = -1$; reduce by $(2x+1)$

$$= -1(x^2+6x+9)$$

Multiply as indicated

$$= -x^2 - 6x - 9$$

Multiply by -1

EXERCISES

Find the indicated product or quotient. Write your answer in simplest form. Assume all denominators are non-zero.

1. $\frac{24}{35}\cdot\frac{7}{8}$
2. $\frac{3}{8}\cdot\frac{5}{9}$
3. $\frac{7}{10}\div\frac{21}{25}$
4. $\frac{56}{39}\div\frac{8}{13}$
5. $\frac{4a}{5}\cdot\frac{5}{2}$
6. $\frac{16b}{7a}\cdot\frac{5}{4}$
7. $\frac{14}{3a}\div\frac{7}{15a}$
8. $\frac{6x}{5y}\div\frac{21x}{15y}$
9. $\frac{5}{6}\cdot\frac{3x}{10y}$
10. $\frac{7a}{12b}\cdot\frac{9b}{28}$
11. $\frac{9x^2}{8}\cdot\frac{4}{6x}$
12. $\frac{36p^2}{7q}\cdot\frac{14q^2}{28p^3}$
13. $\frac{24a}{35x}\div 6a$
14. $\frac{14y}{23x}\div 7y$
15. $6a\div\frac{24a}{35x}$
16. $7y\div\frac{14y}{23}$
17. $\frac{21ab}{16c}\cdot\frac{8c^2}{3ab^2}$
18. $\frac{18x^2y^2}{5ab}\cdot\frac{25a^2b}{12xy}$
19. $\frac{5x^2}{9y^3}\div\frac{20x}{6y}$
20. $\frac{28m}{15n}\div\frac{7m^2}{3n^3}$
21. $\frac{24abc}{7xyz^2}\cdot\frac{14x^2yz}{9a^2}$
22. $\frac{80x^2yz^3}{11mn^2}\cdot\frac{33mn^2}{25xyz}$
23. $\frac{3ab}{8x^2}\div\frac{15b^3}{16x}$
24. $\frac{20mn^3}{9x^2}\div\frac{4mn}{3xy^2}$
25. $\frac{x+y}{3}\cdot\frac{12}{(x+y)^2}$
26. $\frac{5(a-b)}{8}\cdot\frac{12}{10(a-b)}$
27. $\frac{9-p}{7}\div\frac{4(p-9)}{21}$
28. $\frac{4x-2}{15}\div\frac{1-2x}{27}$
29. $\frac{3b-6}{4b+8}\cdot\frac{5b+10}{2-b}$
30. $\frac{8y+16}{3-y}\cdot\frac{4y-12}{3y+6}$
31. $\frac{4a+12}{a-5}\div(a+3)$
32. $\frac{9-3z}{2z+8}\div(6-2z)$

33. $(x^2-4x+4)\cdot\frac{18}{x^2-4}$
34. $\frac{21}{a^2-9}\cdot(a^2+a-12)$
35. $\frac{x^2-4}{25y}\cdot\frac{24y^2}{x+2}$
36. $\frac{16a^2}{b^2-9}\cdot\frac{b-3}{12a^2}$
37. $\frac{r^2-16}{r+1}\div\frac{r+4}{r^2-1}$
38. $\frac{p^2+2p+1}{4p-1}\div\frac{p^2-1}{16p^2-1}$
39. $\frac{9-x^2}{x+y}\cdot\frac{4x+4y}{x-3}$
40. $\frac{b^2-a^2}{2a+4b}\cdot\frac{a+b}{a-b}$
41. $\frac{a^2-5a+6}{a^2-9a+20}\cdot\frac{a^2-5a+4}{a^2-3a+2}$
42. $\frac{a^2-5a-14}{a^2-9a-36}\cdot\frac{a^2+10a+21}{a^2+4a-77}$
43. $\frac{x^2-2x-3}{x^2+3x-4}\div\frac{x^2-x-6}{x^2+x-12}$
44. $\frac{y^2+3y+2}{y^2+5y+4}\div\frac{y^2+5y+6}{y^2+10y+24}$
45. $\frac{2x^2-15x+7}{x^2-9x+8}\cdot\frac{x^2-2x+1}{x^2-49}$

46. $\dfrac{4x^2 - 4}{3x^2 - 13x - 10} \cdot \dfrac{x^2 - 6x + 5}{4x + 4}$

$\boxed{47.}$ $\dfrac{6r^2 - r - 7}{12r^2 + 16r - 35} \div \dfrac{r^2 - r - 2}{2r^2 + r - 10}$

48. $\dfrac{4x^2 - 9}{x^2 - 9x + 18} \div \dfrac{2x^2 - 5x - 12}{x^2 - 10x + 24}$

$\boxed{49.}$ $(3x^2 - 2x - 8) \div \dfrac{x^2 - 4}{x + 2}$

50. $(8a^2 - 16a) \div \dfrac{a^3 - 16a}{a - 4}$

51. $\dfrac{3x - 4}{2x + 1} \div (6x^2 - 5x - 4)$

52. $\dfrac{m^2 - 3m - 10}{m^2 - 4} \div (2m^2 - 9m - 5)$

$\boxed{53.}$ $\dfrac{10}{a^3 - 27} \cdot \dfrac{a^2 + 3a - 18}{15}$

54. $\dfrac{x^3 - 8}{16} \cdot \dfrac{24}{x^2 + 2x - 8}$

55. $\dfrac{z^2 - 5z - 14}{z - 4} \div \dfrac{5z^3 + 40}{z^2 - z - 12}$

56. $\dfrac{3b^3 + 3}{b - 2} \div \dfrac{b^2 + 2b + 1}{b^2 + 6b - 16}$

57. $\dfrac{y^2 + 8y + 16}{y + 4} \cdot \dfrac{y^2 - 25}{y^2 + 9y + 20} \cdot \dfrac{y^2 + 5y}{y^2 - 5y}$

58. $\dfrac{6m^2 - 7m + 2}{6m^2 + 5m + 1} \cdot \dfrac{2m^2 + m}{4m^2 - 1} \cdot \dfrac{12m^2 - 5m - 3}{12m^2 - 17m + 6}$

Addition and subtraction of rational expressions

Recall that to add or subtract fractions having the same denominator, we add, or subtract, the numerators and place this sum, or difference, over the same denominator.

Addition and subtraction properties for fractions

If a, b, and c are real numbers, $b \neq 0$, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

EXAMPLE

Add or subtract as indicated.

1. $\dfrac{3}{11} + \dfrac{4}{11} = \dfrac{3+4}{11}$ Add numerators

$= \dfrac{7}{11}$ $3 + 4 = 7$

2. $\dfrac{3}{7} - \dfrac{1}{7} = \dfrac{3-1}{7}$ Subtract numerators

$= \dfrac{2}{7}$ $3 - 1 = 2$

We use the following similar procedure to add or subtract rational expressions.

Addition and subtraction properties for rational expressions

If $\dfrac{P}{R}$ and $\dfrac{Q}{R}$ are rational expressions, $R \neq 0$, then

$$\frac{P}{R} + \frac{Q}{R} = \frac{P+Q}{R} \quad \text{and} \quad \frac{P}{R} - \frac{Q}{R} = \frac{P-Q}{R}$$

Note Rational expressions having common denominators are called *like* rational expressions.

Addition and subtraction of like rational expressions

1. Add or subtract the numerators.
2. Place the sum or difference over the common denominator.
3. Reduce the resulting rational expression to lowest terms.

EXAMPLES

Find the indicated sum or difference. Assume all denominators are nonzero.

1. $\dfrac{3}{x-2} + \dfrac{5}{x-2} = \dfrac{3+5}{x-2}$ Add numerators and place over $x - 2$

$= \dfrac{8}{x-2}$ $3 + 5 = 8$

2. $\dfrac{5y}{3y+5} - \dfrac{9y}{3y+5} = \dfrac{5y-9y}{3y+5}$ Subtract numerators and place over $3y + 5$

$= \dfrac{-4y}{3y+5}$ $5y - 9y = -4y$

3. $\dfrac{2x-1}{x^2+5x+6} - \dfrac{4-x}{x^2+5x+6}$

$= \dfrac{(2x-1)-(4-x)}{x^2+5x+6}$ Place numerators in parentheses and subtract

$= \dfrac{2x-1-4+x}{x^2+5x+6}$ Remove parentheses and subtract

$= \dfrac{3x-5}{x^2+5x+6}$ Combine like terms

Note Notice that when we subtracted $4 - x$ from $2x - 1$, we placed parentheses around each polynomial. This step is *important* to avoid the common mistake of failing to change signs in the second expression when subtraction is involved.

4. $\dfrac{2x-1}{x^2+5x+6}+\dfrac{4-x}{x^2+5x+6}$

$= \dfrac{(2x-1)+(4-x)}{x^2+5x+6}$ Place numerators in parentheses and add

$= \dfrac{x+3}{x^2+5x+6}$ Remove parentheses and combine like terms

$= \dfrac{x+3}{(x+3)(x+2)}$ Factor denominator

$= \dfrac{1}{x+2}$ Reduce by common factor $x+3$

Note In the last step, *always* look for a possible reduction to lowest terms as we did in example 4.

When one denominator is the opposite of the other, as in the indicated sum

$$\frac{2x}{3}+\frac{5}{-3},$$

where 3 and -3 are opposites, we first multiply one of the expressions by $\dfrac{-1}{-1}$ to obtain equivalent expressions with the same denominator.

EXAMPLES

Find the indicated sum or difference. Assume all denominators are not zero.

1. $\dfrac{2x}{3}+\dfrac{5}{-3}=\dfrac{2x}{3}+\dfrac{-1}{-1}\cdot\dfrac{5}{-3}$ Multiply $\dfrac{5}{-3}$ by $\dfrac{-1}{-1}$

$= \dfrac{2x}{3}+\dfrac{-1(5)}{-1(-3)}$ Multiply numerators and denominators

$= \dfrac{2x}{3}+\dfrac{-5}{3}$ Common denominator of 3

$= \dfrac{2x+(-5)}{3}$ Add numerators and place over 3

$= \dfrac{2x-5}{3}$ Definition of subtraction

2. $\dfrac{5y-1}{y-4}+\dfrac{2y+3}{4-y}$

$= \dfrac{5y-1}{y-4}+\dfrac{-1}{-1}\cdot\dfrac{2y+3}{4-y}$ Multiply $\dfrac{2y+3}{4-y}$ by $\dfrac{-1}{-1}$

$= \dfrac{5y-1}{y-4}+\dfrac{-1(2y+3)}{-1(4-y)}$

$= \dfrac{5y-1}{y-4}+\dfrac{-2y-3}{y-4}$ Same denominator: $-1(4-y)=y-4$

$= \dfrac{(5y-1)+(-2y-3)}{y-4}$ Add numerators in parentheses

$= \dfrac{5y-1-2y-3}{y-4}$ Remove parentheses

$= \dfrac{3y-4}{y-4}$ Combine like terms

Note We could have multiplied $\dfrac{5y-1}{y-4}$ by $\dfrac{-1}{-1}$. The resulting denominator would then have been $4-y$ and the numerator would have been $4-3y$. We would then multiply this by $\dfrac{-1}{-1}$ to obtain the same form of the answer.

3. $\dfrac{2x+1}{x-5}-\dfrac{x-4}{5-x}$

$= \dfrac{2x+1}{x-5}-\dfrac{-1}{-1}\cdot\dfrac{x-4}{5-x}$ Multiply $\dfrac{x-4}{5-x}$ by $\dfrac{-1}{-1}$

$= \dfrac{2x+1}{x-5}-\dfrac{-1(x-4)}{-1(5-x)}$

$= \dfrac{2x+1}{x-5}-\dfrac{4-x}{x-5}$ Same denominator: $-1(x-4)=4-x$, $-1(5-x)=x-5$

$= \dfrac{(2x+1)-(4-x)}{x-5}$ Place "( )" around numerators and subtract

$= \dfrac{2x+1-4+x}{x-5}$ Definition of subtraction

$= \dfrac{3x-3}{x-5}$ Combine like terms

The least common denominator (LCD)

If the fractions to be added or subtracted do not have the same denominator, we must change at least one of the fractions to an equivalent fraction so the fractions do have a common denominator. There are many such numbers we could use as a common denominator. However, the most convenient denominator to use is the smallest (least) number that is exactly divisible by each of the denominators—called the **least common denominator,** denoted by LCD. For example, the least common denominator (LCD) of the two fractions

$$\frac{5}{6} \text{ and } \frac{2}{9}$$

is 18, since 18 is the smallest (least) number that is exactly divisible by both 6 and 9.

Finding the LCD of a set of denominators

1. Factor each denominator completely. Write each factorization using exponential notation.
2. List each *different* factor that appears in any one of the factorizations in step 1.
3. Raise each factor of step 2 to the *greatest* power that factor has in step 1. Form the product of these factors.

Note The LCD of two or more rational expressions is also called the least common multiple (LCM) of the denominators.

EXAMPLE

Find the LCD of rational expressions having the given denominators.

1. 6 and 9

$$\left.\begin{aligned} 6 &= 2 \cdot 3 \\ 9 &= 3 \cdot 3 = 3^2 \end{aligned}\right\} \quad \text{Factor each denominator}$$

The different factors are 2 and 3. The greatest power of 2 is 2^1 and of 3 is 3^2. The LCD is $2^1 \cdot 3^2 = 2 \cdot 9 = 18$.

2. $16a$ and $8a^3$

$$16a = 2 \cdot 2 \cdot 2 \cdot 2 \cdot a = 2^4 \cdot a$$
$$8a^3 = 2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot a = 2^3 \cdot a^3$$

Since the different factors are 2 and a, the greatest power of 2 is 2^4, and the greatest power of a is a^3, the LCD is $2^4 \cdot a^3 = 16a^3$.

3. $50x^3y^2$ and $20x^2y^3$

$$50x^3y^2 = 2 \cdot 5 \cdot 5 \cdot x^3 \cdot y^2 = 2 \cdot 5^2 \cdot x^3 \cdot y^2$$
$$20x^2y^3 = 2 \cdot 2 \cdot 5 \cdot x^2 \cdot y^3 = 2^2 \cdot 5 \cdot x^2 \cdot y^3$$

The different factors are 2, 5, x, and y. Since the greatest power of 2 is 2^2, of 5 is 5^2, of x is x^3, and of y is y^3, the LCD is $2^2 \cdot 5^2 \cdot x^3 \cdot y^3 = 100x^3y^3$.

4. $x^2 + x - 12$ and $x^2 + 2x - 8$

$$x^2 + x - 12 = (x + 4)(x - 3)$$
$$x^2 + 4x - 8 = (x + 4)(x - 2)$$

The different factors are $x + 4$, $x - 3$, and $x - 2$.
Each factor is carried to the first power so the LCD is $(x + 4)(x - 3)(x - 2)$.

5. $x^2 - 2x + 1$, $x^2 + 11x - 12$, and $1 - x$

$$x^2 - 2x + 1 = (x - 1)^2$$
$$x^2 + 11x - 12 = (x - 1)(x + 12)$$
$$1 - x = -1(x - 1)$$

The different factors are $x - 1$ and $x + 12$. Since the greatest power of $x - 1$ is $(x - 1)^2$ and of $x + 12$ is $(x + 12)^1$, the LCD is $(x - 1)^2(x + 12)$.

Note Factors $x - 1$ and $1 - x$ are opposites and we will change $1 - x$ to $-1(x - 1)$ when working with the denominators. The factor -1 is then carried along in the problem.

EXERCISES

Combine the given rational expressions and reduce the answer to lowest terms. Assume all denominators are nonzero.

1. $\frac{5}{x}+\frac{3}{x}$
2. $\frac{8}{y^2}+\frac{10}{y^2}$
3. $\frac{9}{p}-\frac{2}{p}$
4. $\frac{18}{m^2}-\frac{5}{m^2}$

5. $\frac{5x}{x+2}+\frac{9x}{x+2}$
6. $\frac{8y}{y-1}+\frac{-3y}{y-1}$
7. $\frac{x-1}{2x}-\frac{x+3}{2x}$
8. $\frac{3y-2}{y^2}-\frac{4y-1}{y^2}$
9. $\frac{3x+5}{x^2-1}-\frac{2x+3}{x^2-1}$
10. $\frac{b^2+2}{b+3}-\frac{b^2+2b-3}{b+3}$

11. $\frac{5}{7}+\frac{6}{-7}$
12. $\frac{9}{10}-\frac{3}{-10}$
13. $\frac{4}{z}-\frac{5}{-z}$
14. $\frac{6}{y}+\frac{9}{-y}$
15. $\frac{5}{x-2}+\frac{12}{2-x}$
16. $\frac{1}{x-7}-\frac{5}{7-x}$
17. $\frac{5y}{y-6}-\frac{4y}{6-y}$
18. $\frac{4z}{z-3}+\frac{z}{3-z}$
19. $\frac{x+1}{x-5}+\frac{2x-3}{5-x}$
20. $\frac{4y+3}{y-9}-\frac{2y-7}{9-y}$
21. $\frac{2y-5}{2y-3}-\frac{y+7}{3-2y}$
22. $\frac{z+5}{4z-3}+\frac{4z-1}{3-4z}$
23. $\frac{2x+5}{5-2x}+\frac{x+9}{2x-5}$
24. $\frac{5-y}{6-5y}-\frac{9y+1}{5y-6}$

Find the least common denominator (LCD) of rational expressions having the following denominators.

25. $6x$ and $9x$
26. $8a$ and $12a$
27. $16x^2$ and $24x$
28. $6b^2$ and $14b$
29. $28y^2$ and $35y^3$
30. $9z^3$ and $7z^4$
31. $32a^2$ and $64a^4$
32. $4x^2$, $3x$, and $8x^3$
33. $10a^2$, $12a^3$, and $9a$
34. $4x-2$ and $2x-1$
35. $x-4$ and $3x-12$
36. $6x-12$ and $9x-18$

37. $18y^3$ and $9y - 36$

38. $32z^2$ and $16z - 32$

39. $a^2 + a$ and $a^2 - 1$

40. $(z - 1)^2$ and $z^2 - 1$

41. $8a + 16$ and $a^2 + 3a + 2$

42. $9p - 18$ and $p^2 - 7p + 10$

43. $a^2 - 5a + 6$ and $a^2 - 4$

44. $y^2 - y - 12$ and $y^2 + 6y + 9$

45. $a^2 - 9$ and $a^2 - 5a + 6$

46. $p^2 - 9$, $p^2 + p - 6$, and $p^2 - 4p + 4$

47. $x^2 - 49$, $7 - x$, and $2x + 14$

48. $5 - y$, $y^2 - 25$, and $y^2 - 10y + 25$

Addition and subtraction of rational expressions

Now that we can find the least common denominator (LCD) of a group of rational expressions, let us review the process for changing a fraction (or rational expression) to an equivalent fraction with a new denominator.

EXAMPLES

1. Change $\frac{7}{15}$ to an equivalent fraction having denominator 60.

We want $\frac{7}{15} = \frac{?}{60}$.

Since $60 = 15 \cdot 4$ (factor 4 is found by dividing $60 \div 15 = 4$), we multiply the given fraction by $\frac{4}{4}\left(\frac{4}{4} = 1\right)$.

$$\frac{7}{15} = \frac{7}{15} \cdot \frac{4}{4}$$

$$= \frac{7 \cdot 4}{15 \cdot 4} \quad \begin{array}{l}\text{Multiply numerators} \\ \text{Multiply denominators}\end{array}$$

$$= \frac{28}{60}$$

Thus, $\frac{7}{15} = \frac{28}{60}$.

2. Change $\frac{x+1}{x-4}$ to an equivalent rational expression having denominator $x^2 - 2x - 8$.

We want $\frac{x+1}{x-4} = \frac{?}{x^2 - 2x - 8}$.

Since $x^2 - 2x - 8 = (x - 4)(x + 2)$, we multiply the given rational expression by $\frac{x+2}{x+2}$.

$$\frac{x+1}{x-4} = \frac{x+1}{x-4} \cdot \frac{x+2}{x+2}$$

$$= \frac{(x+1)(x+2)}{(x-4)(x+2)} \quad \text{Multiply numerators / Multiply denominators}$$

$$= \frac{x^2+3x+2}{x^2-2x-8} \quad \text{Perform indicated operations}$$

Thus, $\frac{x+1}{x-4} = \frac{x^2+3x+2}{x^2-2x-8}$.

Once equivalent rational expressions are obtained with the LCD as the denominator, we add or subtract as previously learned. Use the following steps to add or subtract rational expressions having unlike denominators.

Addition and subtraction of rational expressions having different denominators

1. Find the LCD of the rational expressions.
2. Write each rational expression as an equivalent rational expression with the LCD as the denominator.
3. Perform the indicated addition or subtraction as before.
4. Reduce the results to lowest terms.

EXAMPLES

Add the following rational expressions. Assume the denominators are not equal to zero. Reduce all answers to lowest terms.

1. $\frac{5x}{8} + \frac{7x}{12}$

$$\left.\begin{aligned} 8 &= 2 \cdot 2 \cdot 2 \\ 12 &= 2 \cdot 2 \cdot 3 \end{aligned}\right\} \text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 = 24 \quad \text{Find the LCD}$$

Since $\frac{24}{8} = 3$ and $\frac{24}{12} = 2$

$$\frac{5x}{8} + \frac{7x}{12} = \frac{5x}{8} \cdot \frac{3}{3} + \frac{7x}{12} \cdot \frac{2}{2} \quad \text{Multiply } \frac{5x}{8} \text{ by } \frac{3}{3} \text{ and } \frac{7x}{12} \text{ by } \frac{2}{2}$$

$$= \frac{15x}{24} + \frac{14x}{24} \quad \text{Multiply numerators and denominators}$$

$$= \frac{15x+14x}{24} \quad \text{Add numerators}$$

$$= \frac{29x}{24} \quad \text{Combine as indicated}$$

2. $\dfrac{15}{4x^2} + \dfrac{25}{18x}$

$$\left.\begin{aligned} 4x^2 &= 2 \cdot 2 \cdot x^2 \\ 18x &= 2 \cdot 3 \cdot 3 \cdot x \end{aligned}\right\} \text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3 \cdot x^2 = 36x^2$$

Find the LCD

Since $\dfrac{36x^2}{4x^2} = 9$ and $\dfrac{36x^2}{18x} = 2x$

$$\begin{aligned} \frac{15}{4x^2} + \frac{25}{18x} &= \frac{15}{4x^2} \cdot \frac{9}{9} + \frac{25}{18x} \cdot \frac{2x}{2x} \\ &= \frac{135}{36x^2} + \frac{50x}{36x^2} \\ &= \frac{135 + 50x}{36x^2} \end{aligned}$$

Multiply $\frac{15}{4x^2}$ by $\frac{9}{9}$ and $\frac{25}{18x}$ by $\frac{2x}{2x}$

Multiply numerators and denominators

Add numerators

3. $\dfrac{3y + 2}{y^2 - 16} + \dfrac{y - 4}{3y + 12}$

$$\left.\begin{aligned} y^2 - 16 &= (y + 4)(y - 4) \\ 3y + 12 &= 3(y + 4) \end{aligned}\right\} \text{LCD} = 3(y + 4)(y - 4)$$

Find the LCD

Since $\dfrac{3(y + 4)(y - 4)}{(y + 4)(y - 4)} = 3$ and $\dfrac{3(y + 4)(y - 4)}{3(y + 4)} = y - 4$, then

$$\begin{aligned} &\frac{3y + 2}{y^2 - 16} + \frac{y - 4}{3y + 12} \\ &= \frac{3y + 2}{(y + 4)(y - 4)} \cdot \frac{3}{3} + \frac{y - 4}{3(y + 4)} \cdot \frac{y - 4}{y - 4} \\ &= \frac{3(3y + 2)}{3(y + 4)(y - 4)} + \frac{(y - 4)(y - 4)}{3(y + 4)(y - 4)} \\ &= \frac{(9y + 6) + (y^2 - 8y + 16)}{3(y + 4)(y - 4)} \\ &= \frac{y^2 + y + 22}{3(y + 4)(y - 4)} \end{aligned}$$

Multiply $\frac{3y + 2}{(y + 4)(y - 4)}$ by $\frac{3}{3}$ and $\frac{y - 4}{3(y + 4)}$ by $\frac{y - 4}{y - 4}$

Multiply numerators and denominators

Add numerators

Remove parentheses and combine

Note When the numerators have two or more terms as in example 3, we place the quantities in parentheses when we add the numerators. This is a good practice to avoid a *most common* mistake when subtracting.

EXAMPLES

Subtract the following rational expressions. Assume the denominators are not equal to zero. Reduce to lowest terms.

1. $\dfrac{5y}{2y-1} - \dfrac{y+1}{y+2}$

The LCD of the rational expressions is $(2y-1)(y+2)$.

$$\frac{5y}{2y-1} - \frac{y+1}{y+2}$$

$$= \frac{5y}{2y-1}\cdot\frac{y+2}{y+2} - \frac{y+1}{y+2}\cdot\frac{2y-1}{2y-1}$$

Multiply $\dfrac{5y}{2y-1}$ by $\dfrac{y+2}{y+2}$ and $\dfrac{y+1}{y+2}$ by $\dfrac{2y-1}{2y-1}$

$$= \frac{5y(y+2)}{(2y-1)(y+2)} - \frac{(y+1)(2y-1)}{(2y-1)(y+2)}$$

Multiply numerators and denominators

$$= \frac{(5y^2+10y)-(2y^2+y-1)}{(2y-1)(y+2)}$$

Subtract numerators

$$= \frac{5y^2+10y-2y^2-y+1}{(2y-1)(y+2)}$$

Remove parentheses and change signs when subtracting

$$= \frac{3y^2+9y+1}{(2y-1)(y+2)}$$

Combine like terms

Don't forget the parentheses

Note The numerator $3y^2+9y+1$ cannot be factored so we are unable to reduce. We should always check this!

2. $\dfrac{5x-4}{x^2-2x+1} - \dfrac{3x}{x^2+4x-5}$

$$\left.\begin{aligned} x^2-2x+1 &= (x-1)^2 \\ x^2+4x-5 &= (x+5)(x-1) \end{aligned}\right\} \text{LCD} = (x-1)^2(x+5)$$

Find the LCD

$$\frac{5x-4}{x^2-2x+1} - \frac{3x}{x^2+4x-5}$$

Factor denominators

$$= \frac{5x-4}{(x-1)^2} - \frac{3x}{(x+5)(x-1)}$$

$$= \frac{5x-4}{(x-1)^2}\cdot\frac{x+5}{x+5} - \frac{3x}{(x+5)(x-1)}\cdot\frac{x-1}{x-1}$$

Multiply $\dfrac{5x-4}{(x-1)^2}$ by $\dfrac{x+5}{x+5}$ and $\dfrac{3x}{(x+5)(x-1)}$ by $\dfrac{x-1}{x-1}$

$$= \frac{(5x-4)(x+5)}{(x-1)^2(x+5)} - \frac{3x(x-1)}{(x-1)^2(x+5)}$$

Multiply numerators and denominators

$$= \frac{(5x^2+21x-20)-(3x^2-3x)}{(x-1)^2(x+5)}$$

Subtract numerators

$$= \frac{5x^2+21x-20-3x^2+3x}{(x-1)^2(x+5)}$$

Remove parentheses and change signs

$$= \frac{2x^2+24x-20}{(x-1)^2(x+5)}$$

Combine like terms

Don't forget the parentheses

EXERCISES

Perform the indicated addition and reduce the answer to lowest terms. Assume all denominators are not equal to zero.

1. $\frac{x}{6} + \frac{3}{4}$
2. $\frac{3z}{10} + \frac{2z}{15}$
3. $\frac{2x-1}{16} + \frac{x+2}{24}$
4. $\frac{4}{3x} + \frac{5}{2x}$
5. $\frac{3a+1}{a} + \frac{2a-3}{3a}$
6. $\frac{4}{x-1} + \frac{5}{x+3}$
7. $\frac{8}{y+4} + \frac{7}{y-5}$
8. $\frac{x}{x+2} + \frac{3x}{4x-1}$
9. $\frac{15}{5y-10} + \frac{14}{2y-4}$
10. $\frac{21}{6x+12} + \frac{15}{2x+4}$
11. $\frac{12}{x^2-4} + \frac{7}{4x-8}$
12. $\frac{16}{2y+6} + \frac{5}{y^2-9}$
13. $5 + \frac{4x}{x+8}$
14. $9 + \frac{y+9}{y-1}$
15. $\frac{x}{x-1} + \frac{3x}{x^2-1}$
16. $\frac{2y}{y^2-16} + \frac{5y}{2y-8}$
17. $\frac{4}{x^2-x-6} + \frac{5}{x^2-9}$
18. $\frac{6}{x^2-4x-12} + \frac{5}{x^2-36}$
19. $\frac{2y}{y^2-6y+9} + \frac{5y}{y^2-2y-3}$
20. $\frac{4z}{z^2+z-20} + \frac{z}{z^2-8z+16}$
21. $\frac{y-2}{y^2-3y-10} + \frac{y+1}{y^2-y-6}$
22. $\frac{2x+1}{x^2+6x+5} + \frac{4x-3}{x^2-x-30}$

Perform the indicated subtraction and reduce to lowest terms. Assume all denominators are not zero.

23. $\frac{y}{9} - \frac{5}{6}$
24. $\frac{5y}{12} - \frac{y}{8}$
25. $\frac{9}{14y} - \frac{1}{21y}$
26. $\frac{7}{12z} - \frac{10}{9z}$
27. $\frac{5a+3}{12} - \frac{a-4}{10}$
28. $\frac{2x+9}{8} - \frac{x-7}{20}$
29. $\frac{2x+5}{6x} - \frac{x-5}{9x}$
30. $\frac{4y-1}{3y} - \frac{2y-3}{15y}$
31. $\frac{7}{2x-3} - \frac{6}{x-5}$
32. $\frac{7}{4x-6} - \frac{12}{3x+9}$
33. $\frac{12}{3y+6} - \frac{11}{7y+14}$
34. $\frac{14}{5x-15} - \frac{8}{2x-6}$
35. $9 - \frac{6}{x+8}$
36. $12 - \frac{7}{z-12}$
37. $\frac{2x}{3x+1} - 9$
38. $\frac{4y}{5y-4} - 10$
39. $\frac{-3}{a^2-5a+6} - \frac{3}{a^2-4}$
40. $\frac{8}{x^2-25} - \frac{7}{x^2+3x-10}$
41. $\frac{20}{y^2-2y-24} - \frac{8}{y^2+y-12}$
42. $\frac{2p}{p^2-9p+20} - \frac{5p-2}{p-5}$
43. $\frac{2a-3}{a^2-5a+6} - \frac{3a}{a-2}$

Add and subtract as indicated.

44. $\frac{13}{12b} - \frac{2}{9b} + \frac{5}{4b}$

45. $\frac{5}{6z} + \frac{4}{8z} - \frac{1}{4z}$

46. $\frac{3x}{8} - \frac{2x}{5} + \frac{7x}{10}$

47. $\frac{4a}{5} + \frac{7a}{15} - \frac{a}{9}$

48. $\frac{a-1}{5} - \frac{2a+3}{15} + \frac{3a-1}{25}$

49. $\frac{5b+1}{6} + \frac{3b-2}{9} - \frac{b+1}{12}$

50. $\frac{4a}{a^2+2a-15} + \frac{3a}{2a^2+11a+5} - \frac{5a}{2a^2-5a-3}$

51. $\frac{5}{z^2-4} - \frac{z}{z^2-1} + \frac{4}{z^2+z-2}$

53. Women A, B, and C can complete a given job in a, b, and c hours, respectively. Working together they can complete in one hour $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ of the job. By combining, obtain a single expression for what they can do together in one hour.