Discussion 15 – Tuesday, October 12th

topic: Chain Rule

1. Let $f(u, v, w) = u^2 + v^3 e^w$, where $u = \sin(x + y + z)$, $v = x^2 e^y$ and $w = z$. Using the Chain Rule, compute the partial derivative $f_z(1, -1, \pi)$.
Solution:

$$\begin{aligned} f_z(u, v, w) &= f_u \frac{\partial u}{\partial z} + f_v \frac{\partial v}{\partial z} + f_w \frac{\partial w}{\partial z} \\ &= 2u(\cos(x + y + z)(1)) + 3v^2 e^w(0) + v^3 e^w(1) \\ f_z(1, -1, \pi) &= 2(\sin(1 - 1 + \pi))\cos(1 - 1 + \pi) + 0 + (1^2 e^{-1})^3 e^\pi \\ &= e^{\pi - 3}. \end{aligned}$$

2. Let $w = f(u, v)$, where $u = x + y$ and $v = x - y$. Show that

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2$$

Solution:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}(1) + \frac{\partial w}{\partial v}(1) \qquad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}(1) + \frac{\partial w}{\partial v}(-1)$$

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right)\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right) = \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2.$$

topic: Directional Derivative

3. Let $f(x, y, z) = x^2 e^{-yz}$.
(a) What is the domain of f?
Solution: $\mathbb{R}^3$
(b) What is the range of f?
Solution: $\{a \in \mathbb{R} \mid a \geq 0\}$.
(c) Compute the rate of change of f in the direction of the vector $\mathbf{v} = (2, 2, 2)$ at the point $(1, 0, 0)$.
Solution:

$$\begin{aligned} \nabla f(1, 0, 0) \cdot (2, 2, 2) &= \left(2xe^{-yz}, -zx^2 e^{-yz}, -yx^2 e^{-yz}\right)\Big|_{(1,0,0)} \cdot (2, 2, 2) \\ &= (2, 0, 0) \cdot (2, 2, 2) \\ &= 4. \end{aligned}$$

4. Let $\vec{u}$ and $\vec{v}$ be unit vectors and θ be the angle between them.
(a) Determine the value of θ that will ensure that $\vec{u} \cdot \vec{v}$ is as large as possible.
Solution: $\vec{u} \cdot \vec{v} = ||\vec{u}||\, ||\vec{v}|| \cos\theta = \cos\theta$ since the vectors are unit vectors. So this is largest when $\cos\theta = 1$, i.e. $\theta = 0$.
(b) Consider the formula for the directional derivative of f in the direction of $\vec{u}$. In what direction should $\vec{u}$ point so that the value of the directional derivative is largest?
Solution: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$, so from part (a) we see that this is largest when the angle between ∇f and $\vec{u}$ is 0, i.e. when $\vec{u}$ points in the same direction as ∇f.

5. Captain Ray is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x, y, z) will be given by $T(x, y, z) = e^{-x^2-2y^2-2z^2}$, where x, y and z are measured in meters. He is currently at $(1, 1, 1)$.

(a) In what direction should he proceed in order to *decrease* the temperature most rapidly? [Hint: Look at 4b.]

Temperature increases most rapidly in the direction of ∇T, so it decreases most rapidly in the direction of $\nabla(-T) = -\nabla T$. In this case,

$$\begin{aligned} -\nabla T(x, y, z) &= -\left(-2xe^{-x^2-2y^2-2z^2}, -4ye^{-x^2-2y^2-2z^2}, -4ze^{-x^2-2y^2-2z^2}\right) \\ -\nabla T(1, 1, 1) &= \left(2e^{-5}, 4e^{-5}, 4e^{-5}\right). \end{aligned}$$

Since we only care about the direction here, we could take the parallel vector $(1, 2, 2)$.

(b) If the ship travels at e^8 meters per second, how fast will the temperature decrease if he proceeds in that direction?

Solution: Rate of change of T in the direction $(1, 2, 2)$ is given by the directional derivative (notice that we do care about speed here). So our direction vector should be in the direction of $(1, 2, 2)$ but with magnitude e^8. That vector is $\frac{e^8}{\sqrt{1+4+4}}(1, 2, 2) = \frac{e^8}{3}(1, 2, 2)$. So the rate of temperature change is

$$\begin{aligned} \nabla T(1, 1, 1) \cdot \left(\frac{e^8}{3}(1, 2, 2)\right) &= \left(-2e^{-5}, -4e^{-5}, -4e^{-5}\right) \cdot \left(\frac{e^8}{3}(1, 2, 2)\right) \\ &= -2e^{-5}\frac{e^8}{3}(1, 2, 2) \cdot (1, 2, 2) \\ &= -6e^3. \end{aligned}$$

(c) Unfortunately, the metal of the hull will crack if cooled at a rate greater than $\sqrt{14}e^2$ degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.

Solution: We need a vector $\vec{u}$ such that $0 < -\nabla T(1, 1, 1) \cdot \vec{u} \leq \sqrt{14}e^2$. But

$$\begin{aligned} -\nabla T(1, 1, 1) \cdot \vec{u} &= ||\nabla T(1, 1, 1)||\ ||\vec{u}|| \cos\theta \\ &= 6e^{-5}||\vec{u}|| \cos\theta \end{aligned}$$

where θ is the angle between the direction of the negative gradient and the direction of $\vec{u}$. So we require

$$0 < ||\vec{u}|| \cos\theta \leq \frac{\sqrt{14}}{6}e^7.$$

So the set of possible directions can be described as

$$\{\vec{u} : -\pi/2 < \theta < \pi/2, \text{ and } ||\vec{u}|| \cos\theta \leq \frac{\sqrt{14}e^7}{6}\}.$$

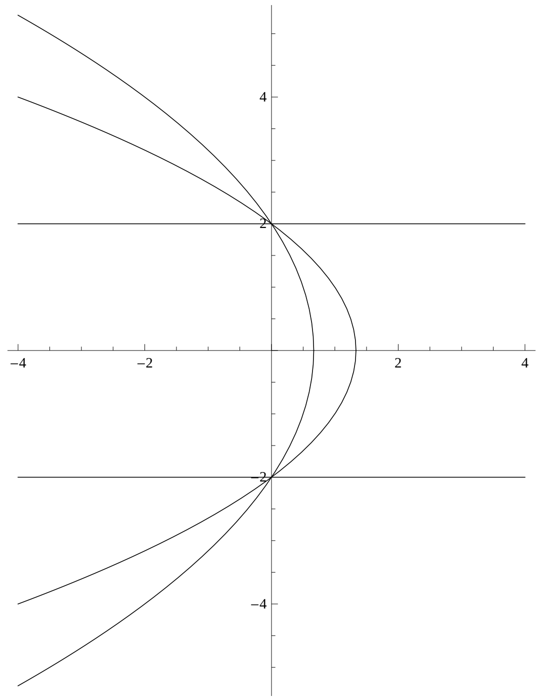
Thinking of these directions geometrically, $||\vec{u}|| \cos\theta$ is the component of the velocity vector parallel to the direction of the negative gradient. So there is a rectangular wedge of velocities available. (If we assume that Ray likes to always go as fast as possible, i.e. e^8 m/s, then we could solve this more exactly.)

6. Let $f(x, y) = \dfrac{4 - y^2}{3x}$.

(a) What is the domain of f?

Solution: $\{(x, y) \in \mathbb{R}^2 : x \neq 0\}$.

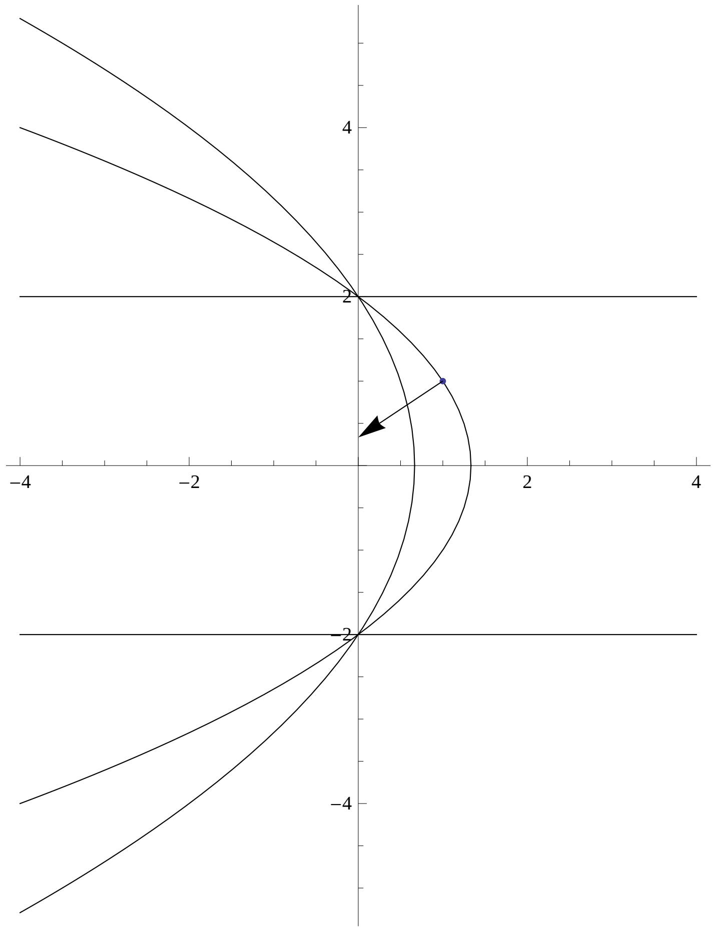
(b) On a large set of coordinate axes, carefully draw the level curves for $z = 0$, $z = 1$ and $z = 2$.
Solution:



(c) Compute the gradient and evaluate ∇f at the point $(1, 1)$.
Solution:

$$\nabla f(1,1) = \left(\frac{y^2-4}{3x^2}, \frac{-2y}{3x}\right)\Bigg|_{(1,1)} = \left(-1, -\frac{2}{3}\right).$$

(d) Graph this gradient vector on your collection of level curves. Is the gradient orthogonal to the level curve? Explain by looking at your sketch why the gradient points in the direction in which the value of f will increase the quickest.
Solution:



The gradient is orthogonal to the level curve. That is, it points in the direction which will take us away from the given level curve most quickly. This direction should be the direction that the function increases most rapidly, to get to the next value for another level curve.