Worksheet 3 January 26, 2011

1. (a) Rewrite the equation $\log_b x = y$ in terms of an exponential function.
   (b) Explain why $b^n \cdot b^m = b^{n+m}$ makes sense when n, m are positive integers.
   (c) Now translate the property in (b) into a property of logarithms.
   (d) Given that $\log(2) \approx 0.30103$, approximate $\log(64000)$.
2. How do you define 2^x when x is a positive integer? Negative integer? Rational number? Real number?
3. Let $f(x) = 2^x$. What is its domain and range? Graph it.
4. Let $d(x) = (-2)^x$. What is its domain and range? Can you graph it?
5. Suppose you record the following data about a population of brewer's yeast:

t	0	15	30	45	60	75
p	20	27	33	41	48	56

   (t is in minutes after the yeast have begun activity, and p is population in billions of cells.) Find an appropriate model for this data. Predict the yeast population at the end of three hours. Do you think this is realistic?
6. Suppose you record the following on another occasion (after properly aerating this time):

t	0	15	30	45	60	75
p	5	9	17	35	70	143

   (t and p are measured as before.) Find an appropriate model for this data. Predict the yeast population at the end of three hours. Do you think this is realistic?
7. Find an equation for an exponential curve that passes through the points $(0, \pi)$ and $(-3, 27)$.
8. Find an equation for an exponential curve that passes through the points $(2, 7)$ and $(5, 56)$. Find y if the curve passes through $(11, y)$. Find x if the curve passes through $(x, 14\sqrt{2})$.
9. How can you get the graph of 2^{x+4} given the graph of 2^x? How about $16 \cdot 2^x$? How are these two functions related? Make a conclusion about exponential graphs.
10. Discuss among your group what you think the value of 0^0 is.
11. Determine the value of $\log_2(3) \log_3(4) \log_4(5) \ldots \log_3 1(32)$. (Hint: try a simpler but similar problem first.)

Okay, that's enough to get us more or less on the same page. Now we'll start talking about the basics of calculus.

11. Suppose a particle's position is given by $p(t) = 3t + 5$. What is its velocity at time $t = 3$?
12. Suppose a particle's position is given by $p(t) = t^2$. What is its velocity at time $t = 3$? (No fair using calculus if you've already seen it!) How can you estimate this value?

13. Suppose a particle's position is given by a function $f(t)$. How can you estimate its velocity at time $t = 3$?

14. Let's go back to our particle with position function t^2. Write an explicit formula that gives the average velocity for the particle in the interval $[3, 3 + h]$ for any $h \neq 0$ (do you need to treat positive and negative h differently?). An old paradox says that velocity *at a given moment* cannot really exist, since velocity is defined by distance per time, and these are both zero in an instant. Explain how you might interpret "instantaneous velocity" at $t = 3$ given the formula you have derived.

15. Now we'll derive the special limit $\lim_{x \to 0} \frac{\sin x}{x}$. The end result is worth memorizing, as it will be needed for computing some derivatives later.

16. If you were around for the first class period, you'll recall that I discussed the definition of π. It is commonly defined to be the ratio of the circumference of a circle to its diameter. But this definition depends on the given circle! Now, a circle is defined by its center and radius, and certainly the center doesn't affect the ratio π, but perhaps the radius does. So really, we should defined π as a *function*, say $\pi(r)$ = the ratio of circumference to diameter of a circle with radius r. Now, if people haven't been lying to you for all your life, we should hope that $\pi(r)$ is a constant function. Let's try to justify this claim.

    (a) Start with an arbitrary circle, and inscribe a regular n-gon. (Make sure everyone in your group knows all these terms!) Break the interior of the polygon into many triangles. What can you say about these triangles?

    (b) Find a formula that gives the perimeter of the polygon in terms of the circle's radius.

    (c) What can you say about the polygons as $n \to \infty$? (This part is a bit fuzzy; to make it rigorous would require some considerably more complicated techniques. For now, just take it on intuition.)

    (d) Compute the ratio of perimeter to radius for the polygons. What happens to this quantity as $n \to \infty$? Is it what you expect? Does it depend on r?