Math 241 X8 **Name(s):** Solutions

Homework 8 supplement

This is a written homework supplement to the homework for Unit 8: Sources, Sinks, Swirls, and Singularities.

(1) Let $\mathbf{F}(x, y) = (e^x \sin y, e^x \cos y)$. Compute $\operatorname{div} \mathbf{F}$ and $\operatorname{rot} \mathbf{F}$. What do these tell you about the net flow of $\mathbf{F}$ across/along closed curves?

$$\operatorname{div}\vec{F} = \nabla \cdot F = \partial_x(e^x \sin y) + \partial_y(e^x \cos y) = e^x \sin y - e^x \sin y = 0$$

$\Rightarrow$ net flow across closed curves $= 0$.

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \partial_x & \partial_y \\ m & n \end{vmatrix} = \partial_x(e^x \cos y) - \partial_y(e^x \sin y) = e^x \cos y - e^x \cos y = 0$$

$\Rightarrow$ net flow along closed curves $= 0$.

(2) Consider the rectangle C with vertices at $(-1, -2)$, $(5, -2)$, $(5, 2)$, and $(-1, 2)$. Measure the net flow of $\mathbf{F}(x, y) = (x^2 + 2y^2, x^2 - 2y^2)$ across C. Is it inside to outside or outside to inside? (Hint: you could parametrize that curve, but you'd rather not.)

$$\int_C \vec{F} \cdot \langle dy, -dx \rangle = \iint_{\text{inside } C} \operatorname{div}\vec{F}\, dA = \int_{-2}^{2}\int_{-1}^{5} (2x - 4y)\, dx\, dy$$

$$= \int_{-2}^{2} \left[x^2 - 4xy\right]_{x=-1}^{5} dy$$

$$= \int_{-2}^{2} (24 - 24y)\, dy = (24)(4) = 96.$$

Net flow is outward.

(3) Consider the same rectangle and the same $\mathbf{F}$ from problem (2). Measure the net flow of $\mathbf{F}$ along C. Is it clockwise or counterclockwise? (Again, find a way to avoid parametrizing the curve.)

$$\int_C \vec{F} \cdot \langle dx, dy \rangle = \iint_{\text{inside } C} \operatorname{rot}\vec{F}\, dA = \int_{-2}^{2}\int_{-1}^{5} (2x \mp 4y)\, dx\, dy$$

$$= \int_{-2}^{2} \left[x^2 + 4xy\right]_{x=-1}^{5} dy$$

$$= \int_{-2}^{2} (24 + 24y)\, dy = 96.$$

Net flow is counterclockwise.

(4) Classify the points of the plane as sources, sinks, or neither, for the vector field

$$\mathbf{G}(x,y) = \left\langle \frac{x^2 - x + y^2}{x^2+y^2}, \frac{x^2+2y+y^2}{x^2+y^2} \right\rangle. = \left\langle 1 - \frac{x}{x^2+y^2}, 1 + \frac{2y}{x^2+y^2} \right\rangle$$

(It may help to manipulate some algebra before jumping into derivatives. Don't forget to check singularities separately.)

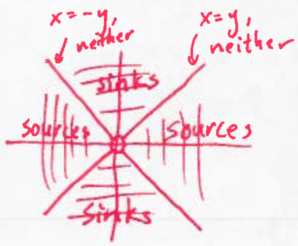
$$\operatorname{div} \vec{G} = -\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2 - y^2 + 2x^2 - 2y^2}{(x^2+y^2)^2} = 3\,\frac{x^2-y^2}{(x^2+y^2)^2}$$

$>0 \iff x^2-y^2>0$

$<0 \iff x^2-y^2<0$

$=0 \iff x^2-y^2=0 \iff x=\pm y$



at origin, singularity.

Check $\displaystyle\int_{\substack{\text{circle}\\ \text{radius}\\ r}} \vec{G}\cdot\langle dy, -dx\rangle$ $\quad x = r\cos t,\ y = r\sin t \quad t\in[0,2\pi)$

$$= \int_0^{2\pi} \left(\left(1 - \frac{\cos t}{r}\right)(r\cos t) + \left(1 + \frac{2\sin t}{r}\right)(r\sin t)\right) dt$$

$$= \int_0^{2\pi} \left(r\cos t - \cos^2 t + r\sin t + 2\sin^2 t\right) dt$$

$$= \int_0^{2\pi} \sin^2 t \, dt = \pi, \text{ regardless of } r.$$

Hence the singularity is also a source.