HW10

1) a) Let x, y be maximum elements. I.e., ① $\forall z\ x \succeq z$ & ② $\forall z\ y \succeq z$.

But in particular (① w/ $z = y$) $x \succeq y$ & (② w/ $z = x$) $y \succeq x$.

By antisymmetry, $x = y$. $\square$

b) Suppose to the contrary that x, y are maximal, $x \neq y$, & z is maximum.

Then since z is max'm, $z \succeq x$. Since x is maximal, $z \not\succ x$, so $z = x$.

Since - - - - - - $z \succeq y$. - - - y - - - - - - - $z \not\succ y$, so $z = y$.

So $x = y$, $\lightning$ $\square$

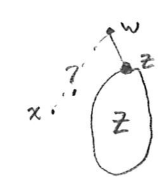
c) Suppose to the contrary that there is a finite poset $(P, \preceq)$ with a unique maximal element x, but x is not maximum.

Let $Z = \{z \in P : z \parallel x\}$. Since x is max'l but not max'm, $Z \neq \emptyset$.

[not max'm $\Rightarrow \exists z\ (z \succ x \vee z \parallel x)$, $z \succ x$ contradicts maximality of x]

By Wkshop 20 #1, the restriction of $\preceq$ to Z is a poset. $Z \subseteq P$ implies Z is finite,

so - - - #2 says that Z has a maximal element z.



Since $z \in Z$, $z \parallel x$.

Since x is the only max'l element of P, z is NOT max'l in P;

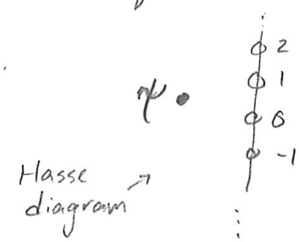
i.e., $\exists w \in P\ w \succ z$. Since z is max'l in Z, $w \notin Z$, i.e. $w \not\parallel x$.

Case 1: $w \succ x$. This contradicts the maximality of x.

Case 2: $w \prec x$. Then $z \prec w \prec x$, so $z \prec x$, contradicting that $z \in Z$.

$\square$

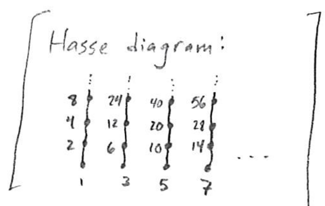
d) Take $(\mathbb{Z}, \leq)$ and add a new element ψ that is incomparable to every element of $\mathbb{Z}$. Then ψ is the unique maximal element, but there is no maximum element.



2) a) reflexive, symmetric. NOT transitive, e.g. $2R6$ & $6R3$ but $2 \not R 3$.

b) reflexive, transitive, neither symmetric — e.g. $2R6$ but $6 \not R 2$
nor antisymmetric — e.g. $2R4$ and $4R2$, but $2 \neq 4$

c) partial order. There are no maximal elements, and every odd integer is minimal.



d) equivalence relation. $[2] = \{2, 4, 8, 16, 32, 64, \dots\}$
$[6] = \{6, 12, 18, 24, 36, 48, \dots\}$

e) equivalence relation. $[2] = \{2, 3, 4, 5, 7, 8, \dots\}$
$[6] = \{6, 10, 12, 14, 15, 18, \dots\}$

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