WORKSHEET SOLUTIONS (11/15/12)

① (a) $\begin{cases} x = a\sin\varphi\cos\theta \\ y = b\sin\varphi\sin\theta \\ z = c\cos\varphi \end{cases}$ $\quad 0 \le \varphi \le \pi$, $\quad 0 \le \theta \le 2\pi$.

(b) $\vec{r}_\varphi = \langle a\cos\varphi\cos\theta, b\cos\varphi\sin\theta, -c\sin\varphi \rangle$

$\vec{r}_\theta = \langle -a\sin\varphi\sin\theta, b\sin\varphi\cos\theta, 0 \rangle$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\varphi\cos\theta & b\cos\varphi\sin\theta & -c\sin\varphi \\ -a\sin\varphi\sin\theta & b\sin\varphi\cos\theta & 0 \end{vmatrix}$$

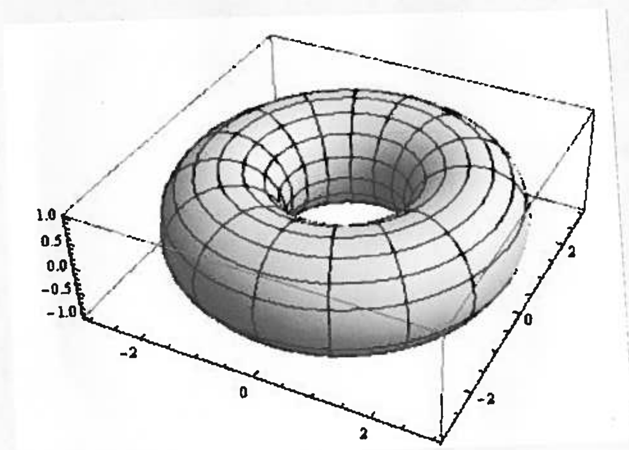
$$= \langle bc\sin^2\varphi\cos\theta, ac\sin^2\varphi\sin\theta, ab\sin\varphi\cos\varphi \rangle.$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = \sqrt{b^2c^2\sin^4\varphi\cos^2\theta + a^2c^2\sin^4\varphi\sin^2\theta + a^2b^2\sin^2\varphi\cos^2\varphi}$$

$$= \sin\varphi\sqrt{b^2c^2\sin^2\varphi\cos^2\theta + a^2c^2\sin^2\varphi\sin^2\theta + a^2b^2\cos^2\varphi}$$

$$A(S) = \int_0^\pi \sin\varphi \left(\int_0^{2\pi} \sqrt{c^2\sin^2\varphi(b^2\cos^2\theta + a^2\sin^2\theta) + a^2b^2\cos^2\varphi}\, d\theta \right) d\varphi.$$

② (a)



(b) $\begin{cases} x = (2+\cos u)\cos v \\ y = (2+\cos u)\sin v \\ z = \sin u \end{cases}$ $\quad \vec{r}_u = \langle -\sin u\cos v, -\sin u\sin v, \cos u \rangle$

$\vec{r}_v = \langle -(2+\cos u)\sin v, (2+\cos u)\cos v, 0 \rangle$

$0 \le u \le 2\pi$, $0 \le v \le 2\pi$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u\cos v & -\sin u\sin v & \cos u \\ -(2+\cos u)\sin v & (2+\cos u)\cos v & 0 \end{vmatrix}$$

$$= \langle -(2+\cos u)\cos u\cos v, -(2+\cos u)\cos u\sin v, -(2+\cos u)\sin u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| \underset{\substack{\uparrow \\ 2+\cos u>0}}{=} (2+\cos u)\sqrt{\underbrace{\cos^2 u\cos^2 v+\cos^2 u\sin^2 v}_{\cos^2 u}+\sin^2 u} = 2+\cos u.$$

$$A(S) = \iint_{[0,2\pi]\times[0,2\pi]} |\vec{r}_u\times\vec{r}_v|\,du\,dv = \int_0^{2\pi} du\left(\int_0^{2\pi}(2+\cos u)\,dv\right) = 2\pi\int_0^{2\pi}(2+\cos u)\,du$$

$$= 2\pi\cdot 2\cdot 2\pi = 8\pi^2.$$

③ (a) $S_1:\ x^2+y^2=1$ can be parameterized as

$$x=\cos\theta,\ y=\sin\theta,\ z=z,\qquad 0\le\theta\le 2\pi,\quad z\in\mathbb{R},$$

$$\vec{r}_\theta\times\vec{r}_z = \langle -\sin\theta,\cos\theta,0\rangle\times\langle 0,0,1\rangle = \langle\cos\theta,\sin\theta,0\rangle,\qquad |\vec{r}_\theta\times\vec{r}_z|=1.$$

(b) $(x,y,z)\in S_2 \Longrightarrow z=0 \Longrightarrow \iint_{S_2} z\,dS = 0.$

(c) $S_3:\ x^2+y^2\le 1$ & $z=x+1$ can be parameterized as

$$x=r\cos\theta,\ y=r\sin\theta,\ z=r\cos\theta+1,\qquad 0\le\theta\le 2\pi,\ 0\le r\le 1$$

$$\vec{r}_r = \langle\cos\theta,\sin\theta,\cos\theta\rangle,\qquad \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, -r\sin\theta\rangle$$

$$\vec{r}_r\times\vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -r\sin\theta & r\cos\theta & -r\sin\theta\end{vmatrix} = \langle -r, 0, r\rangle$$

$$|\vec{r}_r\times\vec{r}_\theta| = \sqrt{2r^2} = r\sqrt{2}.$$

$$\iint_{S_3} z\,dS = \iint_{[0,2\pi]\times[0,1]} r\sqrt{2}(r\cos\theta+1)\,dr\,d\theta = \sqrt{2}\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{0}\int_0^1 r^2\,dr$$

$$+\sqrt{2}\cdot 2\pi\cdot\int_0^1 r\,dr = \pi\sqrt{2}.$$

The subset $\tilde{S}_1$ of S_1 that is part of Σ can be described as

$$x=\cos\theta,\ y=\sin\theta,\ z=z\quad\text{with}\quad 0\le\theta\le 2\pi,\ 0\le z\le 1+\cos\theta$$

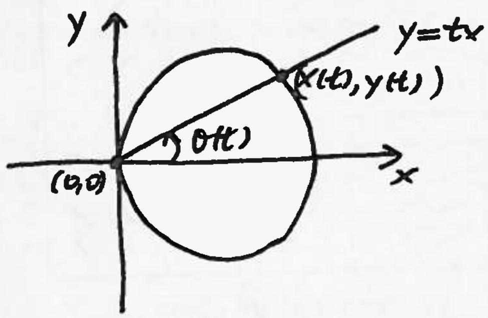
$$\iint_{\tilde{S}_1} z\,\underbrace{dS}_{d\theta\,dz} = \iint_{\substack{0\le\theta\le 2\pi \\ 0\le z\le 1+\cos\theta}} z\,dA = \int_0^{2\pi} d\theta\left(\int_0^{1+\cos\theta} dz\ z\right) = \int_0^{2\pi}\frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+\cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(1+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\cdot\frac{3}{\cancel{2}}\cdot\cancel{2}\pi = \frac{3\pi}{2}.$$

$$\iint_{\Sigma} z\,dS = \iint_{S_3} z\,dS + \iint_{S_2} z\,dS + \iint_{S_1} z\,dS = \pi\sqrt{2} + \frac{3\pi}{2}.$$

④ (a) $x^2+y^2-2x=0 \iff (x-1)^2+y^2=1$ circle of center $(1,0)$ and radius 1.



$$\tan\theta(t) = t = \frac{y(t)}{x(t)} \Longrightarrow y(t) = t\,x(t)$$

$$\Longrightarrow x(t)^2 + t^2x(t)^2 - 2x(t) = 0$$

$$\iff x(t)^2(1+t^2) = 2x(t)$$

$$\underset{x(t)\neq 0}{\Longrightarrow} x(t) = \frac{2}{1+t^2},\quad y(t) = \frac{2t}{1+t^2}$$

$$\Longrightarrow \vec{r}(t) = \left\langle \frac{2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle.$$

(b) $t = \frac{p}{q} \Longrightarrow x(t) = \dfrac{2}{1+\frac{p^2}{q^2}} = \dfrac{2q^2}{p^2+q^2}$

$y(t) = \dfrac{2\frac{p}{q}}{1+\frac{p^2}{q^2}} = \dfrac{2p}{\cancel{q}} \cdot \dfrac{q^{\cancel{2}}}{p^2+q^2} = \dfrac{2pq}{p^2+q^2}$

rational numbers because $2q^2$, $2pq$, and p^2+q^2 are all integers.

$$\left(x\left(\tfrac{p}{q}\right)-1\right)^2 + y\left(\tfrac{p}{q}\right)^2 = \left(\frac{q^2-p^2}{q^2+p^2}\right)^2 + \left(\frac{2pq}{q^2+p^2}\right)^2 = 1 \iff (q^2-p^2)^2 + (2pq)^2 = (q^2+p^2)^2$$

so we can produce triples (U,V,W) of integers with $U^2+V^2=W^2$ (these are called Pythagorean triples) by taking $U=q^2-p^2$, $V=2pq$, $W=q^2+p^2$ for any two integers p and q.

(c) $$\int_C \frac{1}{2}\langle -y, x\rangle \cdot d\vec{r} = \frac{1}{2}\int_C -y\,dx + x\,dy = \text{Area}\left(\{(x,y) \mid (x-1)^2+y^2=1\}\right)$$

$$= \pi$$

(C counterclockwise oriented)