Math 241 X8 **Name(s):**

Homework 11 supplement

This is a written homework supplement to the homework for Unit 12: Surface Integrals.

Consider the surface R_1 given by $y = x^2 + z^2$ for $0 \le y \le 1$; the surface R_2 that is the disk with $y = 1$, $x^2 + z^2 \le 1$; and the vector field $\mathbf{F}(x, y, z) = \langle 0, y, z \rangle$.

(1) Find the net flow of $\mathbf{F}$ across R_1 directly. Which direction is it? (You may use Mathematica to plot the surface for you. Set up and perform the integral by hand.)

Parametrize R_1: $\underbrace{x = r\cos t,\quad z = r\sin t,\quad y = r^2}_{\ell(r,t)}$, $r \in [0,1]$, $t \in [0, 2\pi]$

$$\vec{dS} = \partial_r \ell \times \partial_t \ell = \langle \cos t, 2r, \sin t \rangle \times \langle -r\sin t, 0, r\cos t \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & 2r & \sin t \\ -r\sin t & 0 & r\cos t \end{vmatrix} = \langle 2r^2\cos t, -r, 2r^2 \sin t \rangle$$

$$\text{flow} = \iint_{R_1} \vec{F} \cdot \vec{dS} = \int_0^{2\pi}\int_0^1 \langle 0, r^2, r\sin t \rangle \cdot \langle 2r^2\cos t, -r, 2r^2\sin t \rangle \, dr\,dt$$

$$= \int_0^{2\pi}\int_0^1 (-r^3 + 2r^3\sin^2 t)\, dr\,dt$$

$$= \int_0^{2\pi} \left(-\tfrac{1}{4} + \tfrac{1}{2}\sin^2 t\right) dt = -\frac{\pi}{2} + \frac{\pi}{2} = \boxed{0}$$ No net flow.

OR

$x = u$, $z = v$, $y = u^2 + v^2$, for $u^2 + v^2 \le 1$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 2v & 1 \end{vmatrix} = \langle 2u, -1, 2v \rangle$$

$$\text{flow} = \iint_{u^2+v^2\le 1} \langle 0, u^2+v^2, v \rangle \cdot \langle 2u, -1, 2v \rangle \, du\,dv$$

$$= \iint_{u^2+v^2 \le 1} \left(-(u^2+v^2) + 2v^2\right) du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (-r^2 + 2r^2\sin^2 t)\, r\,dr\,dt$$

(2) Find the net flow of $\mathbf{F}$ across R_2 directly. Which direction is it?

R_2: $x = r\cos t$, $z = r\sin t$, $y = 1$, $r \in [0,1]$, $t \in [0, 2\pi]$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & 0 & \sin t \\ -r\sin t & 0 & r\cos t \end{vmatrix} = \langle 0, -r, 0 \rangle$$

$$\text{flow} = \int_0^{2\pi}\int_0^1 \langle 0, 1, r\sin t \rangle \cdot \langle 0, -r, 0 \rangle \, dr\,dt$$

$$= \int_0^{2\pi}\int_0^1 -r \, dr\,dt$$

$$= -\pi.$$

OR

$x = u$, $z = v$, $y = 1$, $u^2 + v^2 \le 1$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 0, -1, 0 \rangle$$

$$\text{flow} = \iint_{u^2+v^2\le 1} \langle 0, 1, v \rangle \cdot \langle 0, -1, 0 \rangle \, du\,dv$$

$$= \iint_{u^2+v^2 \le 1} -1 \, du\,dv$$

$$= -1 \cdot \text{Area}(u^2+v^2 \le 1)$$

$$= -\pi.$$

Since $\vec{dS}$ is to the left, net flow is $+\pi$ to the right.

(3) Find the volume of the region bounded by R_1 and R_2.

$$= \int_0^1 \Big(\underbrace{\iint_{x^2+z^2\le y} 1 \, dx\,dz}_{\text{Area of disk of radius } \sqrt{y}} \Big) dy$$

$$= \int_0^1 \pi y \, dy$$

$$= \frac{\pi}{2}.$$

OR

$$\iint_{x^2+z^2\le 1} \Big(\int_{x^2+z^2}^{1} 1 \, dy \Big) dx\,dz$$

$$= \int_0^{2\pi}\int_0^1 \Big(\int_{r^2}^{1} 1 \, dy \Big) r\,dr\,d\theta$$

$$= 2\pi \cdot \int_0^1 (1-r^2) r \, dr$$

$$= 2\pi \left(\tfrac{1}{2} - \tfrac{1}{4} \right) = \frac{\pi}{2}.$$

(4) Find the divergence of **F**.

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle 0, y, z \rangle = 0 + 1 + 1 = 2.$$

(5) What relation do you observe between the previous parts? Should you have expected this?

$$0 + \pi \overset{(1)\&(2)}{=} \text{flow across } R_1 + \text{flow across } R_2$$

$$\overset{\text{def}}{=} \text{flow across } R_1 \cup R_2 \text{, outward}$$

$$\overset{\text{Divergence Thm}}{=} \iiint_{\substack{\text{inside}\\ R_1\cup R_2}} \operatorname{div} F \, dV$$

$$\overset{(4)}{=} \iiint_{\substack{\text{inside}\\ R_1\cup R_2}} 2 \, dV \overset{\text{def}}{=} 2 \cdot \operatorname{Vol}\left(\substack{\text{inside}\\ R_1\cup R_2}\right) \overset{(3)}{=} 2 \cdot \frac{\pi}{2}.$$

(6) Find the surface area of R_1.

From (1), $dS = |d\vec{S}|$:

$$dS = |\langle 2r^2\cos t, -r, 2r^2 \sin t \rangle|$$

$$= \sqrt{r^2 + 4r^4} = r\sqrt{1+4r^2}$$

OR

$$dS = |\langle 2u, -1, 2v \rangle| = \sqrt{1+4u^2+4v^2}$$

$$\text{surface area} = \iint_{R_1} 1\, dS = \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}\, dr\,dt$$

$u = 1+4r^2$, $du = 8r\,dr$

$$\text{surface area} = \iint_{R_1} 1\, dS = \iint_{u^2+v^2\le 1} \sqrt{1+4u^2+4v^2}\, du\,dv$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\,dr\,dt$$

$$= \frac{1}{8} \cdot 2\pi \cdot \int_1^5 \sqrt{u}\, du$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 = \frac{\pi}{6}\left(5\sqrt{5} - 1\right).$$