Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have seventy-five (75) minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.

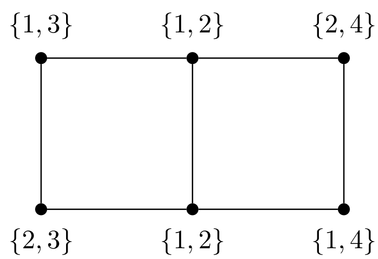
- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| Points: | 8 | 12 | 12 | 8 | 15 | 26 | 24 | 105 |
| Score: | | | | | | | | |

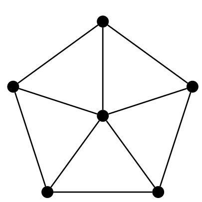
## Short answer

1. (8 points) Find the maximum number of edges in a 16-vertex $K_4$-free graph. Briefly justify.

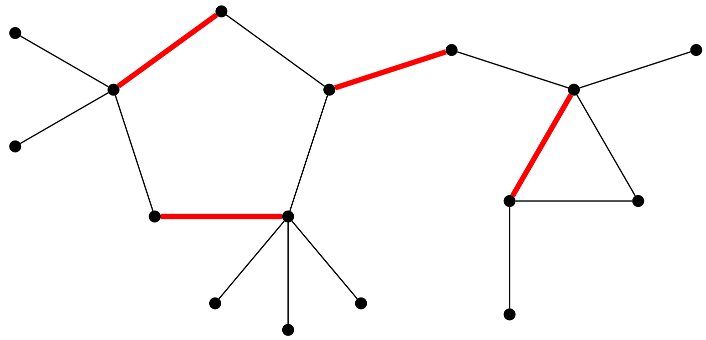
2. (12 points) Prove that the following graph $G$ cannot be properly colored from the displayed list assignment. What does this say about $\chi_\ell(G)$?



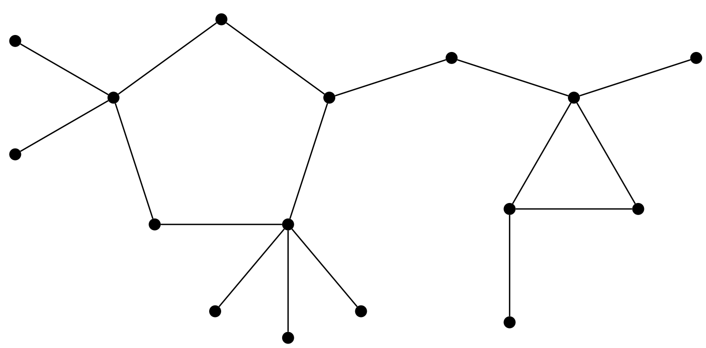

3. (12 points) Prove that the following graph is 4-critical. *(This is very quick with an appropriate theorem, but is not hard to do directly.)*



4. (8 points) The matching displayed in the following graph is *not* maximum. Prove this by finding an augmenting path.



5. (15 points) Find a minimum vertex cover in the graph, and prove that it is minimum.



# Proofs

6. Let $G$ be an $X, Y$-bigraph with $n$ vertices. Please note that throughout this problem we are using $\alpha(G)$, not $\alpha'(G)$.

   (a) (6 points) Prove that $\alpha(G) \geq n/2$.

   (b) (10 points) Prove that if $G$ has no perfect matching, then $\alpha(G) > n/2$. *(Suggestion: first, is $|X| = |Y|$? Then use Hall's Theorem.)*

   (c) (10 points) Prove that if $\alpha(G) > n/2$, then $G$ has no perfect matching. *(Suggestion: use Tutte's 1-Factor Theorem.)*

7. (24 points) Prove **three** of the following four statements. *(All of these have short proofs, some* very *short.)*

   (i) If $G$ is a graph such that $\chi(G - x - y) = \chi(G) - 2$ for every pair of distinct vertices $x, y$, then $G$ is a complete graph.

   (ii) Always $\chi(G) \cdot \chi(\overline{G}) \geq n(G)$.

   (iii) Always $\beta(G) \leq 2\alpha'(G)$.

   (iv) Every maximal matching in a graph $G$ has size at least $\alpha'(G)/2$.

(The statements are repeated here for your convenience.)

(i) If $G$ is a graph such that $\chi(G-x-y)=\chi(G)-2$ for every pair of distinct vertices $x,y$, then $G$ is a complete graph.

(ii) Always $\chi(G)\cdot\chi(\overline{G})\geq n(G)$.

(iii) Always $\beta(G)\leq 2\alpha'(G)$.

(iv) Every maximal matching in a graph $G$ has size at least $\alpha'(G)/2$.

**Scratch Paper -** you may remove this if you find it convenient

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