Irrational, Transcendental, Normal?

What do those mean?

A number is **rational** if it is the ratio of two integers, and **irrational** otherwise.

A number is **algebraic** if it is the root of a polynomial with rational coefficients, and **transcendental** otherwise. For example, every rational number a/b is algebraic because it is the root of the polynomial $bx - a$. But some irrational numbers are also algebraic, for example $\sqrt{2}$ since it is a root of $x^2 - 2$.

A number x is **normal in base b** if for every n, every n-digit string appears equally often in the base b representation of x, and **absolutely normal** if it is normal in every base b. For example, $22/7$ is not normal in base 10, because

$$22/7 = 3.142857142857142857\ldots,$$

and many digit strings never appear, e.g. "143", "999", or even "6". In fact, this argument says that every normal number must be irrational; but the converse doesn't hold, for example the irrational number

$$0.101001000100001000001\ldots$$

clearly fails to be normal (in any base).

π is transcendental

We will use the following theorem.

Lindemann-Weierstrass Theorem, Baker's formulation

If $a_1, \ldots, a_n$ are nonzero algebraic numbers and $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $a_1 e^{\alpha_1} + \cdots + a_n e^{\alpha_n} \neq 0$.

Suppose π is algebraic; then $i\pi$ is also algebraic ($i = \sqrt{-1}$). Let $n = 2$, $a_1 = a_2 = 1$, $\alpha_1 = i\pi$, and $\alpha_2 = 0$. Then $1e^{i\pi} + 1e^0 = 0$, which contradicts the theorem. Hence our assumption that π is algebraic is false. (We won't try to reproduce the proof of the Lindemann-Weierstrass Theorem here. See "Transcendental Number Theory" by Alan Baker.)

Irrationality measure

The **irrationality measure** of a number x, denoted $\mu(x)$, is defined by

$$\mu(x) = \inf\left\{\mu : 0 < \left|x - \frac{p}{q}\right| < \frac{1}{q^\mu} \text{ holds for only finitely many } (p,q) \in \mathbb{Z}^2\right\}.$$

It is a measure of how closely x can be approximated by rational numbers. A rational number x has $\mu(x) = 1$. The Thue-Siegel-Roth Theorem says that every algebraic irrational x has $\mu(x) = 2$. Every transcendental number x has $\mu(x) \geq 2$. Almost every real number x has $\mu(x) = 2$.
We know that $2 \leq \mu(\pi) \leq 7.6063$.

π is irrational

Here is a proof by Miklós Laczkovich, a simplification of an earlier proof by Johann Heinrich Lambert.
Define $f_k(x)$ for $k \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ and $x \in \mathbb{R}$ by

$$f_k(x) = 1 - \frac{x^2}{k} + \frac{x^4}{2!k(k+1)} - \frac{x^6}{3!k(k+1)(k+2)} + \cdots.$$

(Calculus exercise: check that this converges for all x.)
Note that $f_{1/2}(x) = \cos(2x)$.
Claim 1: For $x \in \mathbb{R}$ and $k \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$,

$$\frac{x^2}{k(k+1)} f_{k+2}(x) = f_{k+1}(x) - f_k(x).$$

Proof idea: Compare coefficients.
Claim 2: For fixed $x \in \mathbb{R}$, $\lim_{k\to\infty} f_k(x) = 1$.
Proof idea: Bound $|f_k(x) - 1|$ by a geometric series with ratio $1/k$.
Claim 3: If $x \in \mathbb{Q} \setminus \{0\}$ and $k \in \mathbb{Q} \setminus \{0, -1, -2, \ldots\}$, then $f_k(x) \neq 0$.
Proof: Suppose to the contrary that $f_k(x) = 0$. Let $y = f_{k+1}(x)$. If $y = 0$ then Claim 1 implies that $f_{k+n} = 0$ for all $n \in \mathbb{N}$, which contradicts Claim 2. So $y \neq 0$. Since $k, x \in \mathbb{Q}$, we may choose $c \in \mathbb{N}$ such that c/k, ck/x^2, and c/x^2 are all integers. Define

$$g_n = \begin{cases} f_k(x) & \text{if } n = 0, \\ \frac{c^n}{k(k+1)\cdots(k+n-1)} f_{k+n}(x) & \text{otherwise.} \end{cases}$$

Then $g_0 = 0 \in y\mathbb{Z}$ and $g_1 = \frac{c}{k} y \in y\mathbb{Z}$. From Claim 1 we can show

$$g_{n+2} = \left(\frac{ck}{x^2} + \frac{c}{x^2} n\right) g_{n+1} - \frac{c}{x^2} c\, g_n$$

and so for every n, $g_n \in y\mathbb{Z}$. But from Claim 2 we can show that $g_n \to 0$ and that for large enough n, $g_n > 0$. These three facts form a contradiction.
Conclusion: Suppose π is rational. Apply Claim 3 with $x = \pi/4$, $k = 1/2$ to obtain that $f_{1/2}(\pi/4) \neq 0$. But $f_{1/2}(\pi/4) = \cos(\pi/2) = 0$, so our assumption that π is rational is false.

Is π normal?

No one knows whether π is normal in any base, but it is largely believed to be absolutely normal.

It is not hard to find a number that is normal in base b for a single fixed b, but absolutely normal numbers are much more elusive.

Almost every real number is absolutely normal, but few explicit examples are known. Several examples were known by the 1990s, but many were uncomputable. Sierpinski (1917) gave an example that Becher and Figueira (2002) showed is computable, but to compute the first n digits required time double-exponential in n. Finally in 2013 some examples of absolutely normal numbers were found that require only time polynomial in n to produce n digits.