Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have seventy-five (75) minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wrist-watch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.
- There are seven (7) pages of questions. That gives you a little over ten (10) minutes per page. Later questions are generally longer than earlier ones.

Question	Points	Score
1	12	
2	4	
3	4	
4	4	
5	8	
6	4	
7	6	
8	6	
9	6	
10	8	
11	14	
12	8	
13	8	
Total:	92	

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You may use it for scratch work, but **do not** remove it.

No justification is necessary on this page.

1. (12 points) Circle "True" or "False" as appropriate.
(Some of these are tricky. If you have spare time at the end, come back and think some more.)

(a) True **False** 1 is a prime number.

(b) **True** False There is a prime number divisible by 13.

(c) **True** False If $f : X \to Y$ is injective, then $|X| \leq |Y|$.

(d) True **False** If $f : X \to Y$ is injective, then $|X| \geq |Y|$.

(e) True **False** If $|X| \leq |Y|$ and $f : X \to Y$, then f is injective.

(f) True **False** If $|X| \geq |Y|$ and $f : X \to Y$, then f is injective.

(g) **True** False If $|X| \leq |Y|$ and $f : X \to Y$ is surjective, then $|X| = |Y|$.

(h) **True** False For every $x \in \mathbb{R}$, $\lceil \lfloor x \rfloor + 0.5 \rceil > x$.

(i) True **False** Every integer $n > 1$ has a prime factor that is at most $\sqrt{n}$.

(j) True **False** $10^{10}n^2 = \Omega(n^2 \log n)$

(k) **True** False $(10 + n^2)\,(n \log n + (\log n)^2)\,(2^n + 3^n) = \Theta(n^3 3^n \log n)$

(l) **True** False $P(8) \wedge \forall k \geq 4\ (P(k) \to P(k+1)) \to \forall n \geq 9\ P(n)$ is a valid variation of induction.

2. (4 points) Find the least common multiple of $2^4 \cdot 5 \cdot 13$ and $2^2 \cdot 3 \cdot 5^2$. *(You do not need to simplify.)*

$2^4 \cdot 3 \cdot 5^2 \cdot 13$

3. (4 points) Define $a_1 = 32$ and $a_n = 4 \cdot a_{n-1}$ for $n \geq 2$. Find a_{10}. (It will presumably be faster to find a closed form expression for $\{a_n\}_{n=1}^{\infty}$. *No need to simplify.*)

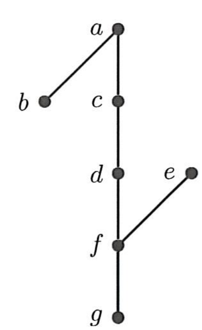
$a_{10} = 32 \cdot 4^9$ $\quad (a_n = 32 \cdot 4^{n-1})$

4. (4 points) Below is the Hasse diagram of a poset. Find all the elements of the following types.

(a) Maximal: a, e

(b) Minimal: b, g

(c) Maximum: none



5. (8 points) Define the relation R on $\{0,1\}^4 - \{0000\}$ (the set of nonzero binary strings of length 4) by

$$xRy \text{ iff the first 1 in } x \text{ appears further to the left than the first 1 in } y.$$

(E.g., 0100 R 0010, but 1001 $\not R$ 1000 and 0011 $\not R$ 0111.) Determine whether this relation has the following properties. For each "No," provide a counterexample.

(a) Yes No reflexive 0100 $\not R$ 0100

(b) Yes No antireflexive

(c) Yes No symmetric 0100 R 0010 but 0010 $\not R$ 0100

(d) Yes No antisymmetric

(e) Yes No transitive

6. (4 points) Define the relation R on $\{0,1\}^4 - \{0000\}$ (the set of nonzero binary strings of length 4) by

$$xRy \text{ iff the first 1 in } x \text{ appears in the same position as the first 1 in } y.$$

(E.g., 0100 R 0110 but 1001 $\not R$ 0001.) **It is true that** R is an equivalence relation. Find the equivalence classes of R.

[0001] = {0001}

[0010] = {0010, 0011}

[0100] = {0100, 0101, 0110, 0111}

[1000] = {1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}

7. (6 points) Use the Extended Euclidean Algorithm to find the multiplicative inverse of 25 in $\mathbb{Z}_{27}$.

$27 = 1(25) + 2$

$25 = 12(2) + 1$

$2 = 1 \cdot 27 - 1 \cdot 25$

$1 = 1 \cdot 25 - 12 \cdot 2$

$= 1 \cdot 25 - 12(1 \cdot 27 - 1 \cdot 25)$

$= \underline{13} \cdot 25 - 12 \cdot 27$

13

8. (6 points) Describe the output of the following recursive function. *(Experimenting is recommended.)*

```
Mystery(x,y)

Input:  x, y, two non-negative integers
Output: ???

If y = 0
   Return( 0 )
End-if

p := Mystery(x,y − 1) // The recursive call

Return( p + x )
```

output $= x \cdot y$

$y=0$:
Return 0

$y=1$: $p = \text{Mystery}(x,0) = 0$
Return $0+x = x$

$y=2$: $p = \text{Mystery}(x,1) = x$
Return $x+x = 2x$

$y=3$: $p = \text{Mystery}(x,2) = 2x$
Return $2x+x = 3x$

⋮

9. (a) (6 points) Find the total number of times the inner **For**-loop is executed in the following algorithm on an input of size n. Then find its asymptotic time complexity using Θ notation.

```
Algorithm: PrefixSums(x_1, ..., x_n)
Input:  (x_1, ..., x_n), a sequence of real numbers
        n, its length
Output: (a_1, ..., a_n), where a_i is the sum of x_1, ..., x_i

For i = 1 to n
   a_i := 0
   For j = 1 to i
      a_i := a_i + x_j
   End-For
End-For

Return( (a_1, ..., a_n) )
```

$\#\text{iterations} = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$\text{time complexity} = \Theta(n^2)$

(b) (6 points (bonus)) Find a more efficient algorithm (that takes the same input and produces the same output).

$a_1 := x_1$

For $i = 2$ to n

$\quad a_i = a_{i-1} + x_i$

End-For

Return($(a_1, \cdots, a_n)$)

10. (8 points) Prove that for every integer n, if n is odd, then $n^2 \bmod 8 = 1$. *(Hint: Use the Division Algorithm with dividend n and divisor 4 to get cases.)*

By the Division Algorithm, $n = 4q + r$ for some $q \in \mathbb{Z}$ & $r \in \{0,1,2,3\}$.

n is odd, so $r \neq 0$ & $r \neq 2$.

Case 1: $n = 4q+1$: $n^2 = (4q+1)^2 = 16q^2 + 8q + 1$;
$n^2 \bmod 8 = 0 + 0 + 1 = 1$.

Case 2: $n = 4q+3$: $n^2 = (4q+3)^2 = 16q^2 + 24q + 9$
$n^2 \bmod 8 = 0 + 0 + 1 = 1$. $\square$

11. Let $f : A \to B$ and $g : B \to C$.

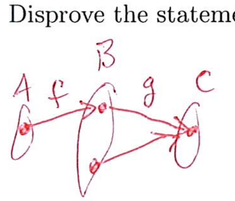
(a) (8 points) Prove the statement "if $g \circ f$ is surjective, then g is surjective." *(A direct proof works.)*

Let $g \circ f$ be surjective.
Let $c \in C$ be arbitrary.
Since $g \circ f$ is surj., $\exists a \in A$ s.t. $g \circ f(a) = c$

$$g \circ f(a) = g(\underbrace{f(a)}_{\in B})$$

So $f(a) \in B$ & $g(f(a)) = c$, so g is surjective. $\square$

(b) (6 points) Disprove the statement "if $g \circ f$ is surjective, then f is surjective."



12. (8 points) Use strong induction to prove that every amount of at least 18¢ can be made using 4¢ and 7¢ coins (without change). Very carefully state your inductive hypothesis and why it applies in the inductive step (which is connected to the number of base cases you will need).

Base Cases:

$18¢ = 2 \cdot 7¢ + 1 \cdot 4¢$

$19¢ = 1 \cdot 7¢ + 3 \cdot 4¢$

$20¢ = 0 \cdot 7¢ + 5 \cdot 4¢$

$21¢ = 3 \cdot 7¢ + 0 \cdot 4¢$

Ind. Hyp. Assume for some $\boxed{k \geq 21}$ that $\forall j \in \{18, \ldots, k\}$, $j¢$ can be made from 4¢ & 7¢ coins.

Ind. Step. $(k+1)¢ = \underbrace{(k-3)¢} + 4¢$

Can be made from 4¢ & 7¢ coins by the I.H., since $18 \leq k-3 \leq k$.

$\Updownarrow$

$21 \leq k$

So $(k+1)¢$ can be made.

By PSMI, the statement holds. $\square$

13. (8 points) Let R be an equivalence relation on a set A. Prove that for any $x, y \in A$, if $x \not R\, y$, then $[x] \cap [y] = \varnothing$. Clearly indicate in your proof whenever you use a property of equivalence relations. *(This result was used to prove that equivalence classes form a partition; you cannot use that result.)*

(Recall that the most usual way to prove that a set is empty is a proof by contradiction: assuming the set is not empty, then there is an element in it; you now seek a contradiction based on the properties of that element.)

Let $x, y \in A$, $x \not R\, y$, but $[x] \cap [y] \neq \varnothing$.

Then there is some $z \in [x] \cap [y]$,

i.e. $z \in [x]$ AND $z \in [y]$,

i.e. zRx AND zRy

$\downarrow$

By symmetry, xRz.

Now xRz & zRy, so by transitivity, xRy,

contradicting our original assumption. $\square$

Scratch Paper - Do Not Remove