HW8

1) Answers will vary.

2) a)

$2543 = 211 \cdot 12 + 11$
$12 = 1 \cdot 11 + 1$
$11 = 11 \cdot 1 + 0$

$$\begin{aligned} \gcd = 1 &= 1 \cdot 12 - 1 \cdot 11 \\ &= 1 \cdot 12 - 1(1 \cdot 2543 - 211 \cdot 12) \\ &= 212 \cdot 12 - 1 \cdot 2543 \end{aligned}$$

$12^{-1} = 212$ in $\mathbb{Z}_{2543}$

b) $2544 = 212 \cdot 12 + 0$

$\gcd = 12 = 1 \cdot 12 + 0 \cdot 2544$
$\uparrow$
>1, so 12^{-1} DNE in $\mathbb{Z}_{2544}$

c)

$55 = 1 \cdot 34 + 21$
$34 = 1 \cdot 21 + 13$
$21 = 1 \cdot 13 + 8$
$13 = 1 \cdot 8 + 5$
$8 = 1 \cdot 5 + 3$
$5 = 1 \cdot 3 + 2$
$3 = 1 \cdot 2 + 1$
$2 = 2 \cdot 1 + 0$

$$\begin{aligned} \gcd = 1 &= 1 \cdot 3 - 1 \cdot 2 \\ &= 1 \cdot 3 - 1 \cdot (1 \cdot 5 - 1 \cdot 3) \\ &= 2 \cdot 3 - 1 \cdot 5 \\ &= 2 \cdot (1 \cdot 8 - 1 \cdot 5) - 1 \cdot 5 \\ &= 2 \cdot 8 - 3 \cdot 5 \\ &= 2 \cdot 8 - 3 \cdot (1 \cdot 13 - 1 \cdot 8) \\ &= 5 \cdot 8 - 3 \cdot 13 \\ &= 5 \cdot (1 \cdot 21 - 1 \cdot 13) - 3 \cdot 13 \\ &= 5 \cdot 21 - 8 \cdot 13 \\ &= 5 \cdot 21 - 8 \cdot (34 - 1 \cdot 21) \\ &= 13 \cdot 21 - 8 \cdot 34 \\ &= 13 \cdot (1 \cdot 55 - 1 \cdot 34) - 8 \cdot 34 \\ &= 13 \cdot 55 - 21 \cdot 34 \end{aligned}$$

$34^{-1} = -21 \bmod 55 = 34$ in $\mathbb{Z}_{55}$

3) a) This is Bézout's Identity (wkshop 15), zyBook Theorem 7.5.2.

b) $g|x$ and $g|y$ (it is a "common factor"),

i.e. $x = cg$ & $y = dg$ for some $c, d \in \mathbb{Z}$.

Then $ax+by = a(cg) + b(dg) = g(ac+bd)$. Since $ac+bd \in \mathbb{Z}$, $g | (ax+by)$.

4) a) Any exchange of coins results in a payment $6a + 22b + 33c$ for some $a, b, c \in \mathbb{Z}$ [positive $\leftrightarrow$ you give vendor, negative $\leftrightarrow$ vendor gives you].

With $b = c = 1$ and $a = -9$, we have $6(-9) + 22(1) + 33(1) = 1$.

[or $a = 2, b = 1, c = -1$, or ......]

If I can make a payment of 1¢, certainly I can pay any amount.

[To pay p¢, give p 22¢ coins, p 33¢ coins, and get 9p 6¢ coins as change.]

b) Now we have $p = 22b + 33c$. Since $\gcd(22, 33) = 11$, by #3 the smallest such p is 11¢. So no, you cannot pay 1¢ exactly.

[In fact, the payments you can make exactly are precisely the multiples of 11¢.]

5) $$\sum_{i=2}^{n} (7 + 3i + 11 \cdot 2^i) = \sum_{i=2}^{n} 7 + 3\sum_{i=2}^{n} i + 11 \sum_{i=2}^{n} 2^i \qquad j = i-2 \leftrightarrow i = j+2$$

$$= 7(n-1) + 3\left(\sum_{i=1}^{n} i - 1\right) + 11 \cdot \sum_{j=0}^{n-2} 2^{j+2}$$

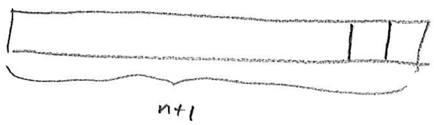
$$= 7n - 7 - 3 + 3\frac{(n+1)n}{2} + 44 \sum_{j=0}^{n-2} 2^j$$

$$= 7n - 10 + \frac{3}{2}(n+1)n + 44 \frac{2^{n-1} - 1}{2-1}$$

6) $t_1 = 1$ (m)

$t_2 = 2$ (m|m), (ccc)

$t_{n+1} = t_n + t_{n-1}$ for $n \geq 2$

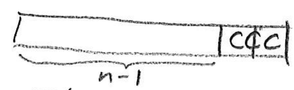


In any strip of length n+1, the last vehicle is either

a motorcycle

Then the rest of the strip can be filled in any fashion; there are t_n many ways to do this.

a car



Then the rest of the strip can be filled in any fashion; there are t_{n-1} many ways to do this.