Worksheet 28 May 4, 2011

1. Limits! Compute:

   (a) $\lim_{x \to 0^+} x \ln(x)$.

   (b) $\lim_{x \to 0} \dfrac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$.

   (c) $\lim_{x \to 0} \lfloor x \rfloor + \lfloor -x \rfloor$. Is this function continuous at zero?

   (d) $\lim_{x \to \infty} \left(1 + \dfrac{1}{2x}\right)^x$.

   (e) $\lim_{x \to 0^+} x^2 \arctan(x^2 3 \ln(x))$.

   (f) $\lim_{x \to \infty} \ln(200x) - x \ln(x)$.

2. Compute the derivative of $x^2 - 2x + 4$ by the definition.

3. Compute the integral of $x^2 - 2x + 4$ between $x = 0$ and $x = 2$ by the definition.

4. Derivatives! Compute $\frac{dy}{dx}$ for the following:

   (a) $x^2 y = \sin(xy)$

   (b) $y = x^{x^2}$

5. Suppose I consume caffeine at a rate of $64\sqrt{t} - t^2$ mg/hour, for $0 \le t \le 16$ hours past waking. Suppose that caffeine dissipates from my system at a rate of 100mg/hour. Write a formula that gives

   (a) the amount of caffeine I have consumed during the day up to any given time;

   (b) the rate of change of my caffeine level at each time t;

   (c) the amount of caffeine I have in my system at any given time.

   What is the maximum level of caffeine in me during the day, and when does it occur?

6. Integral calculus is a fundamental part of probability when dealing with situations that have a continuum (or really just infinitely many) possible outcomes. For instance, age or height of people, lifetime of a machine, or time between passing cars all have infinitely many possible outcomes. We compute probabilities of these events by integrating a *probability density function.*

   (a) For example, suppose the amount of time you spend in a line at the DMV has a density function $f(t) = 4t - 4t^3$, where t is measured in hours and $0 \le t \le 1$ (for other times the probability is zero). To find the probability that you wait between 1/4 hour and 1/2 hour, you need to calculate $\int_{1/4}^{1/2} f(t)\, dt$. Do so.

   (b) What should be the probability that you wait at least 0 minutes? Compute the probability that you wait between 0 and 1 hour to verify that this is true for the given density function. (Remember that we're assuming it's impossible to wait more than an hour; perhaps you just give up and leave after an hour.)

(c) Calculus also gives us a way to compute the *expected* waiting time. To do this we just compute $\int t \cdot f(t)\,dt$. Do so, integrating from 0 to 1.

7. An important function to probability is e^{-x^2}. You may recall that this is one of those functions whose antiderivative is not an "elementary function". (Note that it does have an antiderivative, since it is continuous; we just cannot write this function down in terms of our standard collection of functions.) So you won't be able to actually compute the following integrals, but you can numerically estimate them (by finite Riemann sums for example). You are given the numerical results, for $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \qquad \int_0^1 f(x)\,dx = 0.3413, \qquad \int_0^2 f(x)\,dx = 0.4772, \qquad \int_{-\infty}^3 f(x)\,dx = 0.9987$$

(You should treat infinities in integral limits as being area under a curve all the way to the left or right of the plane. You'll get a rigorous treatment of infinite integrals in Calc 2.) Compute

(a) $\int_{-1}^1 f(x)\,dx$

(b) $\int_1^2 f(x)\,dx$

(c) $\int_0^\infty f(x)\,dx$

(d) $\int_3^\infty f(x)\,dx$

(e) $\int_{-1}^2 f(x)\,dx$