WORKSHOP 7: §2.4-2.5 FEBRUARY 2, 2017

Name: ______________________________

(1) Show that $p \to q \equiv (p \wedge \neg q) \to \mathrm{F}$. *[This shows that proof by contradiction is a valid technique. A proof by contradiction assumes $p \wedge \neg q$ and derives from that a contradiction, which is a direct proof of the right hand side.]*

(2) (a) Prove that for all integers m and n, if m and n have the same parity, then $m+n$ is even.
(b) Prove that for all integers m and n, if m and n have opposite parity, then $m+n$ is odd.
(c) Prove that $|x-3| \leq |x|+3$ for every real number x.

(3) Prove that 1 is not even. (Use proof by contradiction, then cases for whether the new integer you introduced is positive or not.)

(4) Prove that no integer is both even and odd. (Use proof by contradiction and the result of problem 3.) *[Hurray! Now we just need to prove that every integer is even or odd...that still has to wait for Chapter 8...]*

In question 2b, you (hopefully) used cases, and those cases were remarkably similar. To save ourselves from similar tedious casework, we can use the "without loss of generality" (WLOG) technique. This technique observes such symmetry in the cases that need to be considered and asserts that the argument for one case works for the other cases with only very minor adjustments. *In this class, you should briefly justify by specifying the nature of those adjustments.*

For example, to prove that the average of any two distinct real numbers is between those numbers, you need to consider two cases. If $x < y$, then you need to show that $x < \frac{x+y}{2} < y$; if $y < x$, then you need to show that $y < \frac{x+y}{2} < x$. *[Are these all the cases?]* But the proofs of those two cases are exactly the same if you swap the names x and y. So, the following is a valid proof:

Proof. Let x and y be two arbitrary real numbers with $x \neq y$. Then $x < y$ or $y < x$; WLOG, assume $x < y$ (otherwise, swap x and y). Then

$$x = \frac{x+x}{2} < \frac{x+y}{2} < \frac{y+y}{2} = y,$$

hence $x < \frac{x+y}{2} < y$ as desired. □

(5) Use the WLOG technique to prove that for all real numbers x, y, it is true that $|x-y| = \max(x,y) - \min(x,y)$.

(6) Prove that for any three real numbers x, y, z, the average of the three is at most the maximum of the three. (When can the average equal the maximum?)

(7) We can sometimes prove something exists without actually finding an example. This is called a *non-constructive proof* of existence. Prove that there are irrational numbers x, y such that x^y is rational by considering $\sqrt{2}^{\sqrt{2}}$ and $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$.