Workshop 19 November 3, 2011

1. Expand $(1+x)^5$.

2. Expand $(1+x)^{2/3}$ (as a series; what is the radius of convergence?).

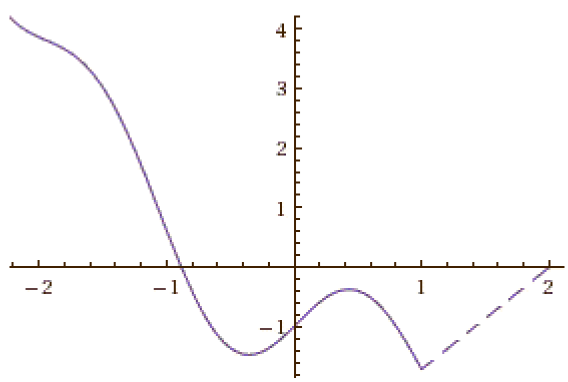
3. It's easy to extend functions like x^2 to having domains in the complex numbers (how do you square $i=\sqrt{-1}$? How do you square $1+i$?). It's not as easy to get functions like e^x to allow complex variables. One way of doing this is just to define it so that the Taylor series (and perhaps other familiar properties) work the way we expect. So, for any real number x, define

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$

Write out the first several terms of this series and simplify powers of i. Then separate the series into those terms containing i and those not. Do you recognize the remaining series? Use this to prove the famous equation

$$e^{i\pi} + 1 = 0.$$

4. Apiarist Joe lives quite a way out of town. There's a road that gets close to his house, and he wants to extend it to reach his new beehives. The road can be modeled by the curve $y = \sin \pi x + x^2 - e^x$, for $x \le 1$. His new beehives are at $(2,0)$. Notice in particular that continuing the road along its current equation will not lead to the beehives. You could just connect the road with a straight-line piece:



but that sharp turn would be bad. Suppose you want the new road to meet the old one with an *order of smoothness* 3 (so the new equation should share the first, second, and third derivatives with the old at $x=1$). Find such an equation. (Don't forget that the road needs to end up at $(2,0)$!)

Choose (as a group) which of the next two problems you want to work on first.

a. *Proving a well-known property*

Using the Maclaurin series for e^x, write series expansions for e^x and e^y, where x, y are arbitrary real numbers. Organizing your work in a table where columns correspond to powers of x and rows correspond to powers of y, write down the first several terms in the product of these series. Now add along the diagonals (so that in diagonal number k, the sum of the powers of x and y are equal to k). If you multiply the coefficients by $k!$ and divide everything by $k!$ (so you haven't changed anything), notice what the coefficients become. In particular, what is a shorter way to express the sums along the diagonals? Hence what is this series in total?

b. *Convergence Issues*

We've been a bit optimistic in our dealings with Taylor series so far. It is possible to start with a function $f(x)$, find its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, and with it some interval of convergence, say $[a-R, a+R]$. So we have two functions, f and the function to which the series converges, but these *may not be the same*, even when $x \in [a-R, a+R]$. Here's an example. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Compute the first few derivatives of $f(x)$ at $x = 0$. While doing this, pretend our shortcut rules work at $x = 0$, and evaluate all divisions by zero as though they were limits. (If we were being fully calc-1 rigorous, this problem would take a lot of work, since the piecewise definition nullifies our usual shortcut rules at zero; however, you can check [with much wailing and gnashing of teeth] that the definition of $f(0) = 0$ makes f infinitely differentiable.)

Make a guess at (or even better, give an argument that proves) what the nth derivative of f is. So what is the Maclaurin series for f? What is its interval of convergence?