Solutions

1. Evaluate the following derivatives.

   (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x)$

   Solution: $-\sin x$

   (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\csc x)$

   Solution: $-\csc x \cot x$

   (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan x)$

   Solution: $\sec^2 x$

   (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right)$

   Solution: $\dfrac{1}{\sqrt{1-x^2}}$

   (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}(5^x)$

   Solution: $5^x \ln(5)$

2. Differentiate the function $y = \dfrac{1 - xe^x}{x + e^x}$.

Solution: $y' = \dfrac{(x+e^x)(1-xe^x)' - (1-xe^x)(x+e^x)'}{(x+e^x)^2} = \dfrac{(x+e^x)(-e^x - xe^x) - (1-xe^x)(1+e^x)}{(x+e^x)^2}$.

3. Differentiate the function $f(\theta) = \sin(\tan(2\theta))$.

Solution: $f'(\theta) = \cos(\tan(2\theta)) \cdot \sec^2(2\theta) \cdot 2.$

4. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the curve $y \sin\left(x^2\right) = x \sin\left(y^2\right)$.

Solution:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sin(x^2)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\sin(y^2)\right) \\
y'\sin(x^2) + y\cos(x^2)\cdot(2x) &= \sin(y^2) + x\cos(y^2)\cdot(2y)\cdot y' \\
y'\sin(x^2) - 2xy\cos(y^2)y' &= \sin(y^2) - 2xy\cos(x^2) \\
y'\left(\sin(x^2) - 2xy\cos(y^2)\right) &= \sin(y^2) - 2xy\cos(x^2) \\
y' &= \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}.
\end{aligned}$$

5. Differentiate the function $f(x) = x^{\cos x}$.

Solution: For ease of notation, let $y = f(x)$.

$$\begin{aligned}
y &= x^{\cos x} \\
\ln y &= \ln\left(x^{\cos x}\right) \\
\ln y &= \cos x \ln x \\
\frac{1}{y} y' &= -\sin x \ln x + \cos x \cdot \frac{1}{x} \\
y' &= y\left(\frac{\cos x}{x} - \sin x \ln x\right) \\
f'(x) &= x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).
\end{aligned}$$

6. A curve has the property that at each point, the slope of its tangent line is half the y-coordinate of that point. If the curve passes through the point $(1, -3)$, find an equation for the curve.

Solution: The information given can be captured by the equations $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2}$ and $y(1) = -3$. We know that the solution to this differential equation has the form $y = Ce^{x/2}$. Using the point given, we have $-3 = Ce^{1/2}$, or $C = \frac{-3}{e^{1/2}}$. So the equation of the curve is given by $y = \frac{-3}{e^{1/2}}e^{x/2}$. (Note that this is equivalent to $y = -3e^{(x-1)/2}$.)

7. If a ball is thrown vertically upwards with a velocity of 80 feet per second, its height after t seconds is $h = 80t - 16t^2$. What is the maximum height of the ball?

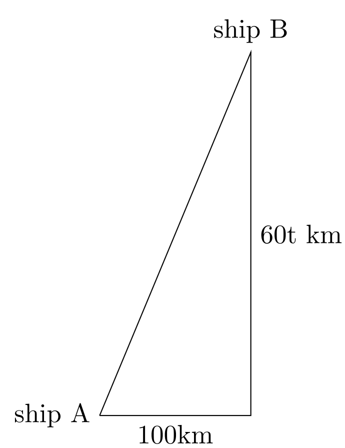
Solution: The maximum height is achieved when the velocity is zero. Velocity is given by $h'(t) = 80 - 32t$, which is zero when $t = 5/2$. The height at this time is $h(5/2) = 80(5/2) - 16(5/2)^2 = 200 - 100 = 100$ feet.

8. At noon, ship A is 100 kilometers west of ship B. Ship A is sailing south at 35 kilometers per hour, and ship B is sailing north at 25 kilometers per hour. How fast is the distance between the ships changing at 4:00 PM?

Solution: If we set our origin at ship A, then ship B starts at $(100, 0)$ and moves northward with a relative speed of 35+25=60 km/hr. So, t hours after noon, ship B is at $(100, 60t)$. Referring to the triangle that this gives (see diagram below), we see that the distance is given by $\ell(t) = \sqrt{100^2 + (60t)^2} = \sqrt{20^2\left(5^2 + (3t)^2\right)} = 20\sqrt{(3t)^2 + 5^2}$. Taking the derivative, we get

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = \frac{20(18t)}{2\sqrt{(3t)^2 + 5^2}}.$$

At $t = 4$, this evaluates to $\dfrac{20(18)(4)}{2(13)} = \dfrac{720}{13}$ km/hr.



9. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other into an equilateral triangle. How should the wire be cut so that the total area enclosed by the square and triangle is maximized?

Solution: Let x be the length cut for the triangle. Then the side length of the triangle is $x/3$. To find the area of the triangle we need the height, which can be found by splitting the triangle into two right triangles (see diagram below).

Note that the angles of the triangle are all 60 degrees, and so the height is given by $\frac{x}{3}\sin(\pi/3) = \frac{x}{2\sqrt{3}}$.

So the area of the triangle is $\frac{1}{2} \cdot \left(\frac{x}{3}\right) \cdot \left(\frac{x}{2\sqrt{3}}\right) = \frac{x^2}{12\sqrt{3}}$. Now the remaining wire has length $10 - x$, so the square has side length $(10-x)/4$, and has area $\frac{(10-x)^2}{16}$. Then the area is given by

$$\begin{aligned} A(x) &= \frac{x^2}{12\sqrt{3}} + \frac{(10-x)^2}{16} \qquad \text{so} \\ A'(x) &= \frac{x}{6\sqrt{3}} - \frac{10-x}{8}. \end{aligned}$$

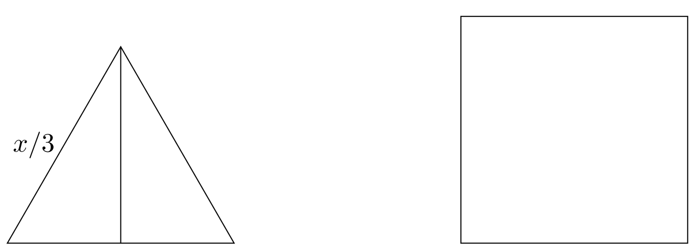
Setting this equal to zero, we get

$$\begin{aligned} 0 &= \frac{x}{6\sqrt{3}} - \frac{10-x}{8} \\ \frac{10-x}{8} &= \frac{x}{6\sqrt{3}} \\ 60\sqrt{3} - 6\sqrt{3}x &= 8x \\ 60\sqrt{3} &= (8+6\sqrt{3})x \\ \frac{60\sqrt{3}}{8+6\sqrt{3}} &= x. \end{aligned}$$

This is our only critical point; it remains to decide whether it is a maximum. Notice that the formula for A' is fairly simple: it is just linear with positive slope. So to the left of this critical point, A' is negative, and to the right it is positive. So this critical point is actually a minimum! (If we had thought of it earlier, notice that A is given by a quadratic with positive coefficient on the x^2 term, so we knew before taking the derivative that we would only find ourselves a minimum.) So the maximum occurs at one of the endpoints:

$$A(0) = 100/16 \qquad\qquad A(10) = 100/(12\sqrt{3}).$$

Since $(12\sqrt{3})^2 = 432 > 256 = 16^2$, we have $100/(12\sqrt{3}) < 100/16$, and so the maximum area is $100/16 = 25/4$ sq. meters, given by creating just a square from the wire.



10. Consider the function $f(x) = x^5 - 2x^3 + x$.
    (a) Find the intervals on which f is increasing and decreasing.
    (b) Find the local maximum and minimum values.
    (c) Find the intervals of concavity and inflection points.
    (d) Use the information from parts (a)-(c) to sketch the graph of f.

Solution: We first compute

$$f'(x) = 5x^4 - 6x^2 + 1 = (5x^2 - 1)(x^2 - 1) \qquad f''(x) = 20x^3 - 12x = 4x(5x^2 - 3).$$

The critical points are $\pm 1, \pm\sqrt{1/5}$. Check the value of f' at some appropriate points between these, say

$$f'(\pm 2) > 0 \qquad f'(\pm 1/2) < 0 \qquad f'(0) = 1 > 0.$$

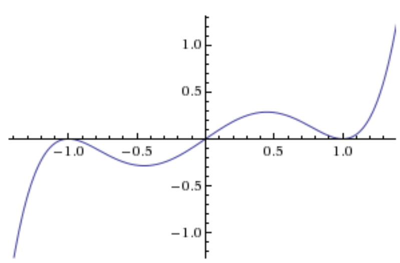
So the function is increasing on $(-\infty, -1) \cup (-\sqrt{1/5}, \sqrt{1/5}) \cup (1, \infty)$ and decreasing on $(-1, -\sqrt{1/5}) \cup (\sqrt{1/5}, 1)$. Furthermore, there are local maxima at $x = -1$ and $x = \sqrt{1/5}$; there are local minima at $x = -\sqrt{1/5}$ and $x = 1$.

The second derivative is zero at $0, \pm\sqrt{3/5}$. Check the value of f'' at some appropriate points, say

$$f''(-1) < 0 \qquad f''(-0.1) > 0 \qquad f''(0.1) < 0 \qquad f''(1) > 0.$$

So the function is concave down on $(-\infty, -\sqrt{3/5}) \cup (0, \sqrt{3/5})$ and concave up on $(-\sqrt{3/5}, 0) \cup (\sqrt{3/5}, \infty)$. There are inflection points at $x = 0, \pm\sqrt{3/5}$.

Note that $f(x) = x(x^4 - 2x^2 + 1) = x(x^2 - 1)^2 = x(x-1)^2(x+1)^2$. So there are zeros at $x = -1, 0, 1$. Note also that f is an odd function. The graph of the function is shown below.



11. Evaluate the following limits.

(a) $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

Solution: This is of the form 0/0. Applying L'Hospital, we get $\lim_{x \to 0} \frac{e^x - 1}{2x}$. This is again of the form 0/0, so we apply L'Hospital again and get $\lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$.

(b) $\lim_{x \to 0^+} x \ln x$

Solution: This is of the form $0 \cdot -\infty$, to which L'Hospital does not directly apply. We can turn it into a form to which L'Hospital does apply, $\lim_{x \to 0^+} \frac{\ln x}{1/x}$. This is now of the form $-\infty/\infty$, so L'Hospital gives $\lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$.

(c) $\lim_{x \to \infty} \frac{x^{3,000}}{e^{0.1x}}$

Solution: We know that, as $x \to \infty$, any exponential grows faster than any fixed power of x, so this goes to 0. (L'Hospital would give the answer here, but we would need to apply it 3,000 times!)