Worksheet 17 March 28, 2011

Some final words on differential calculus.

We've talked a lot about differentiation: the idea behind it, the definition, some computational aids, and applications. There's one last thing to discuss: going backwards in the process, or *antidifferentiation*.

$\mathbb{D}$1. Find an antiderivative for each of the following:

(a) 4

(b) x

(c) $4x^3$

(d) $\sin x$

(e) e^{4x}

$\mathbb{D}$2. As I alluded to before break, any two antiderivatives of a given function differ only by a constant. Let's prove this.

(a) We'll start with the simplest functions. Find all antiderivatives of the constant function 0. Justify that there are no others.

(b) Now use (a) for the general case: Given a function $f(x)$ and two different antiderivatives $F(x)$ and $G(x)$, consider the function $h = F - G$. Show that h is a constant function. (So F and G differ by a constant.)

$\mathbb{D}$3. Find all antiderivatives of each of the following:

(a) $\cos x$

(b) $5x^4 + x^2 - 2$

(c) $(2x)(e^x) + (x^2)(e^x)$

(d) $\dfrac{1 + 2x}{1 + x^2}$

Some of you may have found this (antidifferentiation) section a bit difficult. The next couple of sections ignore it entirely, but it will be back before long. The best way to master antidifferentiation is to know derivatives in your sleep. If you're at all unsure about the derivative of a basic function, you will most likely mix up rules in antidifferentiation.

And now for something completely different...

Integral Calculus

I1. Suppose we want to estimate the area below the graph of the function $f(x)$ and above the x-axis, between say $x = a$ and $x = b$. Suppose further that f is increasing on this interval. Can you say how the left and right endpoint estimations of area compare to the actual area? What if f is decreasing instead? Can you say anything if f is neither increasing nor decreasing?

I2. Let's get to know summations using sigma notation (Σ is the capital of the greek letter σ, sigma). It's just shorthand for writing long sums. When I write

$$\sum_{i=a}^{b} f(i)$$

I mean 1) take each integer value i, starting at a and ending at b, and evaluate $f(i)$; then 2) add all these results. So

$$\sum_{i=a}^{b} f(i) := f(a) + f(a+1) + f(a+2) + \cdots + f(b-1) + f(b).$$

For example, $\sum_{i=0}^{3} i^2 = 0^2 + 1^2 + 2^2 + 3^2$.

(a) Evaluate $\sum_{i=3}^{5} i$.

(b) Evaluate $\sum_{i=0}^{1} i^{25}$.

(c) Evaluate $\sum_{i=a}^{b} 1$ if $b \geq a$.

I3. Here are some useful summations to know when dealing with Riemann sums. (The first one you've already proven, the second is easy enough to prove, and the last one can be proven with mathematical induction, which we may or may not discuss later.)

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Let's do some easy(ish) area calculations.

(a) Compute the area under the graph $y = 3$ between $x = 1$ and $x = 5$ as a limit of sums. (Make sure the answer is what you expect!)

(b) Compute the area under the graph $y = x$ between $x = 0$ and $x = 4$ in the same manner. Again, make sure this is the correct answer with simple geometry.

(c) Compute the area under the graph $y = x^2$ between $x = 0$ and $x = 1$ similarly. Could you compute this with your previous geometry knowledge?

(d) What if I asked for the area under $y = \sin x$ between $x = 0$ and $x = \pi$? Set up an appropriate sum, say with left endpoints as the rectangle heights. Can you evaluate it? Make an estimate of the area using only four rectangles. Improve the estimate by using six rectangles instead.

As you can see, evaluating these areas by the definition can be painful. We'll soon see a much easier way of doing it, but you need to understand the idea behind the definition.