Name:

MATH 231 ED5 / Spring 2010

Quiz 4

Instructions: You have 15 minutes to complete this quiz. No calculators, notes, phones, socializing, or other suspicious behaviors are allowed. Read the problems carefully. Explain your answers and show your work.

Evaluate:

$$\int \frac{x\ln(3x)}{\sqrt{4x^2-1}}dx$$

Method I (as done in Tuesday's lab):

Noticing that $x/\sqrt{4x^2-1}$ is something we know how to integrate with a simple substitution, and $\ln(3x)$ is something we can differentiate, we integrate by parts:

$$\begin{aligned} u &= \ln(3x) & dv &= \frac{x}{\sqrt{4x^2-1}}dx \\ du &= \frac{1}{3x}\cdot 3dx & v &= \frac{1}{4}\sqrt{4x^2-1} \end{aligned}$$

$$= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}\int \frac{\sqrt{4x^2-1}}{x}dx$$

Now there are several ways to evaluate the resulting integral. We can set $t = 4x^2-1$ or $t = \sqrt{4x^2-1}$ or $x = \frac{1}{2}\sec t$; any of these will work out. Let's use the trigonometric substitution:

$$\begin{aligned} &= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}\int \frac{\sqrt{\sec^2 t-1}}{\frac{1}{2}\sec t}\cdot\frac{1}{2}\sec t\tan t\ dt \\ &= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}\int \frac{\tan t}{\sec t}\sec t\tan t\ dt \\ &= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}\int \tan^2 t\ dt \\ &= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}\int (\sec^2 t - 1)\ dt \\ &= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}(\tan t - t) + C \\ &= \frac{1}{4}\sqrt{4x^2-1}\ln(3x) - \frac{1}{4}\left(\sqrt{4x^2-1} - \sec^{-1}(2x)\right) + C \end{aligned}$$

Method II (as suggested as an exercise in Tuesday's lab):

We begin this time with $t = \sqrt{4x^2 - 1}$. Then
$dt = \frac{1}{2}(4x^2-1)^{-1/2}(8x)dx = \frac{4x}{t}dx$, and $x = \frac{1}{2}\sqrt{t^2+1}$, so the integral becomes:

$$\int \frac{\ln\left(\frac{3}{2}\sqrt{t^2+1}\right)}{t}\left(\frac{1}{4}t\ dt\right) = \frac{1}{4}\int \ln\left(\frac{3}{2}\sqrt{t^2+1}\right)\ dt$$

and now we are left with few options. We integrate by parts:

$$u = \ln\left(\frac{3}{2}\sqrt{t^2+1}\right) \qquad dv = dt$$

$$du = \frac{1}{\frac{3}{2}\sqrt{t^2+1}} \cdot \frac{3}{2}\frac{1}{2\sqrt{t^2+1}} \cdot 2tdt \qquad v = t$$

$$= \frac{1}{4}\left(t\ln\left(\frac{3}{2}\sqrt{t^2+1}\right) - \int \frac{t^2}{t^2+1}dt\right)$$

Now again we have to pause to think. Two options present themselves: long divide and find a partial fraction decomposition, or use another trig substitution, say $y = \tan t$. If we work through the long division, we find

$$= \frac{1}{4}\left(t\ln\left(\frac{3}{2}\sqrt{t^2+1}\right) - \int\left(1 + \frac{-1}{t^2+1}\right)dt\right)$$
$$= \frac{1}{4}\left(t\ln\left(\frac{3}{2}\sqrt{t^2+1}\right) - t + \tan^{-1} t\right) + C$$
$$= \frac{1}{4}\left(\sqrt{4x^2-1}\ln(3x) - \sqrt{4x^2-1} + \tan^{-1}\sqrt{4x^2-1}\right) + C$$

Method III:

If we see $\sqrt{4x^2-1}$ and think of a trigonometric substitution, then

$$x = \frac{1}{2}\sec t$$
$$dx = \frac{1}{2}\sec t\tan tdt$$

$$\int \frac{\frac{1}{2}\sec t\ln\left(\frac{3}{2}\sec t\right)}{\tan t} \cdot \frac{1}{2}\sec t\tan tdt = \frac{1}{4}\int \sec^2 t\ln\left(\frac{3}{2}\sec t\right)\ dt.$$

Here we know the integral of $\sec^2 t$, so integrate by parts:

$$u = \ln\left(\frac{3}{2}\sec t\right) \qquad dv = \sec^2 tdt$$

$$du = \frac{1}{\frac{3}{2}\sec t} \cdot \frac{3}{2}\sec t\tan tdt \qquad v = \tan t$$

$$= \frac{1}{4}\left(\tan t\ln\left(\frac{3}{2}\sec t\right) - \int \tan^2 t\ dt\right)$$
$$= \frac{1}{4}\left(\sqrt{4x^2-1}\ln(3x) - \sqrt{4x^2-1} + \sec^{-1}(2x)\right) + C.$$