CHAPTER 1 SECTION 1.9 (Part I)

Principal Root

Definition

For every pair of real numbers a and b, if $a^2 = b$, then a is called a square root of b.

Concept

A square root of a number is one of two equal factors of the number.

To answer a question like the following:

If $x^2 = 9$ then what number is x equal to?

is 3 or -3 since $(3)^2 = 9$ and $(-3)^2 = 9$. To distinguish between the two square roots, we define the *principal square root* of a positive number to be positive. The $\sqrt{\ }$ symbol denotes the principal square root.
Thus, if $x = \sqrt{9}$, then $x = 3$ and we say 3 is the principal square root of 9.

$$\sqrt{9} = 3 \text{ (principal square root)}$$

The parts of the principal square root are

$$\sqrt{a}$$

Radical sign Radicand

The entire expression is called a *radical* and is read "the principal square root of a."

Examples:

Find the principal square root.

1. $\sqrt{16} = 4$, since $4 \cdot 4 = 4^2 = 16$.
2. $\sqrt{49} = 7$, since $7 \cdot 7 = 7^2 = 49$.
3. $\sqrt{25} = 5$, since $5 \cdot 5 = 5^2 = 25$.
4. $\sqrt{36} = 6$, since $6 \cdot 6 = 6^2 = 36$.
5. $\sqrt{0} = 0$, since $0 \cdot 0 = 0^2 = 0$.

Note In the examples, 0, 16, 49, 25, and 36 are called **perfect-square integers** because their square roots are integers.

Whenever we wish to express the negative value of the square root of a number, we use the symbol $-\sqrt{}$. For example, $-\sqrt{9}$ would indicate that we want the negative square root value. That is, $-\sqrt{9} = -3$.

Examples:

Find the indicated root.

1. $-\sqrt{4} = -2$

2. $-\sqrt{49} = -7$

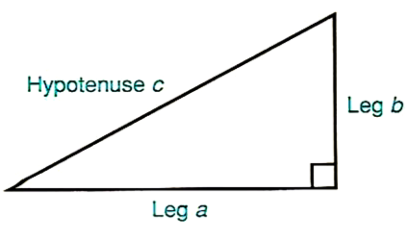
3. $-\sqrt{25} = -5$

4. $-\sqrt{36} = -6$

Pythagorean Theorem

The following is an important property of right triangles called the **Pythagorean Theorem.**

In a right triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the two legs (the sides that form the right angle). If c is the length of the hypotenuse and a and b are the lengths of the legs, this property can be stated as:



$$c^2 = a^2 + b^2 \text{ or } c = \sqrt{a^2 + b^2}$$

also as
$$a^2 = c^2 - b^2 \text{ or } a = \sqrt{c^2 - b^2}$$

and as
$$b^2 = c^2 - a^2 \text{ or } b = \sqrt{c^2 - a^2}$$

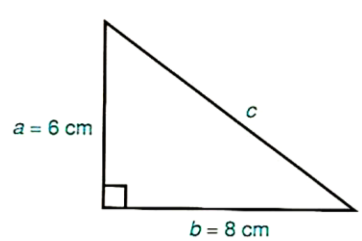
Example:

Find the length of the hypotenuse of a right triangle whose legs are 6 centimeters and 8 centimeters.

We want to find c when $a = 6$ cm and $b = 8$ cm.
By the Pythagorean Theorem,

$c = \sqrt{a^2 + b^2}$
$= \sqrt{6^2 + 8^2}$ Substitute
$= \sqrt{36 + 64}$ Square the values
$= \sqrt{100} = 10$

Hence, $c = 10$ cm.



n th roots

The concept of square root can be extended to find cube roots (third root of a number), fourth roots, fifth roots, and so on. A cube root is one of three equal factors of a number. The symbol that is used to express the principal cube root is $\sqrt[3]{}$. The 3 is called the **index** of the radical expression. The index denotes what root we are looking for. The principal fourth root would be indicated by $\sqrt[4]{}$. General notation for the principal nth root would be $\sqrt[n]{}$, where n is a natural number greater than 1.

The principal *n*th root

The principal nth root of number a, denoted by

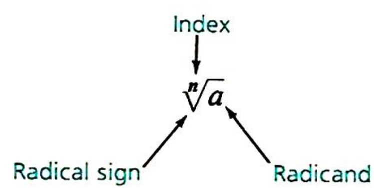
$$\sqrt[n]{a}$$

is one of n equal factors such that $\sqrt[n]{a} = b$ and

$$\overbrace{b \cdot b \cdot b \cdots b}^{n \text{ factors}} = b^n = a$$

where n *is a natural number greater than 1.*

The parts of the principal nth root are



Note If there is no index associated with a radical symbol, it is understood to be 2.

If we exclude even roots of negative numbers, which do not exist in the set of real numbers, we can extend our idea of principal square root to all other roots by saying: **the principal *n*th root** *of a number, denoted by* $\sqrt[n]{}$, *has the same sign as the number itself.*

Examples

Find the indicated root.

1. $\sqrt[4]{16} = 2$, since $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.
2. $\sqrt[3]{-27} = -3$, since $(-3)(-3)(-3) = (-3)^3 = -27$.
3. $\sqrt[3]{-125} = -5$, since $(-5)(-5)(-5) = (-5)^3 = -125$.
4. $\sqrt{-16}$ Does not exist in the set of real numbers.

Exercises:

Find the indicated square root

1. $\sqrt{100}$	**2.** $\sqrt{36}$	**3.** $\sqrt{4}$	**4.** $\sqrt{64}$	**5.** $-\sqrt{144}$	**6.** $-\sqrt{81}$
7. $-\sqrt{121}$	**8.** $-\sqrt{4}$	**9.** $\sqrt{121}$	**10.** $\sqrt{81}$	**11.** $-\sqrt{16}$	**12.** $-\sqrt{64}$

Find the indicated root. See example below.

Example $\sqrt[3]{-64}$

Solution $= -4$ Since $(-4)^3 = -64$

$\boxed{31.}$ $\sqrt[3]{8}$	**32.** $\sqrt[3]{27}$	**33.** $\sqrt[3]{125}$	**34.** $\sqrt[3]{64}$	**35.** $\sqrt[3]{-8}$	**36.** $\sqrt[3]{-64}$
37. $\sqrt[4]{81}$	**38.** $\sqrt[4]{16}$	$\boxed{39.}$ $-\sqrt[4]{81}$	**40.** $-\sqrt[4]{625}$	**41.** $\sqrt[5]{32}$	**42.** $\sqrt[6]{64}$
43. $-\sqrt[5]{243}$	**44.** $\sqrt[5]{-243}$	$\boxed{45.}$ $\sqrt[10]{1}$	**46.** $\sqrt[14]{1}$	**47.** $\sqrt[9]{-1}$	**48.** $\sqrt[15]{-1}$

Product property for radicals

Multiplying square roots

In this section, we are going to develop properties for simplifying radicals. Consider the example

$$\sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10$$

We also observe that

$$\sqrt{4 \cdot 25} = \sqrt{100} = 10$$

From our example, we can conclude that

$$\sqrt{4} \cdot \sqrt{25} = \sqrt{4 \cdot 25}$$

We now can generalize the product property for square roots.

Product property for square roots

For all nonnegative real numbers a and b,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

Concept

The product of two square roots is equal to the square root of their product.

Examples

Perform the indicated operations. Assume that all variables represent nonnegative real numbers.

1. $\sqrt{3}\sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15}$

2. $\sqrt{6}\sqrt{7} = \sqrt{6 \cdot 7} = \sqrt{42}$

3. $\sqrt{3}\sqrt{a} = \sqrt{3a}$

4. $\sqrt{x}\sqrt{y}\sqrt{z} = \sqrt{xyz}$

Simplifying principal square roots

An important use of our product property is in simplifying radicals. Consider the following example:

Since 12 can be factored into $4 \cdot 3$, by our product property, we can write

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$$

We are able to simplify the radical because the radicand contains a perfect-square integer factor, 4. In our example, $2\sqrt{3}$ is called the *simplified form* of $\sqrt{12}$.

Simplifying the principal square root

1. If the radicand is a perfect-square integer, write the corresponding square root.
2. If possible, factor the radicand so that at least one factor is a perfect-square integer. Write the corresponding square root as a coefficient of the radical.
3. The square root is in simplest form when the radicand has no perfect-square integer factors other than 1.

The following property is used when changing radicals involving variables in the radicand to simplest form.

$\sqrt{a^2}$ property

If a is any nonnegative real number, then

$$\sqrt{a^2} = a$$

Examples

Simplify the following expressions. Assume that all variables represent nonnegative real numbers.

1. $\sqrt{50} = \sqrt{25 \cdot 2}$ Factor having a perfect-square integer

$= \sqrt{25}\sqrt{2}$ Product property

$= 5\sqrt{2}$ $\sqrt{25} = 5$

2. $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

3. $\sqrt{9a} = \sqrt{9 \cdot a} = \sqrt{9}\sqrt{a} = 3\sqrt{a}$

4. $\sqrt{a^3} = \sqrt{a^2 \cdot a} = \sqrt{a^2}\sqrt{a} = a\sqrt{a}$

5. $\sqrt{a^3b^4} = \sqrt{a^2 \cdot a \cdot b^2 \cdot b^2} = \sqrt{a^2}\sqrt{a}\sqrt{b^2}\sqrt{b^2} = a\sqrt{a}bb = ab^2\sqrt{a}$

6. $\sqrt{a^2 + b^2}$ will not simplify because we are not able to factor the radicand. *The radicand must always be in a factored form before we can simplify.*

Note $\sqrt{a^2 + b^2} \neq \sqrt{a^2} + \sqrt{b^2}$. For example, $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} \neq \sqrt{9} + \sqrt{16}$. Since $\sqrt{9 + 16} = \sqrt{25} = 5$, whereas $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

7. $\sqrt{2}\,\sqrt{6} = \sqrt{2 \cdot 6} = \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$

8. $\sqrt{10}\sqrt{5} = \sqrt{10 \cdot 5} = \sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

9. $\sqrt{14x}\sqrt{2x} = \sqrt{14x \cdot 2x} = \sqrt{28x^2} = \sqrt{4 \cdot 7 \cdot x^2} = \sqrt{4}\sqrt{7}\sqrt{x^2}$
$= 2\sqrt{7}x = 2x\sqrt{7}$

Multiplying nth roots

In previous section we observed that even roots of negative numbers do not exist in the set of real numbers. Therefore, **we will consider all variables to be representing nonnegative real numbers whenever the index of the radical is even.**

Our product property for square roots can be extended to radicals with any index.

Product property for radicals

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

Concept

When we multiply two radicals *having the same index,* we multiply the radicands and put the product under a radical symbol with the common index.

Examples

Perform the indicated operations.

1. $\sqrt[3]{3}\sqrt[3]{2} = \sqrt[3]{3 \cdot 2} = \sqrt[3]{6}$ **2.** $\sqrt[5]{7}\sqrt[5]{9} = \sqrt[5]{7 \cdot 9} = \sqrt[5]{63}$

3. $\sqrt[3]{3}\sqrt[4]{5}$ These radicals cannot be multiplied together in this form since they do not have the same index.

Simplifying principal nth roots

We simplify *n*th roots, where *n* is greater than 2, as we did square roots. As long as the radicand can be factored so that one or more factors is a

1. perfect cube when the index is 3,
2. perfect fourth-power when the index is 4,
3. perfect fifth-power when the index is 5, and so on,

the radical can be simplified. To do this, we use the property

$$\sqrt[n]{a^n} = a$$

where a is a nonnegative real number.

Examples

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

1.

$\sqrt[3]{81}$	$= \sqrt[3]{27 \cdot 3}$	Factor having a perfect-cube integer
	$= \sqrt[3]{27}\sqrt[3]{3}$	Product property
	$= 3\sqrt[3]{3}$	$\sqrt[3]{27} = 3$

2. $\sqrt[4]{32} = \sqrt[4]{16 \cdot 2} = \sqrt[4]{16}\,\sqrt[4]{2} = 2\sqrt[4]{2}$

3. $\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2} = \sqrt[3]{x^3}\,\sqrt[3]{x^2} = x\sqrt[3]{x^2}$

4. $\sqrt[5]{y^{10}} = \sqrt[5]{y^5 \cdot y^5} = \sqrt[5]{y^5}\,\sqrt[5]{y^5} = y \cdot y = y^2$

Note In example 4, the exponent 10 is evenly divisible by the index 5, and the radical is eliminated. When the exponent of a factor is evenly divisible by the index, that factor will no longer remain under the radical symbol.

5. $\sqrt[4]{x^7y^4} = \sqrt[4]{x^4 \cdot x^3 \cdot y^4} = \sqrt[4]{x^4}\,\sqrt[4]{x^3}\,\sqrt[4]{y^4} = x\sqrt[4]{x^3}y = xy\sqrt[4]{x^3}$

6. $\sqrt[3]{a^7b^2} = \sqrt[3]{a^3a^3ab^2} = \sqrt[3]{a^3}\,\sqrt[3]{a^3}\,\sqrt[3]{ab^2} = a \cdot a\sqrt[3]{ab^2} = a^2\sqrt[3]{ab^2}$

Note No simplification relative to *b* is possible because the exponent of *b* is less than the value of the index.

7. $\sqrt[3]{54x^3y^5} = \sqrt[3]{27 \cdot 2 \cdot x^3 \cdot y^3 \cdot y^2} = \sqrt[3]{27}\,\sqrt[3]{x^3}\,\sqrt[3]{y^3}\,\sqrt[3]{2y^2} = 3xy\sqrt[3]{2y^2}$

Observe from the preceding examples that **we can simplify a radical if the radicand has a factor(s) whose exponent is equal to or greater than the index.**

Examples

Perform the indicated multiplication and simplify. Assume that all variables represent nonnegative real numbers.

1. $\sqrt[3]{a^2}\,\sqrt[3]{a} = \sqrt[3]{a^2 \cdot a}$ Product property

$= \sqrt[3]{a^3}$ Multiply like bases

$= a$ Perfect cube

2. $\sqrt[4]{8a^2}\,\sqrt[4]{4a^3} = \sqrt[4]{8a^2 \cdot 4a^3} = \sqrt[4]{32a^5} = \sqrt[4]{16 \cdot 2 \cdot a^4 \cdot a}$

$= \sqrt[4]{16}\,\sqrt[4]{a^4}\,\sqrt[4]{2a} = 2a\sqrt[4]{2a}$

► ***Quick check*** Simplify. $\sqrt[3]{9a^2}\sqrt[3]{3a^2}$

Note A very common error in problems involving radicals is to forget to carry along the correct index for the radical symbol.

Exercises

Perform any indicated operations and simplify. Assume that all variables represent nonnegative real numbers.

See examples below.

Examples $\sqrt{8a^4}$

Solutions $= \sqrt{4 \cdot 2a^2a^2}$ Factor having perfect squares
$= \sqrt{4}\sqrt{a^2}\sqrt{a^2}\sqrt{2}$ Product property
$= 2aa\sqrt{2}$ $\sqrt{4} = 2$, $\sqrt{a^2} = a$
$= 2a^2\sqrt{2}$ Multiply

$\sqrt{3a}\sqrt{6a}$
$= \sqrt{3a \cdot 6a}$ Product property
$= \sqrt{18a^2}$ Multiply
$= \sqrt{9 \cdot 2 \cdot a^2}$ Factor having perfect squares
$= \sqrt{9}\sqrt{2}\sqrt{a^2}$ Product property
$= 3\sqrt{2}\,a$ $\sqrt{9} = 3$, $\sqrt{a^2} = a$
$= 3a\sqrt{2}$ Multiply

1. $\sqrt{16}$ 2. $\sqrt{63}$ 3. $\sqrt{20}$ 4. $\sqrt{75}$ 5. $\sqrt{45}$ 6. $\sqrt{48}$
7. $\sqrt{32}$ 8. $\sqrt{27}$ 9. $\sqrt{80}$ 10. $\sqrt{54}$ 11. $\sqrt{98}$ 12. $\sqrt{96}$
13. $\sqrt{a^7}$ 14. $\sqrt{a^5}$ $\boxed{15.}$ $\sqrt{4a^2b^3}$ 16. $\sqrt{9ab^4c^3}$ 17. $\sqrt{27a^3b^5}$
18. $\sqrt{24x^5yz^3}$ $\boxed{19.}$ $\sqrt{6}\sqrt{3}$ 20. $\sqrt{27}\sqrt{6}$ 21. $\sqrt{15}\sqrt{15}$ 22. $\sqrt{11}\sqrt{11}$
23. $\sqrt{6}\sqrt{10}$ 24. $\sqrt{18}\sqrt{24}$ 25. $\sqrt{25}\sqrt{15}$ 26. $\sqrt{20}\sqrt{20}$ 27. $\sqrt{5}\sqrt{15}$
28. $\sqrt{2a}\sqrt{3a}$ 29. $\sqrt{5x}\sqrt{15x}$ 30. $\sqrt{6x}\sqrt{14xy}$ 31. $\sqrt{2a}\sqrt{24b^2}$

Perform any indicated operations and simplify. Assume that all variables represent positive real numbers.

Examples $\sqrt[3]{8a^4}$

Solutions $= \sqrt[3]{8 \cdot a^3 \cdot a}$ Factor having perfect cubes
$= \sqrt[3]{8}\,\sqrt[3]{a^3}\,\sqrt[3]{a}$ Product property
$= 2a\,\sqrt[3]{a}$ $\sqrt[3]{8} = 2$, $\sqrt[3]{a^3} = a$

$\sqrt[3]{9a^2}\,\sqrt[3]{3a^2}$
$= \sqrt[3]{9a^2 \cdot 3a^2}$ Product property
$= \sqrt[3]{27a^4}$ Multiply radicands
$= \sqrt[3]{27 \cdot a^3 \cdot a}$ Factor having perfect cubes
$= \sqrt[3]{27}\,\sqrt[3]{a^3}\,\sqrt[3]{a}$ Product property
$= 3a\sqrt[3]{a}$ $\sqrt[3]{27} = 3$, $\sqrt[3]{a^3} = a$

36. $\sqrt[3]{48}$ 37. $\sqrt[5]{64}$ 38. $\sqrt[4]{32}$ 39. $\sqrt[3]{24}$ 40. $\sqrt[5]{a^7}$
41. $\sqrt[3]{b^8}$ $\boxed{42.}$ $\sqrt[3]{x^9}$ 43. $\sqrt[5]{y^{15}}$ 44. $\sqrt[3]{a^{12}}$ 45. $\sqrt[3]{4a^2b^3}$
46. $\sqrt[3]{8r^2s^8}$ 47. $\sqrt[3]{16a^4b^5}$ 48. $\sqrt[5]{64x^{10}y^{14}}$ 49. $\sqrt[3]{81a^5b^{11}}$ 50. $\sqrt[3]{a^2}\,\sqrt[3]{a}$
51. $\sqrt[3]{b^2}\,\sqrt[3]{b^2}$ 52. $\sqrt[5]{b^4}\,\sqrt[5]{b^3}$ 53. $\sqrt[5]{a}\,\sqrt[5]{a^4}$ $\boxed{54.}$ $\sqrt[3]{5a^2b}\,\sqrt[3]{75a^2b^2}$
55. $\sqrt[3]{3ab^2}\,\sqrt[3]{18a^2b^2}$ 56. $\sqrt[4]{8a^3b}\,\sqrt[4]{4a^2b^2}$ 57. $\sqrt[4]{27a^2b^3}\,\sqrt[4]{9ab}$ 58. $\sqrt[3]{25x^5y^7}\,\sqrt[3]{15xy^3}$
59. $\sqrt[3]{16a^{11}b^4}\,\sqrt[3]{12a^4b^6}$ 60. $\sqrt[4]{8xy}\,\sqrt[4]{4x^3y^3}$

Quotient property for radicals

The square root of a fraction

The following example will help us develop a property for division involving radicals.

$$\sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$$

We also observe that

$$\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$$

From our example, we can conclude that

$$\sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}}$$

We can now generalize this idea.

Quotient property for square roots

For any nonnegative real numbers a and b, where $b \neq 0$,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Concept

The square root of a fraction can be written as the square root of the numerator divided by the square root of the denominator.

Examples

Simplify the following expressions. Assume that all variables represent positive real numbers.

1. $\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}}$ — Rewrite as the square root of the numerator over the square root of the denominator and simplify

$= \frac{4}{5}$

2. $\sqrt{\frac{36}{49}} = \frac{\sqrt{36}}{\sqrt{49}} = \frac{6}{7}$

3. $\sqrt{\frac{81}{100}} = \frac{\sqrt{81}}{\sqrt{100}} = \frac{9}{10}$

4. $\sqrt{\frac{x^4}{64}} = \frac{\sqrt{x^4}}{\sqrt{64}} = \frac{x^2}{8}$

5. $\sqrt{\frac{x^3}{y^4}} = \frac{\sqrt{x^3}}{\sqrt{y^4}} = \frac{x\sqrt{x}}{y^2}$

Rationalizing the denominator

When simplifying and evaluating radical expressions containing a radical in the denominator, it is easier if we can eliminate the radical in the denominator. For example,

$$\sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Since $\sqrt{5} \cdot \sqrt{5} = 5$, we can eliminate the radical $\sqrt{5}$ in the denominator by multiplying the numerator and the denominator of the fraction by $\sqrt{5}$.

$$\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{25}} = \frac{2\sqrt{5}}{5}$$

The process of changing the denominator from a radical to a rational number is called **rationalizing the denominator.**

Rationalizing the denominator

1. Multiply the numerator and the denominator by the square root that is in the denominator. The radicand in the denominator will be a perfect-square integer.
2. Simplify the radical expressions in the numerator and the denominator.
3. Reduce the resulting fraction if possible.

Examples

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers.

1. $\frac{5}{\sqrt{7}} = \frac{5}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$ Multiply numerator and denominator by $\sqrt{7}$

$= \frac{5\sqrt{7}}{\sqrt{49}}$ Multiply in numerator and denominator

$= \frac{5\sqrt{7}}{7}$ $\sqrt{49} = 7$

2. $\frac{4}{\sqrt{6}} = \frac{4}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{6}}{\sqrt{36}} = \frac{4\sqrt{6}}{6} = \frac{2\sqrt{6}}{3}$

Note In example 2, we were able to reduce the fraction as a final step. Always check to see that the answer is in reduced form.

3. $\dfrac{a}{\sqrt{a}} = \dfrac{a}{\sqrt{a}} \cdot \dfrac{\sqrt{a}}{\sqrt{a}} = \dfrac{a\sqrt{a}}{\sqrt{a^2}} = \dfrac{a\sqrt{a}}{a} = \sqrt{a}$

4. $\sqrt{\dfrac{a^3}{b}} = \dfrac{\sqrt{a^3}}{\sqrt{b}} = \dfrac{a\sqrt{a}}{\sqrt{b}} \cdot \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{ab}}{\sqrt{b^2}} = \dfrac{a\sqrt{ab}}{b}$

The following is a summary of the conditions necessary for a radical expression to be in **simplest form,** also called **standard form.**

1. The radicand contains no factors that can be written with an exponent greater than or equal to the index. ($\sqrt{a^3}$ violates this.)
2. The radicand contains no fractions. $\left(\sqrt{\dfrac{a}{b}}\right.$ violates this.$\Big)$
3. No radicals appear in the denominator. $\left(\dfrac{1}{\sqrt{a}}\right.$ violates this.$\Big)$

The nth root of a fraction

Our quotient property for square roots can be extended to radicals with any index.

Quotient property for radicals

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad (b \neq 0)$$

Concept

The nth root of a fraction can be written as the nth root of the numerator divided by the nth root of the denominator.

Examples

Simplify the following expressions. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{\dfrac{8}{27}} = \dfrac{\sqrt[3]{8}}{\sqrt[3]{27}}$ Rewrite as the cube root of the numerator over the cube root of the denominator and simplify

$= \dfrac{2}{3}$

2. $\sqrt[5]{\dfrac{32}{a^5}} = \dfrac{\sqrt[5]{32}}{\sqrt[5]{a^5}} = \dfrac{2}{a}$

3. $\sqrt[3]{\dfrac{x^5}{y^6}} = \dfrac{\sqrt[3]{x^5}}{\sqrt[3]{y^6}} = \dfrac{x\sqrt[3]{x^2}}{y^2}$

Rationalizing the denominator (nth root)

The following example will help us develop a general rule for rationalizing a denominator that has a single term.

$$\sqrt[3]{\frac{1}{a}} = \frac{\sqrt[3]{1}}{\sqrt[3]{a}} = \frac{1}{\sqrt[3]{a}}$$

At this point, a radical still remains in the denominator. We must now determine what we can do to the fraction to remove the radical from the denominator.

Observations:

1. We can multiply the numerator and the denominator by the same number and form equivalent fractions.
2. If we multiply by a radical, the indices must be the same to carry out the multiplication.
3. To bring a factor out from under the radical symbol and not leave any of the factor behind, the index must divide evenly into the exponent.

With these observations in mind, we rationalize the fraction as follows:

$= \dfrac{1}{\sqrt[3]{a}} \cdot \dfrac{\sqrt[3]{}}{\sqrt[3]{}}$ Indices are the same

$= \dfrac{1}{\sqrt[3]{a}} \cdot \dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}$ Multiply numerator and denominator by the same number

$= \dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^3}}$ The sum of the exponents of a in the denominator is equal to the index

$= \dfrac{\sqrt[3]{a^2}}{a}$ The index divides evenly into the exponent, the radical is eliminated

To rationalize an nth root denominator

1. Multiply the numerator and the denominator by a radical with the same index as the radical that we wish to eliminate from the denominator.
2. The exponent of each factor under the radical must be such that when we add it to the original exponent of the factor under the radical in the denominator, the sum will be equal to or divisible by the index of the radical.
3. Carry out the multiplication and reduce the fraction if possible.

Examples

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers.

1. $\frac{1}{\sqrt[3]{7}} = \frac{1}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}}$ Multiply numerator and denominator by $\sqrt[3]{7^2}$

$= \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^3}}$ Multiply in numerator and denominator ($\sqrt[3]{7}\sqrt[3]{7^2} = \sqrt[3]{7^3}$)

$= \frac{\sqrt[3]{7^2}}{7}$ $\sqrt[3]{7^3} = 7$

$= \frac{\sqrt[3]{49}}{7}$ $7^2 = 49$

2. $\frac{a}{\sqrt[5]{b^2}} = \frac{a}{\sqrt[5]{b^2}} \cdot \frac{\sqrt[5]{b^3}}{\sqrt[5]{b^3}} = \frac{a\sqrt[5]{b^3}}{\sqrt[5]{b^5}} = \frac{a\sqrt[5]{b^3}}{b}$

3. $\frac{x}{\sqrt[4]{x}} = \frac{x}{\sqrt[4]{x}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{x\sqrt[4]{x^3}}{\sqrt[4]{x^4}} = \frac{x\sqrt[4]{x^3}}{x} = \sqrt[4]{x^3}$

4. $\frac{1}{\sqrt[5]{a^2b}} = \frac{1}{\sqrt[5]{a^2b}} \cdot \frac{\sqrt[5]{a^3b^4}}{\sqrt[5]{a^3b^4}} = \frac{\sqrt[5]{a^3b^4}}{\sqrt[5]{a^5b^5}} = \frac{\sqrt[5]{a^3b^4}}{ab}$

Exercises

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers. See examples

Examples $\sqrt{\dfrac{16}{49}}$ $\qquad \sqrt{\dfrac{a^2}{b}}$ $\qquad \dfrac{6}{\sqrt{6}}$

Solutions

$= \dfrac{\sqrt{16}}{\sqrt{49}} = \dfrac{4}{7}$

Rewrite as the square root of the numerator over the square root of the denominator and simplify

$= \dfrac{\sqrt{a^2}}{\sqrt{b}} = \dfrac{a}{\sqrt{b}}\,\dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$

$= \dfrac{6}{\sqrt{6}}\,\dfrac{\sqrt{6}}{\sqrt{6}} = \dfrac{6\sqrt{6}}{6} = \sqrt{6}$

Multiply the numerator and the denominator by the square root in the denominator and simplify

1. $\sqrt{\dfrac{9}{25}}$	2. $\sqrt{\dfrac{25}{36}}$	3. $\sqrt{\dfrac{25}{49}}$	4. $\sqrt{\dfrac{81}{100}}$	5. $\sqrt{\dfrac{3}{4}}$
6. $\sqrt{\dfrac{5}{9}}$	7. $\sqrt{\dfrac{64}{a^2}}$	8. $\sqrt{\dfrac{y^4}{16}}$	9. $\sqrt{\dfrac{1}{2}}$	10. $\sqrt{\dfrac{1}{3}}$
11. $\sqrt{\dfrac{4}{7}}$	12. $\sqrt{\dfrac{9}{11}}$	13. $\sqrt{\dfrac{1}{15}}$	14. $\sqrt{\dfrac{1}{14}}$	$\boxed{15.}$ $\sqrt{\dfrac{4}{75}}$
16. $\sqrt{\dfrac{5}{12}}$	17. $\dfrac{2}{\sqrt{2}}$	18. $\dfrac{6}{\sqrt{3}}$	19. $\dfrac{10}{\sqrt{8}}$	20. $\dfrac{15}{\sqrt{27}}$
21. $\sqrt{\dfrac{x^2}{y}}$	22. $\sqrt{\dfrac{1}{a}}$	23. $\sqrt{\dfrac{1}{x}}$	24. $\sqrt{\dfrac{a^2}{b^3}}$	25. $\dfrac{\sqrt{a^5}}{\sqrt{a}}$

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers. See examples

Examples $\sqrt[3]{\dfrac{1}{27}}$ $\qquad \sqrt[3]{\dfrac{8a^5}{b^3}}$ $\qquad \dfrac{1}{\sqrt[5]{b^2}}$

Solutions

$= \dfrac{\sqrt[3]{1}}{\sqrt[3]{27}} = \dfrac{1}{3}$

$= \dfrac{\sqrt[3]{8a^5}}{\sqrt[3]{b^3}} = \dfrac{\sqrt[3]{2^3a^5}}{b} = \dfrac{2a\sqrt[3]{a^2}}{b}$

Rewrite as the cube root of the numerator over the cube root of the denominator and simplify

$= \dfrac{1}{\sqrt[5]{b^2}} \cdot \dfrac{\sqrt[5]{b^3}}{\sqrt[5]{b^3}} = \dfrac{\sqrt[5]{b^3}}{\sqrt[5]{b^5}} = \dfrac{\sqrt[5]{b^3}}{b}$

Multiply the numerator and the denominator by $\sqrt[5]{b^3}$

Perfect 5th root

$\sqrt[5]{b^5} = b$

30. $\sqrt[3]{\dfrac{8}{27}}$	31. $\sqrt[3]{\dfrac{1}{8}}$	32. $\sqrt[4]{\dfrac{16}{81}}$	33. $\sqrt[3]{\dfrac{27}{125}}$	34. $\sqrt[3]{\dfrac{a^2}{b^2}}$
$\boxed{35.}$ $\sqrt[3]{\dfrac{3a^6}{b^3}}$	36. $\sqrt[3]{\dfrac{x}{y^{12}}}$	37. $\sqrt[5]{\dfrac{a^4}{b^{10}}}$	38. $\sqrt[5]{\dfrac{32x^4}{y^5}}$	$\boxed{39.}$ $\sqrt[4]{\dfrac{a^4b^9}{c^{11}}}$
40. $\sqrt[4]{\dfrac{a^9b^{13}}{c^8}}$	41. $\sqrt[5]{\dfrac{x^3y^2}{z^{15}}}$	42. $\sqrt[3]{\dfrac{8}{9}}$	43. $\sqrt[3]{\dfrac{4}{25}}$	44. $\sqrt[3]{\dfrac{27}{16}}$
$\boxed{45.}$ $\sqrt[4]{\dfrac{16}{125}}$	46. $\sqrt[4]{\dfrac{3}{4}}$	47. $\sqrt[3]{\dfrac{x^3}{y^2}}$	48. $\sqrt[3]{\dfrac{x^6}{y}}$	$\boxed{49.}$ $\dfrac{ab}{\sqrt[3]{a^2}}$
50. $\dfrac{xy}{\sqrt[5]{y^3}}$	$\boxed{51.}$ $\sqrt[3]{\dfrac{a^3}{b^2c}}$	52. $\sqrt[3]{\dfrac{8}{xy^2}}$	53. $\sqrt[3]{\dfrac{a^2}{b^2c}}$	54. $\dfrac{a}{\sqrt[5]{a^2b^4}}$
55. $\dfrac{ab}{\sqrt[3]{ab^2}}$	56. $\dfrac{xy^2}{\sqrt[5]{x^4y}}$			