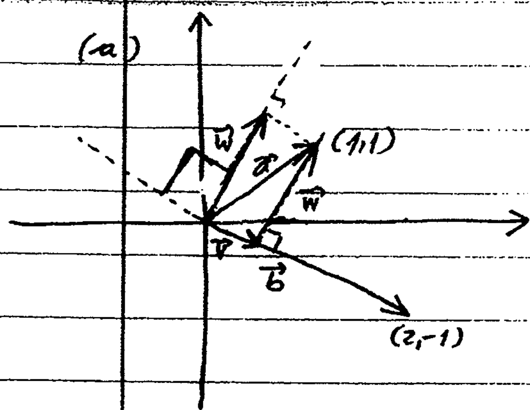
WORKSHEET 3 (09/04/12)

① $\vec{a} = \vec{i} + \vec{j} = \langle 1,1 \rangle$, $\vec{b} = 2\vec{i} - 1\vec{j} = \langle 2,-1 \rangle$

(a)



$$\vec{v} := \operatorname{proj}_{\vec{b}} \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{2-1}{4+1}\vec{b} = \left\langle \frac{2}{5}, -\frac{1}{5} \right\rangle.$$

(b) $\vec{w} := \operatorname{orth}_{\vec{b}} \vec{a} = \vec{a} - \vec{v}$

$$= \langle 1,1 \rangle - \left\langle \frac{2}{5}, -\frac{1}{5} \right\rangle = \left\langle \frac{3}{5}, \frac{6}{5} \right\rangle$$

(c) $\vec{v}\cdot\vec{w} = \frac{2}{5}\cdot\frac{3}{5} + \left(-\frac{1}{5}\right)\cdot\frac{6}{5} = \frac{6-6}{25} = 0 \Longrightarrow \vec{v} \perp \vec{w}.$

(d) distance $= |\vec{w}| = \sqrt{\frac{9}{25} + \frac{36}{25}} = \frac{\sqrt{45}}{5} = \frac{3\sqrt{5}}{5}.$

② We show that $\vec{v} = \operatorname{proj}_{\vec{b}} \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$ is $\perp$ on $\vec{w} = \vec{a} - \vec{v}$.

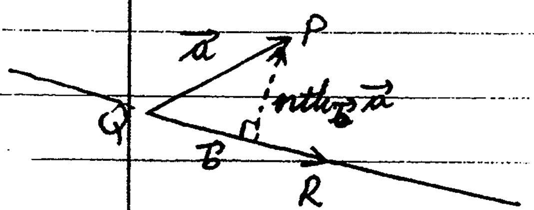
$$\vec{v}\cdot\vec{w} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}\cdot\left(\vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}\right) = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\left(\vec{b}\cdot\vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\frac{\vec{b}\cdot\vec{b}}{|\vec{b}|^2}\right)$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\left(\underbrace{\vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b}}_{0}\right) = 0 \Longrightarrow \vec{v} \perp \vec{w}$$

This computation works in any dimension becaus the formulas for $\operatorname{proj}_{\vec{b}}\vec{a}$ and $\operatorname{orth}_{\vec{b}}\vec{a}$ are given by the same formulas and the properties of the dot product are similar.

$P = (3,4,-1)$

③ $\vec{l}(0) = (2,3,-2) = Q$, $\vec{l}(1) = (3,2,-1) = R$



$\vec{a} := \overrightarrow{QP} = \langle 3-2, 4-3, -1-(-2) \rangle$

$= \langle 1,1,1 \rangle$

$\vec{b} := \overrightarrow{QR} = \langle 1,-1,1 \rangle$

The required distance is given by

$$|\operatorname{orth}_{\vec{b}}\vec{a}| = |\vec{a} - \operatorname{proj}_{\vec{b}}\vec{a}| = \left|\vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}\right| =$$

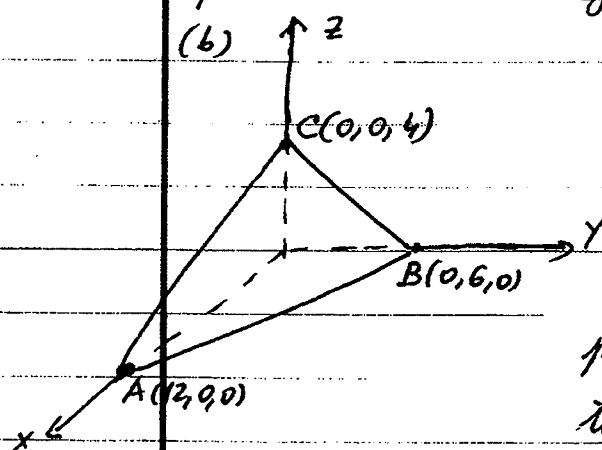
$$= \left| \langle 1,1,1\rangle - \frac{1-1+1}{1+1+1}\cdot\langle 1,-1,1\rangle \right| = \left| \langle 1,1,1\rangle - \langle \tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}\rangle \right|$$

$$= \left| \langle \tfrac{2}{3}, \tfrac{4}{3}, -\tfrac{2}{3}\rangle \right| = \frac{1}{3}\sqrt{4+16+4} = \frac{\sqrt{24}}{3} = \frac{2\sqrt{6}}{3}$$

(4) (a) $\vec{n} = \langle 1,2,3\rangle$ is normal to the plane $x+2y+3z=12$. This can be easily checked: the x-intercept of the plane is the point $A(12,0,0)$, the y-intercept is $B(0,6,0)$, and the z-intercept is $C(0,0,4)$, so $\overrightarrow{AB} = \langle -12, 6, 0\rangle$, $\overrightarrow{BC} = \langle 0,-6,4\rangle$ determine two non-parallel directions in this plane.

But $\vec{n}\cdot\overrightarrow{AB} = -12+12+0=0$ and $\vec{n}\cdot\overrightarrow{BC} = 0-12+12=0$, hence $\vec{n}\perp\overrightarrow{AB}$ and $\vec{n}\perp\overrightarrow{BC}$, showing that $\vec{n}$ is $\perp$ on the plane determined by the triangle ABC.

(b)



(c) $\overrightarrow{AB} = \langle -12, 6, 0\rangle$ and $\overrightarrow{BC} = \langle 0,-6,4\rangle$ are non-parallel and are parallel to the plane ABC.

(d) $\vec{n}' := \overrightarrow{AB}\times\overrightarrow{BC}$ must be perpendicular on both vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, thus $\vec{n}'$ is $\perp$ on the plane ABC

$$\vec{n}' = \langle -12, 6, 0\rangle \times \langle 0,-6,4\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -12 & 6 & 0 \\ 0 & -6 & 4 \end{vmatrix}$$

$$= 24\vec{i} + 48\vec{j} + 72\vec{k} = 24\langle 1,2,3\rangle = 24\cdot\vec{n}$$

so $\vec{n}' \parallel \vec{n}$.

(e) Consider the general equation $\alpha x+\beta y+\gamma z+\delta=0$ of a plane that contains $C(0,0,4)$ (so $4\gamma+\delta=0$) and is perpendicular on $\vec{n}'$! The x-intercept if $x=-\frac{\delta}{\alpha}$ and the y-intercept $y=-\frac{\delta}{\beta}$, so $\vec{n}' = \langle 24,48,72\rangle$ must be $\perp$ on $\overrightarrow{CA'}$ with $A'(-\frac{\delta}{\alpha},0,0)$ and on $\overrightarrow{CB'}$ with $B'(0,-\frac{\delta}{\beta},0)$.

$\overrightarrow{CA'} = \langle \frac{\delta}{\alpha}, 0, -4 \rangle, \quad \overrightarrow{CB'} = \langle 0, -\frac{\delta}{\beta}, -4 \rangle$

$\vec{n}' \cdot \overrightarrow{CA'} = 24 \langle 1,2,3 \rangle \cdot \langle -\frac{\delta}{\alpha}, 0, -4 \rangle = 24 \left(-\frac{\delta}{\alpha} - 12 \right) = 0$

$\Longrightarrow \delta + 12\alpha = 0$

$\vec{n}' \cdot \overrightarrow{CB'} = 24 \langle 1,2,3 \rangle \cdot \langle 0, -\frac{\delta}{\beta}, -4 \rangle = 24 \left(-2\frac{\delta}{\beta} - 12 \right) = 0$

$\Longrightarrow \delta + 6\beta = 0$

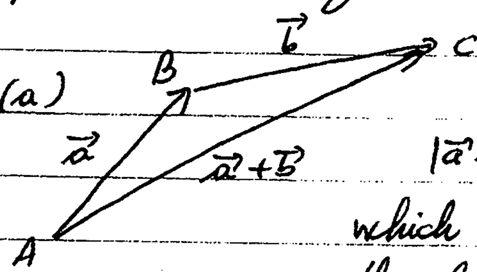
Hence $\delta = -12\alpha = -6\beta = -4\gamma \Longrightarrow \begin{cases} \alpha = -\frac{\delta}{12} \\ \beta = -\frac{\delta}{6} \\ \gamma = -\frac{\delta}{4} \end{cases}$

$\rightarrow$ The equation of the plane is $-\frac{\delta}{12}x - \frac{\delta}{6}y - \frac{\delta}{4}z + \delta = 0$

$\Longleftrightarrow -\frac{\delta}{12}(x + 2y + 3z - 12) = 0 \qquad (1)$

These two equations ((1) and $x + 2y + 3z = 12$) are proportional so they must define the same plane.

5 (a)



$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}| \Longleftrightarrow |AC| \le |AB| + |BC|$ which states the well-known fact that the length of a side of a triangle is less or equal than the sum of lengths of the other two sides.

(b) Done in the lecture.

(c) $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = \underbrace{\vec{a}\cdot\vec{a}}_{|\vec{a}|^2} + \vec{a}\cdot\vec{b} + \underbrace{\vec{b}\cdot\vec{a}}_{\vec{a}\cdot\vec{b}} + \underbrace{\vec{b}\cdot\vec{b}}_{|\vec{b}|^2}$

$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} \qquad (1)$

$\vec{a}\cdot\vec{b} \le |\vec{a}\cdot\vec{b}| \underset{\text{Cauchy-Schwarz}}{\le} |\vec{a}||\vec{b}| \qquad (2)$

$(1)+(2) \Longrightarrow |\vec{a}+\vec{b}|^2 \leq |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}| = (|\vec{a}|+|\vec{b}|)^2$

$\xrightarrow{\sqrt{\ }} |\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|.$