Math 241, Sections BL1 and BL2

Quiz # 1 BDD

September 11, 2012

Solve both exercises. Show work to get credit.

1) [5=4+1pts.] Find the area of the parallelogram with vertices $K(1,2,2)$, $L(1,4,4)$, $M(3,8,4)$, and $N(3,6,2)$. Why is this a parallelogram?

Solution: First we'll compute the vectors that represent the edges and diagonals of the figure:

$\overrightarrow{KL} = (0,2,2)$ $\overrightarrow{KM} = (2,6,2)$ $\overrightarrow{KN} = (2,4,0)$

$\overrightarrow{LM} = (2,4,0)$ $\overrightarrow{LN} = (2,2,-2)$ $\overrightarrow{MN} = (0,-2,-2)$

Since $\overrightarrow{KN} = \overrightarrow{LM}$, the figure is a parallelogram.

We know that the area of a parallelogram can be computed as the magnitude of the cross product of two vectors representing the (different) sides of the parallelogram:

$$\begin{aligned}
\text{Area} &= \left|\overrightarrow{KL} \times \overrightarrow{KN}\right| \\
&= \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 2 \\ 2 & 4 & 0 \end{vmatrix} \right\| \\
&= |(0-8)\mathbf{i} - (0-4)\mathbf{j} + (0-4)\mathbf{k}| \\
&= |-8\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}| \\
&= 4\,|2\mathbf{i} - \mathbf{j} + \mathbf{k}| \\
&= 4\sqrt{2^2 + (-1)^2 + 1^2} \\
&= 4\sqrt{6}.
\end{aligned}$$

2) [5pts.] Find an equation of the plane that passes through the point $(-2,3,2)$ and contains the line of intersection of the planes $x+y-z=3$ and $4x-y+5z=4$.

Solution: There are several methods that will lead to a solution. Here I'll show a cross-product-heavy approach. Let $P=(-2,3,2)$.

Consider the line of intersection of the two given planes. Setting $z=0$, we have the system

$$\begin{cases} x+y-0=3, \\ 4x-y+0=4. \end{cases}$$

The solution to this system is $x=7/5$, $y=8/5$. So the point $Q=(7/5,8/5,0)$ lies on the line of intersection. We can find a direction vector of the line as the cross product of the normal vectors to the given planes:

$$\begin{aligned}
\mathbf{v} :&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 4 & -1 & 5 \end{vmatrix} \\
&= (5-1)\mathbf{i} - (5+4)\mathbf{j} + (-1-4)\mathbf{k} \\
&= 4\mathbf{i} - 9\mathbf{j} - 5\mathbf{k}.
\end{aligned}$$

Since our desired plane contains the line, this vector $\mathbf{v}$ is parallel to our plane. Another vector parallel to our plane is

$$\mathbf{u} := \overrightarrow{PQ} = (17/5, -7/5, -2).$$

So now we can find a normal vector to our plane, (and since all we care about is direction, I'll scale up $\mathbf{u}$ to get rid of fractions)

$$\begin{aligned}
\mathbf{n} :&= \mathbf{5u} \times \mathbf{v} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 17 & -7 & -10 \\ 4 & -9 & -5 \end{vmatrix} \\
&= (35-90)\mathbf{i} - (-85+40)\mathbf{j} + (-153+28)\mathbf{k} \\
&= -55\mathbf{i} + 45\mathbf{j} - 125\mathbf{k}.
\end{aligned}$$

For a simpler formula at the end, I'll take the parallel vector $(11, -9, 25)$. So the desired plane has this as its normal vector, and it contains the point P, so an equation for it is

$$11(x+2) - 9(y-3) + 25(z-2) = 0.$$

That's fine, or we could move the constants around a bit to obtain

$$11x - 9y + 25z = 1.$$