Discussion 3 – Tuesday, August 31

1. Let $\mathbf{a} = (-\sqrt{3}, 0, -1, 0)$ and $\mathbf{b} = (1, 1, 0, 1)$.

(a) Find the distance between the points $(-\sqrt{3}, 0, -1, 0)$ and $(1, 1, 0, 1)$.

Solution: The distance between the points is the length of a vector joining them:

$$\begin{aligned} ||\mathbf{b} - \mathbf{a}|| &= \sqrt{(1-(-\sqrt{3}))^2 + (1-0)^2 + (0-(-1))^2 + (1-0)^2} \\ &= \sqrt{(1 + 2\sqrt{3} + 3) + 1 + 1 + 1} \\ &= \sqrt{7 + 2\sqrt{3}}. \end{aligned}$$

(b) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

Solution: We know that $\mathbf{a} \cdot \mathbf{b} = ||a||||b|| \cos\theta$ where θ is the angle between the vectors. So

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||a||||b||} = \frac{(-\sqrt{3})(1) + (0)(1) + (-1)(0) + (0)(1)}{\sqrt{3+0+1+0}\ \sqrt{1+1+0+1}} = \frac{-\sqrt{3}}{2\sqrt{3}} = -\frac{1}{2}.$$

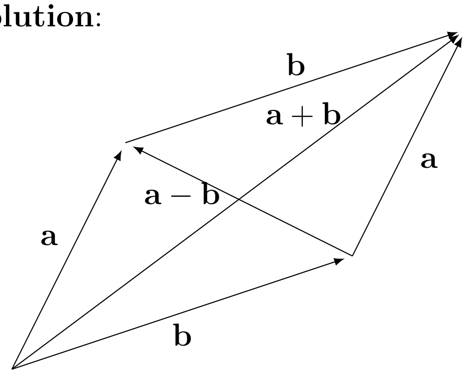
2. The Parallelogram Identity.

Let $\mathbf{a}$ and $\mathbf{b}$ be any vectors in $\mathbb{R}^n$. The parallelogram identity states that

$$||\mathbf{a} + \mathbf{b}||^2 + ||\mathbf{a} - \mathbf{b}||^2 = 2||\mathbf{a}||^2 + 2||\mathbf{b}||^2.$$

(a) Give a geometric interpretation of the parallelogram identity. (Hint: Draw a parallelogram)

Solution:



The left hand side of the equality is the sum of the squares of the lengths of the diagonals. The right hand side is the sum of the squares of the lengths of all four sides.

(b) Prove the parallelogram identity. (Hint: Use the fact that $||\mathbf{a}+\mathbf{b}||^2 = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b})$).

Solution: We have that

$$\begin{aligned} (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = ||\mathbf{a}||^2 + 2\mathbf{a} \cdot \mathbf{b} + ||\mathbf{b}||^2 \text{ and} \\ (\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = ||\mathbf{a}||^2 - 2\mathbf{a} \cdot \mathbf{b} + ||\mathbf{b}||^2 \end{aligned}$$

Adding these two together gives us the result.

3. Let $\mathbf{a} = (2, 1, 4)$ and $\mathbf{b} = (0, 2, 3)$.

(a) Find the projection of $\mathbf{a}$ onto $\mathbf{b}$.

Solution:

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \frac{\mathbf{b}}{||\mathbf{b}||} = \frac{14}{13}(0, 2, 3).$$

(b) The orthogonal complement of the vector $\mathbf{a}$ with respect to the vector $\mathbf{b}$ is defined by

$$\mathrm{orth}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} - \mathrm{proj}_{\mathbf{b}}(\mathbf{a}).$$

Find the orthogonal complement of the vector $\mathbf{a} = (2, 1, 4)$ with respect to $\mathbf{b} = (0, 2, 3)$.
Solution:

$$\mathrm{orth}_{\mathbf{b}}(\mathbf{a}) = (2, 1, 4) - \frac{14}{13}(0, 2, 3) = \frac{1}{13}(26, -15, 10).$$

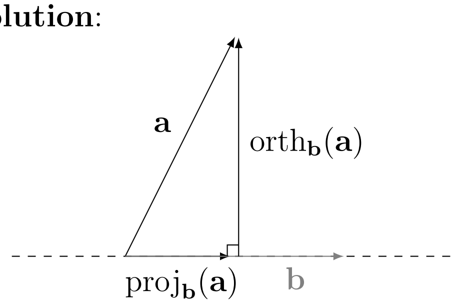
(c) Show that $\mathrm{orth}_{\mathbf{b}}\mathbf{a} \perp \mathrm{proj}_{\mathbf{b}}\mathbf{a}$. Prove algebraically that this is always the case.
Solution: To show that two vectors are orthogonal, we need to show that their dot product is zero:

$$\begin{aligned}
\mathrm{orth}_{\mathbf{b}}(\mathbf{a}) \cdot \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) &= \frac{1}{13}(26, -15, 10) \cdot \frac{14}{13}(0, 2, 3) \\
&= \frac{1}{13}\frac{14}{13}((26)(0) + (-15)(2) + (10)(3)) = \frac{14}{169}\, 0 = 0
\end{aligned}$$

The calculation in general is

$$\begin{aligned}
\mathrm{orth}_{\mathbf{b}}(\mathbf{a}) \cdot \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) &= (\mathbf{a} - \mathrm{proj}_{\mathbf{b}}(\mathbf{a})) \cdot \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) \\
&= \mathbf{a} \cdot \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) - \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) \cdot \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) \\
&= \mathbf{a} \cdot \left(\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\frac{\mathbf{b}}{||\mathbf{b}||}\right) - \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\frac{\mathbf{b}}{||\mathbf{b}||} \cdot \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\frac{\mathbf{b}}{||\mathbf{b}||} \\
&= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\right)^2 \frac{\mathbf{b}\cdot\mathbf{b}}{||\mathbf{b}||^2} \\
&= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\right)^2 - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\right)^2 1 \\
&= 0.
\end{aligned}$$

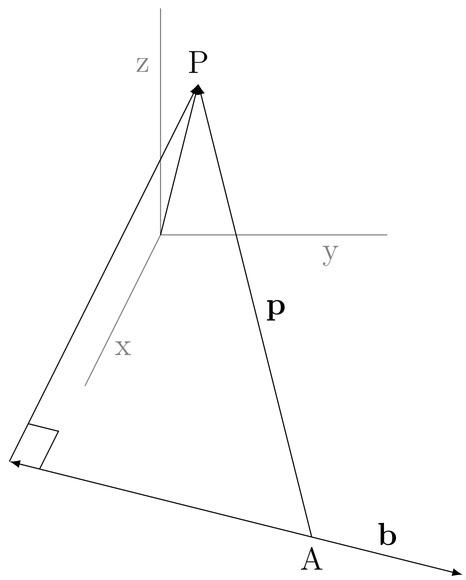
(d) Draw a picture relating $\mathbf{a}$, $\mathrm{orth}_{\mathbf{b}}\mathbf{a}$ and $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$.
Solution:



(e) Consider the line $\mathbf{l}(t) = t\mathbf{b}$ for all t. Find the distance from the point $\mathbf{a}$ to the line $\mathbf{l}$.
Solution: From our picture in (d), the distance should be

$$||\mathrm{orth}_{\mathbf{b}}(\mathbf{a})|| = \left|\left|\frac{1}{13}(26, -15, 10)\right|\right| = \frac{1}{13}\sqrt{26^2 + (-15)^2 + 10^2} = \frac{\sqrt{1001}}{13}.$$

4. Find the distance between the point $P(2, 1, 3)$ and the line $\mathbf{l}(t) = (2, 3, -2) + t(-1, 1, -2)$. (Hint: Your work in prob 3e should help and, in this class, when in doubt, draw a picture.)
Solution: Let $A(2, 3, -2)$, $\mathbf{b} = (-1, 1, -2)$. From the following picture we can see that the distance is given by the magnitude of the orthogonal complement of the vector $\mathbf{p} = \vec{AP} = (0, -2, 5)$ with respect to $\mathbf{b}$:



So we have

$$\begin{aligned}\text{orth}_{(-1,1,-2)}(0,-2,5) &= (0,-2,5) - \frac{(-1)(0)+(1)(-2)+(-2)(5)}{(-1)^2+1^2+(-2)^2}(-1,1,-2)\\ &= (0,-2,5)-(-2)(-1,1,-2) = (-2,0,1).\end{aligned}$$

So the distance is the magnitude of this vector, $\sqrt{(-2)^2+0^2+1^2} = \sqrt{5}$.

5. Consider the points $A(2,-1,4)$ and $B(4,6,1)$.
(a) Find a vector equation of the line containing these two points.
Solution: Since the direction of this line can be given by $(4,6,1)-(2,-1,4) = (2,7,-3)$, and the point A is on the line, such an equation is $\mathbf{l}(t) = (2,-1,4)+t(2,7,-3)$, for all $t \in \mathbb{R}$.

(b) Find a vector equation of the line segment that joins the points A and B. (Hint: How do we restrict the parameter to only include the points on this line segment?)
Solution: We can use the above parameterization, but restrict to $0 \leq t \leq 1$.

6.
(a) Consider the plane formed by the points $(1,1,2)$, $(4,1,2)$ and $(1,2,4)$. Using $(1,1,2)$ as the anchor point, find a set of parametric equations that describe the plane.
Solution: Every point in the plane can be reached by starting at $(1,1,2)$, then following some multiple of the vectors from the anchor point to each of the two other points. So one possible equation for the plane is

$$\mathbf{p}(t,s) = (1,1,2)+t(3,0,0)+s(0,1,2), \text{ for all } t,s \in \mathbb{R}.$$

(b) Determine if the point $(4,2,4)$ lies on the plane described in (a).
Solution: If so, then it satisfies the equation above for some values of t and s. Setting components equal, we have a system of equations

$$\begin{cases} 1+3t+0s=4,\\ 1+0t+1s=2,\\ 2+0t+2s=4.\end{cases}$$

We can check that $t = 1, s = 1$ satisfies all three equations, so the point $(4, 2, 4)$ is on the plane.

(c) Using vector arithmetic, show that the quadrilateral with vertices $(1, 1, 2)$, $(4, 1, 2)$, $(1, 2, 4)$ and $(4, 2, 4)$ is a parallelogram.
Solution: For the quadrilateral to be a parallelogram, we need only check that two of the edges are parallel and of equal length. Well, $(4, 1, 2) - (1, 1, 2) = (3, 0, 0)$ and $(4, 2, 4) - (1, 2, 4) = (3, 0, 0)$, so it is indeed a parallelogram.

(d) Using a dot product, show that this parallelogram is actually a rectangle.
Solution: A parallelogram with a right angle is a rectangle. Since two of the edges have vectors $(3, 0, 0)$ and $(0, 1, 2)$ (from part (a)), we check $(3, 0, 0) \cdot (0, 1, 2) = 0$, so these vectors and hence also the sides are orthogonal.

(b) Obtain a vector description of all points that belong to the rectangle (i.e. that are either inside the rectangle or on its boundary).
Solution: From our descriptions from (a) and (b), we can restrict our parameters to $0 \leq t \leq 1$ and $0 \leq s \leq 1$ to achieve this restricted range.