SECTION 1.6 FACTORING (Part I)

Greatest common factor

To find the solution of certain equations that are not linear, we will need to study a technique called *factoring a polynomial*. Factoring polynomials will also be useful in dealing with algebraic fractions since, as we have seen with arithmetic fractions, we must have the numerator and the denominator of the fractions in a factored form to reduce or to find the least common denominator.

The first type of factoring that we will do involves finding the greatest common factor (GCF) of each term in the polynomial. Recall the statement of the distributive property.

$$a(b + c) = ab + ac$$

$a(b + c)$ is called the *factored form* of $ab + ac$. We now use the distributive property to write $3x + 6$ in the factored form. First we notice that $3x + 6$ can be written as

$$3 \cdot x + 3 \cdot 2 \qquad \text{3 is a common factor in both terms}$$

By applying the distributive property, we have

$$3x + 6 = 3 \cdot x + 3 \cdot 2 = 3(x + 2)$$

$3(x + 2)$ is the factored form of $3x + 6$.

This type of factoring, as its name implies, involves looking for numbers or variables that are **common factors** in *all* of the original terms. In our example, the number 3 was common to all of the original terms, and we were able to factor it out.

When a polynomial is factored, we "factor out" the greatest common factor, GCF. The greatest common factor consists of the following:

Greatest common factor

1. The greatest integer that is a common factor of all the numerical coefficients and
2. the variable factor(s) raised to the least power to which they were raised in any of the terms

← Factoring

Polynomial *(terms)*	Distributive Property *(determine the GCF)*	Factored Form *(factors)*
$3x + 6$	$\mathbf{3} \cdot x + \mathbf{3} \cdot 2$	$3(x + 2)$
$10x^2 + 15y$	$\mathbf{5} \cdot 2x^2 + \mathbf{5} \cdot 3y$	$5(2x^2 + 3y)$
$12a - 42b$	$\mathbf{6} \cdot 2a - \mathbf{6} \cdot 7b$	$6(2a - 7b)$
$18xy + 12xz$	$\mathbf{6x} \cdot 3y + \mathbf{6x} \cdot 2z$	$6x(3y + 2z)$

Multiplying →

In the previous examples, we determined the greatest common factor by inspection. In some problems, this may not be possible, and the following procedure will be necessary. Factor the polynomial $12x^3y + 30x^2y^3$.

Step 1 Factor each term such that it is the product of primes* and variables to powers.

$$12x^3y \qquad\qquad 30x^2y^3$$
$$2^2 \cdot 3 \cdot x^3 \cdot y \qquad\qquad 2 \cdot 3 \cdot 5 \cdot x^2 \cdot y^3$$

Step 2 Write down all the numbers and variables that are common to *every* term.

$$2 \cdot 3 \cdot x \cdot y$$

Note We do not have 5 as part of our greatest common factor since it does not appear as a factor in *all* of the terms.

Step 3 Take the numbers and variables in step 2. Raise them to the *lowest* power to which they were raised in any of the terms.

$$2^1 \cdot 3^1 \cdot x^2 \cdot y^1 = 6x^2y$$

This is the greatest common factor.

Step 4 Find the multinomial factor, the polynomial within the parentheses, by dividing each term of the polynomial being factored by the GCF.

$$\frac{12x^3y}{6x^2y} = 2x \text{ and } \frac{30x^2y^3}{6x^2y} = 5y^2$$

$$6x^2y(2x + 5y^2)$$

Step 5 We can now write the polynomial in its factored form.

$$\begin{aligned} &12x^3y + 30x^2y^3 \\ &= 6x^2y \cdot 2x + 6x^2y \cdot 5y^2 \\ &= 6x^2y(2x + 5y^2) \end{aligned}$$

Completely factored form

In the previous example, $12x^3y + 30x^2y^3$ could also be factored to $3xy(4x^2 + 10xy^2)$ or $12y\left(x^3 + \frac{5}{2}x^2y^2\right)$. This allows room for a given polynomial to be factored in many ways, unless some restrictions are placed on the procedure. We wish to factor each polynomial in a unique manner that will not permit such variations in the results. Thus it is customary to adopt the following criteria for a completely factored polynomial.

A polynomial with integer coefficients will be considered to be in **completely factored form** when it satisfies the following criteria:

Completely factored form

1. The polynomial is written as a product of polynomials with integer coefficients.
2. None of the polynomial factors other than the monomial factor can be further factored.

We see that $6x^2y(2x + 5y^2)$ is the completely factored form of the expression $12x^3y + 30x^2y^3$ since all of the coefficients are integers and, except for the monomial factor $6x^2y$, the remaining factor contains no other factor with integer coefficients.

In general, whenever we factor a monomial out of a polynomial, we factor the monomial so that it has a positive coefficient. Realize that we could also factor out the opposite, or negative, of this common factor. We could have factored out $-6x^2y$ from our example. The completely factored form would have been

$$-6x^2y(-2x - 5y^2)$$

Observe that the only change in our answer when we factor out the opposite of the common factor is that this changes the signs of all terms inside the parentheses.

Example

Write in completely factored form.

1. $7a^3 + 14a$

$= 7a(\quad + \quad)$ — Multinomial factor will have as many terms as the original expression

GCF is $7a$

$= 7a(a^2 + 2)$ — Completely factored form

$\dfrac{14a}{7a}$

$\dfrac{7a^3}{7a}$

If we want to check the answer, we apply the distributive property and perform the multiplication as follows:

$7a(a^2 + 2) = 7a \cdot a^2 + 7a \cdot 2$	Distributive property
$= 7a^3 + 14a$	Carry out the multiplication

2. $9x^5 + 6x^3 - 18x^2$

$= 3^2x^5 + 2 \cdot 3x^3 - 2 \cdot 3^2x^2$ Factor each term

$= 3x^2(\quad + \quad - \quad)$ Determine the GCF

$= 3x^2(3x^3 + 2x - 6)$ Completely factored form

3. $72a^2b - 84a^3b^4 + 48a^4b^2$

$= 2^3 \cdot 3^2 \cdot a^2 \cdot b - 2^2 \cdot 3 \cdot 7 \cdot a^3 \cdot b^4 + 2^4 \cdot 3 \cdot a^4 \cdot b^2$ Factor each term

$= 12a^2b(\quad - \quad + \quad)$ Determine the GCF

$= 12a^2b(6 - 7ab^3 + 4a^2b)$ Completely factored form

4. $3x^3y^2 + 15x^2y^4 + 3xy^2$

$= 3 \cdot x^3 \cdot y^2 + 3 \cdot 5 \cdot x^2 \cdot y^4 + 3 \cdot x \cdot y^2$ Factor each term

$= 3xy^2(\quad + \quad + \quad)$ Determine the GCF

$= 3xy^2(x^2 + 5xy^2 + 1)$ Completely factored form

Note In example 4, the last term in the factored form is 1. This situation occurs when a term and the GCF are the same, that is, whenever we are able to factor all of the numbers and variables out of a given term. For example, $\frac{3xy^2}{3xy^2} = 1$. The number of terms inside the parentheses must be equal to the number of terms in the original polynomial.

► ***Quick check*** Write $9x + 6y + 3$ and $9x^3y^2 - 18x^2y^3$ in completely factored form.

Remember that when an expression is within a grouping symbol, we treat the quantity as just one number. Therefore if we have a quantity common to all of the terms, we can factor it out of the polynomial.

Example

Factor completely.

1. $x(a - 2b) + y(a - 2b)$

The quantity $(a - 2b)$ is common to both terms. We then factor the common quantity out of each term and place the remaining factors from each term in the second parentheses.

$x(a - 2b) + y(a - 2b)$

$(a - 2b)(x + y)$

Common factor Remaining factors

2. $x^2(a + b) + (a + b)$

$= (a + b)(\quad + \quad)$ Determine the GCF

$= (a + b)(x^2 + 1)$ Completely factored form

3. $3x^2(2a - b) - 9x(2a - b)$

$= 3x^2(2a - b) - 3^2x(2a - b)$ Factor each term

$= 3x(2a - b)(\quad - \quad)$ Determine the GCF

$= 3x(2a - b)(x - 3)$ Completely factored form

► ***Quick check*** Factor $x(y + 5) - z(y + 5)$

Four-term polynomials

Consider $ax + ay + bx + by$. We observe that this is a *four-term polynomial* and we will *try* to factor it by grouping.

$$ax + ay + bx + by = (ax + ay) + (bx + by)$$

There is a common factor of a in the first two terms and a common factor of b in the last two terms.

$$(ax + ay) + (bx + by) = a(x + y) + b(x + y)$$

The quantity $(x + y)$ is common to both terms. Factoring it out, we have

$$a(x + y) + b(x + y) = (x + y)(a + b)$$

Therefore we have factored the polynomial by grouping.

Factoring a four-term polynomial by grouping

1. Arrange the four terms so that the first two terms have a common factor and the last two terms have a common factor.
2. Determine the GCF of each pair of terms and factor it out.
3. If step 2 produces a common binomial factor in each term, factor it out.
4. If step 2 does not produce a common binomial factor in each term, try grouping the terms of the original polynomial in a different way.
5. If step 4 does not produce a common binomial factor in each term, the polynomial will not factor by this procedure.

Example

Factor completely.

1. $ax + 2ay + bx + 2by$

$= (ax + 2ay) + (bx + 2by)$ Group in pairs

$= a(x + 2y) + b(x + 2y)$ Factor out the GCF

$= (x + 2y)(a + b)$ Factor out the common binomial

2. $3ac + 6ad - 2bc - 4bd$

$= (3ac + 6ad) - (2bc + 4bd)$ Group in pairs

$= 3a(c + 2d) - 2b(c + 2d)$ Factor out the GCF

$= (c + 2d)(3a - 2b)$ Factor out the common binomial

3. $2ax - 2ay + bx - by$

$= (2ax - 2ay) + (bx - by)$ Group in pairs

$= 2a(x - y) + b(x - y)$ Factor out the GCF

$= (x - y)(2a + b)$ Factor out the common binomial

4. $6ax + by + 3ay + 2bx$

$= 6ax + 3ay + 2bx + by$ Rearrange the terms

$= (6ax + 3ay) + (2bx + by)$ Group in pairs

$= 3a(2x + y) + b(2x + y)$ Factor out the GCF

$= (2x + y)(3a + b)$ Factor out the common binomial

Note As in example 4, sometimes the terms must be rearranged so that the pairs will have a common factor.

It is important to remember to look for the greatest common factor first *when we attempt to determine the completely factored form of any polynomial. If we fail to do this, the answer may not be in a completely factored form or we may not see how to factor the problem by an appropriate procedure.*

EXERCISES

Write in completely factored form.

1. $2y + 6$
2. $3a - 12$
3. $4x^2 + 8y$
4. $8y^2 + 10x^2$
5. $3x^2y + 15z$
6. $5r^2 + 10rs - 20s$
7. $7a - 14b + 21c$
8. $8x - 12y + 16z$
9. $15xy - 18z + 3x^2$
10. $18ab - 27a + 3ac$
11. $42xy - 21y^2 + 7$
12. $15a^2 - 27b^2 + 12ab$
13. $8x - 10y + 12z - 18w$
14. $15L^2 - 21W^2 + 36H$
15. $20a^2b - 60ab + 45ab^2$
16. $4x^2 + 8x$
17. $3x^2y + 6xy$
18. $8x^3 + 4x^2$
19. $2R^4 - 6R^2$
20. $3x^2 - 3xy + 3x$
21. $2x^3 - x^2 + x$
22. $24a^2 + 12a - 6a^3$
23. $15ab + 18ab^2 - 3a^2b$
24. $2x^4 - 6x^2 + 8x$
25. $xy^2 + xyz + xy$
26. $3R^2S - 6RS^2 + 12RS$
27. $2L^3 - 18L + 2L^2$
28. $V^2 + V^3 - V^4 + 2V$
29. $5p^2 + 10p + 15p^3$
30. $16x^3y - 3x^2y^2 + 24x^2y^3$

Supply the missing factor.

Example $-3a - 6b = -3(\quad)$

Solution Since $-3a - 6b = (-3) \cdot a + (-3) \cdot 2b$, then $-3a - 6b = -3(a + 2b)$, the missing factor is $(a + 2b)$.

Example $-a^2b^3 + a^2b^2 = -a^2b^2(\quad)$

Solution $-a^2b^3 + a^2b^2 = (-a^2b^2)(b) + (-a^2b^2)(-1)$ Divide each term by $-a^2b^2$ to find the missing factor
$= -a^2b^2(b - 1)$, the missing factor is $(b - 1)$.

31. $-6x - 9 = -3(\quad)$
32. $-5a + 10b = -5(\quad)$
33. $6x - 8z - 12w = 2(\quad)$
34. $-4a^3 - 36ab + 16ab^2 - 24b^3 = -4(\quad)$
35. $-12L + 15W - 6H = -3(\quad)$
36. $-3a + a^3b = -a(\quad)$
37. $-x + x^2 - x^3 = -x(\quad)$
38. $-x + 2xy + xy^2 = -x(\quad)$

39. $-xyz + x^2yz - xy^2z + xyz^2 = -xyz(\quad)$
40. $-4x^2 + 8x - 12x^3 = -4x(\quad)$
41. $-10a^2b^2 + 15ab - 20a^3b^3 = -5ab(\quad)$
42. $-24RS - 16R + 32R^2 = -8R(\quad)$

Write in completely factored form.

Example $x(y + 5) - z(y + 5)$
Solution $= (y + 5)(\quad - \quad)$ Determine the GCF
$= (y + 5)(x - z)$ Completely factored form

43. $x(a + b) + y(a + b)$
44. $3a(x - y) + b(x - y)$
$\boxed{45.}$ $15x(2a + b) + 10y(2a + b)$
46. $21R(L + 2N) - 35S(L + 2N)$
47. $3x(a + 4b) + 6y(a + 4b)$
48. $4RS(2P + q) - 8RT(2P + q)$
49. $8a(b + 6) - (b + 6)$

Write the following in completely factored form.

Example $3ax + 6bx + 2ay + 4by$

Solution $= (3ax + 6bx) + (2ay + 4by)$ Group in pairs
$= 3x(a + 2b) + 2y(a + 2b)$ Factor out the GCF
$= (a + 2b)(3x + 2y)$ Factor out the common binomial

50. $rt + ru + st + su$
51. $ac + ad + bc + bd$
52. $5ax - 3by + 15bx - ay$
$\boxed{53.}$ $6ax - 2by + 3bx - 4ay$
54. $2ax^2 - bx^2 + 6a - 3b$
55. $4ax + 2ay - 2bx - by$
56. $ac + 3ad - 4bc - 12bd$
57. $20x^2 + 5xz - 12xy - 3yz$
58. $a^2x + 3a^2y - 3x - 9y$
59. $4ax + 12bx - 3ay - 9by$
60. $ac + ad - 2bc - 2bd$
61. $2ac + 6bc - ay - 3by$
62. $2ac + bc - 4ay - 2by$
63. $2ac + 3bc + 8ay + 12by$
64. $5ac - 3by + 15bc - ay$
65. $6ax + by + 2ay + 3bx$
66. $2ax - ad + 4bx - 2bd$
67. $3ax - 2bd - 6ad + bx$
68. $6ax + 3bd - 2ad - 9bx$
69. $2a^3 + 15 + 10a^2 + 3a$
70. $3a^3 - 6a^2 + 5a - 10$
71. $8a^3 - 4a^2 + 6a - 3$

Write in completely factored form.

72. The area of the surface of a cylinder is determined by $A = 2\pi rh + 2\pi r^2$. Factor the right member. (π is the Greek letter pi.)

73. The total surface area of a right circular cone is given by $A = \pi rs + \pi r^2$. Factor the right member.

74. The equation for the distance traveled by a rocket fired vertically upward into the air is given by $S = 560t - 16t^2$, where the rocket is S feet from the ground after t seconds. Factor the right member.

Factoring trinomials of the form $x^2 + bx + c$

Determining when a trinomial will factor

In section 1.5 we learned how to multiply two binomials as follows:

$$\overset{\text{Factors}}{(x+2)(x+6)} = x^2 + 6x + 2x + 12 = \overset{\text{Terms}}{x^2 + 8x + 12}$$

Multiplying →

In this section, we are going to reverse the procedure and factor the trinomial.

$$\overset{\text{Terms}}{x^2 + 8x + 12} = \overset{\text{Factors}}{(x+2)(x+6)}$$

Factoring →

The following group of trinomials will enable us to see how a trinomial factors.

$$12 = 2 \cdot 6 \qquad \text{Product}$$

$$1.\ x^2 + 8x + 12 = (x+2)(x+6)$$

$$8 = 2 + 6 \qquad \text{Sum}$$

$$12 = (-2) \cdot (-6) \qquad \text{Product}$$

$$2.\ x^2 - 8x + 12 = (x-2)(x-6)$$

$$-8 = (-2) + (-6) \qquad \text{Sum}$$

$$-12 = (-2) \cdot 6 \qquad \text{Product}$$

$$3.\ x^2 + 4x - 12 = (x-2)(x+6)$$

$$4 = (-2) + 6 \qquad \text{Sum}$$

$$-12 = 2 \cdot (-6) \qquad \text{Product}$$

$$4.\ x^2 - 4x - 12 = (x+2)(x-6)$$

$$-4 = 2 + (-6) \qquad \text{Sum}$$

In general,

$$(x+m)(x+n) = x^2 + (m+n)x + m \cdot n$$

The trinomial $x^2 + bx + c$ will factor with integer coefficients only if there are two integers, which we will call m and n, such that $m + n = b$ and $m \cdot n = c$.

$$\begin{array}{ccc} \text{Sum} & & \text{Product} \\ m+n & & m \cdot n \end{array}$$

$$x^2 + bx + c = (x+m)(x+n)$$

The signs (+ or −) for m and n

1. If c is positive, then m and n have the same sign as b.
2. If c is negative, then m and n have different signs and the one with the greater absolute value has the same sign as b.

Examples

Factor completely each trinomial.

1. $a^2 + 11a + 18$ $\quad m + n = 11$ and $m \cdot n = 18$

Since $b = 11$ and $c = 18$ are both positive, then m and n are both positive.

List the factorizations of 18	Sum of the factors of 18	
$1 \cdot 18$	$1 + 18 = 19$	
$2 \cdot 9$	$2 + 9 = 11$	← Correct sum
$3 \cdot 6$	$3 + 6 = 9$	

The m and n values are 2 and 9. The factorization is

$$a^2 + 11a + 18 = (a + 2)(a + 9)$$

The answer can be checked by performing the indicated multiplication.

$$(a + 2)(a + 9) = a^2 + 9a + 2a + 18 = a^2 + 11a + 18$$

Note The commutative property allows us to write the factors in any order. That is, $(a + 2)(a + 9) = (a + 9)(a + 2)$.

2. $b^2 - 2b - 15$ $\quad m + n = -2$ and $m \cdot n = -15$

Since $b = -2$ and $c = -15$ are both negative, then m and n have different signs and the one with the greater absolute value is negative.

Factorizations of −15, where the negative sign goes with the factor with the greater absolute value	Sum of the factors of −15	
$1 \cdot (-15)$	$1 + (-15) = -14$	
$3 \cdot (-5)$	$3 + (-5) = -2$	← Correct sum

The m and n values are 3 and −5. The factorization is

$$b^2 - 2b - 15 = (b + 3)(b - 5)$$

3. $5x - 24 + x^2$

It is easier to identify b and c if we write the trinomial in descending powers of the variable, which is called **standard form.**

$x^2 + 5x - 24 \qquad m + n = 5 \quad \text{and} \quad m \cdot n = -24$

Since $b = 5$ is positive and $c = -24$ is negative, m and n have different signs and the one with the greater absolute value is positive.

Factorizations of -24, where the positive factor is the one with the greater absolute value	Sum of the factors of -24
$(-1) \cdot 24$	$(-1) + 24 = 23$
$(-2) \cdot 12$	$(-2) + 12 = 10$
$(-3) \cdot 8$	$(-3) + 8 = 5$ ← Correct sum
$(-4) \cdot 6$	$(-4) + 6 = 2$

The m and n values are -3 and 8. The factorization is

$x^2 + 5x - 24 = (x - 3)(x + 8)$

4. $c^2 - 9c + 14 \qquad m + n = -9 \quad \text{and} \quad m \cdot n = 14$

Since $b = -9$ is negative and $c = 14$ is positive, m and n are both negative.

List the factorizations of 14	Sum of the factors of 14
$(-1)(-14)$	$(-1) + (-14) = -15$
$(-2)(-7)$	$(-2) + (-7) = -9$ ← Correct sum

The m and n values are -2 and -7. The factorization is

$c^2 - 9c + 14 = (c - 2)(c - 7)$

5. $x^2 + 5x + 12 \qquad m + n = 5 \quad \text{and} \quad m \cdot n = 12$

Since $b = 5$ and $c = 12$ are both positive, m and n are both positive.

Factorizations of 12	Sum of the factors of 12
$1 \cdot 12$	$1 + 12 = 13$
$2 \cdot 6$	$2 + 6 = 8$
$3 \cdot 4$	$3 + 4 = 7$

No sum equals 5

Since none of the factorizations of 12 add to 5, there is no pair of integers (m and n) and the trinomial will not factor using integer coefficients. We call this a **prime polynomial.**

6. $x^4 - 4x^3 - 21x^2 = x^2(x^2 - 4x - 21)$ Common factor of x^2

To complete the factorization, we see if the trinomial $x^2 - 4x - 21$ will factor. We need to find m and n that add to -4 and multiply to -21. The values are 3 and -7. The completely factored form is

$x^4 - 4x^3 - 21x^2 = x^2(x + 3)(x - 7)$

Note A common error when the polynomial has a common factor is to factor it out but to forget to include it as one of the factors in the completely factored form.

7. $x^2y^2 + 9xy + 20$

Rewriting the polynomial as $(xy)^2 + 9xy + 20$, we want to find values for m and n that add to 9 and multiply to 20. The numbers are 4 and 5. The factorization is

$$x^2y^2 + 9xy + 20 = (xy + 4)(xy + 5)$$

8. $x^2 - 5ax + 6a^2 \qquad m + n = -5a \quad \text{and} \quad m \cdot n = 6a^2$

We need to find m and n that add to $-5a$ and multiply to $6a^2$. The values are $-2a$ and $-3a$. The factorization is

$$x^2 - 5ax + 6a^2 = (x - 2a)(x - 3a)$$

Factoring a trinomial of the form $x^2 + bx + c$

1. Factor out the GCF. If there is a common factor, make sure to include it as part of the final factorization.
2. Determine if the trinomial is factorable by finding m and n such that $m + n = b$ and $m \cdot n = c$. If m and n do not exist, we conclude that the trinomial will not factor.
3. Using the m and n values from step 2, write the trinomial in factored form.

EXERCISES

Factor completely each trinomial.

Example $z^2 + 8z - 20$

Solution Since $b = 8$ is positive and $c = -20$ is negative, m and n have different signs and the one with the greater absolute value is positive.

Factorizations of -20, where the positive factor is the one with the greater absolute value	Sum of the factors of -20
$(-1) \cdot 20$	$(-1) + 20 = 19$
$(-2) \cdot 10$	$(-2) + 10 = 8 \leftarrow$ Correct sum
$(-4) \cdot 5$	$(-4) + 5 = 1$

The m and n values are -2 and 10. The factorization is

$z^2 + 8z - 20 = (z - 2)(z + 10)$

1. $a^2 + 9a + 18$
2. $c^2 + 9c + 20$
3. $x^2 + 11x - 12$
4. $x^2 + 13x + 12$
5. $y^2 + 13y - 30$
6. $a^2 + 9a + 14$
7. $x^2 - 14x + 24$
8. $b^2 - 10b + 21$
9. $a^2 + 5a - 24$
10. $y^2 + 9y - 36$
11. $x^2 + 8x + 12$
12. $c^2 + 8c + 15$
13. $a^2 - 2a - 24$
14. $z^2 - 5z - 36$
15. $2x^2 + 6x - 20$
16. $2a^2 + 26a + 24$
17. $3x^2 - 18x - 48$
18. $a^2 - 9a + 4$
19. $x^2 + 5x + 7$
20. $x^2 - 4x + 6$
21. $y^2 + 17y + 30$
22. $b^2 + 13b + 40$
23. $4x^2 - 4x - 24$
24. $5y^2 + 5y - 30$
25. $5a^2 - 15a - 50$
26. $x^2y^2 - 4xy - 21$
27. $x^2y^2 - 3xy - 18$
28. $x^2y^2 - xy - 30$
29. $x^2y^2 + 13xy + 12$
30. $4a^2b^2 - 32ab + 28$
31. $3x^2y^2 - 3xy - 36$
32. $3x^2y^2 + 21xy + 36$
33. $x^2 + 3xy + 2y^2$
34. $a^2 - ab - 2b^2$
35. $a^2 - 2ab - 3b^2$
36. $a^2 - 7ab + 10b^2$
37. $a^2 - ab - 6b^2$
38. $x^2 + 2xy - 8y^2$
39. $x^2 - 2xy - 15y^2$
40. $a^2 + 7ab + 12b^2$

Factoring trinomials of the form $ax^2 + bx + c$

How to factor trinomials

In this section, we are going to factor trinomials of the form $ax^2 + bx + c$. This is called the **standard form** of a trinomial, where we have a single variable and the terms of the polynomial are arranged in descending powers of that variable. The a, b, and c in our standard form represent integer constants. For example,

$$2x^2 + 9x + 9$$

is a trinomial in standard form, where $a = 2$, $b = 9$, and $c = 9$.

Consider the product

$$(2x + 3)(x + 3)$$

By multiplying these two quantities together, we get a trinomial.

$$\begin{aligned}(2x + 3)(x + 3) &= 2x^2 + 6x + 3x + 9 \\ &= 2x^2 + 9x + 9\end{aligned}$$

To completely factor the trinomial $2x^2 + 9x + 9$ entails reversing this procedure to get

$$(2x + 3)(x + 3)$$

The trinomial will factor with integer coefficients if we can find a pair of integers (m and n) whose sum is equal to b, and whose product is equal to $a \cdot c$. In the trinomial $2x^2 + 9x + 9$, b is equal to 9, and $a \cdot c$ is $2 \cdot 9 = 18$. Therefore we want $m + n = 9$ and $m \cdot n = 18$. The values for m and n are 3 and 6.

If we observe the multiplication process in our example, we see that m and n appear as the coefficients of the middle terms that are to be combined for our final answer.

$$\begin{aligned}(2x + 3)(x + 3) &= 2x^2 + 6x + 3x + 9 \\ &= 2x^2 + 9x + 9\end{aligned}$$

This is precisely what we do with the m and n values. We replace the coefficient of the middle term in the trinomial with these values. In our example, m and n are 6 and 3 and we replace the 9 with them.

$$2x^2 + 9x + 9 = 2x^2 + \overbrace{6x + 3x}^{9x} + 9$$

Our next step is to group the first two terms and the last two terms.

$$(2x^2 + 6x) + (3x + 9)$$

Now we factor out what is common in each pair. We see that the first two terms contain the common factor $2x$ and the last two terms contain the common factor 3.

$$2x(x + 3) + 3(x + 3)$$

When we reach this point, what is inside the parentheses in each term will be the same. Since the quantity $(x + 3)$ is common to both terms, we can factor it out.

$$2x\underbrace{(x + 3)} + 3\underbrace{(x + 3)}$$

Common to both terms

Having factored out what is common, what is left in each term is placed in the second parentheses.

$$2x(x + 3) + 3(x + 3)$$
$$\underbrace{(x + 3)}_{\text{Common factor}}\underbrace{(2x + 3)}_{\text{Remaining factors}}$$

The trinomial is factored.

A summary of the steps follows:

Factoring a trinomial of the form $ax^2 + bx + c$

Step 1 Determine if the trinomial $ax^2 + bx + c$ is factorable by finding m and n such that $m \cdot n = a \cdot c$ and $m + n = b$. If m and n do not exist, we conclude that the trinomial will not factor.

Step 2 Replace the middle term, bx, by the sum of mx and nx.

Step 3 Place parentheses around the first and second terms and around the third and fourth terms. Factor out what is common to each pair.

Step 4 Factor out the common quantity of each term and place the remaining factors from each term in the second parentheses.

We determine the sign in a similar fashion like we did on page 9

The signs (+ or −) for m and n

1. If $a \cdot c$ is positive, then m and n have the same sign as b.
2. If $a \cdot c$ is negative, then m and n have different signs and the one with the greater absolute value has the same sign as b.

Examples

Factor completely the following trinomials. If a trinomial will not factor, so state.

1. $6x^2 + 13x + 6$

Step 1 $m \cdot n = 6 \cdot 6 = 36$ and $m + n = 13$

We determine by inspection that m and n are 9 and 4.

Step 2	$= 6x^2 + \overbrace{9x + 4x}^{13x} + 6$	Replace bx with mx and nx
Step 3	$= (6x^2 + 9x) + (4x + 6)$	Group the first two terms and the last two terms
	$= 3x(2x + 3) + 2(2x + 3)$	Factor out what is common to each pair
Step 4	$= (2x + 3)(3x + 2)$	Factor out the common quantity

Note The order in which we place m and n into the problem will not change the answer.

(Alternate)

Step 2	$= 6x^2 + \overbrace{4x + 9x}^{13x} + 6$	Replace bx with nx and mx
Step 3	$= (6x^2 + 4x) + (9x + 6)$	Group the first two terms and the last two terms
	$= 2x(3x + 2) + 3(3x + 2)$	Factor out what is common to each pair
Step 4	$= (3x + 2)(2x + 3)$	Factor out the common quantity

We see that the outcome in step 4 is the same regardless of the order of m and n in the problem.

Note The order in which the two factors are written in the answer does not matter. That is, $(2x + 3)(3x + 2) = (3x + 2)(2x + 3)$

2. $3x^2 + 5x + 2$

Step 1 $m \cdot n = 3 \cdot 2 = 6$ and $m + n = 5$

m and n are 2 and 3.

Step 2	$= 3x^2 + \overbrace{2x + 3x}^{5x} + 2$	Replace bx with mx and nx
Step 3	$= (3x^2 + 2x) + (3x + 2)$	Group the first two terms and the last two terms
	$= x(3x + 2) + 1(3x + 2)$	Factor out what is common to each pair

We observe in the last two terms that the greatest common factor is only 1 or -1. We factor out 1 so that we have the same quantity inside the parentheses.

Step 4	$= (3x + 2)(x + 1)$	Factor out the common quantity

3. $4x^2 - 11x + 6$

Step 1 $m \cdot n = 4 \cdot 6 = 24$ and $m + n = -11$

m and n are -3 and -8.

Step 2 $= 4x^2 \overbrace{- 3x - 8x}^{-11x} + 6$ — Replace bx with mx and nx

Step 3 $= (4x^2 - 3x) + (-8x + 6)$ — Group the first two terms and the last two terms

$= x(4x - 3) - 2(4x - 3)$ — Factor out what is common to each pair

We have 2 or -2 as the greatest common factor in the last two terms. We factor out -2 so that we will have the same quantity inside the parentheses.

Step 4 $= (4x - 3)(x - 2)$ — Factor out the common quantity

Note If the third term in step 2 is preceded by a minus sign, we will usually factor out the negative factor.

4. $12x^2 - 4x - 5$

Step 1 $m \cdot n = 12(-5) = -60$ and $m + n = -4$

m and n are 6 and -10.

Step 2 $= 12x^2 \overbrace{+ 6x - 10x}^{-4x} - 5$ — Replace bx with mx and nx

Step 3 $= (12x^2 + 6x) + (-10x - 5)$ — Group the first two terms and the last two terms

$= 6x(2x + 1) - 5(2x + 1)$ — Factor out what is common to each pair

Step 4 $= (2x + 1)(6x - 5)$ — Factor out the common quantity

5. $6x^2 - 9x - 4$

$m \cdot n = 6 \cdot (-4) = -24$ and $m + n = -9$

Our m and n values are not obvious by inspection.

Note If you cannot determine the m and n values by inspection, then you should use the following systematic procedure to list all the possible factorizations of $a \cdot c$. This way you will either find m and n or verify that the trinomial will not factor using integers.

1. Take the natural numbers 1,2,3,4, $\cdot\cdot\cdot$ and divide them into the $a \cdot c$ product. Those that divide into the product evenly we write as a factorization using the correct m and n signs.

Factorization of -24, where the negative sign goes with the factor with the greater absolute value

$1 \cdot (-24)$
$2 \cdot (-12)$
$3 \cdot (-8)$
$4 \cdot (-6)$
$(-6) \cdot 4$
$(-8) \cdot 3$
$(-12) \cdot 2$
$(-24) \cdot 1$

We note that the top four factorizations are the same as the bottom four. Therefore we need only perform this procedure until the factors repeat

$4 \cdot (-6)$
$(-6) \cdot 4$

Factors repeat

2. Find the sum of the factorizations of $a \cdot c$. If there is a sum equal to b, the trinomial will factor. If there is no sum equal to b, then the trinomial will not factor with integer coefficients.

Factorizations of -24	Sum of the factors of -24
$1 \cdot (-24)$	$1 + (-24) = -23$
$2 \cdot (-12)$	$2 + (-12) = -10$
$3 \cdot (-8)$	$3 + (-8) = -5$
$4 \cdot (-6)$	$4 + (-6) = -2$

Passed -9

No sum equals -9

Since none of the factorizations of -24 add to -9, there is no pair of integers (m and n) and the trinomial will not factor.

Note Regardless of the signs of m and n, the column of values of the sum of the factors will either be increasing or decreasing. Therefore, once the desired value has been passed, the process can be stopped and the trinomial will not factor.

Example

6. $24x^2 - 39x - 18$

Before we attempt to apply any factoring rule, recall that we must always factor out what is common to each term. Therefore we have

$24x^2 - 39x - 18 = 3(8x^2 - 13x - 6)$ Common factor of 3

Now we are ready to factor the trinomial $8x^2 - 13x - 6$.

Step 1 $m \cdot n = 8(-6) = -48$ and $m + n = -13$
m and n are 3 and -16.

$$\begin{aligned}
&\textit{Step 2} &&= 3(8x^2 + \overbrace{3x - 16x}^{-13x} - 6) && \text{Replace } bx \text{ with } mx \text{ and } nx\\
&\textit{Step 3} &&= 3[(8x^2 + 3x) + (-16x - 6)] && \text{Group the first two terms and the last two terms}\\
& &&= 3[x(8x + 3) - 2(8x + 3)] && \text{Factor out what is common to each pair}\\
&\textit{Step 4} &&= 3(8x + 3)(\quad - 2) && \text{Factor out the common quantity}
\end{aligned}$$

Note In example 6, we factored out 3 that was common to all the original terms. A common error is to forget to include it as one of the factors in the answer.

Factoring by inspection—an alternative approach

In the beginning of this section, we studied a systematic procedure for determining if a trinomial will factor and how to factor it. In many instances, we can determine how the trinomial will factor by inspecting the problem rather than by applying this procedure.

Factoring by inspection is accomplished as follows: Factor $7x + 2x^2 + 3$.

Step 1 Write the trinomial in standard form.

$2x^2 + 7x + 3$ Arrange terms in descending powers of x

Step 2 Determine the possible combinations of first-degree factors of the first term.

$(2x \quad)(x \quad)$ The only factorization of $2x^2$ is $2x \cdot x$

Step 3 Combine with the factors of step 2 all the possible factors of the third term.

$(2x \quad 3)(x \quad 1)$ The only factorization of 3 is $3 \cdot 1$
$(2x \quad 1)(x \quad 3)$

Step 4 Determine the possible symbol (+ or −) between the terms in each binomial.

$$(2x + 3)(x + 1)$$
$$(2x + 1)(x + 3)$$

The rules of real numbers given in chapter 1 provide the answer to step 4.

1. If the third term is preceded by a + sign and the middle term is preceded by a + sign, then the symbols will be

   $(\quad + \quad)(\quad + \quad)$

2. If the third term is preceded by a + sign and the middle term is preceded by a − sign, then the symbols will be

   $(\quad - \quad)(\quad - \quad)$

3. If the third term is preceded by a − sign, then the symbols will be

   $(\quad + \quad)(\quad - \quad)$

   or

   $(\quad - \quad)(\quad + \quad)$

Note It is assumed that the first term is preceded by a + sign or no sign. If it is preceded by a − sign, these rules could still be used if (−1) is first factored out of all the terms.

Step 5 Determine which factors, if any, yield the correct middle term.

$$(2x + 3)(x + 1)$$

$$+3x \qquad\qquad (+3x) + (+2x) = +5x$$

$$+2x$$

$$(2x + 1)(x + 3)$$

$$+x \qquad\qquad (+x) + (+6x) = +7x \qquad \text{Correct middle term}$$

$$+6x$$

The second set of factors gives us the correct middle term. Therefore

$(2x + 1)(x + 3)$ is the factorization of $2x^2 + 7x + 3$.

Examples

Factor completely the following trinomials by inspection.

1. $6x^2 + 17x + 5$

Step 1 $6x^2 + 17x + 5$ — Standard form

Step 2 $(6x \quad)(x \quad)$ — $6x^2 = 3x \cdot 2x$ or $6x \cdot x$
$(3x \quad)(2x \quad)$

Step 3 $(6x \quad 5)(x \quad 1)$
$(6x \quad 1)(x \quad 5)$ — The only factorization of 5 is $5 \cdot 1$
$(3x \quad 5)(2x \quad 1)$
$(3x \quad 1)(2x \quad 5)$

Step 4 $(6x + 5)(x + 1)$
$(6x + 1)(x + 5)$ — Using the rules of signed numbers, determine the possible signs between the terms
$(3x + 5)(2x + 1)$
$(3x + 1)(2x + 5)$

Step 5 $(6x + 5)(x + 1)$

$+5x$ — $(+5x) + (+6x) = +11x$

$+6x$

$(6x + 1)(x + 5)$

$+x$ — $(+x) + (+30x) = +31x$

$+30x$

$(3x + 5)(2x + 1)$

$+10x$ — $(+10x) + (+3x) = +13x$

$+3x$

$(3x + 1)(2x + 5)$

$+2x$ — $(+2x) + (+15x) = +17x$
Correct middle term

$+15x$

The last set of factors gives us the correct middle term. Hence $(3x + 1)(2x + 5)$ is the factorization of $6x^2 + 17x + 5$.

2. $4x^2 - 5x + 1$

Step 1 $4x^2 - 5x + 1$ — Standard form

Step 2 $(4x \quad)(x \quad)$ — $4x^2 = 2x \cdot 2x$ or $4x \cdot x$
$(2x \quad)(2x \quad)$

Step 3 $(4x \quad 1)(x \quad 1)$ — The only factorization of 1 is $1 \cdot 1$
$(2x \quad 1)(2x \quad 1)$

Step 4 $(4x - 1)(x - 1)$ — Determine the possible signs between the terms
$(2x - 1)(2x - 1)$

Step 5 $(4x - 1)(x - 1)$

$-x$

$-4x$

$(-x) + (-4x) = -5x$ — Correct middle term

$(2x - 1)(2x - 1)$

$-2x$

$-2x$

$(-2x) + (-2x) = -4x$

The first set of factors gives us the correct factorization.

$4x^2 - 5x + 1 = (4x - 1)(x - 1)$

3. $13x - 5 + 6x^2$

Step 1 $6x^2 + 13x - 5$ — Write in standard form

Step 2 $(6x \quad)(x \quad)$ — $6x^2 = 6x \cdot x$ or $2x \cdot 3x$
$(2x \quad)(3x \quad)$

Step 3 $(6x \quad 5)(x \quad 1)$
$(6x \quad 1)(x \quad 5)$ — The only factorization of 5 is $5 \cdot 1$
$(2x \quad 5)(3x \quad 1)$
$(2x \quad 1)(3x \quad 5)$

Step 4 $(6x + 5)(x - 1)$ or $(6x - 5)(x + 1)$ — Determine the possible signs between the terms
$(6x + 1)(x - 5)$ or $(6x - 1)(x + 5)$
$(2x + 5)(3x - 1)$ or $(2x - 5)(3x + 1)$
$(2x + 1)(3x - 5)$ or $(2x - 1)(3x + 5)$

Step 5 $(6x + 5)(x - 1)$ or $(6x - 5)(x + 1)$

$+5x$ $\qquad$ $-5x$

$-6x$ $\qquad$ $+6x$

$(+5x) + (-6x) = -x$ or $(-5x) + (+6x) = +x$

$(6x + 1)(x - 5)$ or $(6x - 1)(x + 5)$

$+x$ $\qquad$ $-x$

$-30x$ $\qquad$ $+30x$

$(+x) + (-30x) = -29x$ or $(-x) + (+30x) = +29x$

$(2x + 5)(3x - 1)$ or $(2x - 5)(3x + 1)$

$+15x$ $\qquad$ $-15x$

$-2x$ $\qquad$ $+2x$

$(+15x) + (-2x) = +13x$ or $(-15x) + (+2x) = -13x$

Correct middle term

$(2x + 1)(3x - 5)$ or $(2x - 1)(3x + 5)$

$+3x$ $\qquad$ $-3x$

$-10x$ $\qquad$ $+10x$

$(+3x) + (-10x) = -7x$ or $(-3x) + (+10x) = +7x$

The factorization of $6x^2 + 13x - 5$ is $(2x + 5)(3x - 1)$. ■

EXERCISES

Factor completely each trinomial. If a trinomial does not factor call it prime.

1. $2x^2 + x - 6$ **2.** $3x^2 + 7x - 6$ **3.** $2x^2 + 3x + 1$ **4.** $4x^2 - 5x + 1$

5. $2R^2 - 7R + 6$ **6.** $R^2 - 4R + 6$ **7.** $5x^2 - 7x - 6$ **8.** $2x^2 - x - 1$

9. $9x^2 - 6x + 1$ **10.** $8x^2 - 17x + 2$ **11.** $5x^2 + 4x + 6$ **12.** $2x^2 - 11x + 12$

13. $6x^2 + 13x + 6$ **14.** $2r^2 + 13r + 18$ **15.** $4x^2 + 20x + 21$ **16.** $7R^2 + 20R - 3$

17. $4x^2 - 2x + 5$ **18.** $4x^2 - 4x - 3$ **19.** $9y^2 - 21y - 8$ **20.** $6x^2 - 23x - 4$

21. $10x^2 + 7x - 6$ **22.** $10x^2 + 9x + 2$ **23.** $2x^2 - 9x + 10$ **24.** $7x^2 - 3x + 6$

$\boxed{\textbf{25.}}$ $4x^2 + 14x + 12$ **26.** $5R^2 - 9R - 2$ **27.** $4x^2 + 10x + 6$ **28.** $6x^2 - 17x + 12$

29. $6x^2 + 7x - 3$ **30.** $3x^2 + 12x + 12$ **31.** $2x^2 + 6x - 20$ **32.** $3a^2 + 8a - 4$

33. $6x^2 + 5x - 6$ **34.** $3x^2 - 19x + 20$ **35.** $4x^2 + 12x + 9$ **36.** $9z^2 - 30z + 25$

37. $7x^2 - 36x + 5$ **38.** $3x^2 + 2x + 4$ **39.** $15P^2 + 2P - 1$ **40.** $12x^2 + 13x - 4$

$\boxed{\textbf{41.}}$ $2x^3 - 6x^2 - 20x$ **42.** $4x^2 + 10x + 4$ **43.** $2a^3 + 15a^2 + 7a$ **44.** $9x^2 + 27x + 8$

45. $8x^2 - 14x - 15$ **46.** $8x^2 - 18x + 9$