SECTION 1.6 FACTORING (Part II)

FACTORING DIFFERENCE of TWO SQUARES and PERFECT SQUARE TRINOMIALS

In section 1.5 we saw that the product of $(a + b)(a - b)$ was $a^2 - b^2$. We refer to the indicated product $(a + b)(a - b)$ as the *product* of the *sum* and *difference* of the same two terms. Notice that in one factor we *add* the terms and in the other we find the *difference* between these same terms. The product will *always* be the *difference of the squares* of the two terms. To factor the **difference of two squares,** we **use the formula from section 1.5**

Factoring the difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

Concept

(first term)^2 − (second term)^2 factors into
(first term + second term)(first term − second term)

To use this factoring technique, we must be able to recognize **perfect squares.**

EXAMPLE

Write the following as a quantity squared, if possible.

1. $16 = 4 \cdot 4$ — 16 is a perfect square
$= (4)^2$ — 16 can be written as 4 squared

2. $x^2 = x \cdot x$ — x is written as a factor twice
$= (x)^2$ — Writing x^2 as $(x)^2$ shows this is a perfect square

3. $25a^2 = 5a \cdot 5a$ — 25 is $5 \cdot 5$ and a^2 is $a \cdot a$
$= (5a)^2$ — It is now rewritten as a square

4. $9y^4 = 3y^2 \cdot 3y^2$ — 9 is $3 \cdot 3$ and y^4 could be written as $y^2 \cdot y^2$
$= (3y^2)^2$ — It is now rewritten as a square

▶ ***Quick check*** Write 64 and $9x^4$ each as a quantity squared.

This is the procedure we use for factoring the difference of two squares.

Factoring the difference of two squares

Step 1 Identify that we have a perfect square *minus* another perfect square.

Step 2 Rewrite the problem as a first term squared minus a second term squared.

$$(\text{first term})^2 - (\text{second term})^2$$

Step 3 Factor the problem into the first term plus the second term times the first term minus the second term.

$$(\text{first term} + \text{second term})(\text{first term} - \text{second term})$$

EXAMPLE

Write the following in completely factored form.

	Step 1 Identify		*Step 2* Rewrite		*Step 3* Factor
	$a^2 - b^2$	$=$	$(a)^2 - (b)^2$	$=$	$(a + b)(a - b)$
1.	$x^2 - 9$	$=$	$(x)^2 - (3)^2$	$=$	$(x + 3)(x - 3)$
2.	$4a^2 - b^2$	$=$	$(2a)^2 - (b)^2$	$=$	$(2a + b)(2a - b)$
3.	$4p^2 - 25v^2$	$=$	$(2p)^2 - (5v)^2$	$=$	$(2p + 5v)(2p - 5v)$
4.	$r^4 - 49$	$=$	$(r^2)^2 - (7)^2$	$=$	$(r^2 + 7)(r^2 - 7)$

▶ ***Quick check*** Write $t^2 - 64$ and $4a^2 - b^2c^2$ in completely factored form.

Our first step in any factoring problem is to look for any common factors. Often an expression that does not appear to be factorable becomes so by taking out the greatest common factor. When we have applied a factoring rule to a problem, *we must inspect all parts of our answer to make sure that nothing will factor further.*

EXAMPLES

Write the following in completely factored form.

1. $2x^2 - 18y^2 = 2(x^2 - 9y^2)$ Factor out what is common, 2

$= 2[(x)^2 - (3y)^2]$ Identify and rewrite as squares

$= 2(x + 3y)(x - 3y)$ Factor and inspect the factors

2. $3w^2 - 48 = 3(w^2 - 16)$ — Common factor of 3
$= 3[(w)^2 - (4)^2]$ — Identify and rewrite
$= 3(w + 4)(w - 4)$ — Factor and inspect the factors

3. $5a^4 - 45a^2b^2 = 5a^2(a^2 - 9b^2)$ — Common factor of $5a^2$
$= 5a^2[(a)^2 - (3b)^2]$ — Identify and rewrite
$= 5a^2(a + 3b)(a - 3b)$ — Factor and inspect the factors

4. $a^4 - 16 = (a^2)^2 - (4)^2$ — Identify and rewrite
$= (a^2 + 4)(a^2 - 4)$ — Factor and inspect the factors
$= (a^2 + 4)[(a)^2 - (2)^2]$ — Identify and rewrite
$= (a^2 + 4)(a + 2)(a - 2)$ — Factor and inspect

Note In example 4, $a^2 + 4$ is called the *sum of two squares*. This will *not* factor using integers.

5. $2x^4 - 162 = 2(x^4 - 81)$ — Common factor of 2
$= 2[(x^2)^2 - (9)^2]$ — Identify and rewrite
$= 2(x^2 + 9)(x^2 - 9)$ — Factor and inspect
$= 2(x^2 + 9)[(x)^2 - (3)^2]$ — Identify and rewrite
$= 2(x^2 + 9)(x + 3)(x - 3)$ — Factor and inspect

Note A common error in examples 1, 2, 3, and 5 is to factor out something that is common but forget to include it as a factor in the final answer.

Perfect square trinomials

In section 1.5 two of the special products that we studied were the squares of a binomial. We will now restate those special products.

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2$$

The right members of the equations are called the squares of binomials, and the left members are called perfect square trinomials. Perfect square trinomials can always be factored by our factoring procedure. However if we observe that the first and last terms of a trinomial are perfect squares, we should see if the trinomial will factor as the square of a binomial. To factor a trinomial as a perfect square trinomial, the following three conditions need to be met.

Necessary conditions for a perfect square trinomial

1. The first term must have a positive coefficient and be a perfect square, a^2.
2. The last term must have a positive coefficient and be a perfect square, b^2.
3. The middle term must be twice the product of the bases of the first and last terms, $2ab$ or $-2ab$.

We observe that

$$\begin{aligned} &9x^2 + 12x + 4 \\ &= \underbrace{(3x)^2}_{\text{Condition 1}} + \underbrace{2(3x)(2)}_{\text{Condition 3}} + \underbrace{(2)^2}_{\text{Condition 2}} \end{aligned}$$

Therefore it is a perfect square trinomial and factors into

$$(3x + 2)^2$$

EXAMPLES

The following examples show the factoring of some other perfect square trinomials.

	Condition 1		*Condition 3*		*Condition 2*		*Square of a binomial*
1. $4x^2 + 20x + 25 =$	$(2x)^2$	$+$	$2(2x)(5)$	$+$	$(5)^2$	$=$	$(2x + 5)^2$
2. $9x^2 - 6x + 1 =$	$(3x)^2$	$-$	$2(3x)(1)$	$+$	$(1)^2$	$=$	$(3x - 1)^2$
3. $16x^2 + 24x + 9 =$	$(4x)^2$	$+$	$2(4x)(3)$	$+$	$(3)^2$	$=$	$(4x + 3)^2$
4. $9y^2 - 30y + 25 =$	$(3y)^2$	$-$	$2(3y)(5)$	$+$	$(5)^2$	$=$	$(3y - 5)^2$

EXERCISES

Write the following as a quantity squared, if possible

Examples	64		$9x^4$		
Solutions	$= 8 \cdot 8$	Identify	$= 3x^2 \cdot 3x^2$	Identify	
	$= (8)^2$	Rewrite	$= (3x^2)^2$	Rewrite	

1. 36 **2.** 25 **3.** c^2 **4.** e^2

5. $16x^2$ **6.** $49b^2$ **7.** $4z^4$ **8.** $25b^2$

Write in completely factored form.

Examples	$t^2 - 64$	Identify	$4a^2 - b^2c^2$	Identify
Solutions	$= (t)^2 - (8)^2$	Rewrite	$= (2a)^2 - (bc)^2$	Rewrite
	$= (t + 8)(t - 8)$	Factor	$= (2a + bc)(2a - bc)$	Factor

9. $x^2 - 1$ **10.** $x^2 - 25$ **11.** $a^2 - 4$ **12.** $r^2 - s^2$

13. $9 - E^2$ **14.** $49 - R^2$ **15.** $1 - k^2$ **16.** $4y^2 - 9$

$\boxed{\textbf{17.}}$ $9b^2 - 16$ **18.** $x^2 - 16z^2$ **19.** $b^2 - 36c^2$ **20.** $16x^2 - y^2$

21. $4a^2 - 25b^2$ **22.** $16a^2 - b^2$ **23.** $25p^2 - 81$ **24.** $r^2 - 4s^2$

$\boxed{\textbf{25.}}$ $8x^2 - 32y^2$ **26.** $3a^2 - 27b^2$ **27.** $5r^2 - 125s^2$ **28.** $20 - 5b^2$

29. $50 - 2x^2$ **30.** $x^2y^2 - 4z^2$ **31.** $r^2s^2 - 25t^2$ **32.** $a^4 - 25$

33. $x^4 - 9$ $\boxed{\textbf{34.}}$ $x^4 - 1$ **35.** $r^4 - 81$ **36.** $16t^4 - 1$

37. $49x^2 - 64y^4$ **38.** $125p^2 - 20v^2$ **39.** $98x^2y^2 - 50p^2c^2$ **40.** $a^2 + 10a + 25$

41. $c^2 - 14c + 49$ **42.** $b^2 + 8b + 16$ **43.** $a^2 + 6a + 9$ **44.** $x^2 - 12x + 36$

45. $y^2 - 6y + 9$ **46.** $a^2 + 6ab + 9b^2$ **47.** $4a^2 - 12ab + 9b^2$ **48.** $x^2 - 16xy + 64y^2$

49. $9c^2 - 12cd + 4d^2$ **50.** $9a^2 - 30ab + 25b^2$

FACTORING SUM and DIFFERENCE of TWO CUBES

In above part /e factored expressions that involved the difference of two squares. 1o ıactor these types of expressions, we identified the two terms as perfect squares and applied the procedure. In this section, we will factor the *sum and difference of two cubes* in a similar fashion.

Consider the indicated product of $(a - b)(a^2 + ab + b^2)$. If we carry out the multiplication, we have

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3\end{aligned}$$

Therefore $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ and $(a - b)(a^2 + ab + b^2)$ is the factored form of $a^3 - b^3$.

The difference of two cubes factors as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Concept

If we are able to write a two-term polynomial as a first term cubed minus a second term cubed, then it will factor as the difference of two cubes.

$$\begin{aligned}&(\text{1st term})^3 - (\text{2nd term})^3 \\ &= (\text{1st term} - \text{2nd term})[(\text{1st term})^2 + (\text{1st term})(\text{2nd term}) + (\text{2nd term})^2]\end{aligned}$$

To use this factoring technique, we must be able to recognize **perfect cubes.**

EXAMPLES

Write the following as a quantity cubed, if possible.

1. $27 = 3 \cdot 3 \cdot 3$ — 27 is a perfect cube
$= (3)^3$ — 27 can be written as 3 cubed

2. $a^3 = a \cdot a \cdot a$ — a is written as a factor three times
$= (a)^3$ — Writing a^3 as $(a)^3$ shows this is a perfect cube

3. $8a^3 = 2a \cdot 2a \cdot 2a$ — 8 is $2 \cdot 2 \cdot 2$ and a^3 is $a \cdot a \cdot a$
$= (2a)^3$ — It is now rewritten as a cube

4. $64x^6 = 4x^2 \cdot 4x^2 \cdot 4x^2$ — 64 is $4 \cdot 4 \cdot 4$ and x^6 is $x^2 \cdot x^2 \cdot x^2$
$= (4x^2)^3$ — It is now rewritten as a cube

This is the procedure we use for factoring the difference of two cubes.

> ***Step 1*** Identify that we have a perfect cube minus another perfect cube.
>
> ***Step 2*** Rewrite the problem as a first term cubed minus a second term cubed.
>
> $$(\text{1st term})^3 - (\text{2nd term})^3$$
>
> ***Step 3*** Factor the expression into the first term minus the second term, times the first term squared plus the first term times the second term plus the second term squared.
>
> $$(\text{1st term} - \text{2nd term})[(\text{1st term})^2 + (\text{1st term} \cdot \text{2nd term}) + (\text{2nd term})^2]$$

EXAMPLES

Factor completely.

1. $x^3 - 27$ We rewrite x^3 as a cube and 27 as a cube.

$x^3 - 27 = (x)^3 - (3)^3$

The first term is x and the second term is 3. Then we write the procedure for factoring the difference of two cubes.

$$(\underset{\substack{\uparrow\\ \text{1st}}}{\quad} - \underset{\substack{\uparrow\\ \text{2nd}}}{\quad})[(\underset{\substack{\uparrow\\ \text{1st}}}{\quad})^2 + (\underset{\substack{\uparrow\\ \text{1st}}}{\quad})(\underset{\substack{\uparrow\\ \text{2nd}}}{\quad}) + (\underset{\substack{\uparrow\\ \text{2nd}}}{\quad})^2]$$

Now substitute x where the first term is in the procedure and 3 where the second term is.

$$(\underset{\substack{\uparrow\\ \text{1st}}}{x} - \underset{\substack{\uparrow\\ \text{2nd}}}{3})[(\underset{\substack{\uparrow\\ \text{1st}}}{x})^2 + (\underset{\substack{\uparrow\\ \text{1st}}}{x})\,(\underset{\substack{\uparrow\\ \text{2nd}}}{3}) + (\underset{\substack{\uparrow\\ \text{2nd}}}{3})^2]$$

Finally we simplify.

$(x - 3)(x^2 + 3x + 9)$

Therefore $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.

2. $8x^3 - y^3 = (2x)^3 - (y)^3$	First term is $2x$ and second term is y.
Then $(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$	Factoring procedure ready for substitution
$= (2x - y)[(2x)^2 + (2x)(y) + (y)^2]$	The first term is $2x$, the second term is y
$= (2x - y)(4x^2 + 2xy + y^2)$	Simplify within the second group
3. $2a^3 - 54b^3 = 2(a^3 - 27b^3)$	Factor out the common quantity of 2
$2(a^3 - 27b^3) = 2[(a)^3 - (3b)^3]$	Rewrite as cubes
Then $2(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$	Factoring procedure ready for substitution
$= 2(a - 3b)[(a)^2 + (a)(3b) + (3b)^2]$	The first term is a, the second term is $3b$
$= 2(a - 3b)(a^2 + 3ab + 9b^2)$	Simplify within the second group
4. $a^{15} - 64b^3 = (a^5)^3 - (4b)^3$	Rewrite as cubes
Then $(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$	Factoring procedure ready for substitution
$= (a^5 - 4b)[(a^5)^2 + (a^5)(4b) + (4b)^2]$	The first term is a^5, the second term is $4b$
$= (a^5 - 4b)(a^{10} + 4a^5b + 16b^2)$	Simplify within the second group

Note In example 4, we observe that a number raised to a power that is a multiple of 3 can be written as a cube by dividing the exponent by 3. The quotient is the exponent of the number inside the parentheses and the 3 is the exponent outside the parentheses. For example, $y^{12} = (y^4)^3$ or $z^{24} = (z^8)^3$.

▶ ***Quick check*** Factor $16R^3 - 54$

The sum of two cubes

If we carry out the indicated multiplication in $(a + b)(a^2 - ab + b^2)$, we have

$$\begin{aligned}(a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\ &= a^3 + b^3\end{aligned}$$

Therefore $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ and $(a + b)(a^2 - ab + b^2)$ is the factored form of $a^3 + b^3$.

The sum of two cubes factors as

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Concept

If we are able to write a two-term polynomial as a first term cubed plus a second term cubed, then it will factor as the sum of two cubes.

$$\begin{aligned}&(\text{1st term})^3 + (\text{2nd term})^3 \\ &= (\text{1st term} + \text{2nd term})[(\text{1st term})^2 - (\text{1st term})(\text{2nd term}) + (\text{2nd term})^2]\end{aligned}$$

This is the procedure we use for factoring the sum of two cubes.

Step 1 Identify that we have a perfect cube plus another perfect cube.

Step 2 Rewrite the problem as a first term cubed plus a second term cubed.

$$(\text{1st term})^3 + (\text{2nd term})^3$$

Step 3 Factor the expression into the first term plus the second term, times the first term squared minus the first term times the second term plus the second term squared.

$$(\text{1st term} + \text{2nd term})[(\text{1st term})^2 - (\text{1st term})(\text{2nd term}) + (\text{2nd term})^2]$$

EXAMPLES

Factor completely.

1. $a^3 + 8 = (a)^3 + (2)^3$

We now write the procedure for the sum of two cubes.

$$(\underset{\underset{\text{1st}}{\uparrow}}{\quad} + \underset{\underset{\text{2nd}}{\uparrow}}{\quad})[(\underset{\underset{\text{1st}}{\uparrow}}{\quad})^2 - (\underset{\underset{\text{1st}}{\uparrow}}{\quad})(\underset{\underset{\text{2nd}}{\uparrow}}{\quad}) + (\underset{\underset{\text{2nd}}{\uparrow}}{\quad})^2]$$

Substituting,

$$(\underset{\underset{\text{1st}}{\uparrow}}{a} + \underset{\underset{\text{2nd}}{\uparrow}}{2})[(\underset{\underset{\text{1st}}{\uparrow}}{a})^2 - (\underset{\underset{\text{1st}}{\uparrow}}{a})\ (\underset{\underset{\text{2nd}}{\uparrow}}{2}) + (\underset{\underset{\text{2nd}}{\uparrow}}{2})^2]$$

simplifying, $(a + 2)(a^2 - 2a + 4)$

therefore $a^3 + 8 = (a + 2)(a^2 - 2a + 4)$

2. $x^3 + 125 = (x)^3 + (5)^3$	Rewrite as cubes
Then $(\quad+\quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$	Factoring procedure ready for substitution
$= (x + 5)[(x)^2 - (x)(5) + (5)^2]$	The first term is x, the second term is 5
$= (x + 5)(x^2 - 5x + 25)$	Simplify within the second group
3. $8a^3 + b^{21} = (2a)^3 + (b^7)^3$	Rewrite as cubes
Then $(\quad+\quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$	Factoring procedure ready for substitution
$= (2a + b^7)[(2a)^2 - (2a)(b^7) + (b^7)^2]$	The first term is $2a$, the second term is b^7
$= (2a + b^7)(4a^2 - 2ab^7 + b^{14})$	Simplify within the second group

4. $x^3y^3 + z^3 = (xy)^3 + (z)^3$ — Rewrite as cubes

Then $(\quad + \quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= (xy + z)[(xy)^2 - (xy)(z) + (z)^2]$ — The first term is xy, the second term is z

$= (xy + z)(x^2y^2 - xyz + z^2)$ — Simplify within the second group

▶ ***Quick check*** Factor $27x^3 + y^3$ ■

EXERCISES

Write the following as quantity cubed if possible.

1. 64	**2.** 8	**3.** 125	**4.** 1	**5.** $27x^3$
6. $64a^3$	**7.** a^6	**8.** x^9	**9.** $8b^{15}$	**10.** $64c^{21}$

Factor completely. If it does not call it prime.

11. $r^3 + s^3$	**12.** $L^3 + 8$	**13.** $8x^3 + y^3$	**14.** $27r^3 + 8$
15. $h^3 - k^3$	**16.** $p^3 - q^3$	**17.** $a^3 - 8$	**18.** $b^3 + 64$
19. $x^3 - 8y^3$	**20.** $27a^3 - b^3$	**21.** $64x^3 - y^3$	**22.** $r^3 - 27$
23. $27x^3 - 8y^3$	**24.** $64a^3 - 8$	**25.** $8a^3 + 27b^3$	**26.** $64s^3 + 1$
$\boxed{\textbf{27.}}$ $2a^3 + 16$	**28.** $3x^3 + 81$	**29.** $2x^3 - 16$	**30.** $81a^3 - 3b^3$
31. $x^5 + 27x^2y^3$	**32.** $16a^3 + 2b^3$	**33.** $x^6 + y^3$	**34.** $x^3 + y^9$
35. $a^9 - b^3$	**36.** $a^6 - 8$	**37.** $x^{12} - 27$	**38.** $x^{15} + 64$
39. $8a^2b^3 - a^5$	**40.** $2x^3 - 54y^3$	**41.** $54r^3 + 2s^3$	**42.** $b^5 + 64b^2c^3$
43. $x^3y^3 - z^3$	$\boxed{\textbf{44.}}$ $x^3y^9 - 1$	**45.** $a^{15}b^6 - 8c^9$	**46.** $x^{18}y^9 - 27z^3$
47. $a^3b^3 + 8$	**48.** $x^3y^6 + z^3$	**49.** $x^9y^{12} + z^{15}$	**50.** $a^{12}b^{15} + c^{24}$