Name: ______________________________

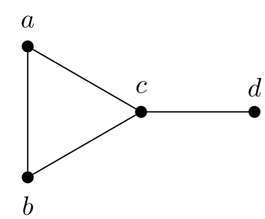
- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have two hours to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.

- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	20	16	18	10	20	10	12	12	20	20	20	178
Score:												

Short answer

1. (20 points) Find all maximal paths, independent sets, cliques, paths, and matchings in the below graph.



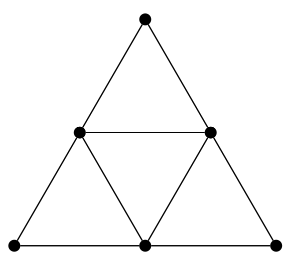
   (i) Maximal independent sets:

   (ii) Maximal cliques:

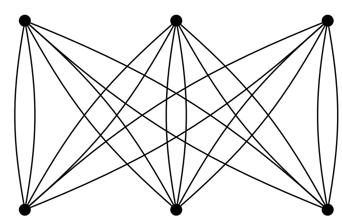
   (iii) Maximal paths:

   (iv) Maximal matchings:

2. (16 points) Determine $\chi'(G)$ and $\chi'(H)$. *(A theorem gives the answer for H quickly, but G requires some argument.)*



G



H

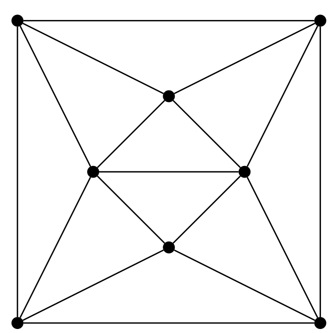
3. (18 points) For each of the following, determine whether such a graph exists. If it does, give an example; if not, give a brief reason why not.

   (a) a 3-connected 4-regular planar graph

   (b) a 4-connected 3-regular planar graph

   (c) a 6-connected planar graph

4. (10 points) Decompose the following planar graph into two bipartite graphs. *(Hint: this can always be done, using the Four Color Theorem and considering, for each edge, the sum its endpoints' colors.)*



Algorithms

Give brief justifications for your answers (but not necessarily full proofs; a computation with each step clearly written may suffice).

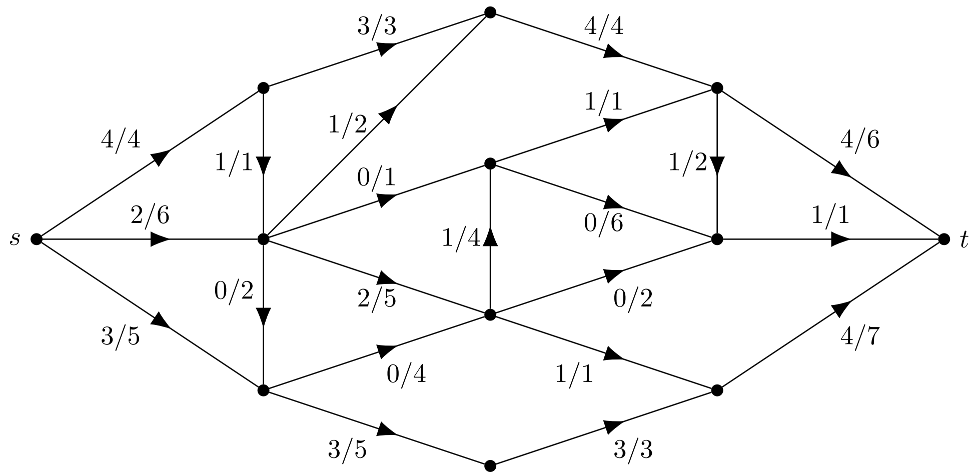
5. Consider the degree sequence $6, 6, 6, 4, 3, 3, 3, 3$.
   (a) (10 points) Prove that every simple graph with this degree sequence is 4-colorable.

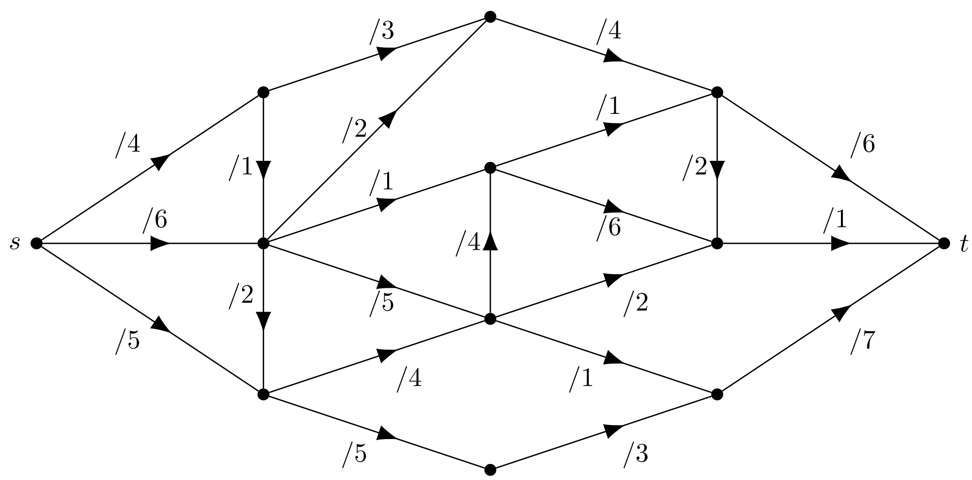
   (b) (10 points) Find one simple graph with this degree sequence and a 4-coloring of it.

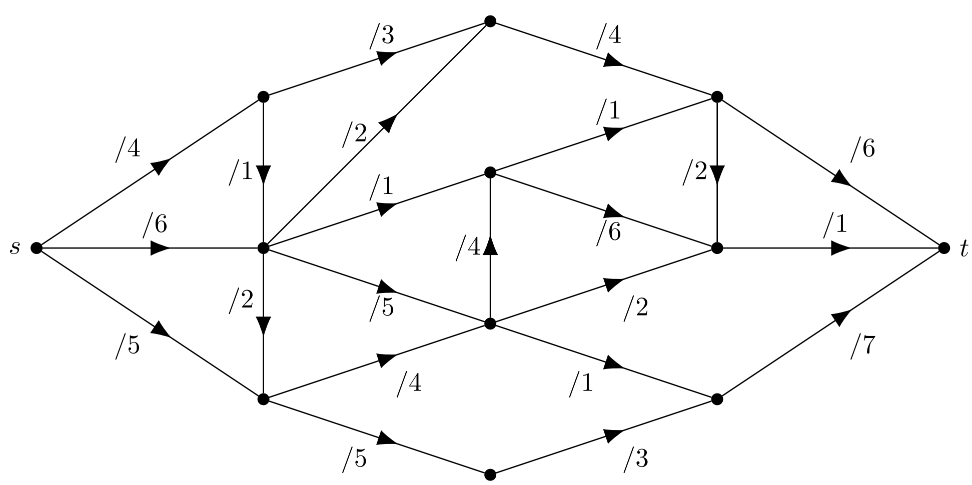
   (c) (8 **bonus**) Prove that every simple graph with this degree sequence has chromatic number at least 3.

6. (10 points) Let G be a weighted graph in which every edge has positive weight, and let T^* be a minimum spanning tree. An edge e with weight zero is added to G; describe how to modify T^* to obtain a minimum spanning tree of $G + e$.

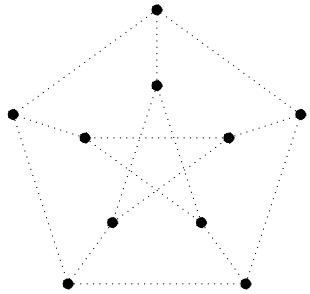
7. (12 points) Apply the Ford-Fulkerson algorithm to augment the feasible flow below to a maximum feasible flow. (Explain how you know the result is maximum.) *(The network is redrawn for your convenience.)*

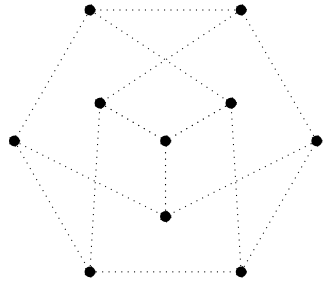


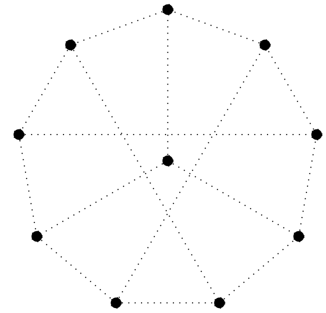




8. (12 points) Find the maximum number of edges in a bipartite planar subgraph of the Petersen graph. *(Hint: use the idea behind the generic planar edge bound; what face lengths can such a subgraph have? For the construction, use that information about face lengths to choose a good drawing of the Petersen graph to modify. The three most common drawings are given at the bottom of the page.)*







Proofs

9. (20 points) Let G be k-connected, $x \in V(G)$, and $U \subseteq V(G) \setminus \{x\}$ with $|U| = k$. Prove that there are k paths from x to U that share only the vertex x. *(Hint: Use the Expansion Lemma and Menger's Theorem.)*

10. (20 points) Prove that every outerplanar graph is 3-colorable. *(Hint: there is an easy proof using the Four Color Theorem (a bit of overkill), and another proof similar to that of the Six Color Theorem.)*

11. (20 points) Prove that for $k \geq 2$, the hypercube Q_k is Hamiltonian.

Scratch Paper - you may remove this if you find it convenient

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