Math 241 X8 **Name(s):** Solutions

Homework 6 supplement

This is a written homework supplement to the homework for Unit 6: Flow Measurements.

For each of the following vector fields, either prove they are not conservative or find a potential function.

(1) $\vec{F}(x,y) = \langle \underbrace{x^3\cos(y) + e^x}_{m}, \underbrace{x^4 y \sin(y)}_{n} \rangle$.

$$\partial_x n = 4x^3 y \sin y$$
$$\not\parallel$$
$$\partial_y m = -x^3 \sin y + 0$$

(2) $\vec{F}(x,y) = \langle x^2 + \frac{3}{2}x^2y^2, \sin(y) + x^3 y \rangle$.

$(\partial_x n = 0 + 3x^2 y = \partial_y m$, no singularities $\Rightarrow$ conservative$)$

$$\vec{F} = \nabla f \Rightarrow \partial_x f = x^2 + \frac{3}{2}x^2y^2$$
$$\Rightarrow f = \frac{1}{3}x^3 + \frac{1}{2}x^3y^2 + g(y)$$
$$\Rightarrow \partial_y f = 0 + x^3 y + g'(y) = \sin y + x^3 y$$
$$\Rightarrow g'(y) = \sin y$$
$$\Rightarrow g(y) = -\cos y + C$$

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}x^3y^2 - \cos y + C$$

is a potential for $\vec{F}$.

Now it's your turn to grade. An imaginary student writes the following solutions, and every one of them is incorrect. In each problem, find and mark where the student went wrong, and correct the computation.

(1) Compute $\int_C y\cos(x)\,dx + 2xy\,dy$ where C is the line going from $(0,0)$ to $(\pi/2, 4)$.

$$\cancel{= y\sin(x)\Big|_{(0,0)}^{(\frac{\pi}{2},4)} + xy^2\Big|_{(0,0)}^{(\frac{\pi}{2},4)}}$$

$$= (4-0) + \left(\tfrac{\pi}{2}\cdot 16 - 0\right) = \boxed{8\pi + 4}$$

this is just nonsense; path integrals don't get computed this way

(2) Compute $\displaystyle\int_C \underbrace{xy}_{m}\,dx + \underbrace{\frac{x^2}{2}}_{n}\,dy$ for the same C.

$\partial_y m = x = \partial_x n,$

so $\left\langle xy, \frac{x^2}{2}\right\rangle$ is a gradient field. ✓

Thus $\displaystyle\int_C xy\,dx + \cancel{\frac{x^2}{2}\,dy} = \boxed{0.}$

C is not a closed curve!

(3) Compute the flow along the unit circle of the vector field $\mathbf{F}(x,y) = \left\langle \underbrace{\frac{x^2 - y + y^2}{x^2+y^2}}_{m}, \underbrace{\frac{x + x^2 + y^2}{x^2+y^2}}_{n} \right\rangle$.

$$\partial_y m = \frac{(x^2+y^2)(-1+2y) - (x^2-y+y^2)(2y)}{(x^2+y^2)^2} = \frac{(x^2+y^2)[(-1+2y)-2y] + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\|$$

$$\partial_x n = \frac{(x^2+y^2)(1+2x) - (x+x^2+y^2)(2x)}{(x^2+y^2)^2} = \frac{(x^2+y^2)[(1+2x)-2x] - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Hence $\vec{F}$ is a gradient ~~field~~, so the flow along the circle is zero.

$\vec{F}$ has a singularity at $(0,0)$

(1) Parametrize C:

$$\ell(t) = (1-t)\langle 0,0\rangle + t\langle \tfrac{\pi}{2}, 4\rangle, \quad t \in [0,1]$$

$$\Rightarrow x(t) = \tfrac{\pi}{2}t, \quad y(t) = 4t$$

$$\int_{t=0}^{1}\Big((4t)\cos(\tfrac{\pi}{2}t)(\tfrac{\pi}{2}dt) + 2(\tfrac{\pi}{2}t)(4t)(4\,dt)\Big)$$

$$= \int_0^1 \left(2\pi t\cos(\tfrac{\pi}{2}t) + 16\pi t^2\right)dt$$

$u = t \quad dv = 2\pi\cos(\tfrac{\pi}{2}t)\,dt$

$du = dt \quad v = 4\sin(\tfrac{\pi}{2}t)$

$$= \overbrace{\Big[4t\sin(\tfrac{\pi}{2}t)\Big]_0^1 - \int_0^1 4\sin(\tfrac{\pi}{2}t)\,dt}\; + \Big[\tfrac{16}{3}\pi t^3\Big]_0^1$$

$$= 4 - 0 + \Big[\tfrac{8}{\pi}\cos(\tfrac{\pi}{2}t)\Big]_0^1 + \tfrac{16}{3}\pi - 0$$

$$= 4 + (0 - \tfrac{8}{\pi}) + \tfrac{16}{3}\pi.$$

(2) Same parametrization.

$$\int_0^1\Big((\tfrac{\pi}{2}t)(4t)(\tfrac{\pi}{2}dt) + \tfrac{1}{2}(\tfrac{\pi}{2}t)^2(4\,dt)\Big)$$

$$= \int_0^1\left(\pi^2t^2 + \tfrac{1}{2}\pi^2t^2\right)dt = \int_0^1 \tfrac{3}{2}\pi^2t^2\,dt = \Big[\tfrac{1}{2}\pi^2t^3\Big]_0^1 = \tfrac{1}{2}\pi^2 - 0.$$

(3) $$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left\langle \frac{1-\sin t}{1}, \frac{1+\cos t}{1}\right\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

$x(t) = \cos t$

$y(t) = \sin t$

$t \in [0, 2\pi)$

$$= \int_0^{2\pi}\left(-\sin t + \sin^2 t + \cos t + \cos^2 t\right)dt$$

$$= \int_0^{2\pi}\left(1 - \sin t + \cos t\right)dt$$

$$= 2\pi - 0 + 0.$$

(2^*) or, find a potential: $\partial_x f = xy \Rightarrow f = \tfrac{1}{2}x^2y + g(y) \Rightarrow \partial_y f = \tfrac{1}{2}x^2 + g'(y) = \tfrac{1}{2}x^2$

$f(x,y) = \tfrac{1}{2}x^2y$ $\qquad \Rightarrow g(y) = c$

Fundamental Thm of Path Integrals $\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(\tfrac{\pi}{2}, 4) - f(0,0)$

$$= \tfrac{1}{2}(\tfrac{\pi}{2})^2(4) - 0 = \pi^2$$