Section 1.2 FRACTIONS

Operations with fractions

Fractions

In day-to-day living, the numbers we use most often are the whole numbers,

0, 1, 2, 3, 4, and so on,

for counting and the fractions, such as

$$\frac{1}{2}, \frac{3}{4}, \frac{9}{10}, \text{ and so on.}$$

In a fraction, the top number is called the **numerator** and the bottom number is called the **denominator**.

$$\begin{array}{ll} 9 \longleftarrow & \text{Numerator} \\ \hline 10 \longleftarrow & \text{Denominator} \end{array}$$

There are two types of fractions:

1. **Proper fractions** where the numerator is less than the denominator; for example, $\frac{9}{10}$.
2. **Improper fractions** where the numerator is greater than or equal to the denominator; for example, $\frac{10}{9}$ or $\frac{9}{9}$.

Reducing fractions to lowest terms

A fraction is **reduced to lowest terms** when the only factor common to the numerator and the denominator is 1.

To reduce a fraction to lowest terms

1. Write the numerator and the denominator as a product of prime factors.
2. Divide the numerator and the denominator by all common factors.

Example

1. $\frac{14}{21} = \frac{2 \cdot 7}{3 \cdot 7}$ Write as a product of prime factors

$= \frac{2}{3}$ Divide numerator and denominator by common factor 7

$\frac{2}{3}$ is the answer since 2 and 3 have only 1 as a common factor.

2. $\frac{45}{60} = \frac{3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 5}$ Write as a product of prime factors

$= \frac{3}{2 \cdot 2}$ Divide numerator and denominator by $3 \cdot 5$

$= \frac{3}{4}$ Multiply in denominator

▶ ***Quick check*** Reduce $\frac{25}{45}$ to lowest terms.

Products and quotients of fractions

To multiply two or more fractions, we use the following procedure.

To multiply fractions

1. Write the numerator and the denominator as an indicated product (do not multiply).
2. Reduce the resulting fraction to lowest terms.

Example

Multiply the following fractions and reduce to lowest terms.

1. $\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7}$ Multiply numerators; Multiply denominators

$= \frac{10}{21}$ Perform multiplications

2. $\frac{5}{6} \cdot \frac{3}{4} = \frac{5 \cdot 3}{6 \cdot 4}$ Multiply numerators; Multiply denominators

$= \frac{5 \cdot 3}{2 \cdot 3 \cdot 2 \cdot 2}$ Factor $6 = 2 \cdot 3$, $4 = 2 \cdot 2$

$= \frac{5}{2 \cdot 2 \cdot 2}$ Divide numerator and denominator by 3

$= \frac{5}{8}$ Multiply in the denominator

Suppose we multiply the fractions

$$\frac{5}{6} \cdot \frac{6}{5} = \frac{5 \cdot 6}{6 \cdot 5} = \frac{30}{30} = 1$$

When the product of two numbers is 1, we call each number the **reciprocal** of the other number. Thus

$\frac{5}{6}$ and $\frac{6}{5}$ are reciprocals,
$\frac{2}{7}$ and $\frac{7}{2}$ are reciprocals,
$\frac{14}{13}$ and $\frac{13}{14}$ are reciprocals.

We can see that the reciprocal of any fraction is obtained by interchanging the numerator and the denominator. The reciprocal of a fraction is used to divide fractions.

To divide two fractions

1. Multiply the first fraction by the **reciprocal** of the second fraction.
2. Reduce the resulting product to lowest terms.

Example

Divide the following fractions and reduce to lowest terms.

1. $\frac{7}{8} \div \frac{6}{7} = \frac{7}{8} \cdot \frac{7}{6}$ — Multiply by the reciprocal of $\frac{6}{7}$

$= \frac{7 \cdot 7}{8 \cdot 6}$ — Multiply numerators / Multiply denominators

$= \frac{49}{48}$

Note The improper fraction $\frac{49}{48}$ can be written as the **mixed number** $1\frac{1}{48}$, which is the sum of a whole number and a proper fraction. This is obtained by dividing the numerator by the denominator.

$$48 \overline{)49} \;\to\; \text{quotient } 1,\ 49 - 48 = 1 \text{ (Remainder)} \qquad = 1\frac{1}{48}$$

Quotient: 1
Remainder: 1
Original denominator: 48

2. $\dfrac{\frac{4}{5}}{\frac{3}{7}} = \frac{4}{5} \div \frac{3}{7} = \frac{4}{5} \cdot \frac{7}{3}$ — Multiply by the reciprocal of $\frac{3}{7}$

$= \frac{4 \cdot 7}{5 \cdot 3}$ — Multiply numerators / Multiply denominators

$= \frac{28}{15}$ or $1\frac{13}{15}$ — Perform indicated operations

The improper fraction answer is usually the one preferred in algebra. The mixed number form is usually preferred in an application problem.

3. $3\frac{1}{4} \div 5\frac{2}{3}$

We change the mixed numbers to improper fractions.

Denominator times whole number

Plus numerator

$$3\frac{1}{4} = \frac{(3 \cdot 4) + 1}{4} = \frac{12 + 1}{4} = \frac{13}{4}$$

Mixed number form

Original denominator

Improper fraction form

$$5\frac{2}{3} = \frac{(3 \cdot 5) + 2}{3} = \frac{15 + 2}{3} = \frac{17}{3}$$

We now divide as indicated.

$$\begin{aligned}
3\frac{1}{4} \div 5\frac{2}{3} &= \frac{13}{4} \div \frac{17}{3} \\
&= \frac{13}{4} \cdot \frac{3}{17} && \text{Multiply by the reciprocal of } \frac{17}{3} \\
&= \frac{13 \cdot 3}{4 \cdot 17} && \begin{array}{l}\text{Multiply numerators} \\ \text{Multiply denominators}\end{array} \\
&= \frac{39}{68} && \text{Perform indicated operations}
\end{aligned}$$

► ***Quick check*** Divide $\dfrac{\frac{5}{6}}{\frac{5}{8}}$ and reduce to lowest terms.

4. The area of a rectangle is found by multiplying the length of the rectangle by the width of the rectangle. Find the area of a rectangle that is $2\frac{1}{2}$ feet long and $1\frac{5}{6}$ feet wide.

$$\begin{aligned}
\text{Area} &= 2\frac{1}{2} \cdot 1\frac{5}{6} && \text{Multiply the given dimensions} \\
&= \frac{5}{2} \cdot \frac{11}{6} && \text{Change mixed numbers to improper fractions} \\
&= \frac{5 \cdot 11}{2 \cdot 6} && \begin{array}{l}\text{Multiply numerators} \\ \text{Multiply denominators}\end{array} \\
&= \frac{55}{12} \text{ or } 4\frac{7}{12} && \text{Perform indicated operations}
\end{aligned}$$

The area of the rectangle is $4\frac{7}{12}$ square feet. ■

Addition and subtraction of fractions

To add or subtract fractions, the fractions must have a *common* (same) *denominator*.

To add or subtract fractions with common denominators

1. Add or subtract the numerators.
2. Place the sum or difference over the common denominator.
3. Reduce the resulting fraction to lowest terms.

Example

Add or subtract the following fractions as indicated. Reduce to lowest terms.

1. $\frac{3}{8}+\frac{1}{8}=\frac{3+1}{8}$ Add numerators

$=\frac{4}{8}$ Combine in numerator

$=\frac{1}{2}$ Reduce to lowest terms

2. $\frac{7}{16}-\frac{5}{16}=\frac{7-5}{16}$ Subtract numerators

$=\frac{2}{16}$ Combine in numerator

$=\frac{1}{8}$ Reduce to lowest terms

When the fractions have different denominators, we must rewrite all of the fractions with a new common denominator. Many numbers can satisfy the condition for any set of denominators, but we want the *least* of these numbers, called the **least common denominator** (denoted by LCD). For example, 24 is the least common denominator of the fractions

$$\frac{7}{8} \text{ and } \frac{5}{6}$$

since it is the least (smallest) number that can be divided by 6 and 8 exactly. The procedure for finding the LCD is outlined next.

To find the least common denominator (LCD)

1. Express each denominator as a product of prime factors.
2. List all the *different* prime factors.
3. Write each prime factor the *greatest* number of times it appears in any of the prime factorizations in step 1.
4. The least common denominator is the product of all factors from step 3.

■ ***Example***

Find the least common denominator (LCD) of the fractions with the following denominators.

1. 24 and 18
   a. Write 24 and 18 as products of prime factors.
      $24 = 2 \cdot 2 \cdot 2 \cdot 3$
      $18 = 2 \cdot 3 \cdot 3$
   b. The different prime factors are 2 and 3.
   c. 2 is a factor three times in 24 and 3 is a factor two times in 18 (the greatest number of times).
   d. The LCD is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$.

2. 6, 8, and 14
   a. $6 = 2 \cdot 3$
      $8 = 2 \cdot 2 \cdot 2$
      $14 = 2 \cdot 7$
   b. The different prime factors are 2, 3, and 7.
   c. 2 is a factor three times in 8, 3 is a factor once in 6, and 7 is a factor once in 14.
   d. The LCD is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 168$.

▶ ***Quick check*** Find the LCD for fractions with denominators of 6, 9, and 12. ■

Building fractions

To write the fraction $\frac{5}{6}$ as an equivalent fraction with new denominator 24, we find the number that is multiplied by 6 to get 24. Since

$$6 \cdot 4 = 24$$

we use the factor 4. Now multiply the given fraction $\frac{5}{6}$ by the fraction $\frac{4}{4}$. The fraction $\frac{4}{4}$ is equal to 1 and is called a unit fraction. Multiplying by the unit fraction $\frac{4}{4}$ will not change the value of $\frac{5}{6}$, only its form. Thus

$$\frac{5}{6} = \frac{5}{6} \cdot \underbrace{\frac{4}{4}}_{\text{Multiplication by 1}} = \frac{5 \cdot 4}{6 \cdot 4} = \frac{20}{24}$$

We use the following procedure to write equivalent fractions.

To find equivalent fractions

1. Divide the original denominator into the new denominator.
2. Multiply the numerator and the denominator of the given fraction by the number obtained in step 1.

Example

Write equivalent fractions having the new denominator.

1. $\frac{3}{5} = \frac{?}{30}$

Since $30 \div 5 = 6$, multiply $\frac{3}{5}$ by $\frac{6}{6}$.

$$\frac{3}{5} = \frac{3}{5} \cdot \frac{6}{6} \quad \text{Multiply by } \frac{6}{6}$$
$$= \frac{3 \cdot 6}{5 \cdot 6} \quad \begin{array}{l}\text{Multiply numerators}\\ \text{Multiply denominators}\end{array}$$
$$= \frac{18}{30}$$

2. $\frac{7}{9} = \frac{?}{72}$

Since $72 \div 9 = 8$, multiply $\frac{7}{9}$ by $\frac{8}{8}$.

$$\frac{7}{9} = \frac{7}{9} \cdot \frac{8}{8} \quad \text{Multiply by } \frac{8}{8}$$
$$= \frac{7 \cdot 8}{9 \cdot 8} \quad \begin{array}{l}\text{Multiply numerators}\\ \text{Multiply denominators}\end{array}$$
$$= \frac{56}{72}$$

To add or subtract fractions having different denominators, we use the following procedure.

To add or subtract fractions having different denominators

1. Find the LCD of the fractions.
2. Write each fraction as an equivalent fraction with the LCD as the new denominator.
3. Perform the addition or subtraction as before.
4. Reduce the resulting fraction to lowest terms.

Example

Add or subtract the following fractions as indicated. Reduce the resulting fraction to lowest terms.

1. $\frac{7}{8} + \frac{5}{6}$

a. The LCD of the fractions is 24.

b. Since $24 \div 8 = 3$, then

$$\frac{7}{8} = \frac{7}{8} \cdot \frac{3}{3} = \frac{7 \cdot 3}{8 \cdot 3} = \frac{21}{24} \quad \text{Multiply by } \frac{3}{3}$$

and since $24 \div 6 = 4$, then

$$\frac{5}{6} = \frac{5}{6} \cdot \frac{4}{4} = \frac{5 \cdot 4}{6 \cdot 4} = \frac{20}{24} \quad \text{Multiply by } \frac{4}{4}$$

c. $\frac{7}{8} + \frac{5}{6} = \frac{21}{24} + \frac{20}{24}$ — Add fractions with LCD

$= \frac{21 + 20}{24}$ — Add numerators

$= \frac{41}{24}$ or $1\frac{17}{24}$

2. $\frac{7}{8} - \frac{1}{3}$

a. The LCD of the fractions is 24.

b. Since $24 \div 8 = 3$, then

$\frac{7}{8} = \frac{7}{8} \cdot \frac{3}{3} = \frac{21}{24}$ — Multiply by $\frac{3}{3}$

Since $24 \div 3 = 8$, then

$\frac{1}{3} = \frac{1}{3} \cdot \frac{8}{8} = \frac{8}{24}$ — Multiply by $\frac{8}{8}$

c. $\frac{7}{8} - \frac{1}{3} = \frac{21}{24} - \frac{8}{24}$ — Subtract fractions with LCD

$= \frac{21 - 8}{24} = \frac{13}{24}$ — Subtract numerators

3. $3\frac{7}{8} - 2\frac{3}{4}$

Change each of the mixed numbers to an improper fraction.

$3\frac{7}{8} = \frac{(8 \cdot 3) + 7}{8} = \frac{31}{8}$; $2\frac{3}{4} = \frac{(4 \cdot 2) + 3}{4} = \frac{11}{4}$

a. The LCD of the fractions is 8.

b. $\frac{31}{8}$ already has the LCD in its denominator.

Since $8 \div 4 = 2$, then

$\frac{11}{4} = \frac{11 \cdot 2}{4 \cdot 2} = \frac{22}{8}$ — Multiply by $\frac{2}{2}$

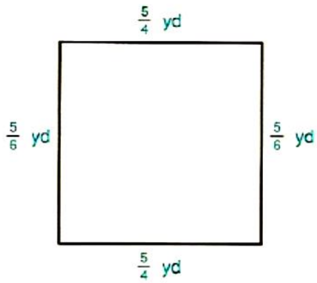
c. $3\frac{7}{8} - 2\frac{3}{4} = \frac{31}{8} - \frac{11}{4}$ — Subtract improper fractions

$= \frac{31}{8} - \frac{22}{8}$ — Subtract fractions with LCD

$= \frac{31 - 22}{8}$ — Subtract numerators

$= \frac{9}{8}$ or $1\frac{1}{8}$

4. The perimeter (distance around) of a rectangle is found by *adding* the lengths of the four sides of the rectangle. Find the perimeter of a rectangle that is $1\frac{1}{4}$ yards long and $\frac{5}{6}$ of a yard wide.



Perimeter $= \frac{5}{6} + 1\frac{1}{4} + \frac{5}{6} + 1\frac{1}{4}$	Add all the sides
$= \frac{5}{6} + \frac{5}{4} + \frac{5}{6} + \frac{5}{4}$	Change to improper fractions
$= \frac{10}{12} + \frac{15}{12} + \frac{10}{12} + \frac{15}{12}$	LCD is 12
$= \frac{10 + 15 + 10 + 15}{12}$	Add numerators
$= \frac{50}{12} = \frac{25}{6}$	Reduce
or $4\frac{1}{6}$	Mixed number form

The perimeter of the rectangle is $4\frac{1}{6}$ yards.

▶ ***Quick check*** $4\frac{1}{2} - 2\frac{3}{4}$

Mastery points

Can you

- Reduce a fraction to lowest terms?
- Multiply and divide fractions?
- Find the least common denominator (LCD) of two or more fractions?
- Add and subtract fractions?

Exercise 1.2

Reduce the following fractions to lowest terms. See example 1–1 B.

Example $\frac{25}{45}$

Solution $\frac{25}{45} = \frac{5 \cdot 5}{3 \cdot 3 \cdot 5}$	Factor numerator Factor denominator
$= \frac{5}{3 \cdot 3}$	Divide numerator and denominator by 5
$= \frac{5}{9}$	Multiply in denominator

1. $\frac{4}{8}$ **2.** $\frac{3}{9}$ **3.** $\frac{10}{12}$ **4.** $\frac{8}{14}$ **5.** $\frac{16}{18}$ **6.** $\frac{14}{21}$

$\boxed{7.}$ $\frac{28}{36}$ **8.** $\frac{50}{75}$ **9.** $\frac{64}{32}$ **10.** $\frac{96}{48}$ **11.** $\frac{100}{85}$ **12.** $\frac{120}{84}$

Multiply or divide the fractions as indicated. Reduce to lowest terms. See examples 1–1 C and D.

Example $\dfrac{\frac{5}{6}}{\frac{5}{8}}$

Solution $\dfrac{\frac{5}{6}}{\frac{5}{8}} = \dfrac{5}{6} \div \dfrac{5}{8} = \dfrac{5}{6} \cdot \dfrac{8}{5}$ Multiply by the reciprocal of $\frac{5}{8}$

$= \dfrac{5 \cdot 8}{6 \cdot 5}$ Multiply numerators
Multiply denominators

$= \dfrac{8}{6}$ Reduce by common factor 5

$= \dfrac{2 \cdot 2 \cdot 2}{2 \cdot 3}$ Write numerator and denominator as the product of prime factors

$= \dfrac{2 \cdot 2}{3}$ Reduce by common factor 2

$= \dfrac{4}{3}$ or $1\dfrac{1}{3}$ Multiply in denominator

13. $\dfrac{5}{6} \cdot \dfrac{3}{5}$ **14.** $\dfrac{2}{3} \cdot \dfrac{5}{6}$ **15.** $\dfrac{7}{8} \cdot \dfrac{7}{12}$ **16.** $\dfrac{7}{5} \cdot \dfrac{3}{2}$ **17.** $\dfrac{7}{9} \cdot \dfrac{3}{4}$

18. $\dfrac{3}{4} \cdot 6$ **19.** $\dfrac{3}{7} \div \dfrac{4}{5}$ **20.** $\dfrac{12}{25} \div \dfrac{8}{15}$ **21.** $\dfrac{6}{7} \div 3$ **22.** $4 \div \dfrac{3}{8}$

23. $\dfrac{15}{17} \div \dfrac{3}{5}$ **24.** $4 \div \dfrac{7}{2}$ **25.** $17 \div 2\dfrac{1}{3}$ **26.** $12 \cdot 1\dfrac{5}{6}$ $\boxed{27.}$ $7\dfrac{1}{3} \cdot 2\dfrac{4}{7}$

28. $1\dfrac{1}{5} \cdot 2\dfrac{1}{2}$ **29.** $4\dfrac{4}{5} \cdot 2\dfrac{1}{2}$ **30.** $7\dfrac{1}{2} \div 5\dfrac{1}{4}$ **31.** $\dfrac{8}{\frac{2}{3}}$ **32.** $\dfrac{17}{\frac{3}{4}}$

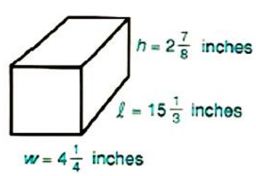
$\boxed{33.}$ $\dfrac{\frac{7}{8}}{\frac{4}{3}}$ **34.** $\dfrac{\frac{15}{64}}{\frac{45}{8}}$ $\boxed{35.}$ $\dfrac{4}{5} \cdot \dfrac{2}{3} \cdot \dfrac{3}{8}$ **36.** $\dfrac{9}{8} \cdot \dfrac{2}{3} \cdot \dfrac{3}{8}$ **37.** $\dfrac{8}{3} \cdot \dfrac{4}{7}$

38. $\dfrac{8}{3} \div \dfrac{15}{14}$ **39.** $\dfrac{8}{3} \cdot \dfrac{15}{14}$

See example 1–1 D–4.

40. What is the total length of 25 pieces of steel, each $5\frac{1}{2}$ inches long?

41. The volume of a rectangular block is found by multiplying the length times the width times the height.



a. What is the volume in cubic inches of a rectangular block of wood $15\frac{1}{3}$ inches long, $4\frac{1}{4}$ inches wide, and $2\frac{7}{8}$ inches high?

b. What is the volume in cubic inches of a block of steel $8\frac{1}{2}$ inches long, $2\frac{1}{8}$ inches wide, and $1\frac{3}{4}$ inches high?

42. A wire $61\frac{1}{2}$ inches long is divided into 14 equal parts. What is the length of each part?

Find the LCD of the fractions with the following groups of denominators. See example 1–1 F.

Example 6, 9, and 12

Solution 1. State 6, 9, and 12 as a product of prime factors.

$$6 = 2 \cdot 3$$
$$9 = 3 \cdot 3$$
$$12 = 2 \cdot 2 \cdot 3$$

2. The different prime factors are 2 and 3.
3. 2 is a factor twice in 12 and 3 is a factor twice in 9.
4. The LCD is $2 \cdot 2 \cdot 3 \cdot 3 = 36$.

43. 3, 8, 10
44. 9, 15, 21
45. 6, 14, 18
46. 5, 10, 12
47. 16, 24, 36
48. 12, 16, 24
49. 5, 7, 11
50. 10, 20, 30
51. 68, 9, 12
52. 10, 14, 18
53. 10, 15, 20
54. 10, 15, 24

Add or subtract the following fractions as indicated. Reduce to lowest terms. See examples 1–1 E and H.

Example $4\frac{1}{2} - 2\frac{3}{4}$

Solution We first change the mixed numbers to improper fractions.

$$4\frac{1}{2} = \frac{(4 \cdot 2) + 1}{2} = \frac{9}{2}; \quad 2\frac{3}{4} = \frac{(4 \cdot 2) + 3}{4} = \frac{11}{4}$$

$$4\frac{1}{2} - 2\frac{3}{4} = \frac{9}{2} - \frac{11}{4}$$ Replace mixed numbers with improper fractions

The LCD is 4.

$$= \frac{18}{4} - \frac{11}{4}$$ Write $\frac{9}{2}$ as $\frac{18}{4}$

$$= \frac{18 - 11}{4}$$ Subtract numerators

$$= \frac{7}{4} \text{ or } 1\frac{3}{4}$$

55. $\frac{1}{3} + \frac{1}{3}$
56. $\frac{2}{5} + \frac{3}{10}$
57. $\frac{1}{3} + \frac{1}{4}$
58. $\frac{5}{6} - \frac{1}{6}$
59. $\frac{4}{5} - \frac{2}{10}$
60. $\frac{5}{6} - \frac{3}{8}$
61. $1 + \frac{5}{8}$
62. $3 + \frac{5}{6}$

63. $4 - \frac{3}{5}$

64. $\frac{2}{3} + \frac{3}{4}$

65. $\frac{3}{5} + \frac{7}{15}$

66. $\frac{5}{6} - \frac{1}{3}$

67. $\frac{3}{8} - \frac{1}{12}$

68. $\frac{7}{24} - \frac{3}{16}$

69. $\frac{7}{54} + \frac{19}{45}$

70. $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$

71. $\frac{7}{15} + \frac{5}{6} - \frac{3}{4}$

72. $\frac{9}{16} + \frac{5}{18} - \frac{2}{15}$

73. $8\frac{3}{16} - 4\frac{5}{8}$

74. $7\frac{1}{2} + 2\frac{3}{4}$

75. $\frac{2}{7} + \frac{2}{3} + \frac{5}{7}$

See example 1–1 H–4.

76. Jane owed Joan some money. If she paid Joan $\frac{1}{4}$ of the debt on June 15, $\frac{1}{3}$ of the original debt on July 1, and $\frac{3}{8}$ of the original debt on August 10, how much of her debt had Jane paid by August 10?

77. A flower garden in the form of a rectangle has two sides that are $24\frac{1}{2}$ feet long and two sides that are $18\frac{3}{4}$ feet long. Find the perimeter (total distance around) of the rectangle.

78. On a given day, Mrs. Jones purchased $\frac{5}{6}$ yard of one material, $\frac{3}{4}$ yard of another material, and $\frac{2}{3}$ yard of a third material. How many yards of material did she purchase altogether?

79. Butcher John has $32\frac{1}{4}$ pounds of pork chops. If he sells $21\frac{1}{3}$ pounds of the pork chops on a given day, how many pounds of pork chops does he have left?

80. A machinist has a piece of steel stock that weighs $12\frac{7}{8}$ ounces. If he cuts off $5\frac{1}{5}$ ounces, how many ounces does he have left?