Worksheet 10 February 23, 2011

In case you didn't get there last time, remember to NEVER say something like "$x^2 = 2x$" when what you mean is $\frac{d}{dx}x^2 = 2x$.

1. Here are the derivatives for the trigonometrics (this is the last time I'll list them for you):

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$$
$$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$$

   (a) On the chalkboard, make a list of patterns you see. (I've arranged the table suggestively.)

   (b) I used to just memorize the derivatives for tan and sec, and to know which was which I thought to myself that it would be silly for the derivative of a function (sec) to be its own square ($\sec^2$). But actually there are such functions, and you've seen one. Do you remember it?

2. Suppose you have two functions, $f(x)$ and $g(x)$, and that $f(3) = 2$, $g(3) = -5$, $f'(3) = 1$, $g'(3) = 8$. Find an equation for the tangent line to the function $|f + g|$ at $x = 3$.

3. Same question, but now for the function $|5f + 2g|$. (Is something fishy going on?)

4. Practice time! Find the derivatives of the following functions.

   (a) x^{200}

   (b) 2^{3t}

   (c) x^0

   (d) $(w^3 + w^{-1})(\sqrt{w} - 2)$

   (e) $(w^3 + w^{-1} - e^w + \sin w)(\sqrt{w} - 2 + \ln w - \tan w)$

   (f) $\dfrac{u^4 - u^{3/2}}{2u}$

   (g) $\dfrac{x^2}{1 - x^3}$

   (h) $\arctan\left(\sqrt{e^{\pi \cdot \log_{23}(\sqrt{2})}}\right)$

   (i) x^π

   (j) π^x

   (k) e^π

5. We saw on the previous worksheet that $\frac{d}{dx}e^{4x} = 4e^{4x}$. From your reading you now know a more general result, the *Chain Rule*.

   (a) State the chain rule.

   (b) If you used Leibniz notation (i.e. "$\frac{d}{dx}$"), explain why this looks reasonable. (Also explain why that explanation is not a proof of the chain rule, just a mnemonic of sorts.) (If you didn't use Leibniz, redo using Leibniz.)

   (c) Use the chain rule to compute the derivatives of the following

      (i) $\sin(\pi x)$
      (ii) $\tan(\cos(x))$
      (iii) $\sqrt{x^2 - 2x + 7}$
      (iv) $\tan(\cos(\sin(x)))$ (Hint: 5(c)ii is helpful.)

   (d) Later in this chapter we will use the chain rule to compute the derivatives of the logarithm and inverse trig functions. Here's how to get $(\arcsin x)'$:

      (i) Let $y = \arcsin(x)$. Rewrite this equation without any *inverse* trig functions.
      (ii) Now differentiate the resulting equation. (Remember that we are differentiating with respect to x, and y is a function of x; you'll need the chain rule here.)
      (iii) Now solve the resulting equation for y'.
      (iv) Your result will have a y still in it; rewrite this using a right triangle to involve only x.

6. Find an equation for the tangent line to the curve $y = e^x$ at $x = \ln 4$.

7. Find a, b so that the parabola $y = ax^2 + bx$ has a tangent line at $(1, 1)$ with equation $y = 3x - 2$.

8. Find an equation for the tangent line to the curve $y = \tan x$ at the point $x = 0$. Try to do the same at $x = \pi/2$; what goes wrong?

9. Find an equation for the tangent line to the function $f \circ g$ at $x = 3$, where f and g are the functions from problem 2.

* Show that the tangent line L_P to the curve $y = x^3$ at the point P meets the curve again at a point Q such that the slope of the tangent at Q is exactly four times the slope of L_P. (To understand what this means, a picture may be helpful.)

** We're going to consider a cool curve (an "elliptic curve"). At the level of pre-calculus it's neat because, well, it looks neat; at the level of calculus it's neat because it has some interesting tangent lines; at a slightly higher level it's neat because these tangent lines and some secant lines form a very cool structure; on an even higher level this kind of structure can be used to help your credit card information travel the web safely (!).

The curve is given by the equation $y^2 = x^3 - 5x + 4$. Here's its graph:

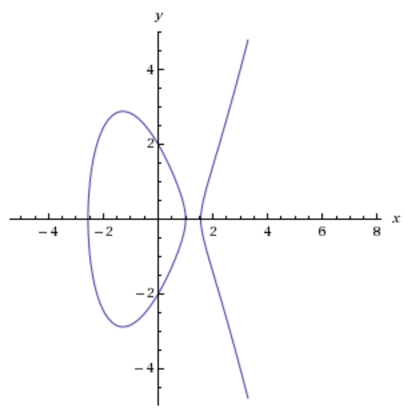

Figure 1: image via Wolfram Alpha

(a) Is y a function of x here? Is x a function of y?

(b) Take the derivative of each side of the given equation with respect to x. This is *implicit differentiation*, which is officially started in section 3.5 of your text, but there's nothing particularly difficult about it. Just keep in mind that when you find a y, you'll need the chain rule.

(c) Now solve the resulting equation for y', in terms of both x and y.

(d) Find the equation for the tangent line to this curve at the point $(0, 2)$.

(e) Find the equation for the tangent line to this curve at the point $(0, -2)$.

(f) Try the same method to find the tangent line at the point $(1, 0)$. What went wrong? Now write down the equation for this tangent line, without resorting to the derivative you computed.

(g) How might you use calculus to find the highest point on the loop of the curve? Do it!