Tuesday, August 28 ** *A review of some important calculus topics*

1. Chain Rule:

   (a) Let $h(t) = \sin\big(\cos(\tan t)\big)$. Find the derivative with respect to t.

   (b) Let $s(x) = \sqrt[4]{x}$ where $x(t) = \ln\big(f(t)\big)$ and $f(t)$ is a differentiable function. Find $\dfrac{ds}{dt}$.

2. Parameterized curves:

   (a) Describe and sketch the curve given parametrically by

$$\begin{cases} x = 5\sin(3t) \\ y = 3\cos(3t) \end{cases} \quad \text{for} \quad 0 \le t < \frac{2\pi}{3}.$$

   What happens if we instead allow t to vary between 0 and 2π?

   (b) Set up, but **do not evaluate** an integral that calculates the arc length of the curve described in part (a).

   (c) Consider the equation $x^2 + y^2 = 16$. Graph the set of solutions of this equation in $\mathbb{R}^2$ and find a parameterization that traverses the curve once counterclockwise.

3. 1st and 2nd Derivative Tests:

   (a) Use the 2nd Derivative Test to classify the critical numbers of the function $f(x) = x^4 - 8x^2 + 10$.

   (b) Use the 1st Derivative Test and find the extrema of $h(s) = s^4 + 4s^3 - 1$.

   (c) Explain why the 2nd Derivative test is unable to classify all the critical numbers of $h(s) = s^4 + 4s^3 - 1$.

4. Consider the function $f(x) = x^2 e^{-x}$.

   (a) Find the best linear approximation to f at $x = 0$.

   (b) Compute the second-order Taylor polynomial at $x = 0$.

   (c) Explain how the second-order Taylor polynomial at $x = 0$ demonstrates that f must have a local minimum at $x = 0$.

5. Consider the integral $\displaystyle\int_0^{\sqrt{3\pi}} 2x\cos(x^2)\,dx$.

   (a) Sketch the area in the xy-plane that is implicitly defined by this integral.

   (b) To evaluate, you will need to perform a substitution. Choose a proper $u = f(x)$ and rewrite the integral in terms of u. Sketch the area in the uv-plane that is implicitly defined by this integral.

   (c) Evaluate the integral $\displaystyle\int_0^{\sqrt{3\pi}} 2x\cos(x^2)\,dx$.

SOLUTIONS TO THE 08/28/12 WORKSHEET

1) (a) $h(t) = \sin(\cos(\tan t))$

Let $f_1(x) = \tan x$, $f_2(y) = \cos y$, $f_3(z) = \sin z$. As $h(t) = f_3(f_2(f_1(t)))$ and f_3 and f_2 are defined on $\mathbb{R}$, the only restriction is on the domain of f_1, which we can take $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or more generally $D = \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$. For $t \in D$ the Chain Rule gives

$$h'(t) = f_3'(f_2(f_1(t))) \cdot \frac{d}{dt}(f_2(f_1(t))) = \cos(\underbrace{f_2(f_1(t))}_{\cos(\tan t)}) \cdot \underbrace{f_2'(f_1(t))}_{-\sin(\tan t)} \underbrace{f_1'(t)}_{\frac{1}{\cos^2 t}}$$

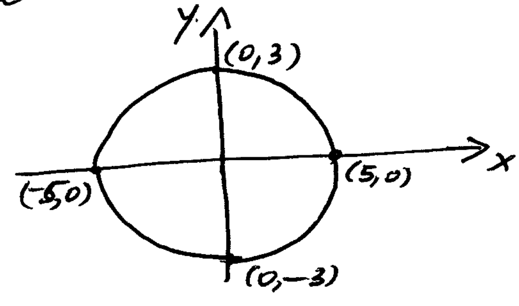
$$= -\cos(\cos(\tan t)) \cdot \sin(\tan t) \cdot \frac{1}{\cos^2 t}.$$

(b) Assuming $f(t) > 1$ we have $x(t) > 0$ so $\sqrt[4]{x}$ exists and

$$\frac{ds}{dt} = \frac{d}{dt}\left(x(t)^{1/4}\right) = \frac{1}{4}\, x(t)^{-3/4} \cdot \frac{dx}{dt} = \frac{1}{4} \cdot x(t)^{-3/4} \cdot \frac{d}{dt}(\ln f(t))$$

$$= \frac{1}{4}(\ln f(t))^{-3/4} \cdot \frac{f'(t)}{f(t)}.$$

2) (a) This is part of the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ because $\sin^2\theta + \cos^2\theta = 1$. Actually since $3t$ varies between 0 and 2π we obtain the whole ellipse.

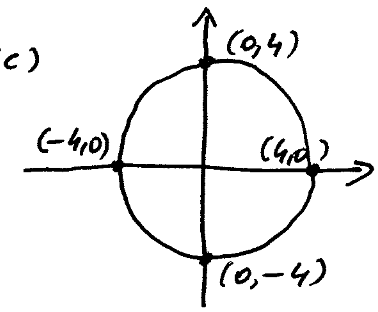


When t varies between 0 and 2π, $3t$ covers the trigonometric circle three times, hence the ellipse is covered exactly three times

(b) $$L = \int_0^{2\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^{2\pi/3} \sqrt{15\cos^2(3t) + 9\sin^2(3t)}\, dt$$

$$\underset{3t = u,\ dt = \frac{du}{3}}{=} \frac{1}{3}\int_0^{2\pi} \sqrt{15\cos^2 u + 9\sin^2 u}\, du$$

(c)



$$\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases} \qquad 0 \le t < 2\pi$$

3) (a) $f(x) = x^4 - 8x^2 + 10$; $f'(x) = 4x^3 - 16x = 4x(x^2-4)$

x critical point of $f \iff f'(x) = 0 \iff x \in \{-2, 0, 2\}$.

$f''(x) = 12x^2 - 16 = 4(3x^2-4)$

$f''(-2) = f''(2) > 0 \implies x=-2$ and $x=2$ local min

$f''(0) < 0 \implies x=0$ local max

(b) $h(s) = s^4 + 4s^3 - 1$; $h'(s) = 4s^3 + 12s^2 = 4s^2(s+3)$

s critical point of $h \iff h'(s) = 0 \iff s \in \{-3, 0\}$.

$s \ge -3 \implies h'(s) \ge 0 \implies h \nearrow$ on $[-3, \infty)$

$s \le -3 \implies h'(s) \le 0 \implies h \searrow$ on $(-\infty, -3]$

$\implies s = -3$ global min

(c) $h''(s) = 12s^2 + 24s = 12s(s+2)$

$h''(-3) > 0 \implies s = -3$ local min

$h''(0) = 0 \implies$ the 2^{nd} Derivative Test is inconclusive for $s=0$

4) (a) $f(x) \approx f(0) + (x-0)f'(0)$ is the best linear approximation of f at 0.

$f(0) = 0$, $f'(x) = 2xe^{-x} - x^2e^{-x} \implies f'(0) = 0 \implies f(x) \approx 0$ best linear approx. of f at 0.

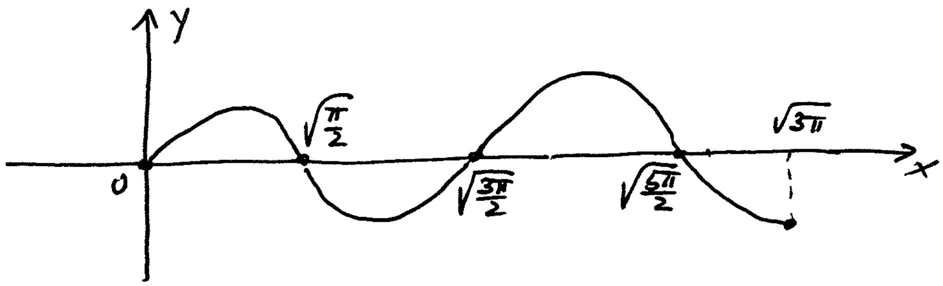
(b) $f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$

$\implies f''(0) = 2 \implies 2^{nd}$ order Taylor polynomial of f at $x=0$ is

$$f(0) + \frac{x-0}{1!}f'(0) + \frac{(x-0)^2}{2!}f''(0) = \frac{x^2}{2}\cdot 2 = x^2$$

$f'(0) = 0$, $f''(0) > 0 \implies x = 0$ local minimum at $x=0$

5) (a) $f(x) = 2x\cos(x^2)$ is ≥ 0 if $x \in [0, \sqrt{\frac{\pi}{2}}] \cup [\sqrt{\frac{3\pi}{2}}, \sqrt{\frac{5\pi}{2}}]$
and $f(x) \leq 0$ if $x \in [\sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{2}}] \cup [\sqrt{\frac{5\pi}{2}}, \sqrt{3\pi}]$.



(b)+(c) Take $u = x^2$ so $2x\cos(x^2)\,dx = \cos(x^2)\,d(x^2) = \cos u\,du$ and

$$\int_0^{\sqrt{3\pi}} 2x\cos(x^2)\,dx = \int_0^{3\pi} \cos u\,du = \sin u \Big|_0^{3\pi} = \sin 3\pi - \sin 0 = 0.$$