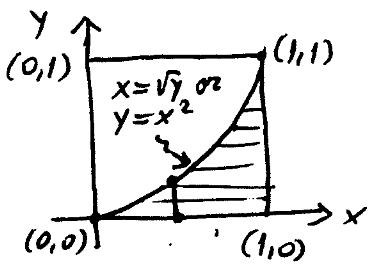
WORKSHEET 10/30/12

① Evaluate the following integral by reversing the order of integration:

$$I = \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$$

Domain of integration: $D = \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$



FUBINI $\Rightarrow I = \int_0^1 dx \left(\int_0^{x^2} dy\ \sqrt{x^3+1} \right) = \int_0^1 x^2 \sqrt{x^3+1}\, dx$

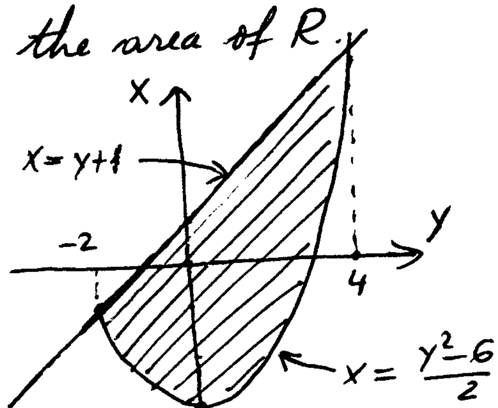
$$= \int_1^2 \frac{1}{3} u^{1/2} du = \frac{1}{3} \cdot \frac{1}{1+\frac{1}{2}} u^{3/2} \Big|_1^2 = \frac{2}{9}(2\sqrt{2}-1)$$

$u = x^3+1$
$du = 3x^2 dx$

② Consider the region bounded by the curves determined by $-2x+y^2=6$ and $-x+y=-1$.

(a) Sketch the region R in the plane.

(b) Set up and evaluate an integral of the form $\iint_R dA$ that calculates the area of R.



$R = \{(x,y) \mid -2 \le y \le 4,\ \frac{y^2-6}{2} \le x \le y+1\}$

$$\begin{cases} x = \frac{y^2-6}{2} \\ x = y+1 \end{cases}$$

$\frac{y^2-6}{2} = y+1$

$\Updownarrow$

$y^2 - 2y - 8 = 0$

$\Updownarrow$

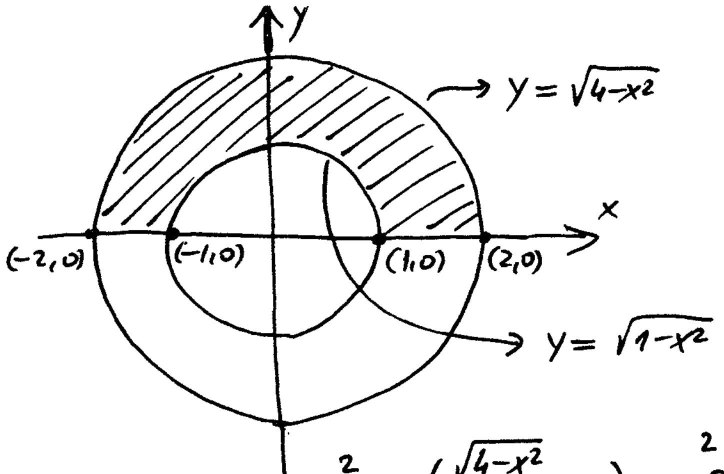
$y \in \{-2, 4\}$

$$\text{Area}(R) = \iint_R 1\, dA = \int_{-2}^4 dy \left(\int_{\frac{y^2-6}{2}}^{y+1} dx \right) = \int_{-2}^4 \left(y+1 - \frac{y^2-6}{2} \right) dy$$

$$\text{Area}(R) = \int_{-2}^4 \left(y - \frac{y^2}{2} + 4 \right) dy = \frac{y^2}{2}\Big|_{-2}^4 - \frac{y^3}{6}\Big|_{-2}^4 + 4 \cdot 6$$

$$= \frac{16-4}{2} - \frac{64-(-2)^3}{6} + 24 = 6 - 12 + 24 = 18$$

③ Consider the region R which lies above the x-axis and between the circles of radius 1 and 2 centered at $(0,0)$. <u>Without using polar coordinates</u> evaluate $\iint y\, dA$.



$$\iint_R y\,dA = \iint_{R_1} y\,dA - \iint_{R_2} y\,dA$$

where

$$R_1 = \{(x,y) \mid -2 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2}\}$$
$$R_2 = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$$

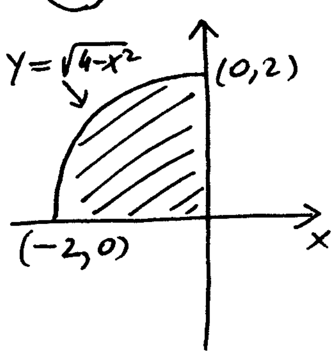
$$I_1 = \iint_{R_1} y\,dA = \int_{-2}^{2} dx \left(\int_0^{\sqrt{4-x^2}} dy\ y \right) = \int_{-2}^{2} \frac{y^2}{2}\Big|_{y=0}^{\sqrt{4-x^2}} dx = \frac{1}{2}\int_{-2}^{2} (4-x^2)\,dx$$
$$= \int_0^2 (4-x^2)\,dx = 8 - \frac{x^3}{3}\Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$x \to 4-x^2$ even

$$I_2 = \iint_{R_2} y\,dA = \int_{-1}^{1} dx \left(\int_0^{\sqrt{1-x^2}} dy\ \ y \right) = \int_{-1}^{1} \frac{y^2}{2}\Big|_{y=0}^{\sqrt{1-x^2}} dx = \frac{1}{2}\int_{-1}^{1} (1-x^2)\,dx$$
$$= \int_0^1 (1-x^2)\,dx = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\iint_R y\,dA = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

(4) Evaluate $I = \displaystyle\int_{-2}^{0}\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$.

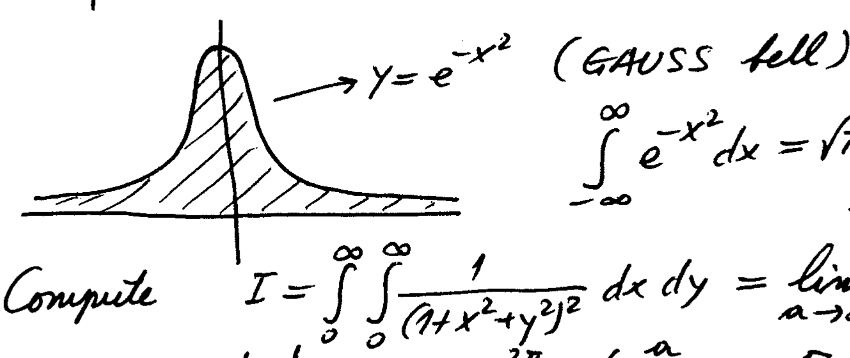


Polar coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad 0 \le r \le 2$, $\ \frac{\pi}{2} \le \theta \le \pi$

$x^2+y^2 = r^2$

$$I = \int_{\pi/2}^{\pi} d\theta \left(\int_0^2 dr\ \ r \cdot r^2 \right) = \frac{\pi}{2} \cdot \frac{r^4}{4}\Big|_0^2 = \frac{\pi}{2} \cdot 4 = 2\pi$$

(5)



$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$ see the lecture notes for details

(6) Compute $I = \displaystyle\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \lim_{a\to\infty} I(a)$, where

$$I(a) = \iint_{x^2+y^2 \le a^2} \frac{dx\,dy}{(1+x^2+y^2)^2} = \int_0^{2\pi} d\theta \left(\int_0^a dr \cdot \frac{r}{(1+r^2)^2} \right) = 2\pi \cdot \frac{1}{2} \int_1^{1+a^2} \frac{du}{u^2} = \pi\left(1 - \frac{1}{1+a^2}\right) \xrightarrow{a\to\infty} \pi$$

Polar coordinates

$1+r^2 = u$
$2r\,dr = du$

$\to \boxed{I = \pi}$