Worksheet 9 February 21, 2011

0. **Exam 1 Redux** (return of the Jedi? or the Sith?)

   (a) Rewrite $\sin 2t$ and $\cos 2t$ using some identities.

   (b) Explain why you do *not* want to divide by x^2 in the limits $\lim_{x\to 4} \frac{x^2-2}{x^2+2x+3}$, $\lim_{x\to 4} \frac{x^2-2}{x^2-16}$, and $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-16}$, but you *do* want to in the limit $\lim_{x\to\infty} \frac{x^2-2x-8}{x^2-16}$.

1. Find an equation for the tangent line to the curve $y = 3x^2 - 2x + 1$ at $(2, 9)$. (Remark: it is important to realize the difference between the derivative *at a point* and the derivative *as a function*. Which one do you care about in this problem? When/why do you care about the other one? [Hint: if you get the tangent line for this problem to be $y = (6x-2)x - 11$, why is this *horribly* wrong? {I have seen this as an answer before (How nested can my bracketing get? [])}])

2. Suppose you have two functions, $f(x)$ and $g(x)$, and that $f(3) = 2$, $g(3) = -5$, $f'(3) = 1$, $g'(3) = 8$. Find an equation of the tangent line to $(f+g)(x)$ at $x = 3$.

3. Suppose you are changing the size of a window on your computer by dragging the top right corner; say the bottom left corner is at the origin. Suppose further that the position of the top right corner is given by $x = f(t) = 5 + 2t$, $y = g(t) = 7 + 3t$.

   (a) What is the original area of the window?

   (b) How fast is the width changing at time t?

   (c) How fast is the height changing at time t?

   (d) Draw a picture demonstrating the size of the picture at times t and $t+h$ (assume $h > 0$). Find the difference in the two areas.

   (e) Divide the expression from (3d) by h. What does this represent?

   (f) Take the limit of the expression from (3e) as $h \to 0$. What does this represent? (Notice that one of the summands disappears. Which area does it correspond to?)

   (g) How does this problem relate to the product rule?

4. Same as 2, but now find the tangent line to $f \cdot g$ at $x = 3$.

5. Compute the rate of change of the area of a circle with respect to its radius. Does your formula look familiar? Can you explain the result with a picture?

6. Compute the rate of change of the volume of a sphere with respect to its radius. Does your formula look familiar? Can you explain the result with a picture?

7. Consider the function $f(x) = e^{4x}$. How does its graph relate to the graph of e^x? Use this relationship to determine the derivative of f.

8. The quotient rule is one of the messier looking derivative rules. There are a handful of mnemonics for it (hi-dee-lo anyone?), but many times you can ignore it altogether. This will be easier once you know the *chain rule*, but for now here's a way to derive the quotient rule without limits:

(a) We want to find the derivative of the function $h(x) = f(x)/g(x)$. Rearrange this equation by clearing the denominator.

(b) Now take the derivative of each side of the equation. What derivative rule(s) do you need to already know for this?

(c) Now solve for the desired derivative, $h'(x)$.

9. Time for some practice and some of my own mnemonics (they tend not to be "cute" ones) for derivative rules.

   (a) State the power rule and the constant multiple rule. These you should be able to recite in your sleep.

   (b) DO NOT EVER write something like $x^2 = 2x$ when you mean $\frac{d}{dx}x^2 = 2x$. (Okay, that wasn't a question.)

   (c) You should also just know that $\frac{d}{dx}e^x = e^x$. Now suppose we want to remember the derivative for a^x (which we tend to use less frequently). You should know that it involves a $\ln a$. If $a > e$, how does the slope of a^x compare to that of e^x? Now can you tell what the derivative of a^x ought to be?

   (d) Let's do that another way; this one is more rigorous. Remember that $a = e^{\ln a}$. Now use the same reasoning as in problem 7 to find the derivative of a^x.

   (e) Here are the derivatives for the trigonometrics:

   $$\begin{array}{cc} (\sin x)' = \cos x & (\cos x)' = -\sin x \\ (\tan x)' = \sec^2 x & (\cot x)' = -\csc^2 x \\ (\sec x)' = \sec x \tan x & (\csc x)' = -\csc x \cot x \end{array}$$

   On the chalkboard, make a list of patterns you see.

   I used to just memorize the derivatives for tan and sec, and to know which was which I thought to myself that it would be silly for the derivative of a function (sec) to be its own square ($\sec^2$). (Actually there are such functions; can you think of any?)

10. Consider the same functions from problem 2. Find an equation for the tangent line to the function $|f + g|$ at $x = 3$.

11. Same question, but now for the function $|5f + 2g|$.

12. Practice time! Find the derivatives of the following functions.

    (a) x^{200}

    (b) 2^{3t}

    (c) x^0

    (d) $(w^3 + w^{-1})(\sqrt{w} - 2)$

    (e) $\dfrac{u^4 - u^{3/2}}{2u}$

    (f) $\dfrac{x^2}{1 - x^3}$