Math 241, Sections BL1 and BL2

Quiz # 3 BDL

October 11, 2012

Solve both exercises. Show work to get credit.

1) [5pts.] Use Lagrange multipliers to find the point of the plane

$$x - y + z = 6$$

that is closest to the point $(2, 5, 4)$.

Solution: We wish to minimize the distance, and it suffices to minimize the square of the distance. That is, we want to minimize $f(x, y, z) = (x-2)^2 + (x-5)^2 + (x-4)^2$, subject to the constraint $g(x, y, z) = 6$, where $g(x, y, z) = x - y + z$. Hence, the method of Lagrange multipliers requires us to solve the system of equations

$$\begin{aligned} 2(x-2) &= \lambda(1) \\ 2(y-5) &= \lambda(-1) \\ 2(z-4) &= \lambda(1) \\ x - y + z &= 6 \end{aligned}$$

One way of solving this is to solve for x, y, z in terms of λ using the first three equations, then substitute into the last:

$$\left(2 + \frac{\lambda}{2}\right) - \left(5 - \frac{\lambda}{2}\right) + \left(4 + \frac{\lambda}{2}\right) = 6.$$

This gives $\lambda = 10/3$, which in turn gives $x = 11/3$, $y = 10/3$, $z = 17/3$. This is the only possibility for a minimum or maximum, and the situation of the problem implies that it must be a minimum.

2) [5pts.] Find the length of the curve

$$\vec{r}(t) = \Big\langle \cos(4t), \sin(4t), 4\ln(\cos t)\Big\rangle, \quad 0 \le t \le \pi/4.$$

Solution: The length is given by

$$\begin{aligned}
\int_0^{\pi/4} |\mathbf{r}'(t)|\, dt &= \int_0^{\pi/4} \sqrt{(-4\sin(4t))^2 + (4\cos(4t))^2 + (4\tan(t))^2}\, dt \\
&= \int_0^{\pi/4} \sqrt{16(\sin^2(4t) + \cos^2(4t)) + 16\tan^2(t)}\, dt \\
&= \int_0^{\pi/4} \sqrt{16(1 + \tan^2 t)}\, dt \\
&= \int_0^{\pi/4} \sqrt{16\sec^2 t}\, dt \\
&= \int_0^{\pi/4} 4\sec t\, dt.
\end{aligned}$$

Now evaluating this integral requires some obscure technique, so getting to this point is worth full credit. You may have an antiderivative already memorized, or you can use the following trick.

$$\begin{aligned}
&= 4\int_0^{\pi/4} \sec t\, \frac{\sec t + \tan t}{\sec t + \tan t}\, dt \\
&= 4\int_0^{\pi/4} \frac{\sec^2 t + \sec t\tan t}{\sec t + \tan t}\, dt \\
&= 4\int_1^{1+\sqrt{2}} \frac{du}{u} \\
&= 4\Big(\ln(1+\sqrt{2}) - \ln(1)\Big) \\
&= 4\ln(1+\sqrt{2}),
\end{aligned}$$

where we've used the substitution $u = \sec t + \tan t$. (There is also a method involving partial fraction decompositions.)