HOMEWORK 4: §2.2-2.5 DUE FEBRUARY 9
SOLUTIONS

(1) Find (with proof) all integers n such that $0 \mid n$.
The only such n is 0.

Theorem. *If n is an integer and $0 \mid n$, then $n = 0$.*

Proof. We use a direct proof. Suppose n is an integer and $0 \mid n$. That is, $n = 0 \cdot k$ for some integer k. But $0 \cdot k = 0$ regardless of the value of k, so $n = 0$. □

Theorem. $0 \mid n$.

Proof. $n = 0 \cdot 1$ and $1 \in \mathbb{Z}$, so $0 \mid n$. □

(2) Find (with proof) all integers n such that $n \mid 0$. Every integer n satisfies $n \mid 0$.

Theorem. *For every integer n, $n \mid 0$.*

Proof. We use a direct proof. Let n be an arbitrary integer. Then $0 = n \cdot 0$, and since $0 \in \mathbb{Z}$, we have $n \mid 0$. □

Prove the following statements.

(3) For all integers a, b, c: if $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof. Direct proof. Let a, b, c be integers such that $a \mid b$ and $b \mid c$. Then $b = ak$ and $c = b\ell$ for some integers k, ℓ. Substituting the former into the latter gives $c = (ak)\ell = a(k\ell)$. Since $k\ell \in \mathbb{Z}$, this means that $a \mid c$. □

(4) For all integers a, b, c: if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Proof. Direct proof. Let a, b, c be integers such that $a \mid b$ and $a \mid c$. Then $b = ak$ and $c = a\ell$ for some $k, \ell \in \mathbb{Z}$. Then $b + c = ak + a\ell = a(k + \ell)$. Since $k + \ell \in \mathbb{Z}$, this means that $a \mid (b + c)$. □

(5) For all integers a, b: if ab is odd, then a and b are both odd. *(Use the WLOG technique.)*

Proof. We prove the contrapositive: "If at least one of a, b is even, then ab is even." Let a, b be arbitrary integers such that at least one of a, b is even. WLOG, assume a is even (otherwise switch the roles of a, b). Then $a = 2k$ for some integer k. So $ab = 2kb$, and since $kb \in \mathbb{Z}$, we have that ab is even. □

(6) The square of any integer has the form $4k$ or $4k + 1$ for some integer k.

Proof. Rephrase the statement: "For any integer n, $n^2 = 4k$ or $n^2 = 4k + 1$ for some integer k." Direct proof, by cases. Suppose n is an arbitrary integer. Then n is either even or odd.
Case 1: n is even. Then $n = 2\ell$ for some integer ℓ, and so $n^2 = 4\ell^2$. Letting $k = \ell^2$ gives $n^2 = 4k$ and $k \in \mathbb{Z}$.
Case 2: n is odd. Then $n = 2\ell + 1$ for some integer ℓ, and so $n^2 = 4\ell^2 + 4\ell + 1$. Letting $k = \ell^2 + \ell$ gives $n^2 = 4k + 1$ and $k \in \mathbb{Z}$. □

(7) The sum of a rational number and an irrational number is irrational.

Proof. Rephrase the statement: "For every rational number r and irrational number x, $r + x$ is irrational." We prove this statement by contradiction. Suppose r is a rational number, x is irrational, but $r + x$ is rational. Then $x = (x + r) - r$. But $x + r$ is rational by assumption, and $-r$ is rational *[$r = a/b$, so $-r = (-a)/b$]*, and we have already proven that the sum of rational numbers is rational. This means x is rational, contradicting our original assumption that x was irrational. □

(8) Every composite number n has a factor d with $1 < d \leq \sqrt{n}$.

Proof. Direct proof. Let n be a composite number. By Workshop 6 Problem 4, $n = ab$ for some positive integers a, b such that both a and b are $< n$. WLOG, assume that $a \leq b$ (otherwise swap their names). We claim that $d = a$ satisfies the theorem. Suppose to the contrary: then either $a = 1$ or $a > \sqrt{n}$.
Case 1: $a = 1$. But since $b < n$, this gives $n = ab = 1 \cdot b = b < n$, a contradiction.
Case 2: $a > \sqrt{n}$. Since $b \geq a$, we have $n = ab \geq a^2 > (\sqrt{n})^2 = n$, a contradiction. $\square$

Challenge: is it possible to make exactly \$3 using exactly 50 coins that are all pennies, dimes, and quarters? (Some of the homework problems may help, but are not necessary. Too hard? Try a warmup: is it possible to make \$3.14 using any combination of nickels, dimes, and quarters?)

No, it is not possible. Suppose to the contrary that it is possible, and let p, d, q represent the number of pennies, dimes, and quarters used, respectively. Then the stated requirements are equivalent to the system of equations

$$p + 10d + 25q = 300$$
$$p + d + q = 50$$

Solving each for p and equating gives $300 - 10d - 25q = 50 - d - q$, and rearranging gives

$$250 = 9d + 24q. \tag{1}$$

But $9d = 3(3d)$ and $3d \in \mathbb{Z}$, so $3 \mid 9d$; and $24q = 3(8q)$ and $8q \in \mathbb{Z}$, so $3 \mid 24q$. By Problem 4, this means $3 \mid (9d + 24q)$. But $3 \nmid 250$. These two statements contradict Equation (1).