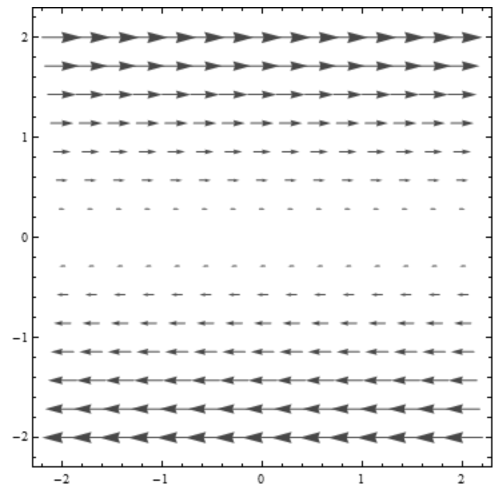
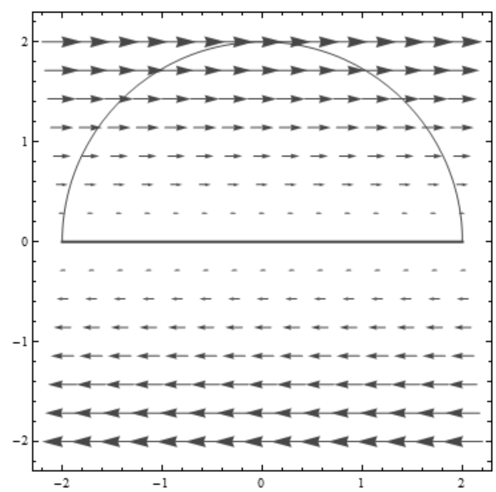
Solutions to the 10/11 Worksheet

1) (a)



(b) $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = 0$ because $\vec{F} = 0$ on C_1.



Parameterization for $-C_2$: $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$, $t \in [0,\pi]$, $dx = -2\sin t$, $dy = 2\cos t$.

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{-C_2} \vec{F}\cdot d\vec{r} = -\int_0^\pi \langle 2\sin t, 0\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt = 4\int_0^\pi \sin^2 t\, dt$$
$$= 4\int_0^\pi \frac{1-\cos(2t)}{2}\, dt = 4\left(\frac{\pi}{2} - \frac{\sin(2t)}{4}\Big|_{t=0}^{\pi}\right) = 2\pi.$$

(c) Based on (b) the vector field $\vec{F}$ cannot be conservative. This is because $\vec{F}$ conservative would imply according to the Fundamental Theorem of Line Integrals that $\displaystyle\int_{C_1\cup(-C_2)} \vec{F}\cdot d\vec{r} = 0$. On the other hand (b) gave the value -2 for this integral, hence $0 = 2$, contradiction. It remains that the field $\vec{F}$ is not conservative.

2) (a) A direction vector for C is given by $\vec{v} = \langle 1, 1\rangle - \langle 3, 2\rangle = \langle -2, -1\rangle$, hence the unit tangent along C is $\vec{T} = \frac{\langle -2,-1\rangle}{\sqrt{5}}$. From the picture $\vec{F} = \langle 1, 1\rangle$ (constant), so $\vec{F}\cdot\vec{T} = \frac{1}{\sqrt{5}}\langle 1, 1\rangle\cdot\langle -2, -1\rangle = -\frac{3}{\sqrt{5}}$, and consequently

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\, ds = \int_C \left(-\frac{3}{\sqrt{5}}\right) ds = -\frac{3}{\sqrt{5}}\,\text{length}(C) = -\frac{3}{\sqrt{5}}\sqrt{(3-1)^2+(2-1)^2} = -3.$$

(b) Parameterization of C: $\vec{r}(t) = \langle 3, 2\rangle + t\vec{v} = \langle 3 - 2t, 2 - t\rangle$, $t \in [0, 1]$, with $dx = -2$ and $dy = -1$. This gives

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 1, 1\rangle\cdot\langle -2, 1\rangle\, dt = -3.$$

3) (a) The unit tangent vector along $[A, B]$ is $\vec{T} = \frac{\langle \pi,-2\rangle}{\sqrt{\pi^2+4}}$, so $\vec{F}\cdot\vec{T} = \langle 0, -mg\rangle\cdot\frac{\langle \pi,-2\rangle}{\sqrt{\pi^2+4}} = \frac{2mg}{\sqrt{\pi^2+4}}$, and we find

$$W = \int_{[AB]} \vec{F}\cdot d\vec{r} = \int_{[AB]} \frac{2mg}{\sqrt{\pi^2+4}}\, ds = \frac{2mg}{\sqrt{\pi^2+4}}\,|AB| = 2mg.$$

(b) A parameterization for C is given by $\vec{r} = \langle t - \sin t, -(1 - \cos t)\rangle$, $t \in [0, \pi]$, and so

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^\pi \langle 0, -mg\rangle\cdot\langle 1 - \cos t, -\sin t\rangle\, dt = mg\int_0^\pi \sin t\, dt = 2mg.$$

(c) Such a function f must satisfy $f_x = 0$ and $f_y = -mg$. The identity $f_x = 0$ gives $f(x, y) = h(y)$, and so $h'(y) = -mg$ gives $f(x, y) = -mgy + C$. The quantity $-f$ represents the potential energy of the vector field $\vec{F}$.

4) $B = |\vec{B}|$ is constant along any circle that lies in a plane perpendicular to the wire and is centered on the wire. Let C be such a circle parameterized by $\vec{r}$, say $\vec{r}(t) = \langle r\cos t, r\sin t, z_0\rangle$, $t \in [0, 2\pi]$. Since $\vec{B}\big(\vec{r}(t)\big)$ is perpendicular on $\vec{r}(t)$ (by hypothesis) and points counterclockwise (by figure), it must be proportional to the unit tangent vector $\vec{T}(t)$ (because both share these properties and such a vector is uniquely determined up to a positive scalar multiple). Hence $\vec{B}\big(\vec{r}(t)\big) = K\vec{T}(t)$ for some constant $K > 0$. Since $|\vec{T}(t)| = 1$ and $B = |\vec{B}(\vec{r}(t))|$, it follows that $K = B$ and we have

$$\int_C \vec{B}\cdot d\vec{r} = \int_C \vec{B}\cdot\vec{T}\, ds = \int_C (B\vec{T})\cdot\vec{T}\, ds = \int_C B\, ds = B\,\text{length}(C) = 2\pi r B.$$

On the other hand Ampère's Law says that this integral is equal to $\mu_0 I$, leading to $B = \frac{\mu_0 I}{2\pi r}$.