Math 241 X8 **Name(s):** Solutions

Homework 12 supplement

This is a written homework supplement to the homework for Unit 13: 3D Flow Along.

(1) Consider the surface R that is the cone $z = \sqrt{x^2+y^2}$ with $z \le 3$. Let $\mathbf{F}(x,y,z) = \langle x^3y, xz, \sin z\rangle$.

(a) Compute $\iint_R \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ directly. Use a downward/outward normal.

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^3y & xz & \sin z \end{vmatrix} = \langle 0-x, -0+0, z-x^3\rangle.$$

Parametrize R: $x = r\cos t$, $y = r\sin t$, $z = r$, $r\in[0,3]$, $t\in[0,2\pi]$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 1 \\ -r\sin t & r\cos t & 0 \end{vmatrix} = \langle -r\cos t, -r\sin t, r\rangle \text{ is upward;}$$

use $\langle r\cos t, r\sin t, -r\rangle$ instead.

$$\int_0^{2\pi}\int_0^3 \langle -r\cos t, 0, r - r^3\cos t\rangle\cdot\langle r\cos t, r\sin t, -r\rangle\,dr\,dt$$

$$= \int_0^{2\pi}\int_0^3 \left(-r^2\underbrace{\cos^2 t}_{\frac{1}{2}(1+\cos(2t))} + 0 - r^2 + r^4\cos t\right) dr\,dt$$

$$= \quad -9\pi + 0 \quad - 18\pi \quad + 0 \quad = -27\pi.$$

(b) Verify your answer to (a) using Stokes's Theorem. Be sure to check that your orientations match.

$$= -\int_{\text{boundary of } R} \vec{F}\cdot\vec{dr}$$

downward normal induces orientation on C that is clockwise when viewed from above; my param. of C is opposite

Boundary of R is circle C at $z=3$, $x^2+y^2=9$

$x = 3\cos t$, $y = 3\sin t$, $z=3$, $t\in[0,2\pi]$

$\vec{dr} = \langle -3\sin t, 3\cos t, 0\rangle$

$$= -\int_0^{2\pi} \langle 81\cos^3 t\sin t, 9\cos t, \sin 3\rangle\cdot\langle -3\sin t, 3\cos t, 0\rangle\,dt$$

$$= -\int_0^{2\pi}\left(-243\cos^3 t\sin^2 t + 27\cos^2 t + 0\right)dt$$

$$= \int_0^{2\pi} 243(1-\sin^2 t)\sin^2 t\cos t\,dt - \int_0^{2\pi} 27\cdot\tfrac{1}{2}(1+\cos(2t))\,dt$$

$u = \sin t$

$du = \cos t\,dt$

$$= \int_0^0 \cdots\, du \qquad - \frac{27}{2}\cdot 2\pi - 0$$

$$= -27\pi.$$

(2) Compute the flow of **G** along C, where $\mathbf{G}(x, y, z) = \langle x^2y, \frac{1}{3}x^3, xy\rangle$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 4$. Which direction is it?

$$\int_C G \cdot d\vec{r} = \iint_R \text{curl}\,\vec{G} \cdot d\vec{S}$$

where R is the piece of the hyperbolic paraboloid inside the cylinder.

Parametrize R: $x = u$, $y = v$, $z = v^2 - u^2$, $u^2 + v^2 \le 4$

$$d\vec{S} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & 2v \end{vmatrix} = \langle 2u, -2v, 1\rangle$$

$$\text{curl}\,\vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & \frac{1}{3}x^3 & xy \end{vmatrix} = \langle x - 0, -y + 0, x^2 - x^2\rangle$$

$$= \langle x, -y, 0\rangle$$

$$= \langle u, -v, 0\rangle$$

$$\iint_R \text{curl}\,\vec{G} \cdot d\vec{S} = \iint_{u^2+v^2\le 4} \langle u, -v, 0\rangle \cdot \langle 2u, -2v, 1\rangle\, du\, dv$$

$$= \iint_{u^2+v^2\le 4} (2u^2 + 2v^2)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^2 2r^2 \cdot r\, dr\, d\theta$$

$$= 2\pi \cdot \frac{1}{2} r^4 \Big]_0^2$$

$$= \boxed{16\pi}.$$

Since $d\vec{S}$ is upward, C must be parametrized counterclockwise when viewed from above.