Worksheet 19 April 4, 2011

1. Let's reason through the Fundamental Theorem of Calculus via linear motion, say of a squirrel on a fence. (If you don't like linear motion, you may be able to think the same way about other functions with interesting derivatives, say populations and population growth.)
   (a) Suppose you have some fixed time interval of interest, say $t \in [a, b]$. Consider the velocity and position functions to find two different expressions for the total displacement of the squirrel during this time. (Pay particularly close attention to how the sign of the velocity affects this calculation.) Thus conclude that the second part of the FTC works for this motion.
   (b) Suppose you have a fixed starting time $t = a$, but want to consider different ending times, say $t = x$ with x variable. Write an expression for the displacement of the squirrel as a function of x. What is the rate of change of this function? Thus conclude that the first part of the FTC works for this motion.
   (c) What changes if you fixed an ending time and let the beginning time vary as x instead? Compute the rate of change as before; what changed? Also show that your formula works using properties of the integral together with the previous question.
   (d) You should have used the position function as an antiderivative of velocity; but remember that there are lots of antiderivatives. The FTC uses the phrase "any antiderivative". Show that, if you used a different antiderivative of velocity, then the answer is unchanged.
   (e) The second part of the FTC is the one most often used. Use it to prove the first part. (So remembering the second part is very important; the first part is nice to remember, but it can be derived from the second.)
2. On the last quiz, you (should have) overestimated the area under $y = x^2 + 1$ between $x = 0$ and $x = 1$ as $47/32$. Now compute the actual area and compare.
3. Compute the area under one hump of the graph of $\sin x$ as an integral. (Remark: from here on out, if we ask you for an area it's probably best to first try a geometric argument, then to try the FTC. Only use Riemann sums if specifically asked.)
4. Compute the integral $\displaystyle\int_0^4 x^3\,dx$ as a limit of Riemann sums. You'll need to use the identity $\displaystyle\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.
5. Now compute the integral $\displaystyle\int_0^4 x^3\,dx$ using the Fundamental Theorem of Calculus.
6. What's wrong with the calculation $\displaystyle\int_{-1}^{e} \frac{1}{x}\,dx = [\ln|x|]_{-1}^{e} = \ln|e| - \ln|-1| = 1$?

7. Use properties of the integral together with the FTC to evaluate the derivative with respect to x of

   (a) $\displaystyle\int_1^x e^{t^4}\,dt$

   (b) $\displaystyle\int_{-4}^{x^2} e^{t^4}\,dt$

   (c) $\displaystyle\int_x^{-2} e^{t^4}\,dt$

   (d) $\displaystyle\int_{x^2}^{x^3} e^{t^4}\,dt$

   (Hint: there isn't a nice way to write down a formula for an antiderivative of e^{t^4}, so don't try.)

8. Define $A(x)$ to be the area beneath the graph of $f(t) = \lfloor t \rfloor$ between $t = 0$ and $t = x$. Find an explicit formula for $A(x)$ (again just assume $x \geq 0$). Your formula will need to be piecewise, and it's reasonable to only write it out for $x \leq 6$ or so. Where is this function continuous? Where is it differentiable, and what is $A'(x)$ where it's defined? Notice in particular that $A'(x)$ is *not* just $f(x)$; why doesn't this contradict the FTC?

9. Not every function is integrable. Suppose we want to compute $\displaystyle\int_0^1 \chi(x)\,dx$, where

$$\chi(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1 & \text{if } x \text{ rational.} \end{cases}$$

   By choosing appropriate sample points x_i^*, show that the Riemann sums can always be made to be 0, but can also be made to be 1.