Worksheet 24 April 20, 2011

1. Suppose the average velocity of your car during a trip is 40 mph. Prove that it must be true that at some point you were traveling at exactly 40 mph. What assumptions about your position and/or velocity function are you making?

2. Show that the equation $\cos t = 2t$ has exactly one solution.

3. Approximate the value of $\sqrt{6}$ in two ways: use linear approximations of $\sqrt{x}$, one based at 4, the other based at 9. Are these over- or under-estimates? (Hint: consider the graph.)

4. Can a function have a linearization at a point where it is not differentiable? Why or why not? You should consider functions such as $|x|$ and $x^{2/3}$.

5. You should have read about Newton's method for finding zeros of a function. Use the method with the function $x^4 - 3x^2 + 1$ with starting point $x = 1$. You should iterate the method twice. (If you are handy with a programmable calculator, consider writing a quick program to iterate Newton's method several times. By the way, you're approximating the value of the (conjugate) golden ratio here.)

6. Approximate $\sqrt{80}$ using the linearization of an appropriate function at an appropriate point.

7. What is an easy-to-differentiate function that is zero at $\sqrt{80}$? Use Newton's method on this function to approximate $\sqrt{80}$. Use only one iteration.

8. Explain why the result from the previous two problems isn't really that surprising. More generally, how is the process of approximating a function's value by a linear approximation related to the process of approximating a root of a function with Newton's method?

9. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal areas.

10. Calculus is a fundamental part of probability when dealing with situations that have a continuum (or really just infinitely many) possible outcomes. For instance, age or height of people, lifetime of a machine, or time between passing cars all have infinitely many possible outcomes. We compute probabilities of these events by integrating a *probability density function*.

    (a) For example, suppose the amount of time you spend in a line at the DMV has a density function $f(t) = 4t - 4t^3$, where t is measured in hours and $0 \leq t \leq 1$ (for other times the probability is zero). To find the probability that you wait between 1/4 hour and 1/2 hour, you need to calculate $\int_{1/4}^{1/2} f(t)\,dt$. Do so.

    (b) What should be the probability that you wait at least 0 minutes? Compute the probability that you wait between 0 and 1 hour to verify that this is true for the given density function. (Remember that we're assuming it's impossible to wait more than an hour; perhaps you just give up and leave after an hour.)

    (c) Calculus also gives us a way to compute the *expected* waiting time. To do this we just compute $\int t \cdot f(t)\,dt$. Do so, integrating from 0 to 1.