Worksheet 7 February 14, 2011

1. Compute the following limits

   (a) $\displaystyle\lim_{x\to\infty} \frac{5x^2 + 100x - 2}{4x^2 - 10000x + 5}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{5x^2 - 4}{\sqrt{9x^4 - x} + x^2 + 1}$

   (c) $\displaystyle\lim_{x\to\infty} \left(x^2 - \sqrt{5x^4 - 2}\right)$

2. The following data was actually collected about a population of yeast:

Time (hr)	0	6	12	18	24	30	36	42	48	72
Volume	0.1	0.2	1.3	3.1	5.1	6.4	8.4	8.3	8.2	8

   Plot this data. Comment on the shape that you see (especially compared to previous worksheets' imaginary data). Can you say anything about limits? Interpret these statements in terms of the yeast population. (Remark: this shape of graph is commonly found for populations.)

3. Compute $f'(x)$ if $f(x) = x^3$. (Using the definition.)

4. Compute $g'(x)$ if $g(x) = 1/\sqrt{x}$.

5. Suppose you have two functions $f(x)$ and $g(x)$. Then we have the new functions $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, $(f/g)(x)$, and $(f \circ g)(x)$. It would be nice if the derivatives of these functions were related to the derivatives of f and g...

   (a) Compute the derivative $(f+g)'$ in terms of f' and g'. (Set up the limit in the definition of $(f+g)'$, then manipulate the expression into a form that includes the limits for f' and g'. Remember that the limit of the sum is the sum of limits.)

   (b) Try to do the same thing with $(f \cdot g)'$. (Don't work too hard yet.)

   (c) What happens if $f(x) = g(x) = x$? That is, what are f', g', and $(f \cdot g)'$? (Remember this example! I will be very upset if you ever say the derivative of a product is the product of derivatives!)

   (d) Let's try to give a proof for the *product rule* of differentiation:

   $$(f \cdot g)' = f \cdot g' + f' \cdot g$$

   Start with the limit definition for the left hand side as before. But now subtract and add the quantity $f(x+h) \cdot g(x)$ to the numerator. Group some terms in the numerator, factor, and continue to prove the product rule.

   (e) By now you've calculated the derivatives for x, x^2, and x^3. Can you guess what the derivative of x^n is? Use the product rule to show that your guess is correct for $n = 4, 5, 6$. (For $n = 4$, write $x^4 = x \cdot x^3$ or $x^2 \cdot x^2$, then apply the rule.)

6. Which of the following are true for a function f at a point a? (You can use some common rules of logic here, or just think about some functions for (counter-)examples).
   - Continuous implies differentiable.
   - Differentiable implies continuous.
   - Not continuous implies not differentiable.
   - Not differentiable implies not continuous.
7. Find the equation of the tangent line to the curve $y = x^2$ at $x = 3$. Draw this curve and the tangent line together.
8. Sketch a graph of the following functions; based on these graphs, sketch the graphs of their derivatives.

   x^3 $\qquad$ $\sin x$ $\qquad$ $\arctan x$ $\qquad$ $\ln x$ $\qquad$ $\sqrt{|\sin 100|}$ $\qquad$ $|x|$
9. Consider the function

   $$f(x) = \begin{cases} 5x+3 & \text{if } x < 1 \\ 9 - x^2 & \text{if } x \geq 1 \end{cases}$$

   (a) What is the domain of f?

   (b) Where is f continuous?

   (c) We'll now decide whether f is differentiable at $x = 0$. You need to know if the limit of the difference quotient exists; compute the left and right limits separately, by hand.
10. Suppose $f(x)$ is even; is $f'(x)$ necessarily even or odd?
11. Suppose $f(x)$ is odd; is $f'(x)$ necessarily even or odd?
12. Consider the functions

    $$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \qquad g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

    Do $\lim_{x\to 0} f(x)$, $\lim_{x\to 0} g(x)$ exist?