Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have one hundred twenty (120) minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.
- There are 10 pages of questions in this exam. That gives you an average of 12 minutes per page. Some pages will take longer than others.
- Some formulas and the tables of common logical equivalences and valid arguments appear on the back page of the exam.

Question	Points	Score
1	12	
2	8	
3	6	
4	6	
5	21	
6	8	
7	5	
8	16	
9	6	
10	4	
11	16	
12	8	
13	16	
14	10	
15	12	
16	26	
Total:	180	

Short answer. No justifications are necessary on this page.

1. (3 points each)
   (a) Find 92 div 13.
   (b) Find 92 mod 13.
   (c) Find (-19) div 4.
   (d) Find (-19) mod 4.

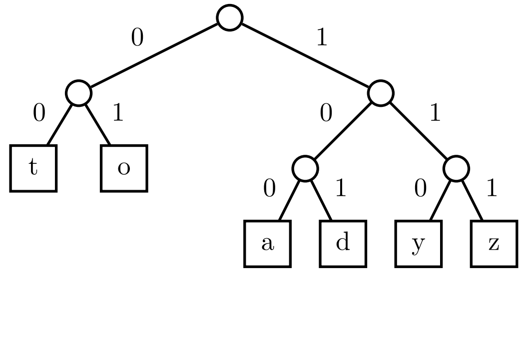
2. (a) (4 points) Find the gcd of $2^7 \cdot 3 \cdot 11^4$ and $2^4 \cdot 5^2 \cdot 11^5 \cdot 13$.

   (b) (4 points) Find the gcd of 2,817,636 and 2,817,632.

3. (6 points) Translate the following into formal logic, using predicates $F(x, y)$ = "x is a friend of y," $B(x)$ = "x likes bananas." The domain is the set of all students.

   "Every student who does not like bananas has a friend who likes bananas."

4. (6 points) Decode the message 11101010001101100110 using the prefix tree below.



5. (3 points each) Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. No justification is necessary, but leave answers unsimplified.

   (a) How many relations are there on X?

   (b) How many reflexive relations are there on X?

   (c) How many functions are there from X to X?

   (d) How many surjective functions are there from X to X?

   (e) How many subsets of size 3 does X have?

   (f) How many subsets of X have at least 2 elements?

   (g) How many subsets of X consist of either 3, 4, or 5 elements, whose product is even?

6. (8 points) Let $f(n) = 3n^4 - n^2 + 7n - 5$. Prove that $f(n) = \Omega(n^4)$.

7. (5 points) Prove that $p \wedge (q \to r) \equiv (p \wedge \neg q) \vee (p \wedge r)$. (Use whatever method you prefer, but fully justify your answer.)

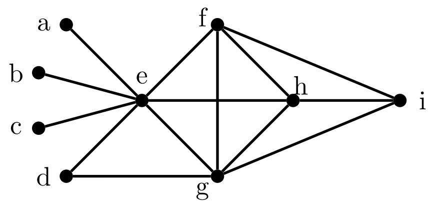
8. (4 points each) For each part, either draw a simple graph with the properties or prove that none exists.
   (a) vertex degrees 3,3,3,3,3,2

   (b) tree, vertex degrees 6,2,2,2,1,1,1,1

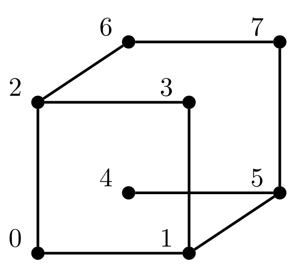
   (c) planar, every vertex has degree 4

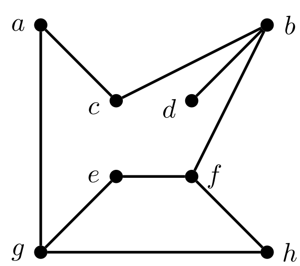
   (d) planar, every vertex has degree 6

9. (6 points) Let G be the graph below. Find, with proof, its chromatic number, $\chi(G)$.



10. (4 points) Are the below graphs isomorphic? If so, provide an isomorphism; otherwise, prove why not.





11. **Theorem.** *For any $x, y \in \mathbb{R}$, if $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $3x - 2y \notin \mathbb{Q}$.*

For each proof method below, state what you assume and what you need to show.

(a) (4 points) Direct

Assume: Our goal is:

(b) (4 points) Contrapositive

Assume: Our goal is:

(c) (4 points) Contradiction

Assume: Our goal is:

Finally,

(d) (4 points) Prove the theorem using the method of your choice. Use only the definition of (no theorems about) $\mathbb{Q}$.

12. (8 points) Let A, B be subsets of some universal set U. Prove that if $A \cap B = \varnothing$, then $A \subseteq \overline{B}$. *(A direct proof works.)*

13. Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+$ be defined by $f((x, y)) = x/y$. (Most of the time, we would drop the extra parentheses, writing just $f(x, y)$ instead of $f((x, y))$.)

    (a) (8 points) Prove that f is **not** injective.

    (b) (8 points) Prove that f is surjective.

14. (10 points) Prove (formally, using induction) the correctness of the following recursive algorithm. *(That is, prove that for every input n, the output really is n^2.)*

```
Algorithm: square(n)
Input:  n, a nonnegative integer
Output: n^2

If n = 0, then Return( 0 )

m := n − 1
x := square(m)
y := x + m + m + 1
Return( y )
```

15. **Theorem.** *If $(P, \preceq)$ is a finite poset with a unique maximal element x, then x is a maximum element.*

 (a) (5 points) How does a proof by contradiction of the above theorem start?

 (b) (7 points) Complete the proof. You may use the fact that every nonempty finite poset has a maximal element, and that restricting $\preceq$ to a subset of P yields a partial order. *(Consider the set of elements incomparable to x.)*

No justification is necessary on this page, and there is no partial credit.

16. (1 point each) Circle "True" or "False" as appropriate.

(a) True False 1 is a prime number.

(b) True False "$x^2 \geq 0$" is a proposition.

(c) True False "Pass me the ketchup" is a proposition.

(d) True False "You passed me the mayo!" is a proposition.

(e) True False The converse of any proposition is logically equivalent to the proposition.

(f) True False For every three sets A, B, C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) True False For every three sets A, B, C, we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(h) True False For every three sets A, B, C, we have $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(i) True False For every two sets B, C, we have $\mathcal{P}(B \cup C) = \mathcal{P}(B) \cup \mathcal{P}(C)$.

(j) True False For every two sets B, C, we have $\mathcal{P}(B \cap C) = \mathcal{P}(B) \cap \mathcal{P}(C)$.

(k) True False For every three sets A, B, C, we have $|A \cup B \cup C| = |A| + |B| + |C|$.

(l) True False $P(3) \wedge (\forall k \geq 5\ (P(k) \rightarrow P(k+1))) \rightarrow \forall n \geq 7\ P(n)$ is a valid variation of induction (all variables have domain $\mathbb{N}$).

(m) True False For $x, y \in \mathbb{N}$, say $x\,R\,y$ if the last digit of x is the same as the last digit of y. This R is an equivalence relation.

(n) True False $n \log n + n^{1.5} = O(n^2)$.

For the next 5 parts, the domain for all variables is $\mathbb{R}$.

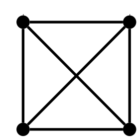
(o) True False $\forall x\ \exists y\ x^2 = y$

(p) True False $\forall x\ \exists y\ x = y^2$

(q) True False $\forall x\ (x > 1 \rightarrow \exists y\ x = y^2)$

(r) True False $\forall x\ (x \in \mathbb{Q}^+ \rightarrow \exists y\ (x = y^2 \wedge y \in \mathbb{Q}^+))$

(s) True False $\exists y\ \forall x\ x^2 = y$



(t) True False The graph at right is a planar graph.

(u) True False Treating the two pictures below as graphs, the graphs are equal.

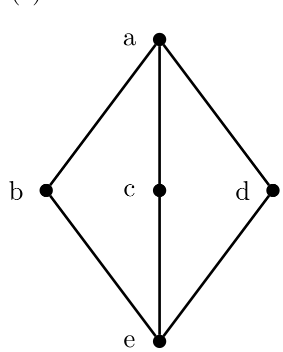
(v) True False Treating the two pictures below as Hasse diagrams, the posets are equal.

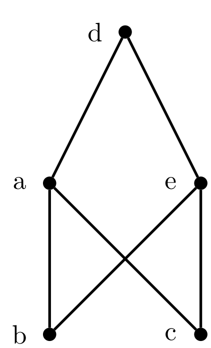
(w) True False Treating the first picture below as a graph, the degree of a is 4.

(x) True False Treating the first picture below as a Hasse diagram, a is a maximal element.

(y) True False Treating the first picture below as a Hasse diagram, a is a maximum element.

(z) True False Treating the first picture below as a Hasse diagram, $e \preceq a$.





Scratch Paper - Do Not Remove.

Logical equivalence(s)	Name
$p \vee \mathrm{F} \equiv p$ $p \wedge \mathrm{T} \equiv p$	Identity
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotence
$p \vee \mathrm{T} \equiv \mathrm{T}$ $p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination
$\neg\neg p \equiv p$	Double-negative
$p \vee \neg p \equiv \mathrm{T}$ $p \wedge \neg p \equiv \mathrm{F}$ $\neg \mathrm{T} \equiv \mathrm{F}$ $\neg \mathrm{F} \equiv \mathrm{T}$	Complement
$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associativity
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutativity
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributivity
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	DeMorgan
$p \rightarrow q \equiv \neg p \vee q$ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Conditional Laws

Argument form	Argument Name
$\begin{array}{ll} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$	Modus Ponens
$\begin{array}{ll} & p \rightarrow q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$	Modus Tollens
$\begin{array}{ll} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$	Disjunctive Syllogism
$\begin{array}{ll} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$	Hypothetical Syllogism
$\begin{array}{ll} & p \wedge q \\ \hline \therefore & p \end{array}$	Simplification
$\begin{array}{ll} & p \\ \hline \therefore & p \vee q \end{array}$	Addition
$\begin{array}{ll} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$	Conjunction
$\begin{array}{ll} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$	Resolution

Argument form	Argument Name
$\begin{array}{ll} & c \text{ is an element (arbitrary or particular)} \\ & \forall x\ P(x) \\ \hline \therefore & P(c) \end{array}$	Universal Instantiation
$\begin{array}{ll} & c \text{ is an arbitrary element} \\ & P(c) \\ \hline \therefore & \forall x\ P(x) \end{array}$	Universal Generalization
$\begin{array}{ll} & \exists x\ P(x) \\ \hline \therefore & (c \text{ is a particular element}) \wedge P(c) \end{array}$	Existential Instantiation
$\begin{array}{ll} & c \text{ is an element (arbitrary or particular)} \\ & P(c) \\ \hline \therefore & \exists x\ P(x) \end{array}$	Existential Generalization

Other Formulas and scratch space. You may remove this if you find it convenient.

For every graph, $\sum_{v \in V} \deg(v) = 2|E|$.

For every planar embedding of a connected graph, $n - m + r = 2$.

For every simple planar graph, $m \leq 3n - 6$.