CHAPTER 2 SECTION 2.4

Solving Linear Inequalities

Inequality Symbols

Here are frequently used inequality symbols and their meanings:

$<$ "is less than"
$\leq$ "is less than or equal to"
$>$ "is greater than"
$\geq$ "is greater than or equal to."

These symbols define the *sense* or *order* of an inequality. Some examples of how we use these symbols would be:

1. If we want to state symbolically that 4 is less than 7, we write $4 < 7$.
2. If we wish to denote that the variable x represents 5 or any number greater than 5, we write $x \geq 5$.

Note $x \geq 5$ represents *any* real number that is greater than or equal to 5, and not just any integer greater than or equal to 5. Remember that 5.1, 5.004, and so on, are all greater than 5.

3. If we wish to denote that the variable T represents any number less than 3, but not 3 itself, we write $T < 3$.

Linear inequalities

When we replace the equal sign in a conditional linear equation with one of these inequality symbols, we form a *conditional linear inequality.*

$$2x \geq 6$$

Left member — $2x$
Inequality symbol — $\geq$
Right member — 6

A major difference between the linear equation and the linear inequality is the solution. The solution of a linear equation has at most one solution, whereas the solution of a linear inequality may consist of an unlimited number of solutions. Consider the inequality $2x \geq 6$. We can, by inspection, see that if we substitute 3, $3\frac{1}{2}$, 4, or 5 for x, the inequality will be true. In fact, we see that if we were to substitute any number greater than or equal to 3, the inequality would be true. This demonstrates the fact that the inequality has an unlimited number of solutions. The values for x that would satisfy the inequality would be $x \geq 3$.

Another way to indicate the solution of an inequality is by graphing. To graph the solution, we simply draw a number line (as we did in chapter 1), place a solid circle at 3 on the number line to signify that 3 is in the solution, and draw an arrow extending from the solid circle to the right (figure 2–1). The solid line indicates that *all* numbers greater than or equal to 3 are part of the graph.

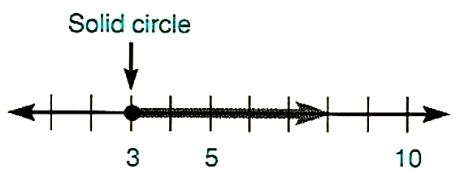
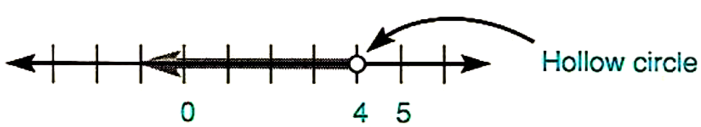


Figure 2–1

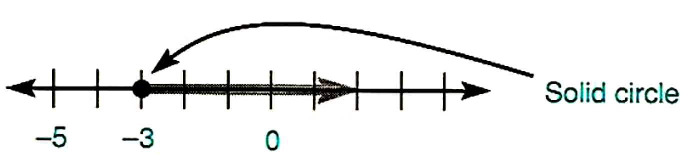
Examples

Graph the following linear inequalities.

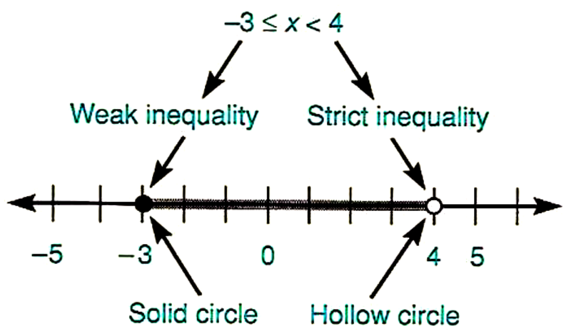
1. $x < 4$ Here x represents all real numbers less than 4, but not 4 itself. To denote the fact that x cannot equal 4, we put a **hollow circle** at 4.



2. $x \geq -3$ The greater than or equal to symbol, $\geq$, indicates that the graph will contain the point -3, and we place a **solid circle** at -3.



3. $-3 \leq x < 4$ This statement is called a *compound inequality*. It is read "-3 is less than or equal to x and x is less than 4." We place a solid circle at -3 to denote that -3 is included and place a hollow circle at 4 to show that 4 is not included. We then draw a line segment between the two circles.



Note When we graph inequalities, a strict inequality ($<$ or $>$) is represented by a hollow circle at the number. A weak inequality ($\leq$ or $\geq$) is represented by a solid circle at the number.

Solving linear inequalities

The properties that we will be using to solve linear inequalities are similar to those that we used to solve linear equations.

Addition and subtraction property of inequalities

For all real numbers a, b, and c, if $a < b$, then

$$a + c < b + c \text{ and } a - c < b - c$$

Concept

The same number can be added to or subtracted from both members of an inequality without changing the direction of the inequality symbol.

Multiplication and division property of inequalities

For all real numbers *a*, *b*, and *c*, if $a < b$, and

1. If $c > 0$ (*c* represents a positive number), then

$$a \cdot c < b \cdot c \text{ and } \frac{a}{c} < \frac{b}{c}$$

Concept

We can multiply or divide *both* members of the inequality by the same positive number without changing the direction of the inequality symbol.

2. If $c < 0$ (*c* represents a negative number), then

$$a \cdot c > b \cdot c \text{ and } \frac{a}{c} > \frac{b}{c}$$

Concept

We can multiply or divide *both* members of an inequality by the same *negative* number, provided that we **reverse** the direction of the inequality symbol.

Note The two properties were stated in terms of the less than ($<$) symbol. The properties apply for any of the other inequality symbols ($>$, $\leq$, or $\geq$).

To demonstrate these operations, consider the inequality $8 < 12$.

1. If we add or subtract 4 in each member, we still have a true statement.

$8 < 12$	or	$8 < 12$	Original true statement
$8 + 4 < 12 + 4$		$8 - 4 < 12 - 4$	Add or subtract 4
$12 < 16$		$4 < 8$	New true statement

2. If we multiply or divide by 4 in each member, we still have a true statement.

$8 < 12$	or	$8 < 12$	Original true statement
$8 \cdot 4 < 12 \cdot 4$		$\frac{8}{4} < \frac{12}{4}$	Multiply or divide by 4
$32 < 48$		$2 < 3$	New true statement

3. **But if we multiply or divide by -4 in each member, we have to reverse the direction of the inequality before we have a true statement.**

$8 < 12$ or	$8 < 12$	Orignial true statement
$8(-4) > 12(-4)$	$\dfrac{8}{-4} > \dfrac{12}{-4}$	Multiply or divide by -4 and reverse direction of the inequality symbol
$-32 > -48$	$-2 > -3$	New true statement

Note When we reverse the direction of the inequality symbol, we say that we **reversed the sense or order** of the inequality.

To summarize our properties, we see that they are the same as the properties for linear equations, with one exception. **Whenever we multiply or divide both members of an inequality by a negative number, we must reverse the direction of the inequality symbol.**

We shall now solve some linear inequalities. The procedure for solving a linear inequality uses the same four steps that we used to solve a linear equation.

Solving a linear inequality

1. Simplify in each member, where necessary, by performing the indicated operations.
2. Add, or subtract, to get all terms containing the unknown in one member of the inequality.
3. Add, or subtract, to get all terms *not* containing the unknown in the other member of the inequality.
4. Multiply, or divide, to obtain a coefficient of 1 for the unknown. *Remember, when multiplying or dividing by a negative number, always change the direction (order) of the inequality symbol.*

Examples

Find the solution.

1. $2x + 5x - 1 < 4x + 2$

Step 1 *We simplify* the inequality by carrying out the indicated addition in the left member.

$$2x + 5x - 1 < 4x + 2$$
$$7x - 1 < 4x + 2$$

Step 2 *We want all the terms containing the unknown,* x, *in one member of the inequality.* Therefore we subtract $4x$ from both members of the inequality.

$$7x - 1 < 4x + 2$$
$$7x - 4x - 1 < 4x - 4x + 2$$
$$3x - 1 < 2$$

Note A negative coefficient of the unknown can be avoided if we form equivalent inequalities where the unknown appears only in the member of the inequality that has the greater coefficient of the unknown.

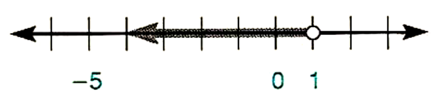
Step 3 *We want all the terms not involving the unknown in the other member of the inequality.* Therefore we add 1 to both members of the inequality.

$$3x - 1 < 2$$
$$3x - 1 + 1 < 2 + 1$$
$$3x < 3$$

Step 4 *We form an equivalent inequality where the coefficient of the unknown is 1.* Hence we divide both members of the inequality by 3.

$$3x < 3$$
$$\frac{3x}{3} < \frac{3}{3}$$
$$x < 1$$

We can also graph the solution.



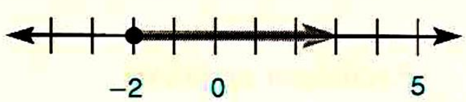
Note We should be careful to observe in step 4 whether we are multiplying or dividing by a positive or negative number so that we will form the correct inequality.

2. $-2x \leq 4$

The only operation we need to perform to solve the inequality is to divide by -2. Since we are dividing by a negative number, we must remember to **reverse** the direction of the inequality symbol.

$$-2x \leq 4$$
$$\frac{-2x}{-2} \geq \frac{4}{-2} \qquad \text{Reverse the direction of the inequality symbol}$$
$$x \geq -2$$

Graph



3.

$$5(2x + 1) \leq 7x - 4x + 3$$

$$10x + 5 \leq 3x + 3$$

Simplify by multiplying in left member and combining like terms in right member

$$10x - 3x + 5 \leq 3x - 3x + 3$$

Subtract $3x$ from both members

$$7x + 5 \leq 3$$

$$7x + 5 - 5 \leq 3 - 5$$

Subtract 5 from both members

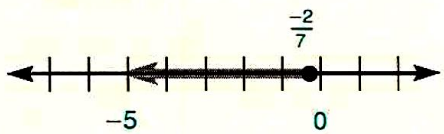
$$7x \leq -2$$

$$\frac{7x}{7} \leq \frac{-2}{7}$$

Divide both members by 7

$$x \leq \frac{-2}{7}$$

Graph



4. $-3 \leq 2x + 1 < 5$

When solving a compound inequality, the solution must be such that the unknown appears only in the middle member of the inequality. We can still use all of our properties, if we apply them to all three members, and we must reverse the direction of *all* inequality symbols when multiplying or dividing by a negative number.

$$-3 \leq 2x + 1 < 5$$

$$-3 - 1 \leq 2x + 1 - 1 < 5 - 1$$

Subtract 1 from all three members

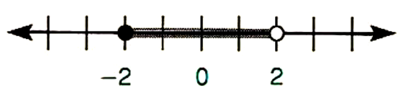
$$-4 \leq 2x < 4$$

$$\frac{-4}{2} \leq \frac{2x}{2} < \frac{4}{2}$$

Divide all three members by 2

$$-2 \leq x < 2$$

Graph



5.

$$-6 < -3x - 3 \leq 9$$

$$-6 + 3 < -3x - 3 + 3 \leq 9 + 3$$

Add 3 to all three members

$$-3 < -3x \leq 12$$

$$\frac{-3}{-3} > \frac{-3x}{-3} \geq \frac{12}{-3}$$

Divide all three members by -3, reversing the direction of *both* inequality symbols

$$1 > x \geq -4$$

Graph

Note From our discussions in chapter 1, we could have written the solution in the previous problem as $-4 \leq x < 1$. This is usually the preferred form.

Problem solving

We are now ready to combine our abilities to write an expression and to solve an inequality and apply them to solve word problems. The guidelines for solving a linear inequality are the same as those for solving a linear equation in section 2-1. The following table shows a number of different ways that an inequality symbol could be written with words.

Symbol	$<$	$\leq$	$>$	$\geq$
In words	is less than is fewer than is almost	is at most is no more than is no greater than is less than or equal to	is greater than is more than exceeds	is at least is no less than is no fewer than is greater than or equal to

Examples

1. Write an inequality for the following statement: A student's test grade, G, must be at least 75 to have a passing grade.

If the student's test grade must be *at least* 75, the grade must be 75 or greater. Thus,

$G \geq 75$

2. Four times a number less 5 is to be no more than three times the number increased by 2. Find the number.

Let x represent the number.

4 times a number	less	5	is no more than	3 times the number	increased by	2
$4x$	$-$	5	$\leq$	$3x$	$+$	2

The inequality is $4x - 5 \leq 3x + 2$

$$
\begin{aligned}
4x - 5 &\leq 3x + 2 \\
4x - 5 - 3x &\leq 3x + 2 - 3x && \text{Subtract } 3x \text{ from each member} \\
x - 5 &\leq 2 && \text{Combine in each member} \\
x - 5 + 5 &\leq 2 + 5 && \text{Add 5 to each member} \\
x &\leq 7 && \text{Combine in each member}
\end{aligned}
$$

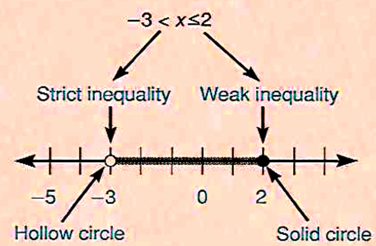
The number is any real number x such that $x \leq 7$.

Exercises

Graph the following. See example

Example $-3 < x \leq 2$

Solution



1. $x > 2$ **2.** $x > -2$ **3.** $x \geq 1$ **4.** $x \geq 4$ **5.** $x < 0$

6. $x < -2$ **7.** $x > 0$ **8.** $x \leq 3$ **9.** $x \leq -4$ **10.** $-2 < x < 0$

11. $-1 < x < 2$ **12.** $-3 \leq x \leq 4$ **13.** $0 \leq x \leq 5$ **14.** $1 \leq x < 4$ **15.** $-1 < x \leq 3$

Find the solution and graph the solution. See example

Example $4x + 5x - 4 < 6x - 1$

Solution

$9x - 4 < 6x - 1$	Combine like terms in left member
$9x - 4 - 6x < 6x - 1 - 6x$	Subtract $6x$ from each member
$3x - 4 < -1$	Combine like terms
$3x - 4 + 4 < -1 + 4$	Add 4 to each member
$3x < 3$	Combine like terms
$x < 1$	Divide each member by 3

Thus $x < 1$ and the solutions are all real numbers less than 1.

16. $4x > 10$ **17.** $2x \leq 5$ **18.** $3x \geq 15$

19. $5x < 30$ **20.** $\frac{3}{4}x < 9$ **21.** $\frac{2}{3}x \geq 12$

$\boxed{22.}$ $-4x < 12$ **23.** $-3x \leq 27$ **24.** $-6x > 18$

25. $-2x < 10$ **26.** $4x + 3x \geq 2x + 7$ **27.** $8x - 2x > 4x - 5$

28. $3x + 2x < x + 6$ **29.** $\frac{3x}{2} > 8$ **30.** $\frac{4x}{3} > 12$

31. $x + 7 \le 9$

32. $x + 4 \ge -12$

33. $x - 5 < 6$

34. $x - 12 < -9$

35. $2x + (3x - 1) > 5 - x$

36. $2(3x + 1) < 7$

37. $3(2x - 1) \ge 4x + 3$

38. $4(5x - 3) \le 25x + 11$

39. $12x - 8 > 5x + 2$

40. $9x + 4 > x - 11$

41. $2 + 5x - 16 < 6x - 4$

42. $3(2x + 5) > 4(x - 3)$

43. $8 - 2(3x + 4) > 5x - 16$

44. $(3x + 2) - (2x - 5) > 7$

45. $(7x - 6) - (4 - 3x) \le 27$

46. $2(x - 4) - 16 \le 3(5 - 2x)$

47. $3(1 - 2x) \ge 2(4 - 4x)$

48. $4(5 - x) > 7(2 - x)$

49. $3(2x + 3) \ge 5 - 4(x - 2)$

50. $6(3x - 2) \le 7(x - 3) - 2$

51. $-1 < 2x + 3 < 4$

52. $-3 < 3x - 4 < 6$

53. $-2 \le 5x + 2 \le 3$

54. $0 \le 7x - 1 \le 7$

55. $-5 < 4x + 3 \le 8$

56. $-2 < -x \le 3$

57. $-1 \le -x < 4$

58. $-4 \le 2 - x < 3$

59. $-3 < 4 - x \le 5$

60. $1 < 3 - 4x < 6$

61. $0 \le 1 - 3x < 7$

62. $-4 \le 3 - 2x \le 0$

Write an inequality to represent the following statements. See Example

Example To complete an order for cement, a company will need at least 3 trucks.

Solution The words *at least* 3 trucks means that the company will need *3 or more* trucks. If x is the number of trucks needed, then

$$x \ge 3$$

63. Mark's score must be at least 72 on the final exam to pass the course.

64. The temperature today will be less than 38.

65. An automobile parts company needs to order at least 8 new lift trucks.

66. An accounting company will hire at least 2 new employees, but not more than 7.

67. The selling price (P) must be at least twice the cost (C).

Write an inequality using the given information and solve. See example 2–9 C.

Example If 4 is subtracted from three times a number, the result is greater than 2 more than twice the number. Find the number.

Solution Let x represent the number. Then

$3x - 4$ is "4 subtracted from three times the number,"
$2x + 2$ is "2 more than twice the number."

Since the two expressions are related by "is greater than," the inequality is $3x - 4 > 2x + 2$

$$\begin{aligned}
3x - 4 &> 2x + 2 \\
3x - 4 - 2x &> 2x + 2 - 2x && \text{Subtract } 2x \text{ from each member} \\
x - 4 &> 2 && \text{Combine in each member} \\
x - 4 + 4 &> 2 + 4 && \text{Add 4 to each member} \\
x &> 6 && \text{Combine in each member}
\end{aligned}$$

The number is any number that is greater than 6.

68. When 7 is subtracted from two times a number, the result is greater than or equal to 9. Find all numbers that satisfy this condition.

69. Five times a number minus 11 is less than 19. Find all numbers that satisfy this condition.

70. The product of 6 times a number added to 2 is greater than or equal to 1 subtracted from five times the number. What are the numbers that satisfy this condition?

71. Twice a number increased by 7 is no more than three times the number decreased by 5. Find the numbers that satisfy this condition.

72. If one-third of a number is added to 23, the result is greater than 30. Find all numbers that satisfy this condition.

73. Eugenia has scores of 7, 6, and 8 on three quizzes. What must she score on the fourth quiz to have an average of 7 or higher?

74. Sam has scores of 72, 67, and 81 on three tests. If an average of 70 is required to pass the course, what is the minimum score he must have on the fourth test to pass?

75. Two times a number plus 4 is greater than 6 but less than 14. Find all numbers that satisfy these conditions.

76. Four times a number minus 7 is greater than 17 but less than 25. Find all numbers that satisfy these conditions.

77. The perimeter of a square must be greater than 20 inches but less than 108 inches. Find all values of a side that satisfy these conditions.

78. The perimeter of a rectangle must be less than 100 feet. If the length is known to be 30 feet, find all numbers that the width could be. (*Note:* The width of a rectangle must be a positive number.)