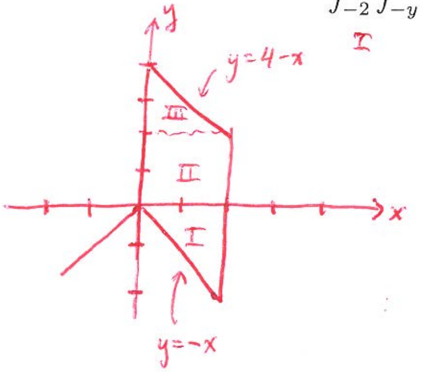
Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

1. Rewrite the following as one double integral, then evaluate.

$$\underbrace{\int_{-2}^{0}\int_{-y}^{2} xy\ dx\ dy}_{\mathrm{I}} + \underbrace{\int_{0}^{2}\int_{0}^{2} xy\ dx\ dy}_{\mathrm{II}} + \underbrace{\int_{2}^{4}\int_{0}^{4-y} xy\ dx\ dy}_{\mathrm{III}}$$



$$= \int_0^2 \int_{-x}^{4-x} xy\ dy\ dx$$

$$= \frac{1}{2}\int_0^2 x\left((4-x)^2-(-x)^2\right) dx$$

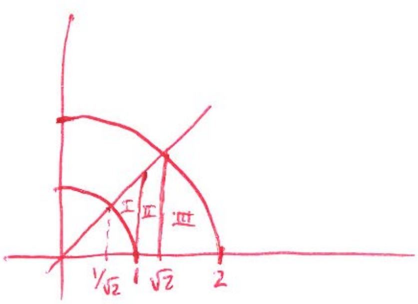
$$16-8x+x^2-x^2$$

$$= \frac{1}{2}\int_0^2 (16x-8x^2)\,dx$$

$$= \frac{1}{2}\left(32-\frac{64}{3}\right) \quad \left(=\frac{16}{3}\right)$$

2. Rewrite the following as one double integral, then evaluate.

$$\underbrace{\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\ dy\ dx}_{\mathrm{I}} + \underbrace{\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\ dy\ dx}_{\mathrm{II}} + \underbrace{\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\ dy\ dx}_{\mathrm{III}}$$



$$= \int_0^{\pi/4}\int_1^2 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} \cos\theta \sin\theta\, d\theta \int_1^2 r^3\, dr$$

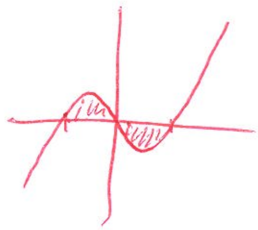
$u=\sin\theta$
$du=\cos\theta\, d\theta$

$$= \frac{1}{2}\sin^2\theta\Big]_0^{\pi/4} \cdot \frac{1}{4}r^4\Big]_1^2$$

$$= \frac{1}{4}\cdot\frac{1}{4}(15)$$

3. Compute $\iint_R x^2\, dA$, where R is the region bounded by $y = x^3 - x$ and $y = 0$. *(Caution: the region has two parts!)*

$y = x(x-1)(x+1)$



$$= 2 \cdot \int_0^1 \int_{x^3-x}^{0} x^2\, dy\, dx$$

$$= 2 \int_0^1 x^2(x-x^3)\, dx$$

$$= 2 \int_0^1 (x^3 - x^5)\, dx$$

$$= 2\left(\frac{1}{4} - \frac{1}{6}\right) \qquad \left(= \frac{1}{6}\right)$$

4. Compute $\iint_R x + y^2 \, dA$, where R is the region in the first quadrant bounded by $y = 1/x$, $y = 4/x$, $y = x$, and $y = 3x$.

Let $u = xy$, $v = \frac{y}{x}$.

Then $uv = y^2$, so $y = \sqrt{uv}$ (since $y \geq 0$)

$\frac{u}{v} = x^2$, so $x = \sqrt{\frac{u}{v}}$ $x \geq 0$

$$\frac{1}{J} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x} = 2v$$

OR

$$\boxed{J = \begin{vmatrix} \frac{1}{2\sqrt{u}\sqrt{v}} & \frac{-\sqrt{u}}{2v^{3/2}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4}\frac{1}{v} + \frac{1}{4}\frac{1}{v} = \frac{1}{2v}}$$

is positive in $[1,4]\times[1,3]$

$$\iint_R (x+y^2)\,dA = \int_1^3 \int_1^4 \left(\sqrt{\frac{u}{v}} + uv\right) \cdot \frac{1}{2v} \, du \, dv$$

$$= \int_1^3 \int_1^4 \left(\frac{u^{1/2}}{2v^{3/2}} + \frac{u}{2}\right) du \, dv$$

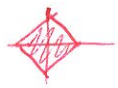
$$= \int_1^3 \frac{1}{3v^{3/2}} \left[u^{3/2}\right]_1^4 + \frac{1}{4}\left[u^2\right]_1^4 \, dv$$

$$= \int_1^3 \left(\frac{7}{3} v^{-3/2} + \frac{15}{4}\right) dv$$

$$= \left[-\frac{14}{3} v^{-1/2} + \frac{15}{4} v\right]_1^3$$

$$= -\frac{14}{3}\left(\frac{1}{\sqrt{3}} - 1\right) + \frac{15}{4}(3-1)$$

5. Let D be the diamond $|x| + |y| \leq 1$.



(a) Evaluate $\iint_D (2 + x^2y^3 - y^2 \sin x)\, dA$.

$$= \iint_D 2\, dA + \iint_D x^2y^3\, dA - \iint_D y^2 \sin x\, dA$$

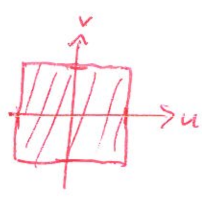
$$= 2 \cdot \text{Area}(D) + \underset{\text{odd w.r.t. } y}{0} - \underset{\text{odd w.r.t. } x}{0}$$

$$= 2 \cdot (\sqrt{2})^2$$

$$= 4$$

(b) Evaluate $\iint_D e^{x+y}\, dA$.

Let $u = x+y$, $v = x-y$



$x = \frac{u+v}{2}$

$y = \frac{u-v}{2}$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

$$\iint_D e^{x+y}\, dA = \int_{-1}^{1}\int_{-1}^{1} e^u \left|-\tfrac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}(2)(e^1 - e^{-1})$$

6. Evaluate $\iint_R y^5 e^{xy^5}\, dA$, where $R = [0,2] \times [0,3]$.

$$= \int_0^3 \int_0^2 y^5 e^{xy^5}\, dx\, dy \qquad u = xy^5, \quad du = y^5\, dx$$

$$= \int_0^3 \int_0^{2y^5} e^u\, du\, dy$$

$$= \int_0^3 e^{2y^5} - 1\, dy$$

$$= \int_0^2 \int_0^3 y^5 e^{xy^5}\, dy\, dx \qquad u = y, \quad dv = y^4 e^{xy^5} dy, \quad du = dy, \quad v = \frac{1}{5x} e^{xy^5}$$

$$= \int_0^2 \left[\frac{y}{5x} e^{xy^5} - \int \frac{1}{5x} e^{xy^5} dy \right]$$

7. Compute the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where a, b, c are positive constants. *(Hint: make an appropriate change of variables.)*

$u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad w = \frac{z}{c}$

$au = x \quad bv = y \quad cw = z$

$$J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\text{Volume} = \iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1} 1\, dV = \iiint_{u^2+v^2+w^2 \le 1} abc\, du\,dv\,dw = abc \cdot \text{Volume}(u^2+v^2+w^2 \le 1)$$

$$= abc \cdot \frac{4}{3}\pi (1)^3$$

8. Find the volume inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 9$. Set up the integral for both cylindrical coordinates and spherical coordinates.

$$\cancel{\iiint r\, dz\, d\theta\, dz}$$

$$2\int_0^{\sqrt{7}}\int_0^{2\pi}\int_3^{\sqrt{16-z^2}} r\, dr\, d\theta\, dz = \int_0^{\sqrt{7}}\int_0^{2\pi} 16-z^2-9\, d\theta\, dz$$

$$= 2\pi\int_0^{\sqrt{7}}(7-z^2)\,dz$$

$$= 2\pi\left(7\sqrt{7} - \frac{7\sqrt{7}}{3}\right)$$

$$= \frac{4\pi}{3}\cdot 7\sqrt{7}$$

OR

$$2\int_0^{2\pi}\int_3^{4}\int_0^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta$$

$$2\int_0^{2\pi}\int_{\tan^{-1}\frac{3}{\sqrt{7}}}^{\pi/2}\int_{3\csc\varphi}^{4} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\parallel$$

$$\frac{2\cdot 2\pi}{3}\cdot\int_{\tan^{-1}\frac{3}{\sqrt{7}}}^{\pi/2}\left(64-27\csc^3\varphi\right)\sin\varphi\, d\varphi$$

$$= \frac{4\pi}{3}\left(64\int_{\tan^{-1}\frac{3}{\sqrt{7}}}^{\pi/2}\sin\varphi\, d\varphi - 27\int_{\tan^{-1}\frac{3}{\sqrt{7}}}^{\pi/2}\csc^2\varphi\, d\varphi\right)$$

$$= \frac{4\pi}{3}\left(-64\cos\varphi\Big|_{\tan^{-1}\frac{3}{\sqrt{7}}}^{\pi/2} + 27\cot\varphi\Big|_{\tan^{-1}\frac{3}{\sqrt{7}}}^{\pi/2}\right)$$

$$= \frac{4\pi}{3}\left(64\cdot\frac{\sqrt{7}}{4} - 27\cdot\frac{\sqrt{7}}{3}\right) = \frac{4\pi}{3}\cdot 7\sqrt{7}$$

9. Find $\iiint_E x^2\, dV$, where E is the region outside the cone $z = \sqrt{3}\sqrt{x^2+y^2}$, inside the cone $z = \sqrt{x^2+y^2}$, and below the plane $z = 2$. Set up the integral for both cylindrical coordinates and spherical coordinates.

$$\int_0^2\int_0^{2\pi}\int_{z/\sqrt{3}}^{z}(r\cos\theta)^2\, r\, dr\, d\theta\, dz$$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^{2\sec\varphi}(\rho\sin\varphi\cos\theta)^2\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^2\int_0^{2\pi}\frac{1}{4}\left(z^4-\frac{z^4}{9}\right)\cos^2\theta\, d\theta\, dz$$

$$= \frac{1}{4}\int_0^2\frac{8}{9}z^4\,dz\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= \frac{2}{9}\cdot\frac{1}{5}\cdot 32\cdot\pi$$

10. Rewrite the following integral as an equivalent iterated integral in the five other orders.

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\; dy\; dz\; dx$$

$z = 1-x^2$ $x = \sqrt{1-z}$
$y = 1-x$ $x = 1-y$
$y = 1-\sqrt{1-z}$
or $1-z = (y-1)^2$
$z = 1-(y-1)^2$

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f\; dy\; dx\; dz$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f\; dz\; dy\; dx$$

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f\; dz\; dx\; dy$$

$$\int_0^1 \int_{1-(y-1)^2}^{1} \int_0^{\sqrt{1-z}} f\; dx\; dz\; dy + \int_0^1 \int_0^{1-(y-1)^2} \int_0^{1-y} f\; dx\; dz\; dy$$

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f\; dx\; dy\; dz + \int_0^1 \int_{1-\sqrt{1-z}}^{1} \int_0^{1-y} f\; dx\; dy\; dz$$

11. Find the average distance from a point inside a ball of radius 1 to its center.

distance $= \sqrt{x^2+y^2+z^2}$ (using center of ball at the origin)

$$\text{average distance} = \frac{1}{\text{vol}} \iiint_{\text{ball}} \sqrt{x^2+y^2+z^2}\, dV$$

$$= \frac{3}{4\pi} \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \frac{3}{4\pi} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{1} \rho^3 d\rho$$

$$= \frac{3}{4\pi} \cdot 2\pi \cdot 2 \cdot \frac{1}{4} = \frac{3}{4}$$

12. Find $\iiint_T xz\, dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $\underbrace{(\frac{1}{3},0,0),\ (0,1,0), \text{ and } (0,0,1)}$.

$3x+y+z=1$

$$= \int_0^{1/3} \int_0^{1-3x} \int_0^{1-3x-z} xz \, dy\, dz\, dx$$

$$= \int_0^{1/3} \int_0^{1-3x} xz(1-3x-z)\, dz\, dx$$

$$\cancel{= \int_0^{1/3} \int_0^{1-3x} (xz - 3x^2 z - xz^2)\, dz\, dx}$$

$$\cancel{= \int_0^{1/3} \left(\frac{x}{2}(1-3x)^2 - \frac{3x^2}{2}(1-3x)^2 - \frac{x}{3}(1-3x)^3 \right) dx}$$

$$= \int_0^{1/3} \int_0^{1-3x} \left((1-3x)xz - xz^2 \right) dz\, dx$$

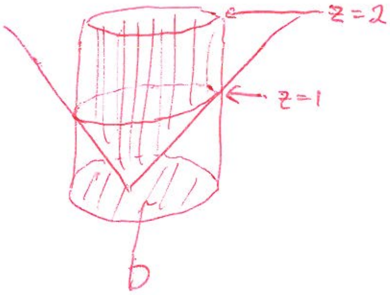
$$= \int_0^{1/3} \left(\frac{x(1-3x)^3}{2} - \frac{x}{3}(1-3x)^3 \right) dx$$

$$= \int_0^{1/3} \frac{x}{6}(1-3x)^3\, dx$$

$$= \frac{1}{6} \int_0^{1/3} \left(x - 9x^2 + 27x^3 - 27x^4 \right) dx$$

$$= \frac{1}{6} \left(\frac{1}{2} \cdot \frac{1}{9} - 3 \cdot \frac{1}{27} + \frac{27}{4} \cdot \frac{1}{81} - \frac{27}{5} \cdot \frac{1}{243} \right)$$

13. Find $\iiint_E z\, dV$, where E is the region inside the cone $z = \sqrt{x^2+y^2}$, inside the cylinder $x^2+y^2=1$, and below the plane $z = 2$.



$$\iint_{\substack{\text{disk}\\ D}} \int_{\sqrt{x^2+y^2}}^{2} z \, dz \, dx \, dy$$

$$= \int_0^{2\pi}\int_0^1\int_r^2 z r \, dz \, dr \, d\theta$$

$$= 2\pi \cdot \int_0^1 \frac{1}{2} r (4 - r^2) \, dr$$

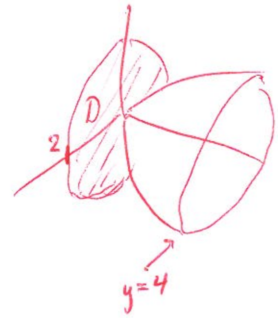
$$= \pi \cdot \int_0^1 (4r - r^3) \, dr$$

$$= \pi \left(2 - \frac{1}{4}\right)$$

14. Change the order of integration to evaluate $\displaystyle\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx.$

$z^2 = y - x^2$

$x^2 + z^2 = y$



$$= \iint_{\substack{\text{disk}\\ D}} \int_{x^2}^{4} \sqrt{x^2+z^2}\, dy\ dx\, dz$$

$x = r\cos\theta$
$z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^{2}\int_{r^2\cos^2\theta}^{4} r\ dy\ r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left(4r^2 - r^4\cos^2\theta\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{4}{3}\cdot 8 - \frac{1}{5}\cdot 32\cos^2\theta\right) d\theta$$

$$= \frac{4}{3}\cdot 8\cdot 2\pi - \frac{32}{5}\cdot \pi$$

15. Use the transformation $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$ to evaluate the integral $\iint_R (x^2 - xy + y^2)\, dA$, where R is the elliptic disk $x^2 - xy + y^2 \le 2$. *(Note: the axes of the ellipse are not parallel to the $x-$ and $y-$ axes, which is why we use the transformation.)*

$$x^2 - xy + y^2 \longleftrightarrow \left(2u^2 + \tfrac{2}{3}v^2 - \tfrac{4}{\sqrt{3}}uv\right) - \left(2u^2 - \tfrac{2}{3}v^2\right) + \left(2u^2 + \tfrac{2}{3}v^2 + \tfrac{4}{\sqrt{3}}uv\right)$$

$$= 2u^2 + 2v^2$$

$$J = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = 4/\sqrt{3}$$

$$\iint_R (x^2 - xy + y^2)\, dA$$

$$= \iint_{u^2+v^2 \le 1} (2u^2 + 2v^2) \cdot \frac{4}{\sqrt{3}}\, du\, dv$$

$u = r\cos\theta$
$v = r\sin\theta$

$$= \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta$$

$$= \frac{8}{\sqrt{3}} \cdot 2\pi \cdot \frac{1}{4}$$

16. True or false?

T (a) If f is continuous on $[0,1]$, then $\int_0^1 \int_0^1 f(x)f(y)\,dx\,dy = \left(\int_0^1 f(x)\,dx\right)^2$.

F (b) The integral $\int_0^{2\pi} \int_0^2 \int_r^2 dz\,dr\,d\theta$ represents the volume of the region inside the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 2$.

F (c) Every triple integral is positive because it measures mass of a solid with density given by the integrand.

F (d) If D is the disk $x^2+y^2 \le 4$, then $\iint_D \sqrt{4-x^2-y^2}\,dA = \frac{\cancel{32}\,16}{3}\pi$ because the integral measures the volume of a sphere of radius 2.

top half of sphere

$\frac{4}{3}\pi(2)^3$