Math 251 **Name(s)*:**

PaperAssign 3

Homework

Due September 27, 2017

*This homework is an exception to the general policy on group work: you may turn in one submission for up to four students. Otherwise, the usual policy applies: you may talk about the problems outside of the group, but what you write in your submission must be done solely within your group.

There are two problems; you may turn in this sheet or just work on lined paper, but be neat in any case.

For each, find the limit if it exists. Your work must justify your answer.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4\cos^4 x}{x^4+y^4}$

Along $x=0$, $\displaystyle\lim_{y\to 0} \frac{y^4 \cos^4 0}{0+y^4} = \lim_{y\to 0} \frac{y^4}{y^4} = 1$

Along $y=0$, $\displaystyle\lim_{x\to 0} \frac{0\cos^4 x}{x^4+0} = \lim_{x\to 0} \frac{0}{x^4} = 0$

different path-limits, so the limit DNE.

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{\frac{y^4\sin^4 x}{x^4+y^4}}_{f(x,y)}$

Given (in class) that $0 \le |\sin x| \le |x|$ for all x,

$0 \le |\sin x|^4 \le |x|^4$,

i.e. $0 \le \sin^4 x \le x^4$.

So

$$0 \le \frac{y^4 \sin^4 x}{x^4+y^4} \le \frac{y^4 \cdot x^4}{x^4+y^4}.$$

And $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4\cdot x^4}{x^4+y^4} = \lim_{r\to 0^+} \frac{r^8 \sin^4\theta \cos^4\theta}{r^4(\cos^4\theta + \sin^4\theta)}$

$\displaystyle = \lim_{r\to 0^+} r^4 \cdot \frac{\sin^4\theta\cos^4\theta}{\cos^4\theta+\sin^4\theta} = 0$

since $\cos^4\theta \ge 0$, with equality only when $\sin^4\theta = 1$ & $\sin^4\theta \ge 0$, so the denominator $\ne 0$.

So by the Squeeze Theorem, our limit is also 0.

Since $x^4 \ge 0$ & $f(x,y) \ge 0$ always,

$$0 \le \frac{y^4 \sin^4 x}{x^4+y^4} \le \frac{(y^4+x^4)\sin^4(x)}{x^4+y^4} = \sin^4 x.$$

And $\displaystyle\lim_{(x,y)\to(0,0)} \sin^4 x = 0$.

OR

Since $y^4 \ge 0$ & $f(x,y) \ge 0$ always,

$$0 \le \frac{y^4\sin^4 x}{x^4+y^4} \le \frac{y^4 \sin^4 x}{x^4}.$$

And $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4 \sin^4 x}{x^4} = \lim_{(x,y)\to(0,0)} y^4 \left[\lim_{(x,y)\to(0,0)} \left(\frac{\sin x}{x}\right)\right]^4 = 0\cdot 1 = 0$

OR

Polar coordinates on the original limit also works, but I won't work out the details here.