Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:50am to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

Some possibly useful formulas:

$$\iint_R (\partial_x n - \partial_y m)\, dx\, dy = \int_a^b (mx' + ny')\, dt$$

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$

$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

1. Consider the parametric equations $x = t^2$, $y = \sin(\pi t)$.

(a) Find the slope(s) of the tangent line(s) to the curve at the point(s) where $x = 9$.

$X = t^2 = 9 \Rightarrow t = \pm 3$

slope of tangent $= \frac{y'}{x'} = \frac{\pi \cos(\pi t)}{2t}$

at $t=-3$: $\frac{\pi \cos(-3\pi)}{-6} = \frac{-\pi}{-6} = \frac{\pi}{6}$

at $t=3$: $\frac{\pi \cos(3\pi)}{6} = \frac{-\pi}{6}$

(b) What are the minimum and maximum values taken by $x$ and $y$ in this curve?

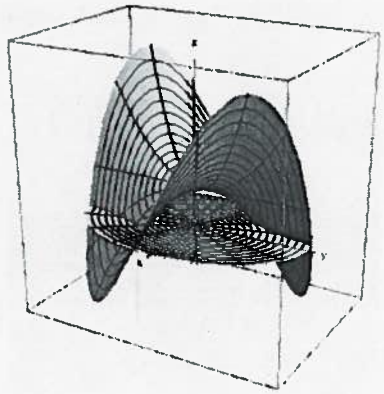
$x' = 2t = 0 \Rightarrow t = 0 \Rightarrow x = 0$; this is the minimum $x$, and there is no maximum $x$.

$y' = \pi \cos(\pi t) = 0 \Rightarrow \cos(\pi t) = 0 \Rightarrow \sin(\pi t) = \pm 1$

max $y = 1$

min $y = -1$

2. Consider the surface plotted below. It is given by $z = f(x,y)$ over the region $R$ inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, where $f(x,y) = xy + 1$.




(a) (8 points) Without doing any computations, do you think the integral $\iint_R f(x,y)\,dx\,dy$ is positive or negative? Why?

Positive; there's much more space below the surface & above R than above - - - - - below R.

(b) (15 points) Compute $\iint_R f(x,y)\,dx\,dy$.

The region makes Gauss-Green attractive. Parametrize the ellipse by $x = 3\cos t$, $y = 2\sin t$, $t \in [0, 2\pi]$. (counterclockwise, traces exactly once)

Put $m=0$, $n = \frac{1}{2}x^2y + x$ (so $\partial_x n = f$),

so $$\iint_R f\,dx\,dy = \iint_R (\partial_x n - \partial_y m)\,dx\,dy$$

$$= \int_0^{2\pi} (m x' + n y')\,dt$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2}(3\cos t)^2(2\sin t) + 3\cos t\right)(2\cos t)\,dt$$

$$= 18\int_0^{2\pi} \cos^3 t \sin t\,dt + 3\int_0^{2\pi} \cos^2 t\,dt$$

$u = \cos t$ $du = -\sin t\,dt$

$$= -18\int_1^1 u^3\,du + \frac{3}{2}\int_0^{2\pi}(1+\cos(2t))\,dt$$



$$= 0 + \frac{3}{2}[t]_0^{2\pi} + \frac{3}{4}[\sin(2t)]_0^{2\pi}$$

0

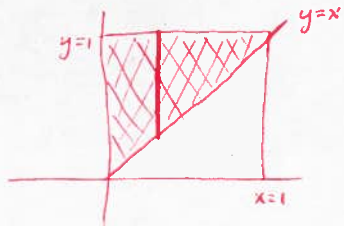
$$= \boxed{3\pi}$$

3. By switching the order of integration (careful about the bounds!), compute

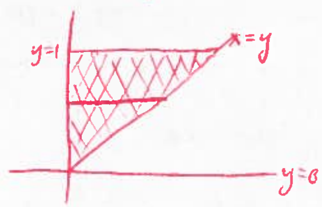
$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx.$$

(Note that you don't know an antiderivative for $e^{y^2}$ with respect to $y$, so you can't do this integral directly.)

The region of integration is $x \le y \le 1,\ 0 \le x \le 1$:




Changing the order means slicing the other way:




$0 \le x \le y,\ 0 \le y \le 1$

So we get $\displaystyle\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^y e^{y^2}\,dx\,dy$

$$= \int_0^1 x e^{y^2}\Big|_{x=0}^{y}\,dy$$

$$= \int_0^1 y e^{y^2}\,dy \qquad u = y^2 \quad du = 2y\,dy$$

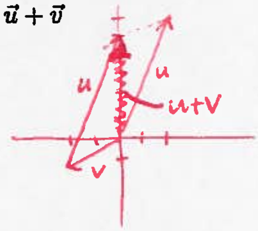
$$= \frac{1}{2}\int_0^1 e^u\,du$$

$$= \frac{1}{2}\left[e^u\right]_0^1$$

$$= \boxed{\frac{1}{2}(e-1).}$$

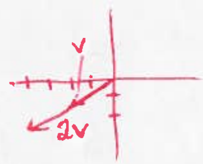
4. Let $\vec{u} = \langle 2, 6\rangle$ and $\vec{v} = \langle -2, -1\rangle$. Compute and plot the following together with $u$ and $v$.

(a) $\vec{u} + \vec{v}$




$u+v = \langle 0, 5\rangle$

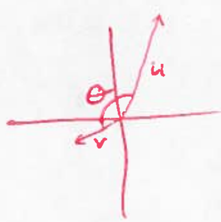
(b) $2\vec{v}$




$2v = \langle -4, -2\rangle$

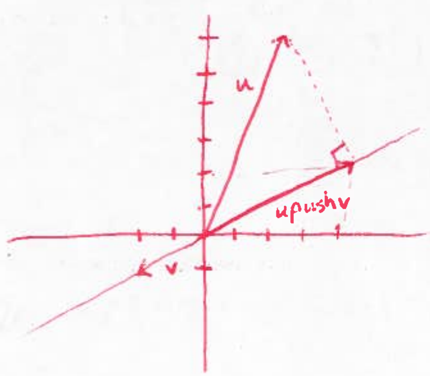
(c) the angle between $\vec{u}$ and $\vec{v}$

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-4-6}{\sqrt{4+36}\cdot\sqrt{4+1}} = \frac{-10}{\sqrt{40}\sqrt{5}} = \frac{-10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}}$$




$$\Rightarrow \theta = \frac{3\pi}{4}$$

(d) the push/projection of $\vec{u}$ in the direction of $\vec{v}$




$$u \text{ push } v = \frac{u \cdot v}{v \cdot v} v$$

$$= \frac{-10}{5} v$$

$$= -2v$$

$$= \langle 4, 2\rangle$$

5. Consider the lines $\ell_1(t) = (0, 1, 3) + t(2, 1, 5)$ and $\ell_2(t) = (-5, 2, 3) + t(-9, 9, 30)$. Are they parallel, perpendicular, or neither? Do they intersect or not?

Oops, wrong numbers...

parallel? $(2,1,5) = c(-9,9,30) \Rightarrow \begin{cases} 2 = -9c \Rightarrow c = -2/9 \\ 1 = 9c \Rightarrow c = 1/9 \\ 5 = 30c \end{cases}$ ※ no solution, so not parallel

perpendicular? $(2,1,5) \cdot (-9,9,30) = -18 + 9 + 150 \neq 0$, so not perpendicular

Neither

intersect? $(0+2t, 1+t, 3+5t) = (-5-9s, 2+9s, 3+30s)$

$\Rightarrow \begin{cases} 0+2t = -5-9s & ① \\ 1+t = 2+9s & ② \\ 3+5t = 3+30s & ③ \end{cases}$ $2\cdot② - ①: 7 = 9 + 27s$

$\Rightarrow s = -\frac{7}{27}$

① & $\Rightarrow t = \frac{1}{2}(-5 + \frac{7}{3})$

in ③, contradiction, so don't intersect

6. Consider the lines $\ell_1(t) = (0, 1, 3) + t(2, 1, 5)$ from above and $\ell_3(t) = (3, 0, 1) + t(-2, -1, 1)$. Are they parallel, perpendicular, or neither? Do they intersect or not?

parallel? $(2,1,5) = c(-2,-1,1)$ No.

$\Rightarrow c = -1$, doesn't work;

perpendicular? $(2,1,5) \cdot (-2,-1,1) = -4 - 1 + 5 = 0$ Yes

intersect? $\begin{cases} 2t = 3 - 2s \Rightarrow t = \frac{3}{2} - s \\ 1 + t = -s \Longrightarrow 1 + \frac{3}{2} - s = -s \Rightarrow \frac{5}{2} = 0 \text{ ※} \\ 3 + 5t = 1 + s \end{cases}$

Don't intersect

7. Consider the two planes given by equations

$$3x - y + z = 4$$
$$2x + y - 2z = 6.$$

Find an equation of the line that is the intersection of these planes.

A direction vector for the line is parallel to both planes, hence perpendicular to both their normals.

$$\vec{v} = \langle 3, -1, 1 \rangle \times \langle 2, 1, -2 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \vec{k}$$

$$= (2-1)\vec{i} - (-8)\vec{j} + (5)\vec{k}$$

$$= 1\vec{i} + 8\vec{j} + 5\vec{k}$$

$$= \langle 1, 8, 5 \rangle.$$

To get a point on the line, arbitrarily choose $x=0$,

so $\begin{cases} -y + z = 4 & \text{①} \\ y - 2z = 6 & \text{②} \end{cases}$ ①+②: $-z = 10$
$\Rightarrow z = -10$
$\overset{①}{\Rightarrow} y = -14$

so the line is $(0, -14, -10) + t(1, 8, 5)$.

8. Give parametric equations for the unit circle in the plane $3x - y + z = 4$ centered at $(1, 0, 1)$.

We want to plot nicely in a "slanted" plane, so having a pair of perpendicular unit vectors would be good. Let $\vec{n} := \langle 3, -1, 1 \rangle$, a normal vector for the plane.

Pick arbitrarily $\vec{a} = \langle 1, 1, 1 \rangle$.

Then $\vec{u}_0 := \vec{n} \times \vec{a}$ is $\perp$ to $\vec{n}$, so ~~is~~ parallel to our plane.

$$\vec{u}_0 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2\vec{i} - 2\vec{j} + 4\vec{k}.$$

Also, $\vec{v}_0 := \vec{n} \times \vec{u}$ is $\perp$ to both $\vec{n}$ — so is parallel to our plane & $\vec{u}$ — as desired

$$\vec{v}_0 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ -2 & -2 & 4 \end{vmatrix} = -2\vec{i} - 14\vec{j} - 8\vec{k}.$$

(Check: $\vec{u} \cdot \vec{v} = 0$ ✓)

We also wanted unit vectors, so divide by lengths:

$$\vec{v} := \frac{\vec{v}_0}{\|\vec{v}_0\|} = \frac{\vec{v}_0}{\sqrt{4+196+64}} = \left\langle -\frac{2}{\sqrt{264}}, -\frac{14}{\sqrt{264}}, -\frac{8}{\sqrt{264}} \right\rangle$$

$$\vec{u} := \frac{\vec{u}_0}{\|\vec{u}_0\|} = \frac{\vec{u}_0}{\sqrt{4+4+16}} = \left\langle -\frac{2}{\sqrt{24}}, -\frac{2}{\sqrt{24}}, \frac{4}{\sqrt{24}} \right\rangle$$

Now the circle is given by $(1,0,1) + \cos(t)\cdot\vec{u} + \sin(t)\cdot\vec{v}$, $t \in [0, 2\pi]$

9. Find the maximum and minimum values of $f(x,y) = xy$ on the disk $x^2 + y^2 \leq 4$. Hint: consider the interior of the disk and its boundary (the circle) separately.

Interior max's & min's can only occur when $\nabla f = 0$:

$$\nabla f = \langle y, x \rangle = \langle 0,0 \rangle \Rightarrow x = y = 0$$

$$\boxed{f(0,0) = 0}$$

Boundary max's & min's can be found most easily by Lagrange multipliers: want min/max $f(x,y) = xy$ subject to $g(x,y) = x^2+y^2 = 4$:

solve $\begin{cases} \nabla f = \lambda \cdot \nabla g \\ g = 4 \end{cases}$, i.e. $\begin{cases} y = \lambda(2x) & (1) \\ x = \lambda(2y) & (2) \\ x^2+y^2 = 4 & (3) \end{cases}$

$(1) \& (3) \Rightarrow x^2 + (2\lambda x)^2 = 4$

$\Rightarrow x^2(1+4\lambda^2) = 4$

$\Rightarrow x^2 = \dfrac{4}{1+4\lambda^2}$

$(2) \& (3) \Rightarrow (2\lambda y)^2 + y^2 = 4$

$\Rightarrow y^2(1+4\lambda^2) = 4$

$\Rightarrow y^2 = \dfrac{4}{1+4\lambda^2}$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = \pm y.$

$(3) \Rightarrow x = \pm\sqrt{2}, \; y = \pm\sqrt{2}.$

So 4 points to check:

$$\boxed{\begin{aligned} f(\sqrt{2}, \sqrt{2}) &= 2 = f(-\sqrt{2}, -\sqrt{2}) \\ f(-\sqrt{2}, \sqrt{2}) &= -2 = f(\sqrt{2}, -\sqrt{2}) \end{aligned}}$$

min $= -2$

max $= 2$

10. Given the motion of a particle, why is ~~acceleration~~ velocity$(t)$ = speed$(t)$ unittangent$(t)$? Use this and the chain rule to find the tangential and normal components of acceleration in terms of the unit tangent, unit normal, and unit binormal vectors.

speed is the magnitude of velocity, and velocity's direction is tangent to the position curve.

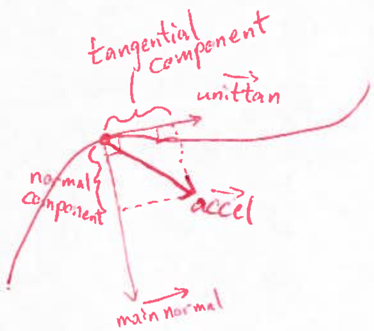
Then

$$\overrightarrow{\text{accel}}(t) = \frac{d}{dt}\,\text{velocity}(t)$$

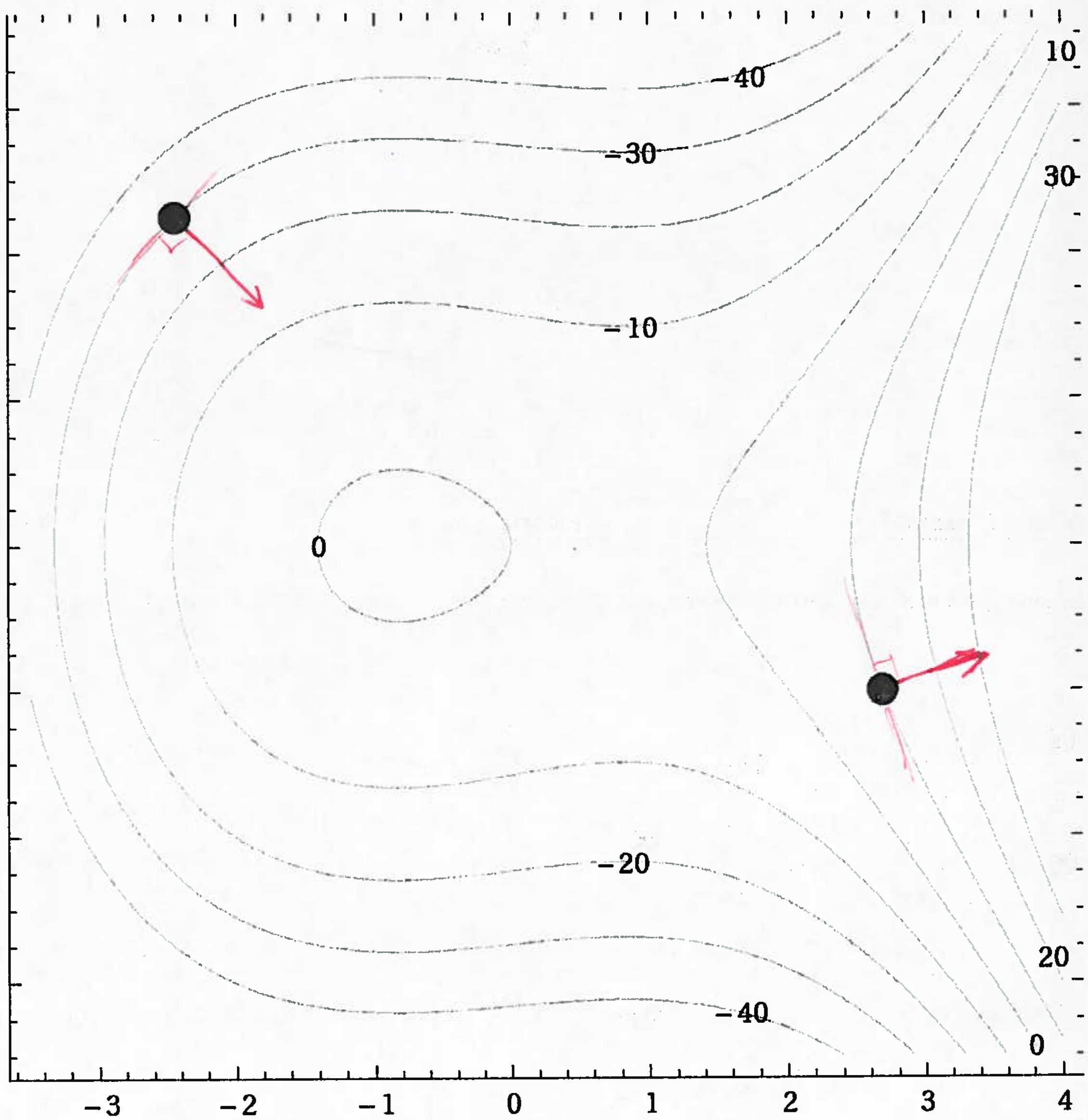
$$= \frac{d}{dt}\left(\text{speed}(t)\cdot \overrightarrow{\text{unittan}}(t)\right)$$

$$= \underbrace{\left(\frac{d}{dt}\,\text{speed}(t)\right)\overrightarrow{\text{unittan}}(t)}_{1} + \underbrace{\text{speed}(t)\cdot\left(\frac{d}{dt}\,\overrightarrow{\text{unittan}}(t)\right)}_{2}.$$

So we have written $\overrightarrow{\text{accel}}$ as the sum of two vectors. The first is in the tangential direction ($\overrightarrow{\text{unittan}}$), and we know that $\frac{d}{dt}(\overrightarrow{\text{unittan}}) = \overrightarrow{\text{mainnormal}}$ is perpendicular to the tangential direction. Hence this expression is the decomposition of $\overrightarrow{\text{accel}}$ into its tangential & normal components.

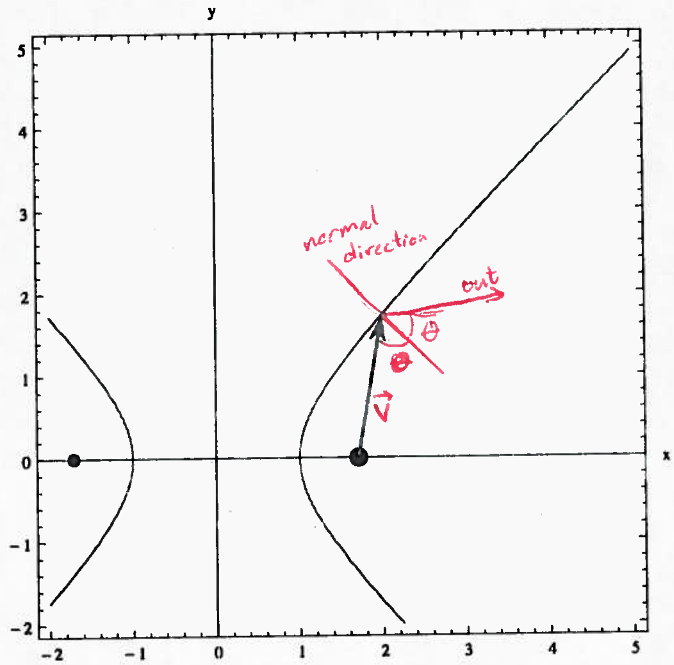


11. Below is a plot of several level curves of a function $f(x, y)$. At the indicated points, sketch in the gradient vectors.




Gradient vectors point in the direction of greatest <u>increase</u> of the function. (In particular, they are perpendicular to the level curve they originate from.)

12. Consider the situation below, in which a ray of light is being sent from the focus $(1+\frac{1}{\sqrt{2}}, 0)$ of the hyperbola $x^2 - y^2 = 1$ toward the point $(2, \sqrt{3})$ on the hyperbola. Provide a system of equations that could be solved to find the vector representing the path of the light after reflection off of the hyperbola. (Your system should have a unique solution.) Briefly explain the theory behind your answer. (Adding to the picture may be useful.)




$\vec{v} = \langle 1 - \frac{1}{\sqrt{2}}, \sqrt{3} \rangle$

The angle θ that $-\vec{v}$ makes w/ the normal vector at $(2, \sqrt{3})$ should be the same as the outgoing vector makes w/ the normal.

We need a normal vector; in this case, thinking of the hyperbola as a level curve $f(x,y) = x^2 - y^2 = 1$, the gradient will serve:

$$\nabla f(2,\sqrt{3}) = \langle 2x, -2y \rangle\Big|_{(2,\sqrt{3})} = \langle 4, -2\sqrt{3} \rangle.$$

If we further make our incoming & outgoing vectors the same length, then equal angles $\Longleftrightarrow$ equal dot product. So our system of equations is

$$\begin{cases} -v \cdot \text{normal} = \text{out} \cdot \text{normal} \\ v \cdot v = \text{out} \cdot \text{out} \end{cases}, \text{ or } \boxed{\begin{cases} (\frac{1}{\sqrt{2}} - 1) \cdot 4 + (\sqrt{3})(2\sqrt{3}) = 4x - 2\sqrt{3}y \\ (1 - \frac{1}{\sqrt{2}})^2 + (\sqrt{3})^2 = x^2 + y^2 \end{cases}}$$