Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have seventy-five (75) minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.

- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.

Question:	1	2	3	4	5	6	7	Total
Points:	8	12	12	8	15	26	24	105
Score:								

Short answer

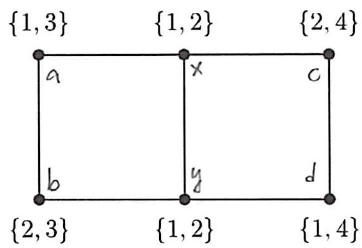
1. (8 points) Find the maximum number of edges in a 16-vertex K_4-free simple graph. Briefly justify.

Turán's Theorem ⇒ balanced complete 3-partite graph

partite sets have size 5, 5, 6 ($\Sigma = 16$)

$\#\text{edges} = 5\cdot 5 + 5\cdot 6 + 5\cdot 6 = 85$

2. (12 points) Prove that the following graph G cannot be properly colored from the displayed list assignment. What does this say about $\chi_\ell(G)$?



If x is colored 1,

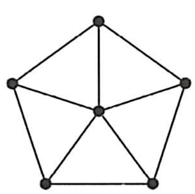
then a needs 3 ⇒ b is 2 ⇒ y is 1 ↯

If x is colored 2,

then c needs 4 ⇒ d is 1 ⇒ y is 2 ↯

Hence $\chi_\ell(G) > 2$.

3. (12 points) Prove that the following graph is 4-critical. *(This is very quick with an appropriate theorem, but is not hard to do directly.)*



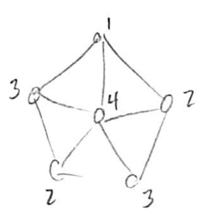
This graph is $C_5 \vee K_1$;

C_5 is 3-critical & K_1 is 1-critical,

so $C_5 \vee K_1$ is (3+1)-critical.

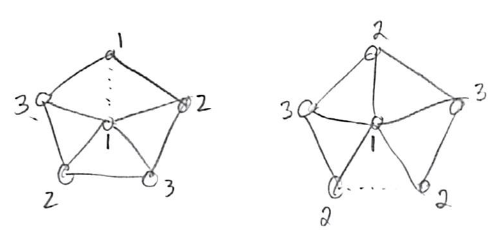
OR

$\chi \le 4$:

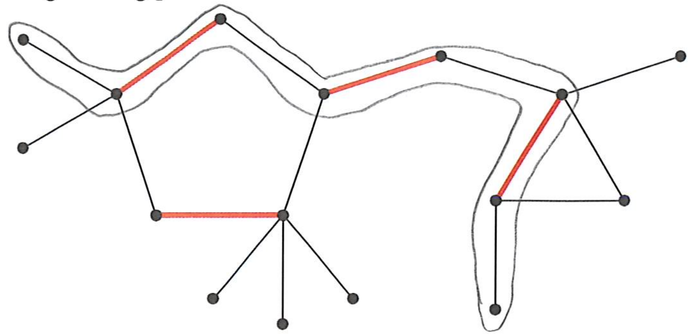


$\chi \ge 4$: the outer odd cycle needs 3 colors, & the center vtx (being adjacent to all others) needs an additional one.

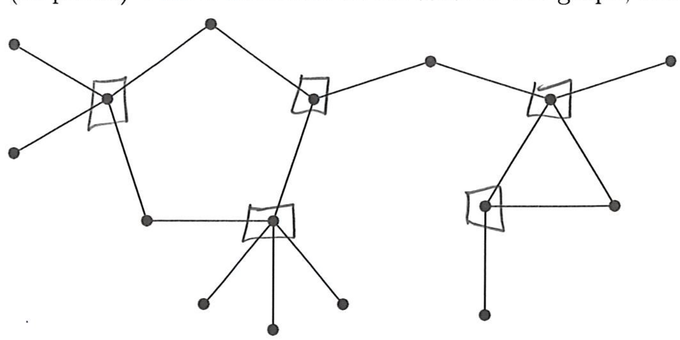
color-critical: there are two kinds of edges, up to symmetry



4. (8 points) The matching displayed in the following graph is *not* maximum. Prove this by finding an augmenting path.



5. (15 points) Find a minimum vertex cover in the graph, and prove that it is minimum.



The boxed vts form a vtx cover of size 5.

Augmenting along the path in #4 gives a matching of size 5, so both of these are optimal.

Proofs

6. Let G be an X, Y-bigraph with n vertices. Please note that throughout this problem we are using $\alpha(G)$, not $\alpha'(G)$.

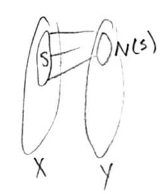
   (a) (6 points) Prove that $\alpha(G) \geq n/2$.

   X & Y are both independent sets; one has to have size $\geq \frac{n}{2}$.

   (b) (10 points) Prove that if G has no perfect matching, then $\alpha(G) > n/2$. *(Suggestion: first, is $|X| = |Y|$? Then use Hall's Theorem.)*

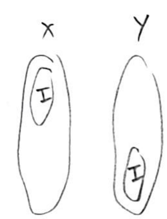
   If $|X| \neq |Y|$, then the larger is an indep. set of size $> \frac{n}{2}$.

   If $|X| = |Y|$, then no p.m. $\overset{\text{Hall}}{\Longrightarrow}$ $\exists\ S \subseteq X$ s.t. $|S| > |N(S)|$.

   Then $S \cup (Y - N(S))$ is an indep. set, w/size $|S| + |Y| - |N(S)| > |Y| = \frac{n}{2}$.

   (c) (10 points) Prove that if $\alpha(G) > n/2$, then G has no perfect matching. *(Suggestion: use Tutte's 1-Factor Theorem.)*

   Let I be an indep. set of size $> \frac{n}{2}$.

   Let $S = V(G) - I$; then $G - S$ is just the isolated vertices I, each of which is an odd cpnt (size 1).

   So $o(G-S) = |I| > \frac{n}{2}$

   & $|S| = n - |I| < \frac{n}{2}$, so Tutte $\Rightarrow$ no p.m.

7. (24 points) Prove **three** of the following four statements. *(All of these have short proofs, some very short.)*

   (i) If G is a graph such that $\chi(G - x - y) = \chi(G) - 2$ for every pair of distinct vertices x, y, then G is a complete graph.

   (ii) Always $\chi(G) \cdot \chi(\overline{G}) \geq n(G)$.

   (iii) Always $\beta(G) \leq 2\alpha'(G)$.

   (iv) Every maximal matching in a graph G has size at least $\alpha'(G)/2$.

(i) Suppose G is not complete, i.e. $\exists\, x, y \in V(G)$ w/ $xy \notin E(G)$.

Take an optimal proper coloring of $G - x - y$; it can be extended to G by giving x & y the same new color;

hence $\chi(G) \leq \chi(G - x - y) + 1$ ↯

(ii) Pf 1: $\chi(G) \geq \dfrac{n(G)}{\alpha(G)} = \dfrac{n(G)}{\omega(\overline{G})} \geq \dfrac{n(G)}{\chi(\overline{G})}$.

Pf 2: Consider optimal proper colorings of G & $\overline{G}$. For each vertex, consider the ordered pair of colors it is given: for any two vertices, they are adjacent in either G or $\overline{G}$, so their colors in that graph are different, so their pair is different. Hence we have defined an injection: $V(G) \hookrightarrow [\chi(G)] \times [\chi(\overline{G})]$.

(iii) Let M be a maximum matching (so $|M| = \alpha'$). Let Q denote the set of vertices saturated by M. $|Q| = 2|M| = 2\alpha'$ b/c M is a matching. Q is also a vertex cover: if some edge were not covered, then it could be added to M, contradicting maximality.

Hence $\beta(G) \leq |Q| = 2\alpha'(G)$.

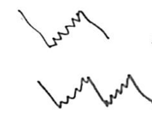
(The statements are repeated here for your convenience.)

(i) If G is a graph such that $\chi(G-x-y) = \chi(G) - 2$ for every pair of distinct vertices x, y, then G is a complete graph.

(ii) Always $\chi(G) \cdot \chi(\overline{G}) \geq n(G)$.

(iii) Always $\beta(G) \leq 2\alpha'(G)$.

(iv) Every maximal matching in a graph G has size at least $\alpha'(G)/2$.



$\#M$ edges $\geq \frac{1}{2}\#M^*$ edges

(iv) Pf 1: Let M be maximal & M^* maximum matchings.

Each cpnt of $M \Delta M^*$ is a path or an even cycle, & each of these has at most 1 more M^* edge than M edges; if $|M^*| > 2|M|$, there must be a cpnt of just one M^* edge. But then that edge of M^* could be added to M, contradicting its maximality.

Pf 2: Contrapositive: M^* maximum, M any matching. Suppose that $|M^*| > 2|M|$. Then M saturates fewer than $|M^*|$ vertices, & hence some edge of M^* has neither endpt saturated by M, so it can be added to M. Thus M is not maximal.

Pf 3: M maximal. The endpts of the edges in M form a vertex cover (else an uncovered edge could be added to M), so $2|M| \geq \beta(G) \geq \alpha'(G)$.