Worksheet 18 March 30, 2011

1. Give an upper and a lower bound on the value of $\int_1^5 \frac{1}{x}\,dx$ by Riemann sums with 4 subintervals. (Make sure to justify that your answers are really an upper bound and a lower bound.) Improve your bounds by taking 8 subintervals.

2. If $f(x)$ is an odd function, what is $\int_{-1}^{1} f(x)\,dx$?

3. Evaluate $\displaystyle\int_5^0 3\,dx$.

4. Rewrite the following as a definite integral on the interval $[1, 4]$: $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^*)^2 \sqrt{x_i^* + 2}\,\Delta x$.

5. Rewrite the following as a definite integral: $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{8}{n} \sin\left(2 + \frac{8i}{n}\right) dx$. (Remark: there are many correct answers; just try to find a simple one.)

6. Evaluate the following definite integrals using geometric arguments:

   (a) $\displaystyle\int_{-2}^{4} \sqrt{9 - (x-1)^2}\,dx$.

   (b) $\displaystyle\int_{-1}^{3} x\,dx$.

   (c) $\displaystyle\int_{0}^{4.5} \lfloor x \rfloor\,dx$. (Recall that $\lfloor x \rfloor$ is the *floor of* x, the greatest integer less than or equal to x.)

7. Given that $\displaystyle\int_0^1 x^3\,dx = \frac{1}{4}$ and $\displaystyle\int_0^2 x^3\,dx = 4$, find

   (a) $\displaystyle\int_1^2 x^3\,dx$.

   (b) $\displaystyle\int_{-2}^0 x^3\,dx$.

   (c) $\displaystyle\int_{-2}^2 x^3\,dx$.

   (d) $\displaystyle\int_{-2}^2 |x^3|\,dx$.

   (e) $\displaystyle\int_{-2}^1 x^3\,dx$.

   (f) $\displaystyle\int_0^1 (-12x^3 + 4)\,dx$.

   (g) $\displaystyle\int_0^1 \sqrt[3]{x}\,dx$. (Hint: sketch the situation.)

8. Consider the function $A(x)$ given as the area in the t,y-plane bounded by $y = 3t - 1$, $y = 0$, $t = 1$, and $t = x$. Find an explicit formula for $A(x)$ (for $x \geq 1$ say). What is $A'(x)$?

9. Compute the integral $\displaystyle\int_0^4 x^3\,dx$ as a limit of Riemann sums. You'll need to use the identity $\displaystyle\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

10. Define $A(x)$ to be the area beneath the graph of $f(t) = \lfloor t \rfloor$ between $t = 0$ and $t = x$. Find an explicit formula for $A(x)$ (again just assume $x \geq 0$). Your formula will need to be piecewise, and it's reasonable to only write it out for $x \leq 6$ or so. What is $A'(x)$?

11. Not every function is integrable. Suppose we want to compute $\displaystyle\int_0^1 \chi(x)\,dx$, where

$$\chi(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1 & \text{if } x \text{ rational.} \end{cases}$$

By choosing appropriate sample points x_i^*, show that the Riemann sums can always be made to be 0, but can also be made to be 1.