Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1+\cos(2t)) \qquad \overline{x} = \frac{\iint_R x \cdot \rho(x,y)\,dA}{\text{mass}} \qquad \overline{x} = \frac{\iiint_E x \cdot \rho(x,y,z)\,dV}{\text{mass}}$$

$$\sin^2 t = \frac{1}{2}(1-\cos(2t)) \qquad \overline{y} = \frac{\iint_R y \cdot \rho(x,y)\,dA}{\text{mass}} \qquad \overline{y} = \frac{\iiint_E y \cdot \rho(x,y,z)\,dV}{\text{mass}}$$

$$\overline{z} = \frac{\iiint_E z \cdot \rho(x,y,z)\,dV}{\text{mass}}$$

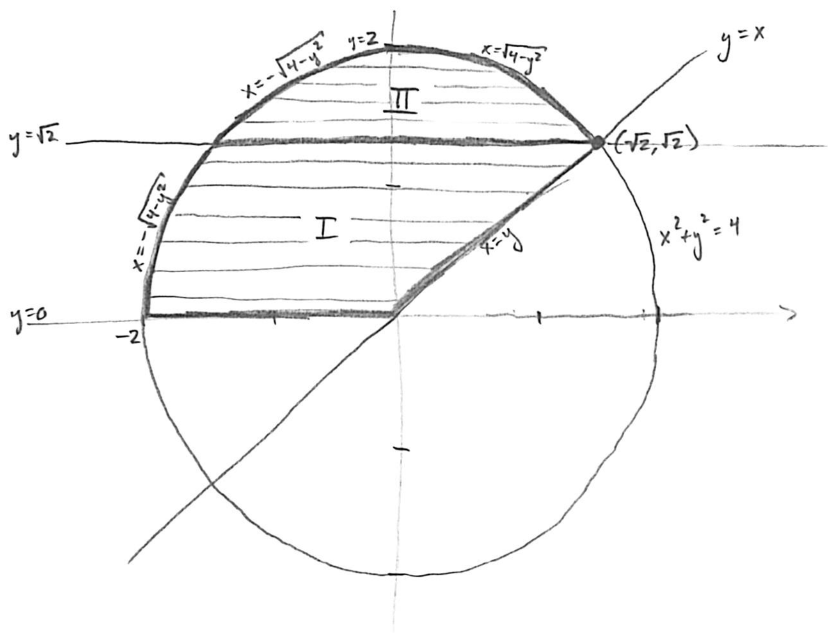
Question:	1	2	3	4	5	6	Total
Points:	17	15	20	15	10	12	89
Score:							

1. (a) (12 points) Draw the region of integration for the following integral. Indicate in your drawing the equation of curves and the coordinates of points of interest.

$$\underbrace{\int_0^{\sqrt{2}} \int_{-\sqrt{4-y^2}}^{y} f(x,y)\,dx\,dy}_{\mathrm{I}} + \underbrace{\int_{\sqrt{2}}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy}_{\mathrm{II}}$$

$x = \pm\sqrt{4-y^2} \longleftrightarrow x^2+y^2=4$

$x=y$



(b) (5 points) Rewrite the integral using a more appropriate method for the region.

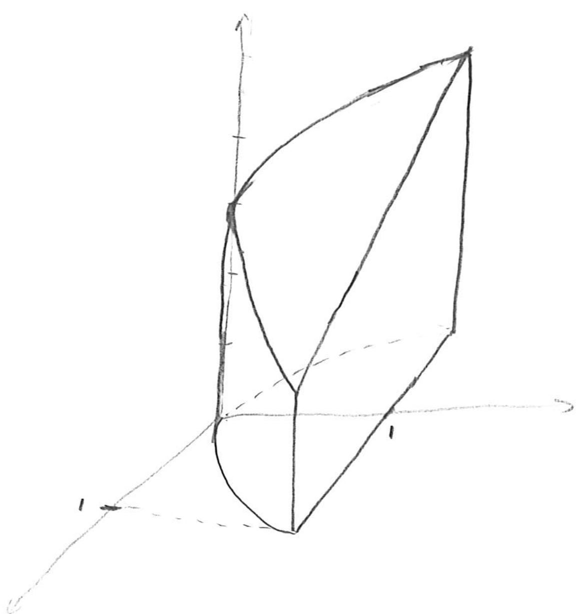
$$= \int_{\frac{\pi}{4}}^{\pi} \int_0^2 f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

2. (15 points) Set up (as an iterated integral, in whichever order you decide) $\iiint_E f(x,y,z)\,dV$, where E is the region bounded by the surfaces $y = x^2$, $z = 3 - x$, $z = 0$, and $y = 1$. Do not evaluate.

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{3-x} f\,dz\,dy\,dx \quad \text{OR} \quad \int_{0}^{1}\int_{0}^{3-x}\int_{x^2}^{1} f\,dy\,dz\,dx$$

$$\text{OR} \quad \int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{3-x} f\,dz\,dx\,dy.$$

[The three other orders require splitting into two pieces.]



3. (a) (15 points) Evaluate $\iiint_E x\,dV$ where E is the part of the ball $x^2 + y^2 + z^2 \leq 9$ in the first octant.

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} (\rho \sin\varphi \cos\theta)(\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, d\theta \int_0^{\pi/2} \underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos(2\varphi))}\, d\varphi \int_0^{3} \rho^3\, d\rho$$

$$= 1 \cdot \frac{\pi}{4} \cdot \frac{81}{4}$$

(b) (5 points) Give an interpretation of the integral in (a) in the context of mass. *(There are infinitely many answers, but one "simplest" answer.)*

It is the mass of a piece of metal in the shape of E if the density at every point is x.

4. (15 points) Use an appropriate change of variables to evaluate $\iint_R e^{3x+2y}\, dA$, where R is the parallelogram bounded by $x+3y=1$, $x+3y=4$, $2x-y=5$, and $2x-y=7$. *(Hint: Use the transformation to make the region nice, and worry about the integrand afterward.)*

$u = x+3y$
$v = 2x-y$

$$J = \frac{1}{\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}} = \frac{1}{-7}$$

$u+v = 3x+2y$

$$\int_5^7 \int_1^4 e^{u+v} \cdot \frac{1}{7}\, du\, dv$$

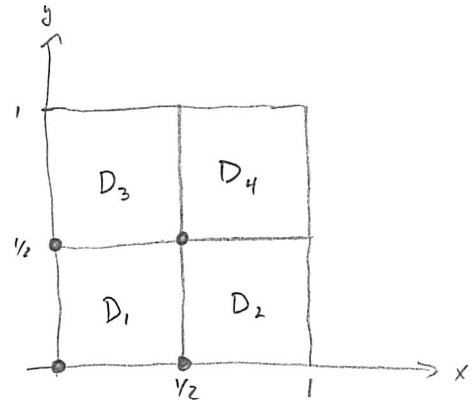
$$= \frac{1}{7} \int_1^4 e^u\, du \int_5^7 e^v\, dv$$

$$= \frac{1}{7}(e^4 - e)(e^7 - e^5)$$

5. Consider the integral

$$\int_0^1 \int_0^1 \underbrace{16^{x^2+y^2}}_{f(x,y)} \, dx \, dy.$$

(a) (10 points) Use a Riemann sum with four terms (two rectangles in each of the x and y directions; WebAssign might say $m = n = 2$) and using lower-left corners to estimate the integral.



$$\approx dA \cdot \left(\underset{D_1}{f(0,0)} + \underset{D_2}{f(\tfrac{1}{2},0)} + \underset{D_3}{f(0,\tfrac{1}{2})}, \underset{D_4}{f(\tfrac{1}{2},\tfrac{1}{2})} \right)$$

$$= \frac{1}{4}\left(1 + 2 + 2 + 4 \right)$$

(b) (3 points (bonus)) Can you guarantee that this estimate is an over- or under-estimate? Why?

It is an underestimate:

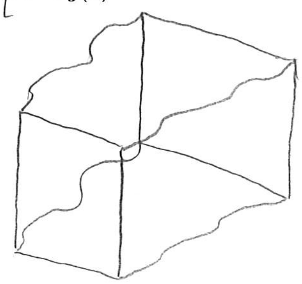
the volume under the graph above each of the four squares is larger than our estimate, because f is an increasing function of both x & y

[so $f(x,y) \geq f(0,0)$ for all $(x,y) \in D_1$,
$f(x,y) \geq f(\frac{1}{2},0)$ for all $(x,y) \in D_2$, etc.].

6. (12 points) Circle 'True' or 'False' and give a brief justification. Throughout, assume f is "very nice."* Frequently, a good picture can serve as the justification. The questions continue onto the next page.

(a) True False Assume that $g(x) \leq h(x)$ always, that both of these functions are "very nice," and $a < b$ and $c < d$. It must be true that

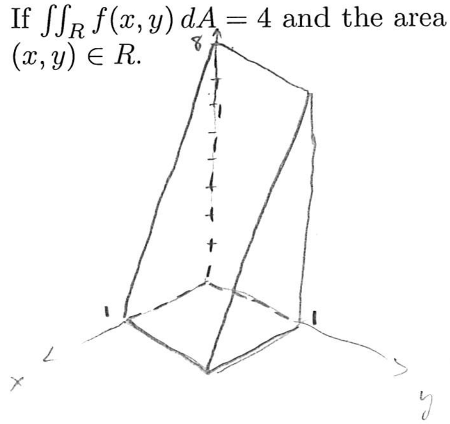
$$\int_a^b \left[\int_c^d \int_{g(x)}^{h(x)} f(x,y,z)\,dy\,dx\right] dz = \int_c^d \int_{g(x)}^{h(x)} \left[\int_a^b f(x,y,z)\,dz\right] dy\,dx.$$



(b) True False If $f(x,y) \leq 4$ for every $(x,y) \in R$ and the area of R is 1, then $\iint_R f(x,y)\,dA \leq 4$.

$$\iint_R f(x,y)\,dA \leq \iint_R 4\,dA = 4 \cdot \text{Area}(R) = 4.$$

(c) True False If $\iint_R f(x,y)\,dA = 4$ and the area of R is 1, then $f(x,y) \leq 4$ for every $(x,y) \in R$.

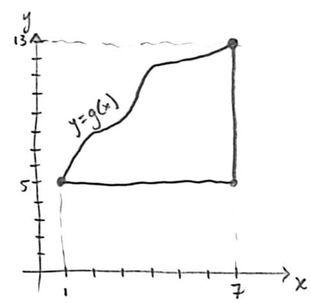


*Specifically, let's say f has continuous partial derivatives of all orders.

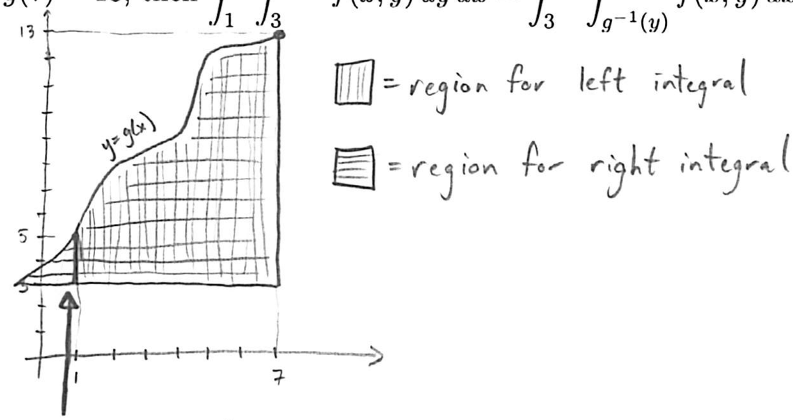
(d) True **False** Every triple integral is positive because it measures either volume or mass.

$$\iiint_E -1\,dV = -1 \cdot \mathrm{Vol}(E) < 0$$

(e) **True** False If $g(x)$ is any increasing continuous function with $g(1) = 5$ and $g(7) = 13$, then $\int_1^7 \int_5^{g(x)} f(x, y)\, dy\, dx = \int_5^{13} \int_{g^{-1}(y)}^7 f(x, y)\, dx\, dy$.



(f) True **False** If $g(x)$ is any increasing continuous function with $g(1) = 5$ and $g(7) = 13$, then $\int_1^7 \int_3^{g(x)} f(x, y)\, dy\, dx = \int_3^{13} \int_{g^{-1}(y)}^7 f(x, y)\, dx\, dy$.



the right integral is over a larger region