

Tuesday, October 30 ** Introduction to multiple integrals

1. Evaluate the following integral by reversing the order of integration:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy.$$

(Hint: When you change to $dx \, dy$, be sure to also change the bounds of integration.)

2. Consider the region bounded by the curves determined by $-2x + y^2 = 6$ and $-x + y = -1$.

(a) Sketch the region R in the plane.

(b) Set up and evaluate an integral of the form $\iint_R dA$ that calculates the area of R .

3. Consider the region R which lies above the x -axis and between the circles of radius 1 and 2 centered at $(0,0)$. Without using polar coordinates, evaluate

$$\iint_R y \, dA.$$

4. Evaluate

$$\int_{-2}^0 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx.$$

Hint: don't do it directly.

5. The function $P(x) = e^{-x^2}$ is fundamental in probability.

(a) Sketch the graph of $P(x)$. Explain why it is called a "bell curve."

(b) Compute $I = \int_{-\infty}^{\infty} e^{-x^2} \, dx$ using the following brilliant strategy of Gauss.

i. Instead of computing I , compute $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} \, dy \right)$.

ii. Rewrite I^2 as an integral of the form $\iint_R f(x, y) \, dA$ where R is the entire Cartesian plane.

iii. Convert that integral to polar coordinates.

iv. Evaluate to find I^2 . Deduce the value of I .

Amazingly, it can be mathematically proven that there is NO elementary function $Q(x)$ (that is, function built up from sines, cosines, exponentials, and roots using "usual" operations) for which $Q'(x) = P(x)$.

6. Compute $\int_0^{\infty} \int_0^{\infty} \frac{1}{(1+x^2+y^2)^2} \, dx \, dy$.