Worksheet 22 April 13, 2011

1. Find the area bounded by the curves $y = x^2$ and $y = 4$.
2. Let $h(x) = \sec x$. Do the Intermediate Value Theorem or Mean Value Theorem guarantee the existence of a $c \in (0, \pi)$ such that $h'(c) = 0$? Why or why not? If neither theorem guarantees this existence, then decide (by hand) whether there is one.
3. Keeping $h(x) = \sec x$, do either the IVT or MVT guarantee the existence of a $c \in (0, \pi)$ such that $h'(c) = -2/\pi$? Why or why not? If neither theorem guarantees this existence, then decide (by hand) whether there is one.
4. Find the area bounded by the curves $y = x^2$ and $y = \sqrt{x}$.
5. Show that there is exactly one real root of $p(x) = x^5 + 7x^3 + 22x + 13$.
6. Show that there are exactly two real roots of $g(x) = x^6 - 2x^2 - 1$. (Hint: make use of symmetry.)
7. How many real roots does $h(x) = e^{x^2} - \frac{1}{2}$ have? Prove your answer.
8. Consider the region in the plane between the x-axis, $y = 1/x$, $x = 1$, and $x = M$, where $M > 1$.
   (a) Find the area of this region.
   (b) What happens to the area as $M \to \infty$?
   (c) Now suppose the region is rotated about the x-axis. Compute the resulting solid's volume.
   (d) What happens to the volume as $M \to \infty$?
9. Find the area bounded by the curves $y = x^3 - x$ and $y = 3x$. (If you're not careful, the answer won't make any sense!)
10. Compute $\displaystyle\lim_{n\to\infty} \frac{\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n}}{n^{4/3}}$ by evaluating an integral of the form $\displaystyle\int_0^1 f(x)\,dx$.
11. Not every function is integrable. Suppose we want to compute $\displaystyle\int_0^1 \chi(x)\,dx$, where

$$\chi(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1 & \text{if } x \text{ rational.} \end{cases}$$

By choosing appropriate sample points x_i^*, show that the Riemann sums can always be made to be 0, but can also be made to be 1.