Section 1.3 Order of Operations

In the previous section (1.2) we introduced some of the properties of real number and we also saw how these properties are used when performing fundamental operations with numbers. Since variables represent numbers, we will be using these and other properties throughout our study of algebra. The properties that we have covered so far are listed and the page number where the property was first introduced is given for reference.

Properties of real numbers

If a, b, and c are any real numbers, then

$a + b = b + a$, commutative property of addition
$a \cdot b = b \cdot a$, commutative property of multiplication
$(a + b) + c = a + (b + c)$, associative property of addition
$(a \cdot b)c = a(b \cdot c)$, associative property of multiplication
$a \cdot 1 = 1 \cdot a = a$, identity property of multiplication
$a + 0 = 0 + a = a$, identity property of addition
$a + (-a) = 0$, additive inverse property
$a \cdot 0 = 0 \cdot a = 0$, zero factor property

Exponents

Consider the indicated products

$$4 \cdot 4 \cdot 4 = 64$$

and

$$3 \cdot 3 \cdot 3 \cdot 3 = 81$$

A more convenient way of writing $4 \cdot 4 \cdot 4$ is 4^3, which is read "4 to the third power" or "4 cubed." We call the number 4 the **base** of the expression and the number 3, to the upper right of 4, the **exponent.**

Thus

$$\text{Exponential form} \longrightarrow \underset{\text{Base}}{4}^{\overset{\text{Exponent}}{3}} = \underbrace{4 \cdot 4 \cdot 4}_{\text{Expanded form}} = 64 \longleftarrow \text{Standard form}$$

In like fashion, $3 \cdot 3 \cdot 3 \cdot 3$ may be written 3^4, where 3 is the base and 4 is the exponent. The expression is read "3 to the fourth power." Then

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$$

Notice that *the exponent tells how many times the base is used as a factor in an indicated product.* We call this form of a product the **exponential form.** That is, the exponential form of the product $3 \cdot 3 \cdot 3 \cdot 3$ is 3^4.

Note The exponent is understood to be 1 when a number has no exponent. That is, $5 = 5^1$.

Remember that when we have a negative number, we place it inside parentheses. With this fact in mind, we can see that there is a definite difference between $(-2)^4$, which is read "-2 to the fourth power," and -2^4, which is read "the opposite of 2 to the fourth power." In the first case, the parentheses denote that this is a negative number to a power: $(-2)^4 = (-2)(-2)(-2)(-2) = +16$. In the second case, since there are no parentheses around the number, we understand that this is *not* (-2) to a power. It is, rather, the opposite of the answer when we raise 2^4: $-2^4 = -(2)^4 = -(2 \cdot 2 \cdot 2 \cdot 2) = -(16)$.

Example:

Perform the indicated multiplication.

1. $(-3)^3 = (-3)(-3)(-3) = -27$

2. $-3^3 = -(3 \cdot 3 \cdot 3) = -27$

3. $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

4. $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

► ***Quick check*** -3^2

When we are performing several different types of arithmetic operations within an expression, we need to agree on an order in which the operations will be performed. To show that this is necessary, consider the following numerical expression.

$$3 + 4 \cdot 5 - 3$$

More than one answer is possible, depending on the order in which we perform the operations. To illustrate,

$$3 + 4 \cdot 5 - 3 = 7 \cdot 2 = 14$$

if we add and subtract as indicated before we multiply. However

$$3 + 4 \cdot 5 - 3 = 3 + 20 - 3 = 20^{*}$$

if we multiply before we add or subtract. A third possibility would be

$$3 + 4 \cdot 5 - 3 = 3 + 4 \cdot 2 = 3 + 8 = 11$$

if we subtract, then multiply, and finally add. To standardize the answer, we agree to the following order of operations, or priorities.

Order of operations, or priorities

1. **Groups:** Perform any operations within a grouping symbol such as ( ) parentheses, [ ] brackets, { } braces, | | absolute value, and above or below the fraction bar.
2. **Exponents:** Perform operations indicated by exponents.
3. **Multiply and divide:** Perform multiplication and division in order from left to right.
4. **Add and subtract:** Perform addition and subtraction in order from left to right.

Note

a. Within a grouping symbol, the order of operations will still apply.
b. If there are several grouping symbols intermixed, remove them by starting with the innermost one and working outward.

*This is the correct answer.

To illustrate this order, consider the numerical expression

$$6 + 5(7 - 3) - 2^2$$

We first evaluate within the grouping symbol, in this case parentheses, to get

$$6 + 5(4) - 2^2$$

We then perform the indicated power and have

$$6 + 5(4) - 4$$

Our third step is to carry out the multiplication, resulting in

$$6 + 20 - 4$$

Our last step is to perform the addition and subtraction in order from left to right, giving

$$\begin{aligned} &26 - 4 \\ &= 22 \end{aligned}$$

Examples:

Perform the indicated operations in the proper order and simplify.

1. $7 + 8 \cdot 3 \div 2 = 7 + 24 \div 2$ — Priority 3, multiply

$= 7 + 12$ — Priority 3, divide

$= 19$ — Priority 4, add

2. $(7 - 1) \div 2 + 3 \cdot 4 = 6 \div 2 + 3 \cdot 4$ — Priority 1, parentheses

$= 3 + 12$ — Priority 3, divide and multiply

$= 15$ — Priority 4, add

3. $\frac{1}{2} + \frac{3}{4} \div \frac{5}{8} = \frac{1}{2} + \frac{3}{\underset{1}{\cancel{4}}} \cdot \frac{\overset{2}{\cancel{8}}}{5}$ — Priority 3, invert, divide out common factors

$= \frac{1}{2} + \frac{6}{5}$ — Priority 3, multiply

$= \frac{5}{10} + \frac{12}{10}$ — Least common denominator

$= \frac{5 + 12}{10}$ — Priority 4, add

$= \frac{17}{10}$ or $1\frac{7}{10}$ — Priority 4, add

4. $2^2 \cdot 3 - 3 \cdot 4 = 4 \cdot 3 - 3 \cdot 4$ Priority 2, exponent
$= 12 - 12$ Priority 3, multiply
$= 0$ Priority 4, subtract

5. $\frac{3}{4} - \frac{1}{2} \cdot \frac{2}{3} = \frac{3}{4} - \frac{1}{\cancel{2}_1} \cdot \frac{\cancel{2}^1}{3}$ Priority 3, divide out common factors

$= \frac{3}{4} - \frac{1}{3}$ Priority 3, multiply

$= \frac{9}{12} - \frac{4}{12}$ Least common denominator

$= \frac{9 - 4}{12}$ Priority 4, subtract

$= \frac{5}{12}$ Priority 4, subtract

6. $(7.28 + 1.6) \div 2.4 - (6.1)(3.8)$
$= (8.88) \div 2.4 - (6.1)(3.8)$ Priority 1, parentheses
$= 3.7 - 23.18$ Priority 3, division and multiplication
$= -19.48$ Priority 4, subtract

7. $\left(\frac{2}{3} + \frac{7}{8}\right) \div \frac{5}{6} = \left(\frac{16}{24} + \frac{21}{24}\right) \div \frac{5}{6}$ Priority 1, parentheses

$= \left(\frac{16 + 21}{24}\right) \div \frac{5}{6}$ Priority 1, parentheses

$= \frac{37}{24} \div \frac{5}{6}$ Priority 1, parentheses

$= \frac{37}{\cancel{24}_4} \cdot \frac{\cancel{6}^1}{5}$ Priority 3, invert, divide out common factors

$= \frac{37}{20}$ or $1\frac{17}{20}$ Priority 3, multiply

8. $(5.4)^2 - 4(3.1)(2.8)$
$= 29.16 - 4(3.1)(2.8)$ Priority 2, exponent
$= 29.16 - 34.72$ Priority 3, multiply
$= -5.56$ Priority 4, subtract

9. $\dfrac{3(2+4)}{4-2} - \dfrac{4+6}{5} = \dfrac{3(6)}{4-2} - \dfrac{4+6}{5}$ Priority 1, groups: numerator and denominator

$= \dfrac{18}{2} - \dfrac{10}{5}$ Priority 1, numerator and denominator

$= 9 - 2$ Priority 3, divide

$= 7$ Priority 4, subtract

10. $5[7 + 3(10 - 4)]$

We first evaluate within the grouping symbol, applying the order of operations.

$5[7 + 3(6)] = 5[7 + 18]$ Priority 1, groups

$= 5[25]$ Priority 1, groups

$= 125$ Priority 3, multiply

More Examples of Different Grouping Symbols:

If there are several parentheses in a problem we will start with the inner most parenthesis and work our way out. Inside each parenthesis we simplify using the order of operations as well. To make it easier to know which parenthesis goes with which parenthesis, different types of parenthesis will be used such as { } and [ ] and ( ), these parenthesis all mean the same thing, they are parenthesis and must be evaluated first

$2\{8^2 - 7[32 - 4(3^2 + 1)](-1)\}$	**Inner most parenthesis, exponents first**
$2\{8^2 - 7[32 - 4(9 + 1)](-1)\}$	**Add inside those parenthesis**
$2\{8^2 - 7[32 - 4(10)](-1)\}$	**Multiply inside inner most parenthesis**
$2\{8^2 - 7[32 - 40](-1)\}$	**Subtract inside those parenthesis**
$2\{8^2 - 7[-8](-1)\}$	**Exponents next**
$2\{64 - 7[-8](-1)\}$	**Multiply left to right, sign with the number**
$2\{64 + 56(-1)\}$	**Finish multiplying**
$2\{64 - 56\}$	**Subtract inside parenthesis**
$2\{8\}$	**Multiply**
16	**Our Solution**

There are several types of grouping symbols that can be used besides parenthesis.
One type is a fraction bar. If we have a fraction, the entire numerator and the entire denominator must be evaluated before we reduce the fraction. In these cases we can simplify in both the numerator and denominator at the same time.

$$\frac{2^4 - (-8) \cdot 3}{15 \div 5 - 1} \quad \textbf{Exponent in the numerator, divide in denominator}$$

$$\frac{16 - (-8) \cdot 3}{3 - 1} \quad \textbf{Multiply in the numerator, subtract in denominator}$$

$$\frac{16 - (-24)}{2} \quad \textbf{Add the opposite to simplify numerator, denominator is done.}$$

$$\frac{40}{2} \quad \textbf{Reduce, divide}$$

$$20 \quad \textbf{Our Solution}$$

Another type of grouping symbol that also has an operation with it is absolute value. When we have absolute value we will evaluate everything inside the absolute value, just as if it were a normal parenthesis. Then once the inside is completed we will take the absolute value, or distance from zero, to make the number positive.

$$\begin{array}{rl}
1 + 3|-4^2 - (-8)| + 2|3 + (-5)^2| & \text{Evaluate absolute values first, exponents} \\
1 + 3|-16 - (-8)| + 2|3 + 25| & \text{Add inside absolute values} \\
1 + 3|-8| + 2|28| & \text{Evaluate absolute values} \\
1 + 3(8) + 2(28) & \text{Multiply left to right} \\
1 + 24 + 2(28) & \text{Finish multiplying} \\
1 + 24 + 56 & \text{Add left to right} \\
25 + 56 & \text{Add} \\
81 & \text{Our Solution}
\end{array}$$

The above example also illustrates an important point about exponents. Exponents only are considered to be on the number they are attached to. This means when we see $\mathbf{-4^2}$, only the 4 is squared, giving us $-(\mathbf{4^2})$ or $\mathbf{-16}$. But when the negative is in parentheses, such as $\mathbf{(-5)^2}$ the negative is part of the number and is also squared giving us a positive solution, **25**.

Exercises:
SET I

Perform the indicated operations. See example

Example -3^2

Solution $= -(3^2)$ $\quad$ -3^2 is not the same as $(-3)^2$, it is the opposite of 3^2

$= -9$

1. $(-4)^2$ $\quad$ $\boxed{2.}$ $(-5)^4$ $\quad$ $\boxed{3.}$ $(-3)^3$ $\quad$ 4. -4^2 $\quad$ 5. -6^2

6. -2^4 $\quad$ 7. -1^2 $\quad$ $\boxed{8.}$ -2^2 $\quad$ 9. $(-1)^2$ $\quad$ 10. $(-2)^2$

Perform the indicated operations and simplify. See the first example

Example $18 \div 6 \cdot 3 + 10 - (4 + 5)$

Solution

$= 18 \div 6 \cdot 3 + 10 - 9$	Priority 1, parentheses
$= 3 \cdot 3 + 10 - 9$	Priority 3, division
$= 9 + 10 - 9$	Priority 3, multiplication
$= 19 - 9$	Priority 4, addition
$= 10$	Priority 4, subtraction

11. $\dfrac{4+2}{3} + 2$ $\quad$ 12. $-6 \cdot 7 + 8$ $\quad$ 13. $6 + 5 \cdot 4$

14. $\dfrac{1}{5} \cdot 5 + 6$ $\quad$ 15. $-2 + 10 \cdot \dfrac{1}{5}$ $\quad$ 16. $4(3 - 2)(2 + 1)$

$\boxed{17.}$ $0(5 + 2) + 3$ $\quad$ 18. $\dfrac{24 \cdot 3}{9} - 6$ $\quad$ 19. $(24 - 6) \div 3$

20. $(37 - 4) \div 11$ $\quad$ 21. $\dfrac{2}{3} \div \left(\dfrac{5}{6} - \dfrac{4}{9}\right)$ $\quad$ 22. $12 \cdot 4 + 2$

23. $2 + 3(8 - 5)$ $\quad$ 24. $5 + 2(11 - 6)$ $\quad$ 25. $6 + 4(8 + 2)$

26. $7 + 3(9 - 4)$ $\quad$ 27. $8 - 3(6 - 4)$ $\quad$ 28. $10 - 2(7 - 11)$

29. $15 \cdot 3^2 - 14$ $\quad$ 30. $(8 - 3)(5 + 3)$ $\quad$ 31. $\dfrac{7}{8} - \dfrac{1}{2} \div \dfrac{3}{4}$

32. $\dfrac{3}{8} + \dfrac{7}{12} \cdot \dfrac{3}{14}$ $\quad$ 33. $3(6 - 2)(7 + 1)$ $\quad$ 34. $12 + 3 \cdot 16 \div 4^2 - 2$

35. $9 - 3(12 + 3) - 4 \cdot 3$ $\quad$ 36. $15 - 2(8 + 1) - 6 \cdot 4$ $\quad$ 37. $50 - 4(6 - 8) + 5 \cdot 4$

38. $18 - 5(7 + 3) - 6$ $\quad$ 39. $10 - 3 \cdot 4 \div 6 - 5$ $\quad$ 40. $8 - (12 + 3) - 4 \cdot 3$

$\boxed{41.}$ $4(2 - 5)^2 - 2(3 - 4)$ $\quad$ 42. $6(-8 + 10) - 5(4 - 7)$ $\quad$ $\boxed{43.}$ $\dfrac{5(3 - 5)}{2} - \dfrac{27}{-3}$

44. $\dfrac{3(8 - 6)}{2} - \dfrac{8}{-2}$ $\quad$ 45. $\dfrac{5(6 - 3)}{3} - \dfrac{(-14)}{2}$

46. $(14.13 + 11.4) \div 3.7 - (2.4)(7.8)$ $\quad$ 47. $(5.1 + 2.2)(4.8) - (6.3)(8.1)$

48. $(5.1)^2 \cdot 3 - (14.64) \div (6.1)$ $\quad$ 49. $(1.9)^2 + 4(3.3)^2 - 8.7$

$\boxed{50.}$ $5[10 - 2(4 - 3) + 1]$ $\quad$ 51. $18 + [14 - 5(6 - 4) + 7]$

52. $(8 - 2)[16 + 4(5 - 7)]$ $\quad$ 53. $(9 - 6)[21 + 5(4 - 6)]$

$\boxed{54.}$ $\left(\dfrac{6 - 3}{7 - 4}\right)\left(\dfrac{14 + 2 \cdot 3}{5}\right)$ $\quad$ 55. $\left(\dfrac{3}{12} - \dfrac{1}{6}\right)\left(\dfrac{2}{3} + \dfrac{1}{8}\right)$

56. $\left(\dfrac{1}{4} - \dfrac{1}{6}\right) \div \left(\dfrac{2}{3} - \dfrac{1}{8}\right)$

SET II

Solve.

1) $-6\cdot 4(-1)$

2) $(-6\div 6)^3$

3) $3+(8)\div|4|$

4) $5(-5+6)\cdot 6^2$

5) $8\div 4\cdot 2$

6) $7-5+6$

7) $[-9-(2-5)]\div(-6)$

8) $(-2\cdot 2^3\cdot 2)\div(-4)$

9) $-6+(-3-3)^2\div|3|$

10) $(-7-5)\div[-2-2-(-6)]$

11) $4-2|3^2-16|$

12) $\frac{-10-6}{(-2)^2}-5$

13) $[-1-(-5)]|3+2|$

14) $-3-\{3-[-3(2+4)-(-2)]\}$

15) $\frac{2+4|7+2^2|}{4\cdot 2+5\cdot 3}$

16) $-4-[2+4(-6)-4-|2^2-5\cdot 2|]$

17) $[6\cdot 2+2-(-6)](-5+\left|\frac{-18}{6}\right|)$

18) $2\cdot(-3)+3-6[-2-(-1-3)]$

19) $\frac{-13-2}{2-(-1)^3+(-6)-[-1-(-3)]}$

20) $\frac{-5^2+(-5)^2}{|4^2-2^5|-2\cdot 3}$

21) $6\cdot\frac{-8-4+(-4)-[-4-(-3)]}{(4^2+3^2)\div 5}$

22) $\frac{-9\cdot 2-(3-6)}{1-(-2+1)-(-3)}$

23) $\frac{2^3+4}{-18-6+(-4)-[-5(-1)(-5)]}$

24) $\frac{13+(-3)^2+4(-3)+1-[-10-(-6)]}{\{[4+5]\div[4^2-3^2(4-3)-8]\}+12}$

25) $\frac{5+3^2-24\div 6\cdot 2}{[5+3(2^2-5)]+|2^2-5|^2}$