Workshop 20 November 8, 2011

1. Use an appropriate series to estimate the value of $\sin(1)$ to within 10^{-3}.

2. Use an appropriate series to estimate the value of $\sqrt{17}$ to within 10^{-3}.

3. Find the Maclaurin series for $f(x) = \dfrac{1}{1+x^2}$. What is its interval of convergence? What is the Maclaurin series for $\arctan x$? What is its interval of convergence?

4. Consider the function $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin(k^3 x)}{k^2}$. What is the interval of convergence for this series? Take the term-by-term derivative of the series; what is the interval of convergence of the result?

5. Determine the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n4^n}$.

6. Use the definition to find the Maclaurin series for $f(x) = 2^x$. What is its interval of convergence? (How could you find this series without using the definition?)

7. Write out the first 5 (or 6) terms of the Maclaurin series for $f(x) = \sqrt{1+x}$. Use this to write down the first 5 (or 6) terms of the Maclaurin series for $[f(x)]^2$. (What *should* you end up with?)

8. Euler did most of his calculus without the level of rigor we now use. One of his computations proceeded thusly:

$$\begin{aligned}
\frac{x}{x-1} + \frac{x}{1-x} &= 0 \\
\frac{1}{1-1/x} + \frac{x}{1-x} &= 0 \\
\sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n + x \sum_{k=0}^{\infty} x^n &= 0 \\
\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + \cdots &= 0.
\end{aligned}$$

Choose x to be any positive number to see that this is absurd. What went wrong? (Note that he hasn't used any limits for the sums, which not good; but there's something even worse here.)

9. Compute $\displaystyle\lim_{x\to 0} \frac{e^{-x^2} - \cos x + \frac{x^2}{2}}{e^{x^4} - 1}$.

10. If $a_n > 0$ and $\sum_{n=0}^{\infty} a_n$ converges, can you say whether $\sum_{n=0}^{\infty} a_n^2$ converges or diverges? Justify your answer.

11. A common probability distribution is the *Poisson distribution.* It is a good approximation to almost anything random that can be counted, for example the number of webpage hits during a day. This distribution is discrete but not finite: the possible outcomes are all nonnegative integers. Specifically, the probability of having exactly k occurrences is given by $\dfrac{\lambda^k e^{-\lambda}}{k!}$, where λ is a parameter depending on the specific situation.

    (a) Show that this is really a probability distribution, i.e. show that $\displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = 1.$

    (b) Find the expected number of occurrences by computing $\displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \cdot k.$

12. Consider the power series $\sum_{n=0}^{\infty} c_n x^n$, where $c_n = 3$ if n is even and $c_n = 5$ if n is odd. What is the interval of convergence for this series? (Why does your usual approach fail? What else can you try?)

13. If $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r (with $0 < r < \infty$), what is the radius of convergence of $\displaystyle\sum_{k=0}^{\infty} a_k (x-c)^{2k}$, where c is any constant?

14. If a power series centered at $x = 3$ is known to converge at $x = 5$ and diverges at $x = -1$, what can be said about the radius of convergence? Is it possible that the a power series centered at $x = 3$ converges at $x = -1$ and diverges at $x = 5$?

15. *Remember Matrices?* We knew, pre-calculus, how to raise e to rational number exponents. Using Calc-1 techniques we can make sense of raising e to real number exponents. We saw last time how to raise e to complex number exponents. Sometimes it turns out to be very useful to be able to raise e to matrix exponents (!). We do it just the same way as last time:

    $$\exp\{A\} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

    (We've used "exp" just so we don't have to write matrices in exponents.) Of course we need to know how to get A^n, but this is just repeated matrix multiplication; and we need to know what it means to have a limit of matrices, but this works pretty much how you would expect. First, compute $\exp\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ (hint: the powers of this matrix are very easy). Next, mimic your work from last time on e^{ix} to evaluate $\exp\left\{x\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$.