## HOMEWORK 10: §9.1-9.3, 9.7, 9.9 DUE APRIL 6

Name: ____________________

- Please refer to the syllabus regarding allowed collaboration on this homework assignment.
- All answers should be fully justified.
- Your homework should be neatly written on additional paper; you may attach this cover page if you would like to keep the questions attached to the answers.

(1) *Maximal vs. maximum.*
An element $m$ of a poset $(P, \preceq)$ is called **maximum** if $m \succeq x$ for every $x \in P$. Note that a maximum element must also be maximal, but the converse is not necessarily true.
  (a) Prove that no poset can have two different maximum elements. *(Start with two maximum elements, then use the properties of posets to prove that the two elements are actually equal.)*
  (b) Prove that if a poset has two different maximal elements, then it has no maximum element. *(Proof by contradiction?)*

Given the note at the beginning of the problem and the result of 1b, it is fairly natural to make the following conjecture: "If a poset has a unique maximal element $x$, then $x$ is a maximum element."
  (c) Prove that the above conjecture holds for all *finite* posets. *(You will probably want to use #1 and #2 from Workshop 20. There is an outline for this proof on the next page, but I encourage you to think for a little while before viewing it.)*
  (d) Provide a counterexample to the conjecture. *(By 1c, your example will have to be infinite. Your proof in 1c can help you to build an example; #3 from Workshop 20 may also help.)*

(2) Each of the following relations is defined on $\mathbb{Z}^+$. For each, determine which of the five properties ((anti)reflexive, (anti)symmetric, transitive) are satisfied. If the relation is an equivalence relation, list the first six elements in [2] and in [6]. If the relation is a partial order, find the minimal and maximal elements. *(In the last two parts, the set of prime factors does not count multiplicity. For example, the set of prime factors of 24 is $\{2, 3\}$.)*
  (a) $xRy$ iff $x$ has a prime factor that is also a factor of $y$
  (b) $xRy$ iff every prime factor of $x$ is also a factor of $y$
  (c) $xRy$ iff $y = 2^n x$ for some $n \in \mathbb{N}$
  (d) $xRy$ iff the sets of their prime factors are equal
  (e) $xRy$ iff the sets of their prime factors have the same cardinality

(3) Draw the Hasse diagram for the divisibility relation restricted to the set $\{1, 2, 3, 5, 6, 14, 30\}$.

"To doubt everything or to believe everything are two equally convenient solutions; both dispense with the necessity of reflection."

*H. Poincaré*

Outline for 1c: Start a proof by contradiction, consider the set $Z = \{z : z||x\}$, and use the results of #1 and #2 from Workshop 20. Finally, take cases for how the maximal element from $Z$ compares to $x$, in each case finding a contradiction.)