Math 241 X8 Name(s): Solutions

Homework 7 supplement

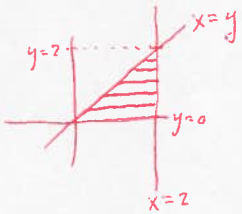
This is a written homework supplement to the homework for Unit 7: 2D Integrals.

Someone set up the following integrals, but chose a poor method to do it. By interpreting the integrals as integrals over a region, change methods to one requiring least effort from you. Then compute the integrals. (Remember $\cos^2 t = \frac{1}{2}(1+\cos(2t))$ and $\sin^2 t = \frac{1}{2}(1-\cos(2t))$.)

(1) $\displaystyle\int_0^2 \int_y^2 e^{x^2}\,dx\,dy$

$y \le x \le 2$

$0 \le y \le 2$



Slice vertically instead

$0 \le y \le x$

$0 \le x \le 2$

$$\int_0^2 \int_0^x e^{x^2}\,dy\,dx$$

$$= \int_0^2 \left[y e^{x^2} \right]_{y=0}^{x} dx$$

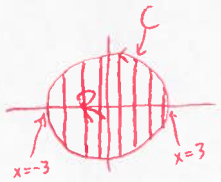
$$= \int_0^2 \left(x e^{x^2} - 0 \right) dx \qquad u = x^2,\ du = 2x\,dx$$

$$= \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}\left[e^u\right]_0^4 = \frac{1}{2}\left(e^4 - 1\right).$$

(2) $\displaystyle\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y^2\,dy\,dx$

$-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

$\left(y = \pm\sqrt{9-x^2} \iff y^2 = 9-x^2 \iff x^2+y^2 = 9 \right)$



Use Gauss-Green ($\iint_R (\partial_x n - \partial_y m)\,dA = \int_C m\,dx + n\,dy$)

with $m = 0,\quad n = xy^2$

(so $y^2 = \partial_x n - \partial_y m$)

$$\Rightarrow \int_C 0\,dx + xy^2\,dy$$

parametrize C as $x = 3\cos t$, $y = 3\sin t$, $t \in [0, 2\pi]$ (counterclockwise ✓)

$$= \int_0^{2\pi} (3\cos t)(3\sin t)^2 (3\cos t\,dt)$$

$$= 81\int_0^{2\pi} \cos^2 t \sin^2 t\,dt = \frac{81}{4}\int_0^{2\pi} (1+\cos(2t))(1-\cos(2t))\,dt = \frac{81}{4}\int_0^{2\pi} \left(1-\cos^2(2t)\right) dt$$

$$= \frac{81}{4}\int_0^{2\pi} \left(1 - \frac{1}{2}(1+\cos(4t))\right) dt = \frac{81}{4}\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(4t)\right) dt = \frac{81\pi}{4} + 0$$