Prequel to Sect 1.5 Power Laws

Section 1 POWERS (Integer Exponents)

In maths we sometimes like to find shorthand ways of writing things. One such shorthand we use is powers. It is easier to write 2^3 than $2 \times 2 \times 2$. The cubed sign tells us to take the number and multiply it by itself 3 times. The 3 is called the index. Then 10^6 means multiply 10 by itself 6 times. This means:

$$10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

We can do calculations with this shorthand. Look at this calculation:

$$3^2 \times 3^3 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$$

because 3 is now being multiplied by itself 5 times. So we could have just written $3^3 \times 3^2 = 3^5$. The more general rule is

$$x^a \times x^b = x^{a+b}$$

where x, a and b are any numbers.

We add the indices when we multiply two powers of the same number.

Example 1 :

$$5^6 \times 5 = 5^7$$

Note that $5 = 5^1$.

Example 2 :

$$x^3 \times x^b = x^{3+b}$$

Example 3 :

$$3^3 \times 3^0 = 3^3$$

so that 3^0 must be equal to 1. Indeed, for any non zero number x, $x^0 = 1$.

$$\boxed{x^0 = 1 \qquad \text{if } x \neq 0}$$

We can only use this trick if we are multiplying powers of the same number. Notice that we can't use this rule to simplify $5^3 \times 8^4$, as the numbers 5 and 8 are different.

This shorthand in powers gives us a way of writing $(3^2)^3$. In words, $(3^2)^3$ means: take 3, multiply it by itself, then take the result, and multiply that by itself 3 times. Then

$$(3^2)^3 = (3 \times 3)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6$$

The general form of the rule in multiplying powers is

$$\boxed{(x^a)^b = x^{a \times b}}$$

Example 4 :

$$\begin{aligned} (5^2)^4 &= 5^8 \\ (x^3)^b &= x^{3 \times b} \end{aligned}$$

Finally, what happens if we have different numbers raised to powers? Say we have $3^2 \times 5^3$. In this particular case, we would leave it as it is. However, in some cases, we can simplify. One case is when the indices are the same. Consider $3^2 \times 6^2$. Then

$$\begin{aligned} 3^2 \times 6^2 &= 3 \times 3 \times 6 \times 6 \\ &= 3 \times 6 \times 3 \times 6 \\ &= (3 \times 6)^2 \\ &= 18^2 \end{aligned}$$

We can get the second line because multiplication is commutative, which is to say that $a \times b = b \times a$. The general rule then when the indices are the same is

$$\boxed{x^a \times y^a = (x \times y)^a}$$

Example 5 :

$$2^2 \times 3^2 \times 5^2 = (2 \times 3 \times 5)^2 = 30^2$$

Exercises:

1. Simplify the following and leave your answers in index form:

(a) $6^3 \times 6^7$

(b) $4^5 \times 4^2$

(c) $x^7 \times x^9$

(d) $m^4 \times m^3$

(e) $(m^4)^3$

(f) $(8^2)^3$

(g) $5^3 \times 5^9$

(h) $x^6 \times x^{12} \times x^3$

(i) $(x^3)^4 \times x^5$

(j) $m^4 \times (m^5)^2 \times m$

Section 2 Negative Powers

We can write $\frac{1}{x}$ as x^{-1}. That is: $x^{-1} = \frac{1}{x}$. Now we can combine this notation with what we have just learnt.

Example 1 :

$$\begin{aligned} \frac{1}{x \times x \times x \times x} &= \frac{1}{x^4} \\ &= (x^4)^{-1} \\ &= x^{-4} \end{aligned}$$

Example 2 :

$$2^{-3} = (2^3)^{-1} = 8^{-1} = \frac{1}{8}$$

We treat negative indices in calculations in the same manner as positive indices. Then

$$\begin{aligned} x^b \times x^{-a} &= x^{b+(-a)} = x^{b-a} \\ (x^b)^{-a} &= x^{-ab} \\ x^{-n} &= \frac{1}{x^n} \end{aligned}$$

Consider this longhand example:

Example 3 :

$$\begin{aligned} 2^{-3} \times 2^5 &= \frac{1}{2 \times 2 \times 2} \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} \\ &= 2 \times 2 \\ &= 2^2 \end{aligned}$$

whereas our shorthand notation gives: $2^{-3} \times 2^5 = 2^{-3+5} = 2^2$.

This concept may be written in the form of a division.

Example 4 : $x^7 \times \frac{1}{x^6} = x^7 \div x^6$. When we divide two powers of the same number, we subtract the indices. Hence,

$$\boxed{x^m \div x^n = x^{m-n}}$$

So

$$\begin{aligned} x^7 \div x^6 &= x^{7-6} \\ &= x^1 \\ &= x \end{aligned}$$

Example 5 :

$$\begin{aligned} 6^8 \div 6^3 &= 6^{8-3} \\ &= 6^5 \end{aligned}$$

Example 6 :

$$\begin{aligned} m^4 \div m^9 &= m^{4-9} \\ &= m^{-5} \\ &= \frac{1}{m^5} \end{aligned}$$

Example 7 :

$$\begin{aligned} x^8 \div x^{-2} &= x^{8-(-2)} \\ &= x^{8+2} \\ &= x^{10} \end{aligned}$$

Exercises:

1. Simplify the following and leave your answers in index form:
   (a) $6^{-4} \times 6^7$
   (b) $10^8 \times 10^{-5}$
   (c) $x^7 \times x^3$
   (d) $(x^{-2})^3$
   (e) $y^{-12} \times y^5$
   (f) $y^8 \div y^3$
   (g) $7^2 \div 7^{-4}$
   (h) $(m^4)^{-2} \times (m^3)^5$
   (i) $y^6 \times y^{14} \div y^5$
   (j) $(8^3)^4 \div (8^2)^3$

Section 3 FRACTIONAL POWERS

What do we mean by $4^{\frac{1}{2}}$? The notation means that we are looking for a number which, when multiplied by itself, gives 4. Then $4^{\frac{1}{2}} = 2$ because $2 \times 2 = 4$. In general, $x^{\frac{1}{a}}$ is asking us to find a number which, when multiplied by itself a times, gives us x. In the case when the index is $\frac{1}{2}$, as above, we also use the square-root sign: $x^{\frac{1}{2}} = \sqrt{x}$. So $8^{\frac{1}{3}}$ means the number which when multiplied by itself 3 times gives us 8. That is, $8^{\frac{1}{3}}$ is the cube root of 8, and is written as $8^{\frac{1}{3}} = \sqrt[3]{8}$.

$$8^{\frac{1}{3}} = 2 \text{ because } 2 \times 2 \times 2 = 8$$

What about $8^{\frac{2}{3}}$? With our previous rule about powers, we end up with this calculation:

$$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (2)^2 = 4$$

Example 1 :

$$8^{\frac{1}{3}} \times 8^{\frac{2}{3}} = 8^{\frac{1}{3}+\frac{2}{3}} = 8^{\frac{3}{3}} = 8^1 = 8$$

And, if we have $8^{\frac{1}{2}} \times 2^{\frac{1}{2}}$, because the indices are the same, then we can multiply the numbers together. Then

$$8^{\frac{1}{2}} \times 2^{\frac{1}{2}} = (8 \times 2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4$$

Another way of writing this is

$$\sqrt{8} \times \sqrt{2} = \sqrt{8 \times 2} = \sqrt{16} = 4$$

The simplification process can often be taken only so far with simple numbers. Consider

$$5^{\frac{1}{3}} \times 3^{\frac{1}{3}} = (5 \times 3)^{\frac{1}{3}} = 15^{\frac{1}{3}}$$

There is no simpler way of writing $15^{\frac{1}{3}}$, so we leave it how it stands.

Example 2 : Recall that $x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}}$. Then

$$\begin{aligned} 9^{-\frac{1}{2}} &= \frac{1}{9^{\frac{1}{2}}} \\ &= \frac{1}{3} \end{aligned}$$

Exercises:

1. Simplify the following:
   (a) $9^{\frac{1}{2}}$
   (b) $27^{\frac{1}{3}}$
   (c) $16^{\frac{1}{2}}$
   (d) $16^{-\frac{1}{2}}$
   (e) $27^{-\frac{2}{3}}$
2. Rewrite the following in index form:
   (a) $\sqrt{8}$
   (b) $\sqrt[3]{m}$
   (c) $(m^6)^{\frac{1}{2}}$
   (d) $(10^{\frac{1}{2}})^3$
   (e) $(16^{\frac{1}{2}})^{-2}$

Power Laws

1. (a) Express $3 \times 3 \times 3 \times 2 \times 2$ using powers.
   (b) Write 27 in index form using base 3.
   (c) Calculate the following. Which are the same?

      i. $2^2 \times 3^2$
      ii. $2^2 + 3^2$
      iii. $(2+3)^2$
      iv. $(2 \times 3)^2$
      v. $\frac{2^2}{3^2}$
      vi. $(\frac{2}{3})^2$

   (d) Express $\frac{1}{5^2}$ in index form with base 5.
   (e) Express $\frac{1}{27}$ in index form with base 3.
   (f) Express $\sqrt{64}$ in index form with base 64.

2. Simplify the following:

   (a) $2^3 \times 2^4$
   (b) $(3^2)^5$ (leave in index form)
   (c) $12^5 \div 12^7$
   (d) $(2.3)^2(2.3)^{-4}$ (leave in index form)
   (e) $8^{-\frac{1}{3}}$
   (f) $\frac{4^8}{4^{12}} \times 4^{-3}$ (leave in index form)
   (g) $\frac{2^1}{2^{-3}} + (2^2 + 2^1)^2$
   (h) $(0.01)^2$
   (i) $10^5 \div (3^2 \times 10^{-2})^3$ (leave in index form)