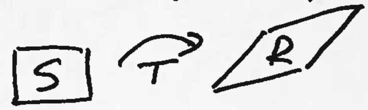
WORKSHEET SOLUTIONS (11/08/12)

① (a) We search for a linear transformation T (because these take lines into lines) with $T(0,0)$, so $T(x,y) = (ax+by, cx+dy)$.

$T(1,0) = (a,c) = (2,1) \Longrightarrow a=2, c=1$



$T(0,1) = (b,d) = (1,3) \Longrightarrow b=1, d=3$

In this case $T(u,v) = (2u+v, u+3v)$ also satisfies $T(1,1) = (3,4)$ and takes the unit square $S = [0,1]^2$ onto the parallelogram with vertices $(0,0) = T(0,0)$, $(2,1) = T(1,0)$, $(1,3) = T(0,1)$, and $(3,4) = T(1,1)$

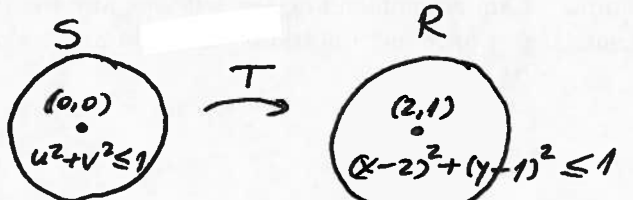
(b) $\left.\begin{array}{l} x = 2u+v \\ y = u+3v \end{array}\right\} \Longrightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$

$T : S \to R$ satisfies the requirements of the Change of Variables Theorem (T is C^1, $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, T is one-to-one) and $f(x,y) = x - 2y$ is continuous, so

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint_{[0,1]^2} (\cancel{2u} + v - \cancel{2u} - 6v)\,du\,dv = -25 \int_0^1 du \int_0^1 v\,dv = -\frac{25}{2}$$

②



$T(u,v) = (u+a, v+b) = (a,b) + (u,v) \Longrightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

(translation)

The area does not change when a translation is applied.

$\left\{\begin{array}{l} x = u+2 \\ y = v+1 \end{array}\right.$ or $T(u,v) = (u+2, v+1)$ takes S onto R and

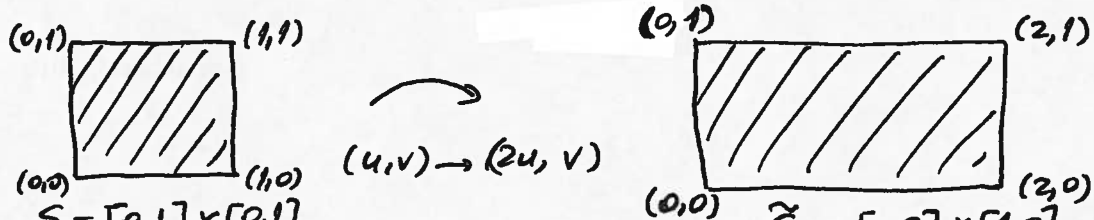
T is one-to-one

$$\iint_R x\,dA = \iint_S (u+2) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \iint_S (u+2)\,du\,dv$$

$$\iint_R x\,dA = \int_0^{2\pi} d\theta\left(\int_0^1 dr\; r(r\cos\theta+2)\right) = \underbrace{\left(\int_0^{2\pi}\cos\theta\, d\theta\right)}_{0}\left(\int_0^1 r^2\,dr\right)$$

$x = r\cos\theta$
$y = r\sin\theta$

$$+\,2\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r\,dr\right) = 4\pi\cdot\frac{1}{2} = 2\pi$$

3) 

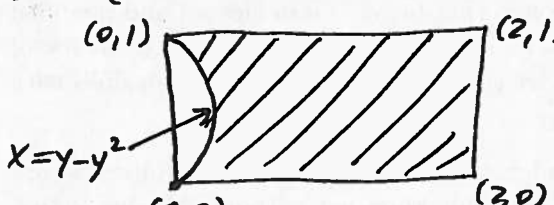
$S = [0,1]\times[0,1]$ $\qquad \tilde{S} = [0,2]\times[1,0]$

$T(u,0) = (2u,0), \quad T(1,v) = (2,v), \quad T(u,1) = (2u,1),$

$T(0,v) = (v-v^2, v), \qquad u,v \in [0,1].$

Now we try to "fill in" between $T(0,v) = (v-v^2, v)$ and $T(1,v) = (2,v)$ with a horizontal line ("linear interpolation") and define $T(t,v) = \big((1-t)(v-v^2)+2t,\ v\big), \quad 0 \le t \le 1.$

$T(u,v) = \big(\underbrace{(1-u)(v-v^2)+2u}_{x},\ \underbrace{v}_{y}\big)$ maps $S = [0,1]^2$ onto R!



T is one-to-one (why?)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -(v-v^2)+2 & 1-u-2v(1-u) \\ 0 & 1 \end{vmatrix} = 2-v+v^2 \neq 0$$

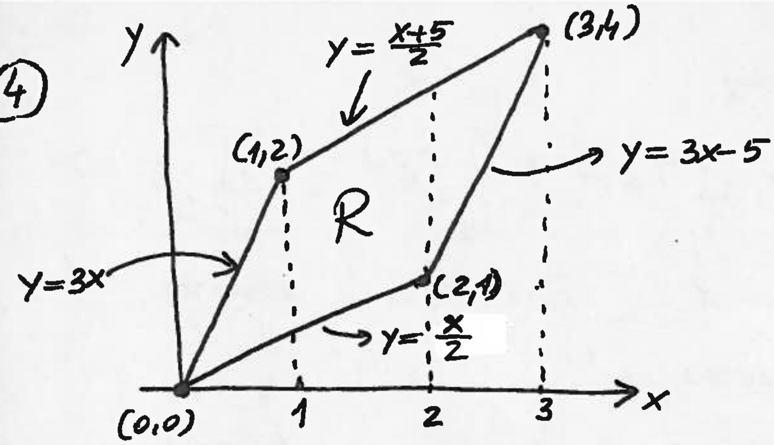
$$\text{Area}(R) = \iint_R 1\,dA = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^1 du \int_0^1 (2-v+v^2)\,dv$$

$$= 2 - \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

A direct computation gives

$$\text{Area}(R) = \int_0^1 dy\left(\int_{y-y^2}^{2} dx\right) = \int_0^1 (2-y+y^2)\,dy = 2-\frac{1}{2}+\frac{1}{3} = \frac{11}{6}.$$

④



$$I = \iint_R (x-2y)\,dA = I_1 + I_2 + I_3 \quad \text{with}$$

$$I_1 = \int_0^1 dx \left(\int_{x/2}^{3x} dy\ (x-2y) \right) = \int_0^1 (xy-y^2)\Big|_{y=\frac{x}{2}}^{3x} dx = \int_0^1 \left(3x^2 - 9x^2 - \frac{x^2}{2} + \frac{x^2}{4}\right) dx$$

$$= -\frac{25}{4} \int_0^1 x^2\,dx = -\frac{25}{12}$$

$$I_2 = \int_1^2 dx \left(\int_{x/2}^{(x+5)/2} dy\ (x-2y) \right) = \int_0^1 (xy-y^2)\Big|_{y=\frac{x}{2}}^{\frac{x+5}{2}} dx$$

$$= \int_1^2 \left(\frac{x^2+5x}{2} - \frac{x^2+10x+25}{4} - \frac{x^2}{2} + \frac{x^2}{4} \right) dx = \int_1^2 \left(-\frac{25}{4}\right) dx = -\frac{25}{4}$$

$$I_3 = \int_2^3 dx \left(\int_{3x-5}^{\frac{x+5}{2}} dy\ (x-2y) \right) = \int_2^3 (xy-y^2)\Big|_{y=3x-5}^{\frac{x+5}{2}} dx$$

$$= \int_2^3 \left(\frac{x^2+\cancel{5x}}{2} - \frac{x^2+\cancel{10x}+25}{4} - 3x^2 + 5x + 9x^2 - 30x + 25 \right) dx$$

$$= \int_2^3 \left(\frac{25}{4}x^2 - 25x + \frac{75}{4} \right) dx = \frac{25}{12}(\underbrace{3^3-2^3}_{19}) - \frac{25}{2}(\underbrace{3^2-2^2}_{5}) + \frac{75}{4}$$

$$= \frac{475}{12} - \overset{6/}{\frac{125}{2}} + \overset{3/}{\frac{75}{4}} = -\frac{50}{12} = -\frac{25}{6}$$

$$I = -\frac{25}{12} - \frac{25}{4} - \frac{25}{6} = -25\left(\overset{1/}{\frac{1}{12}} + \overset{3/}{\frac{1}{4}} + \overset{2/}{\frac{1}{6}} \right) = -25 \cdot \frac{6}{12} = -\frac{25}{2} \quad \checkmark$$