Worksheet 21 April 11, 2011

1. Compute the following integrals:
   (a) $\int (2x+7)^{22}\, dx$
   (b) $\int_1^4 \frac{2x^3}{1+x^4}\, dx$
   (c) $\int_1^4 \frac{2x}{1+x^4}\, dx$
   (d) $\int_0^{\pi^2} t\sin(t^2)\, dt$
   (e) $\int \frac{1}{1+16x^2}\, dx$
   (f) $\int_{-1}^{1} xe^{-x^2}\, dx$
   (g) $\int \frac{1}{x\ln x}\, dx$
2. Define $A(x)$ to be the area beneath the graph of $f(t) = \lfloor t \rfloor$ between $t = 0$ and $t = x$. Find an explicit formula for $A(x)$ for $0 \leq x \leq 6$. (Your formula will need to be piecewise.) Where is this function continuous? Where is it differentiable, and what is $A'(x)$ where it's defined? Notice in particular that $A'(x)$ is *not* just $f(x)$; why doesn't this contradict the FTC?
3. Let $f(x) = x^3$. Is there a theorem that guarantees the existence of a $c \in (1,3)$ such that $f'(c) = 13$? Find all such c.
4. Let $g(x) = \sin x + e^x$. Can you guarantee the existence of a $c \in (0,\pi)$ such that $g'(c) = (e^\pi - 1)/\pi$? Can you find such a c explicitly?
5. Let $h(x) = \sec x$. Can you guarantee the existence of a $c \in (0,\pi)$ such that $h'(c) = 0$?
6. Show that there is exactly one real root of $p(x) = x^5 + 7x^3 + 22x + 13$.
7. Show that there are exactly two real roots of $g(x) = x^6 - 2x^2 - 1$. (Hint: make use of symmetry.)
8. How many real roots does $h(x) = e^{x^2} - \frac{1}{2}$ have? Prove your answer.
9. Find the area bounded by the curve $y = x^2$ and $y = 4$.
10. Find the area bounded by the curve $y = x^2$ and $y = \sqrt{x}$.
11. Find the area bounded by the curves $y = x^3 - x$ and $y = x$. (If you're not careful, the answer won't make any sense!)
12. Compute $\lim_{n\to\infty} \frac{\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n}}{n^{4/3}}$ by evaluating an integral of the form $\int_0^1 f(x)\, dx$.

13. Not every function is integrable. Suppose we want to compute $\int_0^1 \chi(x)\,dx$, where

$$\chi(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1 & \text{if } x \text{ rational.} \end{cases}$$

By choosing appropriate sample points x_i^*, show that the Riemann sums can always be made to be 0, but can also be made to be 1.