Solutions

1. If $f(x) = \sqrt{1-x}$ and $g(x) = \ln(x-1)$, find the domain of the composition $(f \circ g)(x)$.

Solution: For $g(x)$ to make sense, we need $x - 1 > 0$, or $x > 1$. For the radical in f to make sense, we need

$$\begin{aligned} 1 - g(x) &\geq 0 \\ 1 &\geq g(x) \\ 1 &\geq \ln(x-1) \\ e^1 &\geq x - 1 \\ e + 1 &\geq x \end{aligned}$$

Thus the domain of the composition is $(1, e+1]$.

2. (a) State the Intermediate Value Theorem.

Solution: If $f(x)$ is continuous on the closed interval $[a, b]$, then for every y strictly between $f(a)$ and $f(b)$, there is some c in the open interval (a, b) with $f(c) = y$.

(b) Prove that the equation $x^{\frac{3}{2}} = x^{\frac{1}{2}} + 1$ has at least one real solution.

Solution: Let $f(x) = x^{\frac{3}{2}} - x^{\frac{1}{2}} - 1$. Then we want to show that there is some place where $f(x) = 0$. Well, $f(0) = -1$ and $f(4) = 8 - 2 - 1 = 5$, and $-1 < 0 < 5$. So, by the Intermediate Value Theorem, there is some $c \in (0, 4)$ such that $f(c) = 0$. This c is a solution to the original equation.

3. If

$$f(x) = \begin{cases} \ln\left(x^2 - 2x + 4\right) & \text{if } x < 1 \\ C\cos(\pi x) & \text{if } x \geq 1 \end{cases},$$

find the value of C that makes f continuous everywhere.

Solution: First note that f is continuous everywhere except perhaps at the point $x = 1$, since polynomials, logarithms, and trigonometric functions as well as their compositions are continuous on their domains. To decide whether f is continuous at $x = 1$, we need to compute its limit and value there:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \ln\left(x^2 - 2x + 4\right) = \ln(3).$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} C\cos(\pi x) = C \cdot -1 = -C.$$

For the two-sided limit to exist then, we need $-C = \ln(3)$, or $C = -\ln(3)$. Notice that this value of C also makes the value of $f(1)$ equal to the limit.

4. Find the inverse of the function $f(x) = e^{x^3+1}$.

Solution: We want to solve for y in

$$\begin{aligned} x &= e^{y^3+1} \\ \ln(x) &= \ln\left(e^{y^3+1}\right) \\ \ln(x) &= y^3 + 1 \\ \ln(x) - 1 &= y^3 \\ \sqrt[3]{\ln(x) - 1} &= y \end{aligned}$$

That is, $f^{-1}(x) = \sqrt[3]{\ln(x) - 1}$.

5. If the point $(-2, \pi)$ is on the graph of an even function, what other point must be on its graph?

Solution: The graphs of even function have symmetry about the $y - axis$. So the mirror image of this point, $(2, \pi)$, must also be on the graph.

6. If $\cos\theta = 0.8$ and $-\frac{\pi}{2} \leq \theta \leq 0$, compute $\tan\theta$.

Solution: To compute $\tan\theta$, we need to know $\sin\theta$. By the pythagorean identity,

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \sin^2\theta + 0.64 &= 1 \\ \sin^2\theta &= 0.36 \\ \sin\theta &= \pm 0.6 \end{aligned}$$

Since $-\frac{\pi}{2} \leq \theta \leq 0$, $\sin\theta$ must be negative, so $\sin\theta = -0.6$. Then $\tan\theta = -\frac{0.6}{0.8} = -\frac{3}{4}$.
(Remark: you might recognize that these numbers are from the 3-4-5 right triangle.)

7. Evaluate

$$\lim_{\theta \to 0} \theta^2 \cos\left(\frac{e^\theta + 12}{\theta^8}\right).$$

Solution: We could try to evaluate the expression inside the cosine, but the point is that we don't need to. As $\theta \to 0$, $\theta^2 \to 0$ and the cosine cannot get too big to ruin this convergence. Explicitly,

$$-1 \leq \cos\left(\frac{e^\theta + 12}{\theta^8}\right) \leq 1 \quad \Longrightarrow \quad -\theta^2 \leq \theta^2 \cos\left(\frac{e^\theta + 12}{\theta^8}\right) \leq \theta^2$$

and $\lim\limits_{\theta \to 0} -\theta^2 = \lim\limits_{\theta \to 0} \theta^2 = 0$. Hence by the Squeeze Theorem, the limit in question is zero also.

8. Evaluate the following limits.

(a) $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x-1}$

Solution: Note first that this looks like $\frac{0}{0}$. Then with some algebra,

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} = \lim_{x\to 1} x+3 = 4.$$

(b) $\lim\limits_{x\to\infty} \dfrac{2x^2+5}{\sqrt{9x^4+2x+6}}$

Solution: Note that this looks like $\frac{\infty}{\infty}$. Then dividing top and bottom by x^2 (remember that this changes under the radical!),

$$\lim_{x\to\infty} \frac{2x^2+5}{\sqrt{9x^4+2x+6}} = \lim_{x\to\infty} \frac{2+\frac{5}{x^2}}{\sqrt{9+\frac{2}{x^3}+\frac{6}{x^4}}} = \frac{2}{\sqrt{9}} = \frac{2}{3}.$$

(c) $\lim\limits_{x\to\frac{1}{2}^+} \dfrac{\ln x}{2x-1}$

Solution: Note that this looks like $\frac{c}{0}$, and so is infinite. Near $x=\frac{1}{2}$, $\ln x$ is negative. When x is just to the right of $\frac{1}{2}$, $2x-1$ is positive. So the quotient approaches $-\infty$.

(d) $\lim\limits_{x\to\infty} \tan^{-1} x$

Solution: This limit is $\frac{\pi}{2}$.
To get this (without just memorizing it), note that this is equivalent to asking for what values of θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is $\tan\theta$ growing toward ∞. Well, $\tan\theta$ is positive for θ in the first quadrant, and $\cos\theta$ goes to 0 as θ goes to $\frac{\pi}{2}$, so $\tan\theta$ looks like $\frac{1}{0}$, or infinity.

9. Use the limit definition to compute the following derivatives.

(a) $f'(1)$, where $f(x) = x^2 + x + 1$

Solution:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{((1+h)^2 + (1+h) + 1) - (3)}{h} \\ &= \lim_{h\to 0} \frac{1 + 2h + h^2 + 1 + h + 1 - 3}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 3h}{h} \\ &= \lim_{h\to 0} (h+3) \\ &= 3. \end{aligned}$$

(b) $g'(x)$, where $g(x) = \sqrt{x+3}$

Solution:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}} \\ &= \lim_{h\to 0} \frac{(x+h+3) - (x+3)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}} \\ &= \frac{1}{2\sqrt{x+3}}. \end{aligned}$$

(c) $h'(0)$, where $h(\theta) = \sin\theta$.

Solution:

$$\begin{aligned} h'(0) &= \lim_{a\to 0} \frac{h(0+a) - h(0)}{a} \\ &= \lim_{a\to 0} \frac{\sin(0+a) - \sin 0}{a} \\ &= \lim_{a\to 0} \frac{\sin a}{a} \\ &= 1. \end{aligned}$$