Worksheet 13 March 7, 2011

1. Solve the differential equation $\dfrac{dy}{dx} = 2y$, $y(0) = 5$.

2. Consider the curve defined by $x^4 + y^4 = 16$.

   (a) Compute $\dfrac{dy}{dx}$ implicitly.

   (b) Can you find the equation of the tangent line at $(0, 2)$? At $(2, 0)$? At $(\sqrt[4]{15}, 1)$? At $(\sqrt{2}, \sqrt{2})$? Do so or explain why not for each of these.

3. Scientists use a technique called *carbon dating* to determine how old fossils are. Every living thing has a fixed proportion of a particular radioactive isotope of carbon (called C-14) while it's alive. Once it dies, that radioactive carbon starts decaying exponentially with a *half life* of about 5500 years. (This means that after 5500 years, half of the carbon has decayed.)

   (a) After 11000 years, how much of the carbon has decayed? How much is left?

   (b) Keeping in mind the general formula for exponential functions, Ce^{kt}, write an equation that captures the statement "after 5500 years, half of the carbon has decayed."

   (c) Solve that equation for whatever variable is left.

   (d) Suppose an organism started with 1 gram of C-14 when it died, and it's measured to have 0.1 grams now. How long ago did it die?

4. A pirate fires a cannonball with initial vertical speed 15 meters/second (forget horizontal motion for this problem). The acceleration due to gravity is about -10 meters/second/second (assume it's constant; this is reasonable for most projectile motion problems).

   (a) Find a formula for the vertical velocity of the cannonball after t seconds.

   (b) Find a formula for the vertical position of the cannonball after t seconds. (The initial position can be assumed to be the origin.)

   (c) When does the cannonball reach its highest point? How high is it?

5. A metal disk is heating in the sun. As it heats, it expands. If the radius is expanding at a rate of 1cm/hr, how fast is the area increasing? (Hint: start with a formula relating radius and area; note that both of these are functions of time in this situation, so implicit differentiation is more complicated this time.)

6. Consider the cubic function $f(x) = x(x - 2)(x - 6)$. What roots does it have? Choose any two roots and take thier average; then find the tangent line to f at that point. Where does the tangent line meet the x-axis?

7. Find a point where the curves $y = x^3 - 3x + 4$ and $y = 3x^2 - 3x$ are tangent to each other, i.e. share the same tangent line.

8. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution? (For some insight, consider the graphs of the functions on either side of the equation; for differing values of k, how many solutions *can* the equation have?)

9. Let $f(t) = e^{at}$, where a is some constant. Determine $f^{(n)}(t)$, the nth derivative of f.

10. Let $g(t) = \cos(at)$; what is $g^{(n)}(t)$? (Hint: you'll need some cases.)
11. Compute all the derivatives of $P(x) = x^5 + x^4 + x^3 + x^2 + x + 1$.
12. If $f(x) = x^n$, what are $f^{(n)}(x)$ and $f^{(n+1)}(x)$?