Math 241 C8 **Name:**

Quiz # 7

April 17, 2013

No electronic devices, notes, or interpersonal communication allowed.

Show work to get credit.

Find the flow of $\mathbf{F}(x,y,z) = \langle e^{yz}, \cos z, x^2\rangle$ across the top hemisphere of the unit sphere. Which direction is it? (The surface has equation $z = \sqrt{1-x^2-y^2}$, or $x^2+y^2+z^2=1$ with $z \geq 0$.) Big Hint: that surface isn't too bad, but it behaves badly with the field; can you replace it? Remember $\sin^2 t = (1-\cos(2t))/2$ and $\cos^2 t = (1+\cos(2t))/2$.

$\operatorname{div} F = 0 + 0 + 0 = 0$,

so we can replace the surface by any surface w/ the same boundary.

Use the unit disk, $x^2+y^2 \leq 1$, $z=0$.

Parametrize: $x = r\cos\theta \quad 0 \leq r \leq 1$

$y = r\sin\theta \quad 0 \leq \theta \leq 2\pi$

$z = 0$

$$\vec{dS} = \partial_r\langle x,y,z\rangle \times \partial_\theta\langle x,y,z\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0,0,r\rangle.$$

Note: upward normal

$$\iint_{disk} F\cdot\vec{dS} = \int_0^{2\pi}\int_0^1 \langle e^{0}, \cos(0), (r\cos\theta)^2\rangle \cdot \langle 0,0,r\rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (0+0+r^3\cos^2\theta)\, dr\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{1}{8}\int_0^{2\pi}(1+\cos(2\theta))\,d\theta = \frac{1}{8}\Big[\theta + \frac{1}{2}\sin(2\theta)\Big]_0^{2\pi}$$

$$= \boxed{\frac{\pi}{4}}.$$

This is flow upward across the disk, so flow across the hemisphere is outward ("upwardish")

Math 241 C8 **Name:**

Quiz # 7

April 17, 2013

No electronic devices, notes, or interpersonal communication allowed.

Show work to get credit.

Find the flow of $\mathbf{F}(x,y,z) = \langle e^{yz}, \cos z, x^2\rangle$ across the top hemisphere of the unit sphere. Which direction is it? (The surface has equation $z = \sqrt{1-x^2-y^2}$, or $x^2+y^2+z^2=1$ with $z \geq 0$.) Big Hint: that surface isn't too bad, but it behaves badly with the field; can you replace it? Remember $\sin^2 t = (1-\cos(2t))/2$ and $\cos^2 t = (1+\cos(2t))/2$.

Bad (direct) attempts:

1) $x = r\cos\theta$ $\quad 0 \le r \le 1$

$y = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$z = \sqrt{1-r^2}$

$$\vec{dS} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\theta & \sin\theta & \frac{r}{\sqrt{1-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \underbrace{\left\langle -\frac{r^2\cos\theta}{\sqrt{1-r^2}}, \frac{r^2\sin\theta}{\sqrt{1-r^2}}, r\right\rangle}$$

$$\int_0^{2\pi}\int_0^1 \langle e^{r\sqrt{1-r^2}\sin\theta}, \cos(\sqrt{1-r^2}), r^2\cos^2\theta\rangle \cdot \quad dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(-\frac{r^2\cos\theta}{\sqrt{1-r^2}} e^{r\sqrt{1-r^2}\sin\theta} + \frac{r^2\sin\theta}{\sqrt{1-r^2}}\cos(\sqrt{1-r^2}) + r^3\cos^2\theta\right) dr\,d\theta$$

eek!

2) $x = \sin\varphi\cos\theta$ $\quad 0 \le \varphi \le \frac{\pi}{2}$

$y = \sin\varphi\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$z = \cos\varphi$

$$\vec{dS} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi\rangle$$

eek!