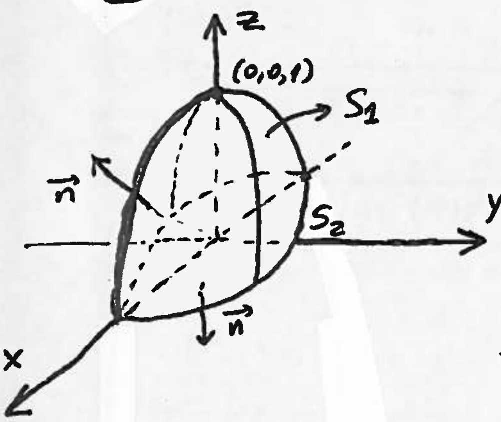
WORKSHEET SOLUTIONS (11/27/12)

① (a) $z = 1-(x^2+y^2)$ Elliptic paraboloid



(b) $S_1 : \begin{cases} x = u \\ y = v \\ z = 1-u^2-v^2 \geq 0, \quad u^2+v^2 \leq 1 \end{cases}$

$$\vec{r} : D = \{(u,v) \mid u^2+v^2 \leq 1\} \longrightarrow \mathbb{R}^3$$

$$\vec{r}(u,v) = (u, v, 1-u^2-v^2).$$

$$\begin{array}{rl} & \quad \vec{i} \quad \vec{j} \quad \vec{k} \\ \vec{r}_u = & \langle 1, \ 0, \ -2u\rangle \end{array}$$

$$\vec{r}_v = \langle 0, 1, -2v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1\rangle$$

$\vec{n}$ points outward (upward in this case) $\longrightarrow$ can take $\vec{n}(u,v,1-u^2-v^2) = \dfrac{\langle 2u, 2v, 1\rangle}{\sqrt{1+4(u^2+v^2)}}$.

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA = \iint_{u^2+v^2 \leq 1} \langle 0,0,1-u^2-v^2\rangle \cdot \langle 2u, 2v, 1\rangle \, dA$$

$$= \iint_{u^2+v^2\leq 1} (1-u^2-v^2)\, dA \underset{\substack{\uparrow \\ \text{Polar coordinates}}}{=} \int_0^{2\pi} d\theta \int_0^1 dr \ r\cdot(1-r^2)\, dr$$

$$= 2\pi \int_0^1 (r-r^3)\, dr = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}.$$

(c) $S_2 : \begin{cases} x = u \\ y = v \\ z = 0 \end{cases} \quad u^2+v^2 \leq 1$

$$\begin{array}{rl} & \ \vec{i} \ \vec{j} \ \vec{k} \\ \vec{r}_u = & \langle 1,0,0\rangle \end{array}$$

$$\vec{r}_v = \langle 0,1,0\rangle$$

$\vec{r}_u \times \vec{r}_v = \langle 0,0,1\rangle$ points upward

The flux through S_2 with downward pointing normal is given by

$$\iint_{S_2} \vec{F} \cdot dS = \iint_{u^2+v^2\leq 1} \underbrace{\langle 0,0,0\rangle \cdot (-\langle 0,0,1\rangle)}_{0}\, dA = 0.$$

(d) Outward pointing normals on $\partial D = S_1 \cup S_2$ = upward pointing normal on S_1 & downward pointing normal on S_2

$$\Longrightarrow \iint_{\partial D} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot dS = \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

(e) $\vec{F} = \langle 0,0,z\rangle \Rightarrow \mathrm{div}(\vec{F}) = \frac{\partial z}{\partial z} = 1$

Divergence Theorem $\Rightarrow \iint\limits_{\partial D} \vec{F}\cdot d\vec{S} = \iiint\limits_{D} 1\, dV = Vol(D) = \frac{\pi}{2}$

② $\vec{F} = \langle -y, x, z\rangle$

(a) $\mathrm{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \left\langle \frac{\partial z}{\partial y} - \frac{\partial x}{\partial z}, -\frac{\partial z}{\partial x} + \frac{\partial y}{\partial z}, \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right\rangle$

$= \langle 0,0,2\rangle$

(b) $\iint\limits_{S_1} \mathrm{curl}(\vec{F})\cdot\vec{n}\, dS = \iint\limits_{S_1} \langle 0,0,2\rangle \cdot \frac{\langle 2u,2v,1\rangle}{\sqrt{1+4(u^2+v^2)}}\, dS$

$= \iint\limits_{D} \langle 0,0,2\rangle \cdot \frac{\langle 2u,2v,1\rangle}{\cancel{\sqrt{1+4(u^2+v^2)}}} \cancel{\sqrt{1+4(u^2+v^2)}}\, dA$

$= 2\iint\limits_{D} dA = 2\, Area(D) = 2\pi.$

③ (a) $Vol(E) = \iint\limits_{D} (1-x^2-y^2)\, dA = \iint\limits_{x^2+y^2\le 1} (1-x^2-y^2)\, dA$

$= \int_0^{2\pi} d\theta \int_0^1 dr\, r(1-r^2) = 2\pi \int_0^1 (r-r^3)\, dr = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}.$

(b) $\iint\limits_{S_2} \mathrm{curl}(\vec{F})\cdot\vec{n}\, dA = \iint\limits_{S_2} \langle 0,0,2\rangle \cdot \left(-\langle 0,0,1\rangle\right) dS = -2\iint\limits_{S_2} dS$

$= -2\, Area(S_2) = -2\pi$; $\iint\limits_{S} \mathrm{curl}(\vec{F})\cdot\vec{n}\, dA = 2\pi - 2\pi = 0$

(c) $\mathrm{div}(\mathrm{curl}(\vec{F})) = \frac{\partial(0)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(2)}{\partial z} = 0+0+0 = 0$

$\Rightarrow \iiint\limits_{E} \mathrm{div}(\mathrm{curl}(\vec{F}))\, dV = 0$ so our calculation in 3(b) is consistent with the Divergence Theorem.

$\vec{F} = \langle F_1, F_2, F_3\rangle \Rightarrow \mathrm{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

$\Rightarrow \mathrm{div}(\mathrm{curl}(\vec{F})) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y} \underset{\uparrow}{=} 0$

Clairaut's Theorem