Worksheet 27 May 2, 2011

1. Find the average velocity of a squirrel in the interval $t \in [0, 3]$ if his
   (a) position is given by $p(t) = \ln(1+t) + \dfrac{1}{1+t^2}$.
   (b) velocity is given by $v(t) = \sin(\pi t)e^{\cos(\pi t)}$.
   (c) acceleration is given by $a(t) = \dfrac{t}{(1+t^2)^2}$ and his initial velocity is 5.
2. Prove that a linear function cannot have more than one root. Use this to prove that a quadratic function cannot have more than two roots. Use this to prove that a cubic function cannot have more than three roots. (All of this is assuming your function isn't the constant zero function.)
3. Compute the volume of a circular cone with height h and radius r by realizing the cone as a solid of revolution.
4. Compute the total distance traveled by a particle with velocity $v(t) = \sin t$ for $0 \leq t \leq 5\pi/2$.
5. Let's consider the volume of balls of different dimensions (to a mathematician, a *ball* consists of all the points in an n-dimensional space whose distance from a fixed center point is at most some fixed radius r.)
   (a) The two-dimensional ball is just a standard disc. We can compute its area (or "volume") by integrating its cross-sectional length. Verify that this is $\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx$, then compute this integral. You should make the change of variables $x = r\cos t$.
   (b) The three-dimensional ball is what is normally called a ball. We can compute its volume by integrating its cross-sectional area. Verify that this is $\int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$, then compute this integral.
   (c) The four-dimensional ball is something that's harder to visualize. But we can compute its (hyper-)volume by integrating its cross-sectional volume. The cross-sections are just 3-balls of radius $\sqrt{r^2 - x^2}$, so this is computed by $\int_{-r}^{r} \frac{4}{3}\pi \left(\sqrt{r^2 - x^2}\right)^3 dx$. Compute this integral making use of $x = r\cos t$ again.
6. For a given function $f(x)$, define the following using limits. Then show an example and a nonexample.
   (a) f is continuous at $x = a$.
   (b) f is differentiable at $x = a$.
   (c) f is integrable on the interval $[a, b]$.
7. Give a good sketch of the curve $y^2 = x^3 - 2x + 1$ (using the derivative, intercepts, limits at infinity, etc.). What is the maximum y-value on the x-interval $[-2, 1]$? What about on $[-2, 2]$?
8. Where is the following function continuous? Where is it differentiable?

$$f(x) = \begin{cases} \arctan x & \text{if } x < -1 \\ e^{x+1} & \text{if } -1 \leq x \leq 0 \\ e^{x^2+1} & \text{if } 0 < x \leq 2 \\ e^5(4x - 7) & \text{if } 2 < x. \end{cases}$$