The Steady Magnetic field

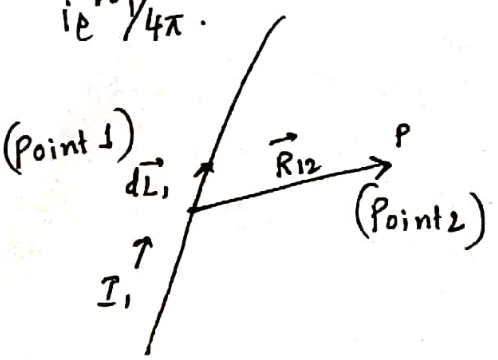
In this unit we study the magnetic field and existence of magnetic field from current distribution. The source of the steady magnetic field may be the permanent magnet, an electric field changing with time or a direct current.

Biot - Savart Law.

Consider the magnetic field produced by a differential dc element in free space.

The Biot - Savart's law states that ~~at~~ any point P the magnitude of magnetic field intensity produced by differential element is proportional to the product of the current, the magnitude of the differential length. and sine of the angle lying between the filament and the line connecting the filament to the point P at which field is desired, also the magnitude of magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P. The constant of proportionality is $1/4\pi$.

$$d\vec{H}_2 = \frac{I\, d\vec{L}_1 \times \hat{a}_{R_{12}}}{4\pi|\vec{R}_{12}|^2}$$



The unit of magnetic field intensity H is A/m.

The direction of magnetic field intensity is normal to the plane containing the differential filament and the line drawn from filament to the point P. Of the two possible normals the direction of progress of right handed screw turned from $d\vec{L}$ through the smaller angle to the line from the filament to P should be considered.

The integral form of Biot-Savart's law can be verified experimentally & is given by

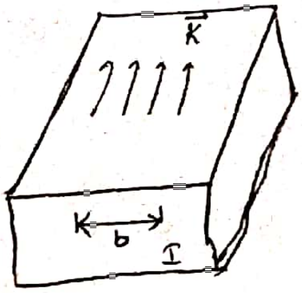
$$\vec{H} = \oint \frac{I\, d\vec{L} \times \hat{a}_R}{4\pi R^2}$$

The Biot-Savart's law may also be expressed in terms of distributed sources, such as current density $\vec{J}$ & surface current density $\vec{K}$. Surface current flows in a sheet of vanishingly small thickness, and the current current density $\vec{J}$ measured in amperes per square meter is therefore infin -ite. $\vec{K}$ is measured in amperes per meter width.

The total current I in any width b is

$$I = Kb$$

where it is assumed that width b is measured perpendicular to the direction in which the current is flowing.



Thus the differential current element $I\vec{dL}$. where $\vec{dL}$ is in the direction of current can be expressed in terms of $\vec{K}$ & $\vec{J}$ as

$$I\,d\vec{L} = \vec{K}\,dS = \vec{J}\,dv$$

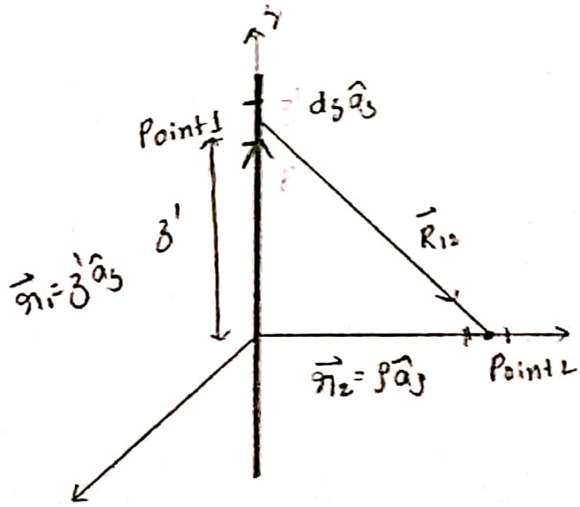
Alternate forms of Biot-Savart's law can be obtained as

$$\vec{H} = \int_S \frac{\vec{K} \times \hat{a}_R\, dS}{4\pi R^2}$$

$$\vec{H} = \int_{vol} \frac{\vec{J} \times \hat{a}_R\, dv}{4\pi R^2}$$

$\vec{H}$ due to Infinitely Long Straight Conductor

Consider an infinitely long straight conductor as shown in the figure. It carries current I amperes. The conductor is placed in z-axis



At point 2 $\vec{H}$ is desired. Point is chosen in $z=0$ plane where

Let point 2 is at $\vec{r}_2$ distance from origin

$$\vec{r}_2 = \rho\,\hat{a}_\rho$$

Consider a small differential element of conductor at point 1 at distance $\vec{R}_1$ from origin. where $\vec{r}_1 = z'\hat{a}_z$ Hence

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 = \rho\hat{a}_\rho - z'\hat{a}_z$$ and $\hat{a}_{12}$ is unit vector along $\vec{R}_{12}$

$$\hat{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\rho\hat{a}_\rho - z'\hat{a}_z}{\sqrt{\rho^2+z'^2}}$$ & $d\vec{L} = dz'\hat{a}_z$

From Biot-Savart's law at point 2 the magnitude of field intensity due to dz' element is given by

$$d\vec{H}_2 = \frac{I\,d\vec{L}\times\hat{a}_R}{4\pi R^2} = \frac{I\,d\vec{L}\times\vec{R}}{4\pi R^3}$$

$$d\vec{H}_2 = \frac{I\,(dz'\,\hat{a}_z)\times(\rho\hat{a}_\rho - z'\hat{a}_z)}{4\pi(\rho^2+z'^2)^{3/2}}$$

$$d\vec{H}_2 = \frac{I\,dz'\rho\,\hat{a}_\phi - 0}{4\pi(\rho^2+z'^2)^{3/2}}$$

Since the current is directed toward increasing values of z' the limits are $-\infty$ to ∞ on the integral

$$\vec{H}_2 = \frac{I}{4\pi}\int_{-\infty}^{\infty}\frac{dz'\,\rho\,\hat{a}_\phi}{(\rho^2+z'^2)^{3/2}}$$

$$\vec{H}_2 = \frac{I\,\hat{a}_\phi}{4\pi}\int_{-\infty}^{\infty}\frac{\rho\,dz'}{(\rho^2+z'^2)^{3/2}}$$

Since it is even function

$$\vec{H}_2 = \frac{I\,\hat{a}_\phi}{4\pi}\int_{0}^{\infty}\frac{\rho\,dz'}{(\rho^2+z'^2)^{3/2}}$$

Substituting $z' = \rho \tan\theta$ $\quad dz' = \rho \sec^2\theta \, d\theta$

$$(\rho^2 + \rho^2 \tan^2\theta)^{3/2} = \rho^3 \sec^3\theta$$

$$z' = 0 \Rightarrow \theta = \tan^{-1} 0 = 0$$

$$z' = \infty \Rightarrow \theta = \tan^{-1} \infty = \pi/2$$

$$\vec{H}_2 = \frac{2I\,\hat{a}_\phi}{4\pi} \int_0^{\pi/2} \frac{\rho^2 \sec^2\theta \, d\theta}{\rho^3 \sec^3\theta}$$

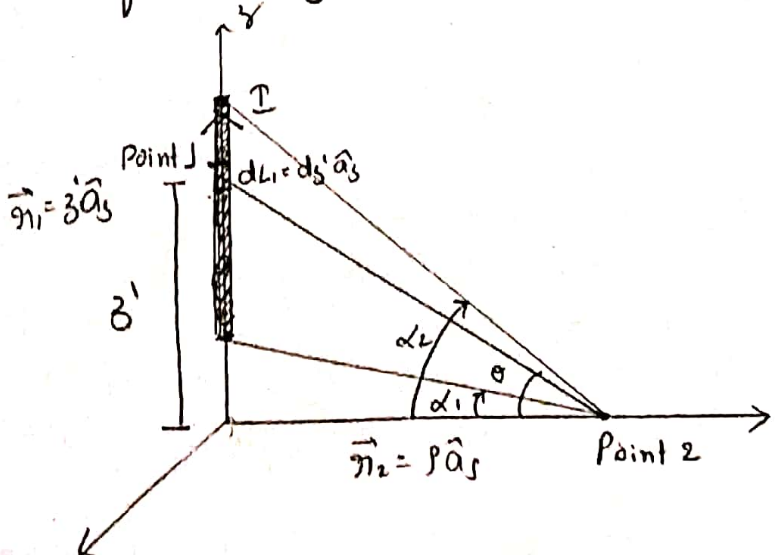
$$\vec{H}_2 = \frac{2I\,\hat{a}_\phi}{4\pi} \int_0^{\pi/2} \cos\theta \, d\theta$$

$$\boxed{\vec{H}_2 = \frac{I\,\hat{a}_\phi}{2\pi\rho}}$$

Magnitude of magnetic field intensity is function of ρ (inversely proportional) which is perpendicular distance from the point to conductor.

Magnetic field intensity due to finite length current element

Consider a finite length current element.



From Biot Savart's law

$$d\vec{H} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3}$$

$$d\vec{H} = \frac{I\, dz'\, \hat{a}_z \times (\rho \hat{a}_\rho - z' \hat{a}_z)}{4\pi\, (\rho^2 + z'^2)^{3/2}}$$

$$d\vec{H} = \frac{I \rho\, dz'\, \hat{a}_\phi}{4\pi (\rho^2 + z'^2)^{3/2}}$$

Substitute $z' = \rho \tan\theta$

$dz' = \rho \sec^2\theta \, d\theta$

$(\rho^2 + z'^2)^{3/2} = \rho^3 \sec^3\theta$

$$d\vec{H} = \frac{I \rho \cdot \rho \sec^2\theta \, d\theta \, \hat{a}_\phi}{4\pi (\rho^3 \sec^3\theta)} = \frac{I \hat{a}_\phi}{4\pi\rho} \cos\theta \, d\theta$$

$$\vec{H} = \frac{I \hat{a}_\phi}{4\pi\rho} \int_{\theta=\alpha_1}^{\alpha_2} \cos\theta \, d\theta = \frac{I \hat{a}_\phi}{4\pi\rho} \sin\theta \Big]_{\alpha_1}^{\alpha_2}$$

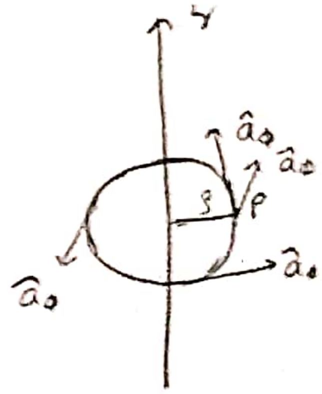
$$\boxed{\vec{H} = \frac{I}{4\pi\rho} [\sin\alpha_2 - \sin\alpha_1] \ A/m}$$

Ampere's Circuit Law.

This law is used to solve complex problems in magnetostatics. It states that the line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

Mathematically $\oint \vec{H} \cdot d\vec{L} = I$

Proof: Consider a long straight conductor carrying direct current I placed along z-axis as shown



Point P is at $\perp^{lar}$ distance ρ from conductor. Consider $d\vec{L}$ at point P in $\hat{a}_\phi$ direction

$$d\vec{L} = \rho \, d\phi \, \hat{a}_\phi$$

$\vec{H}$ obtained at Point P from Biot-Savart's law due to infinitely long conductor is

$$\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$$

Then $\vec{H} \cdot d\vec{L} = \frac{I}{2\pi\rho} \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = \frac{I}{2\pi} d\phi$.

Integrating over the closed path

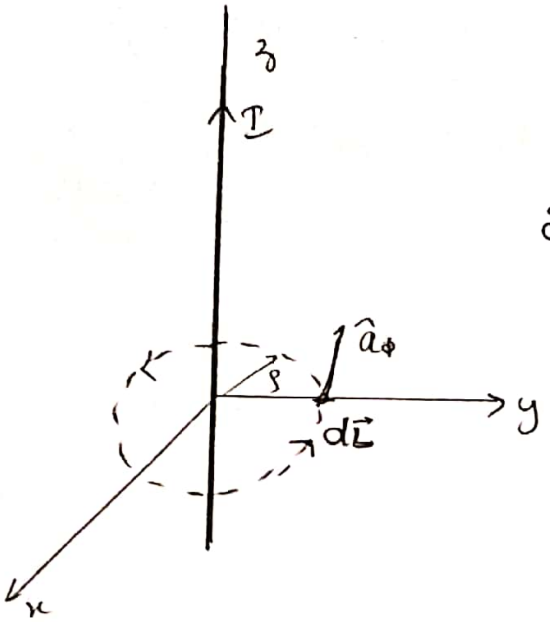
$$\oint \vec{H} \cdot d\vec{L} = \int_0^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi}(2\pi - 0) = I$$

$\oint \vec{H} \cdot d\vec{L} = I$ Current carried by conductor.

Ampere's law doesn't depend upon the shape of the path but the path must be enclosed & is called Amperian path.

Applications of Ampere's Circuit Law.

$\vec{H}$ due to Infinite long Straight Conductor



From figure at point P

$$\vec{H} = H_\phi \hat{a}_\phi$$

Since the $\vec{H}$ depends on ρ and the direction is always tangential to the closed path i.e., $\hat{a}_\phi$

$$dL = \rho \, d\phi \, \hat{a}_\phi$$

$$\vec{H} \cdot d\vec{L} = H_\phi \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = H_\phi \rho \, d\phi$$

From Ampere's ckt Law $\oint \vec{H} \cdot d\vec{L} = I$

$$\int_{\phi=0}^{2\pi} H_\phi \, \rho \, d\phi = I$$

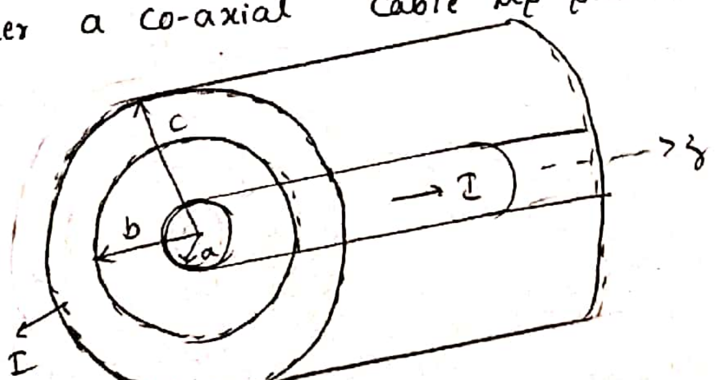
$$H_\phi \rho (2\pi) = I$$

$$H_\phi = I/2\pi\rho$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi \ A/m}$$

$\vec{H}$ due to a co-axial Cable

Consider a co-axial cable as shown in figure



The inner conductor radius is 'a' carrying current I. The outer conductor is the form of concentric cylinder whose inner radius b and outer radius c. Let this cable be placed along z-axis.

The current I is uniformly distributed in the inner conductor. While −I is uniformly distributed in the outer conductor.

The space between inner and outer conductor is with dielectric say air. The calculation of $\vec{H}$ is divided corresponding to various regions of the cable

Region 1 : within the inner conductor $\rho < a$

Consider a closed path having radius $\rho < a$. The area of cross section enclosed is $\pi \rho^2 \ m^2$

The total current flowing is I through the area πa^2

Hence the current enclosed by the closed path is

$$I' = \frac{\pi \rho^2}{\pi a^2} I = \frac{\rho^2}{a^2} I$$

$\vec{H}$ is only in $\hat{a}_\phi$ direction and depends on ρ

$$\vec{H} = H_\phi \hat{a}_\phi \quad \& \quad d\vec{L} = \rho d\phi \hat{a}_\phi$$

$$\therefore \vec{H} \cdot d\vec{L} = H_\phi \rho \, d\phi$$

According to Ampere's circuit law

$$\oint \vec{H} \cdot d\vec{L} = I'$$

$$\int_0^{2\pi} H_\phi \rho \, d\phi = \frac{\rho^2}{a^2} I$$

$$\int_0^{2\pi} H_\phi \rho \, d\phi = \frac{\rho^2}{a^2} I$$

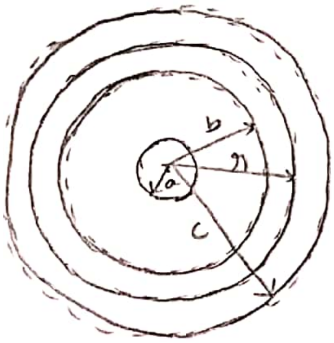
$$H_\phi \rho (2\pi) = \frac{\rho^2}{a^2} I$$

$$\boxed{\vec{H} = \frac{I\rho}{2\pi a^2}\ \hat{a}_\phi \ A/m}$$

Region 2: Within $a<\rho<b$ consider a circular path which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis. Hence $\vec{H}$ in this region is

$$\boxed{\vec{H} = \frac{I}{2\pi\rho}\ \hat{a}_\phi \ A/m}$$

Region 3: Within outer conductor $b<\rho<c$ consider a closed path. The current enclosed by the closed path is only the part of current $-I$ in the outer conductor. The total current $-I$ is flowing through the cross section $\pi(c^2-b^2)$. The closed path encloses the cross section $\pi(r^2-b^2)$. Hence the current enclosed by the closed path of outer conductor is



$$I' = \frac{\pi(\rho^2-b^2)}{\pi(c^2-b^2)}(-I) = -\frac{(\rho^2-b^2)}{(c^2-b^2)} I$$

$I'' = I =$ Current in inner conductor enclosed.

As the closed path also encloses the inner conductor and hence the current I flowing through it.

$$I_{enc} = I' + I'' = -\frac{(\rho^2-b^2)}{(c^2-b^2)} I + I$$

$$= I\left[1 - \frac{(\rho^2-b^2)}{(c^2-b^2)}\right]$$

$$I_{enc} = I\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

According to Ampere's ckt law

$$\oint \vec{H} \cdot d\vec{L} = I_{enc}$$

$$H = H_\phi \hat{a}_\phi \quad \& \quad d\vec{L} = \rho \, d\phi \, \hat{a}_\phi$$

$$\vec{H} \cdot d\vec{L} = H_\phi \rho \, d\phi$$

$$\int_{\phi=0}^{2\pi} H_\phi \rho \, d\phi = I_{enc}$$

$$2\pi\rho \, H_\phi = I \left[\frac{c^2 - \rho^2}{c^2 - b^2} \right]$$

$$H_\phi = \frac{I}{2\pi\rho} \left[\frac{c^2 - \rho^2}{c^2 - b^2} \right]$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} \left[\frac{c^2 - \rho^2}{c^2 - b^2} \right] \hat{a}_\phi \ A/m}$$

Region 4: Outside the cable $r > c$

$$I_{enc} = +I - I = 0$$

$$\oint \vec{H} \cdot d\vec{L} = 0$$

$$\boxed{\vec{H} = 0 \ A/m}$$

The magnetic field does not exists outside the cable

$\vec{H}$ due to Infinite Sheet of Current

Consider an infinite sheet of current in $z=0$ plane. The surface current density is $\vec{K}$. The current is flowing in positive y direction hence

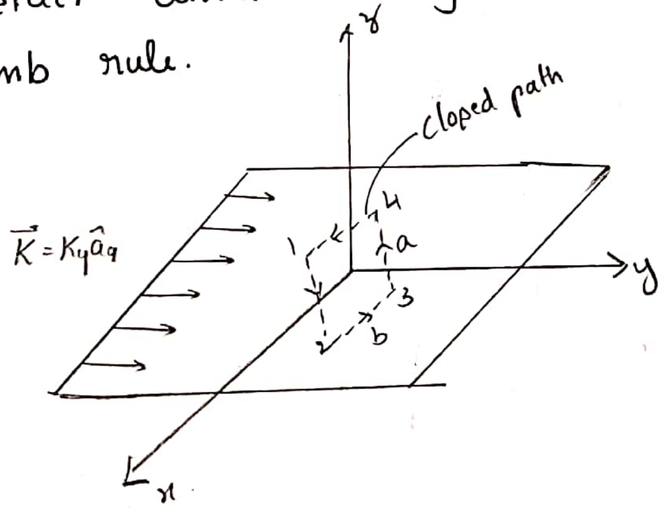
$$\vec{K} = K_y \hat{a}_y$$

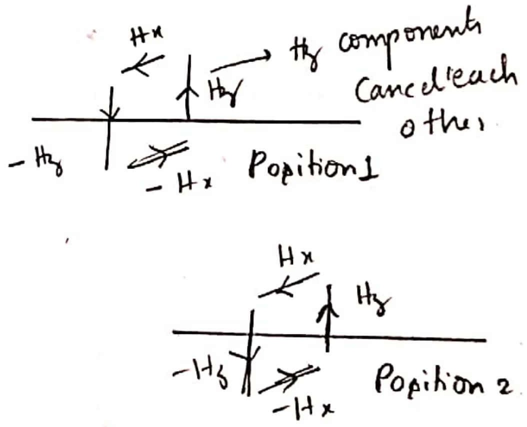
Consider a closed path 1-2-3-4 as shown in the figure. The width of the path is 'b' and the height 'a'.

The current flowing across the distance b is given by

$$I_{enc} = K_y b$$

Consider current in $\hat{a}_y$ direction according to right hand thumb rule.





Between any two very closely spaced conductors, the components of $\vec{H}$ in z direction are oppositely directed. All such components cancel each other and $\vec{H}$ cannot have any component in $\hat{a}_z$ direction.

As current flowing in y direction $\vec{H}$ cannot have any component in y direction. So $\vec{H}$ has only component in x direction.

$\therefore \quad \vec{H} = H_x \hat{a}_x$ for $z > 0$

$= -H_x \hat{a}_x$ for $z < 0$

Applying Ampere's ckt law $\oint \vec{H} \cdot d\vec{L} = I_{enc}$

Evaluating integral over the path 1-2-3-4-1

For path 1-2, $d\vec{L} = dz\, \hat{a}_z$

For path 3-4, $d\vec{L} = dz\, \hat{a}_z$ as $\vec{H}$ is in x direction $\hat{a}_x \cdot \hat{a}_z = 0$

Hence along paths 1-2 and 3-4 $\oint \vec{H} \cdot d\vec{L} = 0$

Consider path 2-3 along which $d\vec{L} = dx\, \hat{a}_x$

$$\therefore \int_2^3 \vec{H} \cdot d\vec{L} = \int_2^3 (-H_x \hat{a}_x) \cdot (dx\, \hat{a}_x) = H_x \int_2^3 dx = bH_x$$

Consider path 4-1 along which $d\vec{L} = dx\, \hat{a}_x$

$$\int_4^1 \vec{H} \cdot d\vec{L} = \int_4^1 (H_x \hat{a}_x) \cdot (dx\, \hat{a}_x) = H_x \int_4^1 dx = b\, H_x.$$

$$\therefore \oint \vec{H} \cdot d\vec{L} = 2bH_x$$

Equating this to $I_{enc} = K_y b$

$2bH_x = K_y b$

$H_x = 1/2\, K_y$

$\vec{H} = \frac{1}{2} K_y \hat{a}_x$ for $z > 0$

$\vec{H} = -\frac{1}{2} K_y \hat{a}_x$ for $z < 0$

In general for an infinite sheet of current density

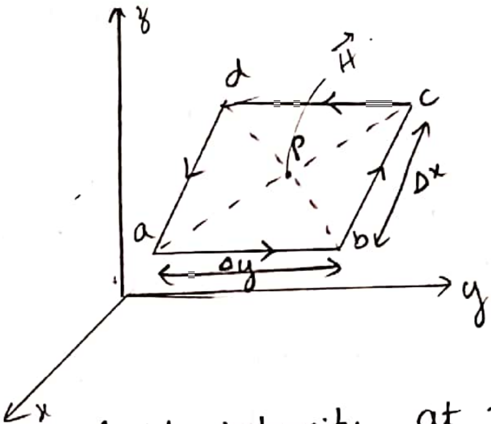
$\vec{K}$ A/m $\quad \boxed{\vec{H} = \frac{1}{2} \vec{K} \times \hat{a}_N}$

$\hat{a}_N$ = Unit vector normal from the current sheet to the point at which $\vec{H}$ is to be obtained.

Curl.

Ampere's circuit law is to be applied to the differential surface element to develop the concept of curl

Consider a differential surface element having sides Δx and Δy plane. The unknown current has produced $\vec{H}$ at the centre of this incremental closed path.



The total magnetic field intensity at the point P which is centre of the small rectangle is

$$\vec{H} = H_{x0}\,\hat{a}_x + H_{y0}\,\hat{a}_y + H_{z0}\,\hat{a}_z$$

To apply Ampere's ckt law to this closed path. Let us evaluate the closed line integral of $\vec{H}$ about this path in the direction abcda. According to right hand thumb rule the current is in $\hat{a}_z$ direction.

Along path a-b : $\vec{H} = \vec{H}_{y(a-b)}$ and $d\vec{L} = \vec{\Delta y}$

$$\int_a^b \vec{H}\cdot d\vec{L} = H_{y(a-b)}(\Delta y) = H_{y(a-b)}\Delta y$$

The intensity H_y along a-b can be expressed in terms of H_{y0} existing at P and the rate of change of H_y in x direction with x

$$\therefore (\vec{H}\cdot d\vec{L})_{a-b} = \left[H_{y0} + \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}\right]\Delta y$$

Along path b-c : $\vec{H} = \vec{H}_{y(b-c)}$ and $d\vec{L} = -\Delta\vec{x}$

$$\therefore \int_b^c \vec{H}\cdot d\vec{L} = H_{y(b-c)}(-\Delta x) \quad H_{y\,b,c}\,(-\Delta x)$$

$$(\vec{H}\cdot d\vec{L})_{b-c} = -\left[H_{x0} + \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}\right]\Delta x$$

Along path c-d : $\vec{H} = \vec{H}_{y(c-d)}$ and $d\vec{L} = -\Delta\vec{y}$

$$\therefore \int_c^d \vec{H}\cdot d\vec{L} = H_{y(c-d)}(-\Delta y) \quad H_{y\,c,d}\,(-\Delta y)$$

$$(\vec{H}\cdot d\vec{L})_{c-d} = -\left[H_{y0} - \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}\right]\Delta y$$

Along path d-a : $\vec{H} = \vec{H}_{x(d-a)}$ and $d\vec{L} = (\Delta x)$

$$\therefore \int_d^a \vec{H}\cdot d\vec{L} = H_{x(d-a)}(\Delta x) \quad H_{x\,d,a}\,(\Delta x)$$

$$(\vec{H}\cdot d\vec{L})_{d-a} = \left[H_{x0} - \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}\right]\Delta x$$

The total $\vec{H}\cdot d\vec{L}$ along abcda path is

$$\vec{H}\cdot d\vec{L} = H_{y0}\,\Delta y + \frac{\partial H_y}{\partial x}\frac{\Delta x\,\Delta y}{2} + \frac{\partial H_y}{\partial x}\frac{\Delta x\,\Delta y}{2} - H_{y0}\,\Delta y - H_{x0}\,\Delta x - \frac{\partial H_x}{\partial y}\frac{\Delta x\,\Delta y}{2} - \frac{\partial H_x}{\partial y}\frac{\Delta x\,\Delta y}{2} + H_{x0}\,\Delta x$$

$$\vec{H}\cdot d\vec{L} = \frac{\partial H_y}{\partial x}\Delta x\,\Delta y - \frac{\partial H_x}{\partial y}\Delta x\,\Delta y$$

$$\vec{H}\cdot d\vec{L} = \Delta x\,\Delta y\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]$$

This integral must be current enclosed by the differential element according to Ampere's ckt law

Current enclosed = Current Density normal to closed path × Area of the closed path

$$I_{enc} = J_z \, \Delta x \, \Delta y$$

$J_z \rightarrow$ Current density in $\hat{a}_z$ direction as the current enclosed in $\hat{a}_z$ direction.

$$\oint \vec{H} \cdot d\vec{L} = \Delta x \, \Delta y \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = J_z \, \Delta x \, \Delta y$$

$$\frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \, \Delta y} = \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = J_z$$

This gives accurate result as closed path shrinks to a point i.e., $\Delta x \, \Delta y$ area tends to zero

$$\lim_{\Delta x \Delta y \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \, \Delta y} = \frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial x} = J_z$$

Considering incremental closed path in yz plane get the current density normal to it i.e., in x direction

i.e.,
$$\lim_{\Delta y \Delta z \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta y \, \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

Considering incremental closed path in zx plane we get the current density normal to it i.e., in y direction

i.e.,
$$\lim_{\Delta z \Delta x \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta z \, \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

In general

$$\lim_{\Delta S_N \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S_N} = J_N$$

where

$J_N \rightarrow$ Current density normal to the surface ΔS

$\Delta S_N \rightarrow$ Area enclosed by closed line integral.

The term on L·H·S of equation is known as curl of H

The total current density is given by at point P

$$\vec{J} = J_x \hat{a}_x + J_y \hat{a}_y + J_z \hat{a}_z$$

$$\vec{J} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] \hat{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] \hat{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \hat{a}_z$$

$$\vec{J} = \text{Curl } \vec{H} = \nabla \times \vec{H}$$

Point form of Ampere's ckt law

$$\boxed{\text{Curl } \vec{H} = \nabla \times \vec{H} = \vec{J}}$$

This is Second Maxwell's equation

The third Maxwell's equation is point form of $\oint \vec{E} \cdot d\vec{L} = 0$

$$\boxed{\nabla \times \vec{E} = 0}$$

Curl in Various Coordinate Systems

1. Cartesian Coordinate System

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] \hat{a}_x - \left[\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right] \hat{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \hat{a}_z$$

Cylindrical Coordinate System

$$\nabla \times \vec{H} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \left[\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_\rho + \left[\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right]\hat{a}_\phi + \left[\frac{\partial \rho H_\phi}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right]\frac{\hat{a}_z}{\rho}$$

Spherical Coordinate System

$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\, \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{r\sin\theta}\left[\frac{\partial (H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right]\hat{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial (r H_\phi)}{\partial r}\right]\hat{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\hat{a}_\phi$$

Magnetic flux and Magnetic flux density

In free space magnetic flux density is $\vec{B} = \mu_0 \vec{H}$ where $\vec{B}$ is measured in wb/m^2 or a new unit Tesla (T) where $1\,T = 1\,wb/m^2$. The constant $\mu_0 = 4\pi \times 10^{-7}$ H/m where μ_0 is permeability of free space. If magnetic flux is represented by ϕ i.e., flux passing through designated area

$$\phi = \int_S \vec{B} \cdot d\vec{S} \quad \text{webers}$$

Electric flux measured in coulomb is

$$\psi = \oint_S \vec{D} \cdot d\vec{S} = Q$$

The magnetic flux lines are closed and do not terminate on a magnetic charge. For a closed surface the number of flux lines entering must be equal to the number of flux lines leaving. The single magnetic pole cannot exist as single charge. Hence integral $\vec{B} \cdot d\vec{S}$ evaluated over a closed surface is zero

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$

Applying divergence theorem to above equation

$$\oint_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B}\, dv = 0$$

where dv is the volume enclosed by closed surface

$$\nabla \cdot \vec{B} = 0$$

The divergence of magnetic flux density is always zero. This is called Gauss's law in differential form for magnetic fields.

The Above equation is 4^{th} Maxwell's equation

Thus for static electric fields & steady magnetic fields

$$\boxed{\begin{aligned} \nabla\cdot\vec{D} &= \rho_v \\ \nabla\times\vec{E} &= 0 \\ \nabla\times\vec{H} &= \vec{J} \\ \nabla\cdot\vec{B} &= 0 \end{aligned}}$$

Represent Maxwell's equations

The corresponding set of four integral equations are

$$\boxed{\begin{aligned} &\oint_S \vec{D}\cdot d\vec{S} = Q = \int_{Vol} \rho_v\, dv \\ &\oint \vec{E}\cdot d\vec{L} = 0 \\ &\oint \vec{H}\cdot d\vec{L} = I = \int_S \vec{J}\cdot d\vec{S} \\ &\oint_S \vec{B}\cdot d\vec{S} = 0 \end{aligned}}$$

Magnetic Potentials

In electrostatics there exists a scalar electric potential V which is related to the electric field intensity $\vec{E}$ as $\vec{E} = -\nabla V$

In case of magnetic fields two types of fields can be defined

1. Scalar magnetic Potential denoted as V_m
2. Vector magnetic Potential denoted as $\vec{A}$

Two Vector identities are

$$\nabla\times\nabla V = 0$$

$$\nabla\cdot(\nabla\times\vec{A}) = 0 \quad \text{for any scalar } V \text{ \& vector } \vec{A}$$

Scalar Magnetic Potential

If V_m is scalar magnetic potential then it must satisfy the equation $\nabla \times \nabla V_m = 0$

But $\vec{H} = -\nabla V_m$

Then from above two expressions $\nabla \times (-\vec{H}) = 0$ i.e., $\nabla \times \vec{H} = 0$

But $\nabla \times \vec{H} = \vec{J}$ hence $\vec{J} = 0$

Thus scalar magnetic potential V_m can be defined for region where $\vec{J}$ the current density is zero

$$\vec{H} = -\nabla V_m \quad \text{only for } \vec{J} = 0$$

Similar to $\vec{E}$ & V the relation between $\vec{H}$ & V_m is

$$V_{m\,a,b} = -\int_b^a \vec{H} \cdot d\vec{L}$$

W.K.T

$$\nabla \cdot \vec{B} = \nabla \cdot \mu_0 \vec{H} = 0$$

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

$$\boxed{\nabla^2 V_m = 0} \quad \text{for } \vec{J} = 0$$

This is known as Laplace's equation for scalar magnetic potential. In many magnetic problems involve geometries in which current-carrying conductor occupy a relatively small fraction of the total region of interest. And also in permanent magnets

Vector Magnetic Potential

Vector magnetic potential is denoted as $\vec{A}$ measured in wb/m

Then any vector has to satisfy

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\text{But } \nabla \cdot \vec{B} = 0$$

$$\vec{B} = \nabla \times \vec{A}$$

Thus curl of vector magnetic potential is flux density

$$\nabla \times \vec{H} = \vec{J}.$$

$$\nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$$

Using vector identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \vec{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right]$$

Thus if vector magnetic potential is known the current density can be obtained.

Consider a differential current element $d\vec{L}$ carrying current I

Then according to Biot Savart's law the vector magnetic potential $\vec{A}$ at a distance R from the differential current element is given by

$$\vec{A} = \oint \frac{\mu_0 I \, d\vec{L}}{4\pi R} \text{ wb/m}$$

From distributed current source $I d\vec{L}$ can be replaced by $\vec{K} ds$ where $\vec{K}$ is surface current density

$$\vec{A} = \oint_S \frac{\mu_0 \vec{K} ds}{4\pi R} \quad wb/m$$

If volume current density $\vec{J}$ is given in A/m^2 then $I d\vec{L}$ can be replaced by $\vec{J} dv$

$$\vec{A} = \oint_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R} \quad wb/m$$

Magnetic Force

The electric field causes a force to be exerted on a charge which may be either stationary or in motion. The steady magnetic field is capable of exerting force only on moving charge.

Magnetic field may be produced by moving charges & may exert forces on moving charges. Magnetic field cannot arise from stationary charges and cannot exert any force on a stationary charge.

Force on a moving charge

In an electric field the force on a charged particle is

$$\vec{F}_e = Q\vec{E}$$

The force is in the same direction as direction of field.

The force is directly proportional to $\vec{E}$ & Q

If the charge is in motion, the force at any point in its trajectory is given by above equation.

A charged particle in motion in a magnetic field of flux density $\vec{B}$ is found experimentally to experience a force, whose magnitude is proportional to the product of magnitude of charge, its velocity $\vec{V}$, and the flux density $\vec{B}$ and to the sine of the angle between vectors $\vec{V}$ & $\vec{B}$.

Thus force may be expressed as

$$\vec{F}_m = Q\vec{V} \times \vec{B}$$

The direction of force is perpendicular to the plane containing $\vec{V}$ & $\vec{B}$ both.

From force equation in electric field it is clear that the force $\vec{F_e}$ is independent of the velocity of the moving charge. Thus the electric force performs work on the charge

$\vec{F_m}$ cannot perform the work on the charge (moving charge) as it is at right angles to the direction of motion of charge $(\vec{F} \cdot d\vec{L} = 0)$

The total force on moving charge in the presence of both electric & magnetic field is

$$\vec{F} = \vec{F_e} + \vec{F_m} = Q\left[\vec{E} + \vec{V} \times \vec{B}\right] \text{ N}$$

Force on a differential Current Element

The force on a differential element of charge dQ moving in a steady magnetic field is given by

$$d\vec{F} = dQ\, \vec{V} \times \vec{B} \quad N$$

The current density $\vec{J}$ can be expressed as

$$\vec{J} = \rho_v \vec{V}$$

But the differential element of charge can be expressed in terms of volume charge density as

$$dQ = \rho_v\, dv$$

Hence

$$d\vec{F} = \rho_v\, dv\, \vec{V} \times \vec{B}$$

$$d\vec{F} = \vec{J} \times \vec{B}\, dv$$

The distributed source of current relationships are

$$\vec{J}\, dv = \vec{K}\, ds = I\, d\vec{L}$$

Then the force exerted on surface current density

$$d\vec{F} = \vec{K} \times \vec{B}\, ds$$

The force exerted on differential current element

$$d\vec{F} = I(d\vec{L} \times \vec{B})$$

Hence the force expression can be written as

$$\vec{F} = \int_{vol} (\vec{J} \times \vec{B})\, dv$$

$$\vec{F} = \int_{s} (\vec{K} \times \vec{B})\, ds$$

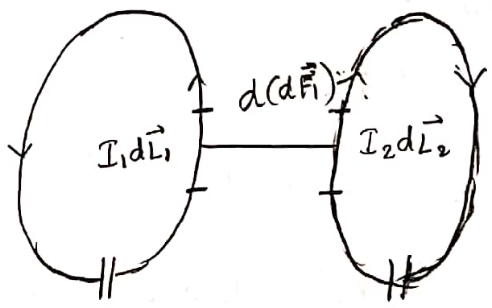
$$\vec{F} = \oint (I\, d\vec{L} \times \vec{B})$$

If the conductor is straight and the field $\vec{B}$ is uniform along it, $\vec{F} = I\vec{L} \times \vec{B}$ the magnitude of the force is given by $F = ILB \sin\theta$ where θ is the angle between the direction of current flow and the direction of magnetic flux density.

Force between differential current elements

Consider two current carrying conductors placed parallel to each other. Each conductor produces its own flux around it. When two such conductors are placed closed to each other there exists a force due to interaction of two fluxes.

Consider two current elements $I_1 d\vec{L}_1$ & $I_2 d\vec{L}_2$ as shown



Both current elements produces their own magnetic fields. As the currents are flowing in the same direction through the elements, the force exerted on element $I_1 d\vec{L}_1$ due to magnetic field $d\vec{B}_2$ produced by current element $I_2 d\vec{L}_2$ is force of attraction.

The differential force on a differential current element is

$$d\vec{F} = I\,d\vec{L} \times \vec{B}$$

The differential amount of differential force on $I_1 d\vec{L_1}$ is

$$d(d\vec{F_1}) = I_1 d\vec{L_1} \times d\vec{B_2}$$

From Biot Savart's law

$$d\vec{B_2} = \mu_0 d\vec{H_2} = \mu_0 \left[\frac{I_2\, d\vec{L_2} \times \hat{a}_{R21}}{4\pi |\vec{R_{21}}|^2} \right]$$

Hence $$d(d\vec{F_1}) = \mu_0 \left[\frac{I_1 d\vec{L_1} \times (I_2 d\vec{L_2} \times \hat{a}_{R21})}{4\pi |\vec{R_{21}}|^2} \right]$$

This equation represents force between two current elements

By integrating above equation twice the total force $\vec{F_1}$ on current element 1 due to current element 2 is given by

$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{L_1} \times (d\vec{L_2} \times \hat{a}_{R21})}{|\vec{R_{21}}|^2}$$

Similarly

$$\vec{F_2} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L1} \oint_{L2} \frac{d\vec{L_2} \times (d\vec{L_1} \times \hat{a}_{12})}{|\vec{R}_{12}|^2}$$

1) Given the following values of P_1, P_2 and $I_1 \Delta L_1$, calculate ΔH_2 :

a) $P_1\,(0,0,2)$, $P_2\,(4,2,0)$, $2\pi \hat{a}_z\ \mu A\cdot m$

b) $P_1\,(0,2,0)$, $P_2\,(4,2,3)$, $2\pi \hat{a}_z\ \mu A\cdot m$

c) $P_1\,(1,2,3)$, $P_2\,(-3,-1,2)$, $2\pi\,(-\hat{a}_x + \hat{a}_y + 2\hat{a}_z)\ \mu A\cdot m$

a)

$$\Delta \vec{H}_2 = \frac{I_1 \Delta L_1 \times \hat{a}_{R12}}{4\pi\, |\vec{R}_{12}|^3}$$

$$= \frac{(2\pi \hat{a}_z) \times (4\hat{a}_x + 2\hat{a}_y - 2\hat{a}_z) \times 10^{-6}}{4\pi\left(\sqrt{16+4+4}\right)^3}$$

$$\Delta \vec{H}_2 = \frac{(2\hat{a}_y - \hat{a}_x)}{(24)^{3/2}}\ \mu A/m$$

$$\Delta \vec{H}_2 = -8.505\,\hat{a}_x + 17.01\,\hat{a}_y\ nA/m$$

b)

$$\Delta \vec{H}_2 = \frac{(2\pi \hat{a}_z) \times (4\hat{a}_x + 3\hat{a}_z) \times 10^{-6}}{4\pi\left(\sqrt{16+9}\right)^3}$$

$$= \frac{1}{2}\,\frac{[4\hat{a}_y + 0]}{(25)^{3/2}}\ \mu A/m$$

$$\Delta \vec{H}_2 = 16\,\hat{a}_y\ nA/m$$

c)

$$\Delta \vec{H}_2 = \frac{2\pi(-\hat{a}_x + \hat{a}_y + 2\hat{a}_z) \times (-4\hat{a}_x - 3\hat{a}_y - 1\hat{a}_z)}{4\pi\left(\sqrt{16+9+1}\right)^3}$$

$$= \frac{\frac{1}{2}\left[\hat{a}_x(-1+6) - \hat{a}_y(1+8) + \hat{a}_z(3+4)\right]}{(16+9+1)^{3/2}}$$

$$\Delta \vec{H}_2 = 18.85\,\hat{a}_x - 33.94\,\hat{a}_y + 26.4\,\hat{a}_z\ nA/m$$

2) A current filament carrying 15 A in the $\hat{a}_z$ direction lies along the entire z axis. Find $\vec{H}$ in rectangular coordinates at a) $P_A(\sqrt{20}, 0, 4)$ b) $P_B(2, -4, 4)$

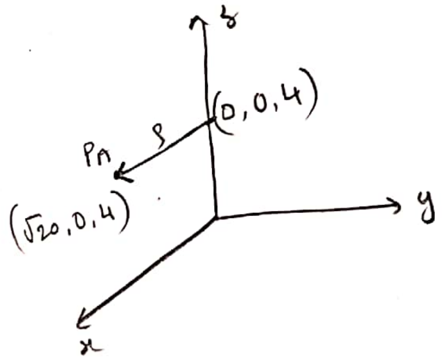
a) $P_A(\sqrt{20}, 0, 4)$

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

$\rho \to \perp^{lar}$ distance from current element to point

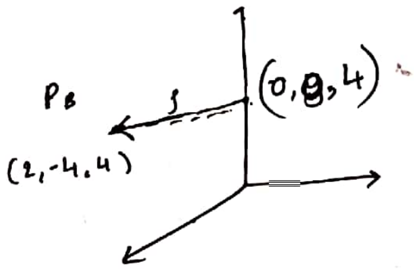
$\hat{a}_\phi$ is resultant direction of $(d\vec{L} \times \hat{a}_R)$

$$d\vec{L} \to \hat{a}_z \qquad \hat{a}_R \to \hat{a}_x$$



$$\vec{H} = \frac{15}{2\pi\sqrt{20}}(\hat{a}_z \times \hat{a}_x) = \underline{\underline{0.53\,\hat{a}_y \text{ A/m}}}$$

b) $P_B(2, -4, 4)$



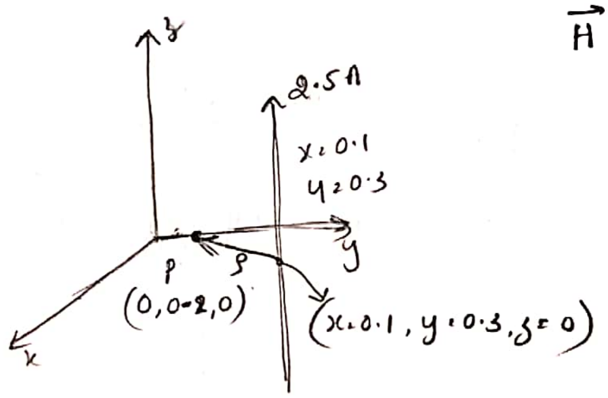
$$\vec{H} = \frac{15}{2\pi\sqrt{4+16}}\frac{\hat{a}_z \times (2\hat{a}_x - 4\hat{a}_y)}{\sqrt{4+16}}$$

$$\vec{H} = 0.1193\left[2\hat{a}_y + 4\hat{a}_x\right]$$

$$\vec{H} = \underline{\underline{0.477\,\hat{a}_x + 0.238\,\hat{a}_y}}$$

3) Express the $\vec{H}$ (rectangular components) at $P(0, 0.2, 0)$ in the field of a) A current filament $2.5A$ in $\hat{a}_z$ direction at $x = 0.1, y = 0.3$; b) A coax centered on the z-axis with $a = 0.3$, $b = 0.5$, $C = 0.6$, $I = 2.5A$ in the $\hat{a}_z$ direction in the center coordinate c) three current sheets, $2.7\,\hat{a}_x$ A/m at $y = 0.1$, $-1.4\,\hat{a}_x$ A/m at $y = 0.15$ & $-1.3\,\hat{a}_x$ A/m at $y = 0.25$

a)

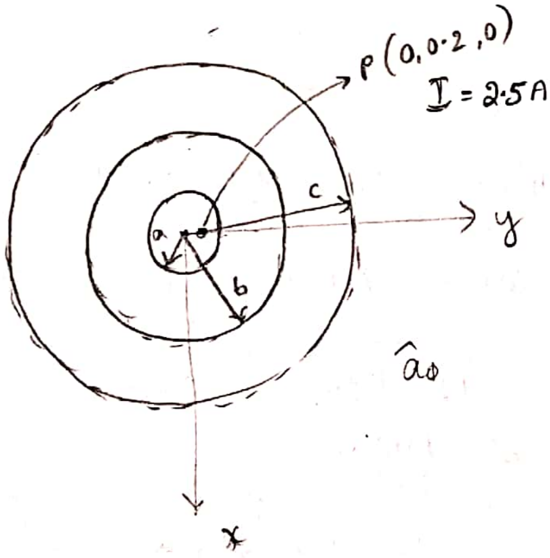


$$\vec{H} = \frac{(2.5)\left[\hat{a}_z \times (-0.1\hat{a}_x - 0.1\hat{a}_y)\right]}{2\pi\left[(0.1)^2 + (0.1)^2\right]}$$

$$\vec{H} = 1.989\,\hat{a}_x - 1.989\,\hat{a}_y$$

$(\hat{a}_z) \times (-0.1\hat{a}_x - 0.1\hat{a}_y)$

b)

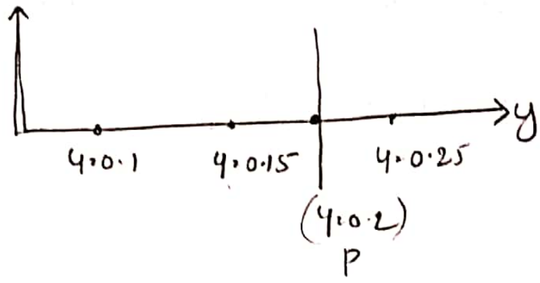


$$\vec{H} = \frac{I\rho}{2\pi a^2}\,\hat{a}_\phi$$

$$= \frac{2.5\,(0.2)}{2\pi(0.3)^2}\left[-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y\right]$$

$$\vec{H} = -0.884\,\hat{a}_x \text{ A/m.}$$

c)

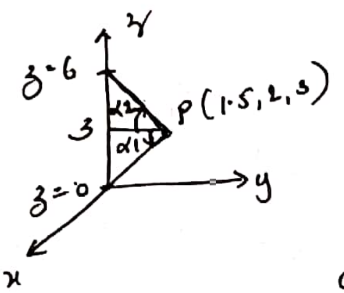


$$\vec{H} = \frac{1}{2}\left\{(2\cdot7\,\hat{a}_x)\times(\hat{a}_y) + (-1\cdot4\,\hat{a}_x)\times(\hat{a}_y) + (-1\cdot3\,\hat{a}_x)(-\hat{a}_y)\right]$$

$$= \frac{1}{2}\left\{2\cdot7\,\hat{a}_z - 1\cdot4\,\hat{a}_z + 1\cdot3\,\hat{a}_z\right\}$$

$$\vec{H} = 1\cdot3\,\hat{a}_z \ \text{A/m}$$

4) Find the magnetic field intensity at (1·5, 2, 3) due to a conductor carrying current of 24 A along z-axis extending from z=0 to z=6.



$$\alpha_2 = \tan^{-1} 3/r$$

$$\alpha_2 = \tan^{-1}(3/r)$$

$$r = \sqrt{x^2+y^2} = 2\cdot5$$

$$\alpha_2 = 50\cdot194^\circ \qquad \alpha_1 = -50\cdot194^\circ$$

$$\vec{H} = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a}_\phi$$

$$\vec{H} = 1\cdot1757\,\hat{a}_\phi \ \text{A/m}$$

Converting to rectangular coordinate system

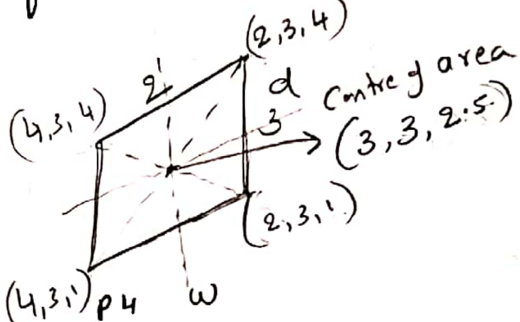
$$\vec{H} = H_x\hat{a}_x + H_y\hat{a}_y = (\vec{H}\cdot\hat{a}_x)\hat{a}_x + (\vec{H}\cdot\hat{a}_y)\hat{a}_y$$

$$\vec{H} = -0\cdot94\,\hat{a}_x + 0\cdot704\,\hat{a}_y \ \text{A/m}$$

3

5 a) Evaluate the closed line integral of $\vec{H}$ about the rectangular path $P_1(2,3,4)$ to $P_2(4,3,4)$ to $P_3(4,3,1)$ to $P_4(2,3,1)$ to P_1 given $\vec{H} = 3z\,\hat{a}_x - 2x^3\,\hat{a}_z$ A/m b) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \vec{H})_y$

c) Determine $(\nabla \times \vec{H})_y$ at the center of the area.



a) $\vec{H} = 3z\,\hat{a}_x - 2x^3\,\hat{a}_z$ A/m

$d\vec{L} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$$\int_{P1}^{P2} (3z\,dx - 2x^3 dz) + \int_{P2}^{P3} (3z\,dx - 2x^3 dz) + \int_{P3} (3z\,dx - 2x^3 dz) +$$

$$\int_{P4}^{P1} (3z\,dx - 2x^3 dz) = I$$

$$\int_{(2,3,4)}^{(4,3,4)} (3zx - 2x^3 z) + \int_{(4,3,4)}^{(4,3,1)} (3zx - 2x^3 z) + \int_{(4,3,1)}^{(2,3,1)} (3zx - 2x^3 z) + \int_{(2,3,1)}^{(2,3,4)} (3zx - 2x^3 z) = I$$

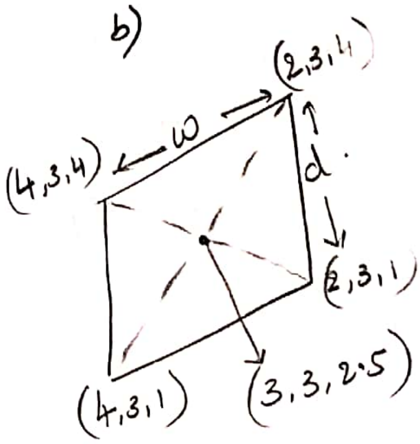
$$3zx\Big]_{(2,3,4)}^{(4,3,4)} + (-2x^3 z)\Big]_{(4,3,4)}^{(4,3,1)} + (3zx)\Big]_{(4,3,1)}^{(2,3,1)} + (-2x^3 z)\Big]_{(2,3,1)}^{(2,3,4)} = I$$

$$3z(2) - 2x^3(-3) + 3z(-2) - 2x^3(3) = I$$

$$\underline{\underline{I = 354\text{ A}}}$$

From Stokes theorem

b)



$$(\nabla\times\vec{H})_y = \lim_{wd\to 0} \frac{\oint \vec{H}\cdot d\vec{L}}{wd} = \lim_{wd\to 0} \frac{(354)}{(wd)}$$

$$(\nabla\times\vec{H})_y = \frac{354}{(3)(2)} = \underline{\underline{59\ A/m^2}}$$

c) Centre of Area $(\nabla\times\vec{H})_y$

$$(\nabla\times\vec{H})_y = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 0 & -2x^3 \end{vmatrix}$$

$$= \hat{a}_x[0-0] - \hat{a}_y[-6x^2-3] + \hat{a}_z[0-0]$$

$$(\nabla\times\vec{H})_y = \hat{a}_y[6x^2+3]$$

The center of area = (3, 3, 2.5)

$$\underline{\underline{(\nabla\times\vec{H})_y = 57\,\hat{a}_y\ A/m^2}}$$

④

6) Calculate the value of vector current density

a) In rectangular coordinates at $P_A(2,3,4)$ if

$$\vec{H} = x^2 z\,\hat{a}_y - y^2 x\,\hat{a}_z$$

b) In cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if

$$\vec{H} = \frac{2}{\rho}\left(\cos 0.2\phi\right)\hat{a}_\rho$$

c) In spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if

$$\vec{H} = \frac{1}{\sin\theta}\,\hat{a}_\theta$$

a) $$\nabla \times \vec{H} = \vec{J}$$

$$\vec{J} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 z & -y^2 x \end{vmatrix} =$$

$$= \hat{a}_x\left[-2yx - x^2\right] - \hat{a}_y\left[-y^2 - 0\right] + \hat{a}_z\left[2xz - 0\right]$$

$$= \left(-2yx - x^2\right)\hat{a}_x + (2xz)\,\hat{a}_z - \hat{a}_y\left[-y^2\right]$$

$$\vec{J} = (-12 - 4)\,\hat{a}_x + 16\,\hat{a}_z + 9\,\hat{a}_y$$

$$\vec{J} = -16\,\hat{a}_x + 9\,\hat{a}_y + 16\,\hat{a}_z \quad A/m^2$$

b) $$\nabla \times \vec{H} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 2/\rho\cos(0.2\phi) & & 0 \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{2}{\rho} \cos(0.2\phi) & 0 & 0 \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{\rho} \left[\hat{a}_\rho (0-0) - \rho \hat{a}_\phi (0-0) + \hat{a}_z \left(0 - \frac{\partial}{\partial \phi} \left[\frac{2}{\rho} \cos(0.2\phi) \right] \right) \right]$$

$$\nabla \times \vec{H} = \frac{-1}{\rho^2} \times 2 - [\sin(0.2\phi)] \times 0.2.$$

$$\nabla \times \vec{H} \text{ at } P_B = \underline{\underline{0.055}} \; \hat{a}_z \; A/m^2$$

c)
$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta \, \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & r H_\theta & r\sin\theta \, H_\phi \end{vmatrix} = \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta \, \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & \frac{r}{\sin\theta} & 0 \end{vmatrix}$$

$$= \left\{ \hat{a}_r [0] - r\hat{a}_\theta [0-0] + r\sin\theta \, \hat{a}_\phi \left[\frac{1}{\sin\theta} \right] \right\}$$

$$= \frac{1}{r^2 \sin\theta} [r \hat{a}_\phi] = \frac{1}{r \sin\theta} \hat{a}_\phi$$

At P_c $\quad \underline{\underline{\nabla \times \vec{H} = \hat{a}_\phi \; A/m^2}}$

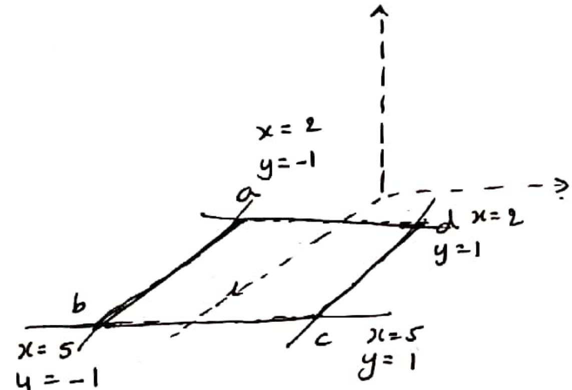
7) Evaluate both sides of Stoke's theorem for the field $\vec{H} = 6xy\,\hat{a}_x - 3y^2\,\hat{a}_y$ A/m and the rectangular path around the region $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\vec{s}$ be $\hat{a}_z$

According to Stoke's theorem

$$\oint \vec{H}\cdot d\vec{L} = \int_s (\nabla \times \vec{H})\cdot d\vec{s}$$

Consider LHS

$$\int_a^b \vec{H}\cdot d\vec{L} + \int_b^c \vec{H}\cdot d\vec{L} + \int_c^d \vec{H}\cdot d\vec{L} + \int_d^a \vec{H}\cdot d\vec{L}$$



$$\int_a^b \vec{H}\cdot d\vec{L} = \int_{\substack{y=-1\\x=2}}^{\substack{y=-1\\x=5}} 6xy\,dx = \frac{6x^2(-1)}{2}\Bigg]_2^5 = -63$$

$$\int_b^c \vec{H}\cdot d\vec{L} = \int_{y=-1}^{y=1} -3y^2\,dy = -\frac{3y^3}{3}\Bigg]_{y=-1}^{+1} = -[1+1] = -2$$

$$\int_c^d \vec{H}\cdot d\vec{L} = \int_{x=5}^{2} 6xy\,dx = \frac{6x^2}{2}\Bigg]_{x=5}^{2} = 3[4-25] = -63$$

$$\int_d^a \vec{H}\cdot d\vec{L} = \int_{y=1}^{-1} -3y^2\,dy = -3\frac{y^3}{3}\Bigg]_1^{-1} = -[-1-1] = 2$$

$$\oint \vec{H}\cdot d\vec{L} = 126\text{ A}$$

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & -3y^2 & 0 \end{vmatrix}$$

$$\nabla \times \vec{H} = -6x\hat{a}_z$$

$$\int_S (\nabla \times \vec{H}) \cdot ds = \int_S (-6x\hat{a}_z) \cdot (dx\,dy\,\hat{a}_z)$$

$$= \int_{y=-1}^{1} \int_{x=2}^{5} -6x\,dx\,dy = \left.\frac{-6x^2}{2}\right]_2^5 \left. y \right]_{-1}^{1}$$

$$= -3(25-4)(1+1)$$

$$\int_S (\nabla \times \vec{H}) \cdot d\vec{s} = -126\text{ A}$$

Hence $\oint \vec{H} \cdot d\vec{L} = \int_S (\nabla \times \vec{H}) \cdot d\vec{s}$

8) A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius $a = 1\text{mm}$, the conductor axis lies on the z-axis, the total current in $\hat{a}_z$ direction is 20 A find a) H_ϕ at $\rho = 0.5\text{ mm}$ b) B_ϕ at $\rho = 0.8\text{ mm}$ c) The total magnetic flux per unit length inside the conductor d) The total flux for $\rho < 0.5\text{mm}$ in unit length outside the conductor.
e) The total magnetic flux

a) $H_\phi = \frac{I\rho}{2\pi a^2}\hat{a}_\phi$

$$H_\phi = \frac{20 \times 0.5 \times 10^{-3}}{2\pi (1 \times 10^{-3})^2} = 1592\text{ A/m}$$

b) $H_\phi = \frac{I\rho}{2\pi a^2} = \frac{20 \times 0.8 \times 10^{-3}}{2\pi (1 \times 10^{-3})^2} = 2.546 \times 10^3\text{ A/m}$
at $\rho = 0.8\text{mm}$

$$B_\phi = \mu_0 H_\phi = 4\pi 10^{-7} \times 2.546 \times 10^3 = 3.199\text{ mT}$$

c) Magnetic field intensity inside the conductor is

$$\vec{H} = \frac{I\rho}{2\pi a^2} \quad (\rho<a)\ \hat{a}_\phi$$

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I \rho}{2\pi a^2}\hat{a}_\phi$$

$$\phi = \int_0^L \int_{\rho=0}^{a} \vec{B}\cdot d\vec{S} = \int_{z=0}^{L}\int_{\rho=0}^{a}\left(\frac{\mu_0 I \rho}{2\pi a^2}\hat{a}_\phi\right)\cdot(d\rho\, dz\, \hat{a}_\phi)$$

$$\phi = \int_{z=0}^{L}\int_{\rho=0}^{a} \frac{\mu_0 I \rho}{2\pi a^2}(d\rho\, dz) = \frac{\mu_0 I L}{2\pi a^2}\left.\frac{\rho^2}{2}\right]_0^a = \frac{\mu_0 I L}{2\pi a^2(2)}(a^2-0)$$

$L = 1\,m$

$$= \frac{\mu_0 I}{4\pi}\ Wb/m$$

$$\phi = 2\,\mu\, Wb/m$$

d). $$\phi = \int_0^L \int_{\rho=0}^{0.5mm} \frac{\mu_0 I \rho}{2\pi a^2} d\rho\, dz = \frac{\mu_0 I L}{2\pi a^2}\left.\frac{\rho^2}{2}\right]_0^{0.5}$$

$$= \frac{\mu_0 I L}{4\pi a^2}(0.5)^2 = \frac{4\pi\times10^{-7}\times 20(0.5)^2}{4\pi(1\times10^{-3})^2}$$

$L = 1\,m.$

$$\phi = 0.5\,\mu Wb$$

e) The magnetic field intensity outside the conductor is

$$H_\phi = \frac{I}{2\pi\rho} \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{a}_\phi \quad \text{where } x>a$$

$$\phi = \int_{z=0}^{L}\int_{\rho=0}^{x}\left(\frac{\mu_0 I}{2\pi\rho}\hat{a}_\phi\right)\cdot(d\rho\, dz)\hat{a}_\phi = \frac{\mu_0 I L}{2\pi}\ln\rho\Big]_{\rho=0}^{x}$$

$$= \frac{\mu_0 I L}{2\pi}\ln\left(\frac{x}{0}\right) = \infty$$

9) A current sheet, $\vec{K} = 2.4\,\hat{a}_z$ A/m is present at the surface $\rho = 1.2$ in free space

a) Find $\vec{H}$ for $\rho > 1.2$. Find V_m at $P(\rho = 1.5, \phi = 0.6\pi, z = 1)$ if

b) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi$

c) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi/2$

d) $V_m = 0$ at $\phi = \pi$ and there is a barrier at $\phi = 0$;

e) $V_m = 5\,V$ at $\phi = \pi$ and there is a barrier at $\phi = 0.8\pi$

a) $$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi = \frac{(2.4)\,[2\pi(a)\,\hat{a}_\phi]}{2\pi\rho}$$

$a = 1.2$

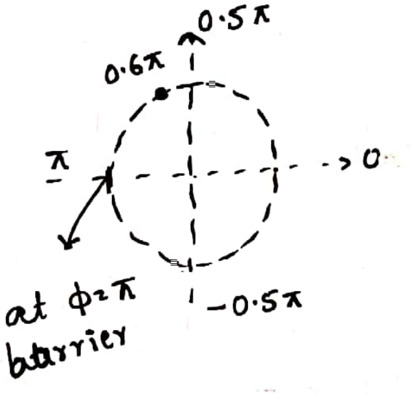
$$\boxed{\vec{H} = \frac{2.88}{\rho}\hat{a}_\phi}$$

b) $V_m = 0$, at $\phi = 0$

Barrier at $\phi = \pi$

$$V_m = -\int \vec{H}\cdot d\vec{L}$$

$$dL = \rho\, d\phi\, \hat{a}_\phi$$



$$V_m = -\int \frac{2.88}{\rho}\hat{a}_\phi \cdot \rho\, d\phi\, \hat{a}_\phi$$

$$V_m = -2.88\int d\phi$$

$$V_m = -2.88\,\phi + C$$

at $\phi = 0$ $\quad V_m = 0$

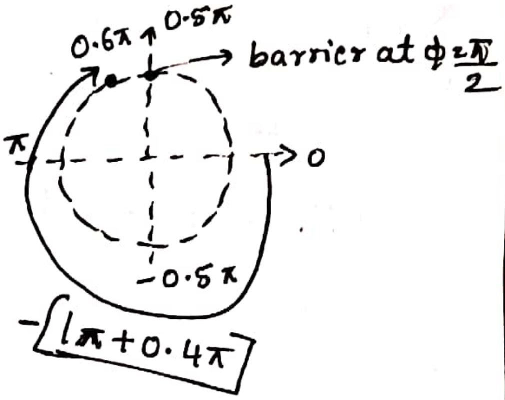
$C = 0$

V_m at 0.6π

$$V_m = -2.88 \times 0.6\pi = \underline{\underline{-5.43\,V}}$$

c) $V_m = 0$ at $\phi = 0$

Barrier at $\phi = \pi/2 = 0.5\pi$



$$V_m = -2.88\,\phi + C$$

$$0 = 0 + C$$

$$C = 0$$

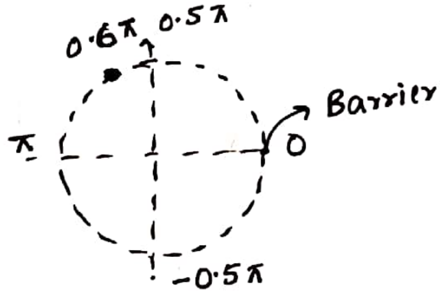
$$V_m = -2.88\,\phi$$

$$V_m = -2.88(-1.4\pi) = 12.7\,V$$

$$\underline{\underline{V_m = 12.7\,V}}$$

d) $V_m = 0$ at $\phi = \pi$

Barrier at $\phi = 0$



$$V_m = -2.88\,\phi + C$$

$$0 = -2.88\pi + C$$

$$C = 2.88\pi$$

$$V_m = -2.88\,\phi + 2.88\pi$$

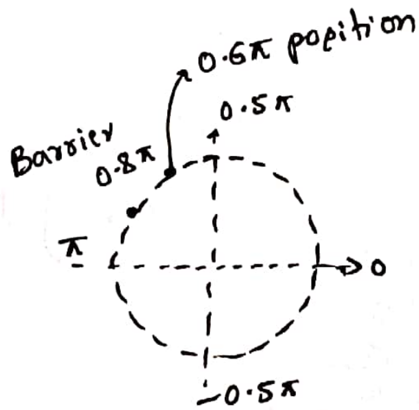
$$V_m = -2.88(0.6\pi) + 2.88\pi$$

$$\underline{\underline{V_m = 3.62\,V}}$$

e) $V_m = -2.88\,\phi + C$

$V_m = 5V$ at $\phi = \pi$

Barrier at $\phi = 0.8\pi$



$$5 = -2.88\pi + C$$

$$C = 5 + 2.88\pi$$

$$V_m = -2.88\,\phi + 5 + 2.88\pi$$

$$V_m = -2.88(+0.6\pi) + 5 + 2.88\pi$$

$$\underline{V_m = 8.619\,V}$$

10) In cylindrical Coordinates $\vec{A} = 50\rho^2 \hat{a}_z$ wb/m is a vector magnetic potential. in a certain region of free space. Find $\vec{H}, \vec{B}, \vec{J}$ and using $\vec{J}$ find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi \le 2\pi$ and $z = 0$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & 50\rho^2 \end{vmatrix}$$

$$\vec{B} = \frac{1}{\rho}\left[-\rho\hat{a}_\phi (100\rho) + 0\right] = -100\rho\,\hat{a}_\phi \ \text{wb/m}^2$$

$$\vec{H} = \vec{B}/\mu_0 = \frac{-100\rho}{\mu_0}\hat{a}_\phi \ \text{A/m}$$

$$\vec{J} = \nabla \times \vec{H}$$

$$\vec{J} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ 0 & \frac{-100\rho}{\mu_0} & 0 \end{vmatrix} = \frac{1}{\rho}\left[\frac{-200\rho}{\mu_0}\hat{a}_z\right] \ \text{A/m}^2$$

$$\vec{J} = -\frac{200}{\mu_0}\hat{a}_z \ \text{A/m}^2$$

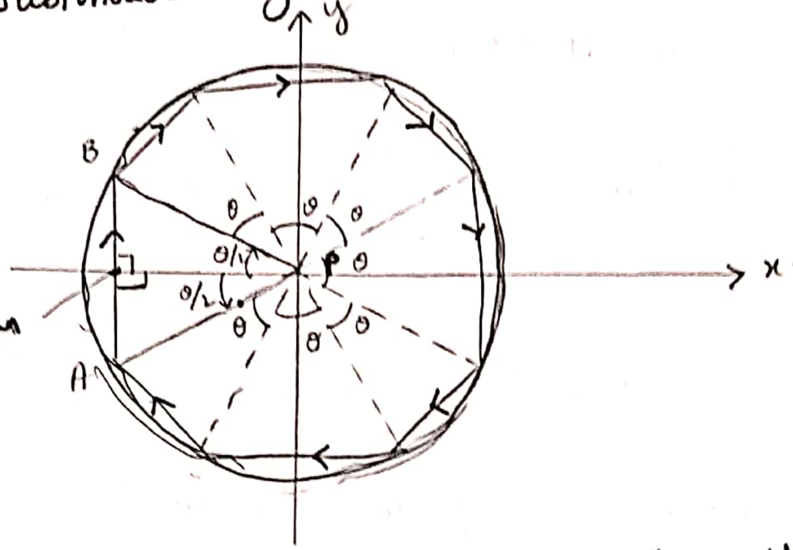
$$I = \int_S \vec{J}\cdot d\vec{s} = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\left(\frac{-200}{\mu_0}\hat{a}_z\right)\cdot\left(\rho\, d\rho\, d\phi\, \hat{a}_z\right)$$

$$= \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\frac{-200}{\mu_0}\rho\, d\rho\, d\phi = \frac{-200}{\mu_0}(2\pi)\left.\frac{\rho^2}{2}\right]_{\rho=0}^{1}$$

$$I = \frac{-200\pi}{4\pi\times10^{-7}} = \underline{-50\times10^{7}\ \text{A}}$$

11) Find the magnetic field intensity at the centre of circle with radius R which carries current I

Let the circle be divided in polygon as show in fig. The angle subtended by each side at the centre is say θ.



Let us consider polygon placed inside circle with n sides

From figure PM is $\perp^{lar}$ to AB. $\angle BPM = \angle AMP = \theta/2$

Now as the sides are n, the angle θ can be written as

$$\theta = \frac{2\pi}{n} = \frac{360^\circ}{n}$$

$$\frac{\theta}{2} = \frac{360^\circ}{2n} = \frac{180^\circ}{n} = \frac{\pi}{n} \text{ rad.}$$

$\vec{H}$ due to finite length conductor (due to AB) can be written as

$$|\vec{H}_{AB}| = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1]$$

$$\alpha_1 = -\theta/2 = -\pi/n \qquad \alpha_2 = +\theta/2 = \pi/n$$

$$|\vec{H}_{AB}| = \frac{I}{4\pi}\left[\sin(\pi/n) + (\sin\pi/n)\right] = \frac{I}{2\pi r}\sin\pi/n$$

r → Perpendicular distance from AB to point P

$$r = MP = R \cos \theta/2 = R \cos \pi/n$$

$$|\vec{H}|_{AB} = \frac{I}{2\pi R \cos \pi/n} \sin \pi/n$$

There are n such sides in a polygon

$$|\vec{H}|_{total} = \frac{I n}{2\pi R \cos \pi/n} \sin \pi/n.$$

$$|\vec{H}|_{total} = \frac{I}{2R \frac{\pi}{n} \cos \pi/n} \sin \pi/n$$

As $n \to \infty$ the polygon becomes the circle of radius R.

By applying the limit as $n \to \infty$

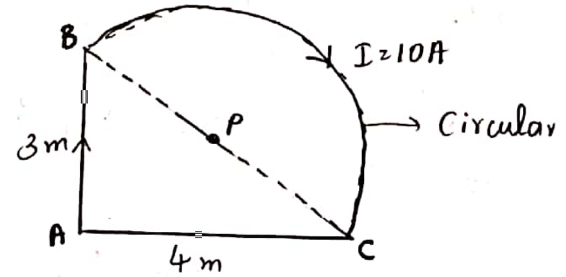
$$\lim_{n\to\infty} \frac{\sin \pi/n}{\pi/n} = 1 \qquad \lim_{n\to\infty} \cos \pi/n = 1$$

$$|\vec{H}|_{total} = \frac{I}{2R} \text{ A/m}$$

$$\boxed{\vec{H} = \frac{I}{2R} \hat{a}_z \text{ A/m}}$$

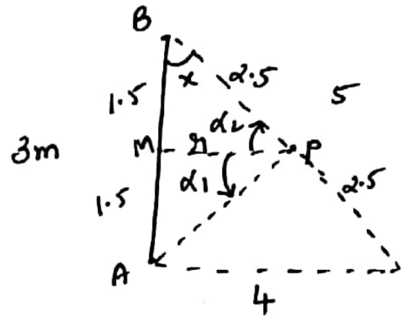
is the magnetic field intensity at the centre of circular conductor carrying current in xy plane ($\perp^{lar}$ to $\hat{a}_z$)

12) Find the magnetic field intensity at the point P for the current circuit shown in figure



There are three sections in the circuit. AB, BC, CA

Section 1 : Consider section AB



From figure $x = \tan^{-1}(4/3) = 53.13°$

$\alpha_2 = 180° - 53.13° - 90° = 36.869°$

Two triangles BMP & AMP are equal

$\alpha_1 = \alpha_2 = 36.869°$

But A is below point M hence $\alpha_1 = -36.869°$

The $\perp^{lar}$ distance $r = 2.5 \sin(x) = 2m$

$$|\vec{H}|_{AB} = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right] = \frac{10}{4\pi(2)}\left[2\sin(36.869°)\right]$$

$$|\vec{H}_{AB}| = 0.4774\ \hat{a}_N\ A/m$$

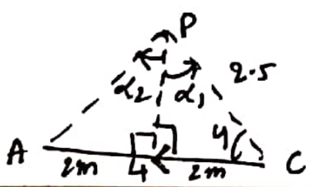
Section 2: Consider circular path BC

$$|\vec{H}|_{BC} = \left[\frac{I}{2R}\ \hat{a}_N\right]\ \frac{1}{2} \quad \left\{ \frac{1}{2} \text{ of } \vec{H} \text{ produced by circle of radius } R \right\}$$

Problem 11

$$|\vec{H}|_{BC} = \frac{1}{2}\frac{10}{\times 2 \times 2.5}\ \hat{a}_N = \hat{a}_N\ A/m$$

Section 3: Consider section CA



$y = \tan^{-1} 3/4 = 36.869°$

$\alpha_1 = 90 - y = 53.13°$

$$\alpha_2 = \alpha_1 = 53.13^\circ$$

But α_1 is negative as point C is below point P

$$r' = PM = PC \cos\alpha_1 = 2.5 \cos(53.13^\circ) = 1.5$$

$$(\vec{H})_{CA} = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a}_N = \frac{10}{4\pi \times 1.5}\left[\sin(53.13^\circ) + \sin(53.13^\circ)\right]$$

$$(\vec{H})_{CA} = 0.8488\ \hat{a}_N\ A/m$$

$$\text{Total } \vec{H} = (\vec{H})_{AB} + (\vec{H})_{BC} + (\vec{H})_{CA}$$

$$\boxed{\vec{H} = 2.3262\ \hat{a}_N\ A/m}$$

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