KARNAUGH MAPS (K-maps):

Graphical method of determining implicants and implicates of a boolean function.

Developed by Veitch. Modified by Karnaugh.

Diagrammatic representation of a boolean function - map

Truth table - 'n' variables
$n+1$ columns
2^n rows

(a) n-tuple → assignment of values to 'n' variables
(b) functional value (output) - last column.

K-map: Geometrical configuration of 2^n cells such that each of the n-tuples which corresponds to the rows of the truth table uniquely locates a cell on the map. Functional values assigned to n-tuples are the entries of the cells.

Functional value = 0 → entry in cell = 0
Functional value = 1 → entry in cell = 1

Two cells are said to be physical adjacent if and only if their respective n-tuples differ in exactly one element.

Ex: $(0,1,\boxed{1})$ and $(0,1,\boxed{0})$ → physically adjacent
$(1,\boxed{0,1})$ and $(1,\boxed{1,0})$ → not physically adjacent

One-variable and two-variable maps:

One-variable K-maps:

$2^1 = 2$ cells

Truth table:

x	$f(x)$
0	$f(0)$
1	$f(1)$

K-map :

	x = 0	x = 1
	$f(0)$	$f(1)$

Two-variable K-maps :

$2^2 = 4$ cells

Truth table :

x	y	$f(x,y)$
0	0	$f(0,0)$
0	1	$f(0,1)$
1	0	$f(1,0)$
1	1	$f(1,1)$

K-map :

x \ y	0	1
0	$f(0,0)$	$f(0,1)$
1	$f(1,0)$	$f(1,1)$

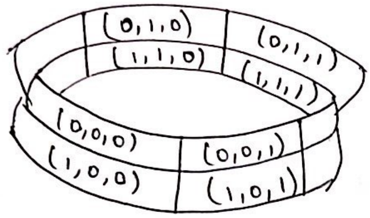
Three-variable K-maps :

$2^3 = 8$ cells

Truth table :

x	y	z	$f(x,y,z)$
0	0	0	$f(0,0,0)$
0	0	1	$f(0,0,1)$
0	1	0	$f(0,1,0)$
0	1	1	$f(0,1,1)$
1	0	0	$f(1,0,0)$
1	0	1	$f(1,0,1)$
1	1	0	$f(1,1,0)$
1	1	1	$f(1,1,1)$

Karnaugh map - 3-dimensional map - constructed on the surface of a cylinder.



2-dimensional K-map:

			yz	
x	00	01	11	10
0	f(0,0,0)	f(0,0,1)	f(0,1,1)	f(0,1,0)
1	f(1,0,0)	f(1,0,1)	f(1,1,1)	f(1,1,0)

Four-variable K-maps:

$2^4 = 16$ cells

Truth table:

w	x	y	z	$f(w,x,y,z)$
0	0	0	0	f(0,0,0,0)
0	0	0	1	f(0,0,0,1)
0	0	1	0	f(0,0,1,0)
0	0	1	1	f(0,0,1,1)
0	1	0	0	f(0,1,0,0)
0	1	0	1	f(0,1,0,1)
0	1	1	0	f(0,1,1,0)
0	1	1	1	f(0,1,1,1)
1	0	0	0	f(1,0,0,0)
1	0	0	1	f(1,0,0,1)
1	0	1	0	f(1,0,1,0)
1	0	1	1	f(1,0,1,1)
1	1	0	0	f(1,1,0,0)
1	1	0	1	f(1,1,0,1)

w	x	y	z	$f(w,x,y,z)$
1	1	1	0	$f(1,1,1,0)$
1	1	1	1	$f(1,1,1,1)$

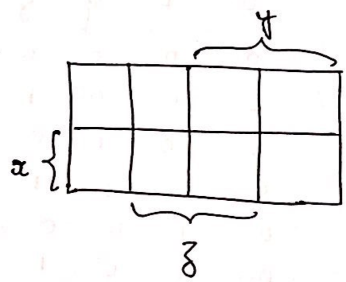
K-map :

wx \ yz	00	01	11	10
00	$f(0,0,0,0)$	$f(0,0,0,1)$	$f(0,0,1,1)$	$f(0,0,1,0)$
01	$f(0,1,0,0)$	$f(0,1,0,1)$	$f(0,1,1,1)$	$f(0,1,1,0)$
11	$f(1,1,0,0)$	$f(1,1,0,1)$	$f(1,1,1,1)$	$f(1,1,1,0)$
10	$f(1,0,0,0)$	$f(1,0,0,1)$	$f(1,0,1,1)$	$f(1,0,1,0)$

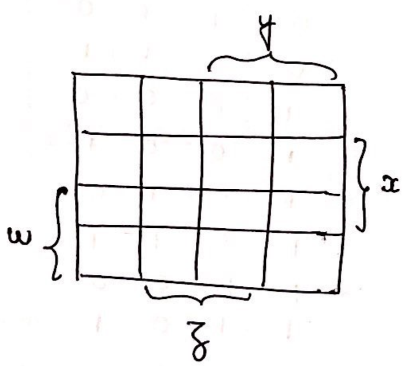
Variation of K-map :

Axes are not labelled with 0 and 1. A bracket is used to indicate those rows and columns with a variable whose value = 1

Three-variable :



Four-variable :



An illustrative 3-variable boolean function:

x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Truth table

x \ yz	00	01	11	10
0	1	0	0	1
1	1	1	0	0

K-map (1)

Karnaugh maps and canonical formulas:

Truth table: Minterm – complemented variables – 0

uncomplemented variables – 1

Ex: (1) $f(x,y,z) = \underset{000}{\bar{x}\bar{y}\bar{z}} + \underset{010}{\bar{x}y\bar{z}} + \underset{100}{x\bar{y}\bar{z}} + \underset{101}{x\bar{y}z} = \Sigma m(0,2,4,5)$

This represents the Boolean function for the above K-map (1).

Value 1 is present in cells – 000, 010, 1,00, 101

(2) Given:

$f(w,x,y,z) = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}y\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}x\bar{y}z +$
$w\bar{x}\bar{y}\bar{z} + w\bar{x}y\bar{z} = \Sigma m(0,1,2,4,5,8,10)$

= 0000 0001 0010 0100 0101

1000 1010

The K-map is as shown below :

wx \ yz	00	01	11	10
00	1	1	0	1
01	1	1	0	0
11	0	0	0	0
10	1	0	0	1

<u>Maxterms</u> :

In the truth table,

(1) uncomplemented variables - 0

(2) complemented variables - 1

For the K-map (1), the maxterm canonical formula is given by,

000 010 100 101 $\longrightarrow$ 1's

001 011 111 110 $\longrightarrow$ 0's

$$f(x,y,z) = (x+y+\bar{z})(x+\bar{y}+\bar{z})(\bar{x}+\bar{y}+\bar{z})(\bar{x}+\bar{y}+z)$$
$$= \Pi M(1,3,6,7)$$

Ex: <u>Given</u> : $f(w,x,y,z) = (w+x+\bar{y}+\bar{z})(\bar{w}+x+\bar{y}+z)$
$(\bar{w}+\bar{x}+y+z)(w+x+y+z)(\bar{w}+\bar{x}+\bar{y}+\bar{z})$

0011 1010 1100 0000 1111 $\longrightarrow$ 0's

$= \Pi M(0,3,10,12,15)$

wx \ yz	00	01	11	10
00	0	1	0	1
01	1	1	1	1
11	0	1	0	1
10	1	1	1	0

K-map

Karnaugh maps with cells designated by decimal numbers:

One-variable map:

	x = 0	x = 1
	0	1

Two-variable map:

x \ y	0	1
0	0	1
1	2	3

Three-variable map:

x \ yz	00	01	11	10
0	0	1	3	2
1	4	5	7	6

Four-variable map:

wx \ yz	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

Product and Sum term representations on K-maps:

Importance of K-maps: We can determine implicants and implicates of a function from the patterns of 0's and 1's.

A cell with a 1 entry - 1-cell
0 entry - 0-cell

Construction of a n-variable map - any set of 1-cells which form a $2^a \times 2^b$ rectangular grouping describes a product term with $n-a-b$ variables where 'a' & 'b' are non-negative integers. Rectangular groupings of these dimensions - subcubes.

Dimensions of a subcube - $2^a \times 2^b$

$\therefore$ Total number of cells $= 2^{a+b}$

For $a=0, b=1$ OR $a=1, b=0$, $2^1 = 2$ cells

For $a=1, b=1$, $2^2 = 4$ cells

For $a=1, b=2$ OR $a=2, b=1$, $2^3 = 8$ cells

Ex: (1)

PRODUCT TERMS

wx \ yz	00	01	11	10
00				
01		1	1	
11				
10				

Subcube dimension $= 2^0 \times 2^1$

The minterms present here are $\bar{w}x\bar{y}z + \bar{w}xyz$

$$f = \bar{w}x\bar{y}z + \bar{w}xyz$$
$$= \bar{w}xz\underbrace{(\bar{y}+y)}_{1} = \bar{w}xz$$

$w=0$ and $x=1$ for both 1-cells.

$z=1$ for both 1-cells

Only value of 'y' changes.

By writing only variables whose values don't change,

$w=0,\ x=1,\ z=1$

$\therefore\ \underline{\underline{\overline{w}xz}}$

K-map is constructed in such a way that two adjacent cells differ by exactly one element.

$$\therefore\ AB+\overline{A}B = B\underbrace{(A+\overline{A})}_{1} = B$$

Here value of A=1 in a cell and in the adjacent cell A=0. The value of 'A' changes. It is not present in the K-map due to the above simplification.
Value of B cannot change since adjacent cells can differ by exactly one element.

(2)

wx \ yz	00	01	11	10
00				
01			1	
11			1	
10				

$(x=1, y=1, z=1)$

Only 'w' changes,

$\therefore\ \underline{\underline{xyz}}$

OR

$f = \overline{w}xyz + wxyz$

$= xyz\underbrace{(w+\overline{w})}_{1} = xyz$

Subcube dimension $= 2^1 \times 2^0$

Subcube consisting of two 1-cells :

'n' variables – input

n-1 variables – subcube simplification (product term)

(3)

wx \ yz	00	01	11	10
00				
01				
11	1			1
10				

$$f = wx\bar{y}\bar{z} + wxy\bar{z}$$
$$= wx\bar{z}(\underbrace{\bar{y}+y}_{1}) = \underline{\underline{wx\bar{z}}}$$

OR

$w=1,\ x=1,\ z=0$

Only value of 'y' changes

$\therefore\ \underline{\underline{wx\bar{z}}}$

Subcube dimension = $2^0 \times 2^1$

(4)

wx \ yz	00	01	11	10
00	$\boxed{1}$			
01				
11				
10	$\boxed{1}$			

$$f = \bar{w}\bar{x}\bar{y}\bar{z} + w\bar{x}\bar{y}\bar{z}$$
$$= \bar{x}\bar{y}\bar{z}(\underbrace{w+\bar{w}}_{1}) = \underline{\underline{\bar{x}\bar{y}\bar{z}}}$$

OR

$x=0,\ y=0,\ z=0$

Only value of 'w' changes

$\therefore\ \underline{\underline{\bar{x}\bar{y}\bar{z}}}$

(5)

wx \ yz	00	01	11	10
00				
01	1	1		
11	1	1		
10				

Subcube dimension = $2^1 \times 2^1$

Subcube consisting of four 1-cells:

'n' variables - input

n-2 variables - subcube simplification (product term)

$$f = \bar{w}x\bar{y}\bar{z} + \bar{w}x\bar{y}z + wx\bar{y}\bar{z} + wx\bar{y}z$$

$$= \bar{w}x\bar{y}(\underbrace{z+\bar{z}}_{1}) + wx\bar{y}(\underbrace{z+\bar{z}}_{1})$$

$$= \bar{w}x\bar{y} + wx\bar{y} = x\bar{y}(\underbrace{w+\bar{w}}_{1})$$

$$= \underline{\underline{x\bar{y}}}$$

OR

$x=1$ and $y=0$

Values of w and z changes

$\therefore \underline{\underline{x\bar{y}}}$

Here number of input variables = 4 ($\underline{n}$)

After simplification = 4-2 = 2 ($\underline{n-2}$) variables

(6)

$wx \backslash yz$	00	01	11	10
00			1	
01			1	
11			1	
10			1	

Subcube dimension $= 2^1 \times 2^1$

$$f = \bar{w}\bar{x}yz + \bar{w}xyz + wxyz + w\bar{x}yz$$

$$= \bar{w}yz(\underbrace{x+\bar{x}}_{1}) + wyz(\underbrace{x+\bar{x}}_{1})$$

$$= \bar{w}yz + wyz = yz(\underbrace{w+\bar{w}}_{1}) = \underline{\underline{yz}}$$

OR

$y=1$ and $z=1$

Value of w and x changes

$\therefore \underline{yz}$

(7)

$wx \backslash yz$	00	01	11	10
00	1	1	1	1
01				
11				
10				

Subcube dimension $= 2^1 \times 2^1$

$$f = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}yz + \bar{w}\bar{x}y\bar{z}$$

$$= \bar{w}\bar{x}\bar{y}\,(\underbrace{z+\bar{z}}_{1}) + \bar{w}\bar{x}y\,(\underbrace{z+\bar{z}}_{1})$$

$$= \bar{w}\bar{x}\bar{y} + \bar{w}\bar{x}y = \bar{w}\bar{x}\,(\underbrace{\bar{y}+y}_{1}) = \underline{\underline{\bar{w}\bar{x}}}$$

OR

$w=0$ and $x=0$

Value of y and z changes

$\therefore \underline{\underline{\bar{w}\bar{x}}}$

(8)

$wx \backslash yz$	00	01	11	10
00				
01				
11	1			1
10	1			1

Subcube dimension $= 2^1 \times 2^1$

$$f = wx\bar{y}\bar{z} + w\bar{x}\bar{y}\bar{z} + wxy\bar{z} + w\bar{x}y\bar{z}$$

$$= w\bar{y}\bar{z}\,(\underbrace{x+\bar{x}}_{1}) + wy\bar{z}\,(\underbrace{x+\bar{x}}_{1})$$

$$= w\bar{y}\bar{z} + wy\bar{z} = w\bar{z}\,(\underbrace{y+\bar{y}}_{1}) = \underline{\underline{w\bar{z}}}$$

OR

$w = 1$ and $z = 0$

Value of x and y changes

$\therefore \; w\bar{z}$

(9)

wx \ yz	00	01	11	10
00	1			1
01				
11				
10	1			1

Subcube dimension $= 2^1 \times 2^1$

$$f = \bar{w}\bar{x}\bar{y}\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + w\bar{x}y\bar{z}$$

$$= \bar{x}\bar{y}\bar{z}\underbrace{(\bar{w}+w)}_{1} + \bar{x}y\bar{z}\underbrace{(\bar{w}+w)}_{1}$$

$$= \bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} = \bar{x}\bar{z}\underbrace{(\bar{y}+y)}_{1} = \bar{x}\bar{z}$$

OR

Value of w and y changes

$x = 0$ and $z = 0$

$\therefore \; \bar{x}\bar{z}$

(10)

wx \ yz	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11				
10				

Subcube dimension $= 2^1 \times 2^2$

Subcube consisting of eight 1-cells:

'n' inputs $\rightarrow$ n-3 variables in product term

$$f = \overline{w}\,\overline{x}\,\overline{y}\,\overline{z} + \overline{w}\,\overline{x}\,\overline{y}z + \overline{w}\,\overline{x}yz + \overline{w}\,\overline{x}y\overline{z}$$
$$+ \overline{w}x\overline{y}\,\overline{z} + \overline{w}x\overline{y}z + \overline{w}xyz + \overline{w}xy\overline{z}$$

$$= \overline{w}\,\overline{x}\,\overline{y}\underbrace{(\overline{z}+z)}_{1} + \overline{w}\,\overline{x}y\underbrace{(z+\overline{z})}_{1} + \overline{w}x\overline{y}\underbrace{(\overline{z}+z)}_{1}$$
$$+ \overline{w}xy\underbrace{(z+\overline{z})}_{1}$$

$$= \overline{w}\,\overline{x}\,\overline{y} + \overline{w}\,\overline{x}y + \overline{w}x\overline{y} + \overline{w}xy$$
$$= \overline{w}\,\overline{x}\underbrace{(\overline{y}+y)}_{1} + \overline{w}x\underbrace{(\overline{y}+y)}_{1}$$

$$= \overline{w}\,\overline{x} + \overline{w}x$$
$$= \overline{w}\underbrace{(\overline{x}+x)}_{1} = \underline{\underline{\overline{w}}}$$

Here number of inputs = 4 (n)

After simplification = n-3 = 4-3 = 1 variable

OR

Value of $w = 0$

Value of x, y and z changes

$\therefore \underline{\underline{\overline{w}}}$

(ii)

wx \ yz	00	01	11	10
00		1	1	
01		1	1	
11		1	1	
10		1	1	

Subcube dimension = $2^2 \times 2^1$

Value of $z = 1$

Values of w, x and y changes

$\therefore \underline{\underline{z}}$

OR

$$f = \bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}yz + \bar{w}x\bar{y}z + \bar{w}xyz + wx\bar{y}z + wxyz + w\bar{x}\bar{y}z + w\bar{x}yz$$

$$= \bar{w}\bar{x}z\underbrace{(\bar{y}+y)}_{1} + \bar{w}xz\underbrace{(\bar{y}+y)}_{1} + wxz\underbrace{(\bar{y}+y)}_{1} + w\bar{x}z\underbrace{(\bar{y}+y)}_{1}$$

$$= \bar{w}\bar{x}z + \bar{w}xz + wxz + w\bar{x}z$$

$$= \bar{w}z\underbrace{(\bar{x}+x)}_{1} + wz\underbrace{(x+\bar{x})}_{1}$$

$$= \bar{w}z + wz = z\underbrace{(\bar{w}+w)}_{1} = \underline{\underline{z}}$$

(12)

$wx \backslash yz$	00	01	11	10
00	1	1	1	1
01				
11				
10	1	1	1	1

Subcube dimension $= 2^1 \times 2^2$

Value of $x = 0$

Values of w, y and z changes $\therefore$ $\underline{\underline{\bar{x}}}$

OR

$$f = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}yz + \bar{w}\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + w\bar{x}\bar{y}z + w\bar{x}yz + w\bar{x}y\bar{z}$$

$$= \bar{w}\bar{x}\bar{y}\underbrace{(\bar{z}+z)}_{1} + \bar{w}\bar{x}y\underbrace{(z+\bar{z})}_{1} + w\bar{x}\bar{y}\underbrace{(\bar{z}+z)}_{1} + w\bar{x}y\underbrace{(z+\bar{z})}_{1}$$

$$= \bar{w}\bar{x}\bar{y} + \bar{w}\bar{x}y + w\bar{x}\bar{y} + w\bar{x}y$$

$$= \bar{w}\bar{x}\underbrace{(\bar{y}+y)}_{1} + w\bar{x}\underbrace{(\bar{y}+y)}_{1}$$

$$= \bar{w}\bar{x} + w\bar{x} = \bar{x}\underbrace{(\bar{w}+w)}_{1}$$

$$= \underline{\underline{\bar{x}}}$$

(13)

wx \ yz	00	01	11	10
00	1			1
01	1			1
11	1			1
10	1			1

Subcube dimension $= 2^2 \times 2^1$

Value of $z = 0$

Values of w, x and y changes $\therefore \underline{\underline{\bar{z}}}$

OR

$$f = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}xy\bar{z}$$
$$+ wx\bar{y}\bar{z} + wxy\bar{z} + w\bar{x}\bar{y}\bar{z} + w\bar{x}y\bar{z}$$
$$= \bar{w}\bar{x}\bar{z}\underbrace{(\bar{y}+y)}_{1} + \bar{w}x\bar{z}\underbrace{(\bar{y}+y)}_{1} + wx\bar{z}\underbrace{(\bar{y}+y)}_{1}$$
$$+ w\bar{x}\bar{z}\underbrace{(\bar{y}+y)}_{1}$$
$$= \bar{w}\bar{x}\bar{z} + \bar{w}x\bar{z} + wx\bar{z} + w\bar{x}\bar{z}$$
$$= \bar{w}\bar{z}\underbrace{(\bar{x}+x)}_{1} + w\bar{z}\underbrace{(x+\bar{x})}_{1} = \bar{w}\bar{z} + w\bar{z}$$
$$= \bar{z}\underbrace{(\bar{w}+w)}_{1} = \underline{\underline{\bar{z}}}$$

SUM TERMS :

(14)

wx \ yz	00	01	11	10
00				
01	0	0		
11				
10				

Value of z changes.

$w = 0,\ x = 1,\ y = 0 \quad \therefore \underline{\underline{w + \bar{x} + y}}$

OR

$$f = (\underbrace{w+\bar{x}+y}_{a}+\underbrace{z}_{b})(\underbrace{w+\bar{x}+y}_{a}+\underbrace{\bar{z}}_{\bar{b}})$$

We know that, $(a+b)(a+\bar{b})$

$$= \underbrace{a \cdot a}_{a} + a \cdot \bar{b} + b \cdot a + \underbrace{b \cdot \bar{b}}_{0}$$

$$= a + a\bar{b} + ab$$

$$= a(\underbrace{1+\bar{b}}_{1}+b)$$

$$= a(\underbrace{1+b}_{1}) = \underline{\underline{a}}$$

Here $a = w+\bar{x}+y$

$\therefore$ $\underline{\underline{f = w+\bar{x}+y}}$

(15)

wx \ yz	00	01	11	10
00	0	0		
01				
11				
10	0	0		

Value of w and z changes

$x=0$ and $y=0$

$\therefore$ $\underline{\underline{x+y}}$

(16)

wx \ yz	00	01	11	10
00			0	0
01			0	0
11			0	0
10			0	0

Value of w, x and z changes

$y = 1 \quad \therefore \quad \underline{\bar{y}}$

Any $2^a \times 2^b$ rectangular grouping or subcube of 0-cells for n-variables represent a sum term with $n-a-b$ variables where 'a' and 'b' are non-negative integers.

0 ⟶ uncomplemented variable

1 ⟶ complemented variable

K-map is constructed in such a way that two adjacent cells differ by exactly one element.

$$\therefore (A+B)(\bar{A}+B)$$

$$= \underbrace{A \cdot \bar{A}}_{0} + AB + \bar{A}B + \underbrace{B \cdot B}_{B}$$

$$= AB + \bar{A}B + B$$

$$= B\underbrace{(A+\bar{A})}_{1} + B = B + B = \underline{\underline{B}}$$

Value of A changes here - so not present in K-map. Value of B is present finally. Value of B can't change since simultaneously values of A and B can't change for adjacent cells.

<u>NOTE</u>: Dimensions of a subcube $= 2^a \times 2^b$

$= 2^{a+b}$

i.e a power of 2

Subcube cannot be formed if it is not a power of 2 → like 3 cells

(3 is not power of 2)

Using Karnaugh maps to obtain minimal expressions for complete boolean functions:

Subcube of 1-cells → product term
Subcube of 0-cells → sum term

Ex:

x \ yz	00	01	11	10
0	0	0	0	1
1	0	0	1	1

A
B
C

$f(x,y,z) = xy + y\bar{z} = y(x+\bar{z})$

xy and $y\bar{z}$ → prime implicants of this function

$xy + y\bar{z}$ → SOP or disjunctive normal formula

$y(x+\bar{z})$ → POS or conjunctive normal formula.

All 1's in the map should be always considered.

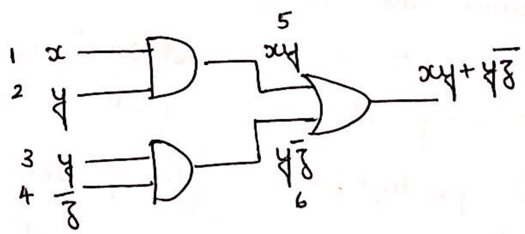
Suppose consider only the term $\bar{x}y\bar{z}$ → subcube A

But subcube A is contained within subcube B

Hence it is not a prime implicant.

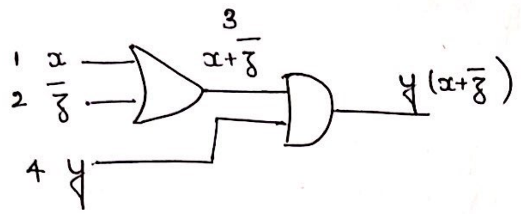
Larger the subcube – smaller is the product term.

$xy + y\bar{z}$



Cost = 6

$y(x+\overline{z})$



Cost = $\underline{\underline{4}}$

<u>NOTE</u>:

If individual 1-cells are selected as $2^0 \times 2^0$ subcubes – resulting expression is the <u>minterm</u> canonical formula or SOP.

<u>General</u> <u>procedure</u>:

'n' variable map

(1) All 2^n entries are 1's $\longrightarrow$ function is identically $= 1$

$1 \longrightarrow$ only prime implicant

(2) All 2^n entries are not 1's,

Search for subcubes of 1-cells with dimension $= 2^a \times 2^b$
$= 2^{n-1}$

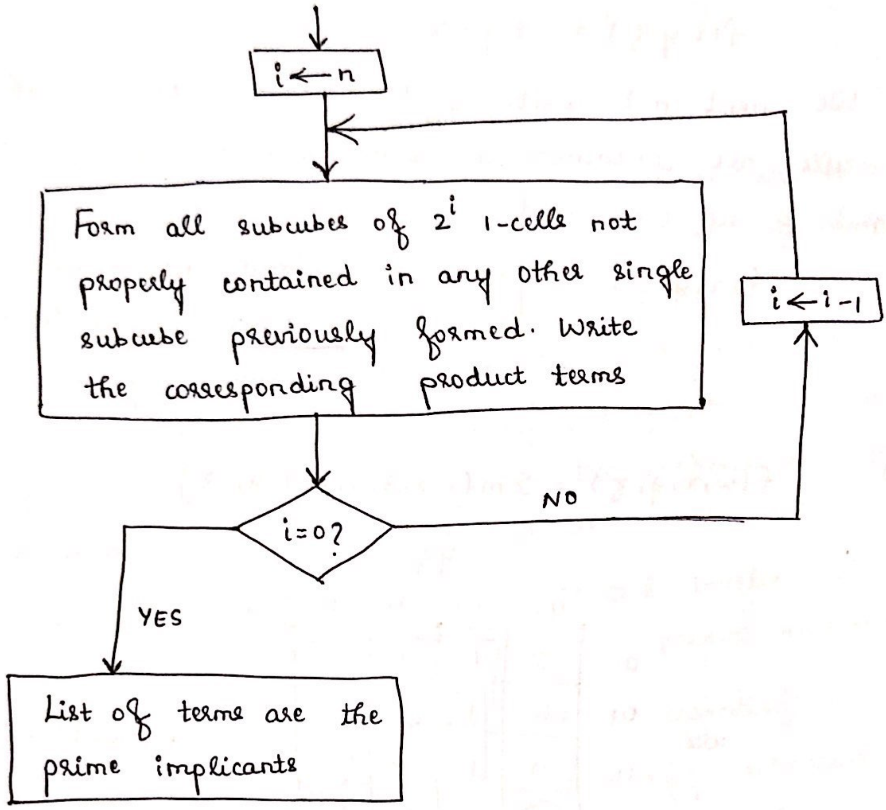
Next search for $2^a \times 2^b = 2^{n-2}$

Similarly for $i = 3, 4, \ldots n$

Continue the search in such a way that no subcube is totally contained within a single previously obtained subcube.

The product terms finally obtained $\rightarrow$ prime implicants.

Algorithm to find prime implicants :

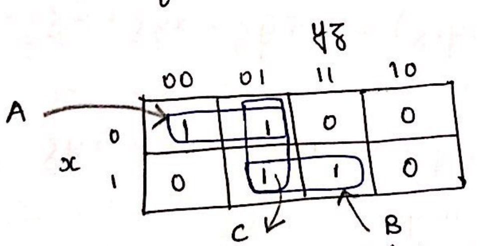


Ex: (1) Given :

$f(x,y,z) = \sum m(0,1,5,7)$

(1) There are no subcubes of 1-cells containing $2^3 = 8$ cells.

(2) No subcubes of 1-cells containing $2^2 = 4$ cells.



K-map(2)

(3) Subcubes of 1-cells containing $2^1 = 2$ cells.

$$\underbrace{1.\ \bar{x}\bar{y} \qquad 2.\ \bar{y}z \qquad 3.\ xz}_{\text{prime implicants}}$$

Subcube C is partially contained in A and B but not totally contained in either of the subcubes.

$$f(x,y,z) = \bar{x}\bar{y} + xz$$

We need not write $\bar{y}z$ (subcube C) because all the 1-cells are contained in some subcube if just A and B are considered.

$$f(x,y,z) = \bar{x}\bar{y} + xz \rightarrow \text{minimal disjunctive normal formula}$$

(2) $f(w,x,y,z) = \Sigma m(1,2,3,5,6,7,8,13)$

$wx \backslash yz$	00	01	11	10
00	0	1	1	1
01	0	1	1	1
11	0	1	0	0
10	1	0	0	0

K-map (3)

Four prime implicants are,

$$w\bar{x}\bar{y}\bar{z}, \quad x\bar{y}z, \quad \bar{w}z, \quad \bar{w}y$$

$$\therefore f(w,x,y,z) = w\bar{x}\bar{y}\bar{z} + x\bar{y}z + \bar{w}z + \bar{w}y$$

(3) Consider $f(w,x,y,z) = x\bar{y} + wxz + w\bar{x}yz$

Draw the K-map

$wx \backslash yz$	00	01	11	10
00				
01	1	1		
11	1	1	1	
10			1	

K-map (4)

$x\bar{y} \rightarrow x=1, y=0$ (4 1-cells)

$wxz \rightarrow w=1, x=1, z=1$ (2 1-cells)

$w\bar{x}yz \rightarrow w=1, x=0, y=1, z=1$ (1 1-cell)

Essential prime implicants:

Minimal sum - minimal disjunctive normal formula describing a function consisting of prime implicants.

Minimal product - minimal conjunctive normal formula describing a function consisting of prime implicates.

A 1-cell that can be in only one prime implicant subcube → essential 1-cell and the corresponding prime implicant → essential prime implicant.

In K-map(2), cells 0 and 7 - essential 1-cells

$\bar{x}\bar{y}$ and xz - essential prime implicants.

In K-map(3), cells 1, 2, 6, 8 and 13 - essential 1-cells

$w\bar{x}\bar{y}\bar{z}$, $\bar{w}z$, $\bar{w}y$ and $x\bar{y}z$ - essential prime implicants.

All essential prime implicants of a function must appear in all irredundant disjunctive normal formula and hence in minimal sum.

Minimal sums:

Sum of all essential prime implicants if all 1-cells of the map are covered.

If 1-cells are remaining, additional subcubes must be as large as possible.

Ex (1) $f(w,x,y,z) = \bar{w}\bar{x} + \bar{w}xz + yz$

		yz			
		00	01	11	10
	00	1	1	1	1
	01	0	1	1	0
wx	11	0	0	1	0
	10	0	0	1	0

		yz			
		00	01	11	10
	00	1	1	1	1
	01	0	1	1	0
wx	11	0	0	1	0
	10	0	0	1	0

Minimal sum $f(w,x,y,z) = \bar{w}\bar{x} + yz + \bar{w}z$

Essential 1-cells $\rightarrow$ 0, 2, 5, 11, 15

(2) $f(w,x,y,z) = \sum m(0,1,2,4,5,7,9,12)$

$f(w,x,y,z) = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}y\bar{z} + \bar{w}x\bar{y}\bar{z}$
$+ \bar{w}x\bar{y}z + \bar{w}xyz + w\bar{x}\bar{y}z + wx\bar{y}\bar{z}$

		yz			
		00	01	11	10
	00	1	1	0	1
	01	1	1	1	0
wx	11	1	0	0	0
	10	0	1	0	0

Minimal sum $f(w,x,y,z) = \bar{w}\bar{x}\bar{z} + \bar{w}xz + x\bar{y}\bar{z} + \bar{x}\bar{y}z$

Essential 1-cells $\rightarrow$ 2, 7, 9, 12

		yz			
		00	01	11	10
	00	1	1	0	1
	01	1	1	1	0
wx	11	1	0	0	0
	10	0	1	0	0

$f(w,x,y,z) = \bar{w}\bar{y} + \bar{w}\bar{x}\bar{z} + \bar{w}xz + x\bar{y}\bar{z} + \bar{x}\bar{y}z$

Essential 1-cells $\rightarrow$ 2, 7, 9, 12

$\therefore f(w,x,y,z) = \bar{w}\bar{x}\bar{z} + \bar{w}xz + x\bar{y}\bar{z} + \bar{x}\bar{y}z$

Consider grouping of 4 cells 0000, 0001, 0100, 0101.
(0) (1) (4) (5)

0 is already grouped with 2.
1 is already grouped with 9.
4 is already grouped with 12.
5 is already grouped with 7.

Hence this subcube is redundant and hence not included in minimal sum (irredundant disjunctive normal formula).

(3) Given:

x \ yz	00	01	11	10
0	1	0	1	0
1	1	0	1	1

$f(w,x,y,z) = \bar{y}\bar{z} + yz + x\bar{z}$ - minimal sum

OR

x \ yz	00	01	11	10
0	1	0	1	0
1	1	0	1	1

$f(w,x,y,z) = \bar{y}\bar{z} + yz + xy$ - minimal sum

(4) Given:

wx \ yz	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

$f(w,x,y,z) = \bar{w}\bar{z} + wx\bar{y} + w\bar{y}z + xyz$ - minimal sum

OR

$wx \backslash yz$	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

$f(w,x,y,z) = \bar{w}\bar{z} + x\bar{y}\bar{z} + w\bar{y}z + xyz$ - minimal sum

(5) <u>Given</u>:

$x \backslash yz$	00	01	11	10
0	1	1	1	0
1	1	0	1	1

$f(x,y,z) = \bar{y}\bar{z} + \bar{x}\bar{y} + yz + xy$

OR

$x \backslash yz$	00	01	11	10
0	1	1	1	0
1	1	0	1	1

$f(x,y,z) = \bar{y}\bar{z} + \bar{x}z + xy$ - minimal sum

OR

$x \backslash yz$	00	01	11	10
0	1	1	1	0
1	1	0	1	1

$f(x,y,z) = \bar{x}\bar{y} + x\bar{z} + yz$ - minimal sum

<u>Minimal products</u>:

Ex: (1) $f(w,x,y,z) = \sum m(1,3,4,5,6,7,11,14,15)$

		yz		
wx	00	01	11	10
00	0	1	1	0
01	1	1	1	1
11	0	0	1	1
10	0	0	1	0

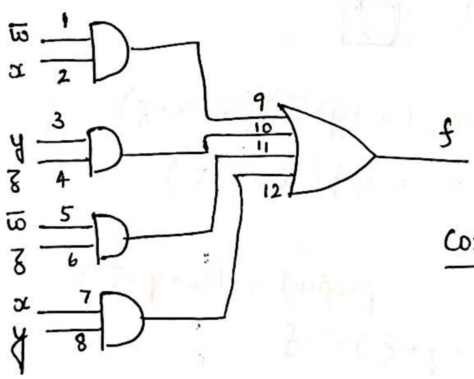
Minimal sum $f(w,x,y,z) = \bar{w}x + yz + \bar{w}z + xy$

		yz		
wx	00	01	11	10
00	0	1	1	0
01	1	1	1	1
11	0	0	1	1
10	0	0	1	0

Minimal product $f(w,x,y,z) = (\bar{w}+y)(x+z)$

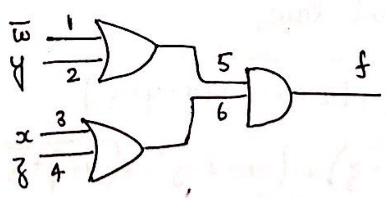
Essential 0-cells $\longrightarrow$ 0, 2, 9, 10, 12, 13

Logic circuit for minimal sum:



Cost = 12

Logic circuit for minimal product:



Cost = 6

To determine a minimal two-level gate network - determine both minimal sum and minimal product. Both are equivalent but one may have a lower cost. Here minimal product - lower cost.

(2) Given :

$wx \backslash yz$	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

Minimal product :

$$f(w,x,y,z) = (w+y+\bar{z})(\bar{w}+\bar{y}+z)(\bar{w}+x+z)(x+\bar{y}+\bar{z})$$

Essential 0-cells : 2, 3, 5, 8., 11, 14

$wx \backslash yz$	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

$$f(w,x,y,z) = (w+x+\bar{z})(w+y+\bar{z})(\bar{w}+x+z)$$
$$(\bar{w}+x+\bar{y})(\bar{w}+\bar{y}+z)$$

We know that minimal product $= (w+y+\bar{z})(\bar{w}+\bar{y}+z)(\bar{w}+x+z)(x+\bar{y}+\bar{z}) = f$

Minimal sum $= \bar{f}$ (complement function)

By applying DeMorgan's law,

$$\bar{f} = \overline{(w+y+\bar{z})(\bar{w}+\bar{y}+z)(\bar{w}+x+z)(x+\bar{y}+\bar{z})}$$
$$= \left(\overline{w+y+\bar{z}}\right)+\left(\overline{\bar{w}+\bar{y}+z}\right)+\left(\overline{\bar{w}+x+z}\right)+\left(\overline{x+\bar{y}+\bar{z}}\right)$$

$$\bar{f} = \bar{w}\bar{y}\bar{\bar{z}} + \bar{\bar{w}}\bar{\bar{y}}\bar{z} + \bar{\bar{w}}\bar{x}\bar{z} + \bar{x}\bar{\bar{y}}\bar{\bar{z}}$$
$$= \bar{w}\bar{y}z + wy\bar{z} + w\bar{x}\bar{z} + \bar{x}yz$$

We can verify the minimal sum expression from the K-map,

wx \ yz	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

$$f(w,x,y,z) = wx\bar{y} + w\bar{y}z + xyz + \bar{w}\bar{z}$$

↓

minimal sum for f

wx \ yz	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

→ 5 terms

OR

wx \ yz	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	1	1	1	0
10	0	1	0	0

→ 4 terms

$$\bar{f}(w,x,y,z) = \bar{w}\bar{y}z + \bar{x}yz + w\bar{x}\bar{z} + \bar{x}yz$$

↓

minimal sum for $\bar{f}$

∴ We can directly write minimal product or write the minimal sum for $\bar{f}$ (by grouping 0-cells) and complement it ($\bar{\bar{f}} = f$) by applying Demorgan's laws.

Minimal expressions of incomplete boolean functions:

Don't care cells - cells containing dashed entries on a K-map.

Prime implicants of an incomplete boolean function are the prime implicants of the complete boolean function obtained by putting a functional value 1 for all don't care conditions.

Prime implicates of an incomplete boolean function are the prime implicates of the complete boolean function obtained by putting a functional value 0 for all don't care conditions.

Minimal sums:

To form a larger group of subcells - don't care cells are considered as 1. Don't care cells are used optionally to obtain best possible groupings.

Only the actual 1-cells in the map can be used for considering essential 1-cells.

Ex: Given:

Consider the truth table,

w	x	y	z	f
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	—

Table 1

w	x	y	z	f
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	–
1	0	1	1	0
1	1	0	0	1
1	1	0	1	–
1	1	1	0	–
1	1	1	1	0

$$f(w,x,y,z) = \sum m(0,1,3,7,8,12) + dc(5,10,13,14)$$

wx \ yz	00	01	11	10
00	1	1	1	0
01	0	–	1	0
11	1	–	0	–
10	1	0	0	–

$f(w,x,y,z) = w\bar{z} + \bar{w}z + \bar{w}\bar{x}\bar{y}$ – minimal sum

OR

wx \ yz	00	01	11	10
00	1	1	1	0
01	0	–	1	0
11	1	–	0	–
10	1	0	0	–

$f(w,x,y,z) = w\bar{z} + \bar{w}z + \bar{x}\bar{y}\bar{z}$ – minimal sum

Here don't care cell 1101 is not considered for grouping.

Minimal products:

Ex: From table 1,

wx \ yz	00	01	11	10
00	1	1	1	0
01	0	-	1	0
11	1	-	0	-
10	1	0	0	-

$$\overline{f}(w,x,y,z) = \overline{w}x\overline{y} + wz + y\overline{z}$$

$\downarrow$

minimal sum of $\overline{f}$

$$f(w,x,y,z) = (w+\overline{x}+y)(\overline{w}+\overline{z})(\overline{y}+z)$$

$\downarrow$

minimal product

OR

wx \ yz	00	01	11	10
00	1	1	1	0
01	0	-	1	0
11	1	-	0	-
10	1	0	0	-

$$f(w,x,y,z) = (w+\overline{x}+z)(\overline{w}+\overline{z})(\overline{y}+z)$$

$\downarrow$

minimal product

In a K-map, (for SOP)

Pair - Two 1-cells grouped together ($2^1=2$)

Quad - Two 1-cells + Two 1-cells
Four 1-cells grouped together ($2^2=4$)

Octet - Eight 1-cells grouped together ($2^3=8$)