CONTROL SYSTEMS

FIRST EDITION

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PREFACE

This text attempts to provide a simple explanation about concepts of control system. The book have been designed as a self study material for students of all disciplines studying in engineering colleges. More attention have been paid to problems than theory and detailed explanation is presented for each solved problem for better understanding.

I will be profusely grateful to the readers for pinpointing any lapses and errors. I would continue to welcome constructive criticism and fruitful suggestions for further improvement of the book.

I thank my wife **Mrs.C.Gnanaparanjothi Nagoor Kani** for her constant encouragement and valuable suggestions. I thank my just born son **N.Bharath Raj** alias **Chandrakani Allaudeen** for allowing to complete this work by entertaining me. I would also like to thank **Ms.P.Annapoorneswari, Mr. J.P. Vivek, Ms.G.Nandhini** for assisting and proof reading and **Mr.T.M.Ramachandran** and **Mr.S.Suresh** for typesetting. I thank **Mr.T.S. Sundar Raman** of Novena Press Ltd., for his timely help and cooperation.

Finally, I thank my sisters, brothers, relatives, friends and the student community for their constant encouragement.

A. Nagoor Kani

PREFACE TO SECOND REPRINT

The author thank the faculty of Engineering Colleges and student community for their overwhelming support to the first edition of this book. The errors in the previous print have been corrected in this second reprint. The author continue to welcome constructive criticism and suggestions for further improvement of the book.

A. Nagoor Kani

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CHAPTER ONE

1 MATHEMATICAL MODELS OF CONTROL SYSTEM

1.1 CONTROL SYSTEM

Control system theory evolved as an engineering discipline and due to universality of the principles involved, it is extended to various fields like economy, sociology, biology, medicine, etc. Control theory has played a vital role in the advance of engineering and science. The automatic control has become an integral part of modern manufacturing and industrial processes. For example, numerical control of machine tools in manufacturing industries, controlling pressure, temperature, humidity , viscosity and flow in process industry.

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system. In a system when the output quantity is controlled by varying the input quantity, the system is called control system. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

OPEN LOOP SYSTEM

Any physical system which does not automatically correct the variation in its output, is called an open loop system, or control system in which the output quantity has no effect upon the input quantity are called open-loop control system. This means that the output is not feedback to the input for correction.

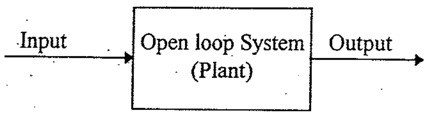


Fig 1.1 : Open loop system

In open loop system the output can be varied by varying the input. But due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct the output. In open loop systems the changes in output are corrected by changing the input manually.

CLOSED LOOP SYSTEM

Control systems in which the output has an effect upon the input quantity in such a manner as to maintain the desired output value are called closed loop systems.

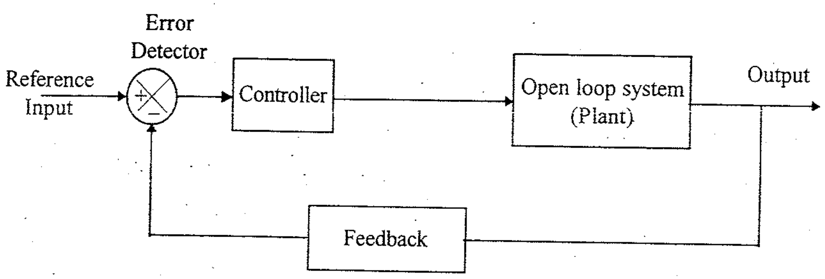


Fig 1.2 : Closed loop system

The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called automatic control system. The general block diagram of an automatic control system is shown in fig 1.2. It consists of an error detector, a controller, plant (open loop system) and feedback path elements.

The reference signal (or input signal) corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

Advantages of open loop systems

1. The open loop systems are simple and economical.
2. The open loop systems are easier to construct.
3. Generally the open loop systems are stable.

Disadvantage of open loop systems

1. The open loop systems are inaccurate and unreliable.
2. The changes in the output due to external disturbances are not corrected automatically.

Advantages of closed loop systems

1. The closed loop systems are accurate.
2. The closed loop systems are accurate even in the presence of non-linearities.
3. The sensitivity of the systems may be made small to make the system more stable.
4. The closed loop systems are less affected by noise.

Disadvantages of closed loop systems

1. The closed loop systems are complex and costlier.
2. The feedback in closed loop system may lead to oscillatory response.
3. The feedback reduces the overall gain of the system.
4. Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

1.2 EXAMPLES OF CONTROL SYSTEMS

EXAMPLE 1 : TEMPERATURE CONTROL SYSTEM

Open loop system

The electric furnace shown in fig 1.3. is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON.

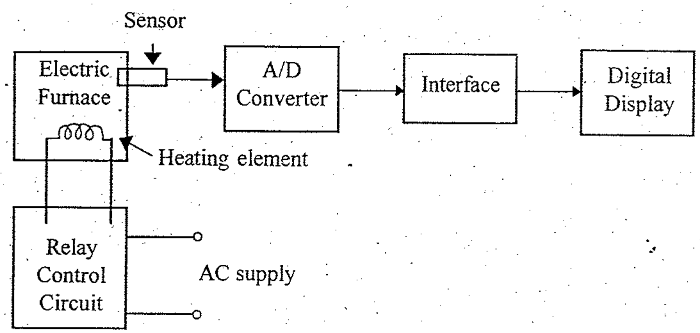


Fig 1.3 : Open loop temperature control system

The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog - to - Digital converter (A/D converter). The digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is not altered automatically.

Closed loop system

The electric furnace shown in fig 1.4 is a closed loop system. The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON.

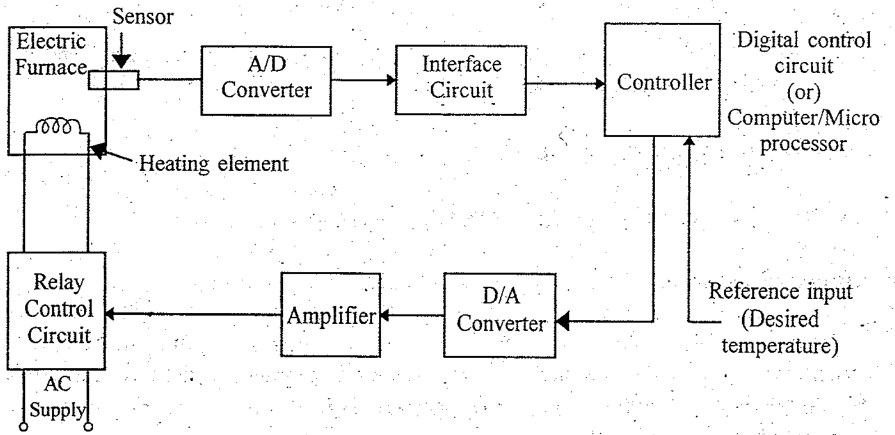


Fig 1.4 : Closed loop temperature control system

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports. The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output. Hence it is a closed loop system.

EXAMPLE 2 : TRAFFIC CONTROL SYSTEM

Open loop system

Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals are based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of the traffic before giving the signals. Since the time slot does not changes according to traffic density, the system is open loop system.

Closed loop system

Traffic control system can be made as a closed loop system if the time slots of the signals are decided based on the density of traffic. In closed loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic. Since the closed loop system dynamically changes the timings the flow of vehicles will be better than open loop system.

EXAMPLE 3 : NUMERICAL CONTROL SYSTEM

Open loop system

Numerical control is a method of controlling the motion of machine components by use of numbers. Here, the position of work head tool is controlled by the binary information contained on a disk.

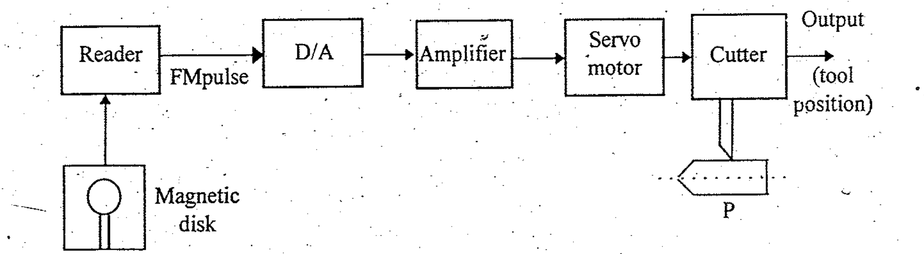


Fig 1.5 : Open loop numerical control system

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). The tool will operate on the desired part P. To start the system, the disk is fed through the reader to the D/A converter. The D/A converter converts the FM(frequency modulated) output of the reader to a analog signal. It is amplified and fed to servomotor which positions the cutter on the desired part P. The position of the cutter head is controlled by the angular motion of the servomotor. This is an open loop system since no feedback path exists between the output and input. The system positions the tool for a given input command. Any deviation in the desired position is not checked and corrected automatically.

Closed loop system

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). To start the system, the disk is loaded in the reader. The controller compares the frequency modulated input pulse signal with the feedback pulse signal. The controller is a computer or microprocessor system. The controller carries out mathematical operations on the difference in the pulse signals and generates an error signal. The D/A converter converts the controller output pulse (error signal) into an analog signal. The amplified analog signal rotates the servomotor to position the tool on the job. The position of the cutterhead is controlled according to the input of the servomotor. The transducer attached to the cutterhead converts the motion into an electrical signal. The analog electrical signal is converted to the digital pulse signal by the A/D converter. Then this signal is compared with the input pulse signal. If there is any difference between these two, the controller sends a signal to the servomotor to reduce it. Thus the system automatically corrects any deviation in the desired output tool position. An advantage of numerical control is that complex parts can be produced with uniform tolerances at the maximum milling speed.

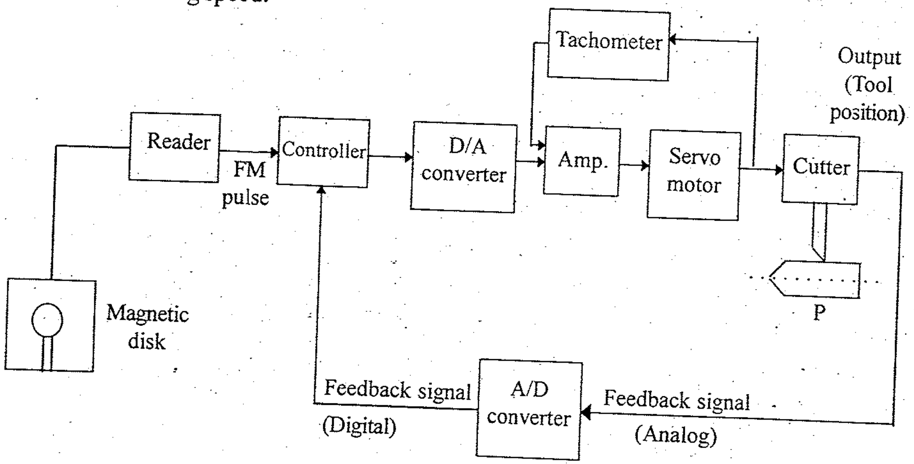


Fig 1.6 : Closed loop numerical control system

EXAMPLE 4 : POSITION CONTROL SYSTEM USING SERVOMOTOR

The position control system shown in fig 1.7 is a closed loop system. The system consists of a servomotor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. Potentiometers are used to convert the mechanical motion to electrical signals. The desired load position (θ_R) is set on the input potentiometer and the actual load position (θ_c) is fed to feedback potentiometer. The difference between the two angular positions generates an error signal, which is amplified and fed to generator field circuit. The induced emf of the generator drives the motor. The rotation of the motor stops when the error signal is zero, i.e. When the desired load position is reached.

This type of control systems are called servomechanisms .The servo or servomechanisms are feedback control systems in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

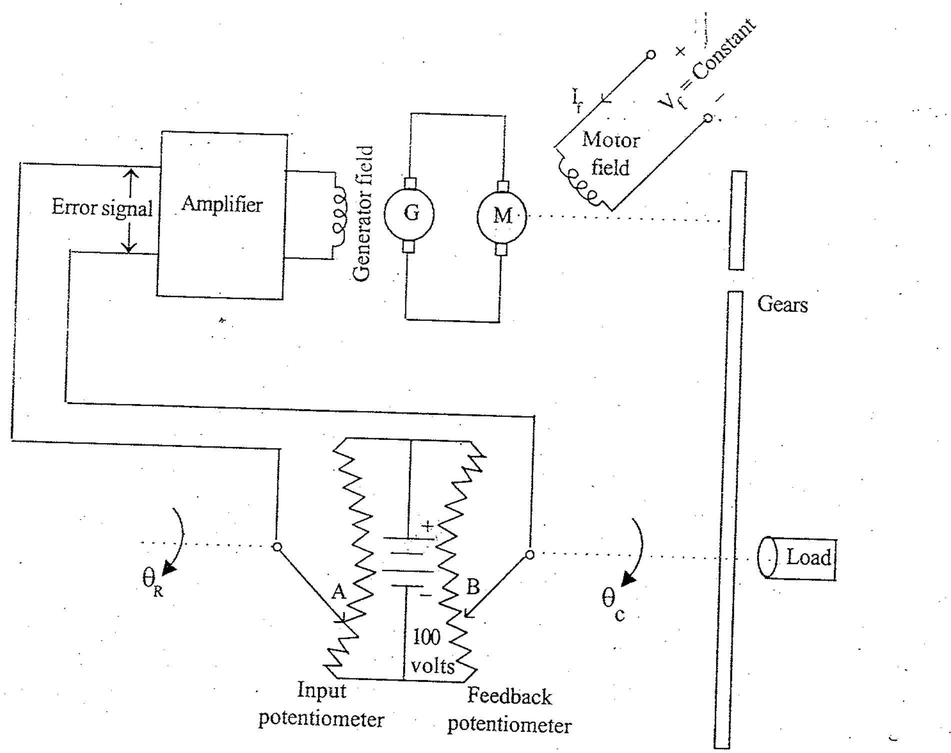


Fig 1.7 : A position control system (servomechanism)

1.3 MATHEMATICAL MODELS OF CONTROL SYSTEMS

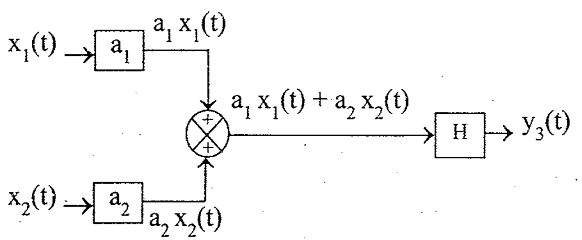
A control system is a collection of physical objects (components) connected together to serve an objective. The input output relations of various physical components of a system are governed by differential equations. The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition and homogenity. This principle implies that if a system model has responses $y_1(t)$ and $y_2(t)$ to any inputs $x_1(t)$ and $x_2(t)$ respectively, then the system response to the linear combination of these inputs $a_1 x_1(t) + a_2 x_2(t)$ is given by linear combination of the individual outputs $a_1 y_1(t) + a_2 y_2(t)$, where a_1 and a_2 are constants.

The principle of superposition can be explained diagrammatically as shown in fig 1.8.

Let H be a linear system.

$x_1(t) \rightarrow \boxed{H} \rightarrow y_1(t)$ $\qquad$ $x_2(t) \rightarrow \boxed{H} \rightarrow y_2(t)$



For linearity $\qquad$ $$y_3(t) = a_1 y_1(t) + a_2 y_2(t) \qquad .....(1.1)$$

Fig 1.8 : Principle of linearity and superposition

A mathematical model will be linear if the differential equations describing the system has constant coefficients (or the coefficients may be functions of independent variables). If the coefficients of the differential equation describing the system are constants then the model is linear time invariant. If the coefficients of differential equations governing the system are functions of time then the model is linear time varying.

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single input and single output system analysis is transfer function of the system. The transfer function of a system is defined as the ratio of laplace transform of output to the laplace transform of input with zero initial conditions.

$$\text{Transfer function} = \left.\frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}\right|_{\text{with zero initial conditions}}$$

The transfer function can be obtained by taking laplace transform of the differential equations governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.4 MECHANICAL TRANSLATIONAL SYSTEMS

The model of mechanical translational systems can be obtained by using three basic elements mass, spring and dash-pot. These three elements represents three essential phenomena which occur in various ways in mechanical systems.

The weight of the mechanical system is represented by the element mass and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a spring. The friction existing in mechanical system can be represented by the dash-pot. The dash-pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by Newton's second law of motion. For translational systems it states that the sum of forces acting on a body is zero. (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m.

$v = \frac{dx}{dt}$ = Velocity, m/sec.

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

***Note :** Lower case letters are functions of time.*

FORCE BALANCE EQUATIONS OF IDEALIZED ELEMENTS

Consider an ideal mass element shown in fig 1.9 which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let f = Applied force

f_m = Opposing force due to mass

Here $f_m \propto \dfrac{d^2x}{dt^2}$ or $f_m = M\dfrac{d^2x}{dt^2}$

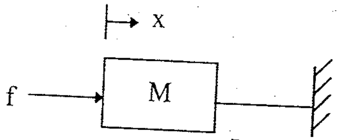


Fig 1.9 : Ideal mass element

By Newton's second law, $f = f_m = M\dfrac{d^2x}{dt^2}$ .....(1.2)

Consider an ideal frictional element dashpot shown in fig 1.10 which has negligible mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

Let f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \dfrac{dx}{dt}$ or $f_b = B\dfrac{dx}{dt}$

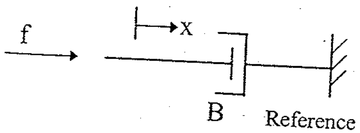


Fig 1.10 : Ideal dashpot with one end fixed to reference

By Newton's second law, $f = f_b = B\dfrac{dx}{dt}$ .....(1-3)

When the dashpot has displacement at both ends as shown in fig 1.11, the opposing force is proportional to differential velocity.

$f_b \propto \dfrac{d}{dt}(x_1 - x_2)$; $f_b = B\dfrac{d}{dt}(x_1 - x_2)$

$\therefore f = f_b = B\dfrac{d}{dt}(x_1 - x_2)$ .....(1-4)

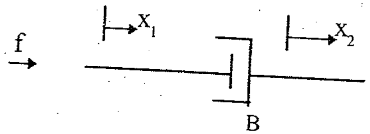


Fig 1.11 : Ideal dashpot with displacement at both ends.

Consider an ideal elastic element spring shown in fig 1.12, which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let f = Applied force

f_k = Opposing force due to elasticity

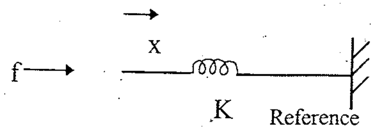


Fig 1.12 : Ideal spring with one end fixed to reference

Here $f_k \propto x$ or $f_k = K\,x$

By Newton's second law, $f = f_k = Kx$ .....(1-5)

When the spring has displacement at both ends as shown in fig 1.13 the opposing force is proportional to differential displacement.

$f_k \propto (x_1 - x_2)$

$f_k = K(x_1 - x_2)$

$\therefore f = f_k = K(x_1 - x_2)$ .....(1-6)

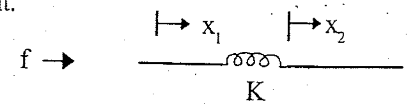


Fig 1.13 : Ideal spring with displacement at both ends

GUIDELINE TO DETERMINE THE TRANSFER FUNCTION OF MECHANICAL TRANSLATIONAL SYSTEM

1. In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as x_1, x_2, x_3, etc. and assign a displacement to each mass(node). The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.

3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass (node). Always the opposing force acts in a direction opposite to applied force. The mass has to move in the direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing force.

4. For each free body diagram write one differential equation by equating the sum of applied forces to the sum of opposing forces.

5. Take laplace transform of differential equations to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. This ratio is the transfer function of the system.

Note : Laplace transform of $x(t) = L[x(t)] = X(s)$

$$\left.\begin{array}{l}\textit{Laplace transform of } \dfrac{dx(t)}{dt} \\ \textit{with zero initial conditions}\end{array}\right\} = L\left[\frac{d}{dt}x(t)\right] = s\,X(s)$$

$$\left.\begin{array}{l}\textit{Laplace transform of } \dfrac{d^2x(t)}{dt^2} \\ \textit{with zero initial conditions}\end{array}\right\} = L\left[\frac{d^2}{dt^2}x(t)\right] = s^2\,X(s)$$

EXAMPLE 1.1

Write the differential equations governing the mechanical system shown in fig 1.1.1. and determine the transfer function.

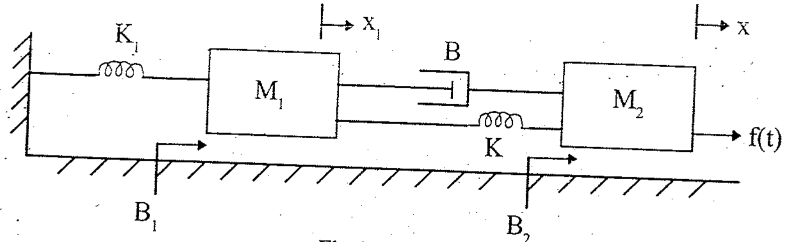


Fig 1.1.1

SOLUTION

In the given system, applied force f(t) is the input and displacement x is the output.

Let Laplace transform of $f(t) = L[f(t)] = F(s)$

and Laplace transform of $x = L[x] = X(s)$

Hence the required transfer function is $\dfrac{X(s)}{F(s)}$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass M_1 be x_1. The free body diagram of mass M_1 is shown in fig 1.1.2. The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.

$$f_{m1} = M_1\frac{d^2x_1}{dt^2}\ ;\quad f_{b1} = B_1\frac{dx_1}{dt}\ ;\ f_{k1} = K_1x_1$$

$$f_b = B\frac{d}{dt}(x_1 - x)\ ;\ f_k = K(x_1 - x)$$

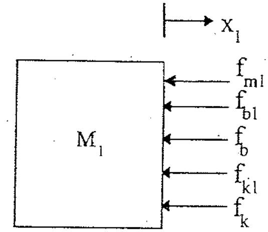


Fig 1.1.2 : Free body diagram of mass M_1 (node 1)

By Newton's second law

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1x_1 + K(x_1 - x) = 0 \qquad .....(1.1.1)$$

On taking Laplace transform with zero initial conditions

$$M_1s^2X_1(s) + B_1sX_1(s) + Bs\,[X_1(s) - X(s)] + K_1X_1(s) + K\,[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

$$\therefore X_1(s) = X(s)\,\frac{Bs + K}{M_1s^2 + (B_1 + B)\,s + (K_1 + K)} \qquad .....(1.1.2)$$

Note : Laplace transform of $x_1 = L[x_1] = X_1(s)$

The free body diagram of mass M_2 is shown in fig 1.1.3. The opposing forces acting on M_2 are marked as f_{m2}, f_{b2}, f_b and f_k

$$f_{m2} = M_2\frac{d^2x}{dt^2}\quad ;\quad f_{b2} = B_2\frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1)\quad ;\quad f_k = K(x - x_1)$$

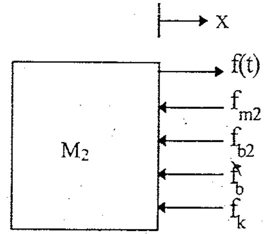


Fig 1.1.3 : Free body diagram of mass M_2 (node 2)

By Newton's second law

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2\frac{d^2x}{dt^2} + B_2\frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t) \qquad .....(1.1.3)$$

On taking laplace transform with zero initial conditions

$$M_2s^2X(s) + B_2sX(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \qquad .....(1.1.4)$$

Substituting for $X_1(s)$ from equation (1.1.2) in equation (1.1.4) we get,

$$X(s)\,[M_2s^2+(B_2+B)s+K]-X(s)\,\frac{(Bs+K)^2}{M_1s^2+(B_1+B)s+(K_1+K)}=F(s)$$

$$X(s)\left[\frac{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}{M_1s^2+(B_1+B)\,s+(K_1+K)}\right]=F(s)$$

$$\therefore\ \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2]}$$

RESULT

The differential equations governing the system are

1. $$M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B\frac{d}{dt}(x_1-x)+K_1x_1+K(x_1-x)=0$$

2. $$M_2\frac{d^2x}{dt^2}+B_2\frac{dx}{dt}+B\frac{d}{dt}(x-x_1)+K(x-x_1)=f(t)$$

The transfer function of the system is

$$\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)\,s+(K_1+K)}{\left[M_1s^2+(B_1+B)\,s+(K_1+K)\right]\left[M_2s^2+(B_2+B)\,s+K\right]-(Bs+K)^2}$$

EXAMPLE 1.2

Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig 1.2.1.

SOLUTION

Let, Laplace transform of $f(t) = L[f(t)]=F(s)$

Laplace transform of $y_1 = L[y_1]=Y_1(s)$

Laplace transform of $y_2 = L[y_2]=Y_2(s)$

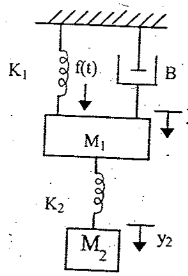


Fig 1.2.1

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes.

The free body diagram of mass M_1 is shown in fig 1.2.2. The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

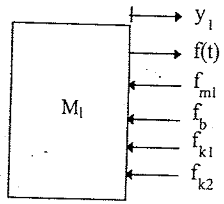


Fig 1.2.2.

$$f_{m1}=M_1\frac{d^2y_1}{dt^2}\ ;\ f_b=B\frac{dy_1}{dt}\ ;\ f_{k1}=K_1y_1\ ;$$
$$f_{k2}=K_2(y_1-y_2)$$

By Newton's second law, $f_{m1}+f_b+f_{k1}+f_{k2}=f(t)$

$$\therefore M_1\frac{d^2y_1}{dt^2}+B\frac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t) \quad .....(1.2.1)$$

On taking laplace transform with zero initial condition

$$M_1s^2Y_1(s)+BsY_1(s)+K_1Y_1(s)+K_2[Y_1(s)-Y_2(s)]=F(s)$$
$$Y_1(s)[M_1s^2+Bs+(K_1+K_2)]-Y_2(s)K_2=F(s) \quad .....(1.2.2)$$

The free body diagram of mass M_2 is shown in fig 1.2.3. The opposing forces acting on M_2 are f_{m2} and f_{k2}

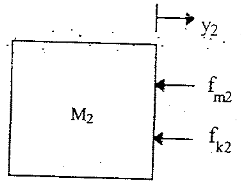


Fig 1.2.3

$$f_{m2}=M_2\frac{d^2y_2}{dt^2}\ ;\ f_{k2}=K_2(y_2-y_1)$$

By Newton's second law, $f_{m2}+f_{k2}=0$

$$\therefore M_2\frac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0 \quad .....(1.2.3)$$

On taking Laplace transform,

$$M_2s^2Y_2(s)+K_2[Y_2(s)-Y_1(s)]=0$$
$$Y_2(s)\,[M_2s^2+K_2]-Y_1(s)\,K_2=0$$
$$\therefore Y_1(s)=Y_2(s)\,\frac{M_2s^2+K_2}{K_2} \quad .....(1.2.4)$$

Substituting for $Y_1(s)$ from equation (1.2.4) in equation (1.2.2)

$$Y_2(s)\left[\frac{M_2s^2+K_2}{K_2}\right]\left[M_1s^2+Bs+(K_1+K_2)\right]-Y_2(s)\,K_2=F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2+K_2)\,(M_1s^2+Bs+(K_1+K_2))-K_2^2}{K_2}\right]=F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1s^2 + Bs + (K_1 + K_2)\right]\left[M_2s^2 + K_2\right] - K_2^2} \qquad .....(1.2.5)$$

RESULT

The differential equations governing the system are

1. $M_1\dfrac{d^2y_1}{dt^2} + B\dfrac{dy_1}{dt} + K_1y_1 + K_2(y_1 - y_2) = f(t)$

2. $M_2\dfrac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$

The transfer function of the system is

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1s^2 + Bs + (K_1 + K_2)\right]\left[M_2s^2 + K_2\right] - K_2^2}$$

EXAMPLE 1.3

Determine the transfer function $\dfrac{X_1(s)}{F(s)}$ and $\dfrac{X_2(s)}{F(s)}$, for the system shown in fig 1.3.1.

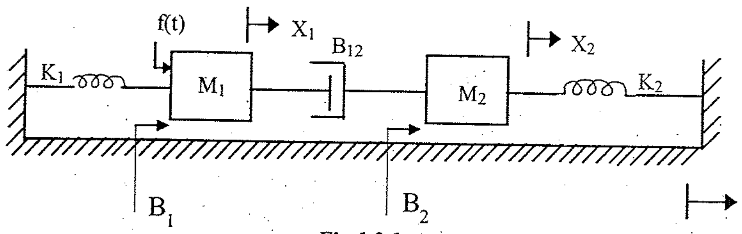


Fig 1.3.1

SOLUTION

Let, Laplace transform of $f(t) = F(s)$

Laplace transform of $x_1 = X_1(s)$

Laplace transform of $x_2 = X_2(s)$

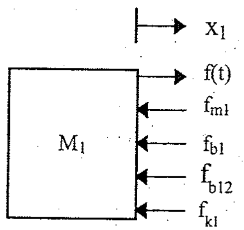


Fig 1.3.2

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes. The free body diagram of mass M_1 is shown in fig 1.3.2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}

$$f_{m1} = M_1\frac{d^2x_1}{dt^2}\ ;\ f_{b1} = B_1\frac{dx_1}{dt}\ ;\ f_{b12} = B_{12}\frac{d}{dt}(x_1 - x_2)\ ;\ f_{k1} = K_1x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_{12}\frac{d(x_1 - x_2)}{dt} + K_1x_1 = f(t) \qquad .....(1.3.1)$$

On taking Laplace transform with zero initial conditions

$$M_1s^2X_1(s) + B_1sX_1(s) + B_{12}s\,[X_1(s) - X_2(s)] + K_1X_1(s) = F(s)$$

$$X_1(s)\ [M_1s^2 + (B_1 + B_{12})\,s + K_1] - B_{12}sX_2(s) = F(s) \qquad .....(1.3.2)$$

The free body diagram of mass M_2 is shown in fig 1.3.3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b12} and f_{k2}

$$f_{m2} = M_2\frac{d^2x_2}{dt^2}\ ;\ f_{b2} = B_2\frac{dx_2}{dt}$$

$$f_{b12} = B_{12}\frac{d}{dt}(x_2 - x_1)\ ;\ f_{k2} = K_2x_2$$

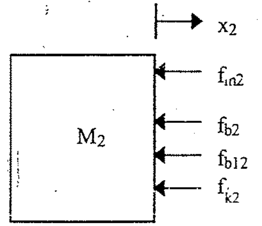


Fig1.3.3.

By Newton's second law, $f_{m2} + f_{b2} + f_{b12} + f_{k2} = 0$

$$M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + B_{12}\frac{d(x_2 - x_1)}{dt} + K_2x_2 = 0 \qquad ......(1.3.3)$$

On taking Laplace transform with zero initial conditions

$$M_2s^2X_2(s) + B_2sX_2(s) + B_{12}s\,[X_2(s) - X_1(s)] + K_2X_2(s) = 0$$

$$X_2(s)\ [M_2s^2 + (B_2 + B_{12})\,s + K_2] - B_{12}\,s\,X_1(s) = 0$$

$$X_2(s)\ [M_2s^2 + (B_2 + B_{12})\,s + K_2] = B_{12}\,s\,X_1(s)$$

$$X_2(s) = \frac{B_{12}s\,X_1(s)}{[M_2s^2 + (B_2 + B_{12})\,s + K_2]} \qquad .....(1.3.4)$$

Substituting for $X_2(s)$ from equation (1.3.4) in equation (1.3.2)

$$X_1(s)\ [M_1s^2 + (B_1 + B_{12})\,s + K_1] - \frac{(B_{12}s)^2\,X_1(s)}{M_2s^2 + (B_2 + B_{12})\,s + K_2} = F(s)$$

$$\frac{X_1(s)\left[[M_1s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2\right]}{M_2s^2 + (B_2 + B_{12})\,s + K_2} = F(s)$$

$$\therefore \frac{X_1(s)}{F(s)} = \frac{M_2s^2 + (B_2 + B_{12})\,s + K_2}{[M_1s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2} \quad .....(1.3.5)$$

From equation (1.3.4), $X_1(s) = \dfrac{[M_2s^2 + (B_2 + B_{12})\,s + K_2]\,X_2(s)}{B_{12}s}$ .....(1.3.6)

Substituting for $X_1(s)$ from equation (1.3.6) in equation (1.3.2)

$$\frac{X_2(s)\,[M_2s^2 + (B_2 + B_{12})\,s + K_2]}{B_{12}s}\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - B_{12}s\,X_2(s) = F(s)$$

$$X_2(s)\left[\frac{[M_2s^2 + (B_2 + B_{12})\,s + K_2]\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - (B_{12}s)^2}{B_{12}s}\right] = F(s)$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{B_{12}s}{[M_2s^2 + (B_2 + B_{12})\,s + K_2]\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - (B_{12}s)^2}$$

RESULT

The differential equation governing the system are

1. $M_1\dfrac{d^2x_1}{dt^2} + B_1\dfrac{dx_1}{dt} + B_{12}\dfrac{d(x_1 - x_2)}{dt} + K_1x_1 = f(t)$

2. $M_2\dfrac{d^2x_2}{dt^2} + B_2\dfrac{dx_2}{dt} + B_{12}\dfrac{d(x_2 - x_1)}{dt} + K_2x_2 = 0$

The transfer functions of the system are

1. $\dfrac{X_1(s)}{F(s)} = \dfrac{M_2s^2 + (B_2 + B_{12})\,s + K_2}{[M_1s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2}$

2. $\dfrac{X_2(s)}{F(s)} = \dfrac{B_{12}\,s}{[M_2s^2 + (B_2 + B_{12})\,s + K_2]\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - (B_{12}s)^2}$

EXAMPLE 1.4

Write the equations of motion in s-domain for the system shown in fig 1.4.1. Determine the transfer function of the system.

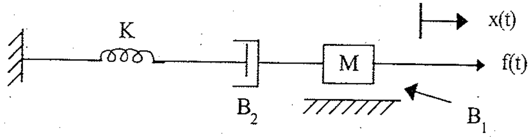


Fig 1.4.1

SOLUTION

Let, Laplace transform of $x(t) = X(s)$

Laplace transform of $f(t) = F(s)$

Let x_1 be the displacement at the meeting point of spring and dashpot. Laplace transform of x_1 is $X_1(s)$.

The system has two nodes and they are mass M and the meeting point of spring and dashpot. The differential equations governing the system are the force balance equations at these nodes. The equations of motion in the s-domain are obtained by taking Laplace transform of the differential equations.

The free body diagram of mass M is shown in fig 1.4.2. The opposing forces are marked as f_m, f_{b1} and f_{b2}:

$$f_m = M\frac{d^2x}{dt^2}\ ;\ f_{b1} = B_1\frac{dx}{dt}$$

$$f_{b2} = B_2\,\frac{d}{dt}(x - x_1)$$

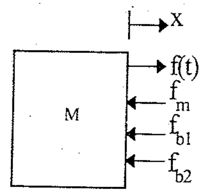


Fig 1.4.2.

By Newton's second law the force balance equation is

$$f_m + f_{b1} + f_{b2} = f(t)$$

$$\therefore M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x - x_1) = f(t) \quad .....(1.4.1)$$

On taking Laplace transform

$$Ms^2\,X(s) + B_1\,s\,X(s) + B_2\,s\,[X(s) - X_1(s)] = F(s)$$

$$\left[Ms^2 + (B_1 + B_2)\,s\right]X(s) - B_2\,s\,X_1(s) = F(s) \quad .....(1.4.2)$$

The free body diagram at the meeting point of spring and dashpot is shown in fig 1.4.3. The opposing forces are marked as f_k and f_{b2}.

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x) \;;\quad f_k = K x_1$$

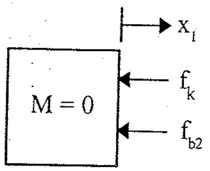


Fig 1.4.3

By Newton's second law, $f_{b2} + f_k = 0$

$$\therefore B_2 \frac{d}{dt}(x_1 - x) + K x_1 = 0 \qquad .....(1.4.3)$$

On taking Laplace transform

$$B_2 s\,[X_1(s) - X(s)] + K X_1(s) = 0$$

$$(B_2 s + K)\, X_1(s) - B_2 s\, X(s) = 0 \qquad .....(1.4.4)$$

$$\therefore X_1(s) = \frac{B_2 s}{B_2 s + K} X(s) \qquad .....(1.4.5)$$

Substituting for $X_1(s)$ from equation (1.4.5) in equation (1.4.2),

$$\left[M s^2 + (B_1 + B_2) s\right] X(s) - B_2 s \left[\frac{B_2 s}{B_2 s + K}\right] X(s) = F(s)$$

$$X(s) \frac{\left[[M s^2 + (B_1 + B_2) s]\,(B_2 s + K) - (B_2 s)^2\right]}{B_2 s + K} = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{B_2 s + K}{[M s^2 + (B_1 + B_2) s]\,(B_2 s + K) - (B_2 s)^2} \qquad .....(1.4.6)$$

RESULT

The differential equations governing the system are

1. $M \frac{d^2x}{dt^2} + B_1 \frac{dx}{dt} + B_2 \frac{d}{dt}(x - x_1) = f(t)$, 2. $B_2 \frac{d}{dt}(x_1 - x) + K x_1 = 0$

The equations of motion in s-domain are

1. $[M s^2 + (B_1 + B_2) s]\, X(s) - B_2 s\, X_1(s) = F(s)$
2. $(B_2 s + K)\, X_1(s) - B_2 s\, X(s) = 0$

The transfer function of the system is

$$\frac{X(s)}{F(s)} = \frac{B_2 s + K}{[M s^2 + (B_1 + B_2) s]\,(B_2 s + K) - (B_2 s)^2}$$

1.5 MECHANICAL ROTATIONAL SYSTEMS

The model of rotational mechanical systems can be obtained by using three elements, moment of inertia [J] of mass, dash-pot with rotational frictional coefficient [B] and torsional spring with stiffness [K].

The weight of the rotational mechanical system is represented by the moment of inertia of the mass. The moment of inertia of the system or body is considered to be concentrated at the centre of gravity of the body. The elastic deformation of the body can be represented by a spring (torsional spring). The friction existing in rotational mechanical system can be represented by the dash-pot. The dash-pot is a piston rotating inside a cylinder filled with viscous fluid.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by Newton's second law of motion for rotational systems. It states that the sum of torques acting on a body is zero (or Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig 1.14 which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \dfrac{d^2\theta}{dt^2}$ or $T_j = J\dfrac{d^2\theta}{dt^2}$

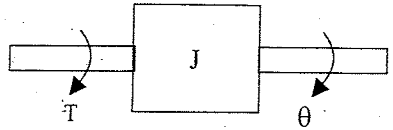


Fig 1.14 : Ideal rotational mass element

By Newton's second law

$$T = T_j = J\frac{d^2\theta}{dt^2} \qquad .....(1.7)$$

Consider an ideal frictional element dash pot shown in fig 1.15 which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let T = Applied torque.

T_b = Opposing torque due to friction.

$T_b \propto \dfrac{d\theta}{dt}$ or $T_b = B\dfrac{d\theta}{dt}$

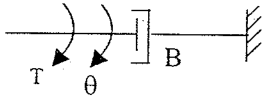


Fig 1.15 : Ideal rotational dash-pot with one end fixed to reference

By Newton's second law

$$T = T_b = B\frac{d\theta}{dt} \qquad .....(1.8)$$

When the dash pot has angular displacement at both ends as shown in fig 1.16, the opposing torque is proportional to the differential angular velocity.

$T_b \propto \dfrac{d}{dt}(\theta_1 - \theta_2)$ $\qquad T_b = B\dfrac{d}{dt}(\theta_1 - \theta_2)$

$$\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2) \qquad .....(1.9)$$

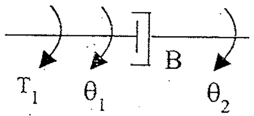


Fig 1.16 : Ideal dash-pot with angular displacement at both ends.

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

Let T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$

$T_k = K\theta$

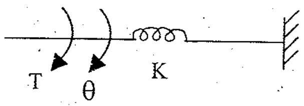


Fig 1.17 : Ideal spring with one end fixed to reference.

By Newton's second law,

$$T = T_k = K\theta \qquad .....(1.10)$$

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to differential angular displacement.

$T_k \propto (\theta_1 - \theta_2)$

$T_k = K(\theta_1 - \theta_2)$

$$\therefore T = T_k = K(\theta_1 - \theta_2) \qquad .....(1.11)$$

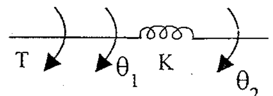


Fig 1.18 : Ideal spring with angular displacement at both ends

GUIDELINE TO DETERMINE THE TRANSFER FUNCTION OF MECHANICAL ROTATIONAL SYSTEM

1. In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.

2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ_1, θ_2, θ_3, etc and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.

3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.

4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.

5. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.

6. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the transfer function of the system.

Note :

Laplace transform of $\theta = L[\theta] = \theta(s)$

Laplace transform of $\frac{d\theta}{dt}$ with zero initial conditions $= L\left[\frac{d\theta}{dt}\right] = s\,\theta(s)$

Laplace transform of $\frac{d^2\theta}{dt^2}$ with zero initial conditions $= L\left[\frac{d^2\theta}{dt^2}\right] = s^2\,\theta(s)$

EXAMPLE 1.5

Write the differential equations governing the mechanical rotational system shown in fig 1.5.1. Obtain the transfer function of the system.

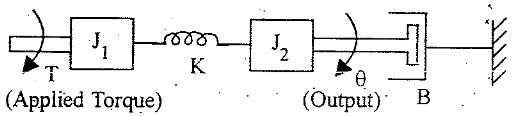


Fig 1.5.1

SOLUTION

In the given system, applied torque T is the input and angular displacement θ is the output.

Let Laplace transform of $T = L[T] = T(s)$

and Laplace transform of $\theta = L[\theta] = \theta(s)$

Hence the required transfer function is $\frac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The Laplace transform of $\theta_1 = L[\theta_1] = \theta_1(s)$. The free body diagram of J_1 is shown in fig 1.5.2. The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_k = K(\theta_1 - \theta)$$

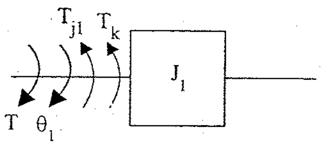


Fig 1.5.2 : Free body diagram of mass with moment of inertia J_1

By Newton's second law,

$$T_{j1} + T_k = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \quad .....(1.5.1)$$

On taking Laplace transform with zero initial conditions,

$$J_1\, s^2\, \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1\, s^2 + K)\, \theta_1(s) - K\, \theta(s) = T(s) \quad .....(1.5.2)$$

The freebody diagram of mass with moment of inertia J_2 is shown in fig 1.5.3. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2}\,; \qquad T_b = B\frac{d\theta}{dt}$$

and $T_k = K(\theta - \theta_1)$

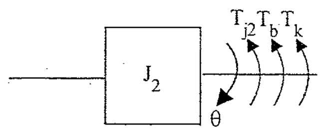


Fig 1.5.3 : Free body diagram of mass with moment of inertia J_2

By Newton's second law,

$$T_{j2} + T_b + T_k = 0$$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0 \quad .....(1.5.3)$$

On taking Laplace transform with zero initial conditions,

$$J_2\, s^2\theta(s) \;+\; B\, s\,\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2\, s^2 + Bs + K)\, \theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \qquad .....(1.5.4)$$

Substituting for $\theta_1(s)$ from equation (1.5.4) in equation (1.5.2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2} \qquad .....(1.5.5)$$

RESULT

The differential equations governing the system are

1. $$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$$

2. $$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

The transfer function of the system is

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

EXAMPLE 1.6

Write the differential equations governing the mechanical rotational system shown in fig 1.6.1. and determine the transfer function $\theta(s)/T(s)$.

SOLUTION

In the given system the torque T is the input and the angular displacement θ is the output.

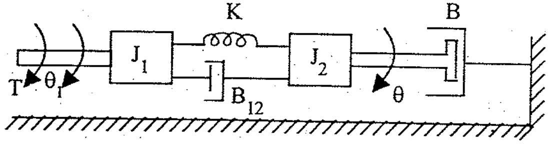


Fig 1.6.1

Let Laplace transform of $T = L[T] = T(s)$

and Laplace transform of $\theta = L[\theta] = \theta(s)$

Hence the required transfer function is $\theta(s)/T(s)$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The Laplace transform of $\theta_1 = L[\theta_1] = \theta_1(s)$. The free body diagram of J_1 is shown in fig 1.6.2. The opposing torques acting on J_1 are marked as T_{j1}, T_{b12} and T_k

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta)$$

$$T_k = K(\theta_1 - \theta)$$

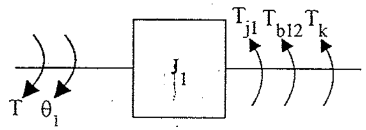


Fig 1.6.2 : Free body diagram of mass with moment of inertia J_1

By Newton's second law

$$T_{j1} + T_{b12} + T_k = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T \qquad .....(1.6.1)$$

On taking Laplace transform with zero initial conditions

$$J_1 s^2 \theta_1(s) + s\,B_{12}\,[\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\,[J_1 s^2 + sB_{12} + K] - \theta(s)\,[sB_{12} + K] = T(s) \qquad .....(1.6.2)$$

The freebody diagram of mass with moment of inertia J_2 is shown in fig 1.6.3. The opposing torques are marked as T_{j2}, T_{b12}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B\frac{d\theta}{dt} \quad \text{and} \quad T_k = K(\theta - \theta_1)$$

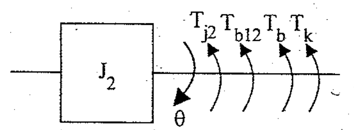


Fig 1.6.3 : Free body diagram of mass with moment of inertia J_2

By Newton's second law

$$T_{j2} + T_{b12} + T_b + T_k = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} - B_{12}\frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K\theta - K\theta_1 = 0 \quad .....(1.6.3)$$

On taking Laplace transform with zero initial conditions,

$$J_2 s^2\theta(s) - B_{12}s\theta_1(s) + s\theta(s)\,[B_{12} + B] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\,[s^2J_2 + s(B_{12} + B) + K] - \theta_1(s)\,[sB_{12} + K] = 0$$

$$\theta_1(s) = \frac{[s^2J_2 + s(B_{12} + B) + K]}{[sB_{12} + K]}\,\theta(s) \quad .....(1.6.4)$$

Substituting for $\theta_1(s)$ from equation (1.6.4) in equation (1.6.2) we get

$$[J_1s^2 + sB_{12} + K]\,\frac{[J_2s^2 + s(B_{12} + B) + K]\,\theta(s)}{(sB_{12} + K)} - (sB_{12} + K)\,\theta(s) = T(s)$$

$$\left[\frac{(J_1s^2 + sB_{12} + K)\,(J_2s^2 + s(B_{12} + B) + K) - (sB_{12} + K)^2}{(sB_{12} + K)}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1s^2 + sB_{12} + K)\,(J_2s^2 + s(B_{12} + B) + K) - (sB_{12} + K)^2} \quad .....(1.6.5)$$

RESULT

The differential equations governing the system are

1. $J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$

2. $J_2 \frac{d^2\theta}{dt^2} - B_{12}\frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K\theta - K\theta_1 = 0$

The transfer function of the system is

$$\frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1s^2 + sB_{12} + K)\,(J_2s^2 + s(B_{12} + B) + K) - (sB_{12} + K)^2}$$

1.6 ELECTRICAL SYSTEMS

The models of electrical systems can be obtained by using resistor, capacitor and inductor. The current-voltage relation of resistor, inductor and capacitor are given in table1. For modelling electrical systems the electrical network or equivalent circuit is formed by using R, L and C and voltage or current source.

The differential equations governing the electrical systems can be formed by writing Kirchoff's current law equations by choosing various nodes in the network or Kirchoff's voltage law equations by choosing various closed path in the network. The transfer function can be obtained by taking Laplace transform of the differential equations and rearranging them as a ratio of output to input.

Table 1 : Current-voltage relation of R, L and C

Element	Voltage across the element	Current through the element
R: i(t), + v(t) −	$v(t) = Ri(t)$	$i(t) = \frac{v(t)}{R}$
L: i(t), + v(t) −	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int v(t)\,dt$
C: i(t), + v(t) −	$v(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

EXAMPLE 1.7

Obtain the transfer function of the electrical network shown in fig 1.7.1

SOLUTION

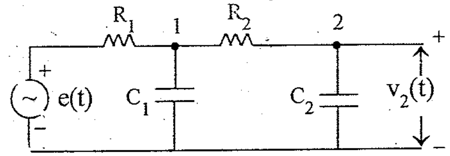


Fig 1.7.1.

In the given network input is e(t) and output is $v_2(t)$.

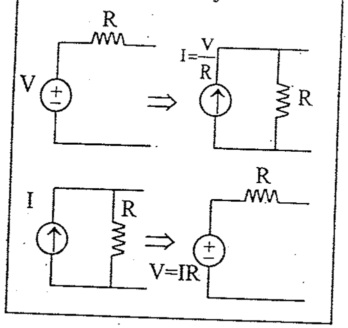
Let Laplace transform of $e(t) = L\,[e(t)] = E(s)$

Laplace transform of $v_2(t) = L[v_2(t)] = V_2(s)$

The transfer function of the network is $\frac{V_2(s)}{E(s)}$

Transform the voltage source in series with resistance R_1 into equivalent current source as shown in figure 1.7.2. The network has two nodes. Let the node voltages be v_1 and v_2. The Laplace transform of node voltages v_1 and v_2 are $V_1(s)$ and $V_2(s)$ respectively. The differential equations governing the network are given by the Kirchoff 's current law equations at these nodes.

Note : Source transformation



At node 1, by Kirchoff's current law (refer fig 1.7.3)

$$\frac{v_1}{R_1}+C_1\frac{dv_1}{dt}+\frac{v_1-v_2}{R_2}=\frac{e}{R_1} \quad .....(1.7.1)$$

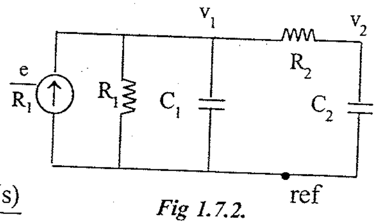


Fig 1.7.2.

On taking Laplace transform with zero initial conditions

$$\frac{V_1(s)}{R_1}+C_1sV_1(s)+\frac{V_1(s)}{R_2}-\frac{V_2(s)}{R_2}=\frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1}+sC_1+\frac{1}{R_2}\right]-\frac{V_2(s)}{R_2}=\frac{E(s)}{R_1} \quad .....(1.7.2)$$

At node 2, by Kirchoff's current law (refer fig 1.7.4)

$$\frac{v_2-v_1}{R_2}+C_2\frac{dv_2}{dt}=0 \quad .....(1.7.3)$$

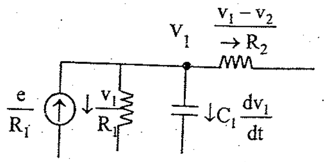


Fig 1.7.3

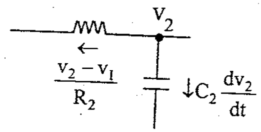


Fig 1.7.4.

On taking Laplace transform of equation (1.7.3) with zero initial conditions

$$\frac{V_2(s)}{R_2}-\frac{V_1(s)}{R_2}+C_2s\,V_2(s)=0$$

$$\frac{V_1(s)}{R_2}=\frac{V_2(s)}{R_2}+C_2s\,V_2(s)=\left[\frac{1}{R_2}+sC_2\right]V_2(s)$$

$$\therefore V_1(s)=\left[1+sC_2R_2\right]V_2(s) \quad .....(1.7.4)$$

Substituting for $V_1(s)$ from equation (1.7.4) in equation (1.7.2) we get,

$$(1+sR_2C_2)\,V_2(s)\left[\frac{1}{R_1}+sC_1+\frac{1}{R_2}\right]-\frac{V_2(s)}{R_2}=\frac{E(s)}{R_1}$$

$$\left[\frac{(1+sR_2C_2)\,(R_2+R_1+sC_1R_1R_2)-R_1}{R_1R_2}\right]V_2(s)=\frac{E(s)}{R_1}$$

$$\therefore \frac{V_2(s)}{E(s)}=\frac{R_2}{\left[(1+sR_2C_2)\,(R_1+R_2+sC_1R_1R_2)-R_1\right]}$$

RESULT

The (node basis) differential equations governing the electrical network are

1. $\frac{v_1}{R_1}+C_1\frac{dv_1}{dt}+\frac{v_1-v_2}{R_2}=\frac{e}{R_1}$ 2. $\frac{v_2-v_1}{R_2}+C_2\frac{dv_2}{dt}=0$

The transfer function of the electrical network is

$$\frac{V_2(s)}{E(s)}=\frac{R_2}{\left[(1+sR_2C_2)\,(R_1+R_2+sC_1R_1R_2)-R_1\right]}$$

1.7 TRANSFER FUNCTION OF ARMATURE CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux in field winding. In armature controlled DC motor the desired speed is obtained by varying the armature voltage. This speed control system is an electro-mechanical control system. The electrical system consists of the armature and the field circuit but for analysis purpose, only the armature circuit is considered because the field is excited by a constant voltage. The mechanical system consist of the rotating part of the motor and load connected to the shaft of the motor. The armature controlled DC motor speed control system is shown in fig 1.19.

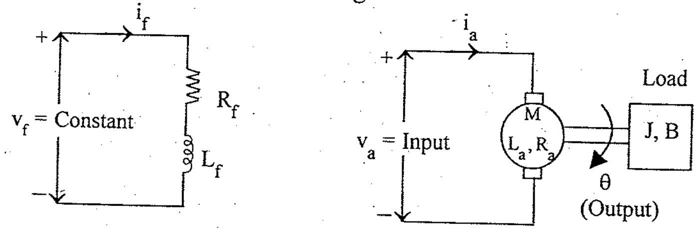


Fig 1.19 : Armature controlled DC motor

Let R_a = Armature resistance, Ω
L_a = Armature inductance, H
i_a = Armature current, A
v_a = Armature voltage, V
e_b = Back emf, V
K_t = Torque constant, N-m/A
T = Torque developed by motor, N-m
θ = Angular displacement of shaft, rad
J = Moment of inertia of motor and load, Kg-m²/rad
B = Frictional coefficient of motor and load, N-m/(rad/sec)
K_b = Back emf constant, V/(rad/sec).

The equivalent circuit of armature is shown in fig 1.20.

By Kirchoff's voltage law, we can write

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \qquad .....(1.12)$$

Torque of DC motor is proportional to the product of flux and current. Since flux is constant in this system, the torque is proportional to i_a alone.

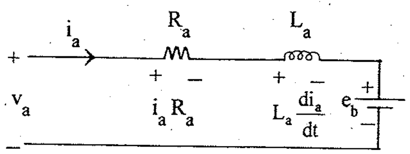


Fig 1.20: Equivalent circuit of armature

$$T \propto i_a$$

$$\therefore \text{Torque, } T = K_t\, i_a \qquad .....(1.13)$$

The mechanical system of the motor is shown in fig 1.21. The differential equation governing the mechanical system of motor is given by

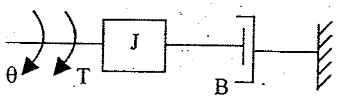


Fig 1.21

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad .....(1.14)$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft

$$\therefore e_b \propto \frac{d\theta}{dt} \quad ; \quad \text{Back emf, } e_b = K_b \frac{d\theta}{dt} \qquad .....(1.15)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$$L[v_a] = V_a(s) \qquad L[e_b] = E_b(s) \qquad L[T] = T(s)$$
$$L[i_a] = I_a(s) \qquad L[\theta] = \theta(s)$$

The differential equations governing the armature controlled DC motor speed control system are

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \qquad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

$$T = K_t\, i_a \qquad e_b = K_b \frac{d\theta}{dt}$$

On taking Laplace transform of the system differential equations with zero initial conditions we get.

$$I_a(s)\, R_a + L_a s\, I_a(s) + E_b(s) = V_a(s) \qquad .....(1.16)$$
$$T(s) = K_t I_a(s) \qquad .....(1.17)$$
$$Js^2\theta(s) + B\, s\, \theta(s) = T(s) \qquad .....(1.18)$$
$$E_b(s) = K_b s\, \theta(s) \qquad .....(1.19)$$

On equating equations (1.17) and (1.18) we get,

$$K_t I_a(s) = (Js^2 + Bs)\, \theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\, \theta(s) \qquad .....(1.20)$$

Equation (1.16) can be written as

$$(R_a + sL_a)\, I_a(s) + E_b(s) = V_a(s) \qquad .....(1.21)$$

Substituting for $E_b(s)$ and $I_a(s)$ from equation (1.19) and (1.20) respectively in equation (1.21),

$$(R_a + sL_a)\, \frac{(Js^2 + Bs)}{K_t}\theta(s) + K_b s\, \theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\theta(s)/V_a(s)$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} \quad .....(1.22)$$

$$= \frac{K_t}{R_a Js^2 + R_a Bs + L_a Js^3 + L_a Bs^2 + K_b K_t s}$$

$$= \frac{K_t}{s\left[JL_a s^2 + (JR_a + BL_a)\, s + (BR_a + K_b K_t)\right]}$$

$$= \frac{K_t / JL_a}{s\left[s^2 + \left(\frac{JR_a + BL_a}{JL_a}\right) s + \left(\frac{BR_a + K_b K_t}{JL_a}\right)\right]} \quad .....(1.23)$$

The transfer function of armature controlled dc motor can be expressed in another standard form as shown below.

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s}$$

$$= \frac{K_t}{R_a \left(\frac{sL_a}{R_a} + 1\right) Bs \left(1 + \frac{Js^2}{Bs}\right) + K_b K_t\, s}$$

$$= \frac{K_t / R_a B}{s\left[(1 + sT_a)(1 + sT_m) + \frac{K_b K_t}{R_a B}\right]} \quad .....(1.24)$$

Where, $L_a/R_a = T_a$ = Electrical time constant

and $J/B = T_m$ = Mechanical time constant

1.8 TRANSFER FUNCTION OF FIELD CONTROLLED DC MOTOR

The speed of a DC motor is directly proportional to armature voltage and inversely proportional to flux. In field controlled DC motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine. Since flux is directly proportional to field current, the flux is varied by varying field current. The speed control system is an electromechanical control system. The electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The field controlled DC motor speed control system is shown in fig 1.22.

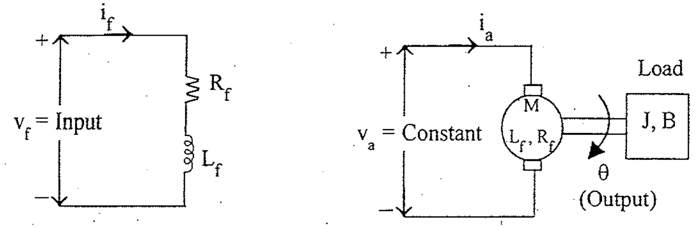


Fig 1.22 : Field controlled DC motor

Let R_f = Field resistance, Ω
L_f = Field inductance, H
i_f = Field current, A
v_f = Field voltage, V
T = Torque developed by motor, N-m
K_{tf} = Torque constant, N-m/A
J = Moment of inertia of motor and load, Kg-m²/rad
B = Frictional coefficient of motor and load, N-m/(rad/sec).

The equivalent circuit of field is shown in fig 1.23.

By Kirchoff's voltage law, we can write

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \quad .....(1.25)$$

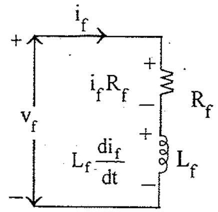


Fig 1.23 : Equivalent circuit of field

The torque of DC motor is proportional to product of flux and armature current. Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current.

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f \quad .....(1.26)$$

The mechanical system of the motor is shown in fig 1.24. The differential equation governing the mechanical system of the motor is given by

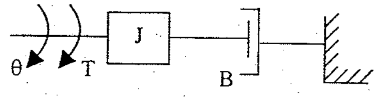


Fig 1.24

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad .....(1.27)$$

The Laplace transform of various time domain signals involved in this system are shown below

$$L[i_f] = I_f(s) \;;\; L[T] = T(s) \;;\; L[v_f] = V_f(s) \;;\; L[\theta] = \theta(s).$$

The differential equations governing the field controlled DC motor are

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \;;\quad T = K_{tf}\, i_f \;;\quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

On taking Laplace transform of the system differential equation, we get

$$R_f I_f(s) + L_f\, s\, I_f(s) = V_f(s) \qquad .....(1.28)$$

$$T(s) = K_{tf}\, I_f(s) \qquad .....(1.29)$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \qquad .....(1.30)$$

Equating equation (1.29) and (1.30) we get,

$$K_{tf}\, I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = s\,\frac{(Js + B)}{K_{tf}}\,\theta(s) \qquad ......(1.31)$$

The equation (1.28) can be written as

$$(R_f + sL_f)\, I_f(s) = V_f(s) \qquad .....(1.32)$$

On substituting for $I_f(s)$ from equation (1.31) in equation (1.32) we get,

$$(R_f + sL_f)\, s\frac{(Js + B)}{K_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + sL_f)\,(B + sJ)}$$

$$= \frac{K_{tf}}{sR_f\left(1 + \frac{sL_f}{R_f}\right)B\left(1 + \frac{sJ}{B}\right)} = \frac{K_m}{s\,(1 + sT_f)\,(1 + sT_m)} \qquad .....(1.33)$$

Where, Motor gain constant, $K_m = K_{tf}/R_f B$

Field time constant, $T_f = L_f/R_f$

Mechanical time constant, $T_m = J/B$

1.9 ELECTRICAL ANALOGOUS OF MECHANICAL TRANSLATIONAL SYSTEMS

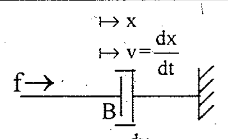
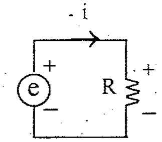
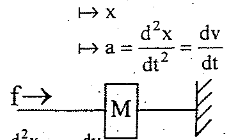
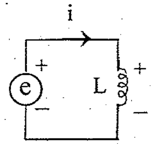
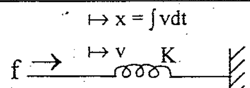
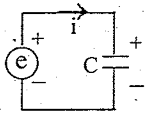
Systems remain analogous as long as the differential equations governing the systems or transfer functions are of identical form. The electric analogue of any other kind of system is of great importance since it is easier to construct electrical models and analyse them.

The three basic elements mass, dash-pot and spring that are used in modelling mechanical translational systems are analogous to resistance, inductance and capacitance of electrical systems. The input force in mechanical system is analogous to either voltage source or current source in electrical systems. The output velocity (first derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system. Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies : force-voltage analogy and force-current analogy.

FORCE-VOLTAGE ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-voltage analogy are shown in table 2.

TABLE 2

Mechanical system	**Electrical system**
Input : Force Output : Velocity	Input : Voltage source Output : Current through the element
 $f = B\frac{dx}{dt} = Bv$	 $e = Ri$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = L\frac{di}{dt}$
 $f = Kx = K\int v\,dt$	 $e = \frac{1}{C}\int i\,dt$

The table 3 shows the list of analogous quantities in force-voltage analogy

TABLE 3

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\Sigma F = 0$	Kirchoff's voltage law $\Sigma V = 0$
Changing the level of independent variable	lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same velocity are said to be in series.
2. The elements having same velocity in mechanical system should have analogous same current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. A mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of velocities of nodes (masses) in mechanical system.

5. The mechanical driving sources (force) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements in a closed loop in analogous electrical system.
6. The element connected between two (nodes) masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

FORCE-CURRENT ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-current analogy are shown in table-4. The table 5 shows the list of analogous quantities in force-current analogy

TABLE 4

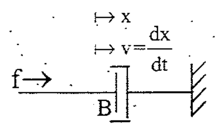
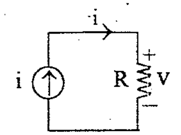
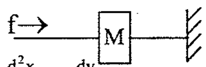
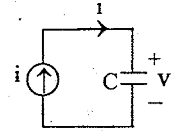
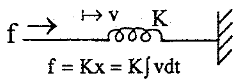
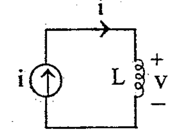
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Current source Output : Voltage across the element
 $\mapsto x$, $\mapsto v = \frac{dx}{dt}$, f, B $f = B\frac{dx}{dt} = Bv$	 i, R, +, v, − $i = \frac{1}{R}v$
 $\mapsto x$, $\mapsto a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$, f, M $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 i, C, +, v, − $i = C\frac{dv}{dt}$
 $\mapsto x = \int v\,dt$, $\mapsto v$, K, f $f = Kx = K\int v\,dt$	 i, L, +, v, − $i = \frac{1}{L}\int v\,dt$

TABLE 5

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, ϕ
Dissipative element	Frictional coefficient of dashpot, B	Conductance G=1/R
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\Sigma F = 0$	Kirchoff's current law $\Sigma i = 0$
Changing the level of independent variable	lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-current analogy.

1. In electrical systems element in parallel will have same voltage, likewise in mechanical systems, the elements having same force are said to be in parallel.
2. The elements having same velocity in mechanical system should have analogous same voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. A mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of velocities of (nodes) masses in mechanical system.
5. The mechanical driving sources (forces) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements connected to a node in electrical system.
6. The element connected between two nodes (masses) in mechanical system is represented as a common element between two nodes in electrical analogous system.

EXAMPLE 1.8

Write the differential equations governing the mechanical system shown in fig 1.8.1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

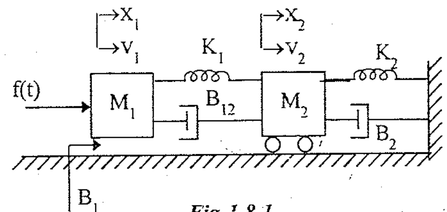


Fig 1.8.1

The free body diagram of M_1 is shown in fig 1.8.2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

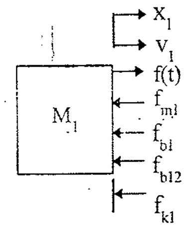


Fig 1.8.2

$f_{m1} = M_1 \frac{d^2x_1}{dt^2}$; $f_{b1} = B_1 \frac{dx_1}{dt}$; $f_{b12} = B_{12}\frac{d}{dt}(x_1 - x_2)$ and

$f_{k1} = K_1(x_1 - x_2)$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12}\frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \quad .....(1.8.1)$$

The free body diagram of M_2 is shown in fig 1.8.3.

The opposing forces are marked as f_{m2}, f_{b2}, f_{b12}, f_{k1} and f_{k2}.

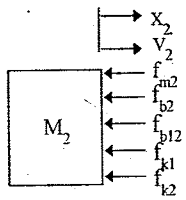


Fig 1.8.3

$f_{m2} = M_2 \frac{d^2x_2}{dt^2}$; $f_{b2} = B_2 \frac{dx_2}{dt}$;

$f_{b12} = B_{12}\frac{d}{dt}(x_2 - x_1)$; $f_{k1} = K_1(x_2 - x_1)$ and $f_{k2} = K_2\, x_2$.

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2x_2 + B_{12}\frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \quad .....(1.8.2)$$

On replacing the displacements by velocity in the differential equations (1.8.1) and (1.8.2) of the mechanical system we get,

$$\left(\text{i.e, } \frac{d^2x}{dt^2}=\frac{dv}{dt} \;;\quad \frac{dx}{dt}=v \;\text{ and } x=\int v\,dt\right)$$

$$M_1\frac{dv_1}{dt}+B_1v_1+B_{12}(v_1-v_2)+K_1\int(v_1-v_2)\,dt=f(t) \qquad .....(1.8.3)$$

$$M_2\frac{dv_2}{dt}+B_2v_2+K_2\int v_2\,dt+B_{12}(v_2-v_1)+K_1\int(v_2-v_1)\,dt=0 \qquad .....(1.8.4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit will have two meshes.

The force applied to mass, M_1 is represented by a voltage source in first mesh. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous element in mesh 1 forming a closed path. The elements K_1, B_{12}, M_2, K_2 and B_2 are connected to second node. Hence they are represented by analogous element in mesh 2 forming a closed path.

The elements K_1 and B_{12} are common between node 1 and 2 and so they are represented by analogous element as common elements between two meshes. The force-voltage electrical analogous circuit is shown in fig 1.8.4. The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$		$B_{12} \rightarrow R_{12}$	

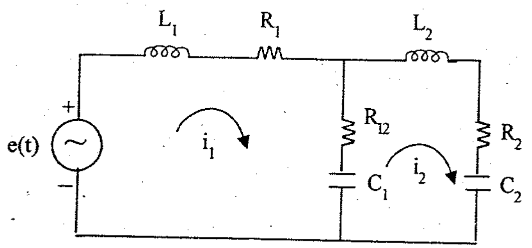


Fig 1.8.4 : Force-voltage electrical analogous circuit

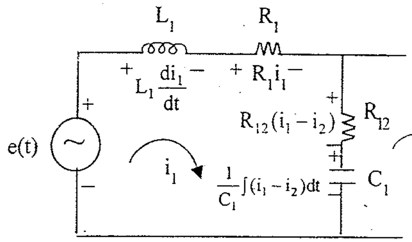


Fig 1.8.5

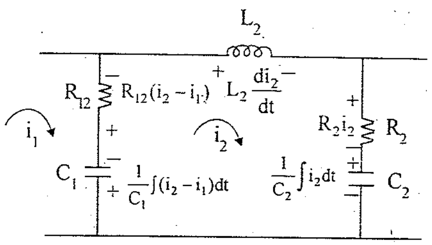


Fig 1.8.6

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 1.8.4 are given below. *(Refer fig 1.8.5. and 1.8.6)*

$$L_1\frac{di_1}{dt}+R_1i_1+R_{12}(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)dt=e(t) \qquad .....(1.8.5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2\,dt+R_{12}(i_2-i_1)+\frac{1}{C_1}\int(i_2-i_1)dt=0 \qquad .....(1.8.6)$$

It is observed that the mesh basis equations (1.8.5) and (1.8.6) are similar to the differential equations (1.8.3) and (1.8.4) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source connected to node 1 in analogous electrical circuit. The elements M_1, B_1, K_1 and B_{12}, are connected to first node. Hence they are represented by analogous elements connected to node 1 in analogous electrical circuit. The elements K_1, B_{12}, M_2, K_2 and B_2 are connected to second node. Hence they are represented by analogous elements as elements connected to node 2 in analogous electrical circuit.

The elements K_1 and B_{12} are common between node 1 and 2 and so they are represented by analogous elements as common element between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 1.8.7.

The electrical analogous elements for the elements of mechanical system are given below

$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$	$B_{12} \rightarrow 1/R_{12}$		

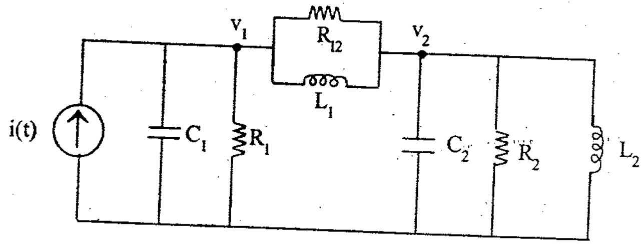


Fig 1.8.7 : Force-current electrical analogous circuit

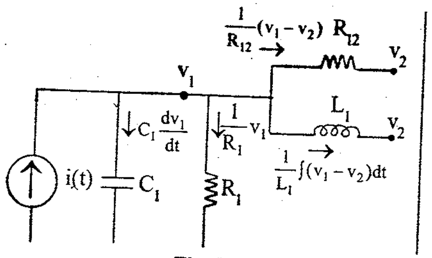


Fig 1.8.8

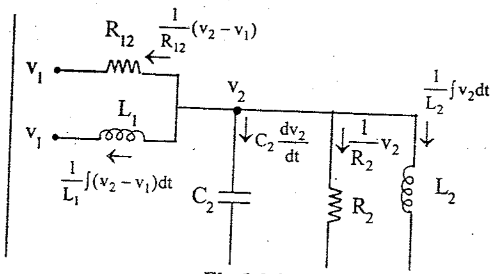


Fig 1.8.9

The node basis equations using Kirchoff's current law for the circuit shown in fig 1.8.7 are given below (refer fig 1.8.8 and 1.8.9)

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{R_{12}}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \qquad .....(1.8.7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_{12}}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=0 \qquad .....(1.8.8)$$

It is observed that the node basis equations (1.8.7) and (1.8.8) are similar to the differential equations (1.8.3) and (1.8.4) governing the mechanical system.

EXAMPLE 1.9

Write the differential equations governing the mechanical system shown in fig 1.9.1. Draw the force voltage and force current electrical analogous circuits and verify by writing mesh and node equations.

SOLUTION

The given mechanical system has three nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

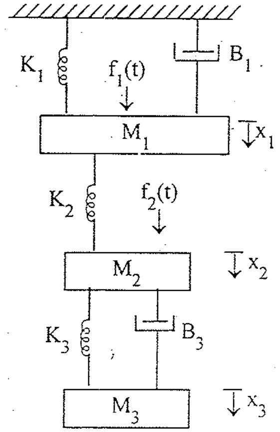


Fig 1.9.1

The free body diagram of M_1 is shown in fig 1.9.2. The opposing forces are marked as f_{m1}, f_{b1}, f_{k2} and f_{k1}.

$$f_{m1}=M_1\frac{d^2x_1}{dt^2}$$

$$f_{b1}=B_1\frac{dx_1}{dt}$$

$$f_{k2}=K_2\,(x_1-x_2)$$

$$f_{k1}=K_1x_1$$

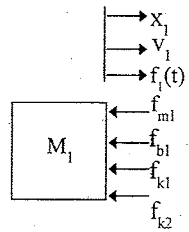


Fig 1.9.2

By Newton's second law, $f_{m1}+f_{b1}+f_{k2}+f_{k1}=f_1(t)$

$$M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+K_1x_1+K_2(x_1-x_2)=f_1(t) \qquad .....(1.9.1)$$

The free body diagram of M_2 is shown in fig 1.9.3. The opposing forces are marked as f_{m2}, f_{b3}, f_{k2}, f_{k3}.

$$f_{m2}=M_2\frac{d^2x_2}{dt^2} \quad ; \quad f_{b3}=B_3\frac{d}{dt}(x_2-x_3)$$

$$f_{k2}=K_2(x_2-x_1) \quad \text{and} \quad f_{k3}=K_3(x_2-x_3)$$

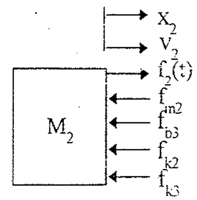


Fig 1.9.3

By Newton's second law, $f_{m2}+f_{b3}+f_{k2}+f_{k3}=f_2(t)$

$$M_2\frac{d^2x_2}{dt^2}+B_3\frac{d}{dt}(x_2-x_3)+K_2(x_2-x_1)+K_3(x_2-x_3)=f_2(t) \qquad .....(1.9.2)$$

The free body diagram of M_3 is shown in fig 1.9.4. The opposing forces are marked as f_{m3}, f_{b3}, f_{k3}.

$$f_{m3}=M_3\frac{d^2x_3}{dt^2} \; ; \; f_{b3}=B_3\frac{d}{dt}(x_3-x_2)$$

$$f_{k3}=K_3(x_3-x_2)$$

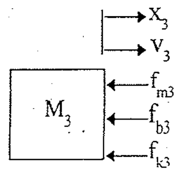


Fig 1.9.4

By Newton's second law, $f_{m3}+f_{b3}+f_{k3}=0$

$$M_3\frac{d^2x_3}{dt^2}+B_3\frac{d}{dt}(x_3-x_2)+K_3(x_3-x_2)=0 \qquad .....(1.9.3)$$

On replacing the displacements by velocity in the differential equations (1.9.1), (1.9.2) and (1.9.3) governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt}\ ;\ \frac{dx}{dt} = v \text{ and } x = \int v\,dt\right)$$

$$M_1\frac{dv_1}{dt} + B_1v_1 + K_1\int v_1dt + K_2\int(v_1 - v_2)dt = f_1(t) \quad .....(1.9.4)$$

$$M_2\frac{dv_2}{dt} + B_3(v_2 - v_3) + K_2\int(v_2 - v_1)dt + K_3\int(v_2 - v_3)dt = f_2(t) \quad .....(1.9.5)$$

$$M_3\frac{dv_3}{dt} + B_3(v_3 - v_2) + K_3\int(v_3 - v_2)dt = 0 \quad .....(1.9.6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes. The force applied to mass, M_1 is represented by a voltage source in first mesh and the force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous element in mesh 1 forming a closed path. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous element in mesh 2 forming a closed path. The elements M_3, K_3 and B_3 are connected to third node. Hence they are represented by analogous element in mesh 3 forming a closed path.

The element K_2 is common between node 1 and 2 and so it is represented by analogous element as common element between mesh 1 and 2. The elements K_3 and B_3 are common between node 2 and 3 and so they are represented by analogous elements as common elements between mesh 2 and 3. The force-voltage electrical analogous circuit is shown in fig 1.9.5.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow e_1(t)$	$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$f_2(t) \rightarrow e_2(t)$	$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$B_3 \rightarrow R_3$	$K_2 \rightarrow 1/C_2$
	$v_3 \rightarrow i_3$	$M_3 \rightarrow L_3$		$K_3 \rightarrow 1/C_3$

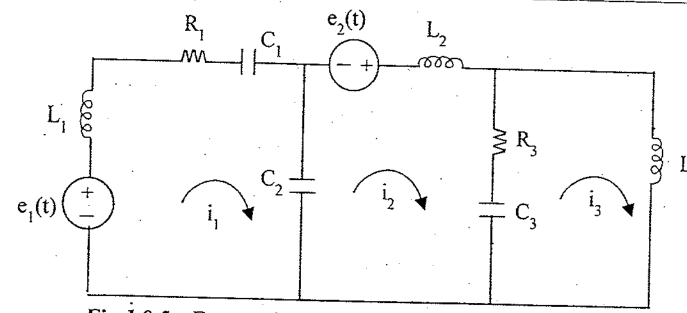


Fig 1.9.5 : Force-voltage electrical analogous circuit

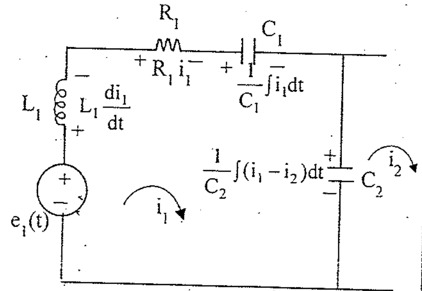


Fig 1.9.6

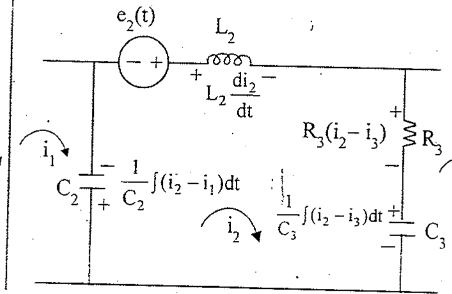


Fig 1.9.7

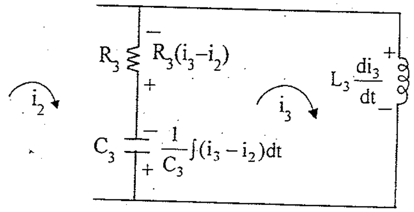


Fig 1.9.8

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 1.9.5 are given below [Refer figures (1.9.6), (1.9.7) and (1.9.8).]

$$L_1\frac{di_1}{dt} + R_1i_1 + \frac{1}{C_1}\int i_1dt + \frac{1}{C_2}\int(i_1 - i_2)dt = e_1(t) \quad .....(1.9.7)$$

$$L_2\frac{di_2}{dt} + R_3(i_2 - i_3) + \frac{1}{C_3}\int(i_2 - i_3)dt + \frac{1}{C_2}\int(i_2 - i_1)dt = e_2(t) \quad .....(1.9.8)$$

$$L_3\frac{di_3}{dt} + R_3(i_3 - i_2) + \frac{1}{C_3}\int(i_3 - i_2)dt = 0 \quad .....(1.9.9)$$

It is observed that the mesh equations (1.9.7), (1.9.8) and (1.9.9) are similar to the differential equations (1.9.4), (1.9.5) and (1.9.6) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

The force applied to mass M_1 is represented as a current source connected to node 1 in analogous electrical circuit. The force applied to mass M_2 is represented as a current source connected to node 2 in analogous electrical circuit.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous elements as elements connected to node 1 in analogous electrical circuit. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous elements as elements connected to node 2 in analogous electrical circuit. The elements M_3, B_3 and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node 3 in analogous electrical circuit.

The element K_2 is common between node 1 and 2 and so it is represented by analogous element as common element between node 1 and 2 in analogous circuit. The elements B_3 and K_3 are common between node 2 and 3 and so they are represented by analogous elements as common elements between node 2 and 3. The force-current electrical analogous circuit is shown in fig 1.9.9.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow i_1(t)$	$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$f_2(t) \rightarrow i_2(t)$	$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$B_3 \rightarrow 1/R_3$	$K_2 \rightarrow 1/L_2$
	$v_3 \rightarrow v_3$	$M_3 \rightarrow C_3$		$K_3 \rightarrow 1/L_3$

The node basis equations using Kirchoff's current law for the circuit shown in fig 1.9.9. are given below. [Refer figures (1.9.10), (1.9.11) and (1.9.12)].

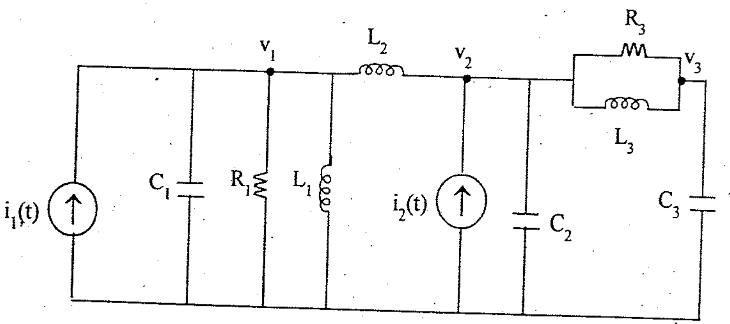


Fig 1.9.9 : Force-current electrical analogous circuit

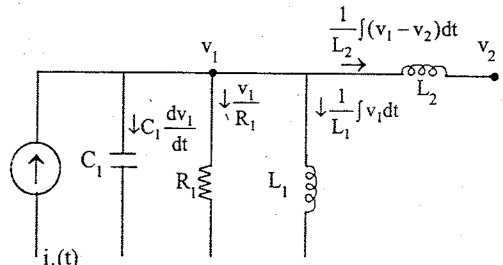


Fig 1.9.10

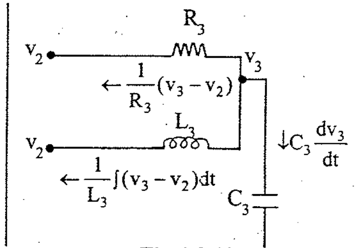


Fig 1.9.12

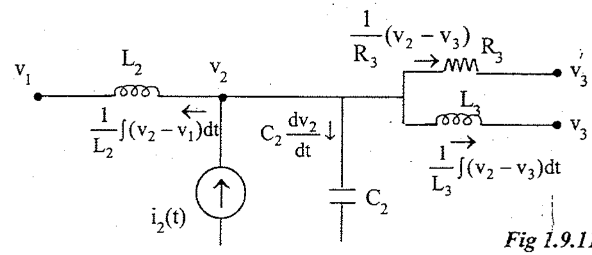


Fig 1.9.11

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1 dt+\frac{1}{L_2}\int(v_1-v_2)dt=i_1(t) \qquad .....(1.9.10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_3}(v_2-v_3)+\frac{1}{L_3}\int(v_2-v_3)dt+\frac{1}{L_2}\int(v_2-v_1)dt=i_2(t) \qquad .....(1.9.11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_2)+\frac{1}{L_3}\int(v_3-v_2)dt=0 \qquad .....(1.9.12)$$

It is observed that node basis equations (1.9.10), (1.9.11) and (1.9.12) are similar to the differential equations (1.9.4), (1.9.5) and (1.9.6) governing the mechanical system.

EXAMPLE 1-10

Write the differential equations governing the mechanical system shown in fig 1.10.1. Draw the force voltage and force current electrical analogous circuits and verify by writing mesh and node equations.

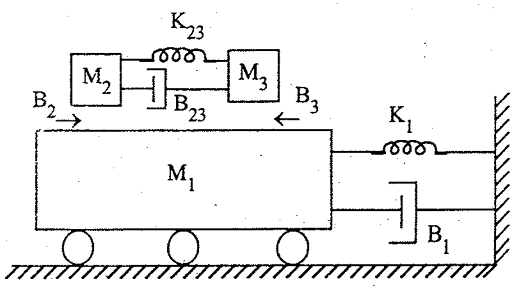


Fig 1.10.1

SOLUTION

The given mechanical system has three nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes.

Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

The free body diagram of M_1 is shown in fig 1.10.2. The opposing forces are marked as f_{b1}, f_{k1}, f_{b2}, f_{b3}, f_{m1}.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}, \quad f_{b1} = B_1 \frac{dx_1}{dt}, \quad f_{k1} = K_1 x_1$$

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x_2) \text{ and } f_{b3} = B_3 \frac{d}{dt}(x_1 - x_3)$$

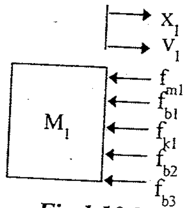


Fig 1.10.2

By Newton's second law, $f_{m1} + f_{b1} + f_{k1} + f_{b2} + f_{b3} = 0$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 + B_2 \frac{d}{dt}(x_1 - x_2) + B_3 \frac{d}{dt}(x_1 - x_3) = 0 \quad .....(1.10.1)$$

The free body diagram of M_2 is shown in fig 1.10.3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b23}, f_{k23}.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2}, \quad f_{b2} = B_2 \frac{d}{dt}(x_2 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_2 - x_3) \text{ and } f_{k23} = K_{23}(x_2 - x_3)$$

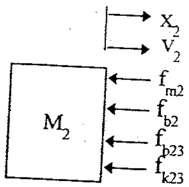


Fig 1.10.3

By Newton's second law, $f_{m2} + f_{b2} + f_{b23} + f_{k23} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{d}{dt}(x_2 - x_1) + B_{23} \frac{d}{dt}(x_2 - x_3) + K_{23}(x_2 - x_3) = 0 \quad .....(1.10.2)$$

The free body diagram of M_3 is shown in fig 1.10.4. The opposing forces are marked as f_{m3}, f_{b3}, f_{b23}, f_{k23}.

$$f_{m3} = M_3 \frac{d^2x_3}{dt^2}, \quad f_{b3} = B_3 \frac{d}{dt}(x_3 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_3 - x_2) \text{ and } f_{k23} = K_{23}(x_3 - x_2).$$

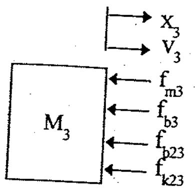


Fig 1.10.4

By Newton's second law, $f_{m3} + f_{b3} + f_{b23} + f_{k23} = 0$

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + B_{23} \frac{d}{dt}(x_3 - x_2) + K_{23}(x_3 - x_2) = 0 \quad .....(1.10.3)$$

On replacing the displacements by velocity in the differential equations 1.10.1, 1.10.2 and 1.10.3 governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt}, \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_1 \int v_1 dt + B_2(v_1 - v_2) + B_3(v_1 - v_3) = 0 \quad .....(1.10.4)$$

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_1) + B_{23}(v_2 - v_3) + K_{23} \int (v_2 - v_3) dt = 0 \quad .....(1.10.5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_1) + B_{23}(v_3 - v_2) + K_{23} \int (v_3 - v_2) dt = 0 \quad .....(1.10.6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes.

The elements M_1, K_1, B_1, B_3 and B_2 are connected to first node. Hence they are represented by analogous elements in mesh 1 forming a closed path. The elements M_2, K_{23}, B_{23} and B_2 are connected to second node. Hence they are represented by analogous elements in mesh 2 forming a closed path. The elements M_3, K_{23}, B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements in mesh 3 forming a closed path.

The elements K_{23} and B_{23} are common between node 2 and 3 and so they are represented by analogous element as common elements between mesh 2 and 3. The element B_2 is common between node 1 and 2 and so it is represented by analogous element as common element between mesh 1 and 2. The element B_3 is common between node 1 and 3 and so it is represented by analogous element between mesh 1 and 3. The force-voltage electrical analogous circuit is shown in fig 1.10.5.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$K_1 \rightarrow 1/C_1$	$B_2 \rightarrow R_2$
$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$K_{23} \rightarrow 1/C_{23}$	$B_3 \rightarrow R_3$
$v_3 \rightarrow i_3$	$M_3 \rightarrow L_3$	$B_1 \rightarrow R_1$	$B_{23} \rightarrow R_{23}$

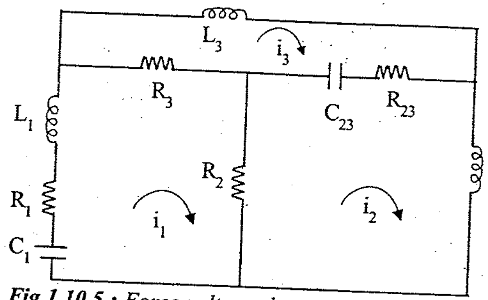


Fig 1.10.5 : Force-voltage electrical analogous circuit

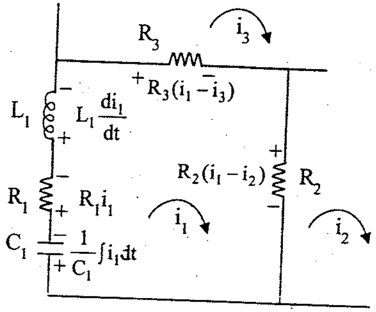


Fig 1.10.6

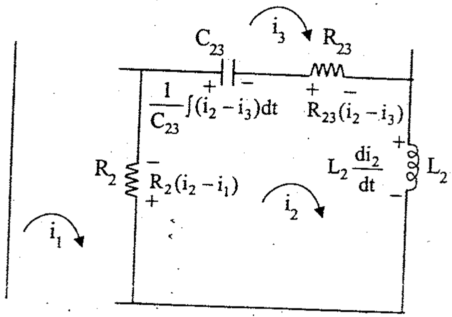


Fig 1.10.7

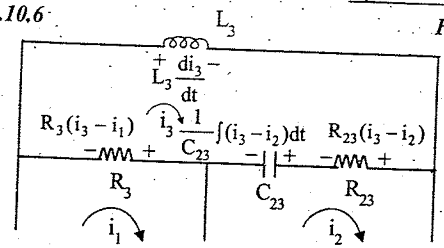


Fig 1.10.8

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 1.10.5 are given below. {Refer figures (1.10.6), (1.10.7) and (1.10.8)}

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int i_1dt+R_2(i_1-i_2)+R_3(i_1-i_3)=0 \quad .....(1.10.7)$$

$$L_2\frac{di_2}{dt}+R_2(i_2-i_1)+\frac{1}{C_{23}}\int(i_2-i_3)dt+R_{23}(i_2-i_3)=0 \quad .....(1.10.8)$$

$$L_3\frac{di_3}{dt}+R_3(i_3-i_1)+\frac{1}{C_{23}}\int(i_3-i_2)dt+R_{23}(i_3-i_2)=0 \quad .....(1.10.9)$$

It is observed that the mesh basis equations (1.10.7), (1.10.8) and (1.10.9) are similar to the differential equations (1.10.4), (1.10.5) and (1.10.6) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

The elements M_1, K_1, B_1, B_2 and B_3 are connected to first node. Hence they are represented by analogous elements as elements connected to node 1 in analogous electrical circuit. The elements M_2, K_{23}, B_{23} and B_2 are connected to second node. Hence they are represented by analogous elements as elements connected to node 2 in analogous electrical circuit. The elements M_3, K_{23}, B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node 3 in analogous electrical circuit.

The elements K_{23} and B_{23} are common between node 2 and 3 and so they are represented by analogous element as common elements between node 2 and 3 in electrical analogous circuit. The element B_2 is common between node 1 and 2 and so it is represented by analogous **element** as common element between node 1 and 2 in electrical analogous circuit. The **element** B_3 is common between node 1 and 3 and so it is represented by analogous element as common element between node 1 and 3 in electrical analogous circuit. The force-current electrical analogous circuit is shown in fig 1.10.9.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$K_1 \rightarrow 1/L_1$	$B_2 \rightarrow 1/R_2$
$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$K_{23} \rightarrow 1/L_{23}$	$B_3 \rightarrow 1/R_3$
$v_3 \rightarrow v_3$	$M_3 \rightarrow C_3$	$B_1 \rightarrow 1/R_1$	$B_{23} \rightarrow 1/R_{23}$

The node basis equations using Kirchoff's current law for the circuit shown in fig 1.10.9 are given below. [Refer figures (1.10.10), (1.10.11) and (1.10.12)]

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1dt+\frac{1}{R_2}(v_1-v_2)+\frac{1}{R_3}(v_1-v_3)=0 \quad .....(1.10.10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}(v_2-v_1)+\frac{1}{L_{23}}\int(v_2-v_3)dt+\frac{1}{R_{23}}(v_2-v_3)=0 \quad .....(1.10.11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_1)+\frac{1}{L_{23}}\int(v_3-v_2)dt+\frac{1}{R_{23}}(v_3-v_2)=0 \quad .....(1.10.12)$$

It is observed that the node basis equations (1.10.10), (1.10.11) and (1.10.12) are similar to the differential equations (1.10.4), (1.10.5) and (1.10.6) governing the mechanical system.

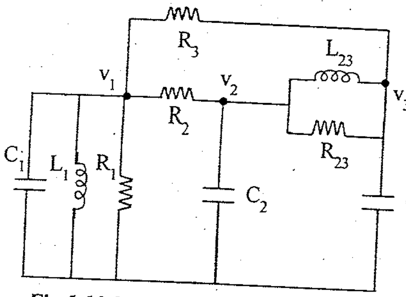


Fig 1.10.9 : Force-current electrical analogous circuit

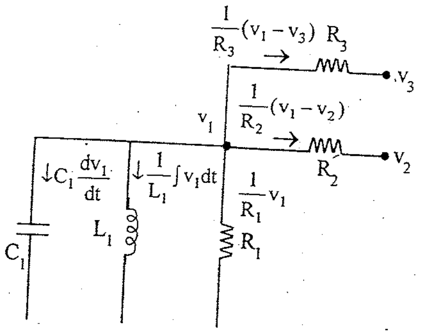


Fig 1.10.10

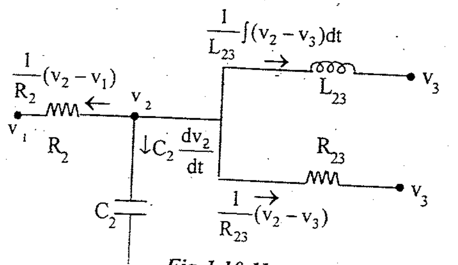


Fig 1.10.11

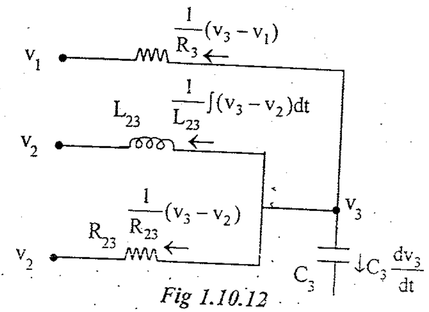


Fig 1.10.12

EXAMPLE 1.11

Write the differential equations governing the mechanical system shown in fig 1.11.1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacement of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

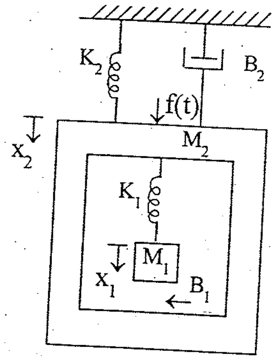


Fig 1.11.1

The free body diagram of M_1 is shown in fig 1.11.2. The opposing forces are marked as f_{m1}, f_{b1} and f_{k1}.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \;;\; f_{b1} = B_1 \frac{d(x_1 - x_2)}{dt} \;;\; f_{k1} = K_1(x_1 - x_2)$$

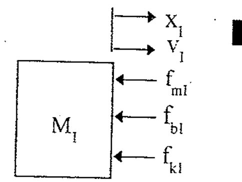


Fig 1.11.2

By Newton's second law, $f_{m1} + f_{b1} + f_{k1} = 0$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{d(x_1 - x_2)}{dt} + K_1(x_1 - x_2) = 0 \quad .....(1.11.1)$$

The free body diagram of M_2 is shown in fig 1.11.3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b1}, f_{k2} and f_{k1}.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \;,\; f_{b2} = B_2 \frac{dx_2}{dt}, \;\; f_{b1} = B_1 \frac{d}{dt}(x_2 - x_1)$$

$$f_{k2} = K_2 x_2 \quad \text{and} \quad f_{k1} = K_1(x_2 - x_1)$$

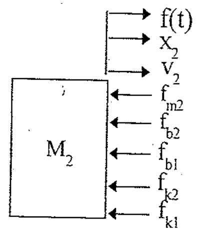


Fig 1.11.3

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b1} + f_{k1} = f(t)$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_1 \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = f(t) \quad .....(1.11.2)$$

On replacing the displacements by velocity in the differential equations 1.11.1 and 1.11.2 governing the mechanical system we get,

$$\left(\text{i.e.,} \quad \frac{d^2x}{dt^2} = \frac{dv}{dt}, \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1(v_1 - v_2) + K_1 \int (v_1 - v_2)dt = 0 \quad .....(1.11.3)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2 dt + B_1(v_2 - v_1) + K_1 \int (v_2 - v_1)dt = f(t) \quad .....(1.11.4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force voltage analogous electrical circuit will have two meshes. The force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh 1 forming a closed path. The elements M_2, K_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous element in mesh 2 forming a closed path.

The elements B_1 and K_1 are common between node 1 and 2 and so they are represented as common elements between mesh 1 and 2. The force-voltage electrical analogous circuit is shown in fig 1.11.4.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$ $\quad v_1 \rightarrow i_1 \quad M_1 \rightarrow L_1 \quad K_1 \rightarrow 1/C_1 \quad B_1 \rightarrow R_1$

$\quad v_2 \rightarrow i_2 \quad M_2 \rightarrow L_2 \quad K_2 \rightarrow 1/C_2 \quad B_2 \rightarrow R_2$

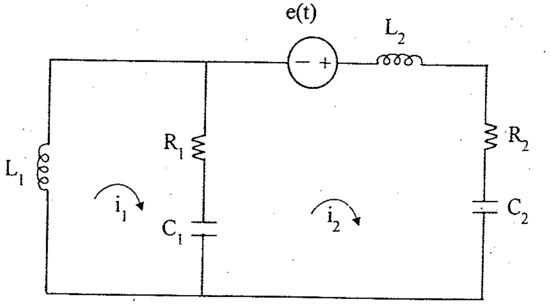


Fig 1.11.4 : Force-voltage electrical analogous circuit

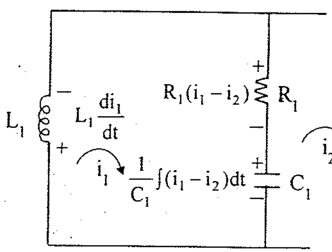


Fig 1.11.5

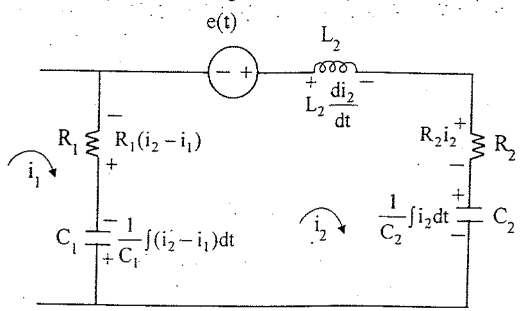


Fig 1.11.6

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 1.11.4. are given below (refer fig 1.11.5 and 1.11.6)

$$L_1\frac{di_1}{dt}+R_1(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)\,dt=0 \quad .....(1.11.5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2dt+\frac{1}{C_1}\int(i_2-i_1)\,dt+R_1(i_2-i_1)=e(t) \quad .....(1.11.6)$$

It is observed that the mesh basis equations 1.11.5 and 1.11.6 are similar to the differential equations 1.11.3 and 1.11.4 governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes. The force applied to mass M_2 is represented as a current source connected to node2 in analogous electrical circuit.

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node 1 in analogous electrical circuit. The elements M_2, K_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node 1 in analogous electrical circuit.

The elements K_1 and B_1 is common to node 1 and 2 and so they are represented by analogous element as common elements between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 1.11.7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$ $\quad v_1 \rightarrow v_1 \quad M_1 \rightarrow C_1 \quad B_1 \rightarrow 1/R_1 \quad K_1 \rightarrow 1/L_1$

$\quad v_2 \rightarrow v_2 \quad M_2 \rightarrow C_2 \quad B_2 \rightarrow 1/R_2 \quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig (1.11.7) are given below [Refer fig (1.11.8) and (1.11.9)].

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)\,dt=0 \quad .....(1.11.7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_1}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=i(t) \quad .....(1.11.8)$$

It is observed that the node basis equations (1.11.7) and (1.11.8) are similar to the differential equations (1.11.3) and (1.11.4) governing the mechanical system.

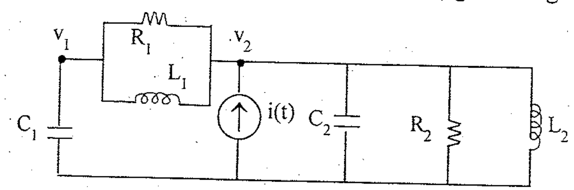


Fig 1.11.7 : Force-current electrical analogous circuit

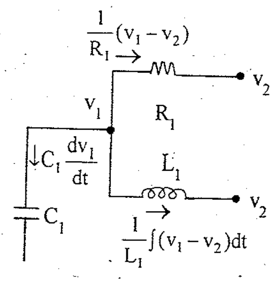


Fig 1.11.8

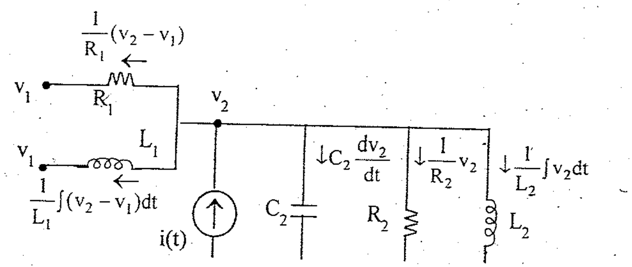


Fig 1.11.9

1.10 ELECTRICAL ANALOGOUS OF MECHANICAL ROTATIONAL SYSTEMS

The three basic elements moment of inertia, rotational dashpot and torsional spring that are used in modelling mechanical rotational systems are analogous to resistance, inductance and capacitance of electrical systems. The input torque in mechanical system is analogous to either voltage source or current source in electrical systems. The output angular velocity (first derivative of angular displacement) in mechanical rotational system is analogous to either current or voltage in an element in electrical system. Since the electrical systems has two types of inputs either voltage source or current source, there are two types of analogies: torque-voltage analogy and torque-current analogy.

TORQUE-VOLTAGE ANALOGY

The torque balance equations of mechanical rotational elements and their analogous electrical elements in torque-voltage analogy are shown in table 6. The table 7 shows the list of analogous quantities in torque-voltage analogy

TABLE 6

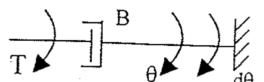
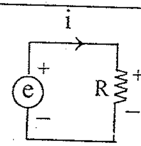
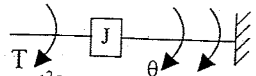
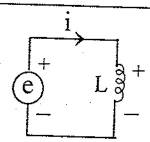
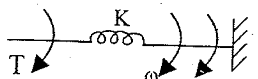
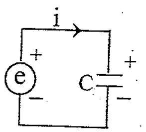
Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Voltage source Output : Current through the element
 $T = B\frac{d\theta}{dt} = B\omega$ $\quad \omega = \frac{d\theta}{dt}$	 $e = Ri$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\quad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $e = L\frac{di}{dt}$
 $T = K\theta = K\int \omega dt$ $\quad \theta = \int \omega dt$	 $e = \frac{1}{C}\int i dt$

TABLE 7

Item	Mechanical rotational system	Electrical system (mesh basis system)
Independent variable (input)	Torque, T	Voltage, e
Dependent variable (output)	Angular Velocity, ω	Current, i
	Angular displacement, θ	Charge, q
Dissipative element	Rotational coefficient of dashpot, B	Resistance, R
Storage element	Moment of inertia, J	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\Sigma T = 0$	Kirchoff's voltage law $\Sigma v = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on torque voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same angular velocity are said to be in series.
2. The elements having same angular velocity in mechanical system should have analogous same current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. The moment of inertia of mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node (moment of inertia of mass) in mechanical system should be

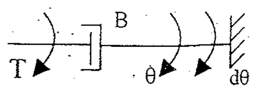
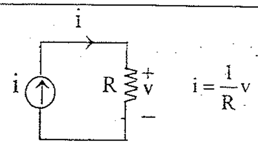
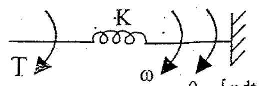
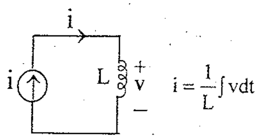
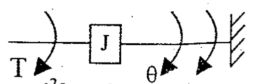
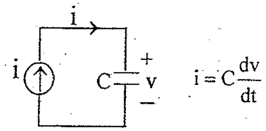
represented by analogous element in a closed loop in analogous electrical system.

6. The element connected between two nodes (moment of inertia) in mechanical system is represented as a common element between two meshes in electrical analogous system.

TORQUE-CURRENT ANALOGY

The torque balance equations of mechanical elements and their analogous electrical elements in torque-current analogy are shown in table-8. The table 9 shows the list of analogous quantities in torque-current analogy.

TABLE 8

Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Current source Output : Voltage across the element
 $T = B\frac{d\theta}{dt} = B\omega$, $\omega = \frac{d\theta}{dt}$	 $i = \frac{1}{R}v$
 $T = K\theta = K\int\omega dt$, $\theta = \int\omega dt$	 $i = \frac{1}{L}\int v dt$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$, $\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $i = C\frac{dv}{dt}$

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on Torque-current analogy.

1. In electrical systems the elements in parallel will have same voltage, likewise in mechanical systems, the elements having same torque are said to be in parallel.

TABLE 9

Item	Mechanical rotational system	Electrical system (node basis system)
Independent variable (input)	Torque, T	Current, i
Dependent variable (output)	Angular Velocity, ω	Voltage, v
	Angular displacement, θ	Flux, ϕ
Dissipative element	Rotational frictional coefficient of dashpot, B	Conductance, G = 1/R
Storage element	Moment of inertia, J	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\Sigma T = 0$	Kirchoff's current law $\Sigma i = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

2. The elements having same angular velocity in mechanical system should have analogous same voltage in electrical analogous system.

3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. The moment of inertia of mass is considered as a node.

4. The number of nodes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.

5. The mechanical driving sources (Torque) and passive elements connected to the node in mechanical system should be represented by analogous element connected to a node in analogous electrical system.

6. The element connected between two nodes (moment of inertia of mass) in mechanical system is represented as a common element between two nodes in electrical analogous system.

EXAMPLE 1.12

Write the differential equations governing the mechanical rotational system shown in fig 1.12.1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

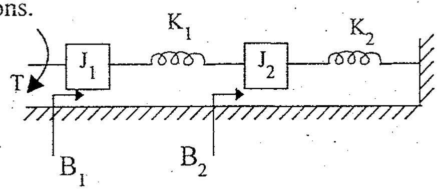


Fig 1.12.1

SOLUTION

The given mechanical rotational system has two nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1 and J_2 be θ_1 and θ_2 respectively. The corresponding angular velocities be ω_1 and ω_2. The free body diagram of J_1 is shown in fig 1.12.2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \;;\; T_{b1} = B_1\frac{d\theta_1}{dt} \quad \text{and}$$

$$T_{k1} = K_1(\theta_1 - \theta_2)$$

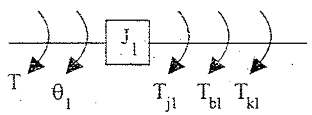


Fig 1.12.2

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1\frac{d^2\theta_1}{dt^2} + B_1\frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T \qquad .....(1.12.1)$$

The free body diagram of J_2 is shown in fig 1.12.3. The opposing torques are marked as T_{j2}, T_{b2}, T_{k2} and T_{k1}.

$$T_{j2} = J_2\frac{d^2\theta_2}{dt^2} \;; \quad T_{b2} = B_2\frac{d\theta_2}{dt}$$

$$T_{k2} = K_2\theta_2 \quad \text{and } T_{k1} = K_1(\theta_2 - \theta_1)$$

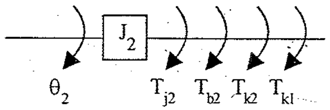


Fig 1.12.3

By Newton's second law, $T_{j2} + T_{b2} + T_{k2} + T_{k1} = 0$

$$J_2\frac{d^2\theta_2}{dt^2} + B_2\frac{d\theta_2}{dt} + K_2\theta_2 + K_1(\theta_2 - \theta_1) = 0 \qquad .....(1.12.2)$$

On replacing the angular displacements by angular velocity in the differential equations (1.12.1) and (1.12.2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \;;\; \frac{d\theta}{dt} = \omega \text{ and } \theta = \int \omega\, dt\right)$$

$$J_1\frac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)\, dt = T \qquad .....(1.12.3)$$

$$J_2\frac{d\omega_2}{dt} + B_2\omega_2 + K_2\int\omega_2 dt + K_1\int(\omega_2 - \omega_1)dt = 0 \qquad .....(1.12.4)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-voltage analogous electrical circuit will have two meshes. The torque applied to J_1 is represented by a voltage source in first mesh.

The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous element in mesh 1 forming a closed path. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous element is mesh 2 forming a closed path.

The element K_1 is common between node 1 and 2 and so it is represented by analogous element as common element between two meshes. The torque-voltage electrical analogous circuit is shown in fig 1.12.4.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$\omega_1 \rightarrow i_1$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$\omega_2 \rightarrow i_2$			

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 1.12.4 are given below [Refer fig (1.12.5) and (1.12.6)].

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int(i_1 - i_2) = e(t) \qquad .....(1.12.5)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + \frac{1}{C_1}\int(i_2 - i_1)dt = 0 \qquad .....(1.12.6)$$

It is observed that the mesh basis equations (1.12.5) and (1.12.6) are similar to the differential equations (1.12.3) and (1.12.4) governing the mechanical system.

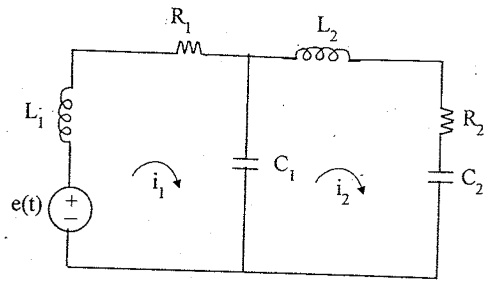


Fig 1.12.4 : Torque-voltage electrical analogous circuit

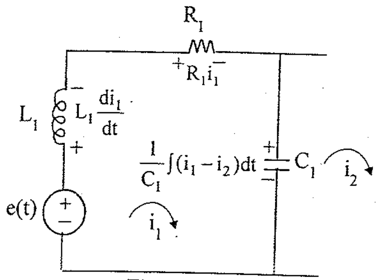


Fig 1.12.5

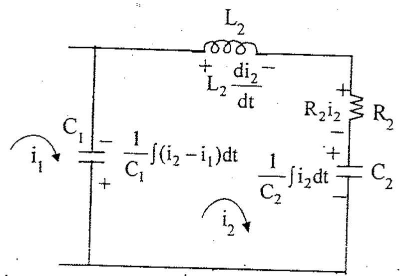


Fig 1.12.6

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-current analogous electrical circuit will have two nodes. The torque applied to J_1 is represented as a current source connected to node 1 in analogous electrical circuit.

The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node 1 in analogous electrical circuit. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node 2 in analogous electrical circuit.

The element K_1 is common between node 1 and 2. So it is represented by analogous element as common element between node 1 and 2. The torque-current electrical analogous circuit is shown in fig 1.12.7.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\quad B_1 \rightarrow 1/R_1$ $\quad \omega_1 \rightarrow v_1$ $\quad J_1 \rightarrow C_1$ $\quad K_1 \rightarrow 1/L_1$

$\quad B_2 \rightarrow 1/R_2$ $\quad \omega_2 \rightarrow v_2$ $\quad J_2 \rightarrow C_2$ $\quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig 1.12.7 are given below [Refer fig (1.12.8) and (1.12.9)].

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \qquad .....(1.12.7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{L_1}\int(v_2-v_1)dt=0 \qquad .....(1.12.8)$$

It is observed that the node basis equations (1.12.7) and (1.12.8) are similar to the differential equations (1.12.3) and (1.12.4) governing the mechanical system.

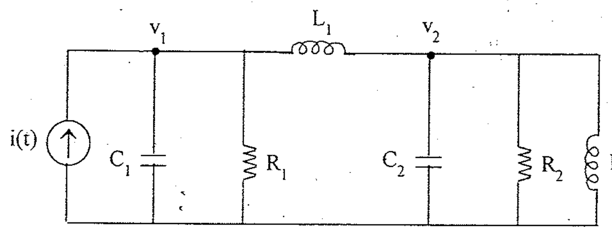


Fig 1.12.7 :Torque-current electrical analogous circuit

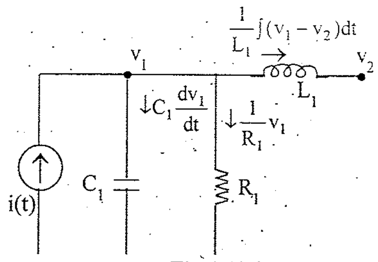


Fig 1.12.8

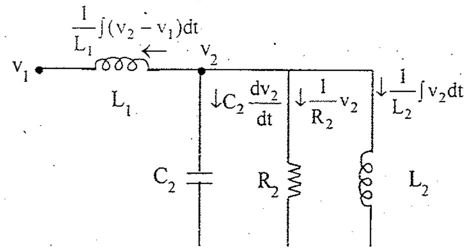


Fig 1.12.9

EXAMPLE 1.13

Write the differential equations governing the mechanical rotational system shown in fig 1.13.1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equ_tions.

SOLUTION

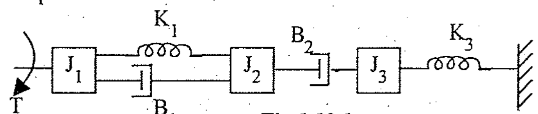


Fig 1.13.1

The given mechanical rotational system has three nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes. Let the angular displacements of J_1, J_2 and J_3 be θ_1, θ_2 and θ_3 respectively. The corresponding angular velocities be ω_1, ω_2 and ω_3.

The free body diagram of J_1 is shown in fig 1.13.2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \; ; \; T_{b1} = B_1 \frac{d(\theta_1 - \theta_2)}{dt}$$

$$T_{k1} = K_1(\theta_1 - \theta_2)$$

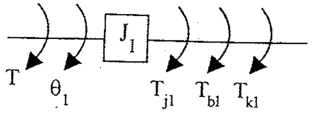


Fig 1.13.2

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d(\theta_1 - \theta_2)}{dt} + K_1(\theta_1 - \theta_2) = T \quad .....(1.13.1)$$

The free body diagram of J_2 is shown in fig 1.13.3. The opposing torques are marked as T_{j2}, T_{b2}, T_{b1} and T_{k1}.

$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2} \; ; \qquad T_{b2} = B_2 \frac{d(\theta_2 - \theta_3)}{dt}$$

$$T_{k1} = K_1(\theta_2 - \theta_1) \; ; \; T_{b1} = B_1 \frac{d(\theta_2 - \theta_1)}{dt}$$

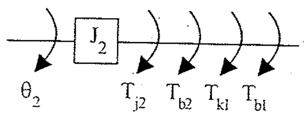


Fig 1.13.3

By Newton's second law, $T_{j2} + T_{b2} + T_{b1} + T_{k1} = 0$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d(\theta_2 - \theta_3)}{dt} + B_1 \frac{d(\theta_2 - \theta_1)}{dt} + K_1(\theta_2 - \theta_1) = 0 \quad .....(1.13.2)$$

The free body diagram of J_3 is shown in fig 1.13.4. The opposing torques are marked as T_{j3}, T_{b2} and T_{k3}.

$$T_{j3} = J_3 \frac{d^2\theta_3}{dt^2} \; ; \qquad T_{b2} = B_2 \frac{d(\theta_3 - \theta_2)}{dt}$$

$$T_{k3} = K_3\theta_3$$

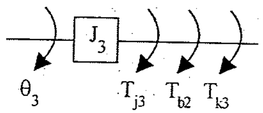


Fig 1.13.4

By Newton's second law, $T_{j3} + T_{b2} + T_{k3} = 0$

$$J_3 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d(\theta_3 - \theta_2)}{dt} + K_3\theta_3 = 0 \quad .....(1.13.3)$$

On replacing the angular displacements by angular velocity in the differential equations (1.13.1), (1.13.2) and (1.13.3) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \; ; \; \frac{d\theta}{dt} = \omega \text{ and } \theta = \int \omega \, dt \right)$$

$$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1 - \omega_2) + K_1 \int (\omega_1 - \omega_2) \, dt = T \quad .....(1.13.4)$$

$$J_2 \frac{d\omega_2}{dt} + B_2(\omega_2 - \omega_3) + B_1(\omega_2 - \omega_1) + K_1 \int (\omega_2 - \omega_1) \, dt = 0 \quad .....(1.13.5)$$

$$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3 \int \omega_3 dt = 0 \quad .....(1.13.6)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-voltage analogous electrical circuit will have three meshes. The torque applied to J_1 is represented by a voltage source in first mesh.

The elements J_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh 1 forming a closed path. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous element in mesh 2 forming a closed path. The element J_3, B_2 and K_3 are connected to third node. Hence they are represented by analogous element in mesh 3 forming a closed path.

The elements K_1 and B_1 are common between the nodes 1 and 2 and so they are represented by analogous element as common between mesh 1 and 2. The element B_2 is common between the nodes 2 and 3 and so it is represented by analogous element as common element between the mesh 2 and 3. The torque-voltage electrical analogous circuit is shown in fig 1.13.5.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$\omega_1 \rightarrow i_1$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
	$\omega_2 \rightarrow i_2$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_3 \rightarrow 1/C_3$
	$\omega_3 \rightarrow i_3$	$J_3 \rightarrow L_3$		

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 1.13.5 are given below *[Refer fig (1.13.6), (1.13.7) and (1.13.8)]*.

$$L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1} \int (i_1 - i_2) dt = e(t) \quad .....(1.13.7)$$

$$L_2 \frac{di_2}{dt} + R_2(i_2 - i_3) + R_1(i_2 - i_1) + \frac{1}{C_1} \int (i_2 - i_1) \, dt = 0 \quad .....(1.13.8)$$

$$L_3 \frac{di_3}{dt} + R_2(i_3 - i_2) + \frac{1}{C_3} \int i_3 dt = 0 \quad .....(1.13.9)$$

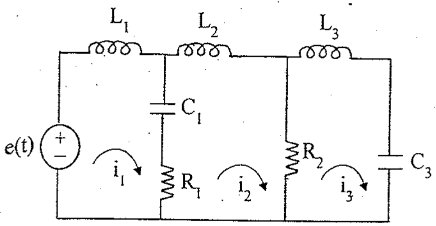


Fig 1.13.5 : Torque-voltage electrical analogous circuit

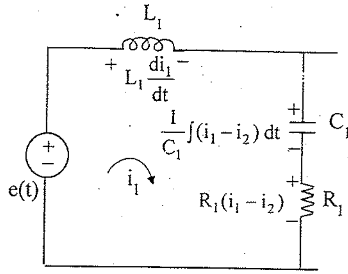


Fig 1.13.6

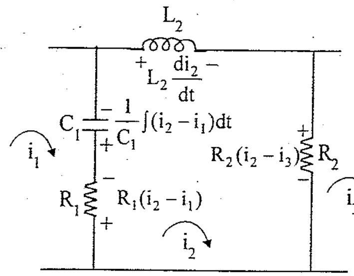


Fig 1.13.7

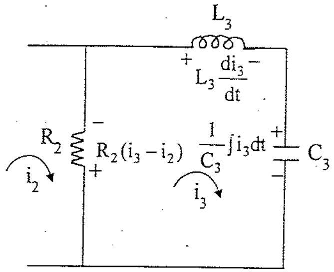


Fig 1.13.8

It is observed that the mesh basis equations (1.13.7), (1.13.8) and (1.13.9) are similar to the differential equations (1.13.4), (1.13.5) and (1.13.6) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-current analogous electrical circuit will have three nodes. The torque applied to J_1 is represented as a current source connected to node 1 in analogous electrical circuit.

The elements K_1, J_1 and B_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node 1 in analogous electrical circuit. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node 2 in analogous electrical circuit. The elements J_3, B_2, and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node 3 in analogous electrical circuit.

The elements K_1 and B_1 are common between node 1 and 2 and so they are represented by analogous element as common elements between node 1 and 2. The element B_2 is common between node 2 and 3 and so it is represented as common element between node 2 and 3 in analogous circuit. The torque-current electrical analogous circuit is shown in fig 1.13.9.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$$\begin{array}{lllll} T \rightarrow i(t) & \omega_1 \rightarrow v_1 & J_1 \rightarrow C_1 & B \rightarrow 1/R_1 & K_1 \rightarrow 1/L_1 \\ & \omega_2 \rightarrow v_2 & J_2 \rightarrow C_2 & B_2 \rightarrow 1/R_2 & K_3 \rightarrow 1/L_3 \\ & \omega_3 \rightarrow v_3 & J_3 \rightarrow C_3 & & \end{array}$$

The node basis equations using Kirchoff's current law for the circuit shown in fig 1.13.9 are given below [Refer fig (1.13.10), (1.13.11) and (1.13.12)].

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)dt = i(t) \qquad .....(1.13.10)$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_1}(v_2 - v_1) + \frac{1}{R_2}(v_2 - v_3) + \frac{1}{L_1}\int (v_2 - v_1)\, dt = 0 \qquad .....(1.13.11)$$

$$C_3 \frac{dv_3}{dt} + \frac{1}{R_2}(v_3 - v_2) + \frac{1}{L_3}\int v_3\, dt = 0 \qquad .....(1.13.12)$$

It is observed that the node basis equations (1.13.10), (1.13.11) and (1.13.12) are similar to the differential equations (1.13.4), (1.13.5) and (1.13.6) governing the mechanical system.

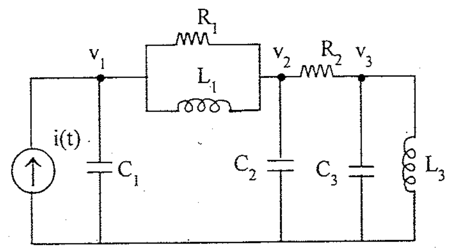


Fig 1.13.9 : Torque-current electrical analogous circuit

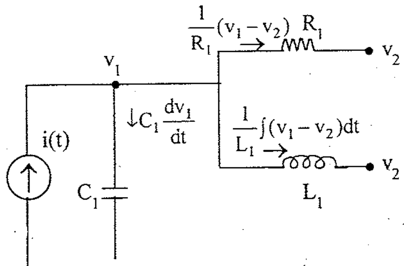


Fig 1.13.10

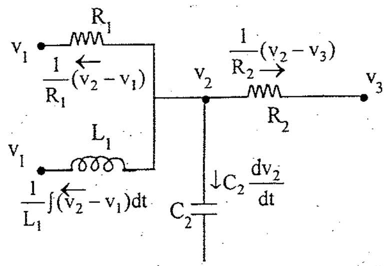


Fig 1.13.11

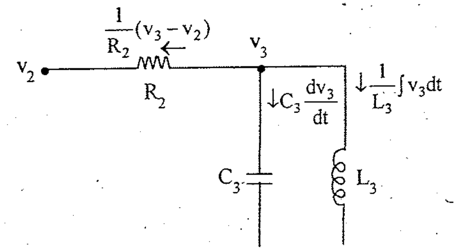


Fig 1.13.12

1.11 BLOCK DIAGRAMS

A control system may consist of a number of components. In control engineering to show the functions performed by each component, we commonly use a diagram called the block diagram. A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. The elements of a block diagram are block, branch point and summing point

BLOCK

In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Figure 1.25 shows the functional block of the block diagram.

The arrowhead pointing towards the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as signals. The output signal from the block is given by the product of input signal and transfer function in the block.

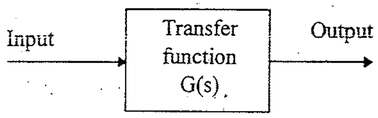


Fig 1.25 : Functional block

SUMMING POINT

Summing points are used to add two or more signals in the system. Referring to figure 1.26, a circle with a cross is the symbol that indicates a summing operation.

The plus or minus sign at each arrowhead indicates whether the signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

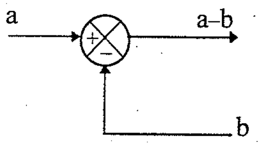


Fig 1.26 : Summing point

BRANCH POINT

A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.

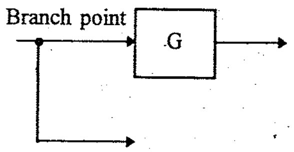


Fig 1.27 : Branch point

CONSTRUCTING BLOCK DIAGRAM FOR CONTROL SYSTEMS

A control system can be represented diagramatically by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebraic equations. The equations will have variables and constants. From the working knowledge of the system the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The output of one section will be input for another section. The various sections are interconnected to obtain the overall block diagram of the system.

EXAMPLE 1.14

Construct the block diagram of armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7)

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \qquad .....(1.14.1)$$

$$T = K_t i_a \qquad .....(1.14.2)$$

$$T = J\frac{d\omega}{dt} + B\omega \qquad .....(1.14.3)$$

$$e_b = K_b\,\omega \qquad .....(1.14.4)$$

$$\omega = \frac{d\theta}{dt} \qquad .....(1.14.5)$$

On taking Laplace transform of equation (1.14.1) we get,

$$V_a(s) = I_a(s)\,R_a + L_a\,s\,I_a(s) + E_b(s) \qquad .....(1.14.6)$$

In equation (1.14.6), $V_a(s)$ and $E_b(s)$ are inputs and $I_a(s)$ is the output. Hence the equation (1.14.6) is rearranged and the block diagram for this equation is shown in fig 1.14.1.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\,L_a]$$

$$\therefore I_a(s) = \frac{1}{R_a + s\,L_a}\,[V_a(s) - E_b(s)]$$

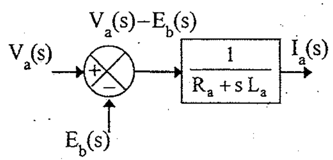


Fig 1.14.1

On taking Laplace transform of equation (1.14.2) we get,

$$T(s) = K_t\, I_a(s) \qquad .....(1.14.7)$$

In equation (1.14.7), $I_a(s)$ is the input and T(s) is the output. The block diagram for this equation is shown in fig 1.14.2.

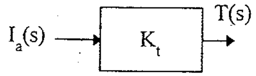


Fig 1.14.2

On taking Laplace transform of equation (1.14.3) we get,

$$T(s) = Js\,\omega(s) + B\,\omega(s) \qquad .....(1.14.8)$$

In equation (1.14.8), T(s) is the input and ω(s) is the output. Hence the equation (1.14.8) is rearranged and the block diagram for this equation is shown in fig (1.14.3).

$$T(s) = (Js + B)\,\omega(s)$$

$$\therefore \omega(s) = \frac{1}{Js+B}\,T(s)$$

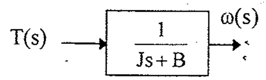


Fig 1.14.3

On taking Laplace transform of equation (1.14.4) we get,

$$E_b(s) = K_b\,\omega(s) \qquad .....(1.14.9)$$

In equation (1.14.9) ω(s) is the input and $E_b(s)$ is the output. The block diagram for this equation is shown in fig 1.14.4.

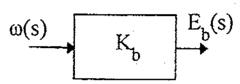


Fig 1.14.4

On taking Laplace transform of equation (1.14.5) we get,

$$\omega(s) = s\,\theta(s) \qquad .....(1.14.10)$$

In equation (1.14.10), ω(s) is the input and θ(s) is the output. Hence equation (1.14.10) is rearranged and the block diagram for this equation is shown in fig 1.14.5.

$$\theta(s) = \frac{1}{s}\,\omega(s)$$

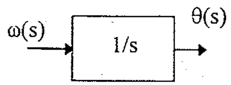


Fig 1.14.5

The overall block diagram of armature controlled dc motor is obtained by connecting the various sections shown in fig 1.14.1 to fig 1.14.5. The overall block diagram is shown in fig 1.14.6.

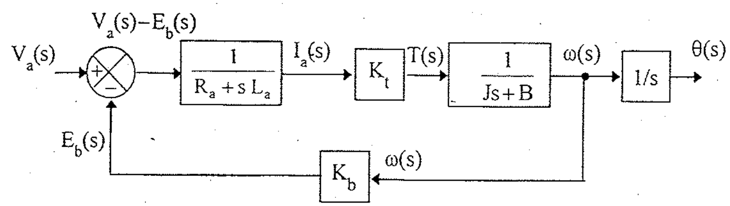


Fig 1.14.6 : Block diagram of armature controlled dc motor

EXAMPLE 1.15

Construct the block diagram of field controlled dc motor.

SOLUTION

The differential equations governing the field controlled dc motor are (refer section 1.8)

$$v_f = R_f\,i_f + L_f\,\frac{di_f}{dt} \qquad .....(1.15.1)$$

$$T = K_{tf}\,i_f \qquad .....(1.15.2)$$

$$T = J\,\frac{d^2\theta}{dt^2} + B\,\frac{d\theta}{dt} \qquad .....(1.15.3)$$

On taking Laplace transform of equation (1.15.1) we get,

$$V_f(s) = R_f\,I_f(s) + L_f\,s\,I_f(s) \qquad .....(1.15.4)$$

In equation (1.15.4), $V_f(s)$ is the input and $I_f(s)$ is the output. Hence the equation (1.15.4) is rearranged and the block diagram for this equation is shown in fig 1.15.1.

$$V_f(s) = I_f(s)\,[R_f + sL_f]$$

$$\therefore I_f(s) = \frac{1}{R_f + s\,L_f}\,V_f(s)$$

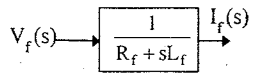


Fig 1.15.1

On taking Laplace transform of equation (1.15.2) we get,

$$T(s) = K_{tf}\;I_f(s) \qquad .....(1.15.5)$$

In equation (1.15.5), $I_f(s)$ is the input and T(s) is the output. The block diagram for this equation is shown in fig (1.15.2).

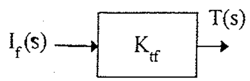


Fig 1.15.2

On taking Laplace transform of equation (1.15.3) we get,

$$T(s) = J\,s^2\,\theta(s) + B\,s\,\theta(s) \qquad .....(1.15.6)$$

In equation (1.15.6), T(s) is the input and θ(s) is the output. Hence the equation (1.15.6) is rearranged and the block diagram for this equation is shown in fig (1.15.3).

$$T(s) = (J\,s^2 + Bs)\,\theta(s)$$

$$\therefore \theta(s) = \frac{1}{Js^2 + Bs}\,T(s)$$

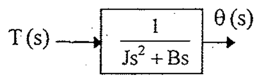


Fig 1.15.3

The overall block diagram of field controlled dc motor is obtained by connecting the various section shown in fig (1.15.1) to fig (1.15.3). The overall block diagram is shown in fig (1.15.4)

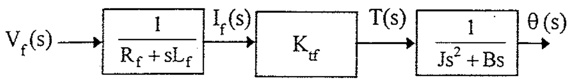


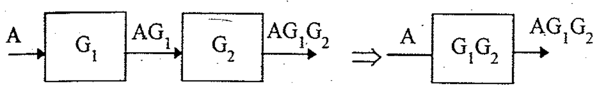
Fig 1.15.4 : Block diagram of field controlled dc motor

BLOCK DIAGRAM REDUCTION

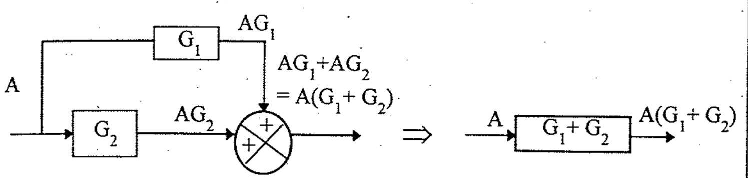
The block diagram can be reduced to find the overall transfer function of the system. The following rules can be used for block diagram reduction. The rules are framed such that any modification made on the diagram does not alter the input output relation.

RULES OF BLOCK DIAGRAM ALGEBRA

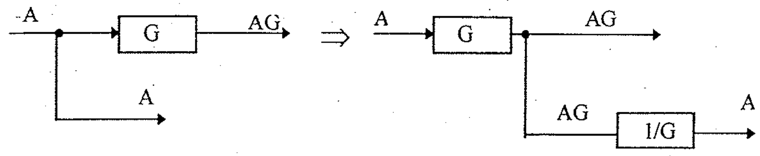
1. ***Combining the blocks in cascade***



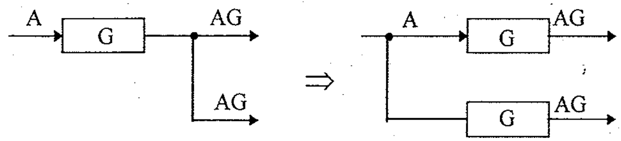
2. ***Combining Parallel blocks (or combining feed forward paths)***



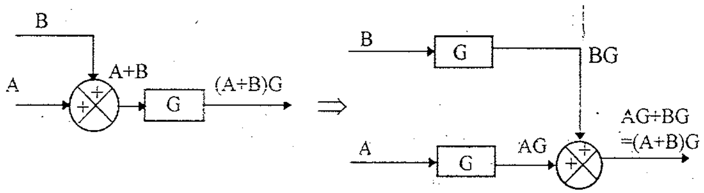
3. ***Moving the branch point ahead of the block***



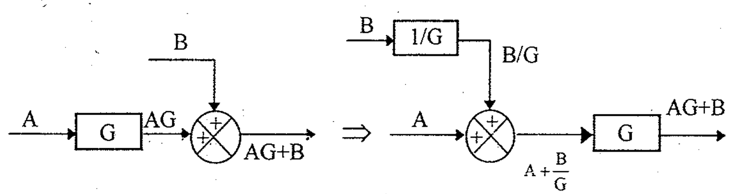
4 ***Moving the branch point before the block***



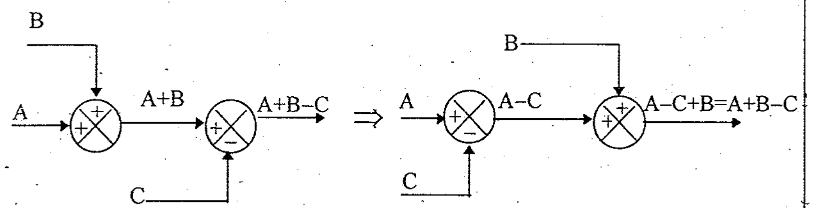
5. ***Moving the summing point ahead of the block***



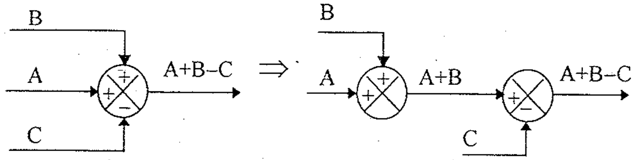
6. ***Moving the summing point before the block***



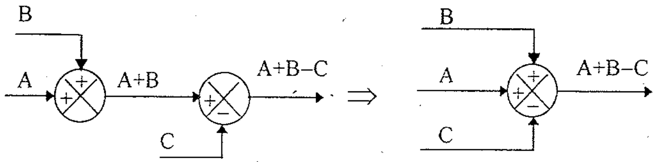
7. ***Interchanging summing point***



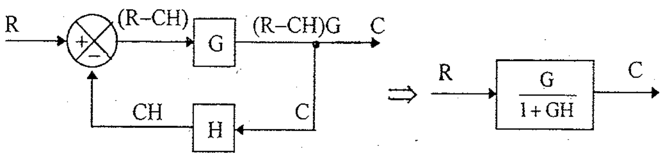
8. Splitting summing points



9. Combining summing points



10. Elimination of feedback loop

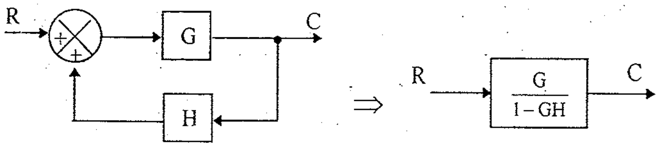


Proof :

$$C = (R - CH)\,G$$
$$C = RG - CHG$$
$$C + CHG = RG$$

$$C(1+HG) = RG$$
$$\frac{C}{R} = \frac{G}{1+GH}$$

Also



CHAPTER 1 - MATHEMATICAL MODELS OF CONTROL SYSTEM

EXAMPLE 1.16

Reduce the block diagram shown in fig 1.16.1 and find C/R

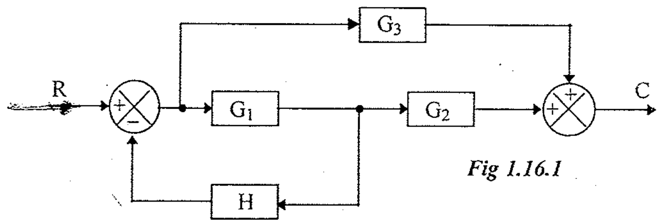
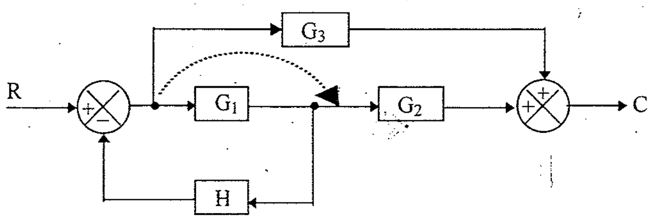


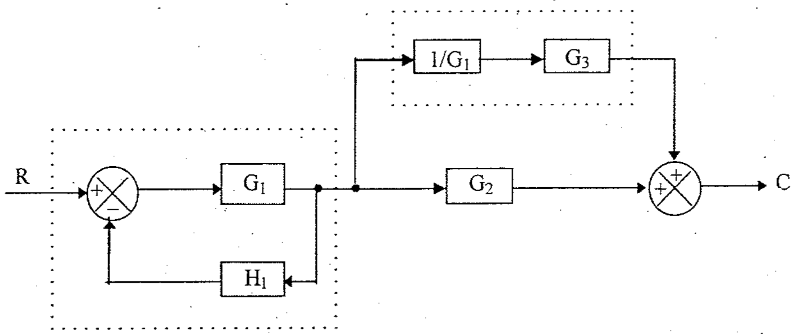
Fig 1.16.1

SOLUTION

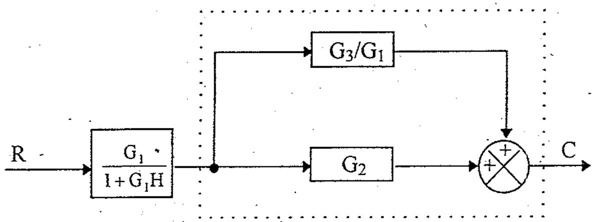
Step 1: Move the branch point after the block



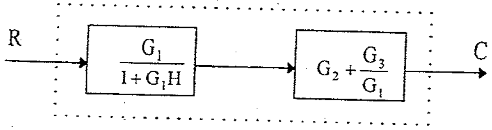
Step 2: Eliminate the feedback path and combining blocks in cascade



Step 3: Combining parallel blocks



Step 4: Combining blocks in cascade



$$\frac{C}{R}=\left(\frac{G_1}{1+G_1H}\right)\left(G_2+\frac{G_3}{G_1}\right)=\left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right)=\frac{G_1G_2+G_3}{1+G_1H}$$

RESULT

The overall transfer function of the system, $\frac{C}{R}=\frac{G_1G_2+G_3}{1+G_1H}$

EXAMPLE 1.17

Using block diagram reduction technique find closed loop transfer function of the system whose block diagram is shown in fig 1.17.1.

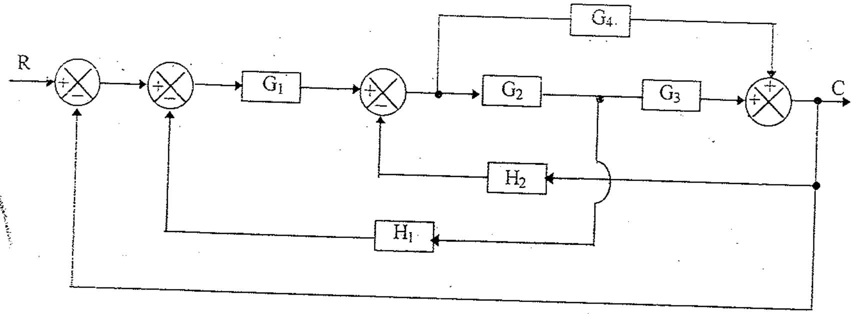
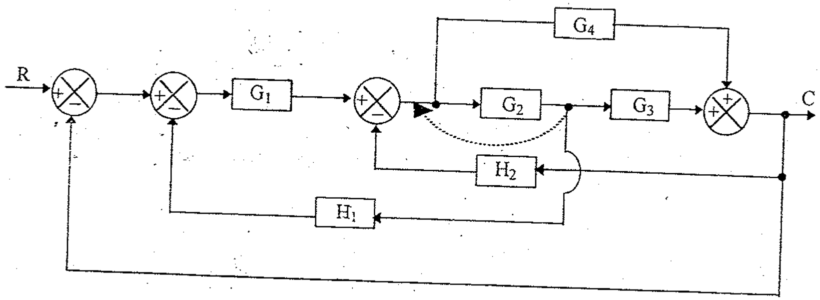


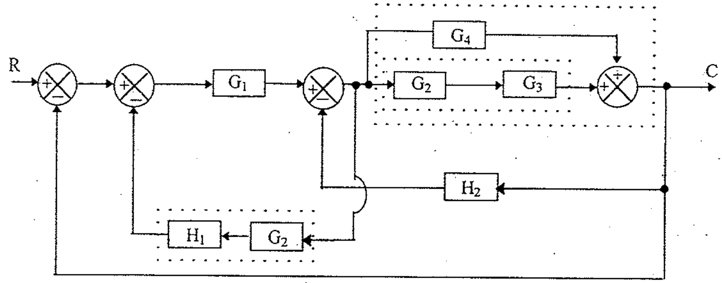
Fig 1.17.1

SOLUTION

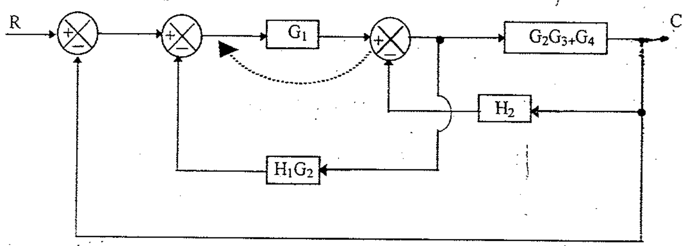
Step 1: Moving the branch point before the block



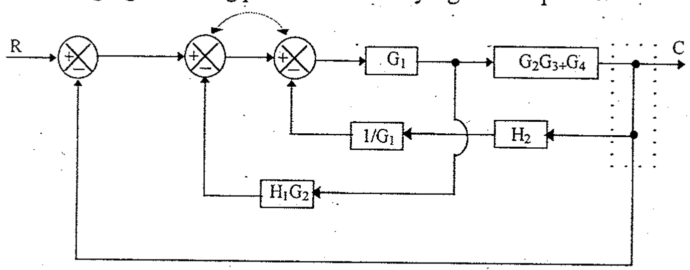
Step 2: Combining the blocks in cascade and eliminating parallel blocks



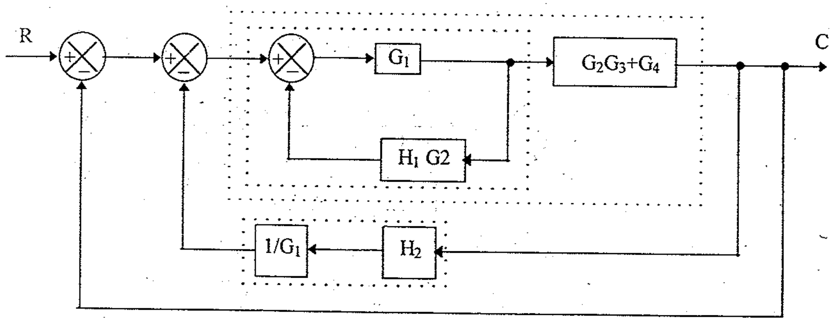
Step 3: Moving summing point before the block



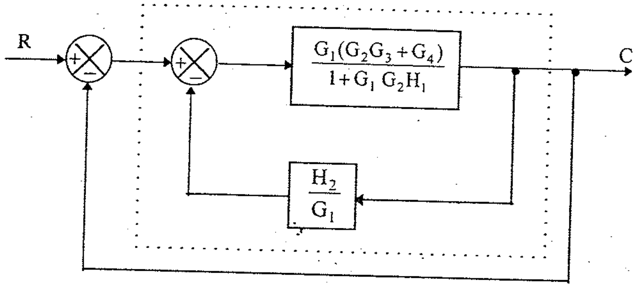
Step 4: Interchanging summing points and modifying branch points.

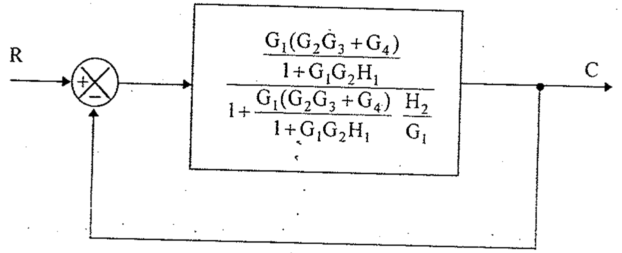


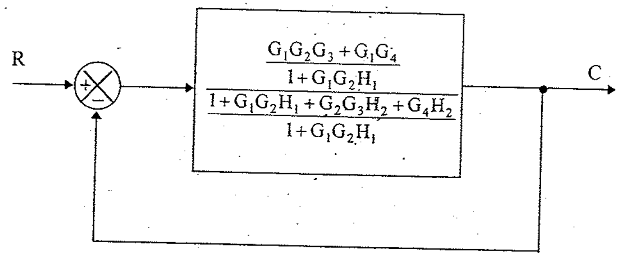
Step 5: Eliminating the feedback path and combining blocks in cascade



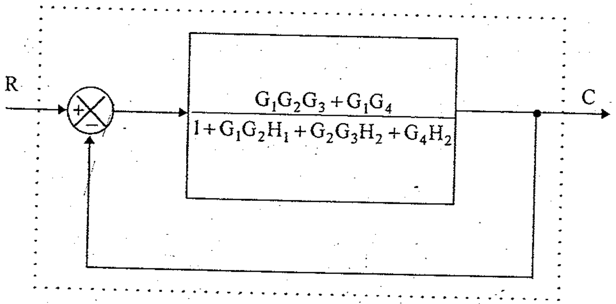
Step 6: Eliminating the feedback path







Step 7: Eliminating the feedback path



$$\frac{C}{R}=\frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}$$

$$=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

RESULT

The overall transfer function is given by

$$\frac{C}{R}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

EXAMPLE 1.18

Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in fig 1.18.1.

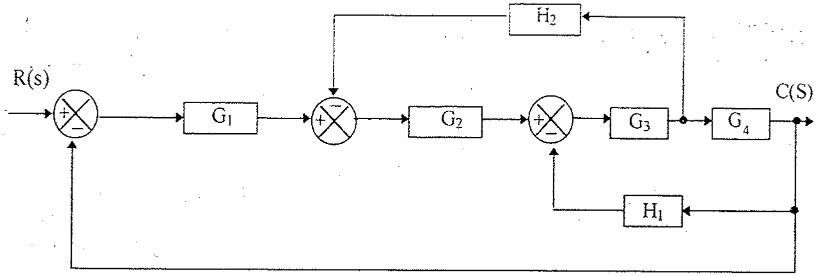
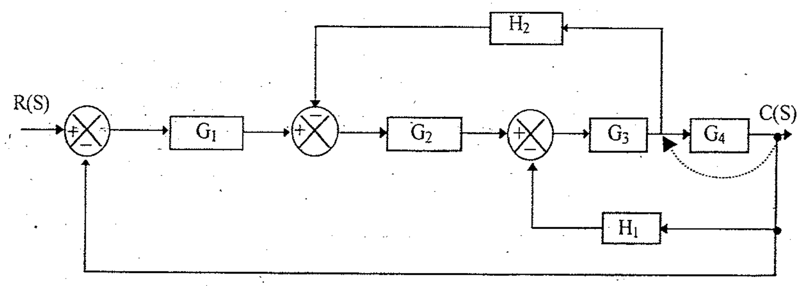


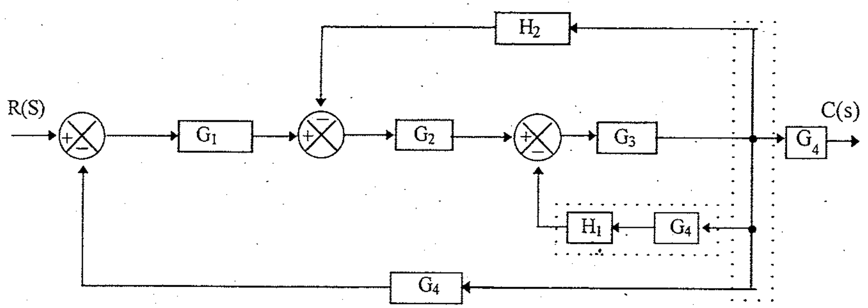
Fig 1.18.1

SOLUTION

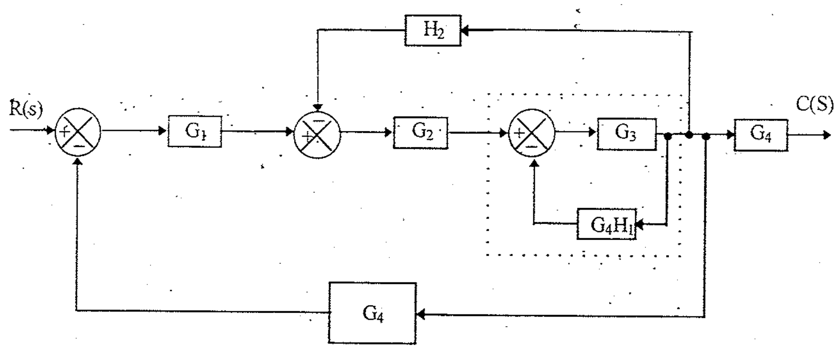
Step 1: Moving the branch point before the block



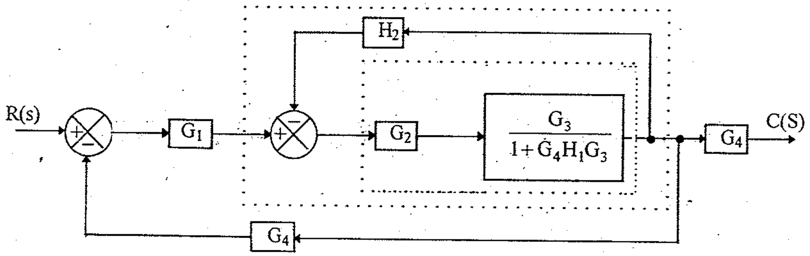
Step 2 : Combining the blocks in cascade and rearranging the branch points



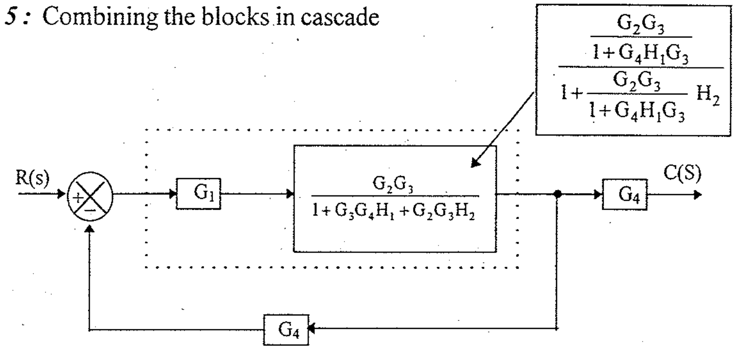
Step 3 : Eliminating the feedback path



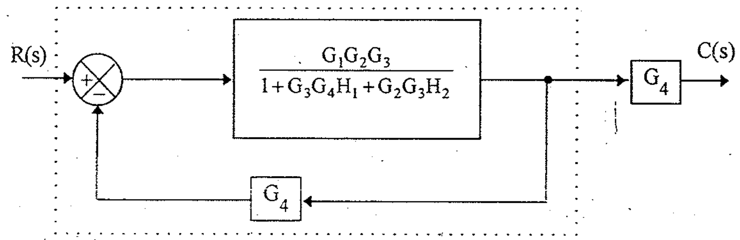
Step 4 : Combining the blocks in cascade and eliminating feedback path



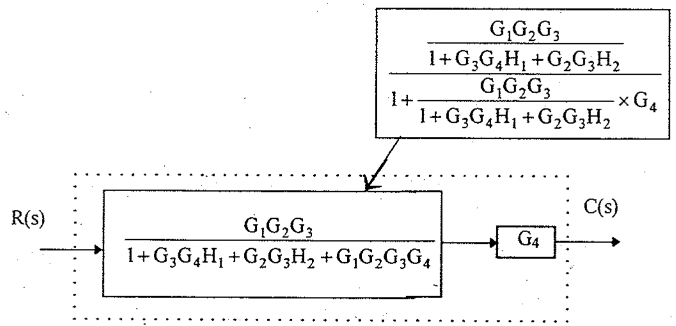
Step 5 : Combining the blocks in cascade



Step 6 : Eliminating the feedback path



Step 7 : Combining the blocks in cascade



$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

RESULT

The overall transfer function of the system is given by

$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

EXAMPLE 1.19

For the system represented by the block diagram shown in fig 1.19.1 Evaluate the closed loop transfer function when the input R is (i) at station I (ii) at station II.

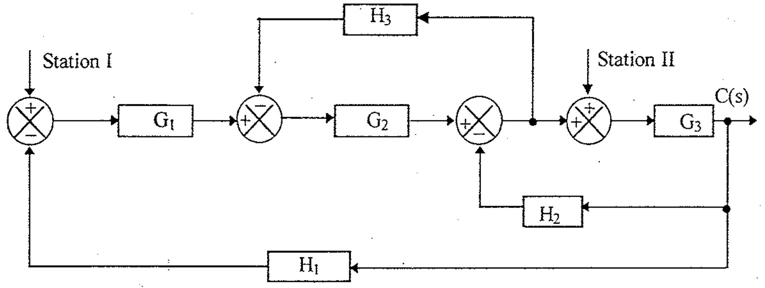


Fig 1.19.1

SOLUTION

(i) Consider the input R is at station I and so the input at station II is made zero. Let the output be C_1. Since there is no input at station II that summing point can be removed and resulting block diagram is shown in fig 1.19.2.

Step 1: Shift the take off point of feedback H_3 beyond G_3 and rearrange the branch points

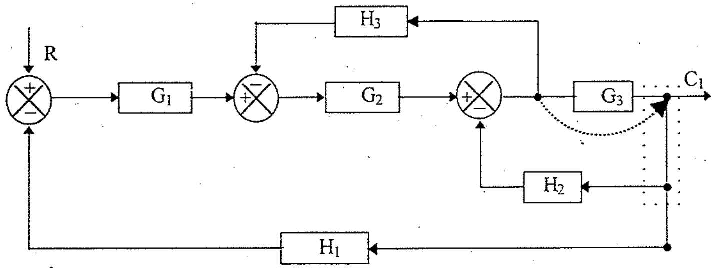
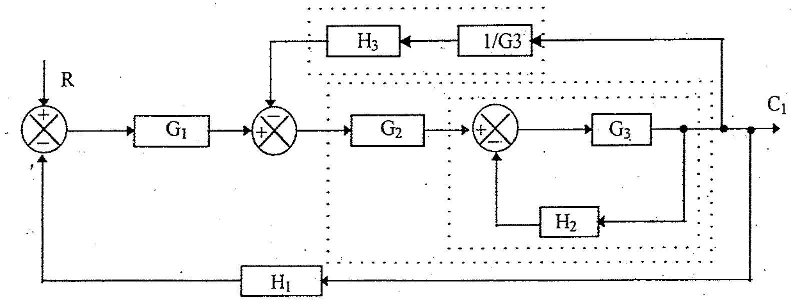
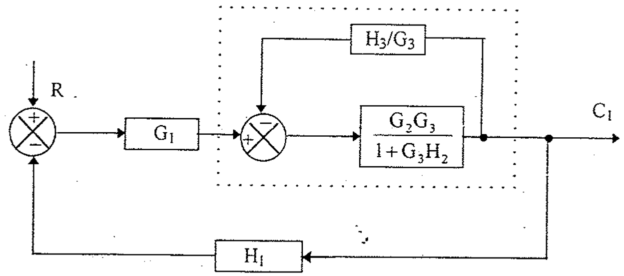


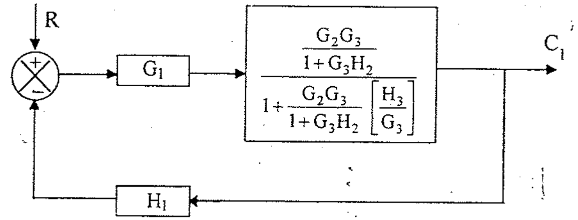
Fig 1.19.2

Step 2 : Eliminating the feedback H_2 and combining blocks in cascade

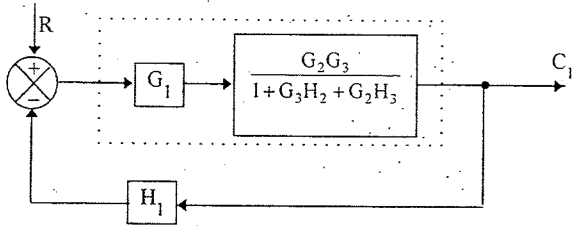


Step 3: Eliminating the feedback path

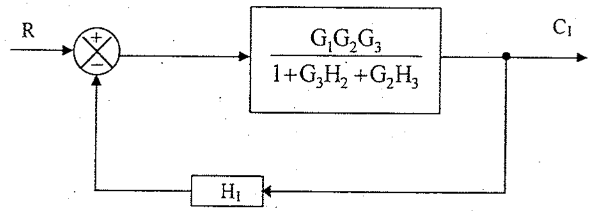


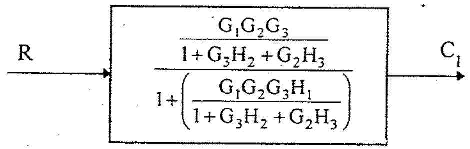


Step 4: Combining the blocks in cascade



Step 5: Eliminating feedback path H_1





$$\frac{C_1(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

(ii) Consider the input R at station II, the input at station I is made zero. Let output be C_2. Since there is no input in station I that corresponding summing point can be removed and a negative sign can be attached to the feedback path gain H_1. The resulting block diagram is shown in fig 1.19.3.

Step 1: Combining the blocks in cascade, shifting the summing point of H_2 before G_2 and rearranging the branch points.

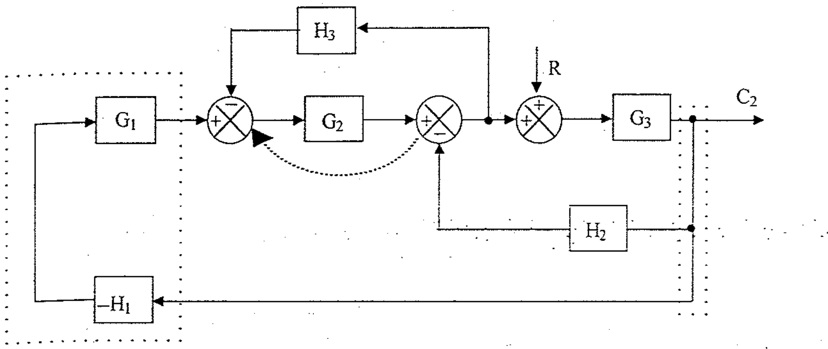
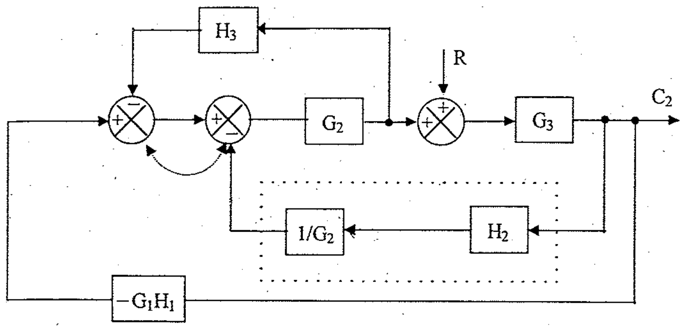
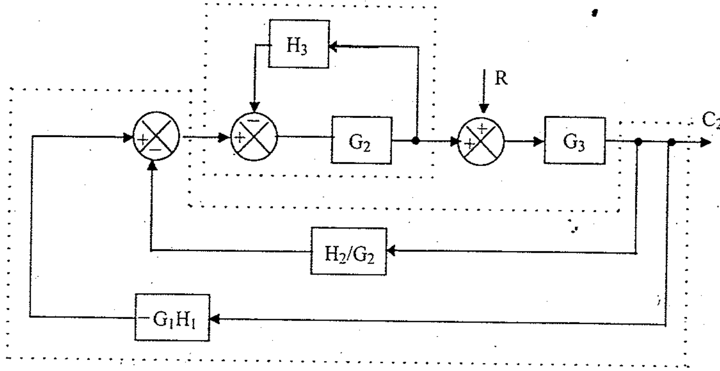


Fig 1.19.3.

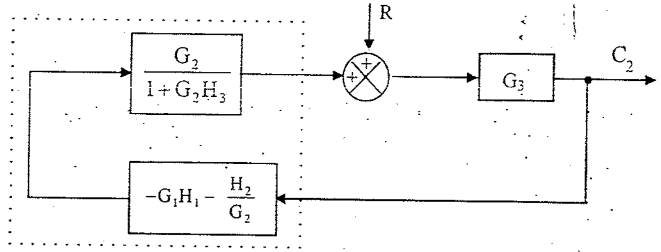
Step 2: Interchanging summing points and combining the blocks in cascade.

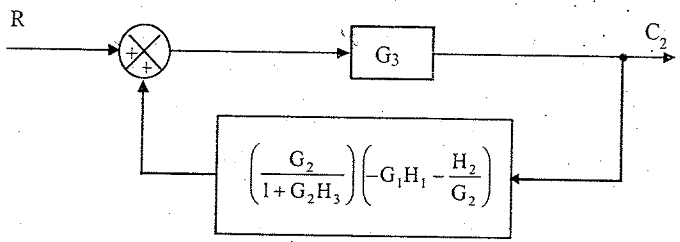


Step 3: Combining parallel blocks and eliminating feedback path

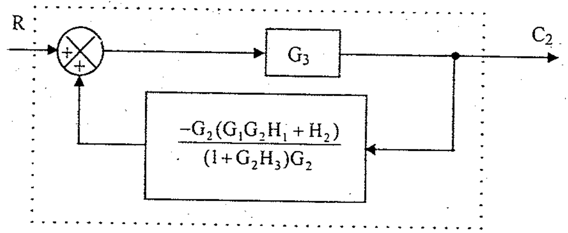


Step 4: Combining the blocks in cascade





Step 5: Eliminating the feedback path



$$\frac{C_2}{R}=\frac{G_3}{1-\left(\frac{-(G_1G_2H_1+H_2)}{1+G_2H_3}\right)G_3}=\frac{G_3}{\frac{1+G_2H_3+G_3(G_1G_2H_1+H_2)}{1+G_2H_3}}$$

$$\frac{C_2}{R}=\frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

RESULT

The transfer function of the system with input at station I is

$$\frac{C_1}{R}=\frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

The transfer function of the system with input at station II is

$$\frac{C_2}{R}=\frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

EXAMPLE 1.20

For the system represented by the block diagram shown in the fig 1.20.1, determine C_1/R_1 and C_2/R_1.

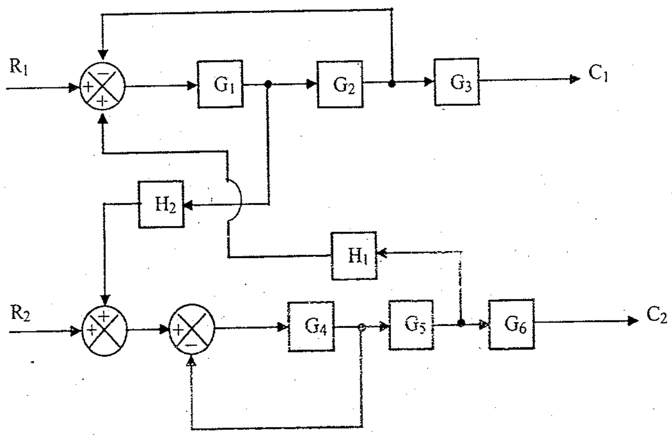


Fig 1.20.1

SOLUTION

Case (i) To find C_1/R_1

In this case set $R_2 = 0$ and consider only one output C_1. Hence we can remove the summing point which adds R_2 and need not consider G_6, since G_6 is on the open path. The resulting block diagram is shown in fig 1.20.2.

Step 1: Eliminating the feedback path

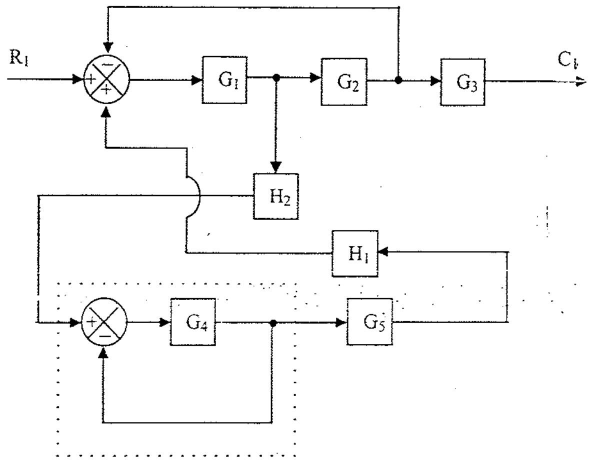
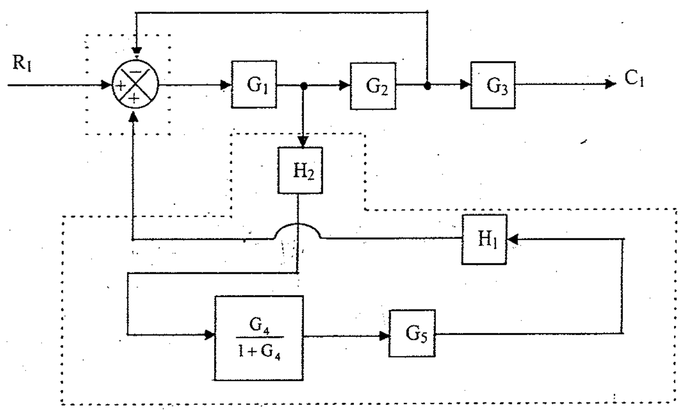
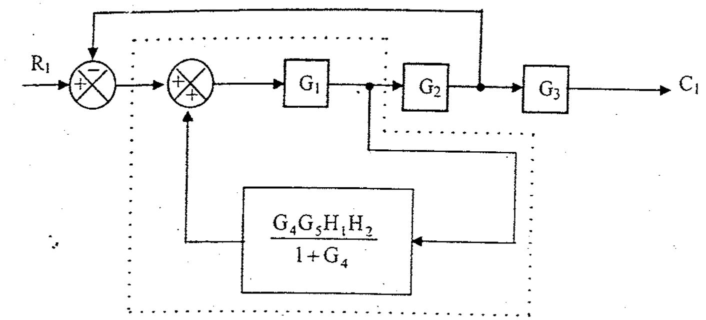


Fig 1.20.2

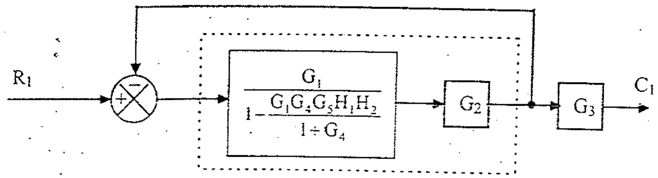
Step 2: Combining the blocks in cascade and splitting the summing point



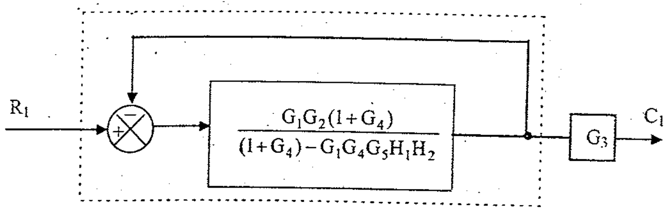
Step 3: Eliminating the feedback path

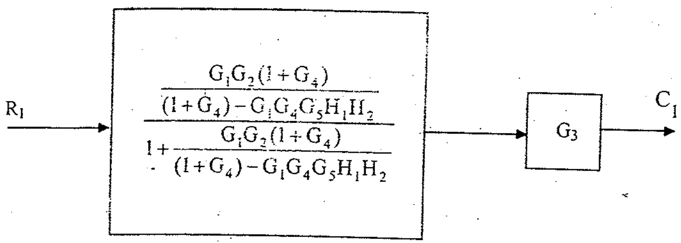


Step 4: Combining the blocks in cascade

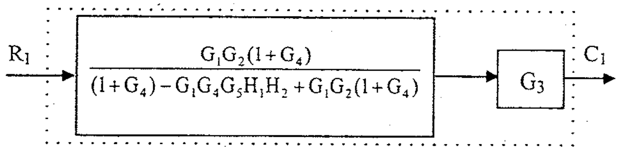


Step 5 : Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_1}{R_1}=\frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)\,(1+G_4)-G_1G_4G_5H_1H_2}$$

Case 2 : To find C_2/R_1

In this case set $R_2 = 0$ and consider only one output C_2. Hence we can remove the summing point which adds R_2 and need not consider G_3, since G_3 is on the open path. The resulting block diagram is shown in fig 1.20.3.

Step 1: Eliminate the feedback path.

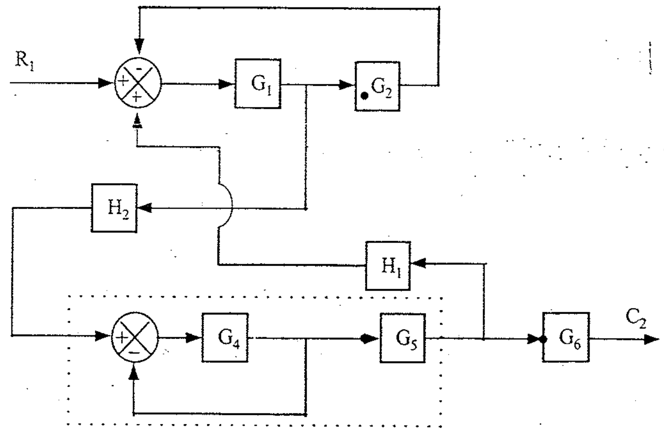
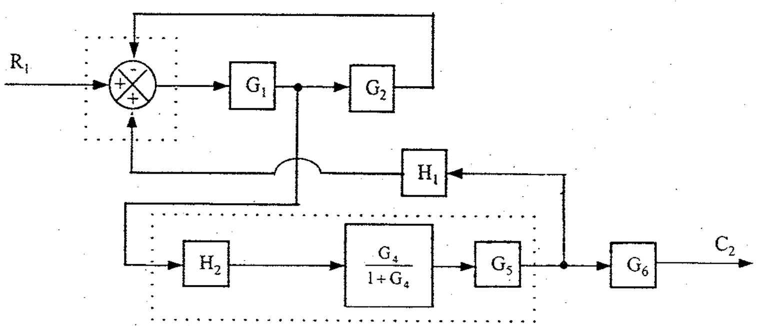
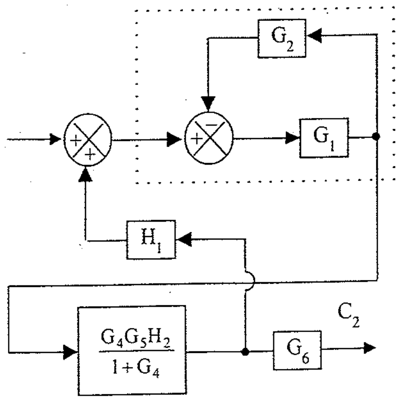


Fig 1.20.3.

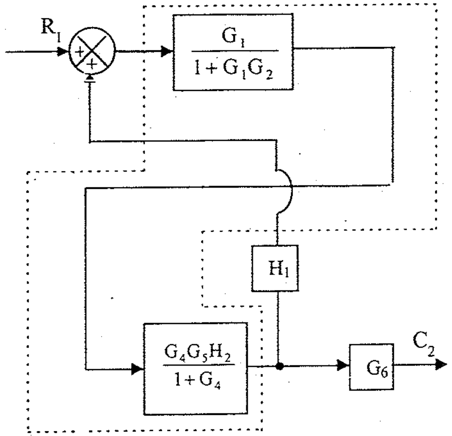
Step 2: Combining blocks in cascade and splitting the summing point



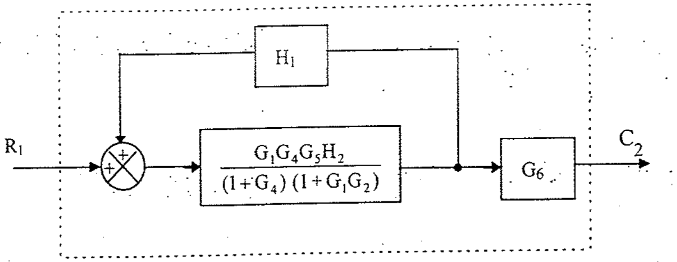
Step 3: Eliminating the feedback path

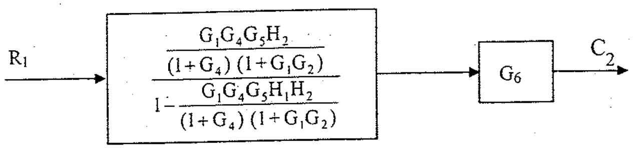


Step 4: Combining the blocks in cascade

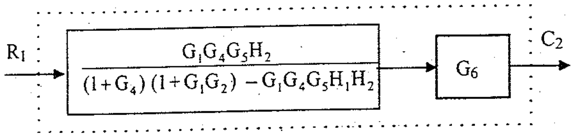


Step 5: Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_2}{R_1}=\frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2)-G_1G_4G_5H_1H_2}$$

RESULT

The transfer function of the system when the input and output are R_1 and C_1 is given by $\frac{C_1}{R_1}=\frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)(1+G_4)-G_1G_4G_5H_1H_2}$

The transfer function of the system when the input and output are R_1 and C_2 is given by $\frac{C_2}{R_1}=\frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2)-G_1G_4G_5H_1H_2}$

EXAMPLE 1.21

Obtain the closed loop transfer function C(s)/R(s) of the system whose block diagram is shown in fig 1.21.1.

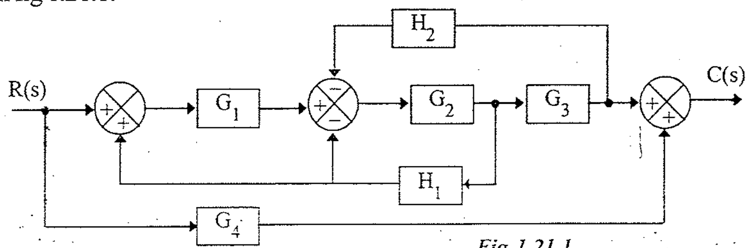
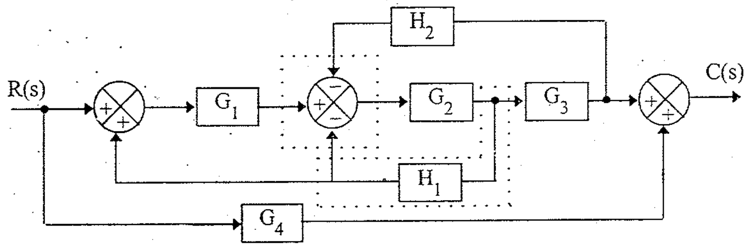


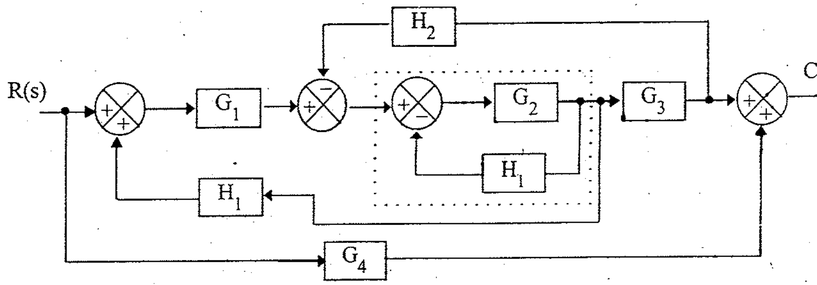
Fig 1.21.1.

SOLUTION

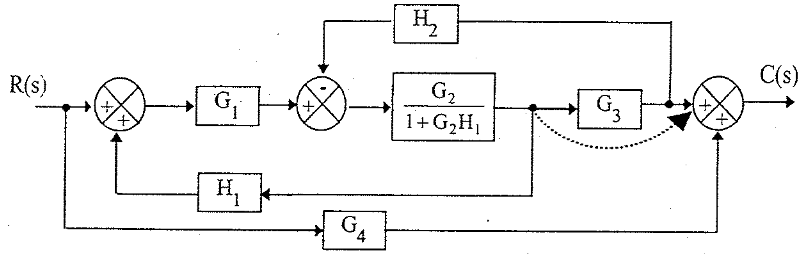
Step 1 : Splitting the summing point and rearranging the branch points



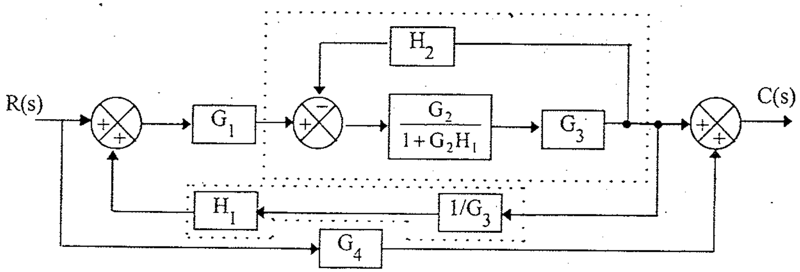
Step 2 : Eliminating the feedback path



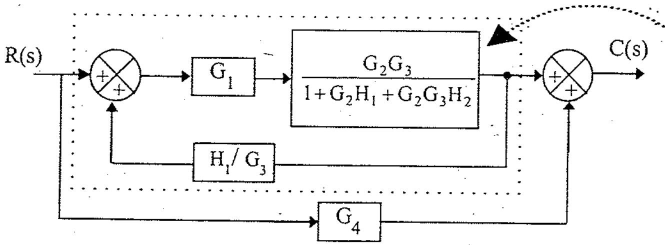
Step 3 : Shifting the branch point after the block.



Step 4 : Combining the blocks in cascade and eliminating feedback path

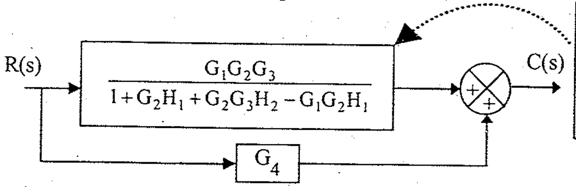


Step 5 : Combining the blocks in cascade and eliminating feedback path



$$\frac{\frac{G_2G_3}{1+G_2H_1}}{1+\frac{G_2G_3H_2}{1+G_2H_1}}$$

Step 6 : Eliminating forward path



$$\frac{\frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2}}{1-\frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2}\left(\frac{H_1}{G_3}\right)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}+G_4$$

RESULT

The transfer function of the system is $\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1}+G_4$

EXAMPLE 1.22

The block diagram of a closed loop system is shown in fig 1.22.1 using the block diagram reduction technique determine the closed loop transfer function C(s)/R(s).

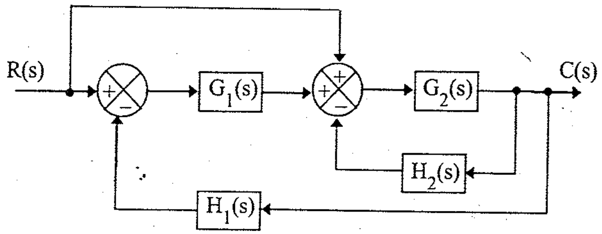
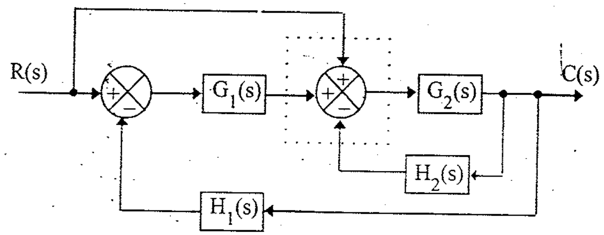


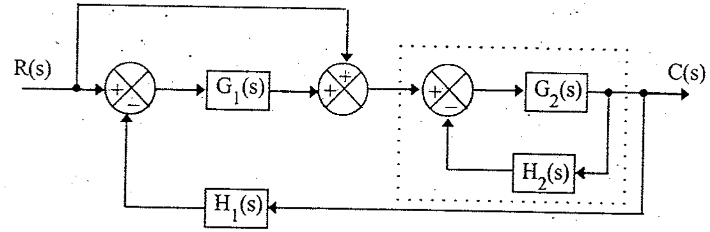
Fig 1.22.1

SOLUTION

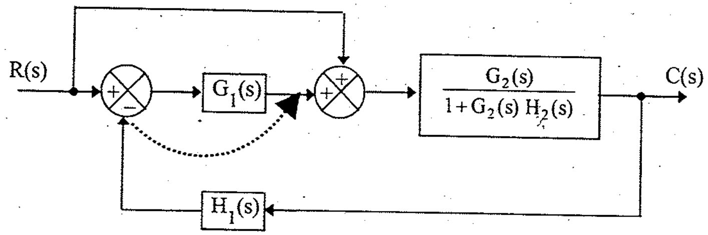
Step 1 : Splitting the summing point.



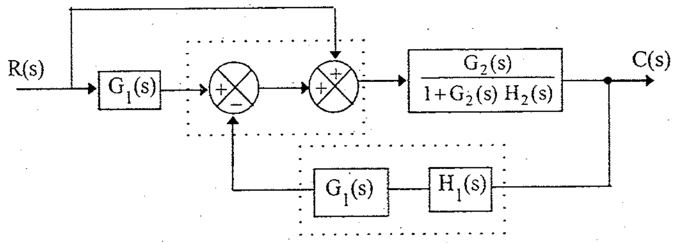
Step 2 : Eliminating the feedback path.



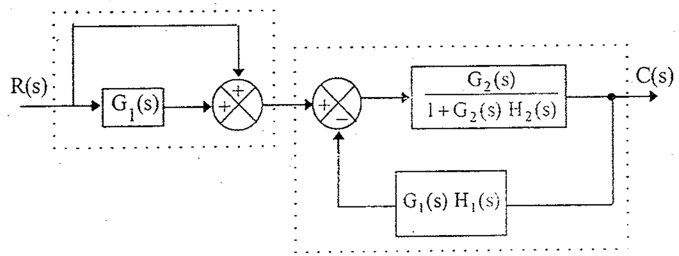
Step 3 : Moving the summing point after the block.



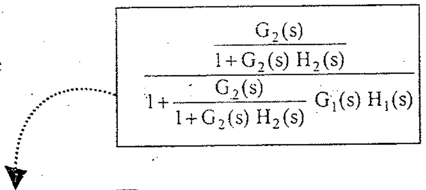
Step 4 : Interchanging the summing points and combining the blocks in cascade

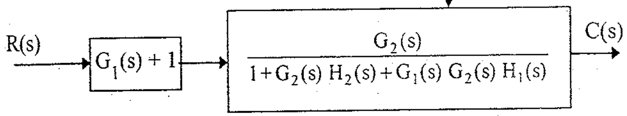


Step 5 : Eliminating the feedback path and feed forward path



Step 6 : Combining the blocks in cascade





$$\therefore \frac{C(s)}{R(s)} = \frac{G_2(s)\,[G_1(s)+1]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

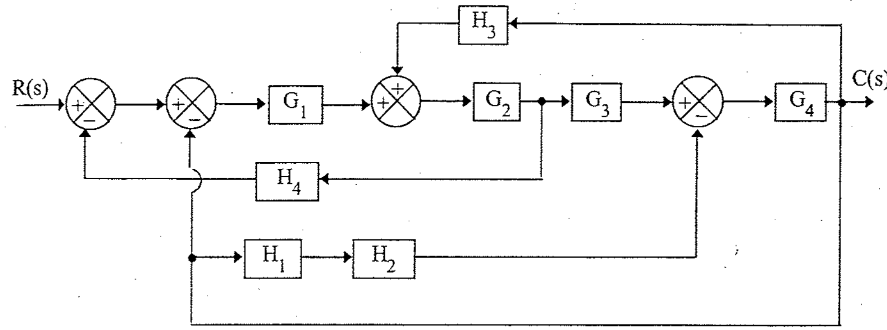
RESULT

The transfer function of the system is

$$\therefore \frac{C(s)}{R(s)} = \frac{G_2(s)\,[G_1(s)+1]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

EXAMPLE 1.23

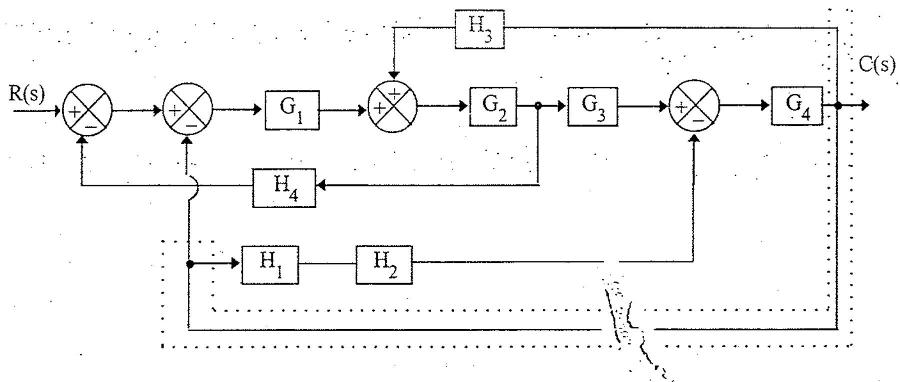
Using block diagram reduction technique find the transfer function C(s)/R(s) for the system shown in fig 1.23.1.



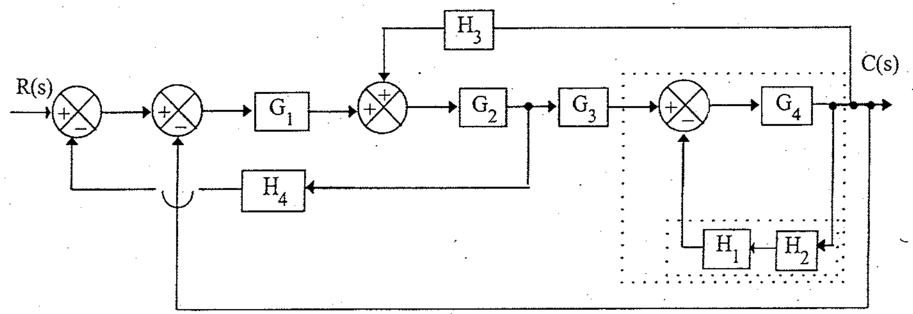
SOLUTION

Fig 1.23.1

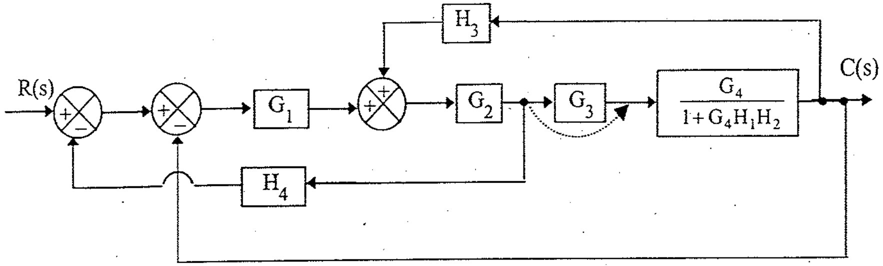
Step 1 : Rearranging the branch points



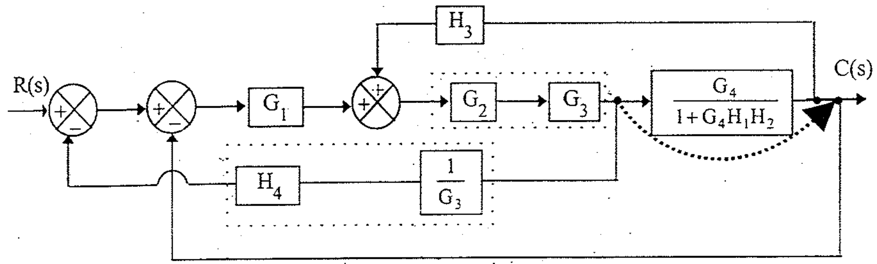
Step 2 : Combining the blocks in cascade and eliminati... ...edback path.



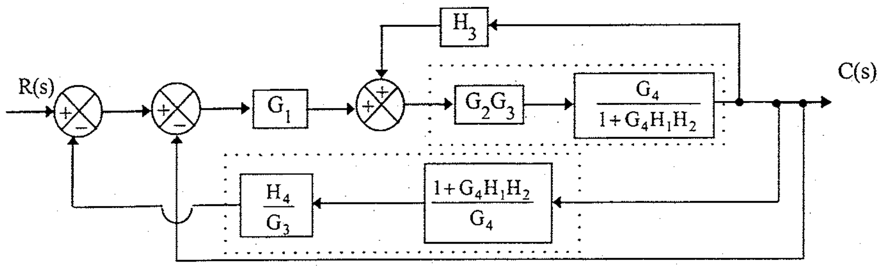
Step 3 : Moving the branch point after the block.



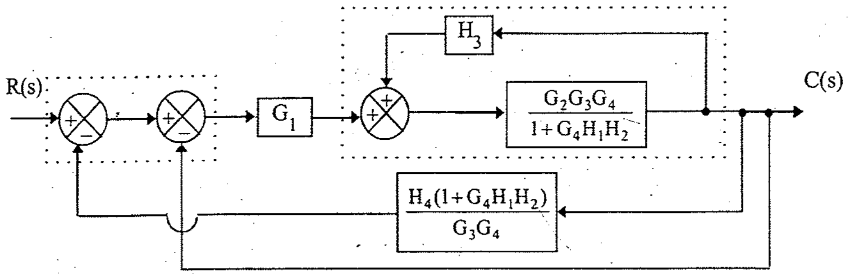
Step 4 : Moving the branch point and combining the blocks in cascade.



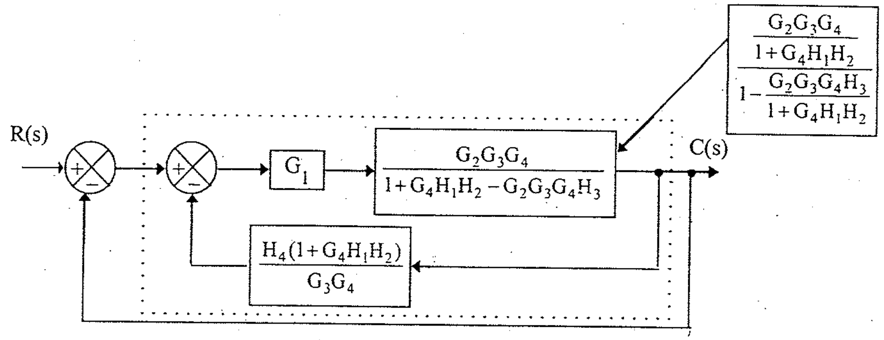
Step 5 : Combining the blocks in cascade



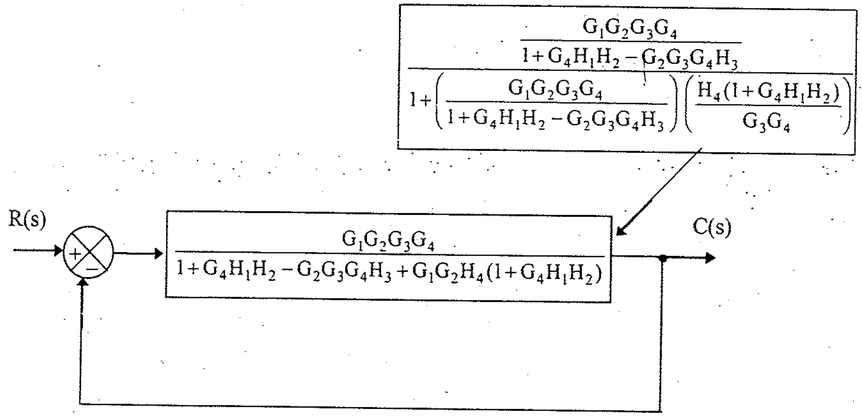
Step 6 : Eliminating feedback path and interchanging the summing points.



Step 7 : Combining the blocks in cascade and eliminating the feedback path



Step 8 : Eliminating the unity feedback path.



$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$= \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)+G_1G_2G_3G_4}$$

$$= \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

RESULT

The transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

1.12 SIGNAL FLOW GRAPH

The signal flow graph is used to represent the control system graphically and it was developed by **S.J. Mason.**

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

It should be noted that the signal flow graph approach and the block diagram approach yield the same information. The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily. This method is simpler than the tedious block diagram reduction techniques.

The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationships among the signals. A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier. Each branch has a gain or transmittance. When the signal pass through a branch, it gets multiplied by the gain of the branch. Note that the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and the multiplication factor is indicated along the branch.

EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH

Node : A node is a point representing a variable or signal.

Branch : A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.

Transmittance : The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.

Input node (or) Source : It is a node that has only outgoing branches.

Output node (or) Sink : It is a node that has only incoming branches.

Mixed node : It is a node that has both incoming and outgoing branches.

Path : A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.

Open path : A open path starts at a node and ends at another node.

Closed path : Closed path starts and ends at same node.

Forward path : It is a path from an input node to an output node that does not cross any node more than once.

***Forward path* gain** : It is the product of the branch transmittances (gains) of a forward path.

Individual loop: It is a closed path starting from a node and after passing through a certain part of a graph arrives at the same node without crossing any node more than once.

Loop gain : It is the product of the branch transmittances (gains) of a loop.

Non-touching Loops : If the loops does not have a common node then they are said to be non-touching loops.

PROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following :

(i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.

(ii) Signal flow graph is applicable to linear systems only.

(iii) A node in the signal flow graph represents the variable or signal.

(iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

(v) A mixed node which has both incoming and outgoing signals can be treated as output node by adding an outgoing branch of unity transmittance.

(vi) A branch indicates functional dependence of one signal on the other.

(vii) The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

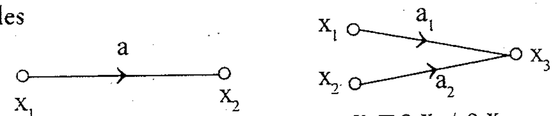
(viii) The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

SIGNAL FLOW GRAPH ALGEBRA

Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at a node is given by sum of all incoming signals.

Rule 1 : Incoming signal to a node through a branch is given by the product of a signal at previou s node and the gain of the branch.

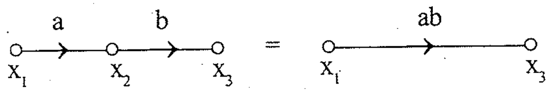
Examples



$x_2 = ax_1$ $\qquad$ $x_3 = a_1x_1 + a_2x_2$

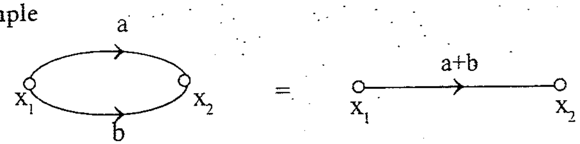
Rule 2 : Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example



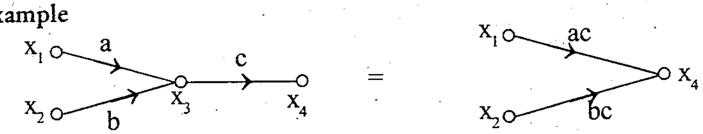
Rule 3 : Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Example



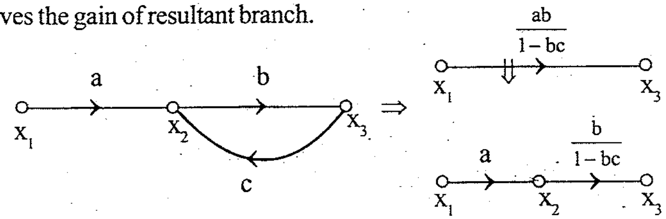
Rule 4 : A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Example



Rule 5 : A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example



Proof :

$x_2 = ax_1 + cx_3$

$x_3 = bx_2$

Put $x_2 = ax_1 + cx_3$ in the equation for x_3

$\therefore x_3 = b\,(ax_1 + cx_3)$

$x_3 = abx_1 + bcx_3$

$x_3 - bc\,x_3 = ab\,x_1$

$x_3(1 - bc) = ab\,x_1$

$$\frac{x_3}{x_1} = \frac{ab}{1-bc}$$

SIGNAL FLOW GRAPH REDUCTION

The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra (i.e.) by writing equations at every node and then rearranging these equations to get the ratio of output and input (transfer function).

The signal flow graph reduction by above method will be time consuming and tedious. **S.J.Mason** has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph. He has developed a formula called by his name **Mason's gain formula** which can be directly used to find the transfer function of the system.

MASON'S GAIN FORMULA

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let $\quad R(s) = \text{Input to the system}$

and $\quad C(s) = \text{Output of the system}$

Transfer function of the system, $T(s) = \dfrac{C(s)}{R(s)}$ .....(1.34)

Mason's gain formula states the overall gain of the system [transfer function] as follows,

Overall gain, $T = \dfrac{1}{\Delta}\sum_{K} P_K \Delta_K$ .....(1.35)

Where, $\quad T \;=\; T(s) = $ Transfer function of the system.

$P_K \;=\;$ Forward path gain of K^{th} forward path

$$\Delta = 1 - (\text{Sum of individual loop gains})$$
$$+ \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of two non-touching loops}\end{pmatrix}$$
$$- \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of three non-touching loops}\end{pmatrix}$$
$$+ \text{.............................}$$

$\Delta_K = \Delta$ for that part of the graph which is not touching K^{th} forward path

CONSTRUCTING SIGNAL FLOW GRAPH FOR CONTROL SYSTEMS

A control system can be represented diagrammatically by signal flow graph. The differential equations governing the system are used to construct the signal flow graph. By taking Laplace transform the differential equations can be converted to algebraic equations. The constants and variables of the equations are identified. From the working knowledge of the system, the variables are identified as input, output and intermediate variables. For each variable a node is assigned in signal flow graph and constants are the gain or transmittance of the branches connecting the nodes. For each equation a signal flow graph is drawn and then they are interconnected to give overall signal flow graph of the system.

PROCEDURE FOR CONVERTING BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

The signal flow graph and block diagram of a system provides the same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reduction technique will be tedious and it is difficult to choose the rule to be applied for simplification. Hence it will be easier if the block diagram is converted to signal flow graph and **Mason's gain formula** is applied to find the transfer function. The following procedure can be used to convert block diagram to signal flow graph.

1. Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2. Draw the nodes seperately as small circles and number the circles in the order 1, 2, 3, 4, ... etc.
3. From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
5. Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

EXAMPLE 1.24

Construct a signal flow graph for armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7)

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad .....(1.24.1)$$

$$T = K_t i_a \quad .....(1.24.2)$$

$$T = J\frac{d\omega}{dt} + B\omega \quad .....(1.24.3)$$

$$e_b = K_b\,\omega \quad .....(1.24.4)$$

$$\omega = d\theta / dt \quad .....(1.24.5)$$

On taking Laplace transform of equations (1.24.1) to (1.24.5) we get,

$$V_a(s) = I_a(s)\,R_a + L_a\,s\,I_a(s) + E_b(s) \quad .....(1.24.6)$$

$$T(s) = K_t\,I_a(s) \quad .....(1.24.7)$$

$$T(s) = J\,s\,\omega(s) + B\,\omega(s) \quad .....(1.24.8)$$

$$E_b(s) = K_b\,\omega(s) \quad .....(1.24.9)$$

$$\omega(s) = s\,\theta(s) \quad .....(1.24.10)$$

The input and output variables of armature controlled dc motor are armature voltage $V_a(s)$ and angular displacement $\theta(s)$ respectively. The variables $I_a(s)$, $T(s)$, $E_b(s)$ and $\omega(s)$ are intermediate variables.

The equations (1.24.6) to (1.24.10) are rearranged and the individual signal flow graph are shown in fig 1.24.1 to fig 1.24.5.

$$V_a(s) - E_b(s) = I_a(s)\left[R_a + s\,L_a\right]$$

$$\therefore I_a(s) = \frac{1}{R_a + s\,L_a}\left[V_a(s) - E_b(s)\right]$$

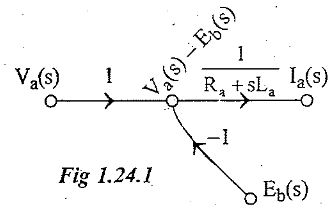


Fig 1.24.1

$$T(s) = K_t\,I_a(s)$$

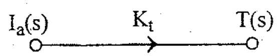


Fig 1.24.2

$$T(s) = \omega(s)\left[Js + B\right]$$

$$\therefore \omega(s) = \frac{1}{Js + B}\,T(s)$$

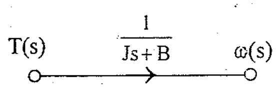


Fig 1.24.3

$E_b(s) = K_b\,\omega(s)$

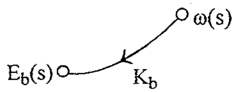


Fig 1.24.4

$\omega(s) = s\theta(s)$

$\therefore \theta(s) = \frac{1}{s}\omega(s)$

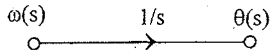


Fig 1.24.5

The overall signal flow graph of armature controlled dc motor is obtained by interconnecting the individual signal flow graphs shown in fig 1.24.1. to fig 1.24.5. The overall signal flow graph is shown in fig 1.24.6.

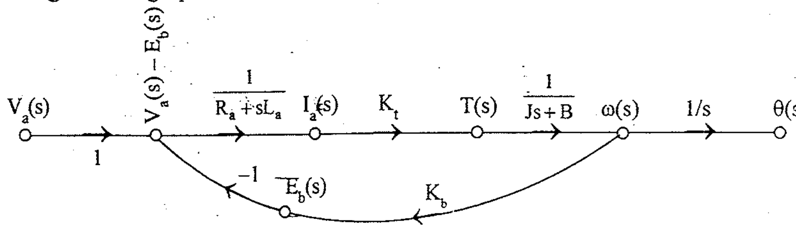


Fig 1.24.6 : Signal flow graph of armature controlled dc motor

EXAMPLE 1.25

Find the overall transfer function of the system whose signal flow graph is shown in fig 1.25.1.

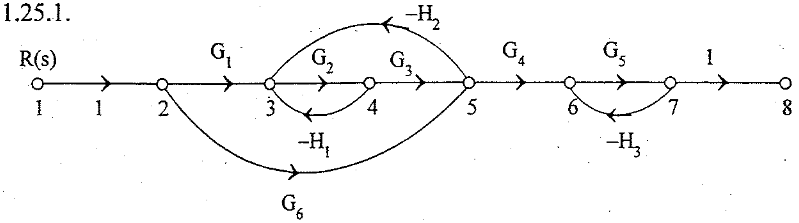


Fig 1.25.1

SOLUTION

I FORWARD PATH GAINS

There are two forward paths. $\therefore K = 2$

Let forward path gains be P_1 and P_2

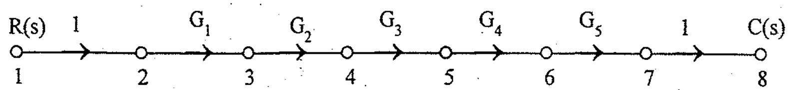


Fig 1.25.2: Forward path - 1

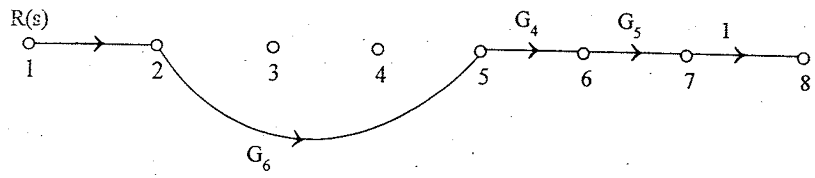


Fig 1.25.3: Forward path - 2

Gain of forward path-1, $P_1 = G_1\ G_2\,G_3\,G_4\,G_5$

Gain of forward path-2, $P_2 = G_4G_5G_6$

II INDIVIDUAL LOOP GAIN

There are three individual loops. Let individual loop gains be P_{11}, P_{21} and P_{31}.

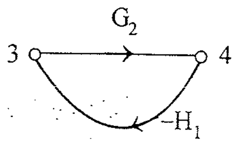


Fig 1.25.4 : Loop-1

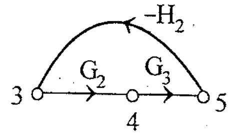


Fig 1.25.5 : Loop-2

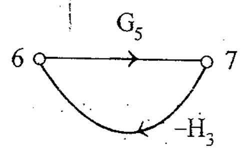


Fig 1.25.6 : Loop-3

Loop gain of individual loop-1, $P_{11} = -\,G_2\,H_1$

Loop gain of individual loop-2, $P_{21} = -\,G_2\,G_3\,H_2$

Loop gain of individual loop-3, $P_{31} = -\,G_5\,H_3$.

III GAIN PRODUCTS OF TWO NON TOUCHING LOOPS

There are two combinations of two non-touching loops. Let the gain products of two non touching loops be P_{12} and P_{22}.

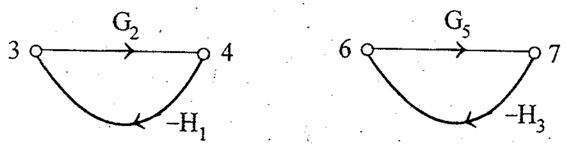


Fig 1.25.7 : First combination of two non touching loops

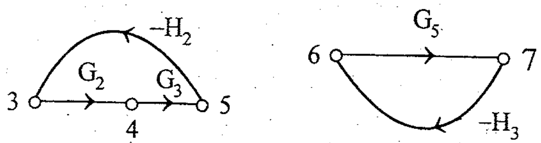


Fig 1.25.8 : Second combination of two non touching loops

Gain product of first combination of two non touching loops } $P_{12} = P_{11}P_{31} = (-G_2H_1)\,(-G_5H_3)$

$= G_2G_5H_1H_3$

Gain product of second combination of two non touching loops } $P_{22} = P_{21}P_{31} = (-G_2G_3H_2)\,(-G_5H_3)$

$= G_2G_3G_5H_2H_3$

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$$

$$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$$

Δ_1 = 1, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non touching with second forward path is shown in fig 1.25.9.

$$\Delta_2 = 1 - P_{11} = 1 - (-G_2H_1)$$

$$= 1 + G_2H_1$$

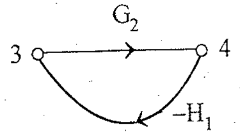


Fig 1.25.9

V TRANSFER FUNCTION, T

By Mason's gain formula the transfer function } $T = \frac{1}{\Delta}\sum_K P_K\Delta_K$

$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$ (Here K = 2, since we have only two forward paths)

$$\therefore T = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6\,(1 + G_2H_1)}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_2G_4G_5G_6H_1}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_2G_4G_5\left[G_1G_3 + G_6/G_2 + G_6H_1\right]}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

EXAMPLE 1.26

Find the overall gain of the system whose signal flow graph is shown in fig 1.26.1.

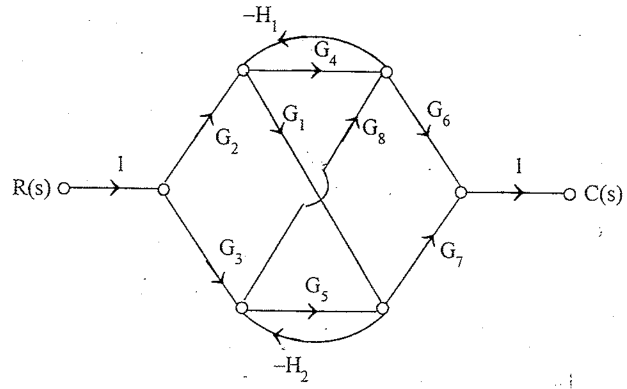


Fig 1.26.1

SOLUTION

Let us number the nodes as shown in fig 1.26.2.

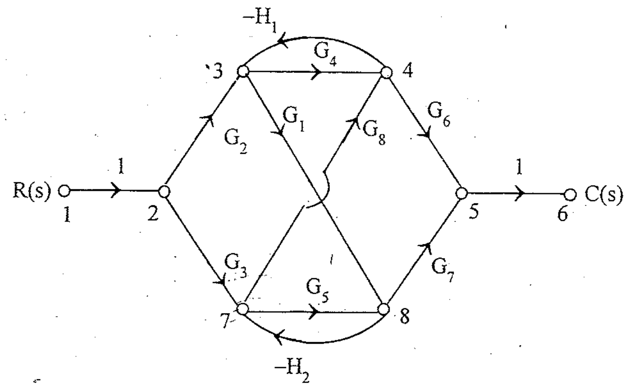


Fig 1.26.2

I FORWARD PATH GAINS

There are six forward paths. $\therefore K = 6$

Let the forward path gains be P_1, P_2, P_3, P_4, P_5 and P_6.

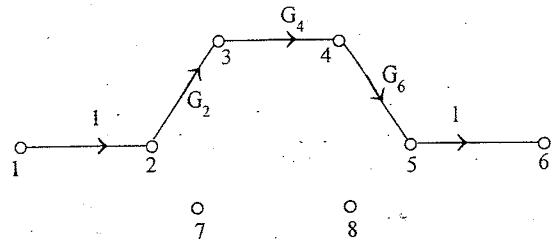


Fig 1.26.3 : Forward path - 1

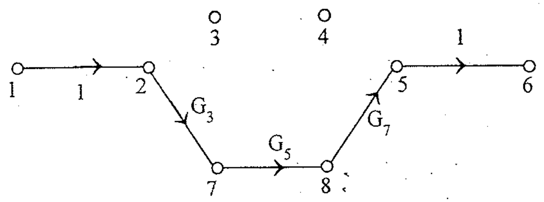


Fig 1.26.4 : Forward path - 2

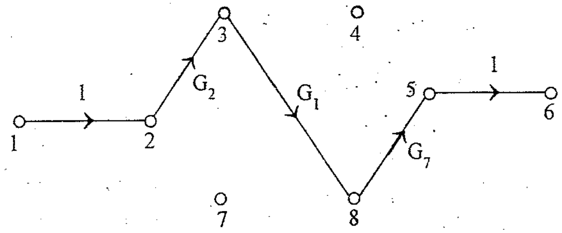


Fig 1.26.5 : Forward path - 3

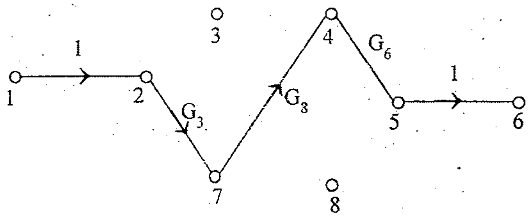


Fig 1.26.6 : Forward path - 4

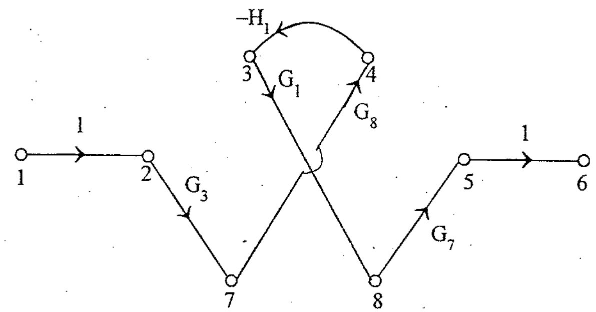


Fig 1.26.7 : Forward path – 5

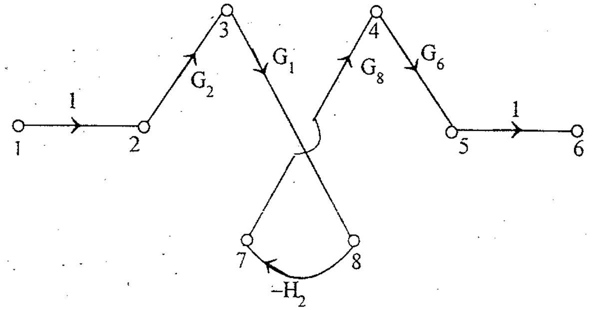


Fig 1.26.8 : Forward path – 6

Gain of forward path-1, $P_1 = G_2\,G_4\,G_6$

Gain of forward path-2, $P_2 = G_3\,G_5\,G_7$

Gain of forward path-3, $P_3 = G_1\,G_2\,G_7$

Gain of forward path-4, $P_4 = G_3\,G_8\,G_6$

Gain of forward path-5, $P_5 = -G_1G_3G_7G_8H_1$

Gain of forward path-6, $P_6 = -G_1G_2G_6G_8H_2$

II INDIVIDUAL LOOP GAIN

There are three individual loops.

Let individual loop gains be P_{11}, P_{21} and P_{31}.

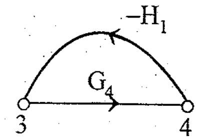


Fig 1.26.9 : Loop–1

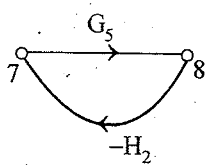


Fig 1.26.10 : Loop–2

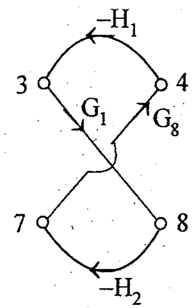


Fig 1.26.11 : Loop–3

Loop gain of individual loop-1, $P_{11} = -G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_5H_2$

Loop gain of individual loop-3, $P_{31} = G_1G_8H_1H_2$

III GAIN PRODUCTS OF TWO NON-TOUCHING LOOPS.

There is only one combination of two non-touching loops. Let the gain product of two non-touching loops be P_{12}.

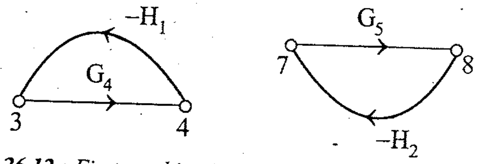


Fig 1.26.12 : First combination of two non-touching loops

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non-touching loops}\end{array}\right\} P_{12} = P_{11}P_{21} = (-G_4H_1)(-G_5H_2)$$

$$= G_4G_5H_1H_2$$

IV CALCULATION OF Δ AND Δ_K

$$\begin{aligned}\Delta &= 1-(P_{11}+P_{21}+P_{31})+P_{12}\\ &= 1-(-G_4H_1-G_5H_2+G_1G_8H_1H_2)+G_4G_5H_1H_2\\ &= 1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2\end{aligned}$$

The part of the graph non-touching forward path – 1 is shown in fig 1.26.13.

$\therefore \Delta_1 = 1-(-G_5H_2) = 1+G_5H_2.$

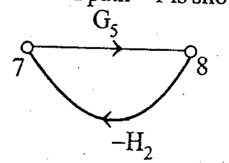


Fig 1.26.13

The part of the graph non-touching forward path –2 is shown in fig 1.26.14.

$\therefore \Delta_2 = 1-(-G_4H_1) = 1+G_4H_1.$

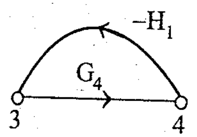


Fig 1.26.14

There is no part of the graph which is non-touching with forward paths 3, 4, 5 and 6.

$\therefore \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1.$

V TRANSFER FUNCTION, T

By Mason's gain formula, $T = \frac{1}{\Delta}\left(\sum_K P_K \Delta_K\right)$

$$= \frac{1}{\Delta}(P_1\Delta_1+P_2\Delta_2+P_3\Delta_3+P_4\Delta_4+P_5\Delta_5+P_6\Delta_6)\left(\begin{array}{l}\text{Here } K=6, \text{ since we}\\ \text{have six forward paths}\end{array}\right)$$

$$= \frac{\begin{array}{c}G_2G_4G_6(1+G_5H_2)+G_3G_5G_7(1+G_4H_1)+G_1G_2G_7+G_3G_6G_8\\ -G_1G_3G_7G_8H_1-G_1G_2G_6G_8H_2\end{array}}{1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2}$$

EXAMPLE 1.27

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.27.1.

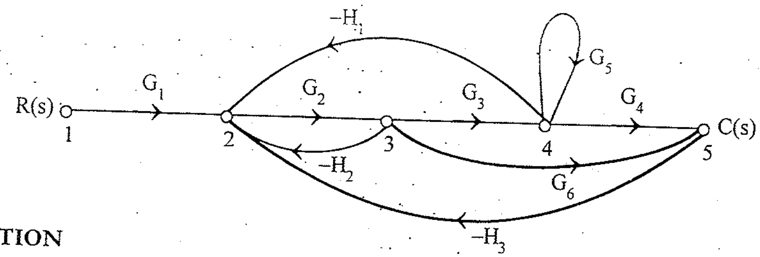


Fig 1.27.1

SOLUTION

I FORWARD PATH GAINS

There are two forward paths. $\therefore K = 2$. Let the forward path gains be P_1 and P_2.

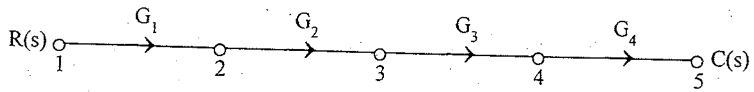


Fig 1.27.2 : Forward path – 1

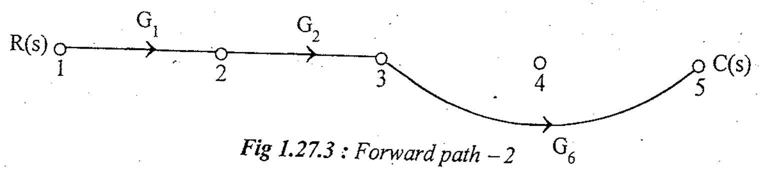


Fig 1.27.3 : Forward path – 2

Gain of forward path-1, $P_1 = G_1G_2G_3G_4.$

Gain of forward path-2, $P_2 = G_1G_2G_6.$

II INDIVIDUAL LOOP GAIN

There are five individual loops. Let the individual loop gains be p_{11}, p_{21}, p_{31}, p_{41} and p_{51}.

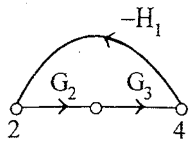


Fig 1.27.4 : loop – 1

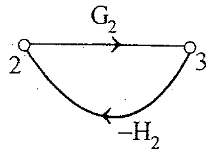


Fig 1.27.5 : loop – 2

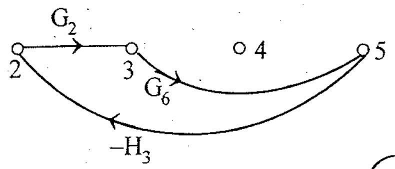
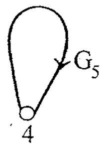


Fig 1.27.6 : loop – 3



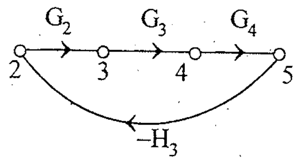


Fig 1.27.7 : loop – 4

Fig 1.27.8 : loop – 5

Loop gain of individual loop-1, $P_{11} = -G_2G_3H_1$

Loop gain of individual loop-2, $P_{21} = -H_2G_2$

Loop gain of individual loop-3, $P_{31} = -G_2G_6H_3$

Loop gain of individual loop-4, $P_{41} = -G_2G_3G_4H_3$

Loop gain of individual loop-5, $P_{51} = G_5$.

III GAIN PRODUCTS OF TWO NON TOUCHING LOOPS

There are two combinations of two non touching loops.

Let the gain products of two non touching loops be P_{12} and P_{22}.

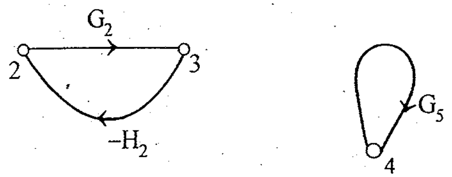


Fig 1.27.9 : First combination of two non touching loops

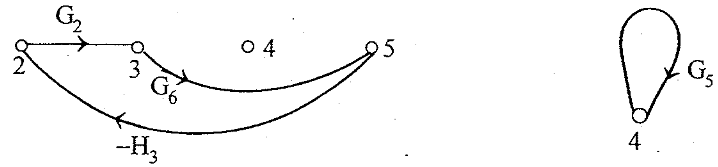


Fig 1.27.10 : Second combination of two non touching loops

Gain product of first combination of two non touching loops $\Big\}$ $P_{12} = P_{21}P_{51} = (-G_2H_2)(G_5)$

$$= -G_2G_5H_2$$

Gain product of second combination of two non touching loops. $\Big\}$ $P_{22} = P_{31}P_{51} = (-G_2G_6H_3)(G_5)$

$$= -G_2G_5G_6H_3$$

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1-(P_{11}+P_{21}+P_{31}+P_{41}+P_{51})+(P_{12}+P_{22})$$
$$= 1-(-G_2G_3H_1 - H_2G_2 - G_2G_3G_4H_3 + G_5 - G_2G_6H_3)$$
$$+(-G_2H_2G_5 - G_2G_5G_6H_3)$$

Since there is no part of graph which is not touching forward path – 1, $\Delta_1 = 1$.

The part of graph which is not touching forward path – 2 is shown in fig 1.27.11.

$$\therefore \Delta_2 = 1 - G_5$$

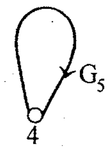


Fig 1.27.11

V TRANSFER FUNCTION, T

By Mason's gain formula

The transfer function, $T = \frac{1}{\Delta}\sum_K P_K\Delta_K$

$$= \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{1}{\Delta}[G_1G_2G_3G_4(1) + G_1G_2G_6(1-G_5)]$$

$$= \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1+G_2G_3H_1+H_2G_2+G_2G_3G_4H_3-G_5-G_2H_2G_5+G_2G_6H_3-G_2G_5G_6H_3}$$

EXAMPLE 1.28

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.28.1.

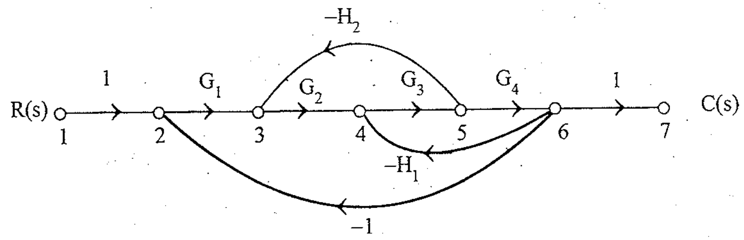


Fig 1.28.1

SOLUTION

I FORWARD PATH GAINS

There is only one forward path. $\therefore K = 1$.

Let the forward path gain be P_1.

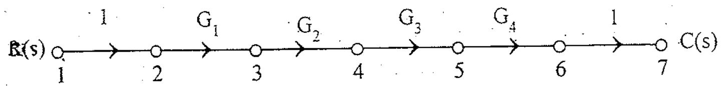


Fig 1.28.2: Forward path – 1

Gain of forward path – 1, $P_1 = G_1G_2G_3G_4$.

II INDIVIDUAL LOOP GAIN

There are three individual loops. Let the loop gains be P_{11}, P_{21}, P_{31}.

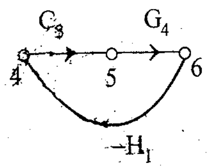


Fig 1.28.3 : loop – 1

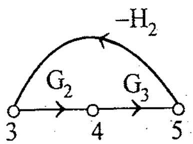


Fig 1.28.4 : loop – 2

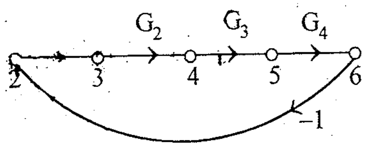


Fig 1.28.5 : loop – 3

Loop gain of individual loop-1, $P_{11} = -G_3G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Loop gain of individual loop-3, $P_{31} = -G_1G_2G_3G_4$

III GAIN PRODUCTS OF NON TOUCHING LOOPS

There are no possible combinations of two non touching loops, three non touching loops, etc.

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1-(P_{11}+P_{21}+P_{31})$$
$$= 1-(-G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4)$$
$$= 1+G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4$$

Since no part of the graph is non touching with forward path –1, $\Delta_1 = 1$.

V TRANSFER FUNCTION, T

By Mason's gain formula,

$$\frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}P_1\Delta_1$$
$$= \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

EXAMPLE 1.29

The signal flow graph for a feedback control system is shown in fig 1.29.1. Determine the closed loop transfer function C(s)/R(s).

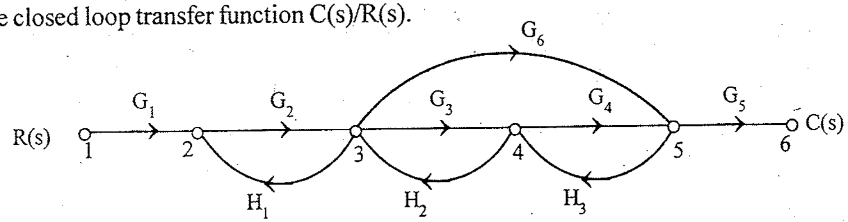


Fig 1.29.1

SOLUTION

I FORWARD PATH GAINS

There are two forward paths, $\therefore K = 2$.

Let forward path gains be P_1 and P_2.

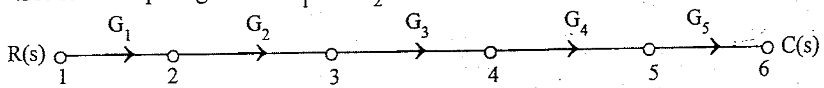


Fig 1.29.2 : Forward path – 1

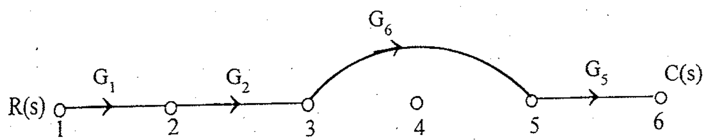


Fig 1.29.3: Forward path – 2

Gain of forward path-1, $P_1 = G_1 G_2 G_3 G_4 G_5$

Gain of forward path-2, $P_2 = G_1 G_2 G_6 G_5$

II INDIVIDUAL LOOP GAIN

There are four individual loops. Let individual loop gains be P_{11}, P_{21}, P_{31} and P_{41}.

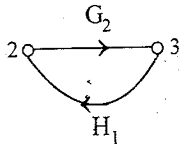


Fig 1.29.4 : loop – 1

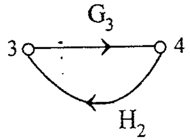


Fig 1.29.5 : loop – 2

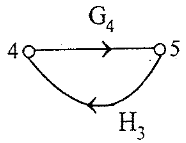


Fig 1.29.6 : loop – 3

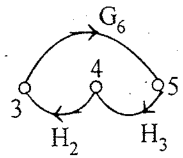


Fig 1.29.7 : loop – 4

Loop gain of individual loop-1, $P_{11} = G_2H_1$

Loop gain of individual loop-2, $P_{21} = G_3H_2$

Loop gain of individual loop-3, $P_{31} = G_4H_3$

Loop gain of individual loop-4, $P_{41} = G_6H_2H_3$

III GAIN PRODUCTS OF TWO NON TOUCHING LOOPS

There is only one combination of two non touching loops. Let the gain products of two non touching loops be P_{12}.

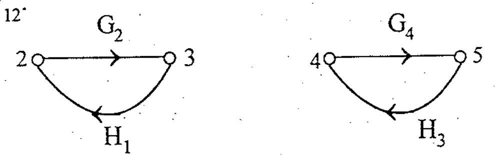


Fig 1.29.8 : First combination of two non touching loops

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non touching loops}\end{array}\right\} P_{12} = (G_2H_1)\,(G_4H_3)$$

$$= G_2G_4H_1H_3$$

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41}) + P_{12}$$
$$= 1 - (G_2H_1 + G_3H_2 + G_4H_3 + G_6H_2H_3) + G_2G_4H_1H_3$$
$$= 1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3$$

Since there is no part of graph which is non touching with forward path 1 and 2, $\Delta_1 = \Delta_2 = 1$.

V TRANSFER FUNCTION, T

By Mason's gain formula

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K \quad \left(\begin{array}{l}\text{Here, } K = 2, \text{ hence we}\\ \text{have two forward paths}\end{array}\right)$$

$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

$$= \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3}$$

EXAMPLE 1.30

Convert the given block diagram to signal flow graph and determine C(s)/R(s).

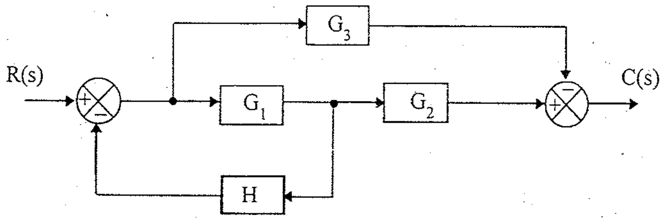


Fig 1.30.1

SOLUTION

The nodes are assigned at input, output, at every summing point and branch point as shown in fig 1.30.2.

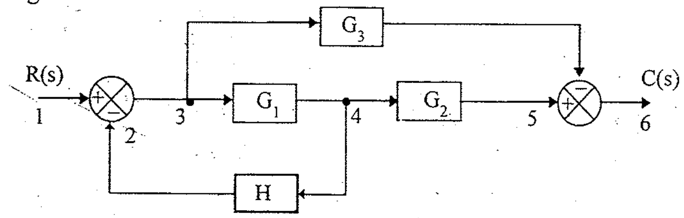


Fig 1.30.2

The signal flow graph of the above system is shown in fig 1.30.3.

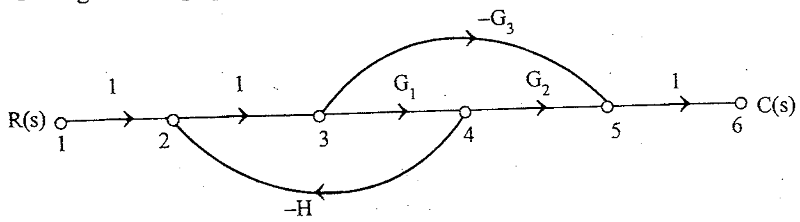


Fig 1.30.3

I FORWARD PATH GAINS

There are two forward paths. $\therefore K=2$

Let the forward path gains be P_1 and P_2.

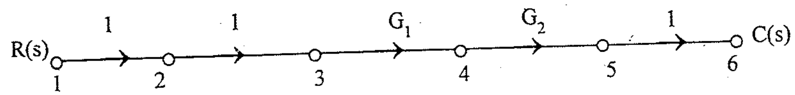


Fig 1.30.4 : *Forward path –1*

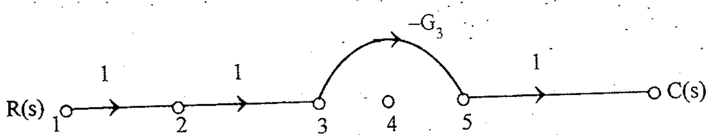
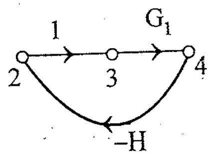


Fig 1.30.5: *Forward path – 2*

Gain of forward path-1, $P_1 = G_1G_2$.
Gain of forward path-2, $P_2 = -G_3$.

II INDIVIDUAL LOOP GAIN

There is only one individual loop. Let the individual loop gain be P_{11}.



Loop gain of individual loop-l, $P_{11} = -G_1H$.

Fig 1.30.6 : *loop – 1*

III GAIN PRODUCTS OF TWO NON TOUCHING LOOPS

There are no combinations of non–touching Loops

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1 - [P_{11}]$$
$$= 1 + G_1H.$$

$\therefore$ Since there are no part of the graph which is non touching with forward path-1 and forward path-2, $\Delta_1 = \Delta_2 = 1$.

V TRANSFER FUNCTION, T

By Mason's gain formula

$$T(s) = \frac{1}{\Delta}\sum_K P_K\Delta_K$$
$$= \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$
$$= \frac{G_1G_2 - G_3}{1 + G_1H}$$

EXAMPLE 1.31

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

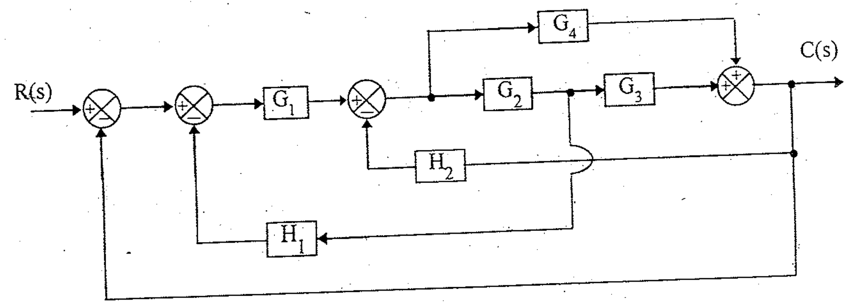


Fig 1.31.1

SOLUTION

The nodes are assigned at input, ouput, at every summing point and branch point as shown in fig 1.31.2.

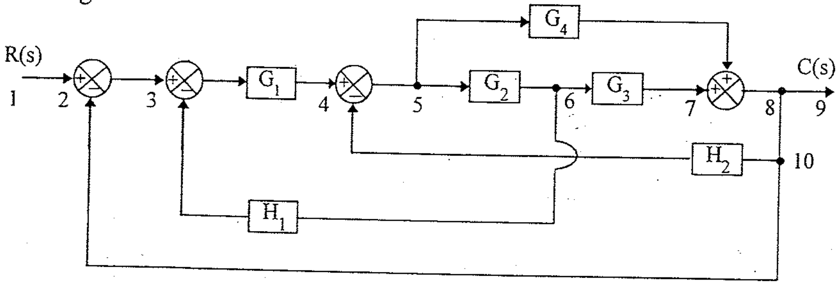


Fig 1.31.2

The signal flow graph for the above block diagram is shown in fig 1.31.3

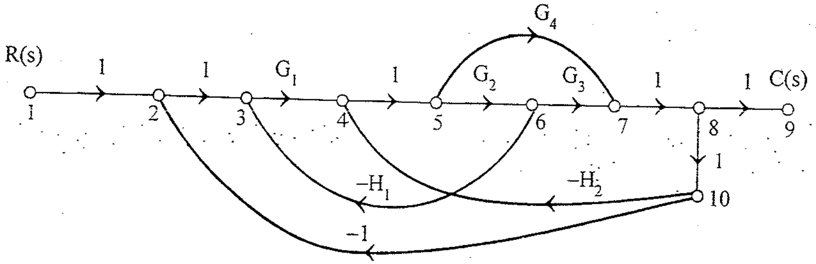


Fig 1.31.3

I FORWARD PATH GAIN

There are two forward paths $\therefore K=2$.

Let the gain of the forward paths be P_1 and P_2.

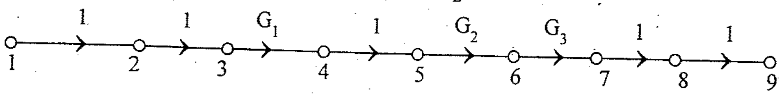


Fig 1.31.4 : Forward path – 1

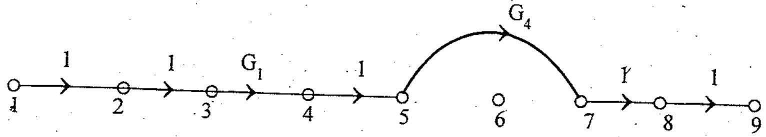


Fig 1.31.5 : Forward path – 2

Gain of forward path-1, $P_1 = G_1G_2G_3$.

Gain of forward path-2, $P_2 = G_1G_4$

II INDIVIDUAL LOOP GAIN

There are five individual loops. Let the individual loop gain be P_{11}, P_{21}, P_{31}, P_{41} and P_{51}.

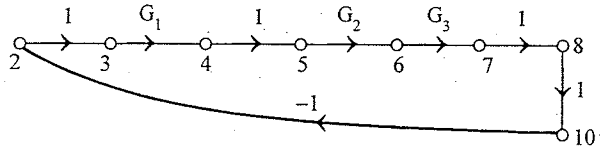


Fig 1.31.6 : Loop – 1

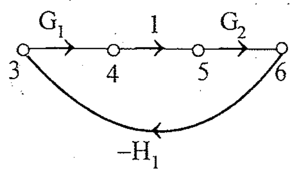


Fig 1.31.7 : loop – 2

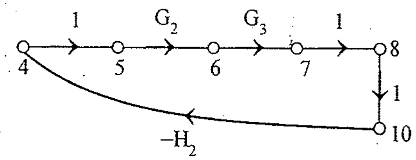


Fig 1.31.8 : Loop – 3

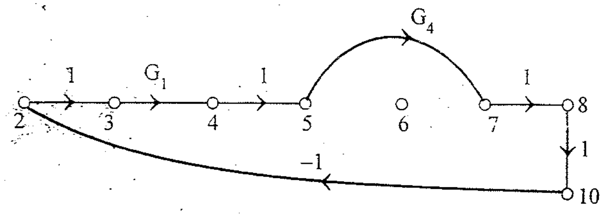


Fig 1.31.9 : Loop – 4

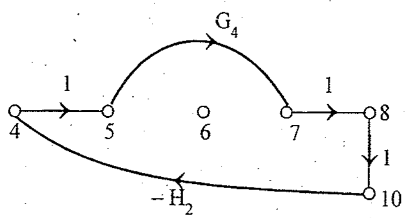


Fig 1.31.10 : Loop – 5

Loop gain of individual loop-1, $P_{11} = -G_1G_2G_3$.

Loop gain of individual loop-2, $P_{21} = -G_2G_1H_1$.

Loop gain of individual loop-3, $P_{31} = -G_2G_3H_2$.

Loop gain of individual loop-4, $P_{41} = -G_1G_4$.

Loop gain of individual loop-5, $P_{51} = -G_4H_2$.

III GAIN PRODUCTS OF TWO NONTOUCHING LOOPS

There are no possible combinations of two non touching loops, three non touching loops, etc.

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1-[P_{11}+P_{21}+P_{31}+P_{41}+P_{51}]$$

$$= 1+G_1G_2G_3+G_1G_2H_1+G_2G_3H_2+G_1G_4+G_4H_2.$$

Since no part of graph is non touching with forward paths 1 and 2, $\Delta_1 = \Delta_2 = 1$.

V TRANSFER FUNCTION, T

By Mason's gain formula

$$T = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

$$= \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1+G_1G_2G_3+G_1G_2H_1+G_2G_3H_2+G_1G_4+G_4H_2}$$

EXAMPLE 1.32

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

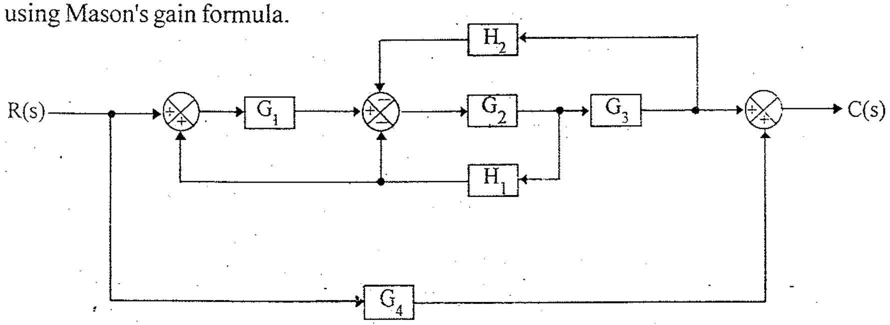


Fig 1.32.1

SOLUTION

The nodes are assigned at input, output, at every summing point and branch point as shown in fig 1.32.2.

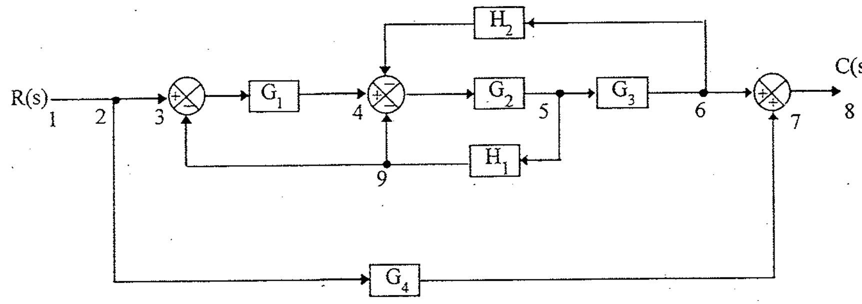


Fig 1.32.2

The signal flow graph of above block diagram is shown in fig 1.32.3.

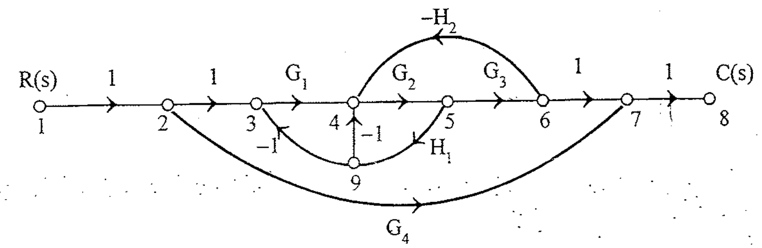


Fig 1.32.3

I FORWARD PATH GAIN

There are two forward path, $\therefore$ K=2.

Let the forward path gains be P_1 and P_2.

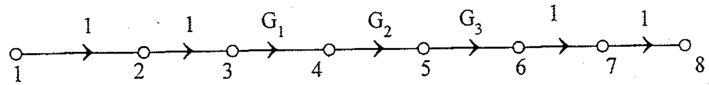


Fig 1.32.4 : Forward path – 1

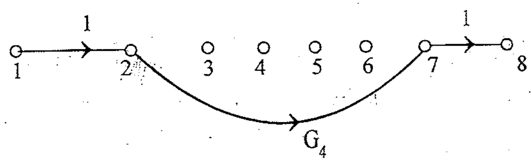


Fig 1.32.5 : Forward path – 2

Gain of forward path-1, $P_1 = G_1G_2G_3$.

Gain of forward path-2, $P_2 = G_4$.

II INDIVIDUAL LOOP GAIN

There are three individual loops with gains P_{11}, P_{21} and P_{31}.

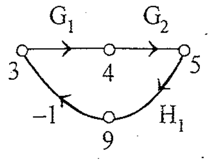


Fig 1.32.6 : loop – 1

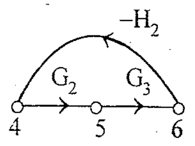


Fig 1.32.7 : loop – 2

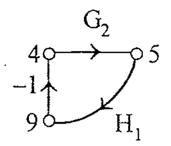


Fig 1.32.8 : loop – 3

Gain of individual loop-1, $P_{11} = -G_1G_2H_1$

Gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Gain of individual loop-3, $P_{31} = -G_2H_1$

III GAIN PRODUCTS OF TWO NON TOUCHING LOOPS

There are no possible combinations of two non touching loops, three non touching loops, etc.

IV CALCULATION OF Δ AND Δ_K

$$\Delta = 1 - [P_{11} + P_{21} + P_{31}]$$
$$= 1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

Since no part of graph touches forward path–1, $\Delta_1 = 1$.

The part of graph non touching forward path–2 is shown in fig 1.32.9.

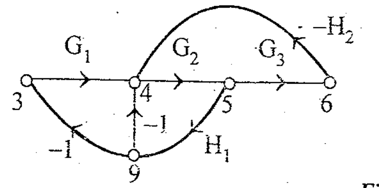


Fig 1.32.9

$$\therefore \Delta_2 = 1 - [-G_1G_2H_1 - G_2G_3H_2 - G_2H_1]$$
$$= 1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

V TRANSFER FUNCTION , T

$$T = \frac{1}{\Delta}\sum_k P_k\Delta_k$$
$$= \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$
$$= \frac{1}{\Delta}[G_1G_2G_3 + G_4(1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1)]$$
$$= \frac{1}{\Delta}[G_1G_2G_3 + G_4 + G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1]$$
$$= \frac{G_1G_2G_3 + G_4 + G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1}{1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1}$$

EXAMPLE 1.33

Draw a signal flow graph and evaluate the closed loop transfer function of a system whose block diagram is shown in fig 1.33.1

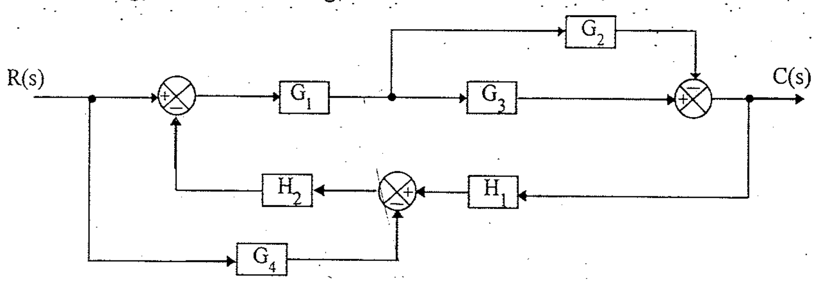


Fig 1.33.1

SOLUTION

The nodes are assigned at input, output, at every summing point and branch point as shown in fig 1.33.2

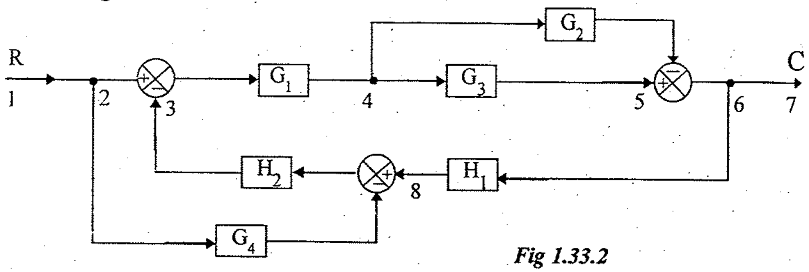


Fig 1.33.2

The signal flow graph for the above block diagram is shown in fig 1.33.3

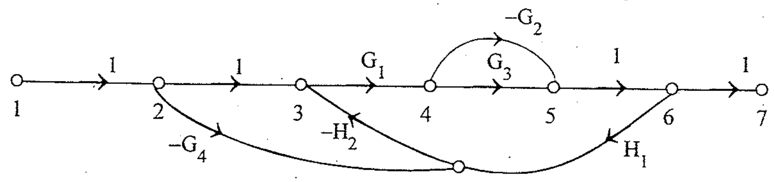


Fig 1.33.3

I FORWARD PATH GAINS

There are four forward paths, $\therefore K = 4$

Let the forward path gains be P_1, P_2, P_3 and P_4.

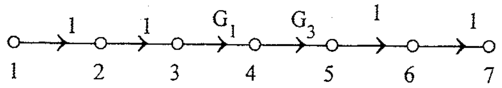


Fig 1.33.4 : Forward path – 1

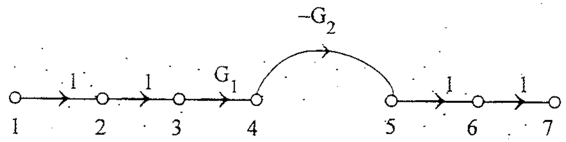


Fig 1.33.5 : Forward path – 2

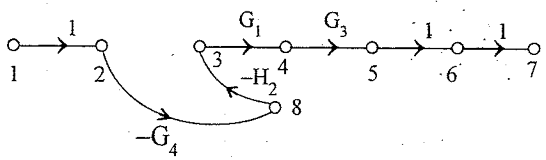


Fig 1.33.6 : Forward path – 3

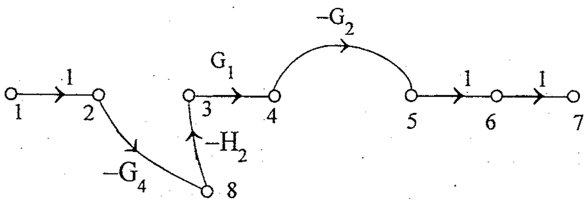


Fig 1.33.7 : Forward path – 4

Gain of forward path-1, $P_1 = G_1G_3$.

Gain of forward path-2, $P_2 = -G_1G_2$.

Gain of forward path-3, $P_3 = G_1G_3G_4H_2$

Gain of forward path-4, $P_4 = -G_1G_2G_4H_2$.

II INDIVIDUAL LOOP GAIN

There are two individual loops, let individual loop gains be P_{11} and P_{21}.

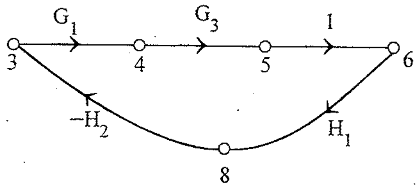


Fig 1.33.8 Loop–1

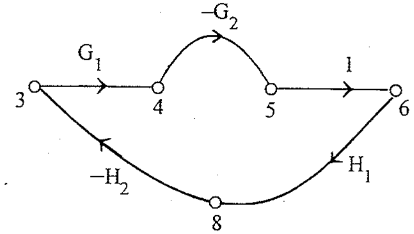


Fig1.33.9 : Loop–2

Loop gain of individual loop-1, $P_{11} = -G_1G_3H_1H_2$

Loop gain of individual loop-2, $P_{21} = +G_1G_2H_1H_2$

III GAIN PRODUCT OF TWO NON TOUCHING LOOPS

There are no possible combinations of two non touching loops, three non touching loops, etc.

IV CALCULATION OF Δ AND Δ_K

$$\begin{aligned}\Delta &= 1-[\text{ sum of individual loop gain }] = 1-(P_{11}+P_{21})\\ &= 1-[-G_1G_3H_1H_2 + G_1G_2H_1H_2]\\ &= 1 + G_1G_3H_1H_2 - G_1G_2H_1H_2\end{aligned}$$

Since no part of graph is non touching with the forward paths, $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$.

V TRANSFER FUNCTION, T

By Mason's gain formula

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{P_1+P_2+P_3+P_4}{\Delta}$$

$$= \frac{G_1G_3 - G_1G_2 + G_1G_3G_4H_2 - G_1G_2G_4H_2}{1+G_1G_3H_1H_2 - G_1G_2H_1H_2}$$

$$= \frac{G_1(G_3-G_2)+G_1G_4H_2(G_3-G_2)}{1+G_1H_1H_2(G_3-G_2)}$$

$$= \frac{G_1(G_3-G_2)(1+G_4H_2)}{1+G_1H_1H_2[G_3-G_2]}$$

1.13 THERMAL SYSTEM

List of symbols used in thermal system

q = Heat flow rate, Kcal/sec

θ_1 = Absolute temperature of emitter, °K

θ_2 = Absolute temperature of receiver, °K

$\Delta\theta$ = Temperature difference, °C

A = Area normal to heat flow, m^2.

K = Conduction or Convection coefficient, Kcal/sec-°C

K_r = Radiation coefficient, Kcal/sec-°C

H = K/A=Convection coefficient, Kcal/m^2-sec-°C

K = Thermal conductivity, Kcal/m-sec-°C

ΔX = Thickness of conductor, m

R = Thermal resistance, °C-sec/Kcal

C = Thermal capacitance, Kcal/°C

HEAT FLOW RATE

Thermal systems are those that involve the transfer of heat from one substance to another. There are three different ways of heat flow from one substance to another. They are conduction, convection and radiation.

For conduction,

$$\text{Heat flow rate, } q = K\,\Delta\theta = \frac{KA}{\Delta X} \quad \text{.....(1.36)}$$

For convection,

$$\text{Heat flow rate, } q = K\,\Delta\theta = HA\,\Delta\theta \quad \text{.....(1.37)}$$

For radiation,

$$\text{Heat flow rate, } q = K_r(\theta_1^4 - \theta_2^4)$$

$$\text{If } \theta_1 >> \theta_2 \text{ then, } q = K_r\bar{\theta}^4 \quad \text{.....(1.38)}$$

$$\text{Where } \bar{\theta}^4 = (\theta_1^4 - \theta_2^4)^{\frac{1}{4}}$$

Note : *$\bar{\theta}_4$ is called effective temperature difference of the emitter and receiver.*

BASIC ELEMENTS OF THERMAL SYSTEM

The model of thermal systems are obtained by using thermal resistance and capacitance which are the basic elements of the thermal system.

The thermal resistance and capacitance are distributed in nature. But for simplicity in analysis lumped parameter models are used. In lumped parameter model it is assumed that the substances that are characterized by resistance to heat flow have negligible heat capacitance and the substances that are characterized by heat capacitance have negligible resistance to heat flow.

The thermal resistance, R for heat transfer between two substances is defined as the ratio of change in temperature and change in heat flow rate.

$$\text{Thermal resistance, } R = \frac{\text{Change in Temperature}, °C}{\text{Change in heat flow rate, Kcal / sec}}$$

For conduction or convection,

$$\text{Heat flow rate, } q = K\,\Delta\theta$$

On differentiating we get,

$$dq = K\,d(\Delta\theta)$$

$$\therefore \frac{d(\Delta\theta)}{dq} = \frac{1}{K}$$

$$\text{But thermal resistance, } R = \frac{d(\Delta\theta)}{dq}$$

$$\therefore \text{Thermal resistance, } R = \frac{1}{K} \quad \text{for conduction}$$

$$= \frac{1}{K} = \frac{1}{HA} \quad \text{for convection}$$

For radiation,

$$\text{Heat flow rate, } q = K_r\bar{\theta}^4$$

On differentiating we get

$$dq = K_r\,4\,\bar{\theta}^3\,d\bar{\theta}$$

$$\therefore \frac{d\bar{\theta}}{dq} = \frac{1}{K_r 4\bar{\theta}^3}$$

But thermal resistance, $R = \frac{d\bar{\theta}}{dq}$

$\therefore$ Thermal resistance, $R = \frac{1}{4K_r\bar{\theta}^3}$ (for radiation) .....(1.39)

Thermal capacitance, C is defined as the ratio of change in heat stored and change in temperature

$$\text{Thermal capacitance, } C = \frac{\text{Change in heat stored , Kcal}}{\text{Change in temperature , °C}}$$

Let M = Mass of substance considered, Kg

c_p = Specific heat of substance, Kcal/Kg-°C

Now, Thermal capacitance, $C = Mc_p$ .....(1.40)

EXAMPLE OF THERMAL SYSTEM

Consider a simple thermal system shown in fig 1.28. Let us assume that the tank is insulated to eliminate heat loss to the surrounding air, there is no heat storage in the insulation and liquid in the tank is kept at uniform temperature by perfect mixing with the help of a stirrer. Thus, a single temperature is used to describe the temperature of the liquid in the tank and of the outflowing liquid. The transfer function of this system can be derived as shown below.

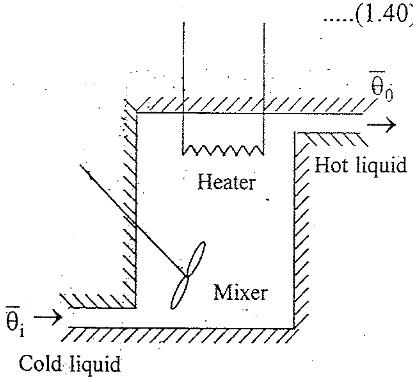


Fig 1.28 : Thermal system

Let $\bar{\theta}_i$ = Steady state temperature of inflowing liquid, °C.

$\bar{\theta}_0$ = Steady state temperature of outflowing liquid, °C.

G = Steady state liquid flow rate, Kg/sec.

M = Mass of liquid in tank, Kg.

c = Specific heat of liquid, Kcal/Kg °C.

R = Thermal resistance, °C - sec/Kcal.

C = Thermal capacitance, Kcal/°C.

Q = Steady state heat input rate, Kcal/sec.

Let us assume that the temperature of inflowing liquid is kept constant. Let the heat input rate to the system supplied by the heater is suddenly changed from $\bar{Q}$ to $\bar{Q}+q_i$. Due to this, the heat output flow rate will gradually change from $\bar{Q}$ to $\bar{Q}+q_0$. The temperature of the outflowing liquid will also be changed from $\bar{\theta}_0$ to $\bar{\theta}_0+\theta$.

For this system the equation for q_0, C and R are obtained as follows,

$$\left.\begin{array}{l}\text{Change in output}\\ \text{heat flow rate,}\end{array}\right\} q_0 = \text{Liquid flow rate, G} \times \text{Specific heat of liquid, c} \times \text{Change in temperature, } \theta$$

$$= Gc\theta \qquad .....(1.41)$$

Thermal capacitance, C = Mass, M x Specific heat of liquid, c

$$= Mc \qquad .....(1.42)$$

$$\text{Thermal resistance, } R = \frac{\text{Change in temperature, } \theta}{\text{Change in heat flow rate, } q_0}$$

$$= \frac{\theta}{q_0} \qquad .....(1.43)$$

On substituting for q_0 from equation (1.41) in equation (1.43) we get,

$$R = \frac{\theta}{Gc\theta} = \frac{1}{Gc} \qquad .....(1.44)$$

In this system, the rate of change of temperature is directly proportional to change in heat input rate.

$$\therefore \frac{d\theta}{dt} \propto q_i - q_0$$

The constant of proportionality in the capacitance C of the system.

$$\therefore C\frac{d\theta}{dt} = q_i - q_0 \qquad .....(1.45)$$

Equation (1.45) is the differential equation governing the system. Since equation (1.45) is of first order equation, the system is first order system.

From equation (1.43), $R = \frac{\theta}{q_0}$, $\quad \therefore q_0 = \frac{\theta}{R}$ .....(1.46)

On substituting for q_0 from equation (1.46) in equation (1.45) we get,

$$C\frac{d\theta}{dt} = q_i - \frac{\theta}{R}$$

$$C\frac{d\theta}{dt} = \frac{Rq_i - \theta}{R}$$

$$RC\frac{d\theta}{dt} = Rq_i - \theta$$

$$RC\frac{d\theta}{dt} + \theta = Rq_i \qquad .....(1.47)$$

Let, $L[\theta] = \theta(s)$; $L\left[\frac{d\theta}{dt}\right] = s\theta(s)$ and $L[q_i] = Q_i(s)$

On taking Laplace transform of equation (1.47) we get,

$$RC\, s\, \theta(s) + \theta(s) = R\, Q_i(s)$$

$$\theta(s)\, [sRC + 1] = R\, Q_i(s)$$

$\frac{\theta(s)}{Q_i(s)}$ is the required transfer function of the system.

$$\therefore \frac{\theta(s)}{Q_i(s)} = \frac{R}{sRC+1} = \frac{R}{RC\left(s+\frac{1}{RC}\right)} = \frac{\frac{1}{C}}{s+\frac{1}{RC}} \qquad .....(1.48)$$

1.14 HYDRAULIC SYSTEM

The Hydraulic system of interest to control engineers may be classified into

1. Liquid Level system and
2. Hydraulic devices

The liquid level system consists of storage tanks and connecting pipes. The variables to be controlled are liquid height in tanks and flow rate in pipes. The driving force is the relative difference of the liquid heights in the tanks.

The Hydraulic devices are devices using incompressible oil as their working medium. These devices are used for controlling the forces and motions. The driving force is the high pressure oil supplied by the Hydraulic pumps.

Liquids are slightly compressible at high pressures. In hydraulic system, the compressibility effects may be neglected and conservation of volume is used as the basic physical law. The variables of hydraulic system are volumetric flow rate, q and pressure, P. The volumetric flow rate, q is through variable and it is analogous to current. The pressure, P is across variable and it is analogous to voltage.

Three basic elements of hydraulic systems are the Resistance, Capacitance and Inertance. The liquid flowing out of a tank can meet the resistance in several ways. Liquid while flowing through a pipe meet with resistance due to the friction between pipe walls and liquid. Presence of valves, bends, coupling of pipe of different diameter also offer resistance to liquid flow.

The capacitance is an energy storage element and it represents storage in gravity field. The inertance represents fluid inertia and is derived from the inertia forces required to accelerate the fluid in a pipe. It is also an energy storage element. But the energy storage due to inertance element is negligible compared to that of capacitance element.

Consider the flow through a short pipe connecting two tanks. The resistance for liquid flow in such a pipe or restriction, is defined as the change in the level difference, necessary to cause a unit change in the flow rate

$$R = \frac{\text{Change in level difference, m}}{\text{Change in flow rate, m}^3/\text{sec}}$$

Capacitance C of a tank is defined to be the change in quantity of stored liquid, necessary to cause a unit change in the potential (head)

$$C = \frac{\text{Change in liquid stored, m}^3}{\text{Change in head, m}}$$

EXAMPLE OF LIQUID LEVEL SYSTEM

A simple liquid level system is shown in figure 1.29 with the steady state flow rate, $\overline{Q}$ and steady state head, $\overline{H}$

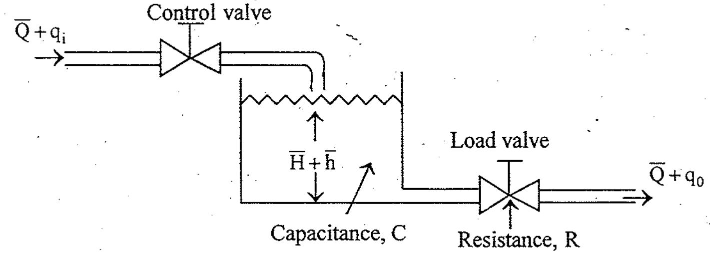


Fig 1.29 : Liquid level system

Let $\overline{Q}$ = Steady-state flow rate (before any change has occured), m^3/sec.
q_i = Small deviation of inflow rate from its steady-state value, m^3/sec.
q_0 = Small deviation of outflow rate from its steady-state value, m^3/sec.
$\overline{H}$ = Steady state head (before any change has occured), m.
h = Small deviation of head from its steady-state value, m.

Let the system be considered linear. The differential equation governing the system is obtained by equating, the change in flow rate to the amount stored in the tank. In a small time interval dt, let the change in flow rate be $(q_i - q_0)$, and the change in height be dh.

Now, Change in storage = Change in flow rate

$$\therefore C\,dh = (q_i - q_0)\,dt \qquad .....(1.49)$$

$$\text{The resistance, } R = \frac{\text{Change in head}}{\text{Change in outflow rate}} = \frac{h}{q_0}$$

$$\therefore q_0 = \frac{h}{R} \qquad .....(1.50)$$

On substituting for q_0 from equation (1.50) in equation (1.49) we get,

$$C\;dh = \left(q_i - \frac{h}{R}\right) dt$$

$$C\;dh = \left(\frac{q_i R - h}{R}\right) dt$$

$$RC\,\frac{dh}{dt} = q_i R - h$$

$$RC\,\frac{dh}{dt} + h = q_i R \qquad .....(1.51)$$

The equation 1.51 is the differential equation governing the system. The term RC is the time constant of the system. On taking Laplace's transform of equation 1.51, we get,

$$RC\;sH(s) + H(s) = Q_i(s)\;R$$

$$(s\,RC + 1)\;H(s) = Q_i(s)\;R$$

$$\therefore \frac{H(s)}{Q_i(s)} = \frac{R}{(sRC+1)} = \frac{R}{RC\,(s + 1/RC)} = \frac{1/C}{s + 1/RC} \qquad .....(1.52)$$

The equation (1.52) is the required transfer function of the system.

HYDRAULIC DEVICES

The hydraulic devices are used in hydraulic feedback systems and in combined electro-mechanical-hydraulic systems. In hydraulic devices, power is transmitted through the action of fluid flow under pressure and the fluid is incompressible. The fluid used are petroleum based oils or non-inflammable synthetic oils.

The hydraulic devices used in control systems are generally classified as hydraulic motors and hydraulic linear actuators.

The output of hydraulic motor is rotary motion and that of linear actuator is translational. The hydraulic motor is physically smaller in size than an electric motor for the same power output. Also, the hydraulic components are more rugged than the corresponding electrical components. The applications of hydraulic devices are power steering and brakes in automobiles, the steering mechanism of large ships, the control of large machine tools, etc.

ADVANTAGES OF HYDRAULIC DEVICES

(i) Hydraulic fluid acts as a lubricant and coolant.

(ii) Comparatively small sized hydraulic actuators can develop large forces or torques.

(iii) Hydraulic actuators can be operated under continuous, intermittent, reversing and stalled conditions without damage.

(iv) Hydraulic actuators have a higher speed of response. They offer fast starts, stops and speed reversals.

(v) With availability of both linear and rotary actuators, the design has become more flexible.

(vi) Because of low leakages in hydraulic actuators, when loads are applied the drop in speed will be small.

(vii) For the same power output, hydraulic motor is much smaller in physical size than an electric motor.

(viii) Hydraulic components are rapidly acting and more rugged compared to the corresponding electrical component.

DISADVANTAGES OF HYDRAULIC DEVICES

(i) Hydraulic power is not readily available compared to electric power.

(ii) They have the inherent problems of leaks and of sealing them against foreign particles.

(iii) Operating noise.

(iv) Costs more when compared to electrical system.

(v) Tendency to become sluggish at low temperature because of increasing viscosity of the fluid.

(vi) Fire and explosion hazards exist.

(vii) Hydraulic lines are not flexible as electric cables.

(viii) Because of the non linear and other complex characteristics involved, it is difficult to design sophisticated hydraulic systems.

EXAMPLE OF HYDRAULIC DEVICE

The most frequently used hydraulic device in control system is hydraulic motor-pump set. It consists of a variable stroke hydraulic pump and a fixed stroke hydraulic motor as shown in fig 1.30. The device accepts a linear displacement (stroke length) as input and delivers a large output torque.

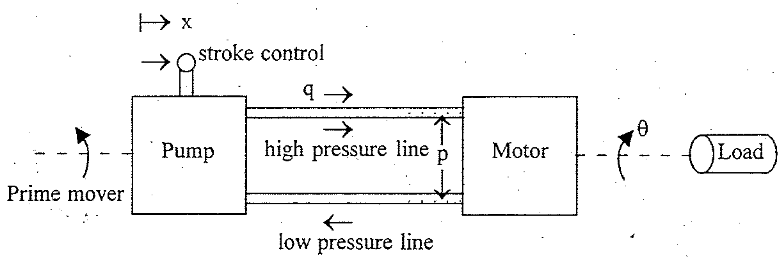


Fig 1.30 : Hydraulic motor-pump set

The hydraulic motor is controlled by the amount of oil delivered by the pump. By mechanically changing the pump stroke, the oil delivered by the pump is controlled. Like in a DC generator and motor, there is no essential difference between hydraulic pump and motor. In a pump the input is mechanical power and output is hydraulic power and in a motor, it is viceversa.

Let q_p = Rate at which the oil flows from the pump

q_m = Oil flow rate through the motor

q_i = Leakage flow rate

q_c = Compressibility flow rate

x = Input stroke length

θ = Output angular displacement of motor

P = Pressure drop across motor.

The rate at which the oil flow from the pump is proportional to stroke angle, i.e., $q_p \alpha x$.

$\therefore$ Oil flow rate from the pump, $q_p = K_p x$ .....(1.53)

Where K_p is a constant and is equal to the ratio of rate of oil flow to unit stroke angle.

The rate of oil flow through the motor is proportional to motor speed, i.e., $q_m \alpha \dfrac{d\theta}{dt}$

$$\therefore \left.\begin{array}{l}\text{Oil flow rate}\\ \text{through motor}\end{array}\right\} q_m = K_m \frac{d\theta}{dt} \quad .....(1.54)$$

Where K_m = Motor displacement constant.

All the oil from the pump does not flow through the motor in the proper channels. Due to back pressure in the motor, a portion of the ideal flow from the pump leaks back past the pistons of motor and pump. The back pressure is the pressure that is built up by the hydraulic flow to overcome the resistance to free movement offered by load on motor shaft.

It is usually assumed that the leakage flow is proportional to motor pressure, i.e. $q_i \alpha P$

$\therefore$ Leakage flow rate, $q_i = K_i P$ .....(1.55)

Where K_i = constant.

The back pressure built up by the motor not only causes leakage flow in the motor and pump but also causes the oil in the lines to compress. Volume compressibility flow is essentially proportional to pressure and therefore the rate of flow is proportional to the rate of change of pressure, i.e. $q_c \alpha \dfrac{dP}{dt}$.

$$\therefore \text{Compressibility flow rate, } q_c = K_c \frac{dP}{dt} \qquad .....(1.56)$$

Where K_c = Coefficient of compressibility.

The rate at which the oil flows from the pump is given by sum of oil flow rate through the motor, leakage flow rate and compressibility flow rate.

$$\therefore q_p = q_m + q_l + q_c \qquad .....(1.57)$$

On substituting from equation (1.53) to (1.56) in equation (1.57) we get,

$$K_p\, x = K_m \frac{d\theta}{dt} + K_l\, P + K_c \frac{dP}{dt} \qquad .....(1.58)$$

The torque T_m developed by the motor is proportional to the pressure drop and balances the load torque.

$$\therefore \text{Hydraulic motor torque, } T_m = K_t\, P \qquad .....(1.59)$$

Where K_t is motor torque constant.

If the load is assumed to consist of moment of inertia J and viscous friction with coefficient B,

$$\text{Then, load torque, } T_l = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \qquad .....(1.60)$$

$$\text{Hydraulic power input} = q_m\, P \qquad .....(1.61)$$

Substitute for q_m, from equation (1.54) in equation (1.61)

$$\text{Hydraulic power input} = K_m \frac{d\theta}{dt}\, P \qquad .....(1.62)$$

$$\text{Mechanical power output} = T_m \frac{d\theta}{dt} \qquad .....(1.63)$$

Substitute for T_m from equation (1.59) in equation (1.63)

$$\therefore \text{Mechanical power output} = K_t\, P \frac{d\theta}{dt} \qquad .....(1.64)$$

If hydraulic motor losses are neglected or included as a part of load, then the hydraulic motor input is equal to mechanical power output of hydraulic motor.

$$\therefore K_m \frac{d\theta}{dt} P = K_t P \frac{d\theta}{dt} \qquad .....(1.65)$$

From equation (1.65), it is clear that, $K_m = K_t$

Hence, equation (1.59) can be written as

$$T_m = K_t\, P = K_m\, P$$

Since the motor torque equals load torque

$$T_m = T_l$$

$$\therefore K_m P = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt}$$

$$\therefore P = \frac{J}{K_m} \frac{d^2\theta}{dt^2} + \frac{B}{K_m} \frac{d\theta}{dt} \qquad .....(1.66)$$

On differentiating with respect to t, we get

$$\frac{dP}{dt} = \frac{J}{K_m} \frac{d^3\theta}{dt^3} + \frac{B}{K_m} \frac{d^2\theta}{dt^2} \qquad .....(1.67)$$

On substituting for P and dP/dt from equation (1.66) and equation (1.67) in equation (1.58) we get,

$$K_p x = K_m \frac{d\theta}{dt} + K_l \left[\frac{J}{K_m} \frac{d^2\theta}{dt^2} + \frac{B}{K_m} \frac{d\theta}{dt} \right] + K_c \left[\frac{J}{K_m} \frac{d^3\theta}{dt^3} + \frac{B}{K_m} \frac{d^2\theta}{dt^2} \right]$$

$$K_p x = \frac{K_c\, J}{K_m} \frac{d^3\theta}{dt^3} + \left(\frac{K_l J}{K_m} + \frac{K_c B}{K_m} \right) \frac{d^2\theta}{dt^2} + \left(K_m + \frac{K_l B}{K_m} \right) \frac{d\theta}{dt}$$

On taking Laplace transform with zero initial conditions, we get,

$$K_p X(S) = \frac{K_c\, J}{K_m} s^3\, \theta(s) + \left(\frac{K_l\, J}{K_m} + \frac{K_c B}{K_m} \right) s^2 \theta(s) + \left(K_m + \frac{K_l\, B}{K_m} \right) s\theta(s)$$

$$\frac{\theta(s)}{X(s)} = \frac{K_p}{s \left[\frac{K_c J}{K_m} s^2 + \left(\frac{K_l J + K_c B}{K_m} \right) s + \frac{{K_m}^2 + K_l B}{K_m} \right]} \qquad .....(1.68)$$

In hydraulic systems, normally $K_c << K_m$, therefore

Put $K_c = 0$, in equation (1.68)

$$\therefore \frac{\theta(s)}{X(s)} = \frac{K_p}{s\left[\frac{K_i J}{K_m}s + \frac{K_m^2 + K_i B}{K_m}\right]} = \frac{K_p \Big/ \frac{K_m^2 + K_i B}{K_m}}{s\left[\frac{K_i J}{K_m^2 + K_i B}s + 1\right]} = \frac{K}{s(\tau s + 1)} \quad .....(1.69)$$

$$\text{Where,}\ K = K_p \Big/ \frac{K_m^2 + K_i B}{K_m} \quad \text{and} \quad \tau = \frac{K_i J}{K_m^2 + K_i B}$$

The equation (1.69) is the required transfer function of the system.

1.15. PNEUMATIC SYSTEM

Pneumatic system uses compressible fluid as working medium usually air. In pneumatic systems, compressibility effects of gas cannot be neglected and hence dynamic equations are obtained using conservation of mass. In pneumatic systems, change in fluid inertia energy and the fluid's internal thermal energy are assumed negligible. In pneumatic system, which employs compressible fluid as working fluid, the mass and volume flow rates are not readily interchangeable and the analysis of gas flow is more complicated.

The pneumatic devices involve the flow of gas or air, through connected pipe lines and pressure vessels. Hence the variables of pneumatic system are mass flow rate, q_m and pressure, P. The mass flow rate is through variable and it is analogous to current. The pressure is across variable and it is analogous to voltage.

The two basic elements of pneumatic system are the resistance and capacitance. The restrictions in the pipes and valves offers resistance to gas flow.

The gas flow resistance, R is defined as the rate of change in gas pressure difference for a change in gas flow rate.

$$R = \frac{\text{Change in gas pressure difference, N/m}^2}{\text{Change in gas flow rate, Kg/sec}}$$

The pnuematic capacitance is defined for a pressure vessel and depends on the type of expansion process involved. The capacitance of a pressure vessel may be defined as the ratio of change in gas stored for a change in gas pressure.

$$C = \frac{\text{Change in gas stored, Kg}}{\text{Change in gas pressure, N/m}^2}$$

Pneumatic devices are employed in guided missiles,air craft systems, automation of production machinery and in many other fields as automatic controllers.

The advantages of pneumatic system are

1. The air or gas used is non-inflammable and so it offers safety from fire hazards.
2. The air or gas has negligible viscosity, compared to the high viscosity of hydraulic fluids.
3. No return pipelines are required since air can be let out, at the end of device work cycle.

The disadvantage in pneumatic system is that the response is slower than that of hydraulic systems, because of the compressibility of the working fluid.

EXAMPLE OF A PNEUMATIC SYSTEM

A simple pneumatic system is shown in fig.1.31 and it consist of a pneumatic bellows in line with the restriction. The pneumatic bellows consists of a hollow chamber with thin pneumatic walls. The side walls of bellows are corrugated and the input and output surface are flat. An increase in pressure within the bellows results in an increase in separation between the input and output surfaces.

Let $\overline{P_i}$ = Steady-state value of input air pressure.

p_i = Increase in the pressure of air-source.

$\overline{P}$ = Steady-state value of pressure inside the bellows.

p = Increase in pressure inside the bellow

$\overline{Q_m}$ = Steady-state value of air flow rate

q_m = Increase in air flow rate

A = Area of each flat surface of the bellows

R = Resistance of the restriction

C = Capacitance of the bellows.

x = Displacement of the movable surface of the bellows.

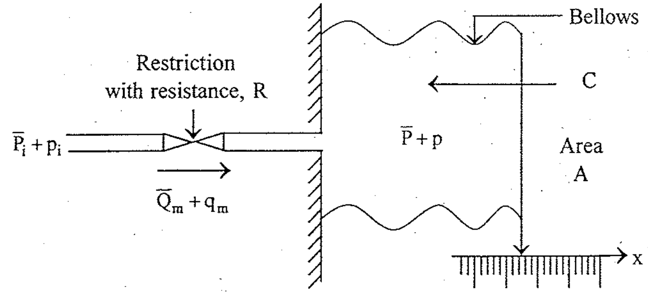


Fig 1.31 : Pneumatic system

Let the pressure of air source be increased from its steady state value by an amount p_i. This results in an increase in air flow by q_m and increase in the pressure inside the bellows by p. Due to increase in pressure, there will be a displacement of the movable surface of the bellows, by an amount x. Here, the terms p_i, q_m, p and x are all functions of time, t and therefore can be expressed as $p_i(t)$, $q_m(t)$, p(t) and x(t).

The force exerted on the movable surface of the bellows is proportional to increase in pressure inside the bellows, i.e, $f_b \alpha p(t)$

$\therefore$ Force exerted on the movable surface of the bellow, $f_b = A\,p(t)$ .....(1.70)

The force opposing the movement of the flat surface of bellow walls is proportional to the displacement i.e, $f_0 \alpha x(t)$.

$\therefore$ Force opposing the motion, $f_0 = K\,x(t)$ .....(1.71)

Where K = a constant representing the stiffness of the bellows.

At steady state the above two forces are balanced,

$$\therefore f_b = f_0 \text{ and so } A\,p(t) = K\,x(t) \quad .....(1.72)$$

$$\text{The resistance, } R = \frac{\text{Difference between change in pressure}}{\text{Change in air flow rate}} = \frac{p_i(t) - p(t)}{q_m(t)}$$

$$\therefore q_m(t) = \frac{p_i(t) - p(t)}{R} \quad .....(1.73)$$

$$\text{The capacitance, } C = \frac{\text{Change in air flow rate}}{\text{Rate of change of pressure}} = \frac{q_m(t)}{dp(t)/dt}$$

$$\therefore q_m(t) = C\frac{dp(t)}{dt} \quad .....(1.74)$$

On equating the two equations of $q_m(t)$ we get

$$C\frac{dp(t)}{dt} = \frac{p_i(t) - p(t)}{R}$$

$$RC\frac{dp(t)}{dt} + p(t) = p_i(t) \quad .....(1.75)$$

From the equation (1.72), we get $p(t) = \frac{K}{A}x(t)$ .....(1.76)

On differentiating equation (1.76) with respect to t, we get,

$$\frac{dp(t)}{dt} = \frac{K}{A}\frac{dx(t)}{dt} \quad .....(1.77)$$

On substituting for p(t) and dp(t)/dt from equation (1.76) and equation (1.77) in equation (1.75) we get

$$RC\left[\frac{K}{A}\frac{dx(t)}{dt}\right] + \frac{K}{A}x(t) = p_i(t) \quad .....(1.78)$$

On taking laplace transform with zero initial conditions we get

$$RC\left(\frac{K}{A}\,s\,X(s)\right) + \frac{K}{A}\,X(s) = P_i(s)\ ;\ \therefore \frac{X(s)}{P_i(s)} = \frac{A/K}{RC\,s+1} = \frac{A/K}{\tau\,s+1} \quad .....(1.79)$$

Where, τ = RC = Time constant of the system.

The equation (1.79) is the required transfer function of the system.

Difference between Hydraulic and Pneumatic system

Pneumatic	Hydraulic
1. Working fluid is compressible	1. Working fluid is incompressible
2. Working fluid lack lubricating property	2. Working fluid acts as lubricant
3. Operating pressure is lower	3. Higher operating pressure.
4. Output power is less	4. More output power
5. Accuracy of actuator is poor	5. More accuracy may be made
6. External leakage is permissible but internal leakage must be avoided	6. Internal leakage is permissible and external leakage must be avoided.
7. No return pipes are required	7. Return pipes are required
8. Insensitive to temperature changes	8. Sensitive to changes in temperature.
9. Fire and explosion proof.	9. Not a fire and explosion proof.

1.16 SHORT QUESTION AND ANSWER

Q1.1 What is system?

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

Q1.2 What is control system?

A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity, then the system is called control system.

The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Q1.3 What are the two major type of control systems?

The two major type of control systems are open loop and closed loop systems.

Q1.4 Define open loop system.

The control system in which the output quantity has no effect upon the input quantity are called open loop control system. This means that the output is not fedback to the input for correction.

Q1.5 Define closed loop system.

The control systems in which the output has an effect upon the input quantity so as to maintain the desired output value are called closed loop control systems.

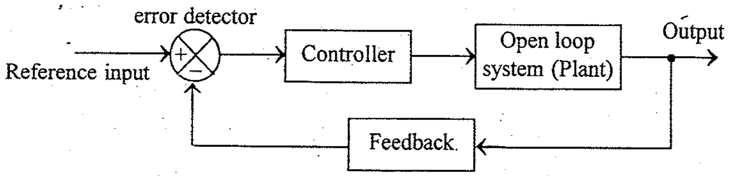
Q1.6 What is feedback? What type of feedback is employed in control system?

The feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in desired output.

Negative feedback is employed in control system.

Q1.7 What are the components of feedback control system?

The components of feedback control system are plant, feedback path elements, error detector and controller.



Q1.8 Why is negative feedback invariably preferred in a closed loop system?

The negative feedback results in better stability in steady state and rejects any disturbance signals. It also has low sensitivity to parameter variations. Hence negative feedback is preferred in closed loop systems.

Q1.9 What are the characteristics of negative feedback?

The characteristics of negative feedback are as follows :

(i) accuracy in tracking steady state value
(ii) rejection of disturbance signals
(iii) low sensitivity to parameter variations
(iv) reduction in gain at the expense of better stability.

Q1.10. What is the effect of positive feedback on stability?

The positive feedback increases the error signal and drives the output to instability. But sometimes the positive feedback is used in minor loops in control systems to amplify certain internal signals or parameters.

Q1.11. Distinguish between open loop and closed loop system.

Open loop	Closed loop
(i) Inaccurate & unreliable	(i) Accurate & reliable
(ii) Simple and economical	(ii) Complex and costlier
(iii) The changes in output due to external disturbances are not corrected automatically.	(iii) The changes in output due to external disturbances are corrected automatically.
(iv) They are generally stable.	(iv) Great efforts are needed to design a stable system.

Q1.12 What is servomechanism?

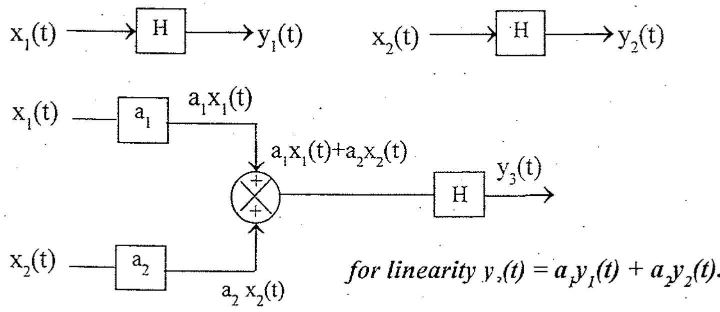
The servomechanism is a feedback control system in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

Q1.13 State the principle of homogenity (or) State the principle of superposition.

The principle of superposition and homogenity states that if the system has responses $y_1(t)$ and $y_2(t)$ for the inputs $x_1(t)$ and $x_2(t)$ respectively then the system response to the linear combination of these input $a_1x_1(t) + a_2x_2(t)$ is given by linear combination of the individual outputs $a_1y_1(t) + a_2y_2(t)$, where a_1 and a_2 are constants.

Q1.14 Define linear system.

A system is said to be linear if it obeys the principle of superposition and homogenity. The principle of superposition states that the response of a system to a weighed sum of signals is equal to the corresponding weighed sum of the responses of the system to each of the individual input signals.



for linearity $y_3(t) = a_1y_1(t) + a_2y_2(t)$.

Q1.15 What is time invariant system?

A system is said to be time invariant if its input output characteristics do not change with time. A linear time invariant system can be represented by constant coefficient differential equations. (In linear time varying systems the coefficients of the differential equation governing the system are function of time).

Q1.16 Define transfer function.

The transfer function of a system is defined as the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions. It is also defined as the Laplace transform of the impulse response of system with zero initial conditions.

Q1.17 State whether transfer function technique is applicable to non-linear system and whether the transfer function is independent of the input of a system.

(i) The transfer function technique is not applicable to non-linear system.

(ii) The transfer function of a system is independent of input and depends only on system parameters but the output of a system depends on input.

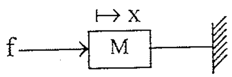
Q1.18 What are the basic elements used for modelling mechanical translational system?

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dashpot.

Q1.19 Write the force balance equation of ideal mass element.

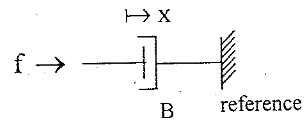
Let a force f be applied to an ideal mass M. The mass will offer an opposing force, f_m which is proportional to acceleration.

Therefore, $f = f_m = M\dfrac{d^2x}{dt^2}$

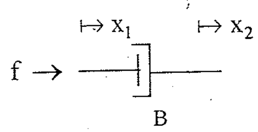


Q1.20 Write the force balance equation of ideal dashpot.

Let a force f be applied to an ideal dashpot, with viscous frictional coefficient B. The dashpot will offer an opposing force, f_b which is proportional to velocity.



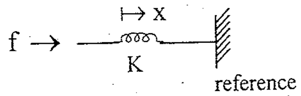
$$f = f_b = B\frac{dx}{dt}$$



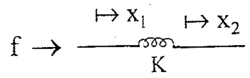
$$f = f_b = B\frac{d}{dt}(x_1 - x_2)$$

Q1.21 Write the force balance equation of ideal spring?

Let a force f be applied to an ideal spring with spring constant K. The spring will offer an opposing force f_k which is proportional to displacement.



$$f = f_k = Kx$$



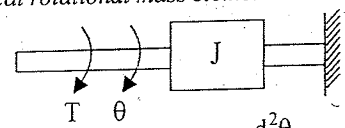
$$f = f_k = K(x_1 - x_2)$$

Q1.22 What are the basic elements used for modelling mechanical rotational system?

The model of mechanical rotational system can be obtained by using three basic elements mass with Moment of Inertia, J, dash-pot with rotational frictional coefficient, B and torsional spring with stiffness, K.

Q1.23 Write the torque balance equation of an ideal rotational mass element.

Let a torque T be applied to an ideal mass with moment of inertia, J. The mass will offer an opposing torque T_j which is proportional to angular acceleration.

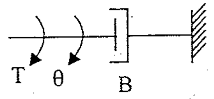


$$T = T_j = J\frac{d^2\theta}{dt^2}$$

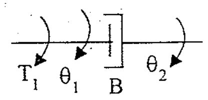
Q1.24 Write the torque balance equation of an ideal rotational dash-pot.

Let a torque T be applied to a rotational dash-pot with frictional coefficient B. The dashpot will offer an opposing torque which is proportional to angular velocity.

$T = T_b = B\frac{d\theta}{dt}$



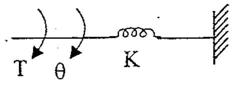
$T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$



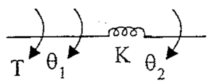
Q1.25 Write the torque balance equation of ideal rotational spring.

Let a torque T be applied to an ideal rotational spring with spring constant K. The spring will offer an opposing torque T_k which is proportional to angular displacement.

$T_k = K\theta$



$T = T_k = K(\theta_1 - \theta_2)$



Q1.26 Name the two types of electrical analogous for mechanical system.

The two types of analogies for the mechanical system are force-voltage and force-current analogy.

Q1.27 Write the analogous electrical elements in force-voltage analogy for the elements of mechanical translational system.

Force, f	→ Voltage, e	Mass, M	→ Inductance, L
Velocity, v	→ Current, i	Stiffness, K	→ Inverse of capacitance, 1/C
Displacement, x	→ Charge, q	Newton's second law, $\Sigma f = 0$	→ Kirchoff's voltage law, $\Sigma v = 0$
Frictional coefficient, B	→ Resistance, R		

Q1.28 Write the analogous electrical elements in force-current analogy for the elements of mechanical translational system.

Force, f	→ Current, i	Mass, M	→ Capacitance, C
Velocity, v	→ Voltage, v	Stiffness, K	→ Inverse of Inductance, 1/L
Displacement, x	→ Flux, ϕ	Newton's second law, $\Sigma f = 0$	→ Kirchoff's current law, $\Sigma i = 0$
Frictional coefficient, B	→ Conductance, G = 1/ R		

Q1.29 Write the analogous electrical elements in torque-voltage analogy for the elements of mechanical rotational system.

Torque, T	Voltage, e
Angular velocity, ω	Current, i
Angular displacement, θ	Charge, q
Frictional coefficient, B	Resistance, R
Moment of inertia, J	Inductance, L
Stiffness of spring, K	Inverse of capacitance, 1/C
Newton's second law, $\Sigma T = 0$	Kirchoff's voltage law, $\Sigma v = 0$.

Q1.30 Write the analogous electrical elements in torque-current analogy for the elements of mechanical rotational system.

Torque, T	Current, i
Angular velocity, ω	Voltage, v
Angular displacement, θ	Flux, ϕ
Frictional coefficient, B	Conductance, G=1/R
Moment of inertia, J	Capacitance, C
Stiffness of spring, K	Inverse of inductance, 1/L
Newton's second law, $\Sigma T = 0$	Kirchoff's current law, $\Sigma i = 0$

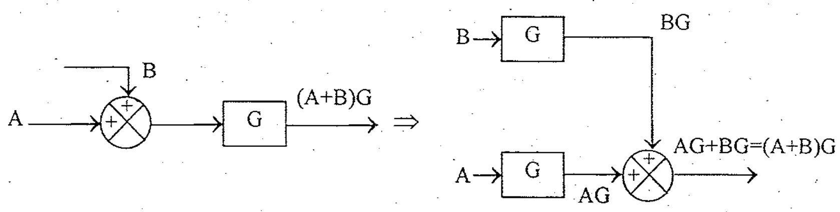
Q1.31 What is block diagram? What are the basic components of block diagram?

A block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals. The basic elements of block diagram are block, branch point and summing point.

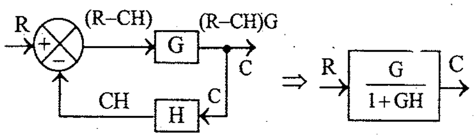
Q1.32 What is the basis for framing the rules of block diagram reduction technique?

The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation.

Q.133 Write the rule for moving the summing point ahead of a block.



Q1.34 *Write the rule for eliminating negative feedback loop.*



Proof

$$C = (R - CH)\,G$$
$$C = RG - CHG$$
$$C + CHG = RG$$
$$C(1+HG) = RG$$
$$\frac{C}{R} = \frac{G}{1+GH}$$

Q1.35 *What is a signal flow graph?*

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

Q1.36 *What is transmittance?*

The transmittance is the gain acquired by the signal when it travels from one node to another node in signal flow graph.

Q1.37 *What is sink and source?*

Source is the input node in the signal flow graph and it has only outgoing branches. Sink is a output node in the signal flow graph and it has only incoming branches.

Q1.38 *Define non-touching loop.*

The loops are said to be non-touching if they do not have common nodes.

Q1.39 *What are the basic properties of signal flow graph?*

The basic properties of signal flow graph are

(i) Signal flow graph is applicable to linear systems.

(ii) It consists of nodes and branches. A node is a point representing a variable or signal. A branch indicates functional dependence of one signal on the other.

(iii) A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

(iv) Signals travel along branches only in the marked direction and is multiplied by the gain of the branch.

(v) The algebraic equations must be in the form of cause and effect relationship.

Q1.40 *Write the Mason's gain formula.*

Mason's gain formula states that the overall gain of the system [transfer function] as follows,

$$\text{Overall gain, } T = \frac{1}{\Delta}\sum_{K} P_K \Delta_K$$

T = T(s) = Transfer function of the system.

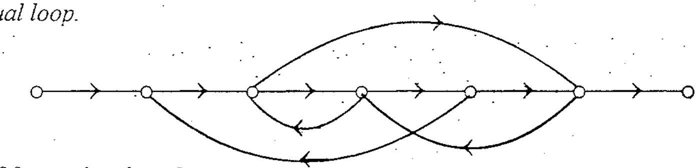
K = Number of forward paths in the signal flow graph.

P_K = Forward path gain of K^{th} forward path.

$$\Delta = 1 - \begin{bmatrix}\text{sum of individual}\\ \text{loop gains}\end{bmatrix} + \begin{bmatrix}\text{sum of gain products of all possible}\\ \text{combinations of two non-touching loops}\end{bmatrix} - \begin{bmatrix}\text{sum of gain products of all possible}\\ \text{combinations of three non-touching loops}\end{bmatrix} + \ldots$$

$$\Delta_K = \begin{bmatrix}\Delta \text{ for that part of the graph which}\\ \text{is not touching } K^{th} \text{ forward path}\end{bmatrix}$$

Q1.41 *For the given signal flow graph, identify the number of forward path and number of individual loop.*



Number of forward paths = 2

Number of individual loops = 4

Q1.42 *What are the basic elements of thermal system.*

The basic elements of thermal system are thermal resistance and thermal capacitance.

Q1.43 *Define thermal resistance.*

The thermal resistance for heat transfer between two substances is defined as the ratio of change in temperature and change in heat flow rate.

$$\text{Thermal resistance, } R = \frac{\text{Change in temperature, } ^\circ C}{\text{Change in heat flow rate, Kcal/sec}}$$

Q1.44 *Define thermal capacitance.*

Thermal capacitance is defined as the ratio of change in heat stored and change in temperature.

$$\text{Thermal capacitance, } C = \frac{\text{Change in heat stored, Kcal}}{\text{Change in temperature, } ^\circ C}$$

Q1.45 *Mention the electrical analogous of simple thermal system.*

The electrical analogous of simple Ist order thermal system is R-C parallel circuit.

Q1.46 *What are the basic elements of hydraulic system.*

The basic elements of hydraulic system are resistance, capacitance and inductance.

Q1.47 *Define hydraulic resistance.*

The resistance for liquid flow is defined as the change in the level difference necessary to cause the unit change in flow rate.

$$R = \frac{\text{Change in level difference, m}}{\text{Change in flow rate, m}^3/\text{sec}}$$

Q1.48 *Define hydraulic capacitance.*

The capacitance C of tank is defined to be the change in quantity of stored liquid necessary to cause a unit change in head (height).

$$C = \frac{\text{Change in liquid stored, m}^3}{\text{Change in head, m}}$$

Q1.49 *What is inertance?*

The inertance represents fluid inertia derived from the inertial forces required to accelerate a fluid in a pipe. It is a energy storing element. The energy storage due to inertance is negligible compared to capacitance element.

Q1.50 *What are the basic elements of pneumatic systems?*

The basic elements of pneumatic systems are pneumatic resistance and pneumatic capacitance.

Q1.51 *Define pneumatic resistance.*

The gas flow resistance, R may be defined as follows

$$R = \frac{\text{Change in gas pressure difference, N/m}^2}{\text{Change in gas flow rate, m}^3/\text{sec}}$$

Q1.52 *Define pneumatic capacitance.*

The capacitance of the pressure vessel may be defined by

$$C = \frac{\text{Change in gas stored, m}^3}{\text{Change in gas pressure, N/m}^2}$$

1.17 EXERCISES

E1.1 For the mechanical system shown in fig E1.1 derive the transfer function. Also draw the force-voltage and force-current analogous circuits.

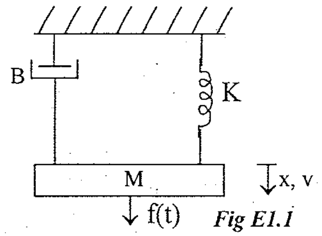


Fig E1.1

E1.2 For the mechanical system shown in fig E1.2 draw the force-voltage and force-current analogous circuits.

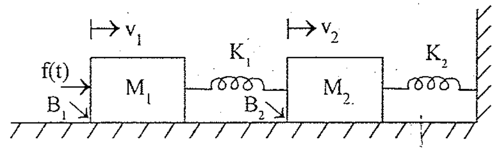


Fig E1.2

E1.3 Write the differential equations governing the mechanical system shown in fig E1.3. Also draw the force-voltage and force-current analogous circuit.

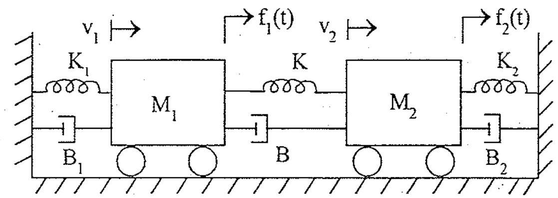


Fig E1.3

E1.4 Write the differential equations governing the mechanical system shown fig E1.4. Draw the force-voltage and force-current analogous circuits.

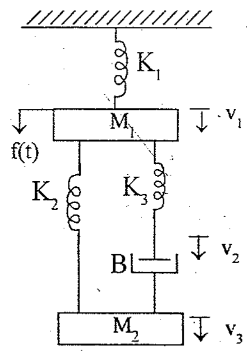


Fig E1.4

E1.5 Consider the mechanical translational system shown in fig E1.5, Draw(a) force-voltage and (b) force-current analogous circuits.

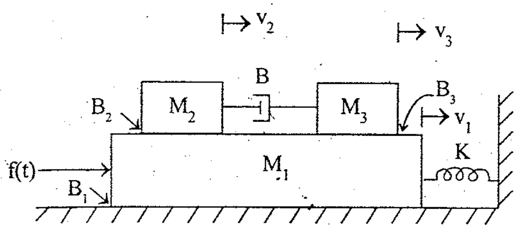


Fig E1.5

E1.6 Write the differential equations governing the rotational mechanical system shown in fig E1.6. Also draw the torque-voltage and torque-current analogous circuits.

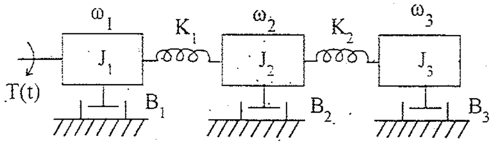


Fig E 1.6

E1.7 In an electrical circuit the elements resistance, capacitance and inductance are connected in parallel across the voltage source E as shown in fig E1.7, Draw(a) Translation mechnical analogous system (b) Rotational mechanical analogous system.

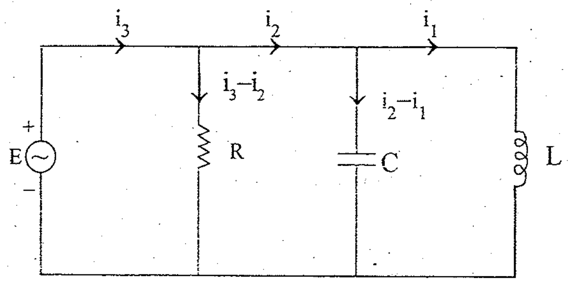


Fig E1.7.

E1.8 Consider the block diagram shown in fig E1.8. Using the block diagram reduction technique, find C/R

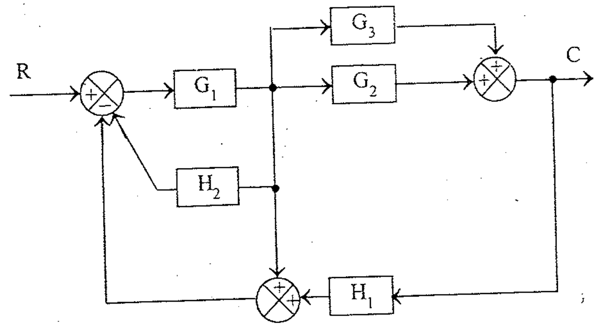


Fig E1.8

E1.9 Consider the block diagram shown in fig E1.9. Using the block diagram reduction technique, obtain C/R.

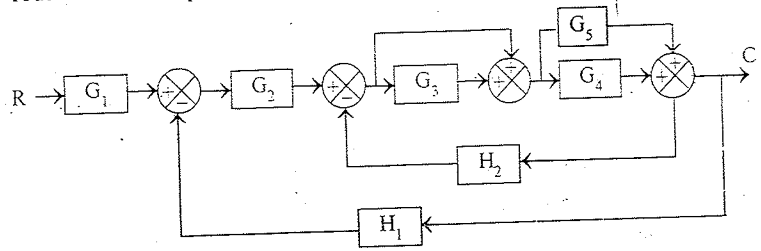


Fig E1.9

E.1.10 For the block diagram shown in fig E1.10. obtain C/R.

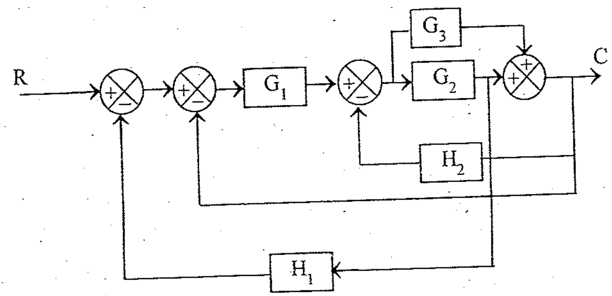


Fig E1.10

E.1.11 For the block diagram shown in fig E1.11.obtain C/R

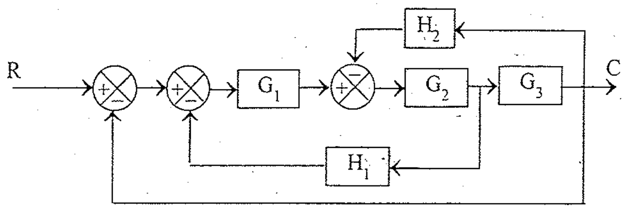


Fig E1.11

E1.12 Consider the system shown in fig E1.12. obtain the transfer function by use of Mason's gain formula.

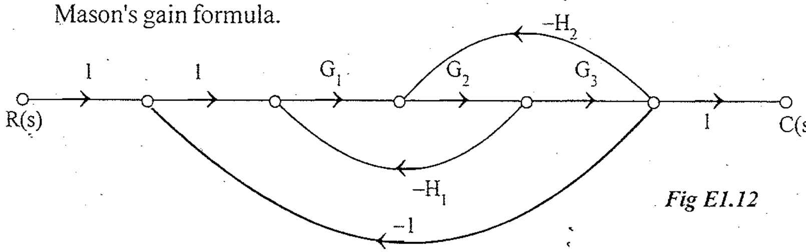


Fig E1.12

E1.13 Convert the block diagram shown in fig E1.13 to signal flow graph and find the transfer function of the system.

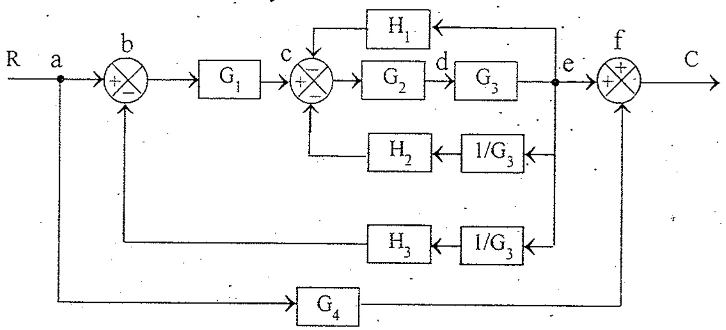


Fig E1.13

E1.14 Find the transfer function of the system whose signal flow graph is shown in fig E1.14.

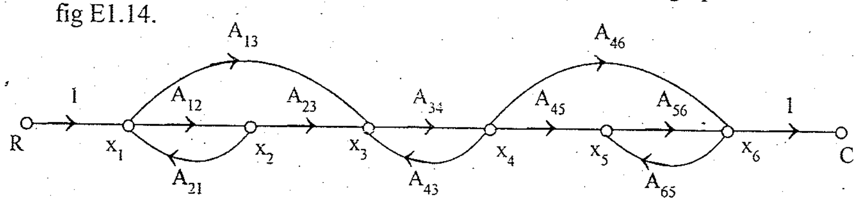


Fig E1.14

E1.15 Find the transfer function of the system having signal flow graph as shown in fig E1.15.

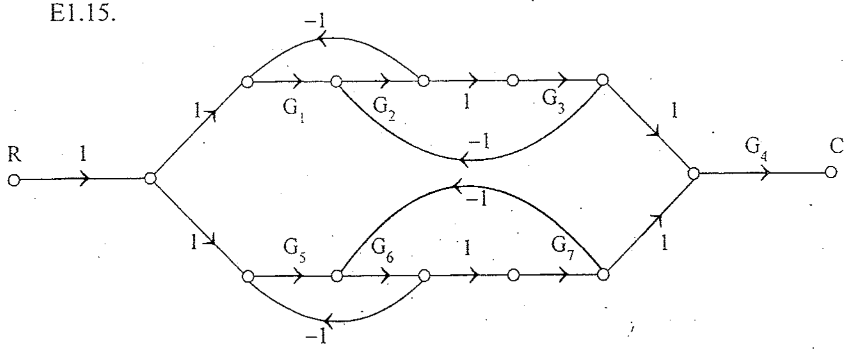


Fig E1.15

E1.16 Consider the signal flow graph shown fig E1.16. Find $\frac{X_8}{X_1}$

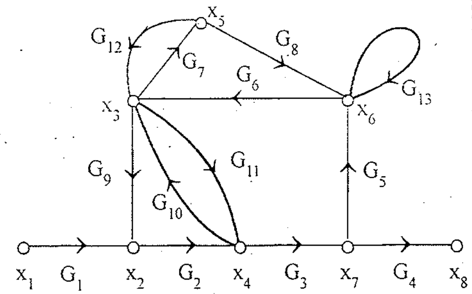


Fig E1.16

E1.17 Consider the signal flow graph shown in fig E1.17 obtain $\frac{x_8}{x_1}$ and $\frac{x_8}{x_2}$

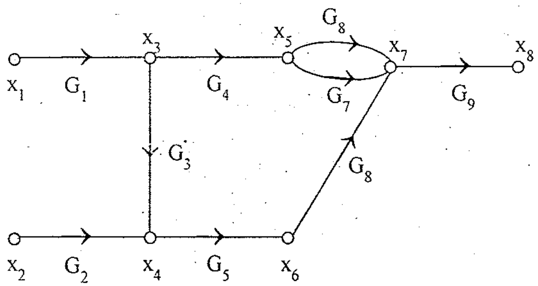


Fig E1.17

E1.18 Find the transfer functions of the networks shown in fig E1.18.

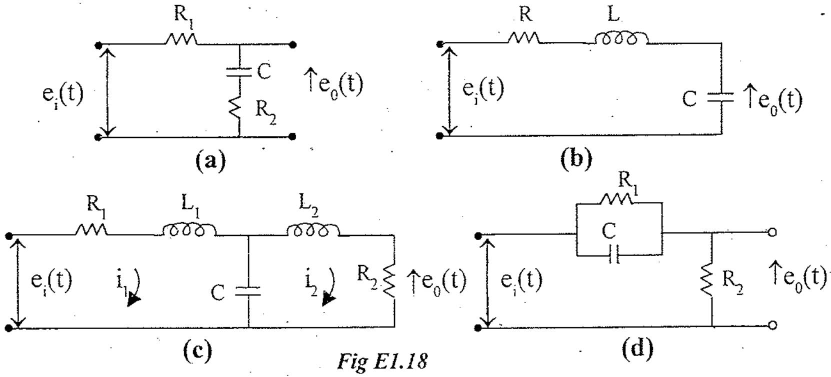


Fig E1.18

E1.19 Find the transfer function of the circuit shown in fig E1.19

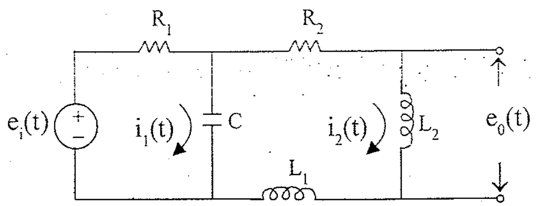


Fig E1.19

E1.20 A process plant consists of two tanks of capacitances C_1 and C_2 respectively. If the flow rate into the top tank is Q_1, find the transfer function relating this flow with the liquid level in the bottom tank. Each tank has a resistance R in its outlet pipe. Assume tanks to be non-interacting.

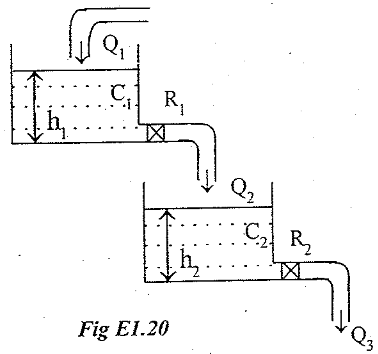
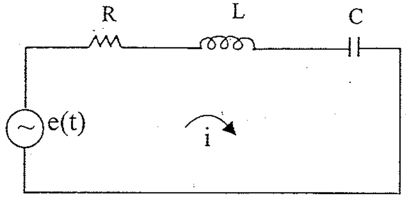


Fig E1.20

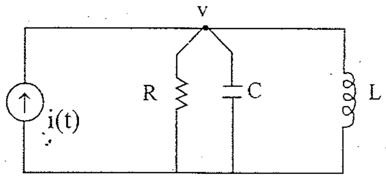
ANSWER FOR EXERCISE PROBLEMS

E1.1 The transfer function is $\dfrac{X(s)}{F(s)} = \dfrac{1}{(Ms^2 + Bs + K)}$



f(t)→e(t) M→L
v→i B→R
K→1/C

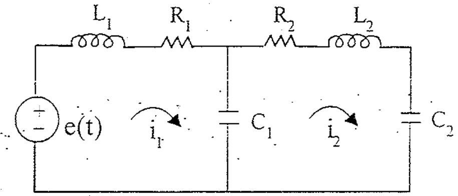
Force-voltage analogous circuit



f(t)→i(t) M→C
v→v B→1/R
K→1/L

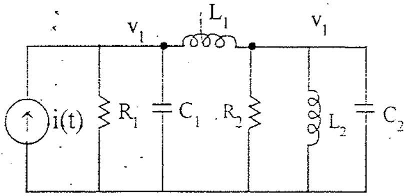
Force-current analogous circuit

E1.2



f(t)→e(t) $M_1 \to L_1$ $B_2 \to R_2$
$v_1 \to i_1$ $M_2 \to L_2$ $K_1 \to 1/C_1$
$v_2 \to i_2$ $B_1 \to R_1$ $K_2 \to 1/C_2$

Force-voltage analogous circuit



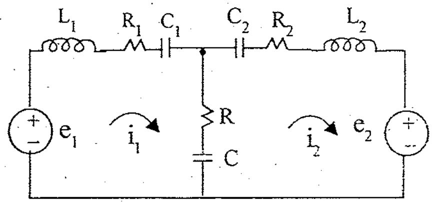
f(t)→i(t) $M_1 \to C_1$ $B_2 \to 1/R_2$
$v_1 \to v_1$ $M_2 \to C_2$ $K_1 \to 1/L_1$
$v_2 \to v_2$ $B_1 \to 1/R_1$ $K_2 \to 1/L_2$

Force-current analogous circuit

E1.3

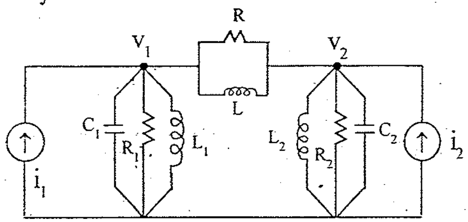
$$M_1\frac{dv_1}{dt} + B_1v_1 + B(v_1 - v_2) + K_1\int v_1dt + K\int (v_1 - v_2)dt = f_1(t)$$

$$M_2\frac{dv_2}{dt} + B_2v_2 + B(v_2 - v_1) + K_2\int v_2dt + K\int (v_2 - v_1)dt = f_2(t)$$



$f_1 \to e_1$ $M_1 \to L_1$ $B \to R$
$f_2 \to e_2$ $M_2 \to L_2$ $K_1 \to 1/C_1$
$v_1 \to i_1$ $B_1 \to R_1$ $K_2 \to 1/C_2$
$v_2 \to i_2$ $B_2 \to R_2$ $K \to 1/C$

Force-voltage analogous circuit



$f_1 \to i_1$ $M_1 \to C$ $B \to 1/R$
$f_2 \to i_2$ $M_2 \to C_2$ $K_1 \to 1/L_1$
$v_1 \to v_1$ $B_1 \to 1/R_1$ $K_2 \to 1/L_2$
$v_2 \to v_2$ $B_2 \to 1/R_2$ $K \to 1/L$

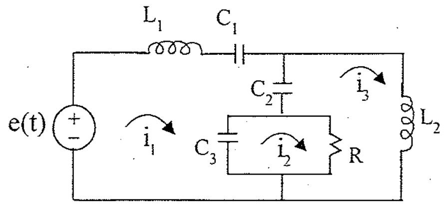
Force-current analogous circuit

E1.4

$$M_1\frac{dv_1}{dt}+K_1\int v_1dt+K_2\int(v_1-v_3)dt+K_3\int(v_1-v_2)dt=f(t)$$

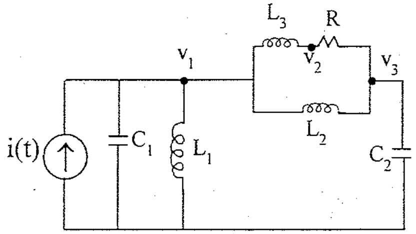
$$K_3\int(v_2-v_1)dt+B(v_2-v_3)=0$$

$$M_2\frac{dv_3}{dt}+B(v_3-v_2)+K_2\int(v_3-v_1)dt=0$$



$f(t)\to e(t)$	$M_1\to L_1$	$K_1\to 1/C_1$
$v_1\to i_1$	$M_2\to L_2$	$K_2\to 1/C_2$
$v_2\to i_2$	$B\to R$	$K_3\to 1/C_3$
$v_3\to i_3$		

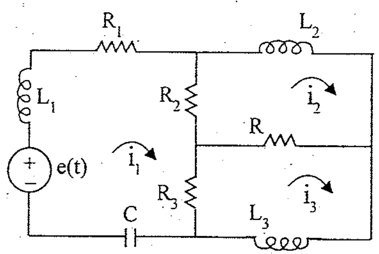
Force-voltage analogous circuit



$f(t)\to i(t)$	$M_1\to C_1$	$K_1\to 1/L_1$
$v_1\to v_1$	$M_2\to C_2$	$K_2\to 1/L_2$
$v_2\to v_2$	$B\to 1/R$	$K_3\to 1/L_3$
$v_3\to v_3$		

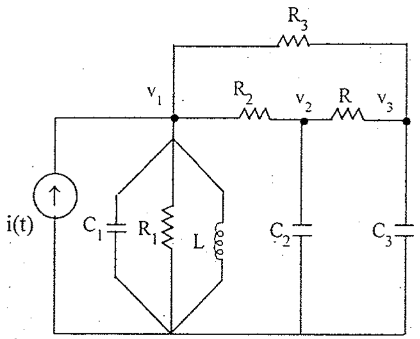
Force-current analogous circuit

E1.5



$f(t)\to e(t)$	$M_1\to L_1$	$B_1\to R_1$
$v_1\to i_1$	$M_2\to L_2$	$B_2\to R_2$
$v_2\to i_2$	$M_3\to L_3$	$B_3\to R_3$
$v_3\to i_3$	$B\to R$	$K\to 1/C$

Force voltage-analogous circuit



$f(t)\to i(t)$	$M_1\to C_1$	$B_1\to 1/R_1$
$v_1\to v_1$	$M_2\to C_2$	$B_2\to 1/R_2$
$v_2\to v_2$	$M_3\to C_3$	$B_3\to 1/R_3$
$v_3\to v_3$	$B\to 1/R$	$K\to 1/L$

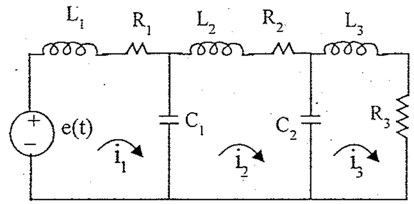
Force-current analogous circuit

E1.6

$$J_1\frac{d\omega_1}{dt}+B_1\omega_1+K_1\int(\omega_1-\omega_2)dt=T(t)$$

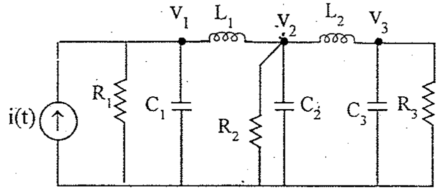
$$J_2\frac{d\omega_2}{dt}+B_2\omega_2+K_1\int(\omega_2-\omega_1)dt+K_2\int(\omega_2-\omega_3)dt=0$$

$$J_3\frac{d\omega_3}{dt}+B_3\omega_3+K_2\int(\omega_3-\omega_2)dt=0$$



$T(t)\to e(t)$	$J_1\to L_1$	$B_1\to R_1$	$K_1\to 1/C_1$
$\omega_1\to i_1$	$J_2\to L_2$	$B_2\to R_2$	$K_2\to 1/C_2$
$\omega_2\to i_2$	$J_3\to L_3$	$B_3\to R_3$	
$\omega_3\to i_3$			

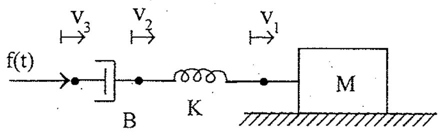
Torque-voltage analogous circuit



$T(t)\to i(t)$	$J_1\to C_1$	$B_1\to 1/R_1$	$K_1\to 1/L_1$
$\omega_1\to v_1$	$J_2\to C_2$	$B_2\to 1/R_2$	$K_2\to 1/L_2$
$\omega_2\to v_2$	$J_3\to C_3$	$B_3\to 1/R_3$	
$\omega_3\to v_3$			

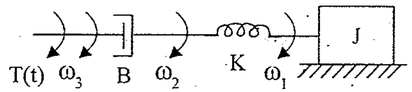
Torque-current analogous circuit

E1.7



$e(t)\to f(t)$	$R\to B$
$i_1\to v_1$	$1/C\to K$
$i_2\to v_2$	$L\to M$
$i_3\to v_3$	

Analogous mechanical translational system



$e(t)\to T(t)$	$R\to B$
$i_1\to\omega_1$	$1/C\to K$
$i_2\to\omega_2$	$L\to J$
$i_3\to\omega_3$	

Analogous mechanical rotational system

E1.8 $$\frac{C}{R}=\frac{G_1G_2+G_1G_3}{1+G_1H_2+G_1+G_1G_2H_1+G_1G_3H_1}$$

E1.9 $$\frac{C}{R}=\frac{G_1G_2(1+G_3)\,(G_4+G_5)}{[1+(1+G_3)\,(G_4+G_5)\,H_2+(1+G_3)\,(G_4+G_5)\,G_2H_1]}$$

E1.10 $$\frac{C}{R}=\frac{G_1G_2+G_1G_3}{[1+G_1G_2H_1+G_1G_2+G_1G_3+G_2H_2+G_3H_2]}$$

E1.11 $\dfrac{C}{R} = \dfrac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3}$

E1.12 $\dfrac{C(s)}{R(s)} = \dfrac{G_1G_2G_3}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3}$

E1.13 $\dfrac{G_1G_2G_3+G_4+G_2G_3G_4H_1+G_2G_4H_2+G_1G_2G_4H_3}{1+G_2G_3H_1+G_2H_2+G_1G_2H_3}$

E1.14

$$\frac{C}{R} = \frac{A_{12}A_{23}A_{34}A_{45}A_{56}+A_{13}A_{34}A_{45}A_{56}+A_{12}A_{23}A_{34}A_{46}+A_{13}A_{34}A_{46}}{1-(A_{12}A_{21}+A_{34}A_{43}+A_{56}A_{65})+(A_{12}A_{21}A_{34}A_{43}+A_{12}A_{21}A_{56}A_{65}+A_{34}A_{43}A_{56}A_{65})-(A_{12}A_{21}A_{34}A_{43}A_{56}A_{65})}$$

E1.15

$$\frac{C}{R} = \frac{G_1G_2G_3G_4\,(1+G_5G_6+G_6G_7)+G_4G_5G_6G_7(1+G_1G_2+G_2G_3)}{1+G_1G_2+G_2G_3+G_5G_6+G_6G_7+G_1G_2G_5G_6+G_5G_6G_2G_3+G_1G_2G_6G_7+G_2G_3G_6G_7}$$

E1.16

$$\frac{x_8}{x_1} = \frac{[G_1G_2G_3G_4]\,[1-(G_7G_{12}+G_6G_7G_8+G_{13})+G_7G_{12}G_{13}]}{1-\{G_2G_9G_{10}+G_{10}G_{11}+G_2G_3G_5G_6G_9+G_3G_5G_6G_{11}+G_7G_{12}+G_6G_7G_8+G_{13}\}+G_2G_9G_{10}G_{13}+G_{10}G_{11}G_{13}+G_7G_{12}G_{13}}$$

E1.17 $\dfrac{x_8}{x_1} = G_1G_4G_6G_9+G_1G_4G_7G_9+G_1G_3G_5G_8G_9$

$\dfrac{x_8}{x_2} = G_2G_5G_8G_9$

E1.18 Transfer function.

a) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1+sR_2C}{1+s(R_1+R_2)C}$ (b) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1}{s^2LC+sRC+1}$

(c) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{sR_2C}{(s^2L_1C+sR_1C+1)\,(s^2L_2C+sR_2C+1)-1}$

(d) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{sR_1R_2C+R_2}{sR_1R_2C+(R_1+R_2)}$

E1.19 $\dfrac{C(s)}{E(s)} = \dfrac{s^2L_2C}{[sR_1C+1]\left[s^2(L_1+L_2)C+sR_2C+1\right]-1}$

E1.20 $\dfrac{H_2(s)}{Q_1(s)} = \dfrac{R_2}{(1+T_1s)(1+T_2s)}$ Where $T_1 = R_1C_1$ and $T_2 = R_2C_2$

CHAPTER TWO

2 COMPONENTS OF CONTROL SYSTEM

2.1 COMPONENTS OF AUTOMATIC CONTROL SYSTEM

The basic components of an automatic control system are Error detector, Amplifier and Controller, Actuator (Power actuator), Plant and Sensor or Feedback system. The block diagram of an automatic control system is shown in fig 2.1.

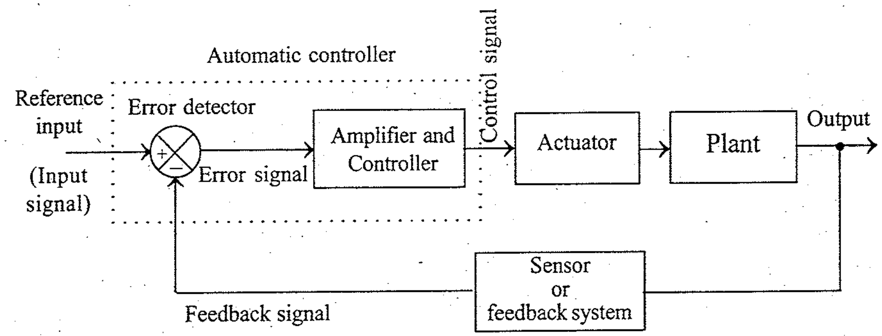


Fig 2.1: Block diagram of automatic control system

The plant is the open loop system whose output is automatically controlled by closed loop system. The combined unit of error detector, amplifier and controller is called automatic controller, because without this unit the system becomes open loop system.

In automatic control systems the reference input will be an input signal proportional to desired output. The feedback signal is a signal proportional to current output of the system. The error detector compares the reference input and feedback signal and if there is a difference it produces an error signal. An amplifier can be used to amplify the error signal and the controller modifies the error signal for better control action.

The actuator amplifies the controller output and converts to the required form of energy that is acceptable for the plant. Depending on the input to the plant, the output will change. This process continues as long as there is a difference between reference input and feedback signal. If the difference is zero, then there is no error signal and the output settles at the desired value.

The function of error detector is to compare the reference input with feedback signal, to produce an error signal if there is a difference between them. The error signal is used to correct the output if there is a deviation from the desired value. Examples of error detector are potentiometer, LVDT (Linearly Variable Differential Transformer), Synchros, etc.

Generally the error signal will be a weak signal and so it has to be amplified and then modified for better control action. In most of the system the controller itself amplifies the error signal and integrates or differentiates to produce a control signal (i.e. modified error signal). The different types of controllers are P,PI, PD and PID controllers.

The controllers employed may be electrical, electronic, hydraulic or pneumatic depending on the nature of error signal. If the error signal is electrical then the controller may be electrical or electronic and they are designed using R-C circuit or operational amplifiers. If the error signal is mechanical then the controller may be hydraulic or pneumatic and they are designed using hydraulic servomotors or pneumatic flapper valves.

The actuator is a power amplifying device that produces the input to the plant according to the control signal. The actuator may be a pneumatic motor/valve, hydraulic motor or electric motor. Examples of electric motors employed as actuator are DC servomotor, AC servomotor and stepper motor.

The feedback system samples the output to produce a feedback signal which is proportional to current output. Also the feedback system should convert the output variable into another suitable variable such as displacement, pressure or voltage, so that it can be used to compare with the reference input. Usually the feedback system consists of sensor and associated circuit/devices. Transducers, Tachogenerators, etc., are used as feedback systems.

Many transducers like thermocouples, photo electric cells produce dc voltages proportional to the quantity to be measured. If this signal is to actuate an ac system, then this dc signal has to be converted to an ac signal before it is applied to ac systems. The device which transfers the information available in a dc signal to ac signal is called a modulator and the convertion process is called **modulation.** In modulation the control

signal is superimposed on a high frequency carrier. Certain control system components like ac tachogenerators produces a modulated output signal. Hence the information is obtained by demodulation. The device which is used to extract the information available on a high frequency carrier is called demodulator and the process is called **demodulation.** In demodulation the control signal is extracted from the carrier signal.

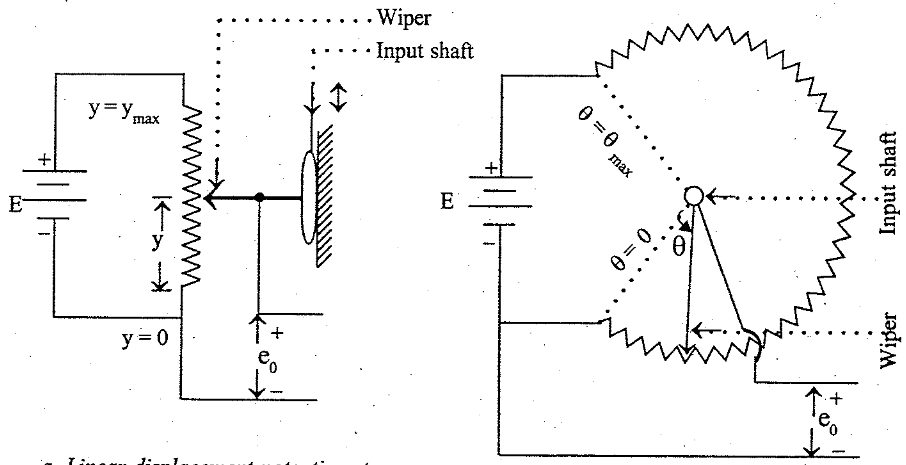
Sometimes power transmitters like transformers, levers and gear trains are employed in control system. The function of transformer is to alter the voltage and current level of the power being transmitted. The function of lever is to alter the force and that of gear train is to alter the torque of the power being transmitted.

2.2 POTENTIOMETERS

A potentiometer is a device that can be used to convert a linear or angular displacement into a voltage. A potentiometer is a variable resistance whose value varies according to the angular/linear displacement of the wiper contact (movable contact).

The resistance element can be constructed by winding resistance wire on a former or by depositing a conducting material on a plastic base. The potentiometer has an input shaft to which a wiper is attached. The displacement is applied to the input shaft. When the shaft moves, the wiper contact slides over the resistance material.

The potentiometer is excited by a dc or ac voltage. The output voltage is measured at wiper contact with respect to reference. The linear and angular displacement potentiometer are shown in fig 2.2.



a. Linear displacement potentiometer *b. Angular displacement potentiometer*

Fig 2.2 : *Potentiometers*

LINEAR DISPLACEMENT POTENTIOMETER

y = Displacement of the wiper contact from reference ($y = 0$)

y_{max} = Maximum displacement of wiper contact

E = Excitation voltage of potentiometer

e_o = Output voltage.

If $y = 0$, then $e_o = 0$

If $y = y_{max}$, then $e_o = E$

$\therefore$ For a displacement y, the output voltage, $e_0 = \dfrac{E}{y_{max}} y = K_p y$ .....(2.1)

Where $K_p = \dfrac{E}{y_{max}}$ = sensitivity of the potentiometer in volts/mm

ANGULAR DISPLACEMENT POTENTIOMETER

θ = Angular displacement of wiper arm from reference ($\theta = 0$)

θ_{max} = Maximum displacement of wiper arm

E = Excitation voltage of potentiometer

e_o = Output voltage.

If $\theta = 0$, then $e_0 = 0$

If $\theta = \theta_{max}$, then $e_o = E$

$\therefore$ For a displacement θ, the output voltage, $e_0 = \dfrac{E}{\theta_{max}}\theta = K_p\theta$ .....(2.2)

Where $K_p = \dfrac{E}{\theta_{max}}$ = sensitivity of the potentiometer in volts/deg.

Note : *K_p is also called gain or gain constant of potentiometer.*

CHARACTERISTICS OF POTENTIOMETER

1. The ideal characteristics of a potentiometer is linear variation of resistance with displacement. This is best realised by having very large radius, more number of turns and high resistance elements.

2. The device which measures the output voltage of potentiometer should have high input impedance to avoid loading error. If necessary an isolation amplifier with high input impedance may be used.

3. When the wiper slides over resistance it makes simultaneous contact with adjacent turns to avoid discontinuity in output. Consequently the output is in the form of staircase steps. Hence we define a term called resolution which specifies the output voltage per step. The resolution of a potentiometer is defined as the ratio of number of steps to total number of turns. The resolution of potentiometer is an important factor in the determination of minimum value of output voltage.

SPECIFICATIONS OF POTENTIOMETER

The specifications of potentiometers used in control system are the following.

1. Turns per unit length is in the range of 5 to 30 turns per mm.
2. Torque required for wiper movement is in the range of 1×10^{-3} Kg-m to 1×10^{-2} Kg-m.
3. The total resistance of the potentiometer is in the range of 25 ohms to 1 mega-ohms.
4. Power rating is 1 to 10 watts.
5. Heat dissipation is 1/2 watts per cm^2.
6. Excitation voltage is 4 to 20 volts.
7. Voltage gradient is 0.01 to 0.05 volt per degree.

AC POTENTIOMETER

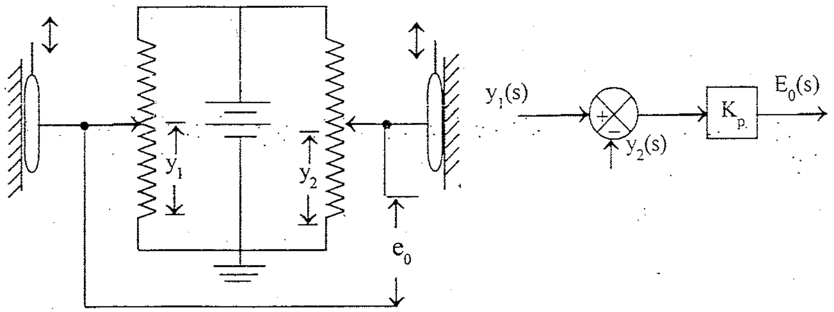
In potentiometers excited by ac supply the output will be a modulated voltage. The carrier is the excitation voltage. The envelope of the carrier is modulated by the movement of the wiper arm. Hence the information is available in the envelope of the carrier. The ac potentiometer will have inductive effect in addition to resistance which leads to difficulty in balancing the potentiometers used as error detectors.

APPLICATIONS OF POTENTIOMETERS

Potentiometers can be used either to convert a mechanical motion to proportional voltage or as an error signal. A single potentiometer excited by dc or ac voltage is used to produce an output voltage proportional to displacement of the input shaft.

When potentiometers are used as error detector, two identical potentiometers are required as shown in fig 2.3. Both the potentiometer are excited by the same source and at same potential. Hence, if the wiper arm of both the potentiometers are in the same position then the voltage between two wiper arms is zero.

The position of one wiper arm is kept as reference input. The displacement to be compared is applied to the wiper arm of another potentiometer. Hence the output voltage which is measured between two wiper arms is proportional to the difference between the displacement of both the wiper arms.



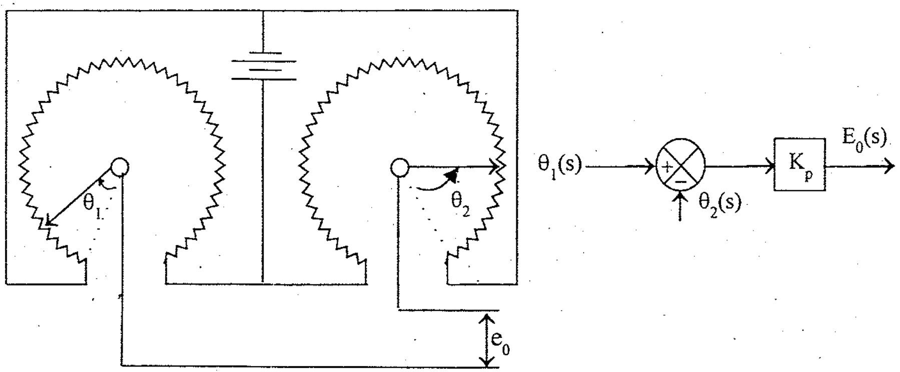


Fig 2.3 : Potentiometers as error detectors

2.3 SYNCHROS

The term synchro is a generic name for a family of inductive devices which works on the principle of a rotating transformer(Induction motor). The trade names for synchros are Selsyn, Autosyn and Telesyn. Basically they are electro-mechanical devices or electromagnetic transducers which produces an output voltage depending upon angular position of the rotor.

A synchro system is formed by interconnection of the devices called the synchro transmitter and the synchro control transformer. They are also called as synchro pair. The synchro pair measures and compares two angular displacements and its output voltage is approximately linear with angular difference of the axis of both the shafts. They can be used in the following two ways,

1. To control the angular position of load from a remote place/long distance.
2. For automatic correction of changes due to disturbance in the angular position of the load.

SYNCHRO TRANSMITTER

Construction

The constructional features, electrical circuit and a schematic symbol of synchro transmitter are shown in fig 2.4. The two major parts of synchro transmitter are stator and rotor. The stator is identical to the stator of three phase alternator. It is made of laminated silicon steel and slotted on the inner periphery to accommodate a balanced three phase winding. The stator winding is concentric type with the axis of three coils 120° apart. The stator winding is star connected (Y-connection)

The rotor is of dumb bell construction with a single winding. The ends of the rotor winding are terminated on two slip rings. A single phase ac excitation voltage is applied to the rotor through the slip rings.

Working principle

When the rotor is excited by ac voltage, the rotor current flows, and a magnetic field is produced. The rotor magnetic field induces an emf in the stator coils by transformer action. The effective voltage induced in any stator coil depends upon the angular position of the coil's axis with respect to rotor axis.

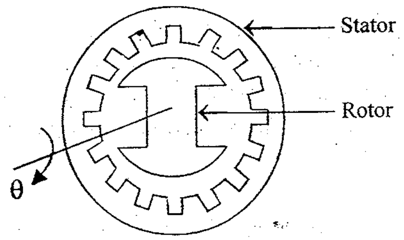


Fig a : Constructional features

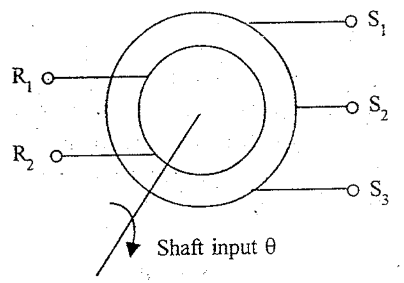


Fig b : Schematic symbol of a synchro transmitter

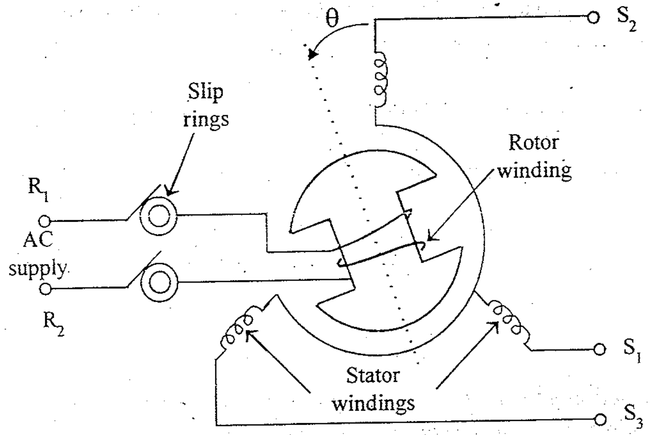


Fig c : Electrical circuit

Fig 2.4 : Synchro transmitter

Let e_r = Instantaneous value of ac voltage applied to rotor

e_{S1}, e_{S2}, e_{S3} = Instantaneous value of emf induced in stator coils s_1, s_2, s_3 with respect to neutral respectively.

E_r = Maximum value of rotor excitation voltage

ω = Angular frequency of rotor excitation voltage.

K_t = Turns ratio of stator and rotor windings.

K_c = Coupling coefficient

θ = Angular displacement of rotor with respect to reference

$\therefore$ The instantaneous value of excitation voltage, $e_r = E_r \sin \omega t$ .....(2.3)

Let the rotor rotates in anticlockwise direction. When the rotor rotates by an angle, θ emfs are induced in stator coils. The frequency of induced emf is same as that of rotor

frequency. The magnitude of induced emfs are proportional to the turns ratio and coupling coefficient. The turns ratio, K is a constant, but coupling coefficient, K_c is a function of rotor angular position.

$\therefore$ Induced emf in stator coil $= K_t K_c E_r \sin\omega t$ .....(2.4)

Let e_{S2} be reference vector. With reference to fig 2.4, when θ = 0, the flux linkage of coil s_2 is maximum and when θ = 90°, the flux linkage of coil s_2 is zero. Hence the flux linkage of coil s_2 is function of cosθ (i.e, $K_c = K_1 \cos\theta$ for coil s_2). The flux linkage of coil s_3 will be maximum after a rotation of 120° in anticlockwise direction and that of s_1 after a rotation of 240°.

$\therefore$ Coupling coefficient, K_c for coil-s_2 = $K_1 \cos\theta$

Coupling coefficient, K_c for coil-s_3 = $K_1 \cos(\theta - 120°)$

Coupling coefficient, K_c for coil-s_1 = $K_1 \cos(\theta - 240°)$

Hence the emfs of stator coils with respect to neutral can be expressed as follows.

$$e_{S2} = K_t K_1 \cos\theta\, E_r \sin\omega t = K\, E_r \cos\theta \sin\omega t \quad .....(2.5)$$

$$e_{S3} = K_t K_1 \cos(\theta - 120°)\, E_r \sin\omega t = K\, E_r \cos(\theta - 120°) \sin\omega t \quad .....(2.6)$$

$$e_{S1} = K_t K_1 \cos(\theta - 240°)\, E_r \sin\omega t = K\, E_r \cos(\theta - 240°) \sin\omega t \quad .....(2.7)$$

Where, $K = K_t K_1$

With reference to fig 2.5 by kirchoff's voltage law the coil-to-coil emf can be expressed as

$$e_{S1S2} = e_{S1} - e_{S2} = \sqrt{3}\, KE_r \sin(\theta + 240°) \sin\omega t \quad .....(2.8)$$

$$e_{S2S3} = e_{S2} - e_{S3} = \sqrt{3}\, KE_r \sin(\theta + 120°) \sin\omega t \quad .....(2.9)$$

$$e_{S3S1} = e_{S3} - e_{S1} = \sqrt{3}\, KE_r \sin\theta \sin\omega t \quad .....(2.10)$$

$$\boxed{\begin{aligned} e_{S1S2} &= e_{S1} - e_{S2} = K\,E_r \cos(\theta - 240°)\sin\omega t - K\,E_r \cos\theta \sin\omega t \\ &= K\,E_r \left[\cos\theta\cos 240° + \sin\theta \sin 240° - \cos\theta\right]\sin\omega t \\ &= K\,E_r \left[\cos\theta\,(-0.5) + \sin\theta\left(-\frac{\sqrt{3}}{2}\right) - \cos\theta\right]\sin\omega t \\ &= \sqrt{3}\,K\,E_r \left[\sin\theta\left(-\frac{1}{2}\right) + \cos\theta\left(-\frac{\sqrt{3}}{2}\right)\right]\sin\omega t \\ &= \sqrt{3}\,K\,E_r \left[\sin\theta\cos 240° + \cos\theta \sin 240\right]\sin\omega t \\ &= \sqrt{3}\,KE_r \sin(\theta + 240°)\sin\omega t \end{aligned}}$$

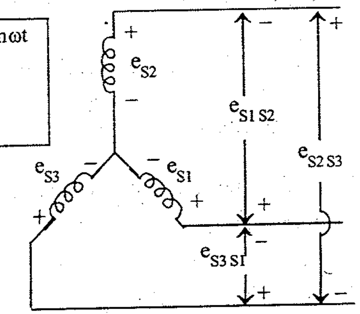


Fig 2.5 : Induced emf in stator coils

$$e_{S2S3} = e_{S2} - e_{S3} = K\,E_r \cos\theta \sin\omega t - K\,E_r \cos(\theta - 120^\circ)\sin\omega t$$
$$= K\,E_r \left[\cos\theta - \cos\theta\cos 120^\circ - \sin\theta \sin 120^\circ\right]\sin\omega t$$
$$= K\,E_r \left[\cos\theta - \cos\theta(-0.5) - \sin\theta\left(\frac{\sqrt{3}}{2}\right)\right]\sin\omega t$$
$$= \sqrt{3}\,K\,E_r \left[\sin\theta\left(-\frac{1}{2}\right) + \cos\theta\left(\frac{\sqrt{3}}{2}\right)\right]\sin\omega t$$
$$= \sqrt{3}\,K\,E_r \left[\sin\theta\cos 120^\circ + \cos\theta \sin 120\right]\sin\omega t$$
$$= \sqrt{3}\,K E_r \sin(\theta + 120^\circ)\sin\omega t$$

$$e_{S3S1} = e_{S3} - e_{S1} = K\,E_r \cos(\theta - 120^\circ)\,\sin\omega t - K\,E_r \cos(\theta - 240^\circ)\sin\omega t$$
$$= K\,E_r \left[\cos\theta\,\cos 120^\circ + \sin\theta \sin 120^\circ - \cos\theta\cos 240^\circ - \sin\theta \sin 240^\circ\right]\sin\omega t$$
$$= K\,E_r \left[\cos\theta(-0.5) + \sin\theta\left(\frac{\sqrt{3}}{2}\right) - \cos\theta(-0.5) - \sin\theta\left(-\frac{\sqrt{3}}{2}\right)\right]\sin\omega t$$
$$= \sqrt{3}\,K\,E_r \sin\theta \sin\omega t$$

When $\theta = 0$, from equation 2.5 we can say that maximum emf is induced in coil s_2. But from equation 2.10 it is observed that the coil-to-coil voltage e_{S3S1} is zero. This position of the rotor is defined as the electrical zero of the transmitter. The electrical zero position is used as reference for specifying the angular position of the rotor.

The input to the synchro transmitter is the angular position of its rotor shaft and the output is a set of three stator coil-to-coil voltages. By measuring and identifying the set of voltages at the stator terminals, it is possible to identify the angular position of the rotor. [A device called synchro/digital converter is available to measure the stator voltages and to calculate the angular measure and then display the direction and angle of rotation of the rotor].

SYNCHRO CONTROL TRANSFORMER

Construction

The constructional features of synchro control transformer is similar to that of synchro transmitter, except the shape of rotor. The rotor of the control transformer is made cylindrical so that the air gap is practically uniform. This feature of the control transformer minimizes the changes in the rotor impedance with the rotation of the shaft. The constructional features, electrical circuit and a schematic symbol of control transformer are shown in fig 2.6.

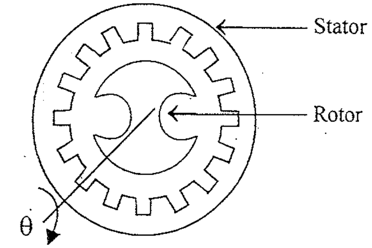


Fig a : Constructional features

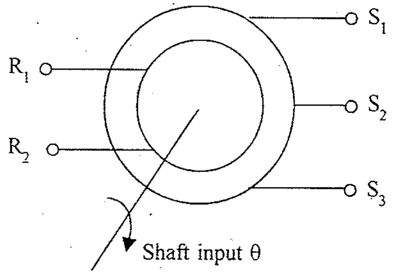


Fig b : Schematic symbol of a synchro control transformer

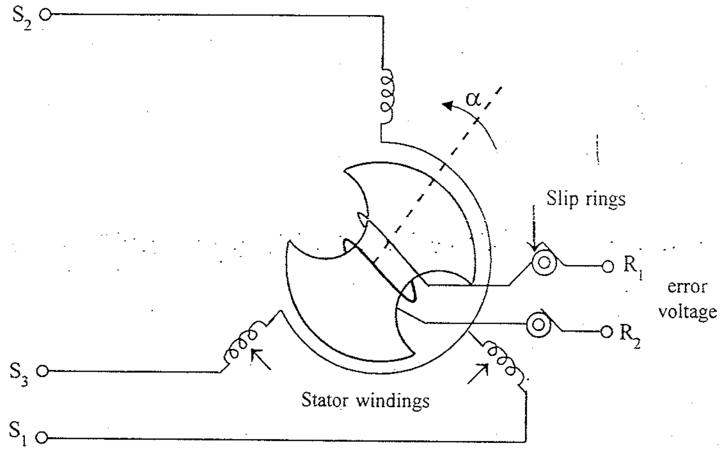


Fig c : Electrical circuit

Fig 2.6 : Synchro control transformer

Working

The generated emf of the synchro transmitter is applied as input to the stator coils of control transformer. The rotor shaft is connected to the load whose position has to be maintained at the desired value. Depending on the current position of the rotor and the applied emf on the stator, an emf is induced on the rotor winding. This emf can be measured and used to drive a motor so that the position of the load is corrected.

SYNCHRO AS ERROR DETECTOR

The synchro error detector is formed by interconnection of a synchro transmitter and synchro control transformer. In this arrangement the stator leads of the transmitter are directly connected to the stator leads of the control transformer. The angular position of the transmitter-rotor is the reference input (or the input corresponding to the desired output) and the rotor is excited by ac supply with frequency ω.

The control transformer rotor is connected to a servo motor and to the shaft of the load, whose position is the desired output. The induced emf (error voltage) available across the rotor slip rings of control transformer is measured by a signal conditioning circuit. The output of signal conditioning circuit is used to drive motor so that desired load position is achieved. A simple schematic diagram of synchros as error detector is shown in fig 2.7.

Initially the shafts of transmitter and control transformer are assumed to be in aligned position. In this position the transmitter rotor will be in electrical zero position and the control transformer rotor will be in null position and the angular separation of both rotor axis in aligned position is 90°. The null position of a control transformer in a servo system is that position of its rotor for which the output voltage on the rotor winding is zero with the transmitter in its electrical zero position.

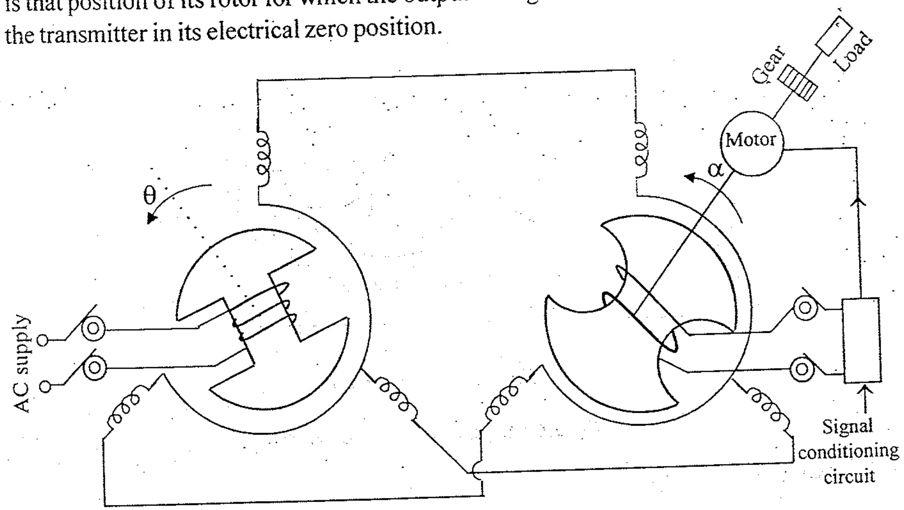


Fig 2.7 : Servo system using synchro error detector

When, the transmitter rotor is excited, the rotor flux is set-up and emfs are induced in stator coils. These induced emfs are impressed on the stator coils of control transformer. The currents in the stator coils set up flux in control transformer. Due to the similarity in the magnetic construction, the flux patterns produced in the two synchros will be the same if all losses are neglected. The flux patterns are shown in fig 2.8.

Let the rotor of the transmitter rotate through an angle θ from its electrical zero position. Now the rotor of the control transformer will rotate in the same direction through an angle α from its null position. The net angular separation of the two rotors is equal to $(90 - \theta + \alpha)$ and the voltage induced in the control transformer rotor is proportional to the cosine of this angle. The error voltage is amplified and used to drive a servo motor. The motor drives the shaft of the synchro control transformer until it comes to a new aligned position at which the error voltage is zero.

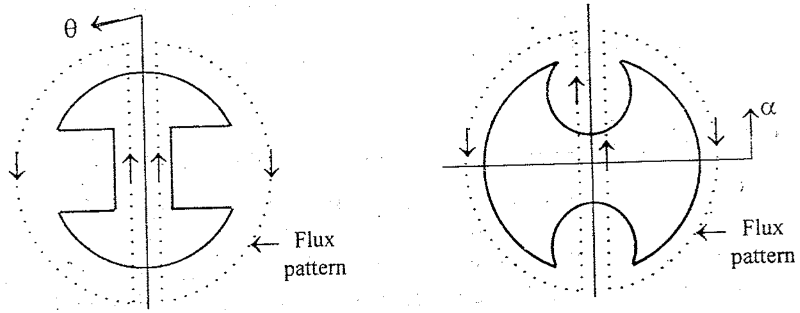


Fig 2.8 : Rotor positions and flux patterns

$\therefore$ Voltage across slip rings of control transformer (modulated error voltage) $\left.\right\}$ $e_m = K'E_r \cos(90 - \theta + \alpha) \sin\omega t$

$$= K' E_r \cos(90 - (\theta - \alpha)) \sin\omega t$$

$$= K'E_r \sin(\theta - \alpha) \sin\omega t \quad .....(2.11)$$

Where K' is a proportional constant

Let $\phi(t) = \theta - \alpha$ .....(2.12)

For small values of $\phi(t)$, $\sin(\theta - \alpha) = \sin\phi(t) \approx \phi(t)$

$$\therefore e_m = K'E_r\phi(t) \sin\omega t \quad .....(2.13)$$

From the equation 2.13 we can say that the output voltage of the synchro error detector is a modulated signal with carrier frequency ω (which is same as supply frequency of the transmitter rotor). The magnitude of the modulated carrier wave is proportional to $\phi(t)$ and the phase reversals of the modulated wave depend on the sign of $\phi(t)$. The signal conditioning circuit demodulates the voltage available across slip rings and develops a demodulated and amplified error voltage to drive the motor.

The demodulated error voltage, $e = K_s\phi(t)$ .....(2.14)

Where K_s = sensitivity of the synchro error detector in volts/deg.

On taking laplace transform of equation (2.14), We get,

$$E(s) = K_s\,\phi(s) \qquad \therefore \frac{E(s)}{\phi(s)} = K_s \qquad .....(2.15)$$

The equation (2.15) is the transfer function of the synchro error detector.

Note : If the motor employed is an ac servomotor then the signal conditioning circuit will not include a demodulator.

2.4 CONTROLLERS

A controller is a device introduced in the system to modify the error signal and to produce a control signal. The manner in which the controller produces the control signal is called the control action. The controller modifies the transient response of the system. The controllers may be electrical, electronic, hydraulic or pneumatic, depending on the nature of signal and the system. Only electronic controllers are presented in this section.

The following six basic control actions are very common among industrial analog controllers. They are

(i) two-position or on-off control action

(ii) proportional control action

(iii) integral control action

(iv) proportional- plus- integral control action

(v) proportional-plus-derivative control action and

(vi) proportional-plus-integral-plus-derivative control action.

Depending on the control actions provided the controllers can be classified as follows.

1. Two position or on-off controllers
2. Proportional controllers
3. Integral controllers
4. Proportional-plus-integral controllers
5. Proportional-plus-derivative controllers
6. Proportional-plus-integral-plus-derivative controllers

ON-OFF (OR) TWO POSITION CONTROLLER

The on-off or two position controller has only two fixed positions. They are either on or off. The on-off control system is very simple in construction and hence less expensive. For this reason, it is very widely used in both industrial and domestic control systems.

The on-off control action may be provided by a relay. There are different types of relay. The most popular one is electromagnetic relay. It is a device which has NO (Normally Open) and NC (Normally Closed) contacts, whose opening and closing are controlled by the relay coil. When the relay coil is excited, the relay operates and the contacts change their positions (i.e., NO $\rightarrow$ NC and NC$\rightarrow$ NO)

Let the output signal from the controller be u(t) and the actuating error signal be e(t). In this controller, u(t) remains at either a maximum or minimum value.

$$u(t) = u_1 \text{ for } e(t) > 0$$
$$= u_2 \text{ for } e(t) < 0 \qquad .....(2.16)$$

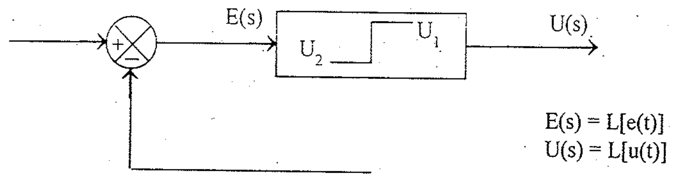


Fig 2.9 : Block diagram of on-off controller

PROPORTIONAL CONTROLLER (P - CONTROLLER)

The proportional controller is a device that produces a control signal, u(t) which is proportional to the input error signal, e(t).

In P-controller, $u(t) \propto e(t)$

$$\therefore u(t) = K_p\,e(t) \qquad .....(2.17)$$

Where, K_p = Proportional gain or constant

On taking laplace transform of equation (2.17)

We get,

$$U(s) = K_p E(s) \qquad .....(2.18)$$

$$\therefore \frac{U(s)}{E(s)} = K_p \qquad .....(2.19)$$

The equation (2.18) gives the output of the P-controller for the input E(s) and equation (2.19) is the transfer function of the P-controller. The block diagram of the P-controller is shown in fig 2.10.

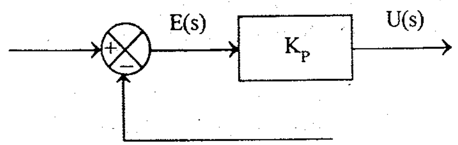


Fig 2.10 : Block diagram of proportional controller

From the equation (2.18) we can conclude that the proportional controller amplifies the error signal by an amount K_P. Also the introduction of the controller on the system increases the loop gain by an amount K_P. The increase in loop gain improves the steady state tracking accuracy, disturbance signal rejection and the relative stability and also makes the system less sensitive to parameter variations. But increasing the gain to very large values may lead to instability of the system. The drawback in P-controller is that it leads to a constant steady state error.

Example of electronic P-Controller

The proportional controller can be realized by an amplifier with adjustable gain. Either the non-inverting operational amplifier or the inverting operational amplifier followed by sign changer will work as a proportional controller. The op-amp proportional controller is shown in fig 2.11 and 2.12

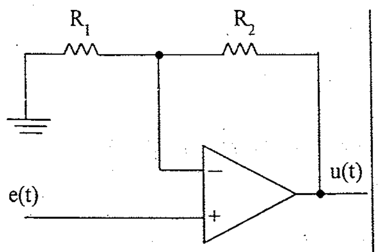


Fig 2.11 : Op-amp P-controller using non inverting amplifier

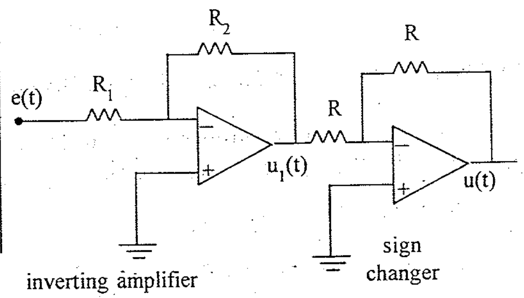


Fig 2.12 : Op-amp P-controller using inverting amplifier

By deriving the transfer function of the controllers shown in fig 2.11 and fig 2.12 and comparing with the transfer function of P- controller defined by equation (2.19), it can be shown that they work as P-controllers.

Analysis of P-controller shown in fig 2.11

In fig 2.11, the input e(t) is applied to positive input. By symmetry of op-amp the voltage of negative input is also e(t). Also we assume an ideal op-amp so that input current is zero. Based on the above assumptions the equivalent circuit of the controller is shown in fig 2.13

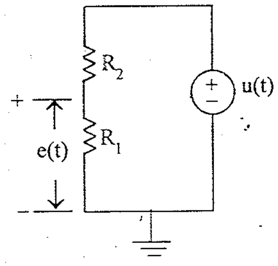


Fig 2.13

By voltage division rule,

$$e(t) = \frac{R_1}{R_1 + R_2} u(t)$$

$$\therefore u(t) = \frac{R_1 + R_2}{R_1} e(t) \qquad .....(2.20)$$

On taking laplace transform of equation 2.20 we get

$$U(s) = \frac{R_1 + R_2}{R_1} E(s) \qquad .....(2.21)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_1 + R_2}{R_1} \qquad ......(2.22)$$

The equation (2.22) is the transfer function of op-amp P-controller. On comparing equation (2.22) with equation (2.19), we get,

$$\text{Proportional gain, } K_P = \frac{R_1 + R_2}{R_1} \qquad .....(2.23)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain, K_p can be varied.

Analysis of P-controller shown in fig 2.12

The assumption made in op-amp circuit analysis are

1. The voltages at both inputs are equal
2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.14 and 2.15

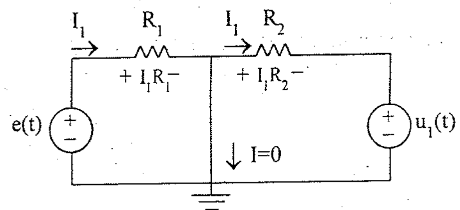


Fig 2.14 : Equivalent circuit of amplifier

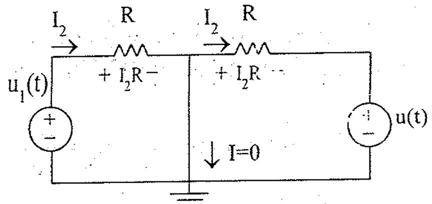


Fig 2.15 : Equivalent circuit of sign changer

From fig 2.14, $e(t) = I_1R_1$, $\therefore I_1 = \frac{e(t)}{R_1}$ .....(2.24)

$$u_1(t) = -I_1R_2 \quad .....(2.25)$$

Substitute for I_1 from equation (2.24) in equation (2.25)

$$\therefore u_1(t) = -\frac{e(t)}{R_1}.R_2 \quad .....(2.26)$$

From fig 2.15, $u(t) = -I_2R$, $\therefore I_2 = -\frac{u(t)}{R}$ .....(2.27)

$$u_1(t) = I_2R \quad .....(2.28)$$

Substitute for I_2 from equation (2.27) in equation (2.28)

$$\therefore u_1(t) = -\frac{u(t)}{R}.R = -u(t) \quad .....(2.29)$$

On equating the equations (2.26) and (2.29) we get

$$-u(t) = -\frac{e(t)}{R_1}.R_2$$

$$u(t) = \frac{R_2}{R_1}\,e(t) \quad .....(2.30)$$

On taking laplace transform of equation (2.30) we get

$$U(s) = \frac{R_2}{R_1}E(s) \quad .....(2.31)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_2}{R_1} \quad .....(2.32)$$

The equation (2.32) is the transfer function of op-amp P-controller. On comparing equation (2.32) with equation (2.19) we get,

$$\text{Proportional gain, } K_p = \frac{R_2}{R_1} \quad .....(2.33)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain K_p can be varied.

INTEGRAL CONTROLLER (I-CONTROLLER)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal, e(t).

In I - controller, $u(t) \propto \int e(t)\,dt$

$$\therefore u(t) = K_i \int e(t)\,dt \quad .....(2.34)$$

Where, K_i = Integral gain or constant

On taking laplace transform of equation (2.34) with zero initial conditions we get

$$U(s) = K_i \frac{E(s)}{s} \quad .....(2.35)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{K_i}{s} \quad .....(2.36)$$

The equation (2.35) gives the output of the I-controller for the input E(s) and equation (2.36) is the transfer function of the I-controller. The block diagram of I-controller is shown in fig 2.16

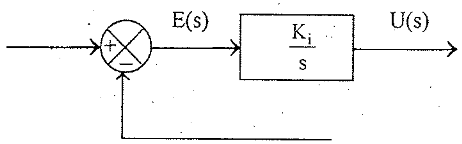


Fig 2.16 : Block diagram of an integral controller

The integral controller removes or reduces the steady error without the need for manual reset. Hence the I-controller is sometimes called automatic reset. The drawback in integral controller is that it may lead to oscillatory response of increasing or decreasing amplitude which is undesirable and the system may become unstable.

Example of electronic I-controller

The integral controller can be realized by an integrator using op-amp followed by a sign changer as shown in fig 2.17

By deriving the transfer function of the controller shown in fig 2.17 and comparing with the transfer function of I-controller defined by equation(2.36), it can be shown that it work as I-controller.

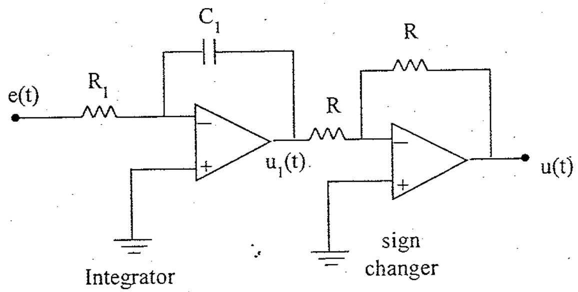


Fig 2.17 : I-controller using op-amp

Analysis of I-controller shown in fig 2.17

The assumptions made in op-amp circuit analysis are,

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.18 and 2.19

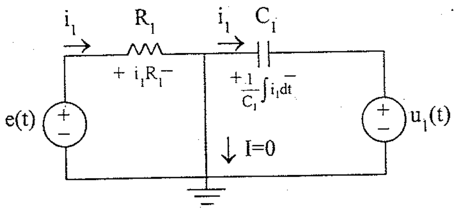


Fig 2.18 : Equivalent circuit of integrator

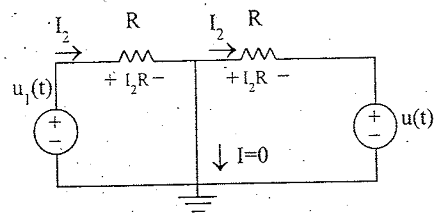


Fig 2.19 : Equivalent circuit of sign changer

From fig 2.18; $e(t) = i_1 R_1$, $\therefore i_1 = \dfrac{e(t)}{R_1}$ .....(2.37)

$$u_1(t) = -\frac{1}{C_1}\int i_1 dt \quad .....(2.38)$$

Substitute for i_1 from equation (2.37) in equation (2.38)

$$\therefore u_1(t) = -\frac{1}{C_1}\int \frac{e(t)}{R_1} dt = -\frac{1}{R_1C_1}\int e(t)dt \quad .....(2.39)$$

From fig 2.19, $u(t) = -I_2R$, $\therefore I_2 = \dfrac{-u(t)}{R}$ .....(2.40)

$$u_1(t) = I_2R \quad .....(2.41)$$

Substitute for I_2 from equation (2.40) in equation (2.41),

$$\therefore u_1(t) = -\frac{u(t)}{R}R = -u(t) \quad .....(2.42)$$

On equating the equation (2.39) and (2.42) we get,

$$-u(t) = -\frac{1}{R_1C_1}\int e(t)\,dt$$

$$\therefore u(t) = \frac{1}{R_1C_1}\int e(t)\,dt \quad .....(2.43)$$

On taking laplace transform of equation (2.43) with zero initial conditions, we get

$$U(s) = \frac{1}{R_1C_1}\frac{E(s)}{s} \quad .....(2.44)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{1}{R_1C_1}\frac{1}{s} \quad .....(2.45)$$

The equation (2.45) is the transfer function of op-amp I-controller. On comparing equation(2.45) with equation (2.36) we get

$$\text{Integral gain, } K_i = \frac{1}{R_1C_1} \quad .....(2.46)$$

Therefore by adjusting the values of R_1 and C_1 the value of gain K_i can be varied.

PROPORTIONAL PLUS INTEGRAL CONTROLLER (PI - CONTROLLER)

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms-one proportional to error signal and the other proportional to the integral of error signal.

In PI – controller, $u(t) \propto \left[e(t) + \int e(t)\,dt\right]$

$$\therefore u(t) = K_p\,e(t) + \frac{K_p}{T_i}\int e(t)dt \quad .....(2.47)$$

Where, K_p = Proportional gain

and T_i = Integral time.

On taking laplace transform of equation (2.47) with zero initial conditions we get,

$$U(s) = K_p\, E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} \qquad .....(2.48)$$

$$\therefore \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right) \qquad .....(2.49)$$

The equation (2.48) gives the output of the PI-controller for the input E(s) and equation (2.49) is the transfer function of the PI controller. The block diagram of PI-controller is shown in fig 2.20.

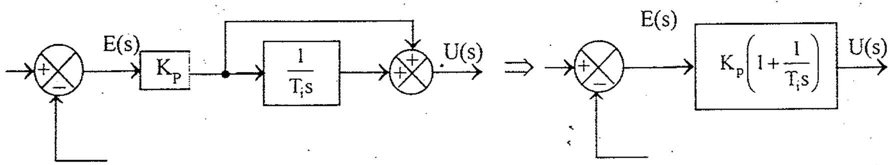


Fig 2.20: Block diagram of PI controller

The advantages of both P-controller and I-controller are combined in PI-controller. The proportional action increases the loop gain and makes the system less sensitive to variations of system parameters. The integral action eliminates or reduces the steady state error.

The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proportional and integral parts of control action. The inverse of the integral time T_i is called the reset rate.

Example of electronic PI-controller

The PI-controller can be realized by an op-amp integrator with gain followed by a sign changer as shown in fig 2.21.

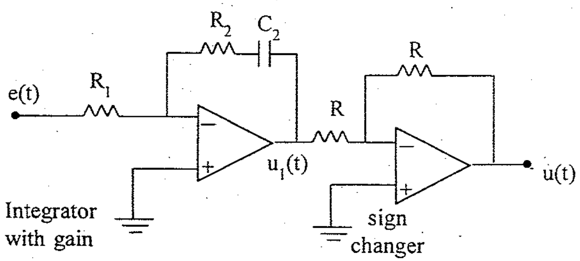


Fig 2.21 : *PI controller using op-amp*

By deriving the transfer function of the controller shown in fig (2.21) and comparing with the transfer function of PI-controller defined by equation (2.49), it can be proved that the circuit shown in fig 2.21, work as PI-controller.

Analysis of PI-controller shown in fig 2.21.

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.22. and 2.23.

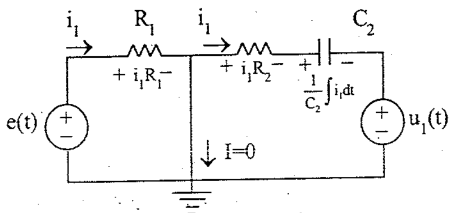


Fig 2.22 : *Equivalent circuit of integrator*

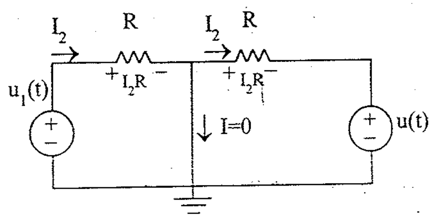


Fig 2.23 : *Equivalent circuit of sign changer*

From fig 2.22, $\quad e(t) = i_1R_1, \qquad \therefore i_1 = \dfrac{e(t)}{R_1} \qquad .....(2.50)$

$$u_1(t) = -i_1R_2 - \frac{1}{C_2}\int i_1 dt \qquad .....(2.51)$$

Substitute for i_1 from equation (2.50) in equation (2.51),

$$\therefore u_1(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt \qquad .....(2.52)$$

From fig 2.23, $\quad u(t) = -I_2R, \qquad \therefore I_2 = \dfrac{-u(t)}{R} \qquad .....(2.53)$

$$u_1(t) = I_2R \qquad .....(2.54)$$

Substitute for I_2 from equation (2.53) in equation (2.54),

$$\therefore u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.55)$$

On equating the equations (2.52) and (2.55) we get,

$$-u(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt$$

$$\therefore u(t) = \frac{R_2}{R_1}e(t) + \frac{1}{R_1C_2}\int e(t)\,dt \qquad .....(2.56)$$

On taking laplace transform of equation (2.56) with zero initial conditions, we get

$$U(s) = \frac{R_2}{R_1}E(s) + \frac{1}{R_1C_2}\frac{E(s)}{s}$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{1}{R_2C_2s}\right) \qquad .....(2.57)$$

The equation (2.57) is the transfer function of op-amp PI-controller. On comparing equation(2.57) with equation (2.49) we get

$$\text{Proportional gain, } K_p = \frac{R_2}{R_1}$$

$$\text{Integral time, } \quad T_i = R_2C_2$$

By varying the values of R_1 and R_2, the value of gain K_p and T_i can be adjusted.

PROPORTIONAL PLUS DERIVATIVE CONTROLLER (PD-CONTROLLER)

The proportional plus derivative controller produces an output signal consisting of two terms-one proportional to error signal and the other proportional to the derivative of error signal.

$$\text{In PD controller, } u(t)\ \alpha \left[e(t) + \frac{d}{dt}e(t)\right]$$

$$\therefore u(t) = K_p\,e(t) + K_pT_d\frac{d}{dt}e(t) \qquad .....(2.58)$$

Where, K_p = Proportional gain

and T_d = Derivative time

On taking laplace transform of equation (2.58) with zero initial conditions we get,

$$U(s) = K_p\,E(s) + K_p\,T_d\,sE(s) \qquad .....(2.59)$$

$$\therefore \frac{U(s)}{E(s)} = K_p(1 + T_ds) \qquad .....(2.60)$$

The equation (2.59) gives the output of the PD-controller for the input E(s) and equation (2.60) is the transfer function of PD-controller.

The block diagram of PD-controller is shown in fig 2.24.

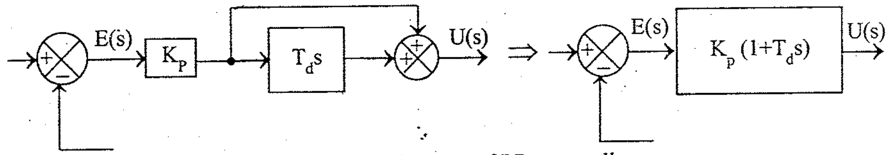


Fig 2.24 : Block diagram of PD- controller

The derivative control acts on rate of change of error and not on the actual error signal. The derivative control action is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers. The derivative controller does not affect the steady-state error directly but antipates the error, initiates an early corrective action and tends to increase the stability of the system. While derivative control action has an advantage of being anticipatory it has the disadvantage that it amplifies noise signals and may cause a saturation effect in the actuator.

The derivative control action is adjusted by varying the derivative time. The change in the value of K_p affects both the proportional and derivative parts of control action. The derivative control is also called rate control.

Example of electronic PD-controller

The PD-controller can be realized by an op-amp differentiator with gain followed by a sign changer as shown in fig 2.25.

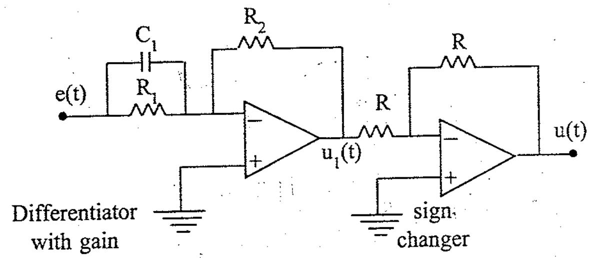


Fig 2.25 : PD controller using op-amp

By deriving the transfer function of the controller shown in fig 2.25 and comparing with the transfer function of PD-controller defined by equation (2.60) it can be proved that the circuit shown in fig 2.25 will work as PD-controller.

Analysis of PD-controller shown in fig 2.25

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp differentiator and sign changer are shown in fig 2.26 and 2.27.

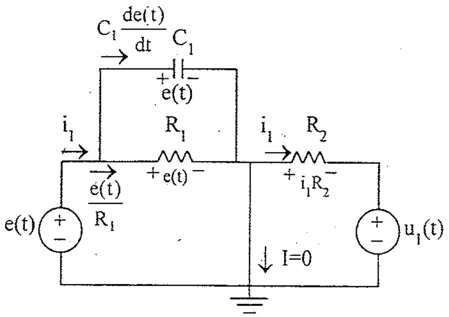


Fig 2.26 : Equivalent circuit of differentiator

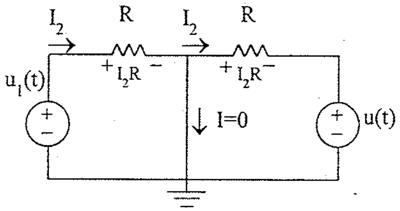


Fig 2.27 : Equivalent circuit of sign changer

From fig 2.26, $i_1 = \frac{e(t)}{R_1} + C_1\frac{de(t)}{dt}$ .....(2.61)

$i_1R_2 = -u_1(t), \quad \therefore i_1 = \frac{-u_1(t)}{R_2}$ .....(2.62)

On equating the equations (2.61) and (2.62) we get,

$$-\frac{u_1(t)}{R_2} = \frac{e(t)}{R_1} + C_1\frac{d}{dt}e(t)$$

$$\therefore u_1(t) = -\left(\frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t)\right) \quad .....(2.63)$$

From fig 2.27, $u(t) = -I_2R, \quad \therefore I_2 = \frac{-u(t)}{R}$ .....(2.64)

$u_1(t) = I_2R$ .....(2.65)

Substitute for I_2 from equation (2.64) in equation (2.65),

$$\therefore u_1(t) = -\frac{u(t)}{R}R = -u(t) \quad .....(2.66)$$

On equating the equations (2.63) and (2.66) we get,

$$-u(t) = -\left(\frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t)\right)$$

$$\therefore u(t) = \frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t) \quad .....(2.67)$$

On taking laplace transform of equation (2.67) with zero initial conditions, we get

$$U(s) = \frac{R_2}{R_1}E(s) + R_2C_1\,s\,E(s) \quad .....(2.68)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_2}{R_1}(1 + R_1C_1\,s) \quad .....(2.69)$$

The equation (2.69) is the transfer function of op-amp PD-controller. On comparing equation(2.69) with equation (2.60) we get,

Proportional gain, $K_p = \frac{R_2}{R_1}$

Derivative time, $T_d = R_1C_1$

By varying the values of R_1 and R_2, the value of gain K_p and T_d can be adjusted.

PROPORTIONAL PLUS INTEGRAL PLUS DERIVATIVE CONTROLLER (PID-CONTROLLER)

The PID-controller produces an output signal consisting of three terms-one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.

In PID – controller, $u(t)\ \alpha \left[e(t) + \int e(t)\,dt + \frac{d}{dt}e(t)\right]$

$$\therefore u(t) = K_p\,e(t) + \frac{K_p}{T_i}\int e(t)dt + K_pT_d\frac{d}{dt}e(t) \quad .....(2.70)$$

Where, K_p = Proportional gain

T_i = Integral time.

T_d = Derivative time

On taking laplace transform of equation (2.70) with zero initial conditions we get,

$$U(s) = K_pE(s) + \frac{K_p}{T_i}\frac{E(s)}{s} + K_p\,T_d\,s\,E(s) \quad .....(2.71)$$

$$\therefore \frac{U(s)}{E(s)} = K_p \left(1 + \frac{1}{T_i s} + T_d s\right) \qquad .....(2.72)$$

The equation (2.71) gives the output of the PID-controller for the input E(s) and equation (2.72) is the transfer function of the PID controller. The block diagram of PID-contoller is shown in fig 2.28.

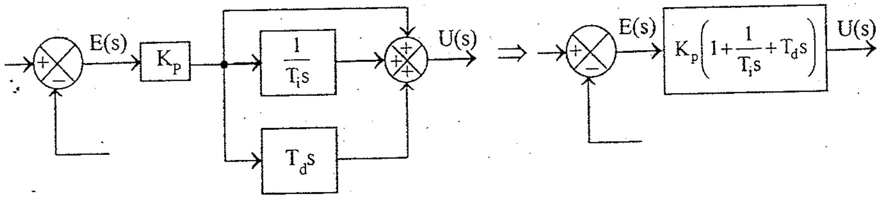


Fig 2.28: Block diagram of PID- controller

The combination of proportional control action, integral control action and derivative control action is called PID-control action. This combined action has the advantages of the each of the three individual control actions.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error.

Example of electronic PID-controller

The PID controller can be realized by op-amp amplifier with integral and derivative action followed by sign changer as shown in fig 2.29.

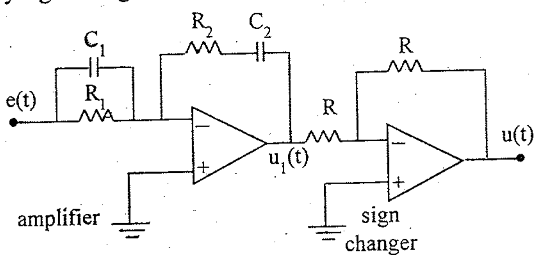


Fig 2.29 : PID controller using op-amp

By deriving the transfer function of the controller shown in fig (2.29) and comparing with the transfer function of PID-controller defined by equation (2.72) it can be proved that the circuit shown in fig 2.29 work as PID-controller.

Analysis of PID-controller shown in fig 2.29

The assumptions made in op-amp circuit analysis are.

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.30 and 2.31.

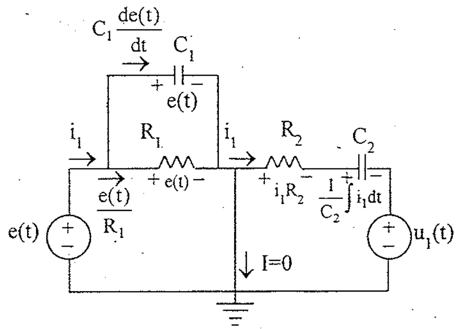


Fig 2.30 : Equivalent circuit of amplifier

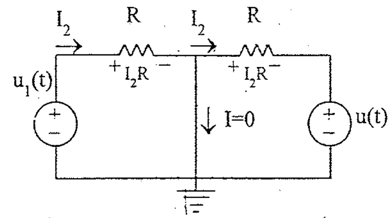


Fig 2.31 : Equivalent circuit of sign changer

From fig 2.30, $$i_1 = \frac{e(t)}{R_1} + C_1\frac{de(t)}{dt} \qquad .....(2.73)$$

On taking laplace transform of equation (2.73) with zero initial conditions we get,

$$I_1(s) = \frac{1}{R_1}E(s) + C_1 s E(s)$$

$$I_1(s) = \left(\frac{1}{R_1} + C_1 s\right)E(s) \qquad .....(2.74)$$

From fig 2.30, $$i_1R_2 + \frac{1}{C_2}\int i_1 dt = -u_1(t) \qquad .....(2.75)$$

On taking laplace transform of equation (2.75) with zero initial conditions we get,

$$I_1(s)\,R_2 + \frac{1}{C_2}\frac{I_1(s)}{s} = -U_1(s)$$

$$\therefore I_1(s)\left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s) \qquad .....(2.76)$$

Substitute for $I_1(s)$ from equation (2.74) in equation (2.76)

$$\therefore \left(\frac{1}{R_1}+C_1 s\right)E(s)\left(R_2+\frac{1}{C_2 s}\right)=-U_1(s)$$

$$-\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}+\frac{1}{R_1C_2 s}+R_2C_1 s\right)E(s)=U_1(s) \qquad .....(2.77)$$

From fig 2.31, $u(t) = -I_2R$, $\quad \therefore I_2 = -\dfrac{u(t)}{R}$ .....(2.78)

$$u_1(t) = I_2R \qquad .....(2.79)$$

Substitute for I_2 from equation (2.78) in equation (2.79)

$$\therefore u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.80)$$

On taking laplace transform of equation (2.80) we get,

$$U_1(s) = -U(s) \qquad .....(2.81)$$

From equation (2.77) and (2.81) we get,

$$U(s)=\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}+\frac{1}{R_1C_2 s}+R_2C_1 s\right)E(s)$$

$$\therefore \frac{U(s)}{E(s)}=\left(\frac{R_2C_2+R_1C_1}{R_1C_2}+\frac{1}{R_1C_2 s}+R_2C_1 s\right)$$

$$=\frac{R_2}{R_1}\left(\frac{R_2C_2+R_1C_1}{R_2C_2}+\frac{1}{R_2C_2 s}+R_1C_1 s\right) \qquad .....(2.82)$$

The equation (2.82) is the transfer function of op-amp PID-controller. On comparing equation(2.82) with equation (2.72) we get,

Proportional gain, $K_p = \dfrac{R_2}{R_1}$

Derivative time, $T_d = R_1C_1$

Integral time, $T_i = R_2C_2$

Also, $\dfrac{R_1C_1+R_2C_2}{R_2C_2} = 1$

By varying the values of R_1 and R_2 the values of K_p, T_d and T_i are adjusted.

2.5 SERVOMOTORS

The motors that are used in automatic control systems are called servomotors. When the objective of the system is to control the position of an object then the system is called Servomechanism. The servomotors are used to convert an electrical signal (control voltage) applied to them into an angular displacement of the shaft. They can either operate in a continuous duty or step duty depending on construction.

There are variety of servomotors available for control system applications. The suitability of a motor for a particular application depends on the characteristics of the system, the purpose of the system and its operating conditions. In general, a servomotor should have the following feature.

1. Linear relationship between the speed and electric control signal.
2. Steady state stability.
3. Wide range of speed control.
4. Linearity of mechanical characteristics throughout the entire speed range.
5. Low mechanical and electrical inertia and
6. Fast response.

Depending on the supply required to run the motor, they are broadly classified as DC servomotors and AC servomotors. The DC motors are expensive than AC motors. But, the DC servomotors have linear characteristics and so it is easier to control.

The advantages of DC servomotors are the following :

1. Higher output than from a 50 Hz motor of same size
2. Linearity of characteristics are achieved easily
3. Easier speed control from zero speed to full speed in both directions.
4. High torque to inertia ratio that gives them quick response to control signals.
5. DC servomotors have light weight, low inertia and low inductance armature that can respond quickly to commands for a change in position or speed.
6. Low electrical time constants(0.1 to 6ms)and low mechanical time constants (2.3 to 40ms).
7. DC motors are capable of delivering over 3 times their rated torque for a short time but AC motors will start at 2 to 2.5 times their rated torque.

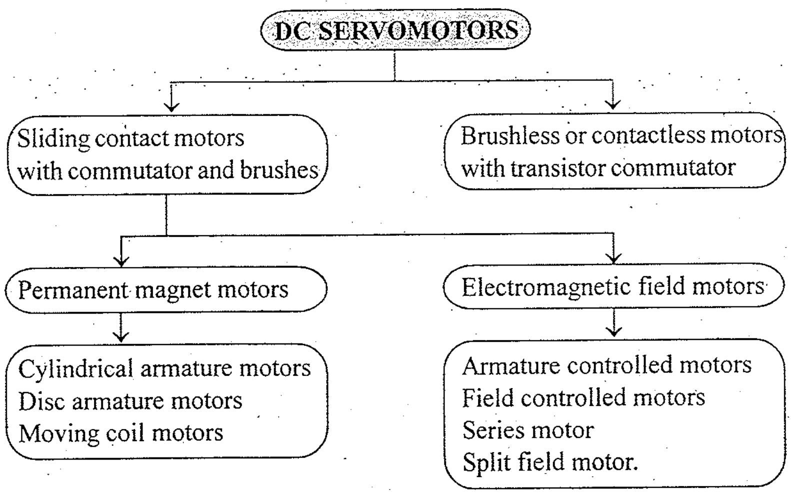
The DC servomotors are generally used for large power applications such as in machine tools and robotics.

Advantages of AC motors are lower cost, higher efficiency and less maintenance since there is no commutator and brushes. The disadvantage of AC motor is that the characteristics are quite non-linear and this motors are more difficult to control especially for positioning applications.(servomechanisms).

The AC motor are best suited for low power applications such as instrument servo (eg. Control of pen in X-Y recorders) and computer related equipment (eg. Disk drives, Tape drives, Printers, etc.,). The three phase induction motors with pulse width modulated power amplifier are currently gaining popularity in high power control application.

2.6 DC SERVOMOTORS

DC servomotors are broadly classified as shown below.



PERMANENT MAGNET DC MOTORS

In this type of motors, the field winding is replaced by permanent magnets to produce the required magnetic field. Permanent magnet motor are economical for power ratings upto a few KW. The following are some of the advantages of permanent-magnet motors.

1. A simpler, more reliable motor because the field power supply is not required.
2. Higher efficiency due to the absence of field losses
3. Field flux is less affected by temperature rise

4. Less heating, making it possible to totally enclose the motor
5. No possibility of over speeding due to loss of field
6. A more linear torque Vs speed curve and
7. Higher power output at the same dimensions and temperature limitations.

The disadvantages of permanent magnet motors are the magnets deteriorate with time and demagnetised by large current transients. These drawbacks are eliminated by high grade magnetic materials such as ceramic magnets and rare earth magnets (samarium cobalt). But the cost of these materials are very high.

In permanent magnet motors, the armature is placed in rotor and permanent magnet poles are fixed to the stator. The rotor employs special type of constructions to reduce the weight and so the inertia of the rotating system. The special type of constructions are cylindrical armature with small diameter and longer axial length, disc armature and hollow armature (moving coil).

ELECTRO-MAGNETIC FIELD MOTORS

These motors are economical for higher power ratings, generally above 1KW. This type of servomotors are similar to conventional dc motors constructionally. But has the following special features.

1. The number of slots and commutator segments is large to improve commutation.
2. Compoles and compensating windings are provided to eliminate sparking.
3. The diameter to length ratio is kept low to reduce inertia.
4. Oversize shafts are employed to withstand the high torque stress.
5. Eddy currents are reduced by complete lamination of the magnetic circuit and by using low-loss steel.

In this type of motor, the torque and speed may be controlled by varying the armature current and/or the field current. Generally, one of these is varied to control the torque while the other is held constant. In armature controlled mode of operation, the field current is held constant and the armature current is varied to control the torque. In the field controlled mode, the armature current is maintained constant and field current is varied to control the torque.

Armature Controlled DC servomotor

It is a DC shunt motor designed to satisfy the requirement of a servomotor. The field is excited by a constant DC supply. If the field current is constant then speed is directly proportional to armature voltage and torque is directly proportional to armature current. Hence the torque and speed can be controlled by armature voltage. Reversible operation is possible by reversing the armature voltage.

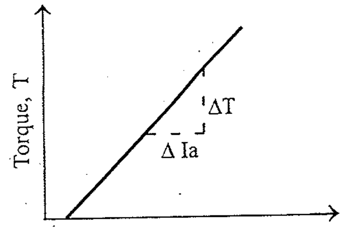
In small motors, the armature voltage is controlled by a variable resistance. But in large motors inorder to reduce power loss, armature voltage is controlled by thyristors. The steady-state operating characteristics of an armature controlled DC servomotor are illustrated in figure 2.32.

For transfer function of armature controlled dc motor, refer chapter 1, section 1.7.

Field controlled DC servomotors

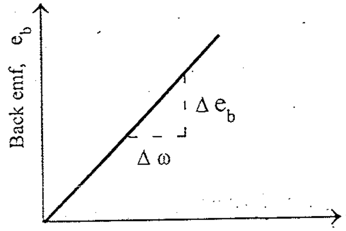
It is a DC shunt motor designed to satisfy the requirement of a servomotor. In this motor, the armature is supplied with a constant current or voltage. When armature voltage is constant the torque is directly proportional to field flux. Since the field current is proportional to flux, the torque of the motor is controlled by controlling the field current. Reversible operation is possible by reversing the field current. The response of field controlled motor is however slowed by field inductance.

For the transfer function of field controlled DC motor refer chapter 1, section 1.8.



Torque constant, $K_t = \Delta T / \Delta Ia$

Fig a: Armature current Vs Torque



Emf constant, $K_b = \Delta e_b / \Delta\omega$

Fig b: Angular speed Vs Back emf

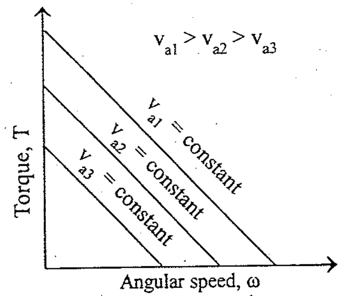


Fig c: Angular speed Vs Torque

Fig 2.32 : Characteristics of armature controlled dc servomotor

2.7 AC SERVOMOTOR

An ac servomotor is basically a two-phase induction motor except for certain special design features. A two-phase servomotor differs in the following two ways from a normal induction motor.

1. The rotor of the servomotor is built with high resistance, so that its X/R (Inductive reactance/Resistance) ratio is small which results in linear speed-torque characteristics. (But conventional induction motors will have high value of X/R which results in high efficiency and non-linear speed-torque characteristics). The speed-torque characteristics of normal induction motor (curve-a) and ac servomotor(curve-b) are shown in fig 2.33.
2. The excitation voltage applied to two stator windings should have a phase difference of 90°.

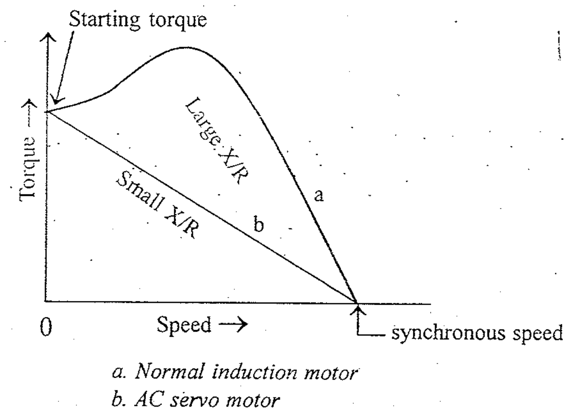
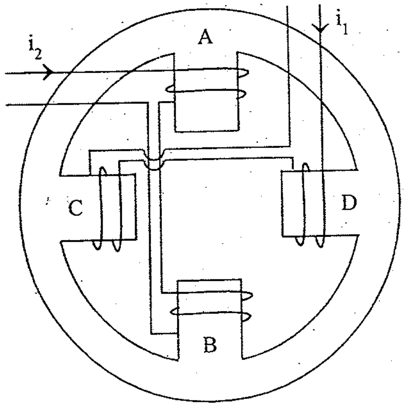


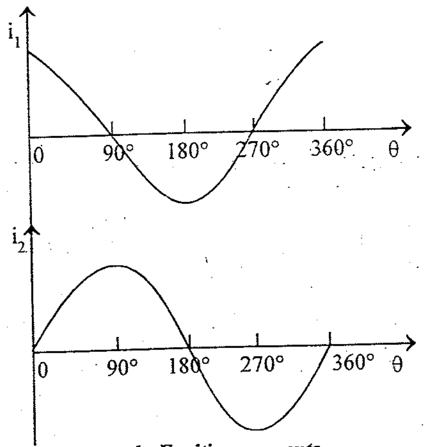
Fig 2.33 : Speed-Torque characteristics of induction motor and ac servomotor

CONSTRUCTION OF AC SERVOMOTOR

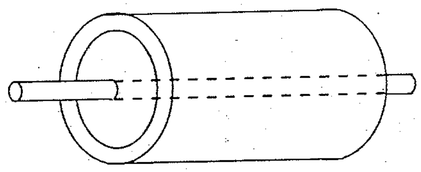
The ac servomotor is basically a two phase induction motor with some special design features. The stator consists of two pole-pairs (A-B and C-D) mounted on the inner periphery of the stator, such that their axes are at an angle of 90° in space. Each pole-pair carries a winding. One winding is called reference winding and the other is called control winding. The exciting current in the winding should have a phase displacement of 90°. The supply used to drive the motor is single phase and so a phase advancing capacitor is connected to one of tne phase to produce a phase difference of 90°. The simple constructional features of ac servomotor is shown in fig 2.34



a. stator



b. Exciting currents



c. Rotor

Fig 2.34 : Simplified constructional features of 2-phase ac servomotor

The rotor construction is usually squirrel cage or drag-cup type. The squirrel cage rotor is made of laminations. The rotor bars are placed on the slots and short circuited at both ends by end rings. The diameter of the rotor is kept small in order to reduce inertia and to obtain good accelerating characteristics.

The Drag-cup construction is employed for very low inertia applications. In this type of construction the rotor will be in the form of hollow cylinder made of aluminium. The aluminium cylinder itself acts as short circuited rotor conductors. (Electrically both the types of rotors are identical)

WORKING PRINCIPLE OF AC SERVOMOTOR

Working of servomotor as ordinary induction motor

The stator windings are excited by voltages of equal rms magnitude and 90° phase difference. This results in exciting currents i_1 and i_2 that are phase displaced by 90° and have equal rms values. These currents give rise to a rotating magnetic field of constant magnitude. The direction of rotation depends on the phase relationship of the two currents (or voltages). The exciting currents shown in fig 2.34 produces a clockwise rotating magnetic field and a phase shift of 180° in i_1 will produce an anticlockwise rotating magnetic field.

The rotating magnetic field sweeps over the rotor conductors. The rotor conductors experience a change in flux and so voltages are induced in rotor conductors. This voltage circulates current in the short circuited rotor conductors and the currents create rotor flux.

Due to the interaction of stator and rotor flux, a mechanical force (or torque) is developed on the rotor and so the rotor starts moving in the same direction as that of rotating magnetic field.

Working of ac servomotor in control systems

The symbolic representation of an ac servomotor as a control system component is shown in fig 2.35. The reference winding is excited by a constant voltage source with a frequency in the range 50 to 1000Hz. By using frequencies of 400 Hz or higher, the system can be made less susceptible to low-frequency noise. Due to this feature, ac devices are extensively used in aircraft and missile control system in which the noise and disturbance often create problems.

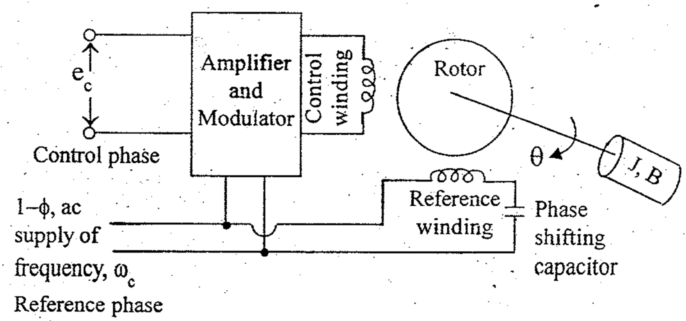


Fig 2.35 : Symbolic representation of an ac servomotor

The control winding is excited by the modulated control signal and this voltage is of variable magnitude and polarity. The control signal of the servo loop (or the system) dictates the magnitude and polarity of this voltage.

The control signals in control systems are usually of low frequency, in the range of 0 to 20Hz. For production of rotating magnetic field, the control-phase voltage must be of the same frequency as the reference-phase voltage and in addition the two voltages must be in time quadrature. Hence the control signal is modulated by a carrier whose frequency is same as that of reference voltage and then applied to control winding. The ac supply itself is used as carrier signal for modulation process. The 90° phase difference between the control-phase and reference-phase voltages is obtained by the insertion of a capacitor in reference winding.

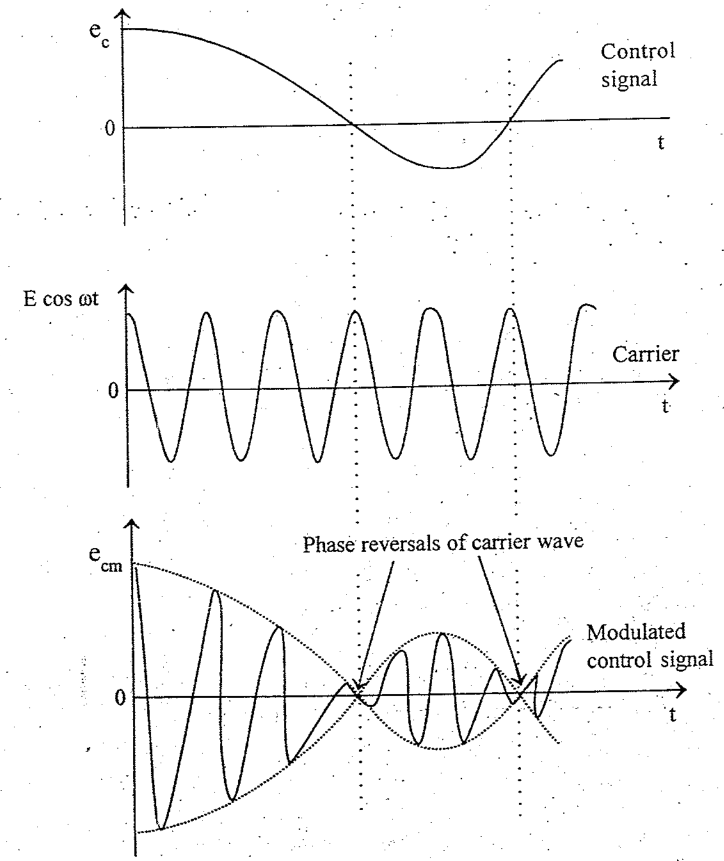


Fig 2.36 : Waveforms of control signal, carrier and modulated control signal

Let e_c = Control signal

e_{car} = $E \cos \omega_c t$ = Carrier signal

e_{cm} = Modulated control signal.

The waveforms of a typical control signal, carrier signal and modulated control signal are shown in fig 2.36. The type of modulation is amplitude modulation and so the information is available on the envelope of the modulated signal. In fig 2.36 it can be observed that the envelope of the modulated wave is identical to control signal.

The polarity of e_c dictates the phase of e_{cm} with respect to that of carrier. If e_c is positive then e_{cm} and e_{car} have the same phase, otherwise they have 180° phase difference.

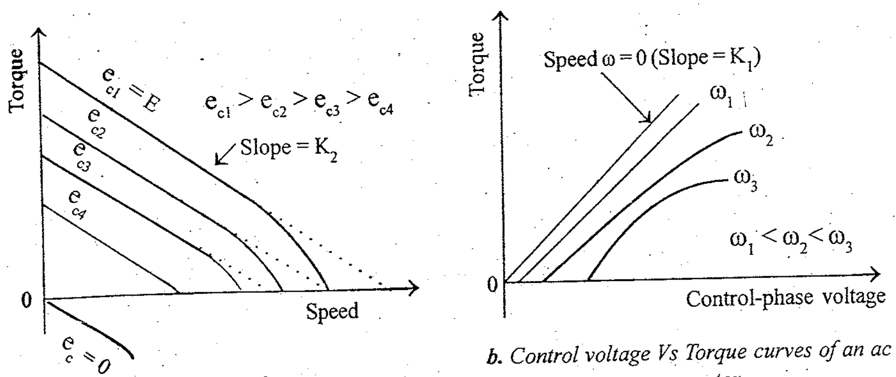
$$\therefore e_{cm} = |E + e_c| \cos \omega_c t \quad \text{for } e_c > 0$$
$$= |E + e_c| \cos(\omega_c t + \pi) \quad \text{for } e_c < 0 \qquad .....(2.83)$$

This means that a reversal in phase of e_{cm} occurs whenever the signal e_c crosses the zero-magnitude axis. This reversal in phase causes a reversal in the direction of rotation of the magnetic field and hence a reversal in the direction of rotation of the motor shaft.

Note: *The control signal is modulated in order to match the frequency to that of reference signal. In the modulated wave the information is stored in the envelope of* e_{cm}*, but not in* e_{cm}*. Hence, we should consider* e_c *as the input and the envelope of* e_{cm} *as the output of the modulator. Therefore the transfer function of the modulator is equal to 1, because the envelope of* e_{cm} *is identical to* e_c*.*

The speed-torque curves of a typical ac servomotor plotted for fixed reference phase voltage $E \cos \omega_c t$ and different values of constant input voltages $e_c \leq E$ are shown in fig 2.37a. All these curves have negative slope. Note that the curve for $e_c = 0$ goes through the origin, this means that when the control-phase voltage becomes zero, the motor develop a decelerating torque and so the motor stops. The curves show a large torque at zero speed. This is a requirement for a servomotor in order to provide rapid acceleration.

The speed-torque curves of ac servomotor are nonlinear except in the low-speed region. In order to derive a transfer function for the motor, some linearizing approximations are necessary. A servomotor seldom operates at high speeds, therefore the linear portions of speed-torque curves can be extended out to the high speed region as shown in fig



a. Speed-torque curves of an ac servomotor

b. Control voltage Vs Torque curves of an ac servomotor

Fig 2.37 : Characteristics of ac servomotor

2.37a, by use of dashed lines. But even with this approximation, the resultant curves are still not parallel to each other. This means that for constant speeds, except near-zero speed,the torque does not vary linearly with respect to input voltage e_c. The curves in fig 2.37 b illustrates this effect.

TRANSFER FUNCTION OF AC SERVOMOTOR

Let, T_m = Torque developed by servomotor

q = Angular displacement of rotor

$w = \frac{d\theta}{dt}$ = Angular speed

T_l = Torque required by the load

J = Moment of inertia of load and the rotor

B = Viscous-frictional coefficient of load and the rotor.

K_1 = Slope of control-phase voltage Vs Torque characteristic

K_2 = Slope of speed-torque characteristic.

With reference to fig(2.37) we can say that for speeds near zero, all the curves are straight lines parallel to the characteristic at rated input voltage (e_c= E) and are equally spaced for equal increments of the input voltage. Under this assumption, the torque developed by the motor is represented by equation(2.84) given below.

$$\text{Torque developed by motor, } T_m = K_1 e_c - K_2 \frac{d\theta}{dt} \qquad .....(2.84)$$

The rotating part of motor and the load can be modelled by the equation (2.85) given below.

$$\text{Load torque , } T_l = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \qquad .....(2.85)$$

At equilibruim the motor torque is equal to load torque.

$$\therefore J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_1 e_c - K_2 \frac{d\theta}{dt} \qquad .....(2.86)$$

On taking laplace transform of equation (2.86) with zero initial conditions, we get,

$$J s^2\theta(s) + B s\theta(s) = K_1 E_c(s) - K_2 s\theta(s)$$

$$\left[Js^2 + Bs + K_2 s\right]\theta(s) = K_1 E_c(s)$$

$$\therefore \frac{\theta(s)}{E_c(s)} = \frac{K_1}{s(Js + B + K_2)} = \frac{K_1/(B+K_2)}{s\left(\frac{J}{B+K_2}s + 1\right)}$$

$$= \frac{K_m}{s(\tau_m s + 1)} \qquad .....(2.87)$$

$$\text{Where, } K_m = \frac{K_1}{B+K_2} = \text{Motor gain constant} \qquad .....(2.88)$$

$$\tau_m = \frac{J}{B+K_2} = \text{Motor time constant} \qquad .....(2.89)$$

The equation (2.87) is the transfer function of ac servomotor.

Note : *The modulation process for running the ac servomotor is not necessary if the control signal in the system is an ac signal of frequency equal to that of the reference phase supply. The transfer function of motor given by equation (2.87) is applicable in these situations also.*

2.8 STEPPER MOTORS

A stepper motor transforms electrical pulses into equal increments of rotary shaft motion called steps. A one-to-one correspondence exists between the electrical pulses and the motor steps. They work in conjunction with electronic switching devices. The function of switching device is to switch the control windings of the stepper motor with a frequency and sequence corresponding to the command issued. It has a wound stator and a non excited rotor. They are classified as variable reluctance, permanent magnet or hybrid, depending on the type of rotor. They are also classified as 2-phase, 3-phase or 4-phase depending on the number of windings (called control windings) on the stator.

The number of teeth or poles on the rotor and the number of poles on the stator determine the size of the step (called the step angle). The step angle is equal to 360 degrees divided by the number of steps per revolution. Motors are available with step rates of 200 steps per revolution. Further increase in step rate is limited by mechanical and physical constraints. This limitation has been overcome by electronic methods of reducing the step size. Half-stepping and microstepping are techniques of electronically dividing each step into two half steps or from 10 to 125 microsteps.

A single pulse advances the shaft position by one angular step and a train of pulses produces a rotation in rapid steps, each with a non cumulative error of say $\pm 3\%$. For example, with a step angle of 1.8 degrees, one pulse will move the rotor position by 1.8 ± 0.05 degrees, a train of 1000 pulses will give a rotation angle of 1800 ± 0.05 degrees equivalent to five complete revolutions. With such an accuracy, a stepper motor does not require closed loop control, since positional precision is determined by the number of pulses.

A single pulse develops a detent torque which turns the rotor by one angular step and it comes to rest after a short damped oscillations which depends upon the system inertia, elasticity and damping. A properly timed pulse sequence sustains a slewing rotation, the rotor passing through a corresponding number of angular steps and develops an effective mean torque and speed (or acceleration) all with positional accuracy.

APPLICATIONS OF STEPPER MOTOR

Stepper motors are used in computer peripherals, X-Y plotters, scientific instruments, robots and machine tools. Stepper motors are used in quartz - crystal watches. The one used in wrist watch has a diameter of 3 mm and takes a few microamperes at 1.5V. Stepper motors are also used for colour registration in printing. Stepper motors can upgrade mechanical systems by replacing cams, complex linkages, and similar mechanisms to give greater precision and production rate. Stepper motors are available with torques in the range from 0.5 micro N-m to 100 N-m, outputs in the range from millwatts to several kilowatts and pulse rates of 1200 or more per second.

ACTIVE ROTOR OR PERMANENT MAGNET ROTOR STEPPER MOTOR

The stator of this type of stepper motor has salient poles carrying control windings. Each pole carries a control winding. A pair of control windings are connected in series and called a phase. Stepper motors may be wound for any number of phases, most popularly being two, three and four-phase stepper motors. The rotor is made in the form of a permanent magnet spider cast integral (fig 2.38a) or assembled of a number of permanent magnets (fig 2.38b).

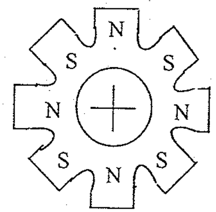


Fig 2.38 a : Cast integral

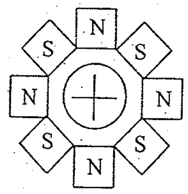


Fig 2.38 b : Assembled separate magnets

Fig 2.38 : Constructions of permanent-magnet stepper motor rotors

Operating principle

Consider a stepper motor having 4-pole stator with two phase-windings. Let the rotor be made of permanent magnet with 2 poles. (fig 2.39). The stator poles are marked A, B, C and D and they are excited with pulses supplied by power transistors. The power transistors are switched by digital controllers or computers. Each control pulse applied by the switching device causes a stepped variation of the magnitude and polarity of voltage fed to the control windings.

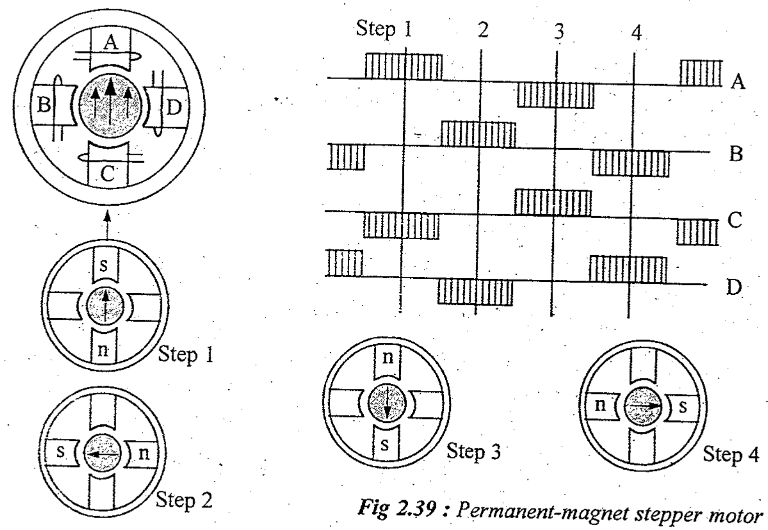


Fig 2.39 : Permanent-magnet stepper motor

If first excitation is applied to A and C, they develop the magnetic polarities indicated for step 1, in fig 2.39, and the rotor sets itself vertically. (The magnetic polarity developed in the stator pole can be determined from right hand rule). If now A and C are switched OFF and B and D excited as in step 2, an alignment torque is developed on the rotor to turn its axis to the horizontal by a 90 deg step. With B and D OFF and A and C re-energised with reverse polarity, the rotor turns a further 90 deg. and so on. The direction of rotation is anticlockwise. The direction can be reversed by changing the current directions suitably. The stator currents with uniform pulse frequency and equal ON/OFF periods are shown to a time base in fig.2.39.

VARIABLE - RELUCTANCE STEPPER MOTOR

It has soft-iron rotor with a number of teeth, giving it the appearance of a gear. The stator also has teeth in addition to a number of wound poles. (Fig.2.40). When electric current is applied to a coil in the stator, magnetic flux is generated that causes teeth in the rotor to line up with teeth in the stator. When the current is switched to the next winding, the rotor moves a distance of one step angle, and a new set of teeth line up.

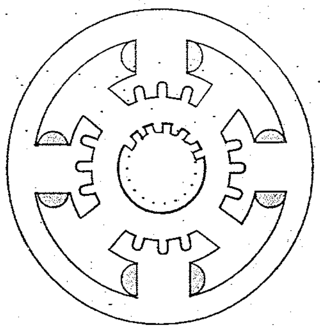


Fig 2.40 : General arrangement of variable reluctance stepper motor

HYBRID STEPPER MOTORS

Hybrid stepper motors have a variable - reluctance rotor with a permanent magnet in its magnetic path. The magnet is usually in the rotor. Hybrid steppers have high torque, high inertia and small step angles. A typical hybrid stepper has 50 teeth in the rotor and 50 teeth in the stator. The stator has eight poles with two center - tapped windings. When the center taps are used, the motor is considered to be a four-phase motor. When the center taps are removed or left open circuited, the motor is considered a two-phase motor.

OTHER TYPES OF STEPPER MOTORS

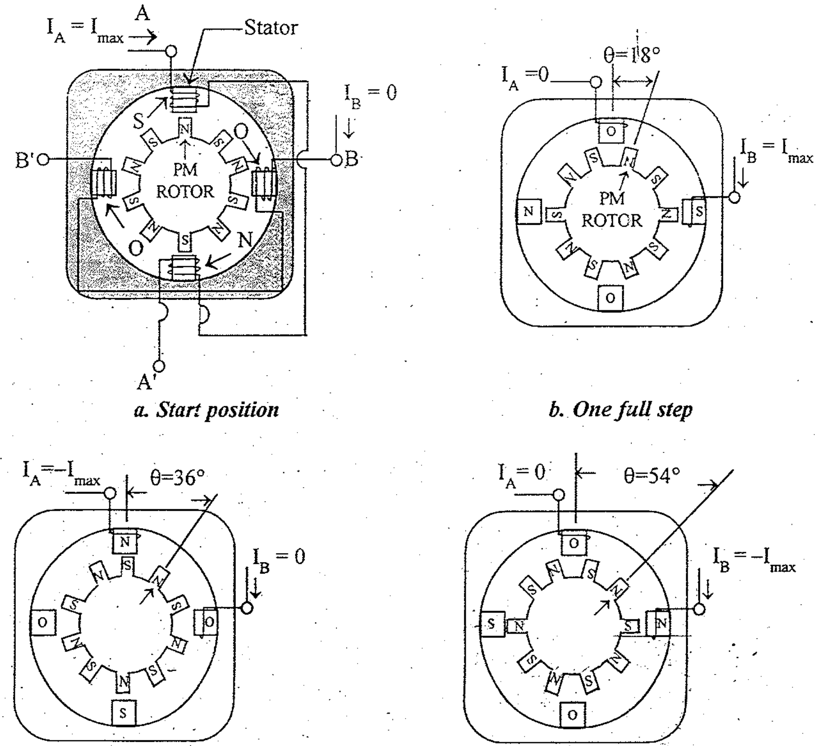
Along with stepper motors discussed herein, harmonic drive stepper motors, servomotors, disk-rotor and printed winding stepper motors are coming into use with a view of decreasing the step angle and improving the dynamic characteristics. There are also linear stepper motors which convert a command issued in the form of pulses into a linear motion.

FULL-STEP OPERATION

Full-step operation of a stepper motor consists of a movement of one full step for each input pulse.

$$\text{Full - step} = \frac{360°}{\text{No.of rotor poles} \times \text{No.of stator pole pairs}} \quad .....(2.90)$$

Consider a two-phase stepper motor with four wound poles in the stator and a 10 pole rotor. Now the step angle will be 18 deg. (Full-step). The arrow on the rotor will enable us to follow the movement of the rotor. In fig.2.41a, the motor is in start position. A current of I_m amp enters the phase A. The direction of this current is such that the top stator pole is a south pole and the bottom stator pole is a north pole. The magnetic attraction between unlike poles will hold the rotor in the position shown in fig 2.41a.



a. Start position *b. One full step*

c. Two full steps *d. Three full steps*

Fig 2.41 : Various positions of stepper motor with step size of 18°

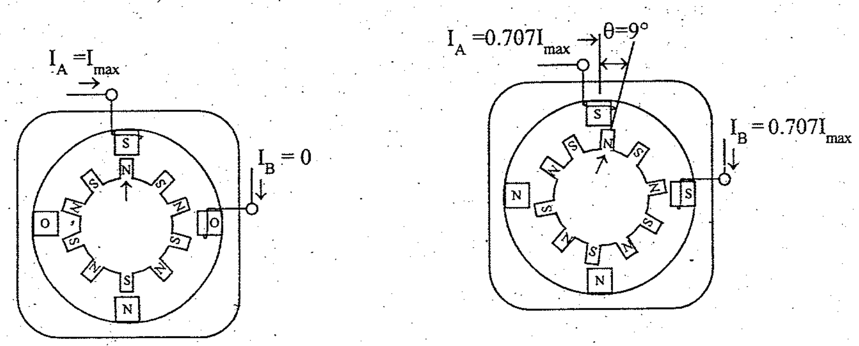
In fig 2.41b, a current of I_m enters the phase B winding and current in phase A is switched OFF. Now the right side stator pole becomes south pole and left side stator pole becomes north pole. The top and bottom stator poles are demagnetised. The magnetic attraction between opposite poles and the repulsion between like poles has caused the rotor to rotate one step in the clockwise direction (fig.2.41b). Table 1 below lists the angle θ and two phase currents for five full steps.

Table 1 : Sequence of phase currents for five full steps.

Step	θ(deg)	I_A/I_m	I_B/I_m
0	0	1	0
1	18	0	1
2	36	–1	0
3	54	0	–1
4	72	1	0
5	90	0	1

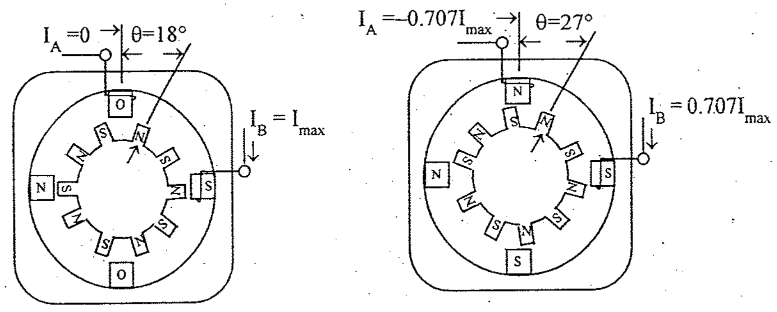
HALF-STEP OPERATION

Half-steps are accomplished by applying partial currents to both phase windings to position the rotor halfway between two full step position. The start conditions in fig.2.42a are identical to the conditions in fig 2.41a. The first half-step is accomplished by reducing the phase A current to 0.707 I_m amperes and increasing the phase B current to 0.707 I_m amperes. (fig 2.42b). Notice that all four stator coils are magnetized. The magnetic forces are such that the rotor is held in the position shown in fig 2.42b. Notice that the position of the marked rotor pole, θ is 9 deg, exactly half of a full 18 deg step. Table 2 below lists the angle θ and the two phase currents for 8 half-steps.



a. Start position

b. One half step



c. Two half steps

d. Three half steps

Fig 2.42 : Half-stepping increases the resolution of the model stepper motor to 40 half-steps per revolution. In half-stepping, the driver first turns on phase A, then both A and B, then B only, then both A and B (with reversed polarity on A).

Half Step	q(deg)	I_A/I_m	I_B/I_m
0	0	1	0
1	9	0.707	0.707
2	18	0	1
3	27	-0.707	0.707
4	36	-1	0
5	45	-0.707	-0.707
6	54	0	-1
7	63	0.707	-0.707
8	72	1	0

MICRO STEP OPERATION

Microstepping simply extends the half-stepping technique to more than one midposition by using different values of current in each phase. The microstep sizes that are most commonly used are 1/10, 1/16, 1/32 and 1/125 of a full step. An obvious advantage of microstepping is the much finer resolution it provides. For example, when 125 microsteps are used in a stepper that has 200 full steps per revolution, the resolution is 200 ´ 125 = 25,000 microsteps per revolution.

The values of the phase currents required for a microstep are given by

$$I_A = \text{Cos}\frac{(90n)}{s}\, I_m \qquad .....(2.91)$$

$$I_B = \sin \frac{(90n)}{s} I_m \qquad .....(2.92)$$

Where, I_m = Maximum value of phase current
I_A = Current in phase A
I_B = Current in phase B
n = n^{th} microstep from start position
s = Number of microsteps in a full step.

CHARACTERISTICS OF STEPPER MOTOR

Response to pulse input

The response of a stepper motor to a single pulse is illustrated in fig 2.43 a. A detent torque is developed which moves the rotor towards the angular position demanded. Initial overshoot reverses the torque and generates an oscillation about the final position. Damping is essential to avoid prolonged oscillation.

In fig 2.43b the response to slow, intermediate and fast stepping rates are graphed. At low rates the motion is a succession of the step responses as shown in fig 2.43a, while at high rates the motor runs at a nearly constant slewing speed. The maximum achievable slew speed is limited by the load torque and inertia. Also depends on the rate at which successive pulses can be established and suppressed without undue reduction of the stator fluxes on which the torque depends.

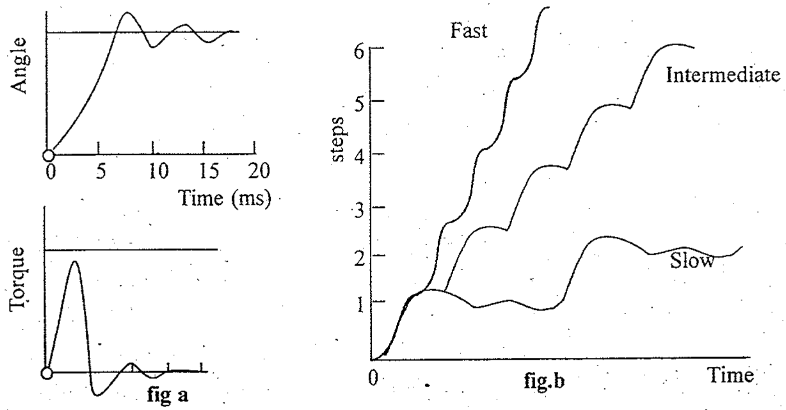


Fig 2.43 : Stepper motor response

TORQUE

A stepper motor is required to start and stop at various stepping rates and with various conditions of load, friction and inertia. There may be fast slewing and settling - time limits to fulfil. The two idealized torque versus pulse-rate curves are shown in fig 2.44a. The curve (i) for starting and curve (ii) for slewing for a system with low inertia. For starting, the load plus friction torque must lie within curve (i). For example a torque demand **OT** cannot be started at a rate exceeding **OQ**.

The curve (ii) shows that the load torque to pull out a motor when slewing is greater for a given stepping rate than would prevent it from starting. At low step rates the curves approach the point M of maximum torque.

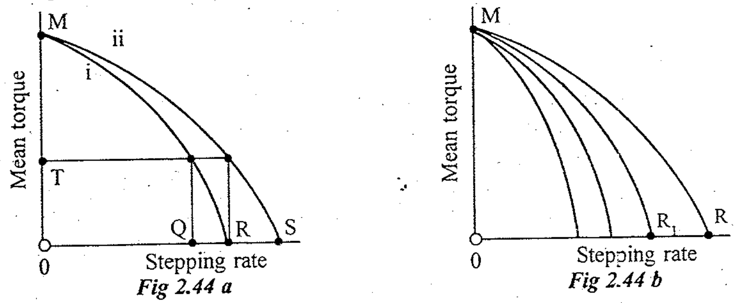


Fig 2.44 : Stepper motor characteristics

The curves in fig 2.44b, indicates the restrictive effects of inertia. The low inertia curve MR is the same as in fig 2.44a. The curve MR_1 corresponds to inertia higher than MR. With increasing inertia the step rate for starting becomes more limited.

The maximum stepping rate may lie between a few hundred and as much as 30000 steps per sec. At high rates several factors become significant. Natural mass-stiffness resonances and drive impedances may appear, phase winding inductance may resonate with stray capacitance and time to establish each working flux may become an appreciable fraction of the time per step. These factors affects the torque/step-rate characteristics to impose "dips" in the available torque (fig 2.45a). In some applications these are trouble some. In others it may be possible to accelerate the stepping motor through them, particularly if adequate damping is provided.

STEADY-STATE TORQUE / DISPLACEMENT CHARACTERISTICS

This is used to predict the shaft angle at any instant following the application of one step or a train of steps and so to find the starting rate. A typical characteristic for one excited phase is shown in fig 2.45b.

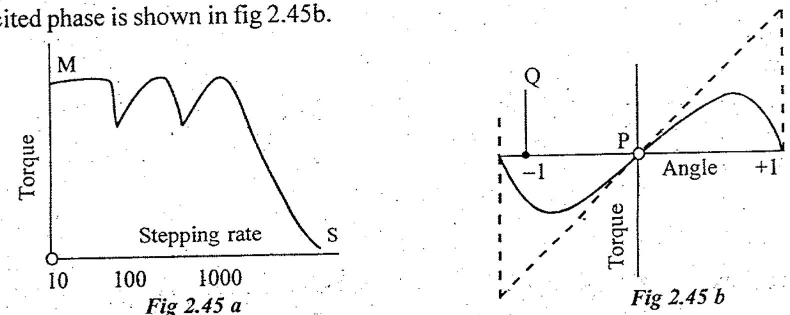


Fig 2.45 : Stepper motor characteristics

The rotor tends to settle at the position P of zero torque. Stepping the drive to the next phase shifts the torque/displacement curve by one step and puts the rotor into a high-torque position, from which it approaches a new zero at Q.

2.9 TACHOGENERATOR

A tachogenerator is an electromechanical device which produces an output voltage proportional to its shaft speed. It can be employed as an analogue speed indicator, velocity feedback device or signal integrator. Two of the most commonly used tachogenerators are dc and ac tachogenerators.

DC TACHOGENERATOR

A dc tachogenerator is a small dc generator with linear characteristics. It consists of stator with a permanent magnet field, a rotating armature circuit and a commutator and brush assembly. The rotor is connected to the shaft whose speed has to be measured.

The output voltage of the tachogenerator is proportional to the angular velocity of the shaft. The polarity of the output voltage is dependent on the direction of rotation of the shaft. The schematic representation of a dc tachogenerator is shown in fig 2.46.

Let $e(t)$ = Output voltage of tachogenerator, volts

θ = Angular displacements, rad

ω = $d\theta/dt$=Angular speed, rad/sec

K_t = Sensitivity of tachogenerator, Volts/(rad/sec).

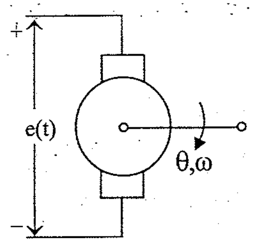


Fig 2.46: Schematic diagram of dc tachogenerator.

Output voltage of tachogenerator, $e(t) \alpha \frac{d\theta}{dt}$

$$\therefore e(t) = K_t \frac{d\theta}{dt} \quad .....(2.93)$$

On taking laplace transform of equation (2.93) we get

$$E(s) = K_t\, s\, \theta(s)$$

$$\therefore \frac{E(s)}{\theta(s)} = s\, K_t \quad .....(2.94)$$

The equation (2.94) is the transfer function of DC tachogenerator. A problem associate with dc tachogenerators is the high-frequency ripple generated by the commutator and brushes. They also suffer from maintenance problems.

AC TACHOGENERATOR

The ac tachogenerator resembles a two phase induction motor. It consists of two stator windings arranged in space quadrature, (the windings are fixed on the inner periphery of the stator such that the angle between their axes is 90°) and the squirrel cage rotor. One stator winding is called reference winding and it is excited by a sinusoidal voltage of frequency, ω_c. The other winding is called output winding, in which an emf is induced when the rotor rotates. The schematic diagram of ac tachogenerator is shown in fig 2.47.

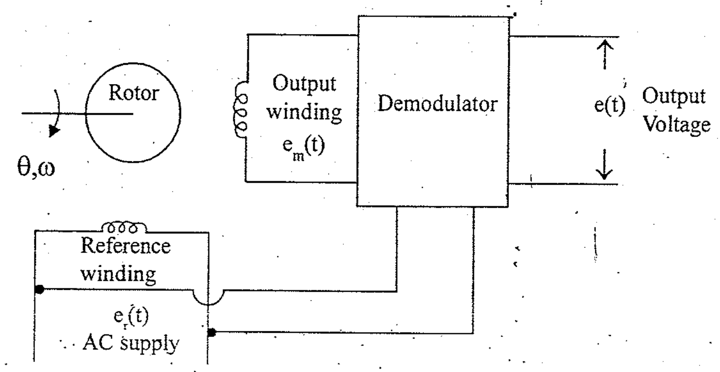


Fig 2.47 : Schematic diagram of ac tachogenerator

When the rotor is stationary, no emf is induced in the output winding and therefore the output voltage is zero. When the rotor turns, a voltage at the reference frequency, ω_c is induced in the output winding. The magnitude of the output voltage is proportional to the rotational speed.

A change in the direction of shaft rotation causes a 180° phase shift in the output voltage. When the output voltage is in phase with the reference, the direction of rotation is said to be positive and when the output voltage is 180° out of phase, the direction is said to be negative. The output voltage of an ac tachogenerator is thus in modulated form.

If the reference winding (excitation)voltage, $e_r(t) = E_r \sin\omega_c t$

then output voltage, $e_m(t) = e_r(t) \sin\omega_c t$

The output voltage, $e_m(t)$ of the tachogenerator is demodulated to get e(t), because the information about shaft speed is available on the envelope of $e_m(t)$. After demodulation, the output voltage of ac tachogenerator is same as that of dc tachogenerator.

$$\text{Output voltage, } e(t) \propto \frac{d\theta}{dt}$$

$$\therefore e(t) = K_t \frac{d\theta}{dt} \quad .....(2.95)$$

Where, K_t= sensitivity of tachogenerator, volts/(rad/sec).

On taking laplace transform of equation(2.95) we get

$$E(s) = K_t \, s\theta(s)$$

$$\therefore \frac{E(s)}{\theta(s)} = s K_t \quad .....(2.96)$$

The equation (2.96) is the transfer function of ac tachogenerator.

2.10 MODULATOR AND DEMODULATOR

In control systems, there may be situations in which the dc signals has to actuate an ac system. In such cases the dc signals has to be converted to ac signal. Many tranducers like thermocouple, photo electric cell, etc., produce dc voltages, (or slowly varying voltages) proportional to the quantity to be measured. If this signal is to actuate an ac system then it should be converted to ac signal.

The device which transfer the information available in a dc or slowly varying signal to ac signal is called a modulator and the conversion process is called modulation. In modulation the dc signal (or control signal or low frequency ac signal) is superimposed on a high frequency carrier.

Certain control system components like ac tachogenerators, ac potentiometers, etc., produces a modulated output signal. Hence the information or control signal is obtained by demodulation. The device which is used to extract the information available on a high frequency carrier is called demodulator and the process is called demodulation. The demodulation is reverse process of modulation and in this process the dc signal (or control signal or low frequency ac signal) is extracted from the carrier signal.

SYNCHRONOUS VIBRATOR AS MODULATOR

This type of modulator consists of a driving coil, a vibrating reed and a centre tapped transformer. The schematic diagram of synchronous vibrator is shown in fig 2.48.

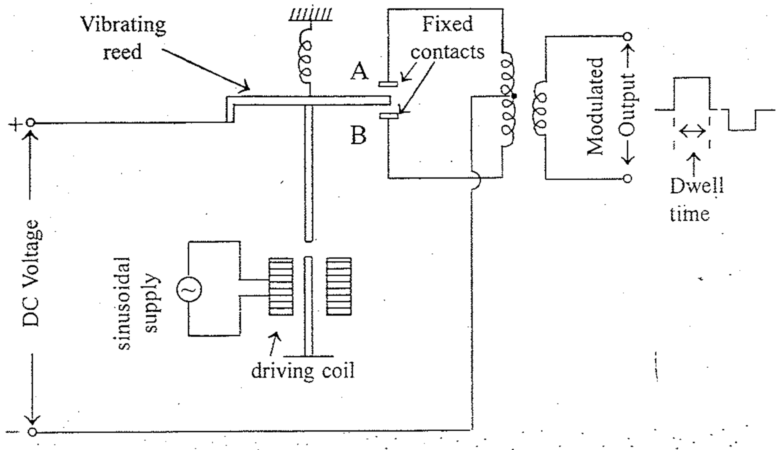


Fig 2.48 : Synchronous vibration as modulator

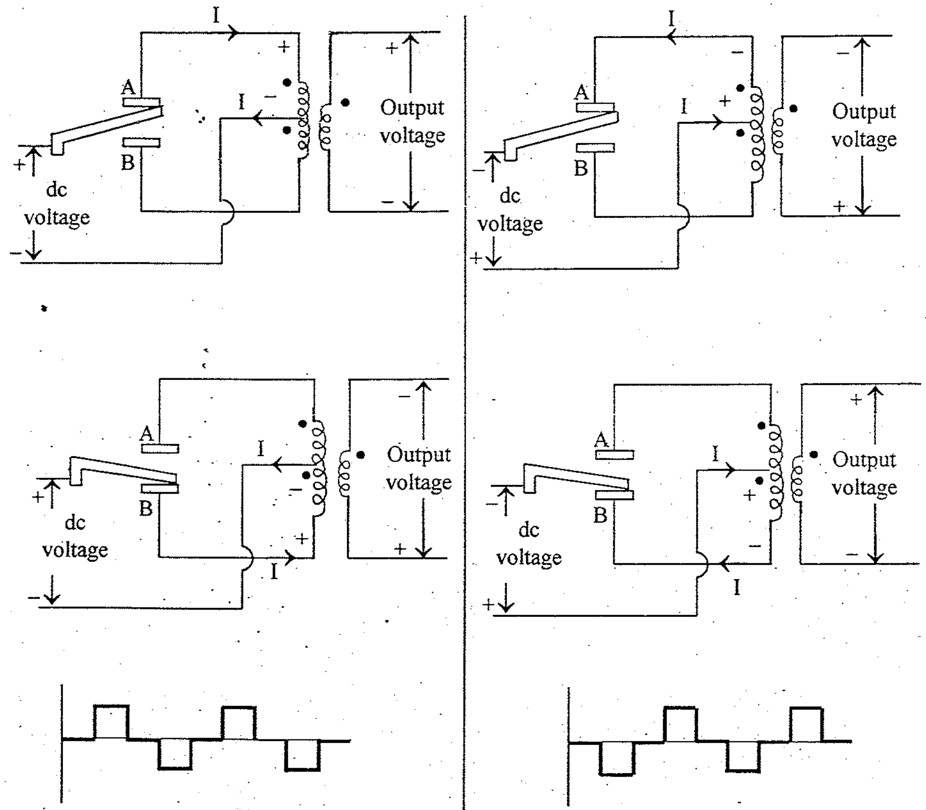
The driving coil of the vibrator is excited by a sinusoidal supply of frequency, ω_c. The magnetic field produced by the coil makes the metal reed to vibrate between two fixed contacts A and B. The fixed contacts are connected to the two ends of the transformer primary winding.

The dc error signal or control signal is applied between the reed and the centretap of the transformer. The vibrating reed will be alternatively touching fixed contact A and B. When the reed is in contact with A, the dc voltage circulates a current in one half of transformer primary. When the reed is in contact with B, the voltage circulates current in the other half of transformer primary but in the opposite direction. Thus the vibrating reed creates current reversals in the primary of the transformer. The frequency of current reversals being same as that of vibrating frequency.

The periodical reversal of current through the primary winding will cause an alternating voltage to be induced in the secondary. Since the vibrator converts dc voltage to ac, it is also called chopper.

The length of the time the vibration reed rest on the fixed contacts is known as the dwell time. During the time the reed moves from one contact to another there is no primary current and so the output voltage is zero.

The output waveform of the modulator is a square wave. The amplitude and phase of the output waveform, depends on the amplitude and polarity of dc voltage. When the dc voltage reverses, the output waveform will have a phase shift of 180°. The output voltage for the two positions of reed contacts are shown in fig 2.49 for both positive and negative dc voltages.



a. Output voltage for positive dc voltage

b. Output voltage for negative dc voltage

Fig 2.49: Polarity of output voltages

The vibrator systems has low life due to wear and tear of mechanical contacts and also it produces noise but the noise level is low.

DIODE MODULATOR

A diode modulator using two diodes is shown in fig 2.50. The primary of centre-tapped transformer is excited by a sinusoidal supply of frequency, ω_c. Since the centre

point is grounded, the emfs induced in the two halves of secondary will have a phase shift of 180°.

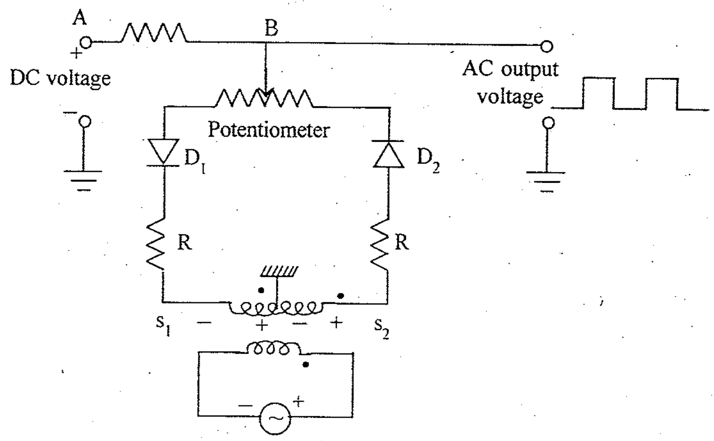


Fig 2.50 : Diode modulator

During positive half cycle of supply voltage, the end s_2 is positive and s_1 is negative. *[This polarity is shown in fig 2.50]*. Now both the diodes D_1 and D_2 are forward biased and a current circulates in the secondary of transformer. If the moving contact of potentiometer is located exactly at the midpoint then the voltage between point B and ground is zero. (It is assumed that diodes are identical and the resistance in series with each diode are identical). Hence the output voltage is zero.

During negative half cycle of supply voltage (i.e., in the next half cycle) the end s_2 is negative and s_1 is positive. Now both the diodes D_1 and D_2 are reverse biased and no current flows in the secondary of transformer. Thus the point B is effectively disconnected from the transformer and the point B is at a potential same as that of point A. Hence the output voltage is equal to dc voltage applied between point A and ground.

Thus the ouput voltage oscillates between two voltage levels-ground potential and dc voltage. The output voltage will be a square wave at the frequency of excitation voltage and having an amplitude proportional to dc voltage applied at the input.

DIODE BRIDGE MODULATOR

A bridge modulator consists of four diodes connected to form a bridge as shown in fig 2.51. The circuit consists of two centre tapped transformers T_1 and T_2. The primary of T_1 is excited by a sinusoidal voltage of frequency, ω_c and it is called reference voltage.

The dc voltage to be modulated is applied between the centre point of secondary of T_1 and ground. In the transformer T_2 the centre tapped winding is primary. The output ac voltage (modulated voltage) is available across the secondary of T_2.

During one half cycle of reference voltage the polarity of the emf induced in the secondary if T_1 will forward bias the diodes D_1 and D_2 and reverse bias the diodes D_3 and D_4. The forward bias diodes acts as short and reverse bias diodes acts as open. Now point A will be at a potential same as that of dc voltage. Hence the dc voltage is available across the upper half primary winding of T_2. Therefore a current flows from P_1 to ground in the primary of T_2. Due to this an emf is induced in the secondary of T_2.

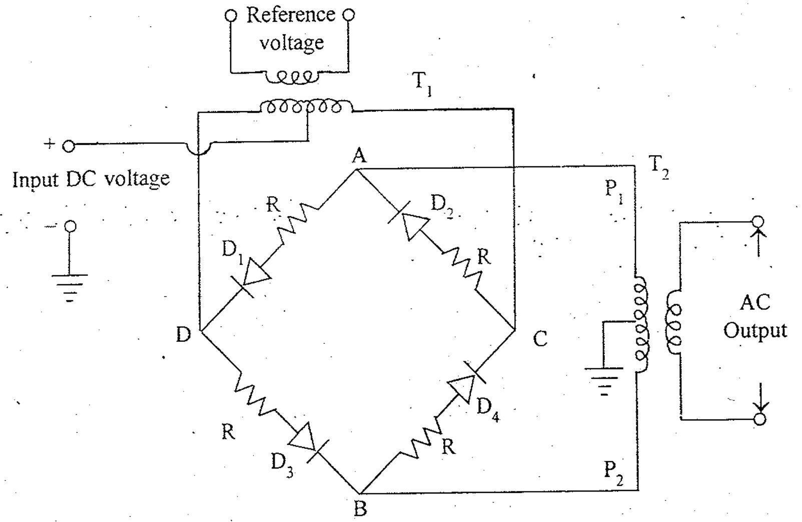


Fig 2.51 : Diode bridge modulator

During next half cycle of reference voltage the polarity of the emf induced in the secondary of T_1 will forward bias the diodes D_3 and D_4 and reverse bias the diodes D_1 and D_2. Now point B will be at a potential same as that of dc voltage. Hence the dc voltage is available across the lower half primary winding of T_2. Therefore a current flows from P_2 to ground in the primary of T_2 and the direction of current is opposite to that of previous half cycle. Due to this an emf is induced in the secondary of T_2, but the polarity of the emf induced will be opposite to that of the emf induced during previous half cycle.

In this manner, the dc input circulates current in the upper half and lower half of primary winding of T_2 in the alternate half cycles of reference voltage. Also the direction of currents are reversed in the alternate half cycles. Therefore an ac voltage is developed across the secondary of T_2. The magnitude and polarity of ac output voltage depends on the magnitude and polarity of dc voltage respectively. The frequency of ac output is same as that of reference voltage.

VIBRATOR AS A DEMODULATOR

Demodulation is reverse process of modulation. A synchronous vibrator will work as demodulator if the modulated ac voltage is applied as input to the transformer. In this arrangement the driving coil of the vibrator is excited by an ac signal whose frequency is same as that of input signal. The centre tapped winding will be the secondary and dc output is obtained between the vibrator and the centre tapped point of the transformer as shown in fig 2.52.

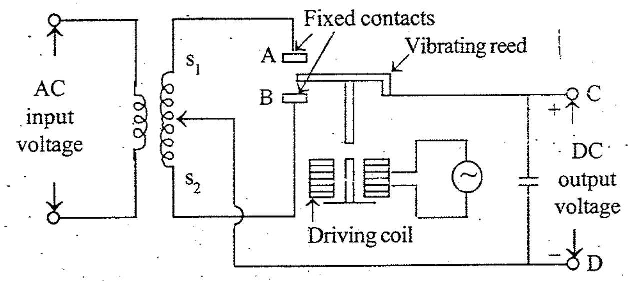


Fig 2.52 : Synchronous vibrator as a demodulator

The vibrator oscillates (vibrates) in synchronism with the excitation voltage. They are designed to operate in the range of 50 to 400 Hz. The choice of excitation frequency depends on signal frequency. During positive half cycle of ac input, the point s_1 of secondary is positive and at the same time, the vibrator will be in contact with fixed contact A. Thus the point C of the output voltage is positive.

During negative half cycle of ac input the point s_2 of secondary is positive and at the same time, the vibrator will be in contact with fixed contact B. Thus the point C of the output voltage is positive. Hence it is observed that, for both half cycles of ac input, the voltage available at the output has same polarity and so it is dc. The magnitude of the output dc voltage depends on the magnitude of input ac voltage. The polarity of the output dc voltage depends on the phase of ac input voltage.

DIODE DEMODULATOR

A diode demodulator is shown in fig 2.53. An ac excitation voltage of frequency same as that of input ac signal is applied to the primary of transformer T_1.

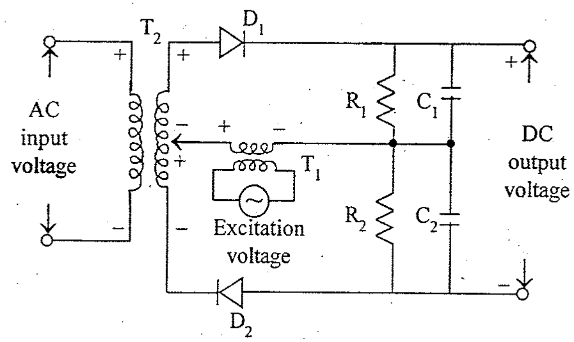


Fig 2.53 : Diode demodulator

Let the ac input is in phase with excitation voltage. During positive half cycle of input ac voltage, the polarities of induced emfs on the secondary of the transformers are as shown in fig 2.53. Now both the diodes D_1 and D_2 will conduct. But voltage available across D_1 is more than that available for D_2. This is due to additive secondary voltages in the closed path of D_1 and subtractive secondary voltages in the closed path of D_2. Therefore the diode D_1 conducts more current than D_2, which results in greater charge storage in C_1 than C_2. Hence the polarity of output voltage is as shown in fig 2.53.

During the next half cycle of ac input, the polarity of the secondary emfs are exactly opposite to the polarity shown in fig 2.53. Hence both the diodes are reverse biased and acts as open circuit. Now the capacitors C_1 and C_2 functioning as filter maintain the dc output essentially constant during these half cycles.

For a successful operation it is essential that the excitation voltage in the secondary of T_1 should be greater in amplitude than the voltage induced in one half of secondary of T_2.

If the ac voltage applied at the input is in phase opposition with excitation voltage then during one half cycle the diode D_2 conducts more current than D_1. As a result the polarity of dc output voltage is opposite to the polarity shown in fig 2.53. During next half cycle the capacitors maintain the same dc voltage levels.

2.11 GEAR TRAINS

Gear trains are used in control systems to alter the speed to torque ratio of the rotational power transmitted from motor to load. This is necessary to match the torque requirement of the load to that of motor. Usually a servomotor operates at high speed but has low torque. To drive a load with high torque and low speed using a servomotor the torque magnification and speed reduction are achieved by gear trains. The gear train in mechanical rotational system is analogous to transformer in electrical system.

Consider a motor driving a load through a gear train as shown in fig 2.54. Let the gear train consists of two gears with teeth N_1 and N_2. The gear connected to motor shaft is called primary gear and the gear connected to load shaft is called secondary gear.

Let
N_1 = Number of teeth in gear-1
r_1 = Radius of gear-1
θ_1 = Angular displacement of motor shaft
J_1 = Moment of inertia of motor and gear-1
B_1 = Viscous friction coefficient of motor and gear-1
T_1 = Load torque on gear-1
T_m = Torque developed by motor
N_2 = Number of teeth in gear-2
r_2 = Radius of gear-2
θ_2 = Angular displacement of load shaft
J_2 = Moment of inertia of load and gear-2
B_2 = Viscous friction coefficient of load and gear-2
T_2 = Torque transmitted to gear-2 from gear-1
T_l = Load torque.

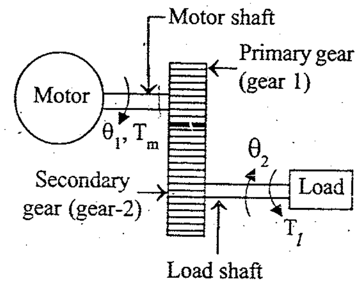


Fig 2.54 : Gear train system

The torque developed by motor is balanced by the sum of load torque requirement on gear-1 and opposing torques due to J_1 and B_1. Hence the torque balance equation for motor shaft is given by,

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + T_1 = T_m \quad .....(2.97)$$

The torque transmitted to gear-2 is balanced by the sum of load torque and opposing torques due to J_2 and B_2. Hence the torque balance equation for load shaft is given by

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + T_l = T_2 \quad .....(2.98)$$

When the motor drives the load, the linear distance travelled by each gear is same. The linear distance travelled by a gear is given by the product of the radius and angular displacement.

Linear distance travelled by the gear $= \theta_1 r_1 = \theta_2 r_2$

$$\therefore \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} \quad .....(2.99)$$

The number of teeth in each gear is proportional to its radius.

$$\therefore N_1 \propto r_1 \quad \text{and} \quad N_2 \propto r_2$$

$$\text{Hence, } \frac{N_1}{N_2} = \frac{r_1}{r_2} \quad .....(2.100)$$

From equation (2.99) and (2.100) we get,

$$\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} \quad .....(2.101)$$

In ideal gear train system, there is no power loss in transmission. Hence work done by both the gears are equal. The work done by a gear is given by the product of torque acting on it and its angular displacement.

$$\text{Work done by the gear} = T_1 \theta_1 = T_2 \theta_2 \quad .....(2.102)$$

$$\therefore \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} \quad .....(2.103)$$

On differentiating equation (2.102) we get,

$$T_1 \frac{d\theta_1}{dt} = T_2 \frac{d\theta_2}{dt} \quad .....(2.104)$$

$$\therefore T_1 \omega_1 = T_2 \omega_2$$

$$\text{and } \frac{T_1}{T_2} = \frac{\omega_2}{\omega_1} \quad .....(2.105)$$

where ω_1 and ω_2 are angular speeds of motor shaft and load shaft respectively.

On differentiating equation (2.104) we get,

$$T_1 \frac{d^2\theta_1}{dt^2} = T_2 \frac{d^2\theta_2}{dt^2}$$

$$\therefore T_1 \alpha_1 = T_2 \alpha_2$$

$$\text{and } \frac{T_1}{T_2} = \frac{\alpha_2}{\alpha_1} \quad .....(2.106)$$

where α_1 and α_2 are angular acceleration of motor shaft and load shaft respectively.

From equations (2.99), (2.100), (2.103), (2.105) and (2.106) we can write,

$$\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1} = \frac{\alpha_2}{\alpha_1} \quad .....(2.107)$$

From equation (2.107) we can say that,

When $N_1 > N_2$, the gear train increase the speed and reduce the torque.

When $N_1 = N_2$, there is no change in speed and torque.

When $N_1 < N_2$, the gear train decrease the speed and increase the torque.

Torque equation referred to motor shaft

In systems with gear trains, it will be easier for analysis if we transfer the torques acting on the system to one of the side.

From equation (2.107) we can write,

$$\alpha_2 = \alpha_1 \frac{N_1}{N_2}, \quad \therefore \frac{d^2\theta_2}{dt^2} = \frac{d^2\theta_1}{dt^2} \frac{N_1}{N_2} \quad .....(2.108)$$

$$\omega_2 = \omega_1 \frac{N_1}{N_2}, \quad \therefore \frac{d\theta_2}{dt} = \frac{d\theta_1}{dt} \frac{N_1}{N_2} \quad .....(2.109)$$

$$T_2 = T_1 \frac{N_2}{N_1} \quad .....(2.110)$$

On substituting for $\frac{d^2\theta_2}{dt^2}$, $\frac{d\theta_2}{dt}$ and T_2 from equation (2.108), (2.109) and (2.110) in equation (2.98) we get,

$$J_2 \frac{d^2\theta_1}{dt^2}\frac{N_1}{N_2} + B_2 \frac{d\theta_1}{dt}\frac{N_1}{N_2} + T_l = T_1 \frac{N_2}{N_1}$$

On multiplying the equation by N_1/N_2 we get,

$$J_2 \frac{d^2\theta_1}{dt^2}\left(\frac{N_1}{N_2}\right)^2 + B_2 \frac{d\theta_1}{dt}\left(\frac{N_1}{N_2}\right)^2 + T_l\left(\frac{N_1}{N_2}\right) = T_1 \qquad .....(2.111)$$

On substituting for T_1 from equation (2.111) in equation (2.97) we get,

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + J_2 \frac{d^2\theta_1}{dt^2}\left(\frac{N_1}{N_2}\right)^2 + B_2 \frac{d\theta_1}{dt}\left(\frac{N_1}{N_2}\right)^2 + T_l\left(\frac{N_1}{N_2}\right) = T_m$$

$$\left(J_1 + J_2\left(\frac{N_1}{N_2}\right)^2\right)\frac{d^2\theta_1}{dt^2} + \left[B_1 + B_2\left(\frac{N_1}{N_2}\right)^2\right]\frac{d\theta_1}{dt} + T_l\left(\frac{N_1}{N_2}\right) = T_m$$

$$J_{eq1}\frac{d^2\theta_1}{dt^2} + B_{eq1}\frac{d\theta_1}{dt} + T_l\left(\frac{N_1}{N_2}\right) = T_m \qquad .....(2.112)$$

Where, $J_{eq1} = J_1 + J_2\left(\frac{N_1}{N_2}\right)^2$ = Total equivalent moment of inertia referred to motor shaft.

$B_{eq1} = B_1 + B_2\left(\frac{N_1}{N_2}\right)^2$ = Total equivalent viscous friction coefficient referred to motor shaft.

The equation (2.112) is the torque equation of the gear train system referred to motor shaft.

Torque equation referred to load shaft

From equation 2.107 we can write,

$$\alpha_1 = \alpha_2 \frac{N_2}{N_1}, \qquad \therefore \frac{d^2\theta_1}{dt^2} = \frac{d^2\theta_2}{dt^2}\frac{N_2}{N_1} \qquad .....(2.113)$$

$$\omega_1 = \omega_2 \frac{N_2}{N_1}, \qquad \therefore \frac{d\theta_1}{dt} = \frac{d\theta_2}{dt}\frac{N_2}{N_1} \qquad .....(2.114)$$

$$T_1 = T_2 \frac{N_1}{N_2} \qquad .....(2.115)$$

On substituting for $\frac{d^2\theta_1}{dt^2}$, $\frac{d\theta_1}{dt}$ and T_1 from equation (2.113), (2.114) and (2.115) in equation (2.97) we get,

$$J_1 \frac{d^2\theta_2}{dt^2}\frac{N_2}{N_1} + B_1 \frac{d\theta_2}{dt}\frac{N_2}{N_1} + T_2\frac{N_1}{N_2} = T_m \qquad .....(2.116)$$

On substituting for T_2 from equation (2.98) in equation (2.116) we get,

$$J_1 \frac{d^2\theta_2}{dt^2}\frac{N_2}{N_1} + B_1 \frac{d\theta_2}{dt}\frac{N_2}{N_1} + \left(J_2\frac{d^2\theta_2}{dt^2} + B_2\frac{d\theta_2}{dt} + T_l\right)\frac{N_1}{N_2} = T_m$$

On multiplying the equation by N_2/N_1 we get,

$$J_1 \frac{d^2\theta_2}{dt^2}\left(\frac{N_2}{N_1}\right)^2 + B_1 \frac{d\theta_2}{dt}\left(\frac{N_2}{N_1}\right)^2 + J_2\frac{d^2\theta_2}{dt^2} + B_2\frac{d\theta_2}{dt} + T_l = T_m\left(\frac{N_2}{N_1}\right)$$

$$\left(J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right)\frac{d^2\theta_2}{dt^2} + \left[B_1\left(\frac{N_2}{N_1}\right)^2 + B_2\right]\frac{d\theta_2}{dt} + T_l = T_m\left(\frac{N_2}{N_1}\right)$$

$$J_{eq2}\frac{d^2\theta_2}{dt^2} + B_{eq2}\frac{d\theta_2}{dt} + T_l = T_m\left(\frac{N_2}{N_1}\right) \qquad .....(2.117)$$

Where,

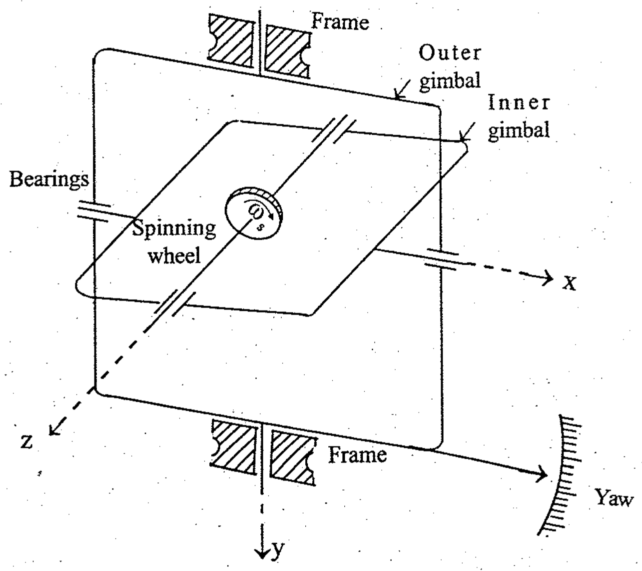
$$J_{eq2} = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2$$ = Total equivalent moment of inertia referred to load shaft.

$$B_{eq2} = B_1\left(\frac{N_2}{N_1}\right)^2 + B_2$$ = Total equivalent viscous friction coefficient referred to load shaft.

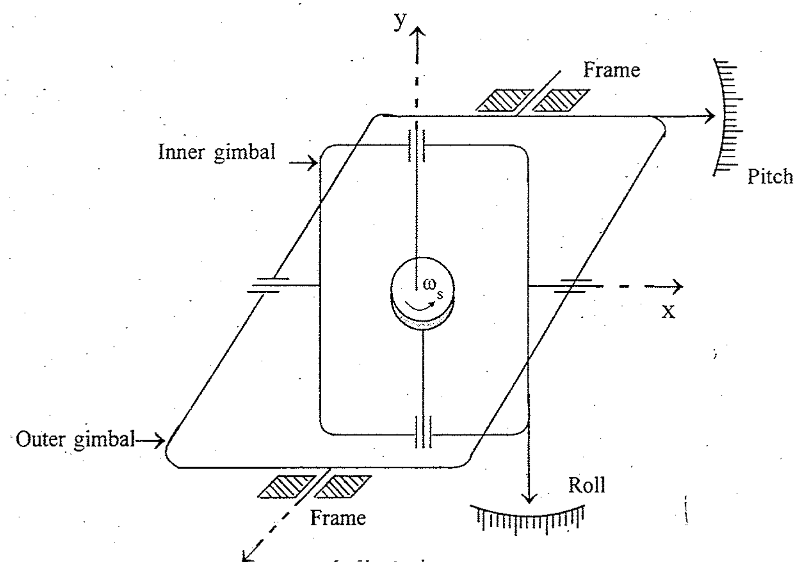
The equation (2.117) is the torque equation of gear-train system referred to load shaft.

2.12. GYROSCOPE

The Gyroscope consists of a wheel or disc (called gyro wheel) mounted on an axle. The axle is supported by bearings fixed on a frame-work called inner gimbal (linkages). The mechanism is made so much flexible, that they can turn in any direction. The inner gimbal is free to rotate in the bearing placed in the outer gimbal which in turn can rotate in the frame to which the gyro is attached.



a. Directional Gyroscope



b. Vertical gyroscope

Fig 2.55 : Gyroscope

The gyro wheel is made to spin by means of a synchronous motor whose rotor is mounted on the axis of the spinning wheel. The position of spin axis relative to the earth determines the basic type of gyro. When the spin axis is in horizontal position, the gyroscope is referred to as positional or directional gyro and when the spin axis is in vertical position it is called vertical gyro. The constructional features of a directional gyroscope and vertical gyroscope are shown in fig 2.55.

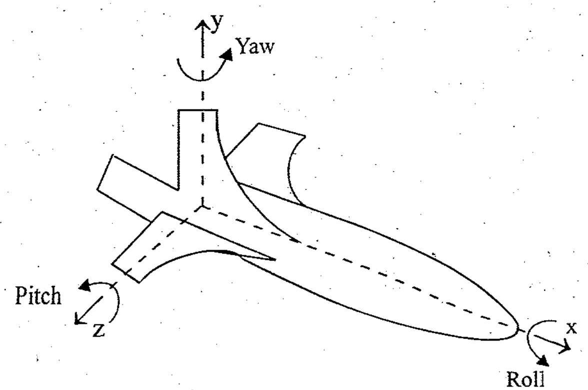


Fig 2.56 : Motions of vechicle flying in air

Vertical and directional gyros are used to measure or indicate the three rotational or angular motions-pitch, roll and azimuth(yaw) of vehicles flying in air. The rotational motions are illustrated in fig 2.56. The roll is angular motion about x-axis called longitudinal axis, azimuth or yaw is angular motion about y-axis called normal axis and pitch is angular motion about z-axis called lateral axis.

PRINCIPLE OF GYROSCOPE

The Gyroscope is developed based on the following two physical principles.

1. A spinning wheel in the absence of external torque, maintains the direction of its spin axis in space. Such a spinning wheel is known as a free gyro.

2. When a torque is applied to an axis inclined to the spin axis of a wheel, the wheel rotates in the plane of the two axes in a direction tending to align the spin axis with the input torque axis. This rotation is termed as precession.

Note : The gyroscope is the technical application of the toy called spinning top. Like the top, a gyroscope resists the changes in the position of the axis about which its is spinning.

A directional gyroscope is used to measure azimuth or yaw of a vehicle. In this the rotor spin axis is maintained in north-south direction. The inner gimbal is maintained in horizontal plane. The outer gimbal moves in vertical plane and carries a pointer to indicate the azimuth or yaw. The yaw scale is fixed on the frame *(Refer fig 2.55a)*. A rotation of the vehicle about the y-axis causes the yaw-scale to rotate. Since the outer gimbal and attached pointer remains fixed in reference direction , the relative motion measures the yaw. The motions about x-axis and z-axis have no effect on the measured angle of yaw.

A vertical gyro is used to measure the pitch and roll of a vehicle. The pitch and roll scale are fixed to the vehicle frame *(Refer fig 2.55 b)*. If there is an angular motion about z-axis(or if there is a pitch motion) then the frame and the pitch scale rotate about the z-axis so that the pointer indicates the pitch. The pitch motion causes the roll scale to move at right angles to the roll pointer and so the roll reading does not change. If there is an angular motion about x-axis (or if there is a roll motion) then the roll scale rotates and the roll pointer indicates the roll.

***Note :** If an electrical signal is desired then pitch, roll and azimuth or yaw scales could be replaced by synchros.*

APPLICATION OF GYROSCOPES

The practical gyros can be classified into two types-free gyro and restrained gyro. In free gyros there are no restraining forces in any direction of the gyro and the gyro wheel remains fixed in space. The angular motion of supporting frame is not transmitted to gyro wheel. Hence the gyro measures the angular motions of the vehicle with respect to the gyro as a reference. This type of gyros are used in automatic plotting, inertial navigational equipment,artifical horizons, stable platform,etc.

The restrained gyros are either rate gyro or integrating gyro. The restrained gyros are employed to provide either derivative or integral feedback from roll, pitch or yaw to damp out vehicle oscillations about these axes.

In rate gyro, a helical spring is connected between outer gimbal and the frame and a synchro is fitted on the outer frame such that the shaft of the synchro transmitter is coupled to the z-axis of the outer gimbal. Now, if the gyro wheel turns by an angle ϕ about z-axis then the signal developed in the synchro mounted on the gyro will be proportional to $d\phi/dt$. Hence this gyro is called rate gyro.

In integrating gyro, only one axis is free to move and the gyro has a single degree of freedom with respect to the frame. The outer gimbal of a gyro is fixed to the frame to convert it into an integrating gyro. In integrating gyros it can be proved that the angular motion about y-axis is proportional to integral of the angle through which gyro wheel turns about the z-axis.

2.13 SOLVED PROBLEMS

EXAMPLE 2.1

A 300 turn, 100KΩ, potentiometer with 1% linearity uses a 20V supply. Find **(a)** Potentiometer constant in volts/turn, **(b)** Range of voltages at the mid-point setting and **(c)** Assuming that the potentiometer is perfectly linear, find the voltage at the midpoint when the potentiometer is loaded with 500KΩ resistance.

SOLUTION

(a) Potentiometer constant, $K_p = \dfrac{\text{Excitation voltage}}{\text{Number of turns}}$

$$= \frac{30}{300} = 0.1 \text{ Volts / turn.}$$

(b) $\left.\begin{array}{r}\text{Number of turns from reference}\\ \text{to midpoint}\end{array}\right\} = \dfrac{\text{Number of turns}}{2}$

$$= \frac{300}{2} = 150 \text{ Turns.}$$

$$\text{Voltage at midpoint} = K_p \times \begin{array}{l}\text{Number of turns from}\\ \text{reference to midpoint}\end{array}$$

$$= 0.1 \times 150 = 15 \text{ Volts}$$

Given that, linearity $= 1\% = 0.01 \times 30 = 0.3$

$\therefore$ Range of voltages at midpoint $= 15 \pm 0.3$

$$= 14.7 \text{ to } 15.3 \text{ Volts}$$

(c) The total resistance of the potentiometer is 100KΩ. ∴ When the potentiometer is perfectly linear, the resistance at the midpoint setting from reference is 50KΩ.

If a load resistance of 500KΩ is connected to the output terminal of the potentiometer, then the 50KΩ resistance will be parallel with 500KΩ resistance as shown in fig 2.1.1.

$\therefore$ $\left.\begin{array}{r}\text{The equivalent resistance}\\ \text{from midpoint to reference}\end{array}\right\} = \dfrac{50 \times 500}{50 + 500}$

$$= 45.45 \text{ K}\Omega$$

By voltage division rule,

$\left.\begin{array}{l}\text{The voltage at midpoint}\\ \text{from reference}\end{array}\right\} V_o = 30 \times \dfrac{45.45}{45.45 + 50}$

$$= 14.3 \text{ Volts}$$

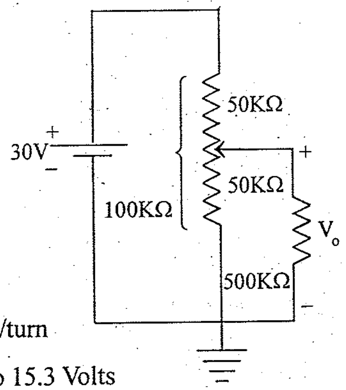


Fig 2.1.1.

RESULT

(a) Potentiometer constant, $K_p = 0.1$ Volts/turn

(b) $\left.\begin{array}{l}\text{Range of voltages at}\\ \text{midpoint with 1\% linearity}\end{array}\right\} = 14.7 \text{ to } 15.3 \text{ Volts}$

(c) $\left.\begin{array}{l}\text{Voltage at midpoint with a}\\ \text{load resistance of 500K}\Omega\end{array}\right\} = 14.3 \text{ Volts}$

EXAMPLE 2.2

A gear train consisting of two gears is used to drive a load. One gear consist of 20 teeth and the other has 10 teeth. (a) What is the ratio of the diameters of the gear?. (b) If gear-1 is rotated by an angle of 40°, then what will be the angular displacement of gear-2?. (c) If the angular speed of gear-1 is 30 rad/sec then what is the value of angular speed of gear-2?. (d) If the angular acceleration of gear-2 is 4 rad/sec^2 then find the angular acceleration of gear-1. (e) It the torque acting on gear-1 is 5 N-m then find the torque on gear-2.

SOLUTION

In gear trains the ratios of radius, teeth, displacement, velocity,etc., are governed by the equation,

$$\text{Gear ratio} = \frac{r_1}{r_2} = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1} = \frac{\alpha_2}{\alpha_1} = \frac{T_1}{T_2}$$

(a) The diameter of the gear is proportional to radius. Hence the ratio of radius is equal to the ratio of diameters.

From the equation of gear ratio, $\dfrac{r_1}{r_2} = \dfrac{N_1}{N_2}$

Where r_1, r_2 = Radius of gear-1 and gear-2

N_1, N_2 = Number of teeth in gear-1 and gear-2.

$$\therefore \text{The ratio of diameters} = \frac{r_1}{r_2} = \frac{N_1}{N_2} = \frac{20}{10} = 2$$

(b) From the equation of gear ratio we get, $\dfrac{\theta_2}{\theta_1} = \dfrac{N_1}{N_2}$

$$\therefore \text{Displacement gear-2,} \quad \theta_2 = \frac{N_1}{N_2} \times \theta_1 = \frac{20}{10} \times 40° = 80°$$

(c) From the equation of gear ratio we get, $\dfrac{\omega_2}{\omega_1} = \dfrac{N_1}{N_2}$

$$\therefore \text{Angular speed of gear-2,} \omega_2 = \frac{N_1}{N_2}\omega_1 = \frac{20}{10} \times 30 = 60 \text{ rad/sec}$$

(d) From the equation of gear ratio we get, $\dfrac{\alpha_2}{\alpha_1} = \dfrac{N_1}{N_2}$

$$\therefore \text{ Angular acceleration of gear-1, } \alpha_1 = \frac{N_2}{N_1}\alpha_2 = \frac{10}{20} \times 4 = 2 \text{ rad / sec}^2$$

(e) From the equation of gear ratio we get, $\dfrac{T_1}{T_2} = \dfrac{N_1}{N_2}$

$$\therefore \text{ Torque acting on gear-2, } T_2 = \frac{N_2}{N_1}T_1 = \frac{10}{20} \times 5 = 2.5 \text{ N - m}$$

RESULT

a)	The ratio of diameters	=	2
b)	The displacement of gear-2	=	80°
c)	Angular speed of gear-2	=	60 rad/sec
d)	Angular acceleration of gear-1	=	2 rad/sec^2
e)	Torque acting on gear-2	=	2.5 N-m

EXAMPLE 2.3

A gear train consists of 7 gear wheels as shown in fig 2.3.1. The number of teeth in each gear wheel is shown in the figure. (a) If the displacement in gear-1 is 2 rad clockwise, then what will be the displacement of gear-4 and gear-7. (b) If the angular velocity of gear-6 is 20 rad/sec then what will be the angular velocities of gear-1 and gear-2. (c) If the torque acting on gear-1 is 10 N-m then what will be the torque on gear-3 and gear-7.

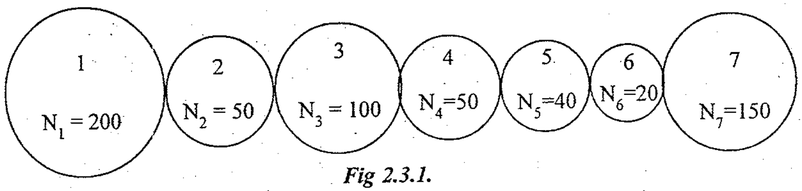


Fig 2.3.1.

SOLUTION

The gear ratio between any two gears x and y in a gear train is given by

$$\text{Gear ratio} = \frac{r_x}{r_y} = \frac{N_x}{N_y} = \frac{\theta_y}{\theta_x} = \frac{\omega_y}{\omega_x} = \frac{\alpha_y}{\alpha_x} = \frac{T_x}{T_y}$$

(a) The gear ratio between gear-1 and gear-4 is given by $\dfrac{N_1}{N_4} = \dfrac{\theta_4}{\theta_1}$

$$\therefore \text{ Angular displacement of gear - 4, } \theta_4 = \frac{N_1}{N_4} \times \theta_1 = \frac{200}{50} \times 2 = 8 \text{ rad}$$
(anticlockwise)

The gear ratio between gear-1 and gear-7 is given by $\dfrac{N_1}{N_7} = \dfrac{\theta_7}{\theta_1}$

$$\therefore \text{ Angular displacement of gear - 7, } \theta_7 = \frac{N_1}{N_7} \times \theta_1 = \frac{200}{150} \times 2 = 2.67 \text{ rad}$$
(clockwise)

***Note :** If gear-1 rotates in clockwise direction then all odd numbered gears will rotate in clockwise direction and even numbered gears will rotate in anticlockwise direction.*

(b) The gear ratio between gear-1 and gear-6 is given by $\dfrac{N_1}{N_6} = \dfrac{\theta_6}{\theta_1} = \dfrac{\omega_6}{\omega_1}$

$$\therefore \text{ Angular velocity of gear - 1, } \omega_1 = \frac{N_6}{N_1} \times \omega_6 = \frac{20}{200} \times 20 = 2 \text{ rad / sec}$$
(clockwise)

The gear ratio between gear-3 and gear-6 is given by $\dfrac{N_3}{N_6} = \dfrac{\theta_6}{\theta_3} = \dfrac{\omega_6}{\omega_3}$

$$\therefore \text{ Angular velocity of gear - 3, } \omega_3 = \frac{N_6}{N_3} \times \omega_6 = \frac{20}{100} \times 20 = 4 \text{ rad / sec}$$
(clockwise)

(c) The gear ratio between gear-1 and gear-3 is given by $\dfrac{N_1}{N_3} = \dfrac{\theta_3}{\theta_1} = \dfrac{T_1}{T_3}$

$$\therefore \text{ Torque acting on gear - 3, } T_3 = \frac{N_3}{N_1} \times T_1 = \frac{100}{200} \times 10 = 5 \text{ N - m}$$

(d) The gear ratio between gear-1 and gear-7 is given by $\dfrac{N_1}{N_7} = \dfrac{\theta_7}{\theta_1} = \dfrac{T_1}{T_7}$

$$\therefore \text{ Torque acting on gear - 7, } T_7 = \frac{N_7}{N_1} \times T_1 = \frac{150}{200} \times 10 = 7.5 \text{ N - m.}$$

RESULT

(a) Angular displacement of gear-4, $\theta_4 = 8$ rad (anticlockwise)
Angular displacement of gear-7, $\theta_7 = 2.67$ rad (clockwise)

(b) Angular velocity of gear-1, $\omega_1 = 2$ rad/sec (clockwise)
Angular velocity of gear-3, $\omega_3 = 4$ rad/sec (clockwise)

(c) Torque acting on gear-3, $T_3 = 5$ N-m
Torque acting on gear-7, $T_3 = 7.5$ N-m.

EXAMPLE 2.4

Consider the speed control system shown in fig 2.4.1., to control the angular speed, ω of the load. The generator field time constant is negligible and it is driven at constant speed giving a generated voltage of K_g Volts/field amp. The generated emf is used to run the separately excited motor which has a back emf of K_b Volts/(rad/sec). The motor develops a torque of K_T N-m/amp. The motor and its load have a combined moment of inertia of J Kg-m^2 and negligible friction. A tachometer is employed for speed feedback which develops a feedback voltage of K_t volts/(rad/sec). The desired speed is set through a potentiometer. The difference between reference voltage, e_r and the feedback voltage, e_t is amplified using an amplifier which produces a field current of K_A amps/volt.

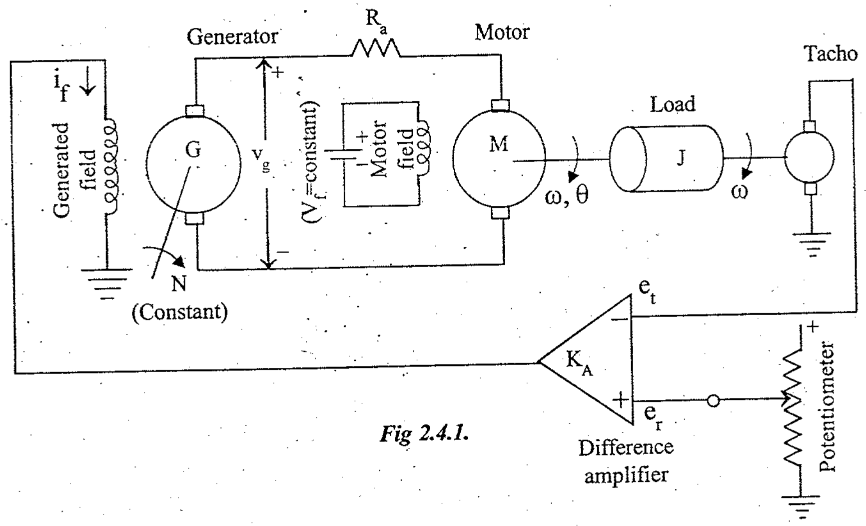


Fig 2.4.1.

Given that: K_A = 4 amps/volts ; K_T = 1.5 N-m/amp,

K_g = 50 volts/amp; K_t = 0.2 volts/(rad/sec).

K_b = 0.75 volts/(rad/sec) ; $R_a = 1\Omega$, $J = 6$ Kg-m^2.

Draw a block diagram of the system and determine the transfer function $\omega(s)/E(s)$.

SOLUTION

From the differential equations governing the individual component of the system, the block diagram of the individual component can be drawn. Then by combining the block diagram of individual component, the overall block diagram of the system is obtained.

DIFFERENCE AMPLIFIER

The difference amplifier compares the reference voltage, e_r (corresponding to desired speed) and the feedback voltage, e_t from tachogenerator. It produces a current proportional to the difference between these two voltages. This current is used to excite the generator field.

Field current, $i_f \propto (e_r - e_t)$

$$\therefore i_f = K_A (e_r - e_t) \quad .....(2.4.1)$$

Where, K_A = amplifier gain in amps/volt

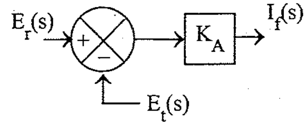


Fig 2.4.2 : Block diagram of difference amplifier

On taking laplace transform of equation (2.4.1) we get

$$I_f(s) = K_A[E_r(s) - E_t(s)] \quad .....(2.4.2)$$

The equation (2.4.2) is represented by a block diagram as shown in fig (2.4.2).

GENERATOR

Since the speed of generator is constant, the generator develops an emf proportional to field current.

Generated emf, $v_g \propto i_f$

$$\therefore v_g = K_g i_f \quad .....(2.4.3)$$

Where K_g = Generator gain in volts/amp

On taking laplace transform of equation(2.4.3) we get

$$V_g(s) = K_g I_f(s) \quad .....(2.4.4)$$

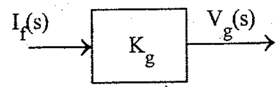


Fig 2.4.3 : Block diagram of generator

The equation(2.4.4) is represented by a block diagram as shown in fig 2.4.3.

MOTOR

The applied emf, v_g on the motor armature is balanced by the armature drop and back emf, e_b

$$v_g = i_a R_a + e_b \quad .....(2.4.5)$$

On taking laplace transform of equation (2.4.5) we get,

$$V_g(s) = I_a(s)R_a + E_b(s)$$

$$\therefore I_a(s) = \left[V_g(s) - E_b(s)\right]\frac{1}{R_a} \qquad .....(2.4.6)$$

Since the flux (or field excitation) of the motor is constant, the back-emf of the motor is directly proportional to angular speed, ω and the torque, T developed by the motor is directly proportional to armature current, i_a.

$\therefore$ Back emf, $e_b \alpha\, \omega$ or $e_b = K_b \omega$ .....(2.4.7)

Torque, $T \alpha\, i_a$ or $T = K_T\, i_a$ .....(2.4.8)

Where K_b = Back emf constant

and K_T = Torque constant

On taking laplace transform of equation (2.4.7) and (2.4.8) we get,

$$E_b(s) = K_b\, \omega(s) \qquad .....(2.4.9)$$

$$T(s) = K_T\, I_a(s) \qquad .....(2.4.10)$$

The torque developed by the motor is used to run the load by overcoming the opposition offered by the moment of inertia, J.

$$\therefore \text{Torque, } T = J\frac{d\omega}{dt} \qquad ....(2.4.11)$$

On taking laplace transform of equation (2.4.11) with zero initial conditions we get

$$T(s) = Js\, \omega(s)$$

$$\therefore \omega(s) = (1/Js)\, T(s) \qquad .....(2.4.12)$$

Using the equations (2.4.6), (2.4.9), (2.4.10) and (2.4.12) the block diagram of the motor with load is obtained as shown in fig 2.4.4.

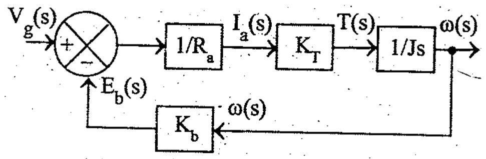


Fig 2.4.4 : Block diagram of motor

TACHOGENERATOR

The tachogenerator develops an emf proportional to angular speed, ω. The emf produced by tachogenerator is used as feedback signal.

Tachogenerator emf, $e_t \alpha\, \omega$

$$\therefore e_t = K_t \omega \qquad .....(2.4.13)$$

Where K_t = Tachogenerator gain constant in volts/(rad/sec)

On taking laplace transform of equation (2.4.13) we get

$$E_t(s) = K_t\, \omega(s) \qquad .....(2.4.14)$$

The equation (2.4.14) is represented by a block diagram as shown in fig 2.4.5.

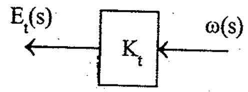


Fig 2.4.5 : Block diagram of tachogenerator

BLOCK DIAGRAM OF THE SPEED CONTROL SYSTEM

The block diagram of the speed control system is obtained by combining the block diagrams shown in figures(2.4.1) to (2.4.5) and the overall block diagram is shown in fig(2.4.6)

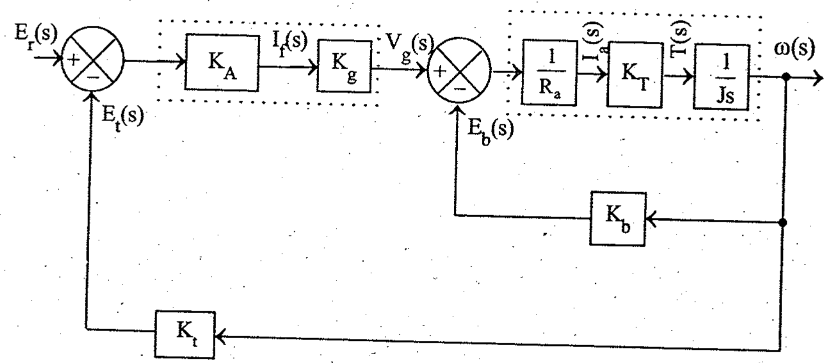
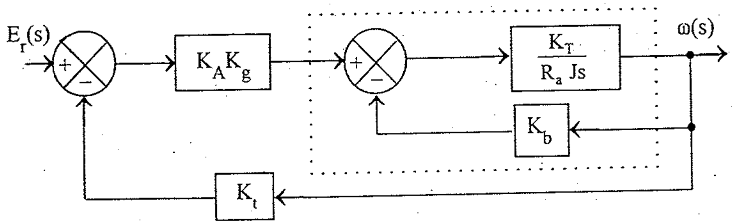


Fig 2.4.6 : Block diagram of speed control system

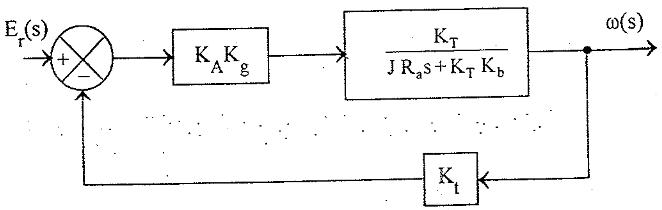
TRANSFER FUNCTION OF THE SPEED CONTROL SYSTEM

The block diagram of the system can be reduced by using the rules of block diagram algebra as shown below.

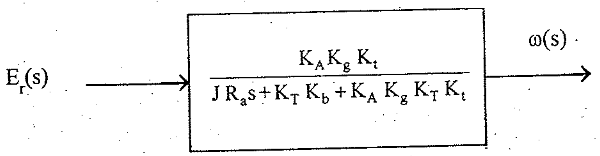
Step 1 : Combining the blocks in cascade



Step 2 : Eliminating the inner feedback loop.



Step 3 : Combining the blocks in cascade and eliminating the feedback path



$\therefore$ The transfer function of the system $\dfrac{\omega(s)}{E_r(s)} = \dfrac{K_A K_g K_T}{J R_a s + K_T K_b + K_A K_g K_T K_t}$

On substituting the values of various constants we get

$$\frac{\omega(s)}{E_r(s)} = \frac{4 \times 50 \times 1.5}{6 \times 1 \times s + 1.5 \times 0.75 + 4 \times 50 \times 1.5 \times 0.2}$$

$$= \frac{300}{6s + 61.125} = \frac{300/6}{s + 61.125/6} = \frac{50}{s + 10.1875}$$

2.14 SHORT QUESTION AND ANSWER

Q2.1. *What are the basic components of an automatic control system ?*

The basic components of an automatic control system are the following

a. Error detector
b. Amplifier and controller
c. Actuator (Power actuator)
d. Plant
e. Sensor or feedback system

Q2.2 *What is automatic controller ?*

The combined unit of error detector, amplifier and controller is called automatic controller.

Q2.3 *What is the need for a controller?*

The controller is provided to modify the error signal for better control action.

Q2.4 *What are the different types of controllers?*

The different type of controller used in control system are P,PI,PD, and PID controllers.

Q2.5 *What is a Potentiometer?*

A potentiometer is a device that can be used to convert linear or angular displacement into a voltage. Basically it is a variable resistance whose value varies according to the angular or linear displacement of the wiper contact.

Q2.6 *What are the differences between AC and DC potentiometers?*

1. The output of AC potentiometer is a modulated voltage and so it has to be demodulated to get the error or control signal. But the output of DC potentiometer need not be demodulated.

2. The AC potentiometer will have inductive effects. But the DC potentiometer will not have any inductive effects.

Q2.7 *What are the applications of Potentiometer?*

The potentiometers are used to convert a linear or angular displacement into a proportional electrical signal. It is also used as error detector in which it produces an output voltage which is proportional to the difference between two linear or angular displacements.

Q2.8 *A 10KΩ potentiometer whose center point is grounded at the two fixed ends are excited by +10V and –10V. The potentiometer has a total angular motion of 350°. What is the gain constant of potentiometer.*

$$\text{Gain constant, } K_p = \frac{\text{Total excitation voltage}}{\text{Total angular motion}}$$

$$= \frac{20}{350° \times \pi / 180} = 3.27 \text{ V / rad.}$$

Q2.9 *Determine the number of turns of wire needed to provide a potentiometer with the resolution of 0.05%.*

$$\% \text{ resolution} = \frac{100}{\text{number of turns}} \qquad \therefore 0.05 = \frac{100}{N}$$

$$N = \frac{100}{0.05} = 2000 \text{ turns}$$

Q2.10 *A helical 5 turn pot has a resistance of 10KΩ and 9000 winding turns. If the measured resistance at its midpoint setting is 5050Ω, then what is the linearity?*

Deviation from nominal at midpoint = 5050–5000=50

$$\text{Linearity} = \frac{\text{Deviation from nominal}}{\text{Total resistance}} \times 100 = \frac{50}{10000} \times 100 = 0.5\%$$

Q2.11 *What is Synchro?*

A synchro is a device used to convert an angular motion to an electrical signal or viceversa. It works on the principle of a rotating transformer (induction motor)

Q2.12 *What is synchro pair?*

A synchro pair is a system formed by interconnection of the devices-synchro transmitter and synchro control transformer. A synchro pair is used to either transmit an angular motion from one place to another or employed to produce an error voltage proportional to the difference between two angular motions.

Q2.13 *What are the differences between synchro transmitter and controlled transformer?*

1. The rotor of transmitter is of dumb bell shape. But the rotor of control transmitter is cylindrical.
2. The rotor winding of transmitter is excited by an AC voltage. In control transformer, the induced emf in the rotor is used as an output signal (error signal)

Q2.14 *What is electrical zero of a synchro?*

The electrical zero of a synchro transmitter is a position of rotor at which one of the coil to coil voltage is zero. Any angular motion of the rotor is measured with respect to the electrical zero position of the rotor.

For the arrangement shown in fig Q2.14, the coil s_2 will have maximum emf induced in it and the coil to coil voltage e_{S3S1} will be zero and the rotor is in electrical zero position.

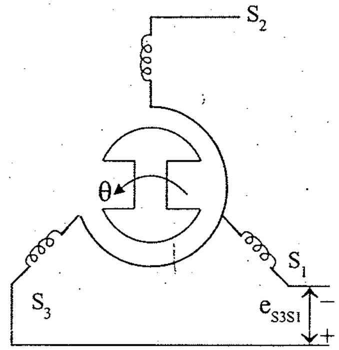


Fig Q2.14 :Electrical zero position of synchro transmitter

Q2.15 *What is Null position in Synchro?*

The Null position of a synchro control transformer in a servo system is that position of its rotor for which the output voltage on the rotor winding is zero, with the transmitter in its electrical zero position.

Q2.16 *What is aligned position of a synchro pair?*

In the aligned position of a synchro pair, the transmitter rotor will be in electrical zero position and the control transformer rotor will be in null position. The angular separation of both rotor axis in aligned position is 90°. The error signal is zero in the aligned position.

Q2.17 *What are the applications of Synchro?*

The synchros are used in positional control systems (servomechanism) as error detector and to convert angular displacements to proportional electrical signals. It is also used in control systems to transmit angular motions from one place to another.

Q2.18 *What are the trade names of Synchros?*

The trade names for synchros are selsyn, Autosyn and Telesyn.

Q2.19 What is Proportional controller and what are its advantages?

The Proportional controller is a device that produces a control signal which is proportional to the input error signal.

The advantages in the proportional controller are improvement in steady-state tracking accuracy, disturbance signal rejection and the relative stability. It also makes a system less sensitive to parameter variations.

Q2.20 What is the drawback in P-controller?

The drawback in P-controller is that it develop a constant steady-state error.

Q2.21 What is integral control action?

In integral control action, the control signal is proportional to integral of error signal.

Q2.22 What is the advantage and disadvantage in integral controller?

The advantage in Integral controller is that it eliminates or reduces the steady-state error. The disadvantage is that it can make a system unstable.

Q2.23 Write the transfer function of P, PI, PD and PID controllers.

The transfer function of P-controller, $\dfrac{U(s)}{E(s)} = K_p$

The transfer function of PI-controller, $\dfrac{U(s)}{E(s)} = K_p\left(1+\dfrac{1}{T_i s}\right)$

The transfer function of PD-controller, $\dfrac{U(s)}{E(s)} = K_p(1+T_d s)$

The transfer function of PID-controller, $\dfrac{U(s)}{E(s)} = K_p\left(1+\dfrac{1}{T_i s}+T_d s\right)$

Where K_p = Proportional gain

T_i = Integral time constant

T_d = Derivative time constant.

Q2.24 What is Reset rate?

The reset rate is the reciprocal of integral time or reset time. The reset rate is the number of times per minute that the proportional part of the control action is duplicated and it is measured in terms of repeats/minute.

Q2.25 Why derivative control is not employed in isolation?

A derivative control mode in isolation produces no corrective efforts for any constant errors. Because, it acts only on rate of change of error.

Q2.26 What is PI controller?

The PI controller is a device which produces a control signal consisting of two terms-one proportional to error signal and the other proportional to the integral of error signal.

Q2.27 What is PD controller?

The PD controller is a device which produces a control signal consisting of two terms-one proportional to error signal and the other proportional to the derivative of error signal.

Q2.28 What is PID controller?

The PID controller is a device which produces a control signal consisting of three terms-one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.

Q2.29 Give an example of electronic PID controller

The electronic PID controller can be realized by an op-amp amplifier with integral and derivative action followed by sign changer, as shown in figure Q2.29.

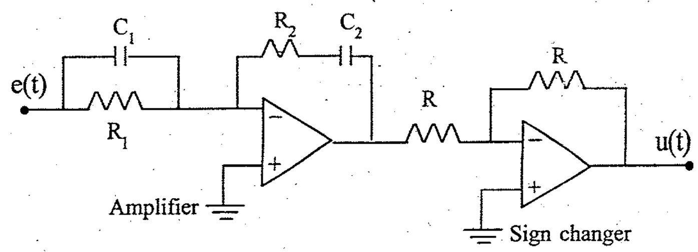


Fig Q2.29

Q2.30 Sketch the step response of a P and PI controller ?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.30.

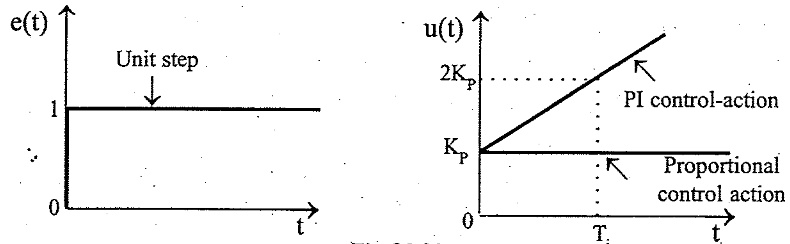


Fig Q2.30

Q2.31 Sketch the ramp response of P,PD and PID controller?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.31.

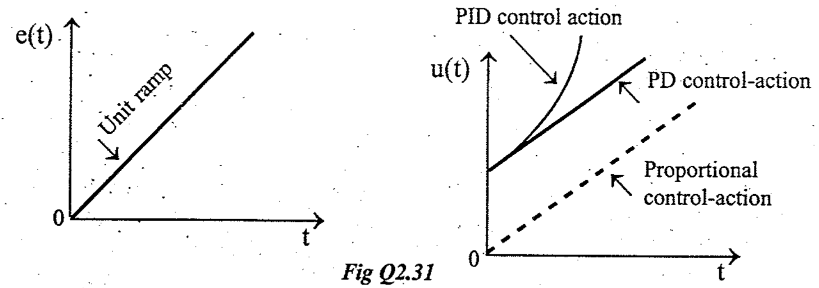


Fig Q2.31

Q2.32 What is Servomotor?

The motors used in automatic control system or in servomechanism are called servomotors. They are used to convert electrical signal to angular motion.

Q2.33 What are the characteristic of servomotors?

1. Linear relationship between the speed and electric control signal
2. Steady-state stability
3. Wide range of speed control.
4. Linearity of mechanical characteristic throughout the entire speed range.
5. Low mechanical and electrical inertia and
6. Fast response.

Q2.34 Compare the AC and DC servomotors?

S.No	DC servomotor	AC servomotor
1.	Higher power output.	Relatively lesser power output than a DC servomotor of same size.
2.	Characteristics are linear.	Characteristics are non-linear.
3.	Fast response due to low electrical and mechanical time constant.	The response is relatively slower than DC servomotors due to higher values of time constants.
4.	Suitable for large power applications.	Suitable for low power applications .

Q2.35 What are the advantages of Permanent magnet DC servomotors?

1. A simpler, more reliable motor because the field power supply is not required.
2. Higher efficiency due to the absence of field losses.
3. Field flux is less affected by temperature rise.
4. Less heating, making it possible to totally enclose the motor.
5. No possibility of over speeding due to loss of field.
6. A more linear torque Vs speed curve and
7. Higher power output at the same dimensions and temperature limitations.

Q2.36. What are the special features of DC servomotors ?

1. The number of slots and commutator segments is large to improve commutation.
2. Compoles and compensating windings are provided to eliminate sparking.
3. The diameter to length ratio is kept low to reduce inertia.
4. Oversize shafts are employed to withstand the high torque stress.
5. Eddy currents are reduced by complete lamination of the magnetic circuit and by using low-loss steel.

Q2.37 What is the difference between ac servomotor and two phase induction motor?

1. The ac servomotor has low value of X/R to achieve linear speed-torque characteristics. But conventional induction motor will have large values of X/R for higher efficiency.
2. The ac servomotor has low inertia rotor. The inertia of the rotor is reduced by reducing the diameter or by drag-cup construction.

Q2.38 What are the different types of rotor that are used in ac servomotor?

The types of rotors of ac servomotor are squirrel-cage rotor and drag-cup rotor.

Q2.39 Write the differential equation governing the ac servomotor.

The differential equation governing the ac servomotor is $T_m = K_1 e_c - K_2 \frac{d\theta}{dt}$

Where, T_m = Torque developed by motor
e_c = Control signal
θ = Angular displacement of rotor
K_1 = Slope of control-phase voltage Vs Torque characteristics
K_2 = Slope of speed Vs Torque characteristics.

Q2.40 What is stepper motor?

A stepper motor is a device which transforms electrical pulses into equal increments of rotary shaft motion called steps.

Q2.41 What are the different types of stepper motor?

The various types of stepper motor are

1. Permanent magnet stepper motor
2. Variable reluctance stepper motor
3. Hybrid stepper motor.

Q2.42 What is full-step, half-step and micro-step?

In stepper motors the number of teeth or poles on the rotor and the number of poles on the stator determine the size of the step or step angle. This basic size of step is called full-step.

$$\text{Full - step} = \frac{360°}{\text{Number of rotor poles} \times \text{Number of stator pole pairs}}$$

Half-step is dividing a full-step by two and micro-step is dividing a full-step into 10, 16, 32 or 125 micro-steps. Half-step and micro-step operations are achieved in stepper motor by passing partial currents to stator windings (or control windings)

Q2.43 What is tachogenerator?

A tachogenerator is an electromechanical device which produces an output voltage proportional to its shaft speed.

Q2.44 Compare the dc tacho and ac tacho.

1. The field poles of dc tacho are made of permanent magnets. But the reference winding (which is equivalent to field winding) of ac tacho is excited by an ac voltage.
2. The output of dc tacho is a dc signal which can be directly used as control signal. But the output of ac tacho is a modulated ac signal and so it has to be demodulated to get the control signal.
3. In dc tacho the commutator and brush arrangement develops high frequency ripples. Also the dc tachos require more maintenance. The ac tachos require less maintenance.

Q2.45. What is a modulator?

The modulator is a device which transfer the information available as a dc signal (or slowly varying signal) to an ac signal. The conversion process is called modulation. In modulation the dc signal (or low frequency ac signal) is superimposed on a high frequency carrier.

Q2.46. What is a demodulator?

The demodulator is a device used to extract the information available in a modulated signal and the process is called demodulation. The demodulation is reverse process of modulation and in this process the dc signal (or low frequency ac signal) is extracted from the carrier signal.

Q2.47 Why gear trains are used in control systems?

Gear trains are used in control systems to alter the speed to torque ratio of the rotational power transmitted from motor to load. This is necessary to meet the torque requirement of the load.

Q2.48 What is gear ratio?

The gear ratio of a gear train decides the ratio of torque, speed, angular displacement, velocity and acceleration between any two gear wheels in that gear train.

The gear ratio between gear wheels x and y in a gear train is given by

$$\frac{N_x}{N_y} = \frac{r_x}{r_y} = \frac{T_x}{T_y} = \frac{\theta_y}{\theta_x} = \frac{\omega_y}{\omega_x} = \frac{\alpha_y}{\alpha_x}$$

Q2.49 A servomotor is connected through a gear ratio of 10 to a load having moment of inertia, J and coefficient of friction, B. Write the equivalent parameters referred to motor shaft side.

Let N_1 = Number of teeth in gear wheel connected to motor shaft.

N_2 = Number of teeth in gear wheel connected to load shaft.

It is given that, $N_1/N_2 = 10$

$$\text{Moment of inertia of load referred to motor shaft} = J\left(\frac{N_1}{N_2}\right)^2 = 100\ J$$

$$\text{Coefficient of friction of load referred to motor shaft} = B\left(\frac{N_1}{N_2}\right)^2 = 100\ B$$

Q2.50 What is Gyroscope?

The Gyrcscope is a device used to measure the three angular motions-pitch, roll and yaw of a vehicle moving in air.

Q2.51 What are the basic types of gyros?

The basic types of gyro are Directional (or Positional) gyroscope and Vertical gyroscope. The position of spin axis relative to the earth determines the basic type of gyro. When the spin is in horizontal position, the gyroscope is called directional gyro and when the spin axis is in vertical position it is called vertical gyro.

Q2.52 What is free gyro?

In free gyro there are no restraining forces in any direction of the gyro and the gyro wheel remains fixed in space.

Q2.53 What are the applications of gyroscope?

The gyroscopes are used in automatic pilotting, inertial navigational equipment, artificial horizons and stable platform.

Q2.54 What is rate gyro?

A rate gyro is a gyro which is used for derivative feedback. For such an action a helical spring is connected between outer gimbal and the frame.

Q2.55 What is integrating gyro?

An intergrating gyro is a gyro which is used for integral feedback. For such an action the outer gimbal is fixed to the frame and so the gyro wheel has a single degree of freedom.

2.15 EXERCISES

E2.1 Consider the positional servomechanism shown in fig E2.1. Assume that the input to the system is reference shaft position θ_R and the system output is the load shaft position θ_L. Draw a block diagram of the system indicating the transfer function of each block. Simplify the block diagram to obtain $\theta_L(s)/\theta_R(s)$. The parameters of the system are given below.

Sensitivity of error detector	K_P	=	10 Volts/rad
Amplifier gain	K_A	=	50 Volts/volt
Motor field resistance	R_f	=	100 Ohms
Motor field inductance	L_f	=	20 Henrys
Motor torque constant	K_T	=	10 Newton-m/amp
Moment of inertia of load	J_L	=	250 Kg-m²
Coefficient of viscous friction of load	B_L	=	2500 Newton-m/(rad/sec)
Motor to load gear ratio	$\dot{\theta}_L / \dot{\theta}_M$	=	1/50
Load to potentiometer gear ratio	$\dot{\theta}_P / \dot{\theta}_L$	=	1

Motor inertia and friction are negligible.

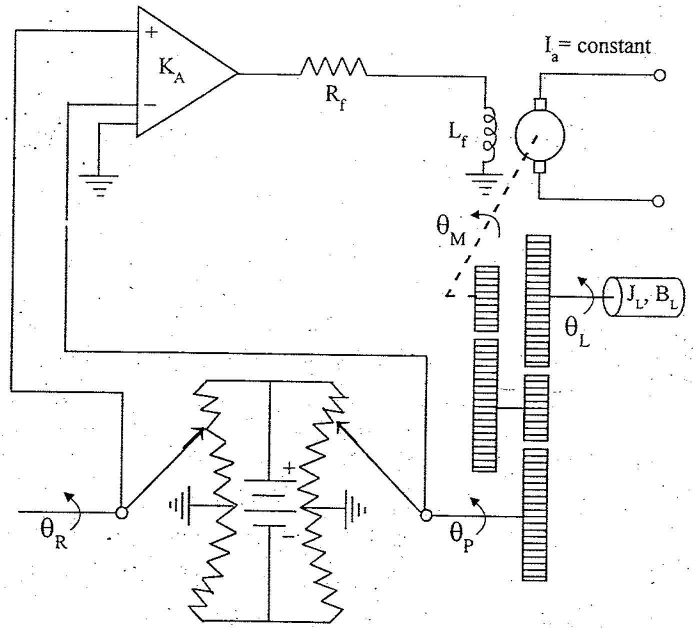


Fig E2.1

E2.2 Consider the system shown in fig E2.2a. The characteristic of ac motor is shown in fig E2.2b. The moment of inertia of the motor is $J_M = 0.003$ N-m/(rad/sec²). The motor drives a load through a gear train. N_1, N_2, N_3 and N_4 are the number of teeth with $N_1/N_2 = 1/2$ and $N_3/N_4 = 1/5$. The moment of inertia of the load is $J_L = 0.02$ N-m/(rad/sec²) and coefficient of viscous friction of the load is $B_L = 0.001$ N-m/(rad/sec). Find the transfer function $\theta_L(s)/E_c(s)$.

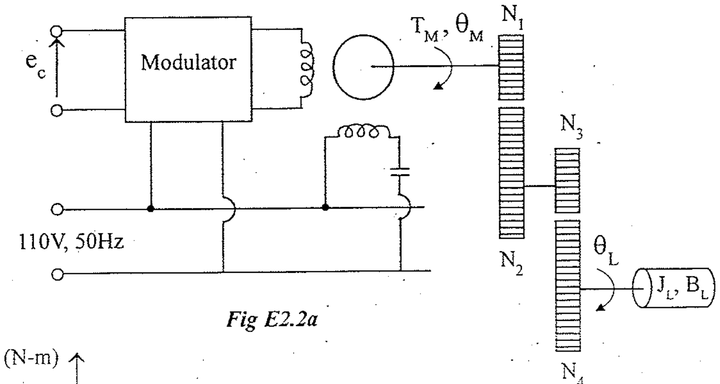


Fig E2.2a

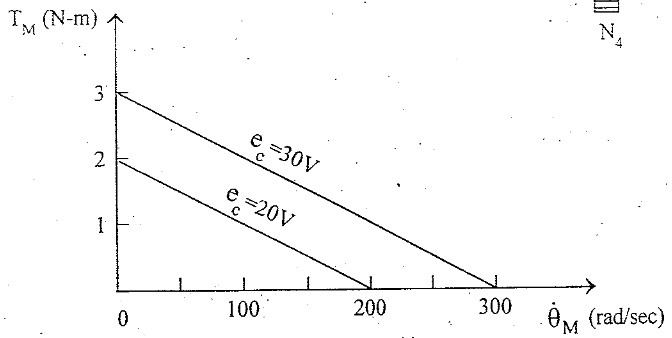


Fig E2.2b

E2.3 Consider the system shown in fig E2.3 with $R_a = 10\Omega$, $L_a = 0.1$H, $K_b = 1$ Volt/(rad/sec), $\dot{\theta}_L / \dot{\theta}_M = 1/2$, $K_t = 0.8$ Volt/(rad/sec), $K_p = 1.5$ Volt/rad. Moment of inertia of load, $J_L = 2$ N-m/(rad/sec²). Moment of inertia of motor shaft, $J_M = 0.1$ N-m/(rad/sec²). Coefficient of viscous friction of load, $B_L = 0.02$ N-m/(rad/sec). Coefficient of viscous friction on motor shaft, $B_M = 0.1$ N-m/(rad/sec). Find the transfer function $\theta_M(s)/E_a(s)$

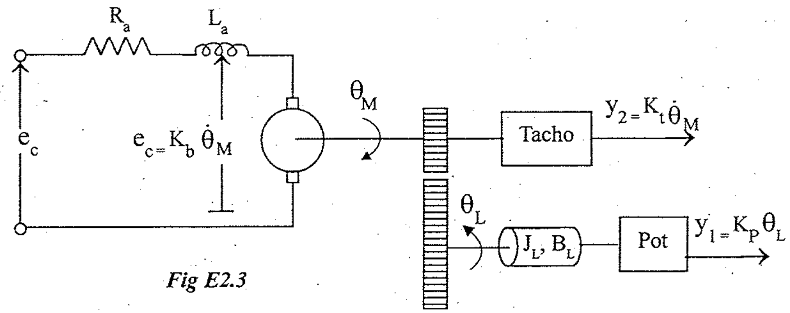


Fig E2.3

E2.4 The schematic diagram of a servo system is shown in fig E2.4. The two-phase servomotor develops a torque in accordance with the equation $T_M = K_1 e_c - K_2 \dot{\theta}_M$, where $K_1 = 1\times10^{-5}$ N-m/volt, $K_2 = 0.25\times10^{-5}$ N-m/(rad/sec). The other parameters of the system are : Synchro sensitivity $K_s = 1$ Volt/rad. Amplifier gain $K_A = 20$ Volt/volt. Tachometer constant $K_t = 0.2$ Volt/(rad/sec). Load inertia $J_L = 1.5\times10^{-5}$ Kg-m². Viscous friction $B_L = 1\times10^{-5}$ N-m/(rad/sec). $\dot{\theta}_M / \dot{\theta}_S = 1$, $\dot{\theta}_M / \dot{\theta}_T = 1$. Motor inertia and friction are negligible. Draw the block diagram of the system and therefrom obtain the transfer function $\theta_M(s)/\theta_R(s)$.

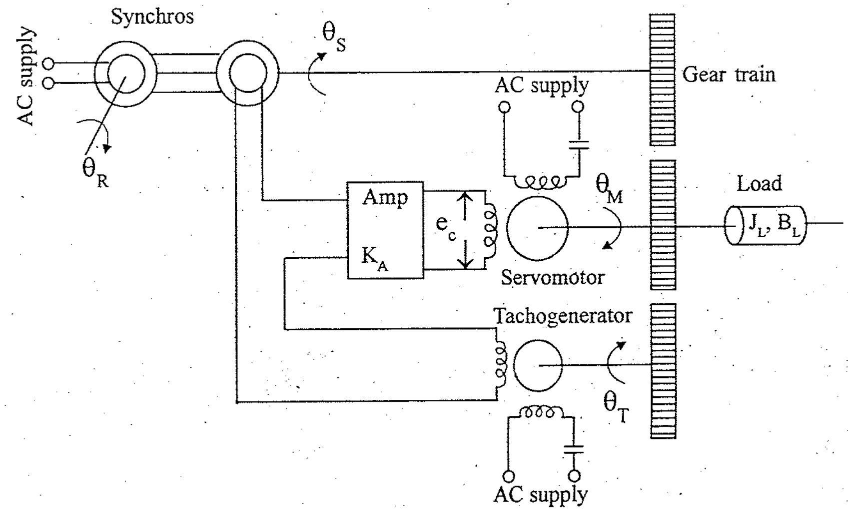


Fig E2.4

E2.5 An ac-dc servo system is shown in fig E2.5. The sensitivity of the synchro error detector is K_s Volt/rad and the gain of the generator is K_g Volts/field amp. The dc motor is separately excited and has a back emf of K_b Volts/(rad/sec) and a torque constant of K_T N-m/amp. Motor inertia and friction are negligible. Draw the block diagram of the system indicating the transfer function of each block. Obtain $\theta_L(s)/\theta_R(s)$.

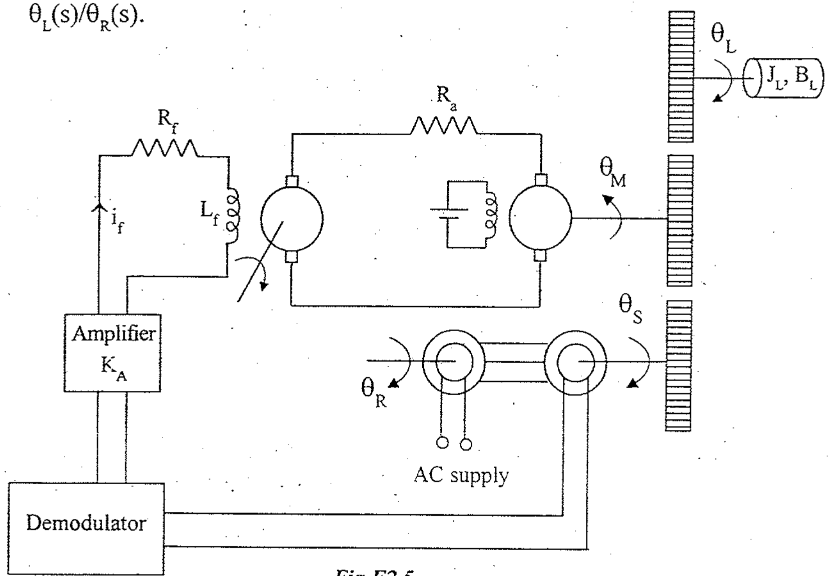


Fig E2.5

The system parameters are given below :

K_s	=	30 Volts/rad	K_A	=	5 Volts/Volts
R_f	=	100 Ohms	L_f	=	2 Henrys
K_g	=	100 Volts/field amp	R_a	=	1 Ohm
K_b	=	1 Volts/(rad/sec)	J_L	=	0.5 Kg-m^2
B_L	=	1 N-m/(rad/sec)	$\dot{\theta}_L / \dot{\theta}_M = \dot{\theta}_S / \dot{\theta}_M = 1$		

ANSWER FOR EXERCISE PROBLEMS

E2.1 $$\frac{\theta_L(s)}{\theta_R(s)} = \frac{1}{s\,(0.1s+1)\,(0.2s+1)+1}$$

E2.3 $$\frac{\theta_M(s)}{E_a(s)} = \frac{16.7}{s\,(s^2+100s+19.2)}$$

E.2.2 $$\frac{\theta_L(s)}{E_C(s)} = \frac{1}{s\,(0.32s+1)}$$

E2.4 $$\frac{\theta_M(s)}{\theta_R(s)} = \frac{13.33}{s^2+3.5s+13.33}$$

E2.5 $$\frac{\theta_L(s)}{\theta_R(s)} = \frac{15\times10^3}{s^3+54\,s^2+200s+15\times10^3}$$

CHAPTER THREE

3 TIME RESPONSE ANALYSIS

3.1 TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

The output in s-domain, C(s) is given by the product of the transfer function and the input, R(s). On taking inverse laplace transform of this product the time domain response, c(t) can be obtained.

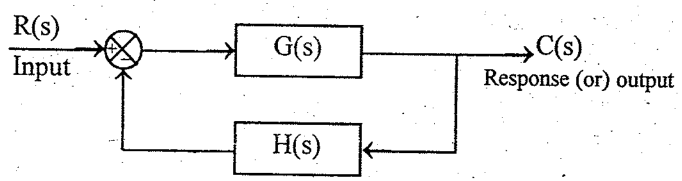


Fig 3.1 : Closed loop system

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$ .....(3.1)

$$\text{Response in s-domain, } C(s) = R(s)\,\frac{G(s)}{1+G(s)\,H(s)} \quad .....(3.2)$$

$$\text{Response in time domain, } c(t) = L^{-1}[C(s)] = L^{-1}\left[R(s)\frac{G(s)}{1+G(s)\,H(s)}\right] \quad .....(3.3)$$

The time response of a control system consists of two parts : the transient and the steady state response. The transient response shows the response of the system when the input changes from one state to another. The steady state response shows the response as time, **t** approaches infinity.

3.2 TEST SIGNALS

The knowledge of input signal is required to predict the response of a system. In most of the systems the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

The standard test signals are

1. a) Step signal
   b) Unit step signal
2. a) Ramp signal
   b) Unit ramp signal
3. a) Parabolic signal
   b) Unit parabolic signal
4. Impulse signal
5. Sinusoidal signal.

Since the test signals are simple functions of time, they can be easily generated in laboratories. The mathematical and experimental analysis of control systems using these signals can be carried out easily. The use of the test signals can be justified because of a correlation existing between the response characteristics of a system to a test input signal and capability of the system to cope with actual input signals.

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at $t=0$ and remains constant at A for $t>0$. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which **A** is unity. The mathematical representation of the step signal is

$$r(t) = A\,u(t) \quad .....(3.4)$$

Where $u(t) = 1;\ t \geq 0$

$u(t) = 0\ ;\ t < 0$

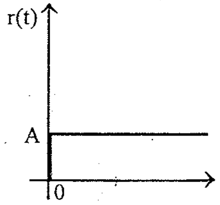


Fig 3.2 : Step signal

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at $t=0$. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of **A** is unity. The mathematical representation of the ramp signal is

$r(t) = A\,t\ ;\ t \geq 0$

$= 0\ ;\ t < 0$ .....(3.5)

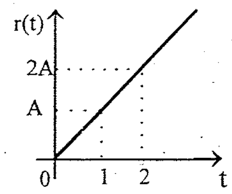


Fig 3.3 : Ramp signal

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at $t=0$. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity. The mathematical representation of the parabolic signal is

$$r(t) = \frac{At^2}{2}\ ;\ t \geq 0 \quad .....(3.6)$$

$$= 0 \quad ;\ t < 0$$

(Note : Integral of step signal is ramp signal.
Integral of ramp signal is parabolic signal.)

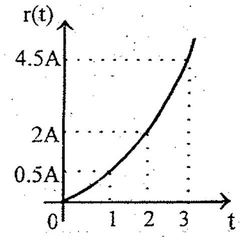


Fig 3.4 : Parabolic signal

IMPULSE SIGNAL

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all times except at t = 0. At t = 0, the magnitude becomes infinite. It is denoted by δ(t) and mathematically it is expressed as

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad .....(3.7)$$

$$\text{and } \underset{t_1 \to 0}{\mathrm{Lt}} \int_{-t_1}^{+t_1} \delta(t)dt = 1$$

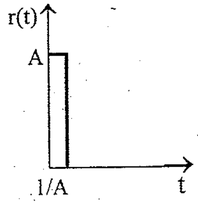


Fig 3.5 : Unit impulse

Since a perfect impulse cannot be achieved in practice it is usually approximated by a pulse of small width but unit area. Mathematically an impulse function is the derivative of a step function. Laplace transform of the impulse function is unity.

The response of the system, with input impulse signal is called weighting function (impulse response) of the system. It is given by the inverse laplace transform of the system transfer function.

$$c(t) = L^{-1}\left[R(s)\frac{G(s)}{1+G(s)\,H(s)}\right] = L^{-1}\left[\frac{G(s)}{1+G(s)\,H(s)}\right] \quad (\because R(s) = 1 \text{ for impulse}) \qquad .....(3.8)$$

TABLE 3-1 : STANDARD TEST SIGNALS

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	A/s
Unit step	1	1/s
Ramp	At	A/s^2
Unit ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
Unit parabolic	$t^2/2$	$1/s^3$
Impulse	δ(t)	1

3.3 ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by a differential equation. The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th} order differential equation, then the system is called n^{th} order system. Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in **s**.

$$\text{Transfer function, } T(s) = K\frac{P(s)}{Q(s)} \qquad .....(3.9)$$

Where, K = Constant
P(s) = Numerator polynomial
Q(s) = Denominator polynomial

The order of the system is given by the maximum power of **s** in the denominator polynomial, Q(s).

$$\text{If } Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots\ldots + a_{n-1}s + a_n \qquad .....(3.9a)$$

then **n** is the order of the system .
If n = 0, then the system is zero order system
If n = 1, then the system is first order system
If n = 2, then the system is second order system and so on.

The order can be specified for both open loop system and closed loop system. Also the value of **n** gives the number of poles in the transfer function. Hence the order is given by the number of poles of the transfer function.

3.4 REVIEW OF PARTIAL FRACTION EXPANSION

The time response of the system is obtained by taking the inverse Laplace transform of the product of input signal and transfer function of the system. Taking inverse Laplace transform requires the knowledge of partial fraction expansion. In control systems three different types of transfer function are encountered. They are,

Case 1 : Functions with separate poles
Case 2 : Functions with multiple poles
Case 3 : Functions with complex conjugate poles.

The partial fraction of all the three cases are explained with an example,

CASE 1: When the transfer function has distinct poles

$$T(s)=\frac{K}{s(s+p_1)\,(s+p_2)}$$

By partial fraction expansion, T(s) can be expressed as

$$T(s)=\frac{K}{s(s+p_1)\,(s+p_2)}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{s+p_2}$$

The residues A, B and C are given by,

$$A=T(s)\,s\big|_{s=0}\qquad B=T(s)\,(s+p_1)\big|_{s=-p_1}\qquad C=T(s)\,(s+p_2)\big|_{s=-p_2}$$

Example :

$$\text{Let } T(s)=\frac{2}{s(s+1)\,(s+2)}$$

By partial fraction expansion, T(s) can be expressed as

$$T(s)=\frac{2}{s(s+1)\,(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A=T(s)\,s=\frac{2s}{s(s+1)\,(s+2)}=\frac{2}{(s+1)\,(s+2)}\bigg|_{s=0}=\frac{2}{1\times 2}=1$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1

$$B=T(s)\,(s+1)=\frac{2(s+1)}{s(s+1)\,(s+2)}=\frac{2}{s\,(s+2)}\bigg|_{s=-1}=\frac{2}{-1(-1+2)}=-2$$

C is obtained by multiplying T(s) by (s +2) and letting s = –2

$$C=T(s)\,(s+2)=\frac{2(s+2)}{s(s+1)\,(s+2)}=\frac{2}{s\,(s+1)}\bigg|_{s=-2}=\frac{2}{-2(-2+1)}=+1$$

$$\therefore T(s)=\frac{2}{s(s+1)\,(s+2)}=\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}$$

CASE 2: When the transfer function has multiple poles

$$T(s)=\frac{K}{s(s+p_1)\,(s+p_2)^2}$$

By partial fraction expansion, T(s) can be expressed as

$$T(s)=\frac{K}{s(s+p_1)\,(s+p_2)^2}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{(s+p_2)^2}-\frac{D}{(s+p_2)}$$

The residues A, B, C and D are given by,

$$A=T(s)\,s\big|_{s=0}\qquad B=T(s)\,(s+p_1)\big|_{s=-p_1}$$

$$C=T(s)\,(s+p_2)^2\big|_{s=-p_2}\qquad D=\frac{d}{ds}\left[T(s)\,(s+p_2)^2\right]\bigg|_{s=-p_2}$$

Example : $\text{Let } T(s)=\dfrac{2}{s(s+1)\,(s+2)^2}$

By partial fraction expansion, T(s) can be expressed as

$$T(s)=\frac{K}{s(s+1)\,(s+2)^2}=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A=T(s)\,s=\frac{2}{s(s+1)\,(s+2)^2}s=\frac{2}{(s+1)\,(s+2)^2}\bigg|_{s=0}=\frac{2}{1\times 2^2}=0.5$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1

$$B=T(s)\,(s+1)=\frac{2}{s(s+1)\,(s+2)^2}(s+1)=\frac{2}{s\,(s+2)^2}\bigg|_{s=-1}=\frac{2}{-1(-1+2)^2}=-2$$

C is obtained by multiplying T(s) by $(s+2)^2$ and letting s = –2

$$C=T(s)\,(s+2)^2=\frac{2(s+2)^2}{s(s+1)\,(s+2)^2}=\frac{2}{s\,(s+1)}\bigg|_{s=-2}=\frac{2}{-2(-2+1)}=1$$

D is obtained by differentiating the product T(s) $(s+2)^2$ with respect to s and then letting s = –2.

$$D=\frac{d}{ds}\left[T(s)\,(s+2)^2\right]=\frac{d}{ds}\left[\frac{2}{s\,(s+1)}\right]=\left.\frac{-2(2s+1)}{s^2(s+1)^2}\right|_{s=-2}=\frac{-2(2(-2)+1)}{(-2)^2(-2+1)^2}=+1.5$$

$$\therefore T(s)=\frac{2}{s\,(s+1)\,(s+2)^2}=\frac{0.5}{s}-\frac{2}{s+1}+\frac{1}{(s+2)^2}+\frac{1.5}{s+2}$$

CASE 3 : When the transfer function has complex conjugate poles

$$T(s)=\frac{K}{(s+p_1)\,(s^2+bs+c)}$$

By partial fraction expansion, T(s) can be expressed as

$$T(s)=\frac{K}{(s+p_1)\,(s^2+bs+c)}=\frac{A}{s+p_1}+\frac{Bs+C}{s^2+bs+c} \quad .....(3.10)$$

The residue A given by, $A=T(s)\,.\,(s+p_1)\big|_{s=-p_1}$

The residues B and C are solved by cross multiplying the equation (3.10) and then equating the coefficient of like power of s.

Finally express T(s) as shown below,

$$T(s)=\frac{A}{s+p_1}+\frac{Bs}{s^2+bs+c}+\frac{C}{s^2+bs+c}$$

$$=\frac{A}{s+p_1}+\frac{Bs}{(s+\frac{b}{2})^2+(c-\frac{b^2}{4})}+\frac{C}{(s+\frac{b}{2})^2+(c-\frac{b^2}{4})}$$

Example : Let $T(s)=\dfrac{1}{(s+2)\,(s^2+s+1)}$

By partial fraction expansion,

$$T(s)=\frac{1}{(s+2)\,(s^2+s+1)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+s+1}$$

A is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$\therefore A=T(s)\,(s+2)=\left.\frac{1}{(s+2)\,(s^2+s+1)}(s+2)\right|_{s=-2}=\frac{1}{(-2)^2-2+1}=\frac{1}{3}$$

To solve B and C, cross multiply the following equation and substitute the value of A. Then equate the like power of s.

$$\frac{1}{(s+2)\,(s^2+s+1)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+s+1}$$

$$1=A(s^2+s+1)+(Bs+C)\,(s+2)$$

$$1=\frac{1}{3}(s^2+s+1)+Bs^2+2Bs+Cs+2C$$

$$1=\frac{s^2}{3}+\frac{s}{3}+\frac{1}{3}+Bs^2+2Bs+Cs+2C$$

On equating the coefficient of s^2 terms, $0=\frac{1}{3}+B \quad \therefore B=-\frac{1}{3}$

On equating the coefficient of s terms,

$$0=\frac{1}{3}+2B+C \quad \therefore C=-\frac{1}{3}-2B=-\frac{1}{3}+\frac{2}{3}=\frac{1}{3}$$

$$T(s)=\frac{\frac{1}{3}}{s}+\frac{\frac{-1}{3}s+\frac{1}{3}}{s^2+s+1}=\frac{1}{3s}-\frac{1}{3}\frac{s}{(s^2+s+1)}+\frac{1}{3}\frac{1}{(s^2+s+1)}$$

$$=\frac{1}{3s}-\frac{1}{3}\frac{s}{(s+0.5)^2+0.75}+\frac{1}{3}\frac{1}{(s+0.5)^2+0.75}$$

3.5 RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

The closed loop transfer function of first order system, $\dfrac{C(s)}{R(s)}=\dfrac{1}{1+Ts}$ .....(3.11)

If the input is unit step then, $r(t)=1$ and $R(s)=\dfrac{1}{s}$

$\therefore$ The response in s-domain, $C(s)=R(s)\dfrac{1}{(1+Ts)}=\dfrac{1}{s}\cdot\dfrac{1}{(1+Ts)}$

$$=\frac{1}{sT(\frac{1}{T}+s)}=\frac{\frac{1}{T}}{s(s+\frac{1}{T})} \quad .....(3.12)$$

By partial fraction expansion,

$$C(s)=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}=\frac{A}{s}+\frac{B}{(s+\frac{1}{T})}$$

A is obtained by multiplying C(s) by s and letting s = 0

$$A=C(s)s=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}s=\left.\frac{\frac{1}{T}}{s+\frac{1}{T}}\right|_{s=0}=\frac{\frac{1}{T}}{\frac{1}{T}}=1$$

B is obtained by multiplying C(s) by (s+1/T) and letting s = –1/T,

$$B=C(s)\ (s+\frac{1}{T})=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}\ (s+\frac{1}{T})=\left.\frac{\frac{1}{T}}{s}\right|_{s=-\frac{1}{T}}=\frac{\frac{1}{T}}{\frac{-1}{T}}=-1$$

$$\therefore C(s)=\frac{1}{s}-\frac{1}{s+\frac{1}{T}}$$

The response in time domain is given by,

$$c(t)=L^{-1}[C(s)]=L^{-1}\left[\frac{1}{s}-\frac{1}{s+\frac{1}{T}}\right]=1-e^{-\frac{t}{T}} \quad .....(3.13)$$

When	$t = 0$,	$c(t)$	$=1-e^{0}=0$
When	$t = 1T$,	$c(t)$	$=1-e^{-1}=0.632$
When	$t = 2T$,	$c(t)$	$=1-e^{-2}=0.865$
When	$t = 3T$,	$c(t)$	$=1-e^{-3}=0.95$
When	$t = 4T$,	$c(t)$	$=1-e^{-4}=0.9817$
When	$t = 5T$,	$c(t)$	$=1-e^{-5}=0.993$
When	$t = \infty$,	$c(t)$	$=1-e^{-\infty}=1$

Here T is called Time constant of the system. In a time of 5T, the system is assumed to have attained steady state. The input and output of the system is shown in fig 3.6(a) and (b).

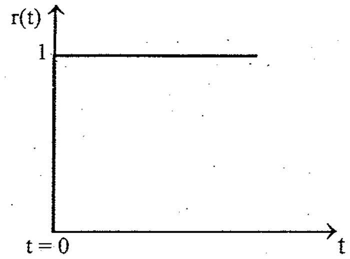


Fig 3.6 (a) : Unit step input

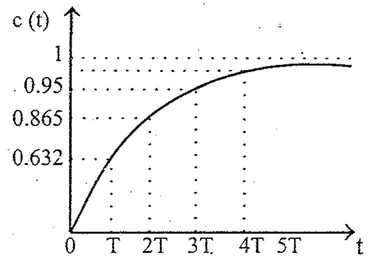


Fig 3.6 (b) : Response of first order system to unit step input

3.6 SECOND ORDER SYSTEM

The standard form of closed loop transfer function of second order system is given by

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad .....(3.14)$$

Where, ω_n = Undamped natural frequency, rad/sec

ζ = Damping ratio.

The damping ratio is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1 :	Undamped system,	$\zeta=0$
Case 2 :	Under damped system,	$0<\zeta<1$
Case 3 :	Critically damped system,	$\zeta=1$
Case 4 :	Over damped system,	$\zeta>1$

The characteristics equation of the 2nd order system is

$$s^2+2\zeta\omega_n s+\omega_n{}^2=0 \quad .....(3.15)$$

It is a quadratic equation and the roots of this equation is given by

$$s_1, s_2=\frac{-2\zeta\omega_n\pm\sqrt{4\zeta^2\omega_n^2-4\omega_n^2}}{2}=\frac{-2\zeta\omega_n\pm\sqrt{4\omega_n^2(\zeta^2-1)}}{2}$$

$$=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1} \quad .....(3.16)$$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$; $\begin{cases}\text{roots are purely imaginary}\\ \text{and the system is undamped}\end{cases}$ .....(3.17)

When $\zeta = 1$, $s_1, s_2 = -\omega_n$; $\begin{cases}\text{roots are real and equal and}\\ \text{the system is critically damped}\end{cases}$ .....(3.18)

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases}\text{roots are real and unequal and}\\ \text{the system is overdamped}\end{cases}$ .....(3.19)

When $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_d \; ; \begin{cases}\text{roots are complex conjugate}\\ \text{the system is underdamped}\end{cases} \quad .....(3.20)$$

Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ .....(3.21)

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for undamped system, $\zeta = 0$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(3.22)$$

When the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$

$$\therefore \text{The response in s-domain, } C(s) = R(s)\,\frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s}\,\frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(3.23)$$

By partial fraction expansion

$$C(s) = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s = 0

$$A = C(s)\,s = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times s = \left.\frac{\omega_n^2}{s^2 + \omega_n^2}\right|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying C(s) by $(s^2 + \omega_n^2)$ and letting $s^2 = -\omega_n^2$ or $s = j\omega_n$

$$B = C(s)\,(s^2 + \omega_n^2) = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times (s^2 + \omega_n^2) = \left.\frac{\omega_n^2}{s}\right|_{s = j\omega_n}$$

$$= \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$$\text{The time domain response, } c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right] = 1 - \cos\omega_n t \quad .....(3.24)$$

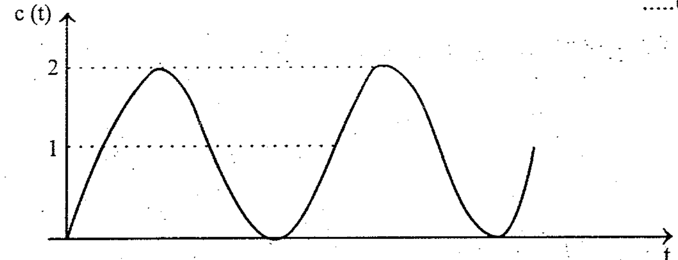


Fig 3.7 : Response of undamped second order system for unit step input

The response of undamped second order system for unit step input is completely oscillatory.

Note : *Every practical system has some amount of damping. Hence undamped system does not exist in practice.*

RESPONSE OF SECOND ORDER SYSTEM FOR UNDERDAMPED CASE AND WHEN THE INPUT IS UNIT STEP

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The roots of the denominator are, $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

Since $\zeta < 1$, $\zeta^2 < 1$

$$\therefore s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain, $C(s) = R(s)\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, $r(t) = 1$ and $R(s) = 1/s$

$$\therefore C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} \qquad .....(3.25)$$

By partial fraction expansion, $C(s) = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$A = s\,C(s)\big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C

$$\frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

On cross multiplication and equating the coefficients of like power of s we get,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get, $0 = 1 + B \quad \therefore B = -1$

Equating coefficient of s we get, $0 = 2\zeta\omega_n + C \quad \therefore C = -2\zeta\omega_n$

$$C(s) = \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(3.26)$$

Adding and subtracting $\zeta^2\omega_n^2$ to the denominator of second term in the equation (3.26)

$$= \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad (\because \omega_d = \omega_n\sqrt{1-\zeta^2})$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad .....(3.27)$$

Multiplying and dividing by ω_d in the third term of the equation (3.27)

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

On taking inverse Laplace transform

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\,e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t.\zeta + \cos\omega_d t.\sqrt{1-\zeta^2}\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\omega_d t \cos\theta + \cos\omega_d t \sin\theta) \qquad \textit{(refer note)}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \qquad .....(3.28)$$

$$\text{Where, } \left(\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The response of underdamped second order system oscillates before settling to a final value. The oscillations depends on the value of damping ratio.

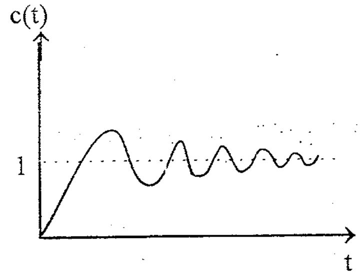


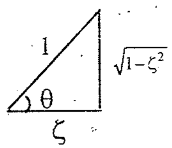
Fig 3.8 : Response of under damped second order system for unit step input.

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$



RESPONSE OF SECOND ORDER SYSTEM FOR CRITICALLY DAMPED CASE AND WHEN INPUT IS UNIT STEP

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \qquad .....(3.29)$$

When input is unit step, $r(t) = 1$ and $R(s) = 1/s$

$\therefore$ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2}$$

$$= \frac{\omega_n^2}{s\,(s+\omega_n)^2} \qquad .....(3.30)$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s\,C(s)\Big|_{s=0} = \frac{\omega_n^2}{(s+\omega_n)^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2\,C(s)\Big|_{s=-\omega_n} = \frac{\omega_n^2}{s}\Bigg|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\left[(s+\omega_n)^2\,C(s)\right]\Bigg|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Bigg|_{s=-\omega_n}$$

$$= \frac{-\omega_n^2}{s^2}\Bigg|_{s=-\omega_n} = \frac{-\omega_n^2}{(-\omega_n)^2} = -1$$

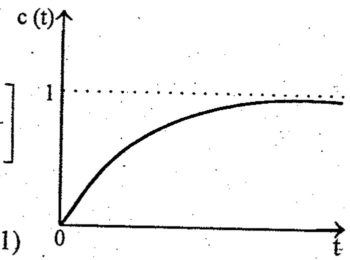
$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

The response in time domain,

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \qquad .....(3.31)$$



The response of critically damped second order system have no oscillations.

Fig 3.9 : Response of critically damped second order system for unit step input

RESPONSE OF SECOND ORDER SYSTEM FOR OVERDAMPED CASE AND WHEN INPUT IS UNIT STEP

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b

$$s_a,\ s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n \mp \omega_n\sqrt{\zeta^2-1}\right] \qquad .....(3.32)$$

Let $s_1 = -s_a$ and $s_2 = -s_b$ $\qquad \therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$ $\qquad .....(3.33)$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \qquad .....(3.34)$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)\,(s+s_2)} \qquad .....(3.35)$$

For unit step input r(t) = 1 and R(s) = 1/s

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)\,(s+s_2)} = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s\,C(s)\big|_{s=0} = s\frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}\bigg|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = 1$$

$$= \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1)\,C(s)\big|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\bigg|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}$$

$$C = C(s)\,(s+s_2)\big|_{s=-s_2} = \frac{\omega_n^2}{s\,(s+s_1)}\bigg|_{s=-s_2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2}$$

$$= \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}$$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,\frac{1}{(s+s_2)}$$

On taking inverse Laplace transform we get,

The response in time domain,

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad .....(3.36)$$

Where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

and $\quad s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

The response of overdamped second order system has no oscillations but it takes longer time for the response to reach the final steady value.

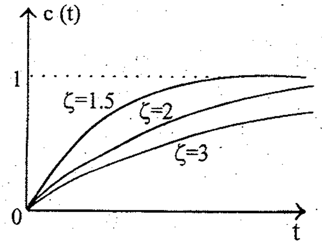


Fig 3.10 : Response of overdamped second order system for unit step input

3.7 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit-step input with various values of damping ratio is shown in fig 3.11.

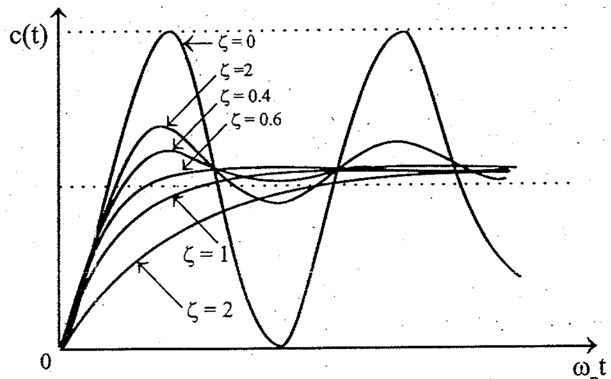


Fig 3.11 : Unit step response of second order system

The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before $t = 0$ will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in fig 3.12.

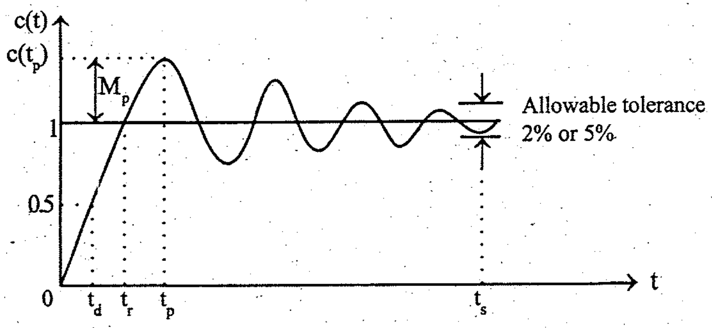


Fig 3.12 : Damped oscillatory response of second order system

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

The time domain specifications are defined as follows.

1. DELAY TIME (t_d)

It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r)

It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p)

It is the time taken for the response to reach the peak value for the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

4. PEAK OVERSHOOT (M_p)

It is defined as the ratio of the maximum peak value measured from final value to the final value.

Let final value $= c(\infty)$

Maximum value $= c(t_p)$

$$\text{Peak overshoot, } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \qquad .....(3.37)$$

$$\%\text{ Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \qquad .....(3.38)$$

5. **Settling time, (t_s)**

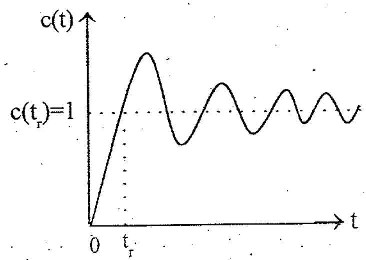
It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

EXPRESSIONS FOR TIME DOMAIN SPECIFICATIONS

Rise time (t_r)

Response of second order system for underdamped case is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$



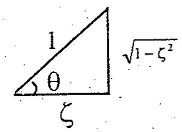
At $t = t_r$, $c(t) = c(t_r) = 1$

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get



$\sin\phi = 0$, when $\phi = 0, \pi, 2\pi, 3\pi...$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\therefore \text{Rise Time}, \; t_r = \frac{\pi - \theta}{\omega_d} \qquad .....(3.39)$$

Here, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ ***(refer note)***

and damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in secs} \qquad .....(3.40)$$

Note : θ or $\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ should be measured in radians.

Peak time (t_p)

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

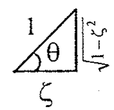
$$\text{i.e., } \left.\frac{d}{dt}c(t)\right|_{t=t_p} = 0$$

The unit step response of second order system is given by

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$



$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to **t**

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t + \theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t + \theta) - \sqrt{1-\zeta^2}\cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\cos\theta\sin(\omega_d t + \theta) - \sin\theta\cos(\omega_d t + \theta)\right]$$

(refer note)

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left[\sin((\omega_d t+\theta)-\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin(\omega_d t)$$

at $t = t_p,\ \frac{d}{dt}c(t) = 0$

$$\therefore \left.\frac{d}{dt}c(t)\right|_{t=t_p} = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_p}\sin(\omega_d t_p) = 0$$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_p}\sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

$\sin\phi = 0$, when $\phi = 0,\ \pi,\ 2\pi,\ 3\pi$

$\therefore \omega_d t_p = \pi$

$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d} \qquad .....(3.41)$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad .....(3.42)$$

Peak overshoot (M_p)

$$\%\text{Peak overshoot, } \%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)}\times 100$$

Where, $c(t_p)$ = Peak response at $t = t_p$

$c(\infty)$ = Final steady state value

The unit step response of second order system is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

$$\text{At } t=\infty,\quad c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) = 1 - 0 = 1 \qquad .....(3.43)$$

$$\text{At } t = t_p,\quad c(t) = c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\theta)$$

$$\text{Put } t_p = \frac{\pi}{\omega_d} \quad \therefore c(t_p) = 1 - \frac{e^{-\zeta\omega_n\frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d\frac{\pi}{\omega_d}+\theta\right)$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

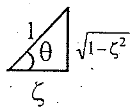
$$c(t_p) = 1 - \frac{e^{-\zeta\omega_n\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}\sin(\pi+\theta)}{\sqrt{1-\zeta^2}}$$

$$c(t_p) = 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\theta \quad (\text{since } (\pi+\theta) = -\sin\theta)$$

$$c(t_p) = 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2}$$

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$



$$c(t_p) = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \qquad .....(3.44)$$

$$\%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)}\times 100$$

$$\%M_p = \frac{1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}-1}{1}\times 100$$

$$\left.\begin{array}{l}\text{Percentage peak}\\ \text{overshoot}\end{array}\right\}\ \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100 \qquad .....(3.45)$$

Settling time (t_s)

The response of second order system has two components. They are,

1. Decaying exponential component, $\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$

2. Sinusoidal component, $\sin(\omega_d t+\theta)$

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2% tolerance error band, at $t = t_s$, $\dfrac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02)$$
$$-\zeta\omega_n t_s = -4$$
$$\therefore \text{Settling time, } t_s = \frac{4}{\zeta\omega_n} = 4T \quad ; \quad \text{for 2\% error} \qquad .....(3.46)$$

$$\text{where, } T = \frac{1}{\zeta\omega_n} = \text{Time constant of the system} \qquad ....(3.47)$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.05)$$
$$-\zeta\omega_n t_s = -3$$
$$\therefore \text{Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \; ; \; \text{for 5\% error} \qquad .....(3.48)$$

In general for a specified percentage error settling time, $t_s = \dfrac{ln(\%\ \text{error})}{\zeta\omega_n}$ ...(3.49)

3.8 RESPONSE WITH P, PI, PD AND PID CONTROLLERS

In feedback control systems a controller may be introduced to modify the error signal and to achieve better control action. The introduction of controllers will modify the transient response and the steady state error of the system.

EFFECT OF PROPORTIONAL CONTROLLER

The proportional controller produces an output signal which is proportional to error signal. The transfer function of proportional controller is K_p. The term K_p is called the gain of the controller. Hence the proportional controller amplifies the error signal and increases the loop gain of the system. The following aspects of system behaviour are improved by increasing loop gain.

(i) Steady state tracking accuracy.

(ii) Disturbance signal rejection.

(iii) Relative stability.

In addition to increase in loop gain it decreases the sensitivity of the system to parameter variations. The drawback in proportional control action is that it produces a constant steady state error.

EFFECT OF PI CONTROLLER

The proportional plus integral controller (PI controller) produces an output signal consisting of two terms-one proportional to error signal and the other proportional to the integral of error signal.

$$\text{Transfer function of PI controller} = K_p\left(1+\frac{1}{T_i s}\right) = K_p\left(\frac{T_i s+1}{T_i s}\right)$$

Where K_p is equal to proportional gain and T_i is equal to integral time.

The block diagram of unity feedback system with PI controller is shown in fig 3.13.

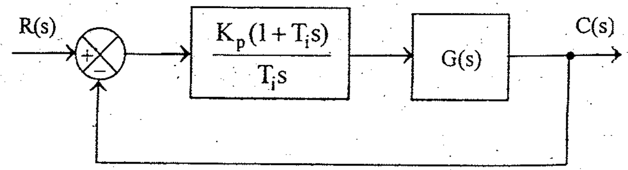


Fig 3.13 : Block diagram of feedback system with PI controller

Let the open loop transfer function G(s) be a second order system with transfer function, $G(s) = \dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$.

Now the closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K_p\left(\dfrac{1+T_i s}{T_i s}\right)\dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}}{1+K_p\left(\dfrac{1+T_i s}{T_i s}\right)\dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}}$$

$$= \frac{K_p\omega_n^2\,(1+T_i s)}{s^2 T_i\,(s+2\zeta\,\omega_n)+K_p\omega_n^2\,(1+T_i s)}$$

$$= \frac{K_p\omega_n^2\,(1+T_i s)}{T_i s^3+2\zeta\omega_n T_i s^2+K_p\omega_n^2\,T_i\,s+K_p\omega_n^2}$$

$$= \frac{(K_p/T_i)\,\omega_n^2(1+T_i s)}{s^3+2\zeta\omega_n s^2+K_p\omega_n^2\,s+\dfrac{K_p}{T_i}\omega_n^2}$$

$$= \frac{K_i\omega_n^2\,(1+T_i s)}{s^3+2\zeta\omega_n s^2+K_p\omega_n^2 s+K_i\omega_n^2}$$

Where $K_i = \dfrac{K_p}{T_i}$

From the closed loop transfer function it is observed that the PI controller introduces a zero in the system and increases the order by one. Also the type number of open loop system is increased by one. The increase in type number results in reducing the steady state error. For example if the steady state error of the original system is constant, then the integral controller will reduce the error to zero.

The increase in the order of the system results in a less stable system than the original one because higher order systems are less stable than lower order systems.

EFFECT OF PD CONTROLLER

The proportional plus derivative controller produces an output signal consisting of two terms-one proportional to error signal and the other proportional to the derivative of error signal.

The transfer function of PD controller $= K_p\,(1+T_d\,s)$

Where K_p = Proportional gain and T_d = Derivative time

The block diagram of unity feedback system with PD controller is shown in fig 3.14.

Let the open loop transfer function G(s) be a second order system with transfer function

$$G(s) = \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$$

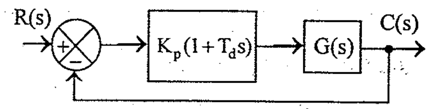


Fig 3.14 : Block diagram of feedback system with PD controller

Now the closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K_p\,(1+T_d s)\,\dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}}{1+K_p\,(1+T_d s)\,\dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)+K_p\omega_n^2\,(1+T_d s)} = \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+2\zeta\omega_n s+K_p\omega_n^2+K_p\omega_n^2 T_d s}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+(2\zeta\omega_n+K_p\omega_n^2 T_d)\,s+K_p\omega_n^2} = \frac{\omega_n^2\,(K_p+K_d s)}{s^2+(2\zeta\omega_n+K_d\omega_n^2)\,s+K_p\omega_n^2}$$

Where $K_d = K_p T_d$

From the closed loop transfer function it is observed that the PD controller introduces a zero in the system and increases the damping ratio. The addition of the zero may increase the peak overshoot and reduce the rise time. But the effect of increased damping ultimately reduces the peak overshoot.

PID CONTROLLER

A suitable combination of the three basic modes - proportional, integral and derivative (PID) can improve all aspects of the system performance.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error. The combined effect of all the three cannot be judged from the parameters K_p, K_i and K_d.

EXAMPLE 3.1

Obtain the response of unity feedback system whose open loop transfer function is

$G(s) = \dfrac{4}{s(s+5)}$ and when the input is unit step.

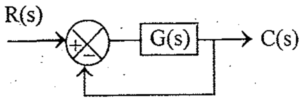


Fig 3.1.1 : Closed loop system

SOLUTION

The closed loop system is shown in fig 3.1.1

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{4/s\,(s+5)}{1+\dfrac{4}{s\,(s+5)}} = \frac{4/s\,(s+5)}{\dfrac{s\,(s+5)+4}{s\,(s+5)}} = \frac{4}{s\,(s+5)+4}$$

$$= \frac{4}{s^2+5s+4} = \frac{4}{(s+4)\,(s+1)}$$

The response in s-domain, $C(s) = R(s)\,\dfrac{4}{(s+1)\,(s+4)}$

Since the input is unit step, $R(s) = \dfrac{1}{s}$

$$\therefore C(s) = \frac{4}{s\,(s+1)\,(s+4)}$$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s)\,s\big|_{s=0} = \left.\frac{4}{(s+1)\,(s+4)}\right|_{s=0} = \frac{4}{1\times 4} = 1$$

$$B = C(s)\,(s+1)\big|_{s=-1} = \left.\frac{4}{s\,(s+4)}\right|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s)\,(s+4)\big|_{s=-4} = \left.\frac{4}{s\,(s+1)}\right|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

The response in time domain, $c(t) = L^{-1}[C(s)] = L^{-1}\left[\dfrac{1}{s} - \dfrac{4}{3}\,\dfrac{1}{s+1} + \dfrac{1}{3}\,\dfrac{1}{s+4}\right]$

$$= 1 - \frac{4}{3}\,e^{-t} + \frac{1}{3}\,e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

RESULT

The response of unity feedback system is $c(t) = 1 - \dfrac{1}{3}\left[4e^{-t} - e^{-4t}\right]$

EXAMPLE 3.2

A positional control system with velocity feedback is shown in fig 3.2.1. What is the response of the system for unit step input.

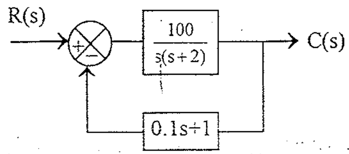


Fig 3.2.1 : Positional control system

SOLUTION

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

Given that $G(s) = \dfrac{100}{s(s+2)}$ and $H(s) = 0.1s+1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{100}{s\,(s+2)}}{1+\left(\dfrac{100}{s\,(s+2)}\right)(0.1s+1)} = \frac{\dfrac{100}{s\,(s+2)}}{\dfrac{s(s+2)+100\,(0.1s+1)}{s\,(s+2)}}$$

$$= \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

Here $(s^2+12s+100)$ is the characteristic polynomial. The roots of the characteristic polynomial are

$$s_1,\ s_2 = \frac{-12 \pm \sqrt{144-400}}{2} = \frac{-12 \pm j16}{2} = -6 \pm j8$$

The roots are complex conjugate. The system is underdamped and so the response of the system will have damped oscillations.

The response in s-domain, $C(s) = R(s)\,\dfrac{100}{s^2+12s+100}$

Since input is unit step, $R(s) = \dfrac{1}{s}$

$$\therefore C(s) = \frac{1}{s}\,\frac{100}{s^2+12s+100} = \frac{100}{s(s^2+12s+100)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s)\,s\Big|_{s=0} = \frac{100}{s^2+12s+100}\Bigg|_{s=0} = \frac{100}{100} = 1$$

The residue B and C are evaluated by cross multiplying the following equation and equating the coefficients of like power of s.

$$\frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

$$100 = A(s^2+12s+100) + (Bs+C)\,s$$

$$100 = As^2 + 12As + 100A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\quad \therefore B = -A = -1$

On equating the coefficients of s we get, $0 = 12A + C$ $\quad \therefore C = -12A = -12$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+12}{s^2+12s+36+64} = \frac{1}{s} - \frac{s+6+6}{(s+6)^2+64}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+64} - \frac{6}{(s+6)^2+64}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain $\Big\}$ $c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\dfrac{1}{s} - \dfrac{s+6}{(s+6)^2+8^2} - \dfrac{6}{8}\,\dfrac{8}{(s+6)^2+8^2}\right]$

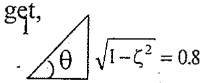
$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\,e^{-6t}\sin 8t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$. The damping ratio ζ is evaluated by comparing the closed loop transfer function of the system with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2} = \frac{100}{s^2+12s+100}$$

On comparing we get, $\quad \omega_n^2 = 100 \quad \therefore \omega_n = 10 \quad \Big| \quad 2\zeta\omega_n = 12 \quad \therefore \zeta = \dfrac{12}{2\omega_n} = \dfrac{12}{2\times 10} = 0.6$

Constructing right angled triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$$\sin\theta = 0.8\ ;\ \cos\theta = 0.6\ ;\ \tan\theta = \frac{0.8}{0.6}$$

$$\therefore \theta = \tan^{-1}\frac{0.8}{0.6} = 53° = 0.925 \text{ rad.}$$

The response in time domain $\Big\}$ $c(t) = 1 - e^{-6t}\left[\dfrac{6}{8}\sin 8t + \cos 8t\right]$

$$= 1 - e^{-6t}\,\frac{10}{8}\left[\frac{6}{10}\sin 8t + \frac{8}{10}\cos 8t\right] = 1 - \frac{10}{8}e^{-6t}[\sin 8t \times 0.6 + \cos 8t \times 0.8]$$

$$= 1 - 1.25\,e^{-6t}[\sin 8t\cos\theta + \cos 8t\sin\theta]$$

$$= 1 - 1.25\,e^{-6t}[\sin\,(8t+\theta)] = 1 - 1.25\,e^{-6t}\,\sin(8t + 0.925)$$

*(**Note :** θ is expressed in radians.)*

RESULT

The response in time domain,

$$c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] \quad \text{or} \quad c(t) = 1 - 1.25\,e^{-6t}\,\sin(8t + 0.925)$$

EXAMPLE 3.3

Measurements conducted on a servomechanism show the system response to be $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

SOLUTION

Given that $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\,\frac{1}{(s+60)} - 1.2\,\frac{1}{(s+10)}$$

$$= \frac{(s+60)\,(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 1.2s^2 - 72s}{s\,(s+60)\,(s+10)}$$

$$C(s) = \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\,\frac{600}{(s+60)\,(s+10)}$$

Since input is unit step, $R(s) = 1/s$.

$$\therefore C(s) = R(s)\,\frac{60}{(s+60)\,(s+10)} = R(s)\,\frac{600}{s^2 + 70s + 600}$$

$\therefore$ The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2 + 70s + 600}$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

On comparing we get,

$\omega_n^2 = 600$	$2\zeta\omega_n = 70$
$\therefore \omega_n = \sqrt{600} = 24.49$ rad / sec	$\therefore \zeta = \dfrac{70}{2\omega_n} = \dfrac{70}{2 \times 24.49} = 1.43$

RESULT

The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2 + 70s + 600}$

Natural frequency of oscillation, $\omega_n = 24.49$ rad/sec

Damping ratio, $\zeta = 1.43$

EXAMPLE 3.4

The unity feedback system is characterized by an open loop transfer function $G(s) = K/s\,(s+10)$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine settling time, peak overshoot and time to peak overshoot for a unit step input.

SOLUTION

The unity feedback system is shown in fig 3.4.1.

$$\left.\begin{matrix}\text{The closed loop}\\ \text{transfer function}\end{matrix}\right\}\ \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

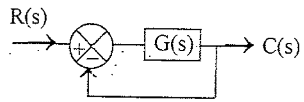


Fig 3.4.1 : Unity feedback system

Given that, $G(s) = K/s\,(s+10)$

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s\,(s+10)}{1 + K/s\,(s+10)} = \frac{K}{s\,(s+10) + K} = \frac{K}{s^2 + 10s + K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + 10s + K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$\therefore K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K = 100

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100$$

$$= 16.3\%$$

Peak time, $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec.

RESULT

The value of gain,	K	=	100
Percentage peak overshoot,	$\%M_p$	=	16.3%
Peak time,	t_p	=	0.363 sec.

EXAMPLE 3.5

The open loop transfer function of a unity feedback system is given by $G(s) = K/s\,(sT + 1)$, where K and T are positive constant. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

SOLUTION

The unity feedback system is shown in fig 3.5.1.

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

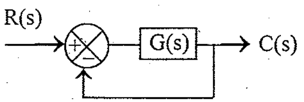


Fig 3.5.1 : Unity feedback system

Given that, $G(s) = K/s\,(sT+1)$

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s\,(sT+1)}{1+K/s(sT+1)} = \frac{K}{s\,(sT+1)+K} = \frac{K}{s^2T+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Expression for ζ and ω_n can be obtained by comparing the transfer function with the standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

On comparing we get,

$$\omega_n^2 = K/T \qquad\qquad 2\zeta\omega_n = 1/T$$

$$\therefore \omega_n = \sqrt{K/T} \qquad\qquad \zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}}.T} = \frac{1}{2\sqrt{KT}}$$

The peak overshoot, M_p is reduced by increasing the damping ratio ζ. The damping ratio ζ is increased by reducing the gain K.

When $M_p = 0.75$, Let $\zeta = \zeta_1$ and $K = K_1$

When $M_p = 0.25$, Let $\zeta = \zeta_2$ and $K = K_2$

Peak overshoot, $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Taking natural logarithm on both sides, $\ln M_p = \dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}$

On squaring we get, $\quad (\ln M_p)^2 = \dfrac{\zeta^2\pi^2}{1-\zeta^2}$

On cross multiplication we get,

$$(1-\zeta^2)\,(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 - \zeta^2(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 = \zeta^2\pi^2 + \zeta^2(\ln M_p)^2$$

$$(\ln M_p)^2 = \zeta^2\left[\pi^2 + (\ln M_p)^2\right]$$

$$\therefore \zeta^2 = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2} \quad .....(1)$$

but $\zeta = \dfrac{1}{2\sqrt{KT}}$, $\quad \therefore \zeta^2 = \dfrac{1}{4KT} \quad .....(2)$

On equating, equation (1) and (2) we get,

$$\frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$\frac{1}{K} = \frac{4T\,(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$K = \frac{\pi^2+(\ln M_p)^2}{4T\,(\ln M_p)^2}$$

When, $K = K_1$, $M_p = 0.75$, $\therefore K_1 = \dfrac{\pi^2+(\ln 0.75)^2}{4T\,(\ln 0.75)^2} = \dfrac{9.952}{0.331T} = \dfrac{30.06}{T}$

When, $K = K_2$, $M_p = 0.25$, $\therefore K_2 = \dfrac{\pi^2 + (ln0.25)^2}{4T\,(ln0.25)^2} = \dfrac{11.79}{7.68T} = \dfrac{1.53}{T}$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\ 30.06}{(1/T)\ 1.53} = 19.6$$

$$\mathbf{K_1 = 19.6\,K_2} \qquad \text{(or)} \qquad K_2 = \frac{\mathbf{1}}{\mathbf{19.6}}\,K_1$$

To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times (approximately 20 times).

RESULT

The value of gain, K should be reduced approximately 20 times to reduce peak overshoot from 0.75 to 0.25.

EXAMPLE 3.6

A positional control system with velocity feedback is shown in fig 3.6.1. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

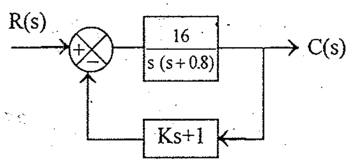


Fig 3.6.1

SOLUTION

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

Given that $G(s) = 16/s(s + 0.8)$ and $H(s) = Ks + 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{16}{s\,(s+0.8)}}{1 + \dfrac{16}{s\,(s+0.8)}\,(Ks+1)} = \frac{16}{s\,(s+0.8) + 16(Ks+1)}$$

$$= \frac{16}{s^2 + 0.8s + 16Ks + 16} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

The values of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

On comparing we get.

$$\omega_n^2 = 16 \qquad\qquad 0.8 + 16K = 2\zeta\omega_n$$

$$\therefore \omega_n = 4 \text{ rad/sec} \qquad \therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2 \times 0.5 \times 4 - 0.8}{16} = 0.2$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\,s + 16} = \frac{16}{s^2 + 4s + 16}$$

Given that the damping ratio, $\zeta = 0.5$. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of characteristic polynomial will be complex conjugate.

The response in s-domain, $C(s) = R(s)\,\dfrac{16}{s^2 + 4s + 16}$

For unit step input, $R(s) = 1/s$,

$$\therefore C(s) = \frac{1}{s}\,\frac{16}{s^2 + 4s + 16} = \frac{16}{s\,(s^2 + 4s + 16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s)\,s\big|_{s=0} = \left.\frac{16}{s^2 + 4s + 16}\right|_{s=0} = \frac{16}{16} = 1$$

The residues B and C are evaluated by cross multiplying the following equation and equating the coefficients of like powers of **s**.

$$\frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

On cross multiplication we get, $16 = A\,(s^2 + 4s + 16) + (Bs + C)\,s$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs.$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\quad \therefore B = -A = -1$

On equating the coefficients of s we get, $0 = 4A + C$ $\quad \therefore C = -4A = -4$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12} = \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\,\frac{\sqrt{12}}{(s+2)^2+12}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain,

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\,\frac{\sqrt{12}}{(s+2)^2+12}\right]$$

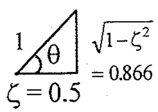
$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\,e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$.

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$ we get,

$\sin\theta = 0.866 = \sqrt{3}/2$; $\cos\theta = 0.5 = 1/2$; $\tan\theta = 1.732$



$\therefore \theta = \tan^{-1} 1.732 = 60° = 1.047$ rad.

$\therefore$ The response in time domain.

$$c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}} \times 2 \times \sin\sqrt{12}\,t \times \frac{1}{2} + \frac{2}{\sqrt{3}} \times \cos\sqrt{12}\,t \times \frac{\sqrt{3}}{2}\right]$$

$$= 1 - e^{-2t}\,\frac{2}{\sqrt{3}}\left[\sin\sqrt{12}\,t\cos\theta + \cos\sqrt{12}\,t\sin\theta\right]$$

$$= 1 - \frac{2}{\sqrt{3}}\,e^{-2t}\left[\sin(\sqrt{12}\,t+\theta)\right] = 1 - \frac{2}{\sqrt{3}}\,e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$$

(Note : θ is expressed in radians)

Damped frequency of oscillation $\Big\}\; \omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464$ rad / sec.

$$\therefore \text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - 1.047}{3.464} = 0.6046 \text{ sec.}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec.}$$

% Maximum overshoot $\Big\}\; \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%.$

Settling time, $t_s = 3T$ for 5% error

$= 4T$ for 2% error

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec.}$$

For 5% error, settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec.

For 2% error, settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec.

RESULT

The time domain response, $\quad c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$

(or) $\quad c(t) = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$

Rise time,	t_r	=	0.6046 sec
Peak time,	t_p	=	0.907 sec
Percentage maximum overshoot ,%M_p		=	16.3%
Settling time,	t_s	=	1.5 sec for 5% error
		=	2 sec for 2% error.

EXAMPLE 3.7

A unity feedback control system is characterized by the following open loop transfer function G(s) = (0.4 s +1)/s(s +0.6). Determine its transient response for unit step input and sketch the response. Evaluate the maximum overshoot and the corresponding peak time.

SOLUTION

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Given that, $G(s) = (0.4\,s+1)/s(s+0.6)$

For unity feedback system, $H(s) = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{0.4\,s+1}{s(s+0.6)}}{1+\frac{0.4\,s+1}{s(s+0.6)}} = \frac{0.4\,s+1}{s\,(s+0.6)+0.4\,s+1}$$

$$= \frac{0.4\,s+1}{s^2+0.6s+0.4s+1} = \frac{0.4\,s+1}{s^2+s+1}$$

The s-domain response, $C(s) = R(s)\,.\,\frac{0.4\,s+1}{s^2+s+1}$

For step input, $R(s) = 1/s$

$$\therefore C(s) = \frac{1}{s}\,\frac{0.4\,s+1}{s^2+s+1} = \frac{0.4\,s+1}{s(s^2+s+1)}$$

By partial fraction expansion C(s) can be expressed as

$$C(s) = \frac{0.4\,s+1}{s\,(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

The residue A is solved by multiplying C(s) by s and letting $s = 0$.

$$\therefore A = C(s)\,s\big|_{s=0} = \left.\frac{0.4\,s+1}{s^2+s+1}\right|_{s=0} = 1$$

The residues B and C are solved by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{0.4\,s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

On cross multiplication we get

$$0.4\,s+1 = A(s^2+s+1)+(Bs+C)\,s$$

$$0.4\,s+1 = As^2+As+A+Bs^2+Cs$$

On equating coefficients of s^2 we get $0 = A+B$ $\quad\therefore B = -A = -1$

On equating coefficients of s we get, $0.4 = A+C$ $\quad\therefore C = 0.4 - A = -0.6$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-0.6}{s^2+s+1} = \frac{1}{s} - \frac{s+0.6}{s^2+s+0.25+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5+0.1}{(s+0.5)^2+0.75} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\,.\,\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

$\therefore$ The response in time domain,

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\,\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}\right]$$

$$= 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}\,e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$= 1 - e^{-0.5t}\left[0.1155\,\sin(\sqrt{0.75}\,t)+\cos(\sqrt{0.75}\,t)\right]$$

The transient response is the part of the output which vanishes as t tends to infinity. Here as t tends to infinity the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by the damped sinusoidal component.

The transient response of $c(t) = e^{-0.5t}\left[0.1155\,\sin(\sqrt{0.75}\,t)+\cos(\sqrt{0.75}\,t)\right]$

The value of ζ and ω_n can be estimated by comparing the characteristic equation of the system transfer function with standard form of second order characteristic equation.

$$\therefore s^2 + 2\zeta\omega_n\,s + \omega_n^{\,2} = s^2 + s + 1$$

On comparing we get,

$\omega_n^2 = 1$	$2\zeta\omega_n = 1$
$\therefore \omega_n = 1$ rad / sec	$\therefore \zeta = \frac{1}{2\omega_n} = \frac{1}{2} = 0.5$

Maximum overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 0.163$

$$\%\text{Maximum overshoot, } \%M_p = M_p \times 100 = 0.163 \times 100$$
$$= 16.3\%$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1\times\sqrt{1-0.5^2}} = 3.628 \text{ sec.}$$

The response of the system is underdamped and it is shown in fig 3.7.1.

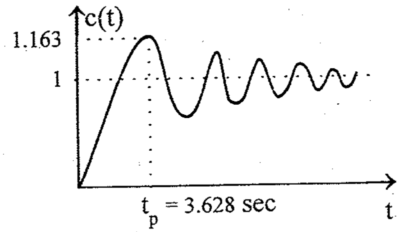


Fig : 3.7.1: Response of underdamped system

RESULT

Transient response of the system = $e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

% Maximum peak overshoot = 16.3 %

Peak time = 3.628 sec.

EXAMPLE 3.8

A unity feedback control system has an amplifier with gain $K_A = 10$ and gain ratio, $G(s) = 1/s(s+2)$ in the feed forward path. A derivative feedback, $H(s) = sK_o$ is introduced as a minor loop around G(s). Determine the derivative feedback constant, K_o so that the system damping factor is 0.6.

SOLUTION

The given system can be represented by the block diagram shown in fig 3.8.1.

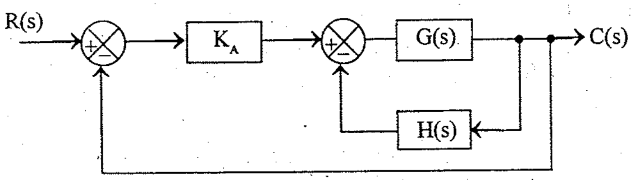
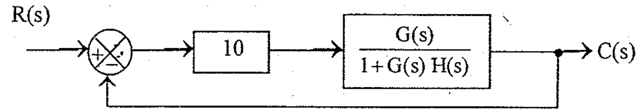


Fig 3.8.1.

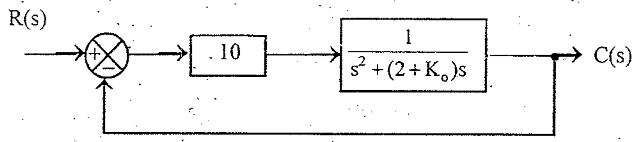
Here, $K_A = 10$; $G(s) = \dfrac{1}{s(s+2)}$ and $H(s) = sK_o$

The closed loop transfer function of the system can be obtained by block diagram reduction techniques.

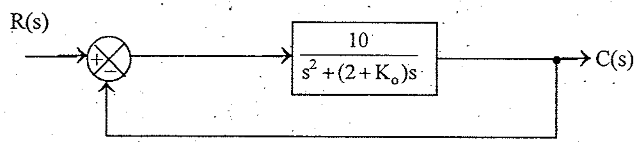
Step 1: Reducing the inner feedback loop.



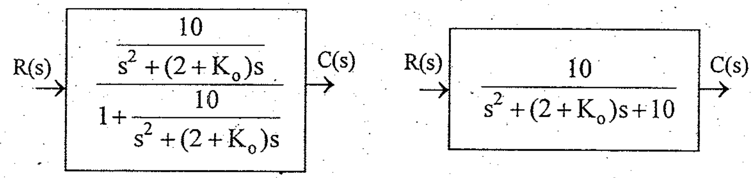
$$\frac{G(s)}{1+G(s)\,H(s)} = \frac{\dfrac{1}{s\,(s+2)}}{1+\dfrac{1}{s\,(s+2)}.sK_o} = \frac{1}{s\,(s+2)+sK_o} = \frac{1}{s^2+2s+sK_o}$$
$$= \frac{1}{s^2+(2+K_o)\,s}$$



Step 2 : combining blocks in cascade



Step 3 : Reducing the unity feedback path



$$\text{The closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{10}{s^2+(2+K_o)\,s+10}$$

The given system is a second order system. The value of K_o can be determined by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{10}{s^2 + (2+K_o)\,s + 10}$$

On comparing we get

$$\omega_n^2 = 10 \qquad \therefore \omega_n = \sqrt{10} = 3.162 \text{ rad / sec.}$$

$$2 + K_o = 2\zeta\omega_n \qquad \therefore K_o = 2\zeta\omega_n - 2 = 2 \times 0.6 \times 3.162 - 2 = 1.7944.$$

RESULT

The value of constant, $K_o = 1.7944$

EXAMPLE 3.9

A unity feedback control system has an open loop transfer function, G(s) = 10/s(s+2). Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.

SOLUTION

(Note : The formulae for rise time, percentage overshoot and peak time remains same for unit step and step input).

The unity feedback system is shown in fig 3.9.1.

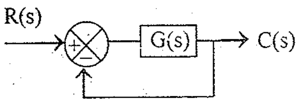


Fig 3.9.1 : Unity feedback system

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s\,(s+2)}}{1 + \frac{10}{s\,(s+2)}} = \frac{10}{s\,(s+2) + 10} = \frac{10}{s^2 + 2s + 10}$$

The values of damping ratio ζ and natural frequency of oscillation ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{10}{s^2 + 2s + 10}$$

On comparing we get,

$$\omega_n^2 = 10 \qquad \therefore \omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$$

$$2\zeta\omega_n = 2 \qquad \therefore \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad / sec.}$$

$$\text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.249}{3} = 0.63 \text{ sec.}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100 = 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot} = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec.}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1 \text{ sec.}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

RESULT

Rise time,	t_r	=	0.63 sec
Percentage overshoot,	$\%M_p$	=	35.12%
Peak overshoot		=	4.2144 units, (for a input of 12 units)
Peak time,	t_p	=	1.047 sec
Settling time,	t_s	=	3 sec for 5% error
		=	4 sec for 2% error

EXAMPLE 3.10

A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\ e$

Where **c** is the displacement of the output shaft, r is the displacement of the input shaft and e = r–c. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input.

SOLUTION

The mathematical equations governing the system are

$$\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\ e \qquad .....(3.10.1)$$

and e = r–c .....(3.10.2)

Put e = r–c in equation (3.10.1)

$$\therefore \frac{d^2c}{dt^2}+8\frac{dc}{dt}=64(r-c) \qquad .....(3.10.3)$$

Let L[c] = C(s) and L[r] = R(s)

On taking Laplace transform of equation (3.10.3) we get,

$$s^2\ C(s) + 8s\ C(s) = 64\ [R(s) - C(s)]$$

$$\therefore s^2\ C(s) + 8s\ C(s) + 64\ C(s) = 64\ R(s)$$

$$(s^2 + 8s + 64)\ C(s) = 64\ R(s)$$

$$\therefore \frac{C(s)}{R(s)}=\frac{64}{s^2+8s+64}$$

The ratio C(s)/R(s) is the closed loop transfer function of the system. On comparing the system transfer function with standard form of second order transfer function, we can estimate the values of ζ and ω_n.

$$\therefore \frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{64}{s^2+8s+64}$$

On comparing we get,

$$\omega_n^2 = 64 \qquad 2\zeta\omega_n = 8$$

$$\therefore \omega_n = 8\ \text{rad / sec} \qquad \zeta=\frac{8}{2\omega_n}=\frac{8}{2\times 8}=0.5$$

$$\text{Percentage peak overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100$$

$$= e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}}\times 100$$

$$= 16.3\%$$

RESULT

Undamped natural frequency of oscillation,	ω_n	=	8 rad/sec
Damping ratio,	ζ	=	0.5
Percentage peak overshoot,	$\%M_p$	=	16.3%

3.9 TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general if **N** is the number of poles at the origin then the type number is **N**.

The loop transfer function can be expressed as a ratio of two polynomials in **s**

$$G(s)\ H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\ (s+z_2)\ (s+z_3)\ ....}{s^N\ (s+p_1)\ (s+p_2)\ (s+p_3)\ ....} \qquad .....(3.50)$$

Where, z_1, z_2, z_3, etc, are zeros of transfer function

p_1, p_2, p_3, etc, are poles of transfer function

K = Constant

N = Number of poles at the origin

The value of **N** in the denominator polynomial decides the type number of the system.

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

EXAMPLE 3.10

A closed loop servo is represented by the differential

Where **c** is the displacement of the outpu shaft and e = r–c. Determine undamped na maximum overshoot for unit step inp

SOLUTION

The mathemati

ring we get,

$2\zeta\omega_n = 8$

$\zeta = \frac{8}{2\omega_n} = \frac{8}{2\times 8} = 0.5$

100

ity. The nature of eady state rror to step,

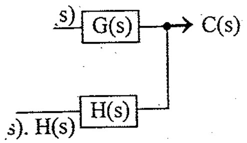


Fig 3.15

.....(3.51)

.....(3.52)

E(

E(s) [1 + G(s)

$$\therefore E(s) = \frac{R(s)}{1+G(s)\,H(s)} \quad .....(3.53)$$

Let e(t) = error signal in time domain

$$\therefore e(t) = L^{-1}[E(s)] = L^{-1}\left[\frac{R(s)}{1+G(s)\,H(s)}\right] \quad .....(3.54)$$

Let e_{ss} = steady state error

The steady state error is defined as the value of e(t) when t tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{Lt}\; e(t) \quad .....(3.55)$$

The final value theorem states that,

$$\text{If } F(s) = L[f(t)] \text{ then, } \underset{t\to\infty}{Lt}\; f(t) = \underset{s\to 0}{Lt}\; s\,F(s) \quad .....(3.56)$$

Using final value theorem,

$$\text{The steady state error, } e_{ss} = \underset{t\to\infty}{Lt}\; e(t) = \underset{s\to 0}{Lt}\; sE(s) = \underset{s\to 0}{Lt}\; \frac{sR(s)}{1+G(s)\,H(s)} \quad .....(3.57)$$

3.11 STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type – 0 system will have a constant steady state error when the input is step signal. Type –1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type–2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above the steady state error is associated with one of the constants defined as follows,

$$\text{Positional error constant, } K_p = \underset{s\to 0}{Lt}\; G(s)\,H(s) \quad .....(3.58)$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{Lt}\; s.G(s)\,H(s) \quad .....(3.59)$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{Lt}\; s^2G(s)\,H(s) \quad .....(3.60)$$

The K_p, K_v and K_a are in general called static error constants.

3.12 STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{Lt}\; \frac{sR(s)}{1+G(s)\,H(s)}$$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; \frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\; \frac{1}{1+G(s)\,H(s)}$$

$$= \frac{1}{1+\underset{s\to 0}{Lt}\; G(s)\,H(s)} = \frac{1}{1+K_p} \quad .....(3.61)$$

Where, $K_p = \underset{s\to 0}{Lt}\; G(s)\,H(s)$. The constant K_p is called positional error constant.

3.10 STEADY STATE ERROR

The steady state error is the value of error signal e(t), when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non linearity of system components. The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig 3.15.

R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

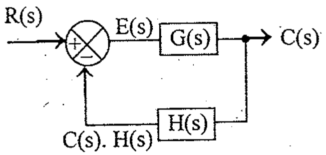


Fig 3.15

The error signal, $E(s) = R(s) - C(s)\,H(s)$ .....(3.51)

The output signal, $C(s) = E(s)\,G(s)$ .....(3.52)

On substituting equation (3.52) in equation (3.51)

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)\,H(s)} \qquad .....(3.53)$$

Let e(t) = error signal in time domain

$$\therefore e(t) = L^{-1}[E(s)] = L^{-1}\left[\frac{R(s)}{1+G(s)\,H(s)}\right] \qquad .....(3.54)$$

Let e_{ss} = steady state error

The steady state error is defined as the value of e(t) when t tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t) \qquad .....(3.55)$$

The final value theorem states that,

$$\text{If } F(s) = L[f(t)] \text{ then, } \underset{t\to\infty}{\mathrm{Lt}}\; f(t) = \underset{s\to 0}{\mathrm{Lt}}\; s\,F(s) \qquad .....(3.56)$$

Using final value theorem,

$$\text{The steady state error, } e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t) = \underset{s\to 0}{\mathrm{Lt}}\; sE(s) = \underset{s\to 0}{\mathrm{Lt}}\; \frac{sR(s)}{1+G(s)\,H(s)} \qquad .....(3.57)$$

3.11 STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type – 0 system will have a constant steady state error when the input is step signal. Type –1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type–2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above the steady state error is associated with one of the constants defined as follows,

Positional error constant, $K_p = \underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s)$ .....(3.58)

Velocity error constant, $K_v = \underset{s\to 0}{\mathrm{Lt}}\; s.G(s)\,H(s)$ .....(3.59)

Acceleration error constant, $K_a = \underset{s\to 0}{\mathrm{Lt}}\; s^2G(s)\,H(s)$ .....(3.60)

The K_p, K_v and K_a are in general called static error constants.

3.12 STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{sR(s)}{1+G(s)\,H(s)}$$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{1}{1+G(s)\,H(s)}$$

$$= \frac{1}{1+\underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s)} = \frac{1}{1+K_p} \qquad .....(3.61)$$

Where, $K_p = \underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s)$. The constant K_p is called positional error constant.

TYPE-0 SYSTEM

$$K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s) = \underset{s\to 0}{Lt}\ K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......}$$

$$= K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type 0 systems when the input is unit step there will be a constant steady state error.

TYPE-1 SYSTEM

$$K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s) = \underset{s\to 0}{Lt}\ K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

3.13 STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{Lt}\ \frac{sR(s)}{1+G(s)\,H(s)}$$

$$\text{When the input is unit ramp, } R(s) = \frac{1}{s^2}$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\ \frac{1}{s+sG(s)\,H(s)}$$

$$= \frac{1}{\underset{s\to 0}{Lt}\ sG(s)\,H(s)} = \frac{1}{K_v}$$

Where, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$. The constant K_v is called velocity error constant.

TYPE-0 SYSTEM

$$K_v = \underset{s\to 0}{Lt}\ sG(s)\,H(s) = \underset{s\to 0}{Lt}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type 0 systems when the input is unit ramp, the steady state error is infinity.

TYPE-1 SYSTEM

$$K_v = \underset{s\to 0}{Lt}\ sG(s)\,H(s) = \underset{s\to 0}{Lt}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......}$$

$$= K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

Hence in type 1 systems when the input is unit ramp there will be a constant steady state error.

TYPE-2 SYSTEM

$$K_v = \underset{s\to 0}{Lt}\ sG(s)\,H(s) = \underset{s\to 0}{Lt}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

3.14 STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{Lt}\ \frac{sR(s)}{1+G(s)\,H(s)}$$

$$\text{When the input is unit parabola, } R(s) = \frac{1}{s^3}$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\frac{1}{s^3}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\ \frac{1}{s^2+s^2\,G(s)\,H(s)}$$

$$= \frac{1}{\underset{s\to 0}{Lt}\ s^2G(s)\,H(s)} = \frac{1}{K_a} \qquad .....(3.62)$$

Where, $K_a = \underset{s\to 0}{Lt}\ s^2G(s)\,H(s)$. The constant K_a is called acceleration error constant.

TYPE-0 SYSTEM

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s \to 0}{Lt}\, s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type 0 systems for unit parabolic input, the steady state error is infinity.

TYPE-1 SYSTEM

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s \to 0}{Lt}\, s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type 1 systems for unit parabolic input, the steady state error is infinity.

TYPE-2 SYSTEM

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s \to 0}{Lt}\, s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots}$$

$$= K \frac{z_1.z_2.z_3\ldots\ldots}{p_1.p_2.p_3\ldots\ldots} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type 2 system when the input is unit parabolic signal there will be a constant steady state error.

TYPE-3 SYSTEM

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s \to 0}{Lt}\, s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

TABLE 3.2 : The steady state error for various types of inputs

Type number	Steady state error when the input signal is		
	Unit step	Unit ramp	Unit parabolic
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$
3	0	0	0

3.15 GENERALIZED ERROR COEFFICIENT

The drawback in static error coefficients is that it does not show the variation of error with time and input should be a standard input. The generalized error coefficients gives the steady state error as a function of time. Also using the generalized error coefficients, the steady state error can be found for any type of input.

The error signal in s-domain, E(s) can be expressed as a product of two s-domain functions.

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)} = \frac{1}{1+G(s)\,H(s)} R(s) = F(s)\,R(s) \qquad \ldots..(3.63)$$

$$\text{Where,}\ F(s) = \frac{1}{1+G(s)\,H(s)}$$

Let $e(t) = L^{-1}[E(s)]$ (error signal in time domain)

$f(t) = L^{-1}[F(s)]$

$r(t) = L^{-1}[R(s)]$ (input signal in time domain)

The convolution theorem states that the inverse Laplace transform of the product of two s-domain functions is equal to convolution of their time domain function.

$$\text{i.e., } L^{-1}[F(s)\,R(s)] = \int_{-\infty}^{\infty} f(T)\, r(t-T)\, dT$$

Hence by convolution theorem the error function in time domain can be written as,

$$e(t) = \int_{-\infty}^{+\infty} f(T)\, r(t-T)\, dT \qquad \text{where T is a dummy variable} \qquad .....(3.64)$$

It is assumed that the input signal starts only at $t=0$ and does not exist before $t=0$. Hence the limit of integral can be changed as 0 to ∞.

$$\therefore e(t) = \int_{0}^{\infty} f(T)\, r(t-T)\, dT \qquad .....(3.65)$$

Using Taylor's series expansion the signal r(t–T) can be expressed as,

$$r(t-T) = r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + ...+(-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)... \qquad .....(3.66)$$

Where, $\dot{r}(t) = 1^{st}$ derivative of r(t)

$\ddot{r}(t) = 2^{nd}$ derivative of r(t)

and $\overset{n.}{r}(t) = n^{th}$ derivative of r(t)

On substituting the Taylor's series expansion of r(t–T), the error e(t) can be written as

$$e(t) = \int_{0}^{\infty} f(T)\left[r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + ...+(-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)...\right] dT$$

$$e(t) = \int_{0}^{\infty} f(T)\, r(t)\, dT - \int_{0}^{\infty} f(T)\, T\,\dot{r}(t)\, dT + \int_{0}^{\infty} f(T)\frac{T^2}{2!}\ddot{r}(t)\, dT$$

$$- \int_{0}^{\infty} f(T)\frac{T^3}{3!}\dddot{r}(t) dT + ... + \int_{0}^{\infty} f(T)\,(-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)\, dT....\infty$$

Since $r(t), \dot{r}(t), \ddot{r}(t), ... \overset{n.}{r}(t)$ are constants when the integration is done with respect to T, the error signal can be written as

$$e(t) = r(t)\int_{0}^{\infty} f(T)\, dT - \dot{r}(t)\int_{0}^{\infty} Tf(T)\, dT + \frac{\ddot{r}(t)}{2!}\int_{0}^{\infty} T^2 f(T)\, dT$$

$$-\frac{\dddot{r}(t)}{3!}\int_{0}^{\infty} T^3 f(T)\, dT + ...+(-1)^n \frac{\overset{n.}{r}(t)}{n!}\int T^n f(T)\, dT...$$

Let $C_0 = +\int_{0}^{\infty} f(T)\, dT$ $\qquad C_3 = -\int_{0}^{\infty} T^3 f(T)\, dT$

$C_1 = -\int_{0}^{\infty} Tf(T)\, dT$

$C_2 = +\int_{0}^{\infty} T^2 f(T)\, dT \qquad C_n = (-1)^n \int_{0}^{\infty} T^n f(T)\, dT \qquad .....(3.67)$

$$e(t) = r(t).C_0 + \dot{r}(t)\, C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + ...+ \overset{n.}{r}(t)\,\frac{C_n}{n!} + ...$$

$$= C_0\, r(t) + C_1\, \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + ...+ \frac{C_n}{n!}\overset{n.}{r}(t)..... \qquad .....(3.68)$$

The coefficients $C_0, C_1, C_2, ...C_n$ are called the generalized error coefficients or dynamic error coefficients.

The steady state error e_{ss} is obtained by taking limit $t \to \infty$ on e(t), $\therefore e_{ss} = \underset{t\to\infty}{Lt}\, e(t)$.

3.16 EVALUATION OF GENERALIZED ERROR COEFFICIENTS

The generalized error coefficient is given by,

$$C_n = (-1)^n \int_{0}^{\infty} T^n f(T)\, dT, \qquad \text{Where } F(s) = \frac{1}{1+G(s)\,H(s)}$$

We know that $L[f(T)] = F(s)$, hence by the definition of Laplace transform,

$$F(s) = \int_{0}^{\infty} f(T)\, e^{-sT} dT \qquad .....(3.69)$$

On taking $\underset{s\to 0}{Lt}$ on both sides

$$\underset{s\to 0}{Lt}\, F(s) = \underset{s\to 0}{Lt} \int_{0}^{\infty} f(T)\, e^{-sT} dT$$

$$= \int_{0}^{\infty} f(T)\, \underset{s\to 0}{Lt}\, e^{-sT} dT = \int_{0}^{\infty} f(T)\, dT = C_0$$

$$\therefore \mathbf{C_0 = \underset{s\to 0}{Lt}\, F(s)} \qquad .....(3.70)$$

On differentiating equation (3.69) with respect to s,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(T)\, e^{-sT}\, dT$$

$$= \int_0^\infty f(T)\, \frac{d}{ds}(e^{-sT})dT = \int_0^\infty f(T)\,(-T)\, e^{-sT} dT$$

$$\frac{d}{ds}[F(s)] = -\int_0^\infty Tf(T)\, e^{-sT} dT \qquad .....(3.71)$$

On taking $\underset{s\to 0}{\text{Lt}}$ on both sides,

$$\underset{s\to 0}{\text{Lt}}\, \frac{d}{ds}F(s) = \underset{s\to 0}{\text{Lt}}\, -\int_0^\infty Tf(T)\, e^{-sT} dT$$

$$= -\int_0^\infty Tf(T)\, \underset{s\to 0}{\text{Lt}}\, e^{-sT} dT = -\int_0^\infty Tf(T)\, dT = C_1$$

$$\therefore \mathbf{C_1 = \underset{s\to 0}{Lt}\, \frac{d}{ds}F(s)} \qquad .....(3.72)$$

Differentiating equation (3.71) on both sides with respect to s

$$\frac{d}{ds}\left[\frac{d}{ds}(F(s))\right] = \frac{d}{ds}\left[-\int_0^\infty Tf(T)\, e^{-sT}\, dT\right]$$

$$\frac{d^2}{ds^2}F(s) = \left[-\int_0^\infty Tf(T)\, \frac{d}{ds}(e^{-sT})dT\right] = -\int_0^\infty Tf(T)\,(-T)\, e^{-sT} dT$$

$$\therefore \frac{d^2}{ds^2}F(s) = \int_0^\infty T^2 f(T)\, e^{-sT} dT \qquad .....(3.73)$$

Applying the limit $s \to 0$ on both sides of the equation (3.73)

$$\underset{s\to 0}{\text{Lt}}\, \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{\text{Lt}}\int_0^\infty T^2\, f(T)\, e^{-st} dT$$

$$= \int_0^\infty T^2 f(T)\, \underset{s\to 0}{\text{Lt}}\, e^{-sT} dT = \int_0^\infty T^2 f(T)\, dT = C_2$$

$$\therefore \mathbf{C_2 = \underset{s\to 0}{Lt}\, \frac{d^2}{ds^2}F(s)} \qquad .....(3.74)$$

Similarly it can be shown that

$$\mathbf{C_n = \underset{s\to 0}{Lt}\, \frac{d^n}{ds^n}F(s)} \qquad .....(3.75)$$

3.17 CORRELATION BETWEEN STATIC AND DYNAMIC ERROR COEFFICIENTS

The values of dynamic error coefficients can be used to calculate static error coefficients. The following expressions shows the relationship between them.

$$C_0 = \frac{1}{1+K_p} \qquad .....(3.76)$$

$$C_1 = \frac{1}{K_v} \qquad .....(3.77)$$

$$C_2 = \frac{1}{K_a} \qquad .....(3.78)$$

PROOF

$$C_0 = \underset{s\to 0}{\text{Lt}}\, F(s) = \underset{s\to 0}{\text{Lt}}\, \frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s\to 0}{\text{Lt}}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

3.18 ALTERNATE METHOD FOR GENERALIZED ERROR COEFFICIENTS

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)\,H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} \qquad .....(3.79)$$

The equation (3.79) can be expressed as a power series of s as shown in equation (3.80)

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} = C_0 + C_1\, s + \frac{C_2}{2!}\, s^2 + \frac{C_3}{3!}\, s^3 + ... \qquad .....(3.80)$$

$$\therefore E(s) = C_0\, R(s) + C_1\, s\, R(s) + \frac{C_2}{2!}\, s^2\, R(s) + \frac{C_3}{3!}\, s^3\, R(s) + \ldots \qquad \ldots..(3.81)$$

On taking inverse Laplace transform of equation (3.81) we get

$$e(t) = C_0\, r(t) + C_1\, \dot{r}(t) + \frac{C_2}{2!}\, \ddot{r}(t) + \frac{C_3}{3!}\, \dddot{r}(t) + \ldots \qquad \ldots..(3.82)$$

The equation (3.82) is same as that of equation (3.68) in section 3.15. The coefficients $C_0, C_1, C_2, \ldots\ldots$ are called generalized error coefficients or dynamic error coefficients. The steady state error e_{ss} is obtained by taking limit $t \to \infty$ on e(t).

$$\therefore e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t)$$

EXAMPLE 3.11

For a unity feedback control system the open loop transfer function $G(s) = 10(s+2)/s^2(s+1)$. Find (a) the position, velocity and acceleration error constants, (b) the steady state error when the input is R(s) where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

SOLUTION

a) **To find static error constants**

For a unity feedback system, H(s)=1

Position error Constant, $K_p = \underset{s\to 0}{\mathrm{Lt}}\; G(s)H(s) = \underset{s\to 0}{\mathrm{Lt}}\; G(s)$

$$= \underset{s\to 0}{\mathrm{Lt}}\; \frac{10(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant, $K_v = \underset{s\to 0}{\mathrm{Lt}}\; s.G(s)H(s) = \underset{s\to 0}{\mathrm{Lt}}\; s.G(s)$

$$= \underset{s\to 0}{\mathrm{Lt}}\; s\frac{10(s+2)}{s^2(s+1)} = \infty$$

Acceleration error constant, $K_a = \underset{s\to 0}{\mathrm{Lt}}\; s^2G(s)H(s) = \underset{s\to 0}{\mathrm{Lt}}\; s^2G(s)$

$$= \underset{s\to 0}{\mathrm{Lt}}\; s^2\frac{10(s+2)}{s^2(s+1)} = \frac{10\times 2}{1} = 20$$

b) **To find steady state error**

I[st] method

The steady state error for non standard input is obtained using generalized error series, given below.

The error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + \ldots\ldots + \overset{n}{r}(t)\frac{C_n}{n!} + \ldots$

Given that, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Input signal in time domain, $r(t) = L^{-1}[R(s)] = L^{-1}\left[\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right]$

$$= \frac{1}{3} - 2t + \frac{1}{3}\frac{t^2}{2!} = \frac{1}{3} - 2t + \frac{t^2}{6}$$

$$\therefore \dot{r}(t) = \frac{d}{dt}r(t) = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

The derivatives of r(t) is zero after second derivative. Hence we have to evaluate only three constants C_0, C_1 and C_2.

The generalized error constants are given by

$$C_0 = \underset{s\to 0}{\mathrm{Lt}}\; F(s)\;;\; C_1 = \underset{s\to 0}{\mathrm{Lt}}\; \frac{d}{ds}F(s)\;;\; C_2 = \underset{s\to 0}{\mathrm{Lt}}\; \frac{d^2}{ds^2}F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)}$$

$$= \frac{s^3+s^2}{s^3+s^2+10s+20}$$

$$C_0 = \underset{s\to 0}{\mathrm{Lt}}\; F(s) = \underset{s\to 0}{\mathrm{Lt}}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right] = 0$$

$$C_1 = \underset{s\to 0}{Lt}\frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)(3s^2+2s)-(s^3+s^2)(3s^2+2s+10)}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right] = 0$$

$$C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{d}{ds}F(s)\right]$$

$$= \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)^2(60s^2+140s+40) - (20s^3+70s^2+40s)\,2\times(s^3+s^2+10s+20)\,(3s^2+2s+10)}{(s^3+s^2+10s+20)^4}\right]$$

$$= \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!}$

$$= \left(\frac{1}{3} - 2t + \frac{t^2}{6}\right)\times 0 + \left(-2+\frac{t}{3}\right)\times 0 + \frac{1}{3}\times\frac{1}{10}\times\frac{1}{2!}$$

$$= \frac{1}{60}$$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\frac{1}{60} = \frac{1}{60}$

IInd method

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

Given that $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3\,s^3}$

$$G(s) = \frac{10(s+2)}{s^2(s+1)} \quad \text{and} \quad H(s) = 1$$

$$\therefore E(s) = \frac{\dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3\,s^3}}{1+\dfrac{10(s+2)}{s^2(s+1)}} = \frac{\dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3\,s^3}}{\dfrac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, s.E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\}$$

$$= 0 - 0 + \frac{1}{60} = \frac{1}{60}$$

$$\therefore e_{ss} = \frac{1}{60}$$

IIIrd method

Error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}, \quad \text{Given that } G(s) = \frac{10(s+2)}{s^2(s+1)} \text{ and } H(s) = 1$$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\dfrac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)}$$

$$= \frac{s^3+s^2}{s^3+s^2+10s+20} = \frac{s^2+s^3}{20+10s+s^2+s^3} = \frac{s^2}{20}+\frac{s^3}{40}+\ldots.$$

$$\begin{array}{r|l}
 & \dfrac{s^2}{20}+\dfrac{s^3}{40} \\
20+10s+s^2+s^3 & s^2+s^3 \\
 & s^2+\dfrac{s^3}{2}+\dfrac{s^4}{20}+\dfrac{s^5}{20} \\
 & \dfrac{s^3}{2}-\dfrac{s^4}{20}-\dfrac{s^5}{20} \\
 & \dfrac{s^3}{2}+\dfrac{s^4}{4}+\dfrac{s^5}{40}+\dfrac{s^6}{40} \\
 & \dfrac{-3}{10}s^4-\dfrac{3}{40}s^5-\dfrac{s^6}{40}
\end{array}$$

$$E(s) = R(s)\left[\frac{s^2}{20}+\frac{s^3}{40}+\ldots\right]$$

$$= \frac{1}{20}s^2R(s)+\frac{1}{40}s^3R(s)+\ldots$$

On taking inverse laplace transform,

$$e(t) = \frac{1}{20}\ddot{r}(t)+\frac{1}{40}\dddot{r}(t)+\ldots$$

Given that $R(s) = \dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3s^3}$

$$\therefore r(t) = L^{-1}[R(s)] = L^{-1}\left[\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}\right] = \frac{1}{3}-2t+\frac{1}{3}\frac{t^2}{2!} = \frac{1}{3}-2t+\frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}r(t) = -2+\frac{1}{6}2t = -2+\frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error signal in time domain, $e(t) = \dfrac{1}{20}\dfrac{d}{dt}\dot{r}(t) = \dfrac{1}{20}\left(\dfrac{1}{3}\right) = \dfrac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\, \dfrac{1}{60} = \dfrac{1}{60}$

RESULT

a) Position error constant, $K_p = \infty$
Velocity error constant, $K_v = \infty$
Acceleration error constant, $K_a = 20$

b) Steady state error When $R(s) = \dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3s^3}$ $\Bigg\}$ $e_{ss} = \dfrac{1}{60}$

EXAMPLE 3.12

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

(i) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; (ii) $G(s) = \dfrac{10}{(s+2)(s+3)}$; (iii) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

SOLUTION

(i) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = 1/K_v$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s).H(s) = \underset{s\to 0}{Lt}\, sG(s)$

$$= \underset{s\to 0}{Lt}\, s\frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{ss} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

(ii) $G(s) = \dfrac{10}{(s+2)(s+3)}$

Let us assume unity feedback system $\therefore$ H(s)=1.

The open loop system has no pole at origin ,Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \dfrac{1}{1+K_p}$

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ G(s)$

$$= \underset{s\to 0}{Lt}\frac{10}{(s+2)(s+3)} = \frac{10}{2\times 3} = \frac{5}{3}$$

Steady state error, $$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{5}{3}} = \frac{3}{3+5} = \frac{3}{8} = 0.375$$

(iii) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore$H(s)=1

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = 1/K_a$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2\ G(s)H(s) = \underset{s\to 0}{Lt}\ s^2\ G(s)$

$$= \underset{s\to 0}{Lt}\ s^2\frac{10}{s^2(s+1)(s+2)} = \frac{10}{1\times 2} = 5$$

Steady state error, $e_{ss} = \dfrac{1}{K_a} = \dfrac{1}{5} = 0.2$

RESULT

1. Steady state error in system (i) with unit velocity input = 0.075
2. Steady state error in system (ii) with unit step input = 0.375
3. Steady state error in system (iii) with unit acceleration input = 0.2

EXAMPLE 3.13

The open loop transfer function of a servo system with unity feedback is G(s) = 10/s(0.1s+1). Evaluate the static error constants of the system. Obtain the steady state error of the system. When subjected to an input given by the polynominal $r(t) = a_0 + a_1 t + \frac{a_2}{2}t^2$.

SOLUTION

To find static error constant

For unity feedback system, H(s) = 1

$\therefore$ Loop transfer function, G(s) H(s) = G(s)

The static error constants are K_p, K_v and K_a

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\dfrac{10}{s(0.1s+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s\dfrac{10}{s(0.1s+1)} = 10$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ s^2\dfrac{10}{s(0.1s+1)} = 0$

To find steady state error

I[st] method

The steady state error for non standard input is obtained using generalized error series, given below.

The error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\dfrac{C_2}{2!} + ... + \overset{n}{r}(t)\dfrac{C_n}{n!} + ...$

Given that, $r(t) = a_0 + a_1 t + \dfrac{a_2}{2}t^2$

$\therefore \dot{r}(t) = a_1 + a_2 t$

$\ddot{r}(t) = a_2$

$\dddot{r}(t) = 0$

The derivatives of r(t) is zero after second derivative. Hence we have to evaluate only three constants C_0, C_1 and C_2.

The generalized error constants are given by

$$C_o = \underset{s\to 0}{Lt}\ F(s)\ ;\ C_1 = \underset{s\to 0}{Lt}\frac{d}{ds}F(s)\ ;\ C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2+s}{0.1s^2+s+10}$$

$$C_o = \underset{s\to 0}{Lt}\ F(s) = \underset{s\to 0}{Lt}\frac{0.1s^2+s}{0.1s^2+s+10} = 0$$

$$C_1 = \underset{s\to 0}{Lt}\ \frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{0.1s^2+s}{0.1s^2+s+10}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(0.1s^2+s+10)(0.2s+1)-(0.1s^2+s)(0.2s+1)}{(0.1s^2+s+10)^2}\right]$$

$$= \underset{s\to 0}{Lt}\frac{2s+10}{(0.1s^2+s+10)^2} = \frac{10}{10^2} = 0.1$$

$$C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{2s+10}{(0.1s^2+s+10)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(0.1s^2+s+10)^2(2)-(2s+10)(2)(0.1s^2+s+10)(0.2s+1)}{(0.1s^2+s+10)^4}\right]$$

$$= \frac{10^2\times 2-10\times 2\times 10\times 1}{10^4} = 0$$

$$\text{Error signal, } e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!}$$

$$= \dot{r}(t)C_1 \quad (\because C_o \text{ and } C_2 \text{ are zeros})$$

$$= (a_1 + a_2 t)\ 0.1$$

Steady state error is obtained by letting t to ∞

$$\therefore \text{ Steady state error, } e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to\infty}{Lt}\left[(a_1+a_2t)\ 0.1\right] = \infty$$

IInd method

$$\text{The error signal in s-domain, } E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$\text{Given that } r(t) = a_o + a_1 t + \frac{a_2}{2}t^2$$

$$G(s) = \frac{10}{s(0.1s+1)}$$

$$H(s) = 1$$

On taking laplace transform of r(t) we get R(s)

$$\therefore R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{a_o}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}}{1+\frac{10}{s(0.1s+1)}} = \frac{\frac{a_o}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}}{\frac{s(0.1s+1)+10}{s(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]$$

The steady state error e_{ss} can be obtained from final value theorem

$$\text{Steady state error, } e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s.E(s)$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{a_o s(0.1s+1)}{s(0.1s+1)+10} + \frac{a_1(0.1s+1)}{s(0.1s+1)+10} + \frac{a_2(0.1s+1)}{s[s(0.1s+1)+10]}\right\} = 0 + \frac{a_1}{10} + \infty = \infty$$

$$\therefore e_{ss} = \infty$$

III[rd] method

Error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

Given that $G(s) = \dfrac{10}{s(0.1s+1)}$ and $H(s) = 1$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\dfrac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2+s}{0.1s^2+s+10}$$

$$= \frac{s+0.1s^2}{10+s+0.1s^2}$$

$$= \frac{s}{10} - \frac{s^3}{1000} + ...$$

$$\therefore E(s) = \frac{s}{10}R(s) - \frac{s^3}{1000}R(s) + ..$$

$$
\begin{array}{r|l}
 & \dfrac{s}{10} - \dfrac{s^3}{1000} \\
10+s+0.1s^2 & s+0.1s^2 \\
 & s + \dfrac{s^2}{10} + \dfrac{s^3}{100} \\
 & -\dfrac{s^3}{100} \\
 & -\dfrac{s^3}{100} - \dfrac{s^4}{1000} - \dfrac{s^5}{10000} \\
 & \dfrac{s^4}{1000} + \dfrac{s^5}{10000}
\end{array}
$$

On taking inverse laplace transform,

$$e(t) = \frac{1}{10}\dot{r} - \frac{1}{1000}\dddot{r}(t) + ...$$

Given that $r(t) = a_0 + a_1 t + \dfrac{a_2}{2}t^2$

$$\therefore \quad \dot{r} = \frac{d}{dt}r(t) = a_1 + a_2 t$$

$$\ddot{r}(t) = \frac{d}{dt}\dot{r}(t) = a_2$$

$$\dddot{r}(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error signal in time domain, $e(t) = \dfrac{1}{10}\dot{r}(t) = \dfrac{1}{10}(a_1 + a_2 t)$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\, \dfrac{1}{10}(a_1 + a_2 t) = \infty$

RESULT

1. Position error constant, $K_p = \infty$
2. Velocity error constant, $K_v = 10$
3. Acceleration error constant, $K_a = 0$
4. Steady state error (when input $r(t) = a_o + a_1 t + \dfrac{a_2 t^2}{2}$), $e_{ss} = \infty$

EXAMPLE 3.14

Consider a unity feedback system with a closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{Ks+b}{s^2+as+b}$. Determine the open loop transfer function G(s). Show that the steady state error with unit ramp input is given by $\dfrac{(a-K)}{b}$.

SOLUTION

For unity feedback system, H(s)=1

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{G(s)}{1+G(s)}$

Let $M(s) = \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$\therefore \frac{G(s)}{1+G(s)} = M(s)$$

On cross multiplication we get,

$$G(s) = M(s)[1+G(s)] = M(s) + M(s).G(s)$$

$$G(s) - M(s).G(s) = M(s)$$

$$G(s)[1 - M(s)] = M(s)$$

$$\therefore G(s) = \frac{M(s)}{1-M(s)}$$

Given that, $M(s) = \dfrac{Ks+b}{s^2+as+b}$

$\therefore$ Open loop transfer function,

$$G(s)=\frac{\dfrac{Ks+b}{s^2+as+b}}{1-\dfrac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{(s^2+as+b)-(Ks+b)}$$

$$=\frac{Ks+b}{s^2+as+b-Ks-b}=\frac{Ks+b}{s^2+(a-K)s}=\frac{Ks+b}{s[s+(a-K)]}$$

Velocity error constant, $K_V=\underset{s\to 0}{Lt}\, s\,G(s)H(s)=\underset{s\to 0}{Lt}\, s\,G(s)$

$$=\underset{s\to 0}{Lt}\; s\frac{Ks+b}{s[s+(a-K)]}=\frac{b}{a-K}$$

Steady state error with velocity input, $e_{ss}=\dfrac{1}{K_V}=\dfrac{a-K}{b}$

RESULT

Open loop transfer function, $G(s)=\dfrac{Ks+b}{s[s+(a-K)]}$

Steady state error with veloctiy input, $e_{ss}=\dfrac{a-K}{b}$

EXAMPLE 3.15

A unity feedback system has the forward transfer function $G(s)=\dfrac{K_1(2s+1)}{s\,(5s+1)(1+s)^2}$

The input $r(t)=1+6t$ is applied to the systems. Determine the minimum value of K_1 if the steady error is to be less than 0.1.

SOLUTION

Given that input, $r(t)=1+6t$

On taking laplace transform of $r(t)$ we get $R(s)$

$$\therefore R(s)=L[r(t)]=L[1+6t]=\frac{1}{s}+\frac{6}{s^2}$$

The error signal in s-domain, $E(s)=\dfrac{R(s)}{1+G(s)H(s)}$

$$\therefore E(s)=\frac{\dfrac{1}{s}+\dfrac{6}{s^2}}{1+\dfrac{K_1(2s+1)}{s(5s+1)(1+s)^2}}\qquad \text{(Here } H(s)=1\text{)}$$

$$=\frac{\dfrac{1}{s}+\dfrac{6}{s^2}}{\dfrac{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}{s\,(5s+1)\,(1+s)^2}}$$

$$=\frac{1}{s}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right]+\frac{6}{s^2}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

$$e_{ss}=\underset{t\to\infty}{Lt}\;e(t)=\underset{s\to 0}{Lt}\;s\,E(s)$$

$$=\underset{s\to 0}{Lt}\; s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]+\frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]\right\}$$

$$=\underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}+\frac{6(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right\}$$

$$=0+\frac{6}{K_1}=\frac{6}{K_1}$$

Given that, $e_{ss}<0.1$,

$$\therefore 0.1=\frac{6}{K_1}\;\text{ or }\;K_1=\frac{6}{0.1}=60$$

RESULT

For steady state error, $e_{ss}<0.1$, the value of K_1 should be greater than 60

3.19 SHORT QUESTION AND ANSWER

Q3.1 What is time response?

The time response is the output of the closed loop system as a function of time. It is denoted by c(t). It is given by inverse Laplace of the product of input and transfer function of the system.

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)\,H(s)}$

Response in s-domain, $C(s) = \dfrac{R(s)\,G(s)}{1+G(s)\,H(s)}$

Response in time domain, $c(t) = L^{-1}[C(s)] = L^{-1}\left[\dfrac{R(s)\,G(s)}{1+G(s)\,H(s)}\right]$

Q3.2 What is transient and steady state response?

The transient response is the response of the system when the input changes from one state to another. The response of the system as $t \to \infty$ is called steady state response.

Q3.3 What is the importance of test signals?

The test signals can be easily generated in test laboratories and the characteristics of test signals resembles, the characteristics of actual input signals. The test signals are used to predetermine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications.

Q3.4 Name the test signals used in control system

The commonly used test input signals in control system are impulse, step, ramp, acceleration and sinusoidal signals.

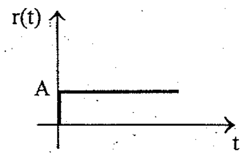
Q3.5 Define step signal.

The step signal is a signal whose value changes from 0 to A and remains constant at A for $t > 0$. The mathematical representation of step signal is

$$r(t) = A\,u(t)$$

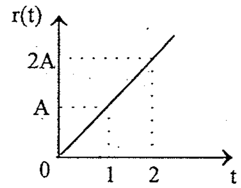
Where, $u(t) = 1, t \geq 0$

$u(t) = 0, t < 0$



Q3.6 Define ramp signal.

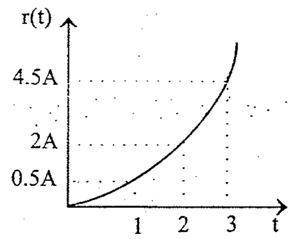
A ramp signal is a signal whose value increases linearly with time from an initial value of zero at $t = 0$. It is mathematically represented as



$$r(t) = At\,,\ t \geq 0$$
$$= 0\,,\ t < 0$$

Q3.7 Define parabolic signal.

It is a signal in which the instantaneous value varies as square of the time from an initial value of zero at $t = 0$. It is mathematically represented as



$$r(t) = \frac{At^2}{2},\ t \geq 0$$
$$= 0\,,\ t < 0$$

Q3.8 What is an impulse signal?

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time except at $t = 0$. At $t = 0$ the magnitude becomes infinite. It is denoted by $\delta(t)$ and mathematically expressed as

$$\delta(t) = 0 \ \text{ for } t \neq 0 \qquad \text{and} \qquad \underset{t_1 \to 0}{\mathrm{Lt}} \int_{-t_1}^{+t_1} \delta(t)\,dt = 1$$

Q3.9 What is weighting function?

The impulse response of system is called weighting function. It is given by inverse laplace transform of system transfer function.

Q3.10 Define pole.

The pole of a function, F(s) is the value at which the function, F(s) becomes infinite, where F(s) is a function of complex variable **s**.

Q3.11 *Define zero.*

The zero of a function, F(s) is the value at which the function, F(s) becomes zero, where F(s) is a function of complex variable s.

Q3.12 *What is the order of a system?*

The order of the system is given by the order of the differential equation governing the system. It is also given by the maximum power of s in the denominator polynomial of transfer function. The maximum power of s also gives the number of poles of the system and so the order of the system is also given by number of poles of the transfer function.

Q3.13 *Define damping ratio.*

The damping ratio is defined as the ratio of the actual damping to critical damping.

Q3.14 *Give the expression for damping ratio of mechanical and electrical system.*

The damping ratio of second order mechanical translational system, $\zeta = \dfrac{B}{2\sqrt{MK}}$

The damping ratio of second order mechanical rotational system, $\zeta = \dfrac{B}{2\sqrt{JK}}$

The damping ratio of second order electrical system, $\zeta = \dfrac{R}{2\sqrt{L/C}}$

Q3.15 *How the system is classified depending on the value of damping?*

Depending on the value of damping, the system can be classified into the following four cases

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$.

Q3.16 *What will be the nature of response of a second order system with different types of damping?*

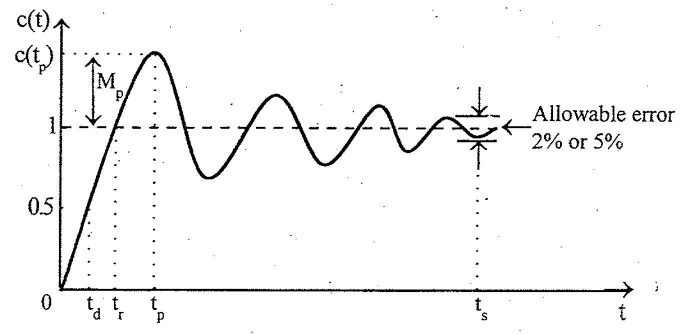
For undamped system the response is oscillatory.

For underdamped system the response is damped oscillatory.

For critically damped system the response is exponentially rising.

For overdamped system the response is exponentially rising but the rise time will be very large.

Q3.17. *Sketch the response of a second order underdamped system.*



Q3.18. *What is damped frequency of oscillation?*

In underdamped system the response is damped oscillatory. The frequency of damped oscillation is given by $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Q3.19. *Give the expression for natural frequency of oscillations of electrical and mechanical system.*

The natural frequency of oscillation of second order mechanical translational system $\Big\}\ \omega_n = \sqrt{\dfrac{K}{M}}$

The natural frequency of oscillation of second order mechanical rotational system $\Big\}\ \omega_n = \sqrt{\dfrac{K}{J}}$

The natural frequency of oscillation of second order electrical system, $\Big\}\ \omega_n = \dfrac{1}{\sqrt{LC}}$

Q3.20. *The closed loop transfer function of second order system is* $\dfrac{C(s)}{R(s)} = \dfrac{10}{s^2+6s+10}$. *What is the type of damping in the system?.*

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{10}{s^2+6s+10}$$

$\therefore \omega_n^2 = 10$

$\omega_n = \sqrt{10}$

$= 3.1622 \text{ rad / sec}$

$2\zeta\omega_n = 6$

$\zeta = \dfrac{6}{2\times\omega_n} = \dfrac{6}{2\times\sqrt{10}}$

$\therefore \zeta = 0.95$

Since $\zeta < 1$, the system is underdamped.

Q3.21 *The closed loop transfer function of a second order system is given by* $\dfrac{200}{s^2+20s+200}$. *Determine the damping ratio and natural frequency of oscillation.*

Let us compare the given transfer function to standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{200}{s^2+20s+200}$$

$\therefore \omega_n^2 = 200$

$\omega_n = \sqrt{200} = 14.14 \text{ rad / sec}$

$2\zeta\omega_n = 20$

$\zeta = \dfrac{20}{2\times\omega_n} = \dfrac{20}{2\times 14.14}$

$\therefore \zeta = 0.707$

Damping ratio, $\zeta = 0.707$

Natural frequency of oscillation, $\omega_n = 14.14$ rad/sec.

Q3.22 *A second order system has a damping ratio of 0.6 and natural frequency of oscillation is 10 rad/sec. Determine the damped frequency of oscillation.*

$$\text{Damped frequency of oscillation, } \omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.6)^2}$$
$$= 10\times 0.8 = 8 \text{ rad / sec}$$

Q3.23 *The open loop transfer function of a unity feedback system is* $G(s) = \dfrac{20}{s(s+10)}$. *What is the nature of response of closed loop system for unit step input.*

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{20/s(s+10)}{1+\dfrac{20}{s(s+10)}} = \frac{20}{s(s+10)+20} = \frac{20}{s^2+10s+20}$$

The standard form of second order transfer function is $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$.

On comparing the system transfer function with standard form of second order transfer function we get,

$\omega_n^2 = 20$

$\therefore \omega_n = \sqrt{20} = 4.47 \text{ rad / sec}$

$2\zeta\omega_n = 10$

$\zeta = \dfrac{10}{2\times\omega_n} = \dfrac{10}{2\times 4.47} = 1.12$

Since damping ratio, $\zeta > 1$, the system is overdamped and the response will be exponentially rising.

Q3.24 *List the time domain specifications.*

The time domain specifications are :

(i) Delay time
(ii) Rise time
(iii) Peak time
(iv) Maximum overshoot
(v) Settling time.

Q3.25 *Define delay time.*

It is the time taken for response to reach 50% of the final value, for the very first time.

Q3.26 *Define rise time.*

It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

Q3.27 *Define peak time.*

It is the time taken for the response to reach the peak value for the very first time (or) It is the time taken for the response to reach peak overshoot, M_p.

Q3.28 Define peak overshoot.

It is defined as the ratio of the maximum peak value measured from final value to final value. Let final value = $c(\infty)$, Maximum value = $c(t_p)$

$$\text{Peak overshoot, } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}.$$

Q3.29 Define settling time.

It is defined as the time taken by the response to reach and stay within a specified error and the error is usually specified as % of final value. The usual tolerable error is 2% or 5% of the final value.

Q3.30 The damping ratio of a system is 0.75 and the natural frequency of oscillation is 12 rad/sec. Determine the peak overshoot and the peak time.

$$\text{Peak overshoot, } M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{0.75\times\pi}{\sqrt{1-(0.75)^2}}}$$

$$\%M_p = 0.028 \times 100 = 2.8\%$$

$$\text{Damped frequency of oscillation, } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= 12\sqrt{1-(0.75)^2} = 7.94 \text{ rad / sec.}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7.94} = 0.396 \text{ sec.}$$

Q3.31 The damping ratio of system is 0.6 and the natural frequency of oscillation is 8 rad/sec. Determine the rise time.

$$\text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.6)^2}}{0.6} = 53.13°$$

$$= \frac{53.13}{180} \times \pi \text{ rad} = 0.927 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 8\sqrt{1-(0.6)^2} = 6.4 \text{ rad / sec}$$

$$\therefore t_r = \frac{\pi - 0.927}{6.4} = 0.34 \text{ sec}$$

Q3.32 What is type number of a system? What is its significance?

The type number is given by number of poles of loop transfer function at the origin. The type number of the system decides the steady state error.

Q3.33 Distinguish between type and order of a system.

(i) Type number is specified for loop transfer function but order can be specified for any transfer function. (open loop or closed loop transfer function).

(ii) The type number is given by number of poles of loop transfer function lying at origin of s-plane but the order is given by the number of poles of transfer function.

Q3.34 For the system with following transfer function, determine type and order of the system.

(i) $G(s)\,H(s) = \dfrac{K}{s(s+1)\,(s^2+6s+8)}$ (ii) $G(s)\,H(s) = \dfrac{20\,(s+2)}{s^2(s+3)\,(s+0.5)}$

(iii) $G(s)\,H(s) = \dfrac{(s+4)}{(s-2)\,(s+0.25)}$ (iv) $G(s)\,H(s) = \dfrac{10}{s^3(s^2+2s+1)}$

Ans: (i) Type - 1, order - 4
(ii) Type - 2, order - 4
(iii) Type - 0, order - 2
(iv) Type - 3, order - 5.

Q3.35 What is steady state error?

The steady state error is the value of error signal e(t), when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Q3.36 What are static error constants?

The K_p, K_v and K_a are called static error constants. These constants are associated with steady state error in a particular type of system and for a standard input.

Q3.37 *Define positional error constant.*

The positional error constant $K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s)$. The steady state error in type - 0 system when the input is unit step is given by $\frac{1}{1+K_p}$

Q3.38 *Define velocity error constant.*

The velocity error constant $K_v = \underset{s\to 0}{Lt}\ sG(s)\,H(s)$. The steady state error in type-1 system for unit ramp input is given by $1/K_v$.

Q3.39 *Define acceleration error constant.*

The acceleration error constant $K_a = \underset{s\to 0}{Lt}\ s^2\,G(s)\,H(s)$. The steady state error in type-2 system for unit parabolic input is given by $1/K_a$.

Q3.40 *A unity feedback system has a open loop transfer function of $G(s) = \frac{10}{(s+1)\,(s+2)}$. Determine the steady state error for unit step input.*

The steady state error for unit step input, $e_{ss} = \frac{1}{1+K_p}$.

Where $K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s)$

For unity feedback system $H(s) = 1$

$$\therefore K_p = \underset{s\to 0}{Lt}\quad G(s) = \underset{s\to 0}{Lt}\quad \frac{10}{(s+1)\,(s+2)} = 5$$

$$\therefore e_{ss} = \frac{1}{1+5} = \frac{1}{6}$$

Q3.41 *A unity feedback system has a open loop transfer function of $G(s) = \frac{25\,(s+4)}{s(s+0.5)\,(s+2)}$. Determine the steady state error for unit ramp input.*

The steady state error for unit ramp input is $e_{ss} = \frac{1}{K_v}$,

Where $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$. For unity feedback system $H(s) = 1$.

$$\therefore K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s\left[\frac{25\,(s+4)}{s(s+0.5)\,(s+2)}\right] = \frac{25\times 4}{0.5\times 2} = 100$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{100} = 0.01$$

Q3.42 *A unity feedback system has a open loop transfer function of $G(s) = \frac{20\,(s+5)}{s(s+0.1)\,(s+3)}$. Determine the steady state error for parabolic input.*

The steady state error for unit parabolic input is $e_{ss} = \frac{1}{K_a}$

Where $K_a = \underset{s\to 0}{Lt}\ s^2G(s)\,H(s)$. For unity feedback system $H(s) = 1$.

$$\therefore K_a = \underset{s\to 0}{Lt}\ s^2\left[\frac{20\,(s+5)}{s^2(s+0.1)\,(s+3)}\right] = \frac{20\times 5}{0.1\times 3} = \frac{100}{0.3} = 333.33$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{333.33} = 0.003$$

Q3.43 *What are generalized error coefficients?*

They are the coefficients of generalized error series. The generalized error series is given by

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \ldots + \frac{C_n}{n!}\overset{n}{r}(t) \ldots$$

The coefficients $C_0, C_1, C_2 \ldots C_n$ are called generalized error coefficients or dynamic error coefficients.

The n^{th} coefficient, $C_n = \underset{s\to 0}{Lt}\ \frac{d^n}{ds^n} F(s)$, Where $F(s) = \frac{1}{1+G(s)\,H(s)}$

Q3.44 *Give the relation between generalized and static error coefficients.*

The following expression shows the relation between generalized and static error coefficient

$$C_0 = \frac{1}{1+K_p}, \quad C_1 = \frac{1}{K_v}, \quad C_2 = \frac{1}{K_a}$$

Q3.45 Mention two advantages of generalized error constants over static error constants.

(i) Generalized error series gives error signal as a function of time.

(ii) Using generalized error constants the steady state error can be determined for any type of input but static error constants are used to determine steady state error when the input is anyone of the standard input.

Q3.46 What is the effect on system performance when a proportional controller is introduced in a system?

The proportional controller improves the steady-state tracking accuracy, disturbance signal rejection and relative stability of the system. It also increases the loop gain of the system which results in reducing the sensitivity of the system to parameter variations.

Q3.47 What is the disadvantage in proportional controller?

The disadvantage in proportional controller is that it produces a constant steady state error.

Q3.48 What is the effect of PI controller on the system performance?

The PI controller increases the order of the system by one, which results in reducing, the steady state error. But the system becomes less stable than the original system.

Q3.49 What is the effect of PD controller on the system performance?

The effect of PD controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

Q3.50 Why derivative controller is not used in control systems?

The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control systems.

3.20 EXERCISES

E3.1 What is the unit-step response of the system shown in fig E3.1

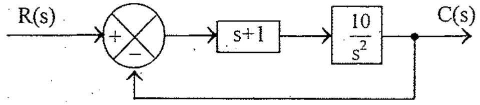


Fig E3.1

E3.2 Obtain the unit-step response of a unity-feedback system whose open-loop transfer function is $G(s) = 5(s+20)/s\,(s+4.59)\,(s^2+3.41s+16.35)$.

E3.3 The open loop transfer function of an unity feedback control system is given by $G(s) = 100/s\,(s+2)\,(s+5)$. For unit step input, find the time response of the closed loop system and determine % over shoot and the rise time.

E3.4 A Servomechanism has its moment of inertia $J = 10\times10^{-6}$ Kg-m^2, retarding friction, $B = 400\times10^{-6}$ N-m/(rad/sec) and elasticity coefficient, $K = 0.004$ N-m/rad. Find the natural frequency and damping factor of the system.

E3.5 For a second order system whose open loop transfer function $G(s) = 4/s(s+2)$, determine the maximum over shoot, the time to reach the maximum overshoot when a step displacement of 18° is given to the system. Find the rise time, time constant and the settling time for an error of 7%.

E3.6 Consider the unity feedback closed loop system where the forward transfer function is $G(s) = 25/s\,(s+5)$. Obtain the rise time, Peak time, Maximum overshoot and the settling time when the system is subjected to a unit-step input.

E3.7 Consider the system shown in fig E3.7, where $\zeta = 0.6$ and $\omega_n = 0.5$ rad/sec. Determine the rise time, peak time, maximum overshoot and settling time, when the system is subjected to a unit-step input

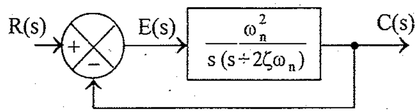


Fig E3.7.

E3.8 For the system shown in fig E3.8, determine the values of K and K_h so that the maximum overshoot in the unit step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain rise time and settling time.

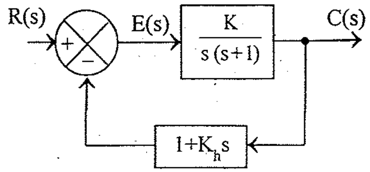


Fig E3.8.

E3.9 The system shown in fig E3.9 subjected to a unit-step input. Determine the values of K and T, where the Maximum overshoot of the system is 25.4% corresponding to $\zeta = 0.4$

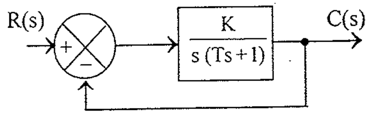
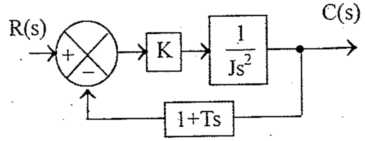


Fig E3.9.

E3.10 Determine the values of K and T of the closed-loop system shown in Fig E3.10 so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J=1 Kg-m^2.



FigE3.10.

E3.11 A unity-feedback system is characterized by the open-loop transfer function $G(s) = 1/s(0.5s+1)\,(0.2s+1)$. a) Determine the steady-state errors to unit-step, unit-ramp and unit parabolic inputs. b) Determine rise time, peak time, peak overshoot and settling time of the unit-step respone of the system.

E3.12 For a system whose $G(s) = 10/s(s+1)\,(s+2)$, find the steady state error when it is subjected to the input, $r(t) = 1+2t+1.5\,t^2$.

E3.13 A unity feedback system has $G(s)=1/s(1+s)$. The input to the system is described by $r(t) = 4+6t+2t^3$. Find the generalized error coefficients and steady state error.

E3.14 A unity feedback system has the forward path transfer function $G(s) = 10/(s+1)$. Find the steady state error and the generalised error coefficient for $r(t) = t$.

E3.15 Find out the position, velocity and acceleration error coefficients for the following unity feedback systems having forward loop transfer function G(s) as.

(a) $100/(1+0.5s)\,(1+2s)$ (b) $K/s(1+0.1s)\,(1+s)$

(c) $K/s^2\,(s^2+8s+100)$ (d) $K\,(1+s)\,(1+2s)/s^2(s^2+4s+20)$

E3.16 The open loop transfer function of a unity feedback control system is $G(s) = 9/(s+1)$, using the generalized error series determine the error signal and steady state error of the system when the system is excited by

(i) $r(t) = 2$ (ii) $r(t) = t$

(iii) $r(t) = 3t^2/2$ (iv) $r(t) = 1+2t+3t^2/2$

E3.17 For unity feedback system having open loop transfer function as $G(s) = K\,(s+2)/s^2(s^2+7s+12)$, determine (i) type of system (ii) error constants K_p, K_v and K_a (iii) steady state error for parabolic input.

ANSWER FOR EXERCISE PROBLEMS

E3.1 $c(t) = -1.1455\,e^{-8.87t} + 0.1455e^{-1.13t} + 1$

E3.2 $c(t) = 1+\frac{3}{8}e^{-t}\cos 3t - \frac{17}{24}e^{-t}\sin 3t - \frac{11}{8}e^{-3t}\cos t - \frac{13}{8}e^{-3t}\sin t$

E3.3 $c(t) = \left[1-0.186\,e^{-7.45t} - 0.88\,e^{0.225t}\cos(3.65t-22°)\right]$

As t tends to infinity, c(t) tends to infinity and so the system is unstable. Therefore % over shoot and rise time are not defined.

E3.4 Natural frequency, $\omega_n = 20$ rad/sec, Damping factor, $\zeta = 1$.

E3.5 Maximum overshoot = 0.16, when input is 18°, $M_p = 2.88°$

Peak time, t_p=1.81sec. Rise time, t_r=1.21 sec

Time constant, T=1 sec, Settling time for 7% error = 2.66 sec.

E3.6 Rise time, t_r = 0.55sec, %Peak overshoot, M_p = 9.5%

Peak time, t_p = 0.785sec, Settling time, t_s = 1.33sec (for 2% error)
= 1 sec(for 5% error)

E3.7 Rise time, t_r = 0.55sec, Maximum overshoot, M_p = 0.095

Peak time, t_p = 0.785sec, Settling time, t_s = 1 sec (for 5% criterion)

E3.8 K=12.5, Rise time, t_r = 0.65sec

K_h = 0.178, Settling time, t_s = 2.48 sec (for 2% error)

t_s = 1.86 sec (for 5% error)

E3.9 K=1.42 T=1.09

E3.10 K=2.95 N-m T=0.471 sec

E3.11 (a) $e_{ss}\big|_{\text{unit step}} = 0$

$e_{ss}\big|_{\text{unit ramp}} = 1$

$e_{ss}\big|_{\text{unit parabola}} = \infty$

(b) Rise time, $t_r = 1.91$ sec

Peak time, $t_p = 2.79$ sec

Peak overshoot. $M_p = 0.1265$

Settling time, $t_s = 5.4$ sec

E3.12 The total steady state error is ∞.

E3.13 $C_0 = 0$ $\quad C_1 = 1$ $\quad e_{ss} = \infty$

$C_2 = 0$ $\quad C_3 = -6$

E3.14 $C_o = 1/11$ $\quad e_{ss} = \infty$.

$C_1 = 10/121$

E3.15

Question	K_p	K_v	K_a
(a)	100	0	0
(b)	∞	K	0
(c)	∞	∞	K/100
(d)	∞	∞	K/20

E3.16 (i) $e(t) = 0.2$; $e_{ss} = 0.2$

(ii) $e(t) = 0.1t + 0.09$; $e_{ss} = \infty$.

(iii) $e(t) = 0.15t^2 + 0.27\,t - 0.054$; $e_{ss} = \infty$.

(iv) $e(t) = 0.15t^2 + 0.77\,t + 0.226$; $e_{ss} = \infty$.

E3.17 (i) It is type 2 system

(ii) $K_p = \infty$.

$K_v = \infty$.

$K_a = K/6$

(iii) $e_{ss} = 6/K$

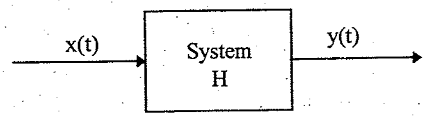
CHAPTER FOUR

4 FREQUENCY RESPONSE ANALYSIS

4.1 FREQUENCY RESPONSE

The frequency response is the steady state response (output) of a system when the input to the system is a sinusoidal signal.

Consider a linear time invariant (LTI) system, H shown in fig 4.1. Let x(t) be an input sinusoidal signal. The response or output y(t) is also a sinusoidal signal of same frequency but with different magnitude and phase angle.



$x(t) = X \sin\omega t$

$y(t) = Y \sin(\omega t + \phi)$

Fig 4.1 : LTI System

The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed the frequency response. In linear time invariant systems the frequency response is independent of the amplitude and phase of the input signal.

The frequency response of a system is normally obtained by varying the frequency of the input signal by keeping the magnitude of the input signal at a constant value.

In the system transfer function T(s), if s is replaced by $j\omega$ then the resulting transfer function $T(j\omega)$ is called sinusoidal transfer function. The frequency response of the system can be directly obtained from the sinusoidal transfer function $T(j\omega)$ of the system. The transfer function $T(j\omega)$ is a complex function of frequency. The magnitude and phase of $T(j\omega)$ are functions of frequency and can be evaluated for various values of frequency.

The frequency response can be evaluated for both open loop system and closed loop system

$$\text{Open loop transfer function, } G(j\omega) = |G(j\omega)|\angle G(j\omega) \quad .....(4.1)$$

$$\text{Loop transfer function, } G(j\omega)\,H(j\omega) = |G(j\omega)\,H(j\omega)|\angle G(j\omega)\,H(j\omega) \quad .....(4.2)$$

$$\text{Closed loop transfer function, } \frac{C(j\omega)}{R(j\omega)} = M(j\omega) = |M(j\omega)|\angle M(j\omega) \quad .....(4.3)$$

The advantages of frequency response analysis are the following

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments.
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.
4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.
5. When the system is designed by use of the frequency response analysis, the effects of noise disturbance and parameters variations are relatively easy to visualize and incorporate corrective measures.
6. The frequency response analysis and designs can be extended to certain nonlinear control systems.

4.2 FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are

1. Resonant peak, M_r
2. Resonant Frequency, ω_r
3. Bandwidth
4. Cut-off rate
5. Gain margin
6. Phase margin.

RESONANT PEAK (M_r)

The maximum value of the magnitude of closed loop transfer function is called the resonant peak, M_r. A large resonant peak corresponds to a large overshoot in transient response.

RESONANT FREQUENCY (ω_r)

The frequency at which the resonant peak occurs is called resonant frequency, ω_r. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

BANDWIDTH (ω_b)

The Bandwidth is the range of frequencies for which the system gain is more than –3 db. The frequency at which the gain is –3 db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The Bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

CUT-OFF RATE

The slope of the log-magnitude curve near the cut off frequency is called the cut-off rate. The cut -off rate indicates the ability of the system to distinguish the signal from noise.

GAIN MARGIN, K_g

The gain margin, K_g is defined as the reciprocal of the magnitude of open loop transfer function at phase cross over frequency. The frequency at which the phase of open loop transfer function is 180° is called the phase cross-over frequency, ω_{pc}.

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega_{pc})|} \quad .....(4.4)$$

The gain margin in db can be expressed as,

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|} = -20 \log |G(j\omega_{pc})| \quad .....(4.5)$$

Note : $|G(j\omega_{pc})|$ *is the magnitude of* $G(j\omega)$ at $\omega = \omega_{pc}$

(Gain margin in db is defined as the negative of the db magnitude of $G(j\omega)$ at phase cross-over frequency). The gain margin indicates the amount by which the gain of the system can be increased without affecting the stability of the system.

PHASE MARGIN (γ)

The phase margin γ, is that amount of additional phase lag at the gain cross over frequency required to bring the system to the verge of instability. The gain cross over frequency ω_{gc} is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero).

The phase margin γ, is obtained by adding 180° to the phase angle ϕ of the open loop transfer function at the gain cross over frequency

Phase margin , $\gamma = 180° + \phi_{gc}$, Where, $\phi_{gc} = \angle G(j\omega_{gc})$ .....(4.6)

Note : $\angle G(j\omega_{gc})$ *is the phase angle of* $G(j\omega)$ at $\omega = \omega_{gc}$

4.3 ESTIMATION OF FREQUENCY DOMAIN SPECIFICATIONS FOR A SECOND ORDER SYSTEM

RESONANT PEAK (M_r)

Consider the closed loop transfer function of second order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad .....(4.7)$$

The sinusoidal transfer function $T(j\omega)$ is obtained by letting $s = j\omega$

$$\therefore T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \quad .....(4.8)$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n} + 1\right)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

Let, Normalized frequency, $u = (\omega / \omega_n)$

$$\therefore T(j\omega) = \frac{1}{(1-u^2) + j2\zeta u} = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2} \angle \tan^{-1}\frac{2\zeta u}{1-u^2}} \quad .....(4.9)$$

Let M = Magnitude of closed loop transfer function

α = Phase of closed loop transfer function.

$$M = |T(j\omega)| = \left[\frac{1}{(1-u^2)^2 + (2\zeta u)^2}\right]^{\frac{1}{2}} = \left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}} \quad .....(4.10)$$

$$\alpha = \angle T(j\omega) = -\tan^{-1}\frac{2\zeta u}{1-u^2}$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting dM/du = 0 when $u = u_r$,

Where, $u_r = \frac{\omega_r}{\omega}$ = Normalized resonant frequency

On differentiating equation (4.10) with respect to u we get,

$$\frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{3}{2}}\left[2(1-u^2)(-2u) + 8\zeta^2u\right]$$

$$= \frac{-\left[-4u(1-u^2) + 8\zeta^2u\right]}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} = \frac{4u(1-u^2) - 8\zeta^2u}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} \quad .....(4.11)$$

Replace u by u_r in equation (4.11) and equate to zero.

$$\frac{4u_r(1-u_r^2) - 8\zeta^2u_r}{2\left[(1-u_r^2)^2 + 4\zeta^2u_r^2\right]^{\frac{3}{2}}} = 0 \quad .....(4.12)$$

The equation (4.12) will be zero if the numerator is zero. Hence, on equating numerator to zero we get,

$$4u_r(1-u_r^2)-8\zeta^2u_r=0$$
$$4u_r-4u_r^3-8\zeta^2u_r=0$$
$$4u_r^3=4u_r-8\zeta^2u_r$$
$$u_r^2=1-2\zeta^2$$
$$u_r=\sqrt{1-2\zeta^2} \quad .....(4.13)$$

The resonant peak occurs when $u_r=\sqrt{1-2\zeta^2}$

Put this condition in the equation for M and solve for M_r.

$$\therefore M_r=\left.\frac{1}{\left[(1-u^2)^2+4\zeta^2u^2\right]^{\frac{1}{2}}}\right|_{u=u_r}$$

$$\therefore M_r=\frac{1}{\left[(1-u_r^2)^2+4\zeta^2u_r^2\right]^{\frac{1}{2}}}=\frac{1}{\left[(1-(1-2\zeta^2))^2+4\zeta^2(1-2\zeta^2)\right]^{\frac{1}{2}}}$$

$$=\frac{1}{\left[4\zeta^4+4\zeta^2-8\zeta^4\right]^{\frac{1}{2}}}=\frac{1}{\left[4\zeta^2-4\zeta^4\right]^{\frac{1}{2}}}=\frac{1}{\left[4\zeta^2(1-\zeta^2)\right]^{\frac{1}{2}}}$$

$$=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$\therefore$ The resonant peak, $M_r=\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ .....(4.14)

RESONANT FREQUENCY (ω_r)

The normalized resonant frequency, $u_r=\dfrac{\omega_r}{\omega_n}=\sqrt{1-2\zeta^2}$

The resonant frequency, $\omega_r=\omega_n\sqrt{1-2\zeta^2}$ .....(4.15)

BANDWIDTH (ω_b)

Let, Normalized bandwidth, $u_b=\dfrac{\omega_b}{\omega_n}$ .....(4.16)

When $u=u_b$, the magnitude M, of the closed loop system is $1/\sqrt{2}$ (or -3db).
Hence in the equation for M (equation 4.10), put $u=u_b$ and equate to $1/\sqrt{2}$

$$M=\frac{1}{\sqrt{2}}=\frac{1}{\left[(1-u_b^2)^2+4\zeta^2u_b^2\right]^{\frac{1}{2}}}$$

On squaring and cross multiplying,

$$(1-u_b^2)^2+4\zeta^2u_b^2=2$$
$$1+u_b^4-2u_b^2+4\zeta^2u_b^2=2$$
$$u_b^4-2u_b^2(1-2\zeta^2)-1=0$$
Let $x=u_b^2$
$$\therefore x^2-2(1-2\zeta^2)x-1=0$$

$$\therefore x=\frac{2(1-2\zeta^2)\pm\sqrt{4(1-2\zeta^2)^2+4}}{2}$$
$$=\frac{2(1-2\zeta^2)\pm2\sqrt{(1+4\zeta^4-4\zeta^2)+1}}{2}$$

By taking only the positive sign,

$$x=1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}$$

But, $u_b=\sqrt{x}=\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{\frac{1}{2}}$

Also, $u_b=\dfrac{\omega_b}{\omega_n}$

$\therefore$ Bandwidth, $\omega_b=\omega_n u_b=\omega_n\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{\frac{1}{2}}$ .....(4.17)

PHASE MARGIN (γ)

The open loop transfer function of second order system,

$$G(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)} \quad .....(4.18)$$

The sinusoidal transfer function $G(j\omega)$ is obtained by letting $s=j\omega$

$$G(j\omega)=\frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}=\frac{\omega_n^2}{-\omega^2+j2\zeta\omega\omega_n} \quad .....(4.19)$$

$$=\frac{\omega_n^2}{\omega_n^2\left(-\dfrac{\omega^2}{\omega_n^2}+j2\zeta\dfrac{\omega}{\omega_n}\right)} \quad .....(4.20)$$

Let normalized frequency, $u = \omega/\omega_n$

On substituting $u = \omega/\omega_n$ in equation (4.20) we get,

$$G(j\omega) = \frac{1}{-u^2 + j\,2\,\zeta u} \quad .....(4.21)$$

$$\text{Magnitude of } G(j\omega) = |G(j\omega)| = \frac{1}{\sqrt{u^4 + 4\zeta^2 u^2}} \quad .....(4.22)$$

$$\text{Phase of } G(j\omega) = \angle G(j\omega) = -\tan^{-1}\frac{2\zeta u}{-u^2} = -\tan^{-1}\left(-\frac{2\zeta}{u}\right) = \tan^{-1}\frac{2\zeta}{u} \quad .....(4.23)$$

At the gain cross-over frequency ω_{gc}, the magnitude of $G(j\omega)$ is unity

Let normalized gain cross over frequency, $u_{gc} = \omega_{gc}/\omega_n$

Substitute u by u_{gc} in the equation for magnitude of $G(j\omega)$ and equate to unity

$$\therefore \text{At } u = u_{gc},\ |G(j\omega)| = 1 = \frac{1}{\sqrt{u_{gc}^4 + 4\zeta^2 u_{gc}^2}}$$

$$\therefore u_{gc}^4 + 4\zeta^2 u_{gc}^2 = 1$$

$$u_{gc}^4 + 4\zeta^2 u_{gc}^2 - 1 = 0$$

let $x = u_{gc}^2$

$$\therefore x^2 + 4\zeta^2 x - 1 = 0$$

$$x = \frac{-4\zeta^2 \pm \sqrt{16\zeta^4 + 4}}{2}$$

$$x = -2\zeta^2 \pm \sqrt{4\zeta^4 + 1}$$

On taking only the positive sign,

$$u_{gc} = \sqrt{x} = \left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}} \quad .....(4.24)$$

$$\text{The phase margin, } \gamma = 180 + \angle G(j\omega)|_{\omega=\omega_{gc},\,u=u_{gc}} \quad .....(4.25)$$

Substituting for $\angle G(j\omega)$ from equation (4.23) in equation (4.25) we get,

$$\gamma = 180^\circ + \tan^{-1}\frac{2\zeta}{u_{gc}} = 180^\circ + \tan^{-1}\left[\frac{2\zeta}{\left(-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right)^{\frac{1}{2}}}\right] \quad .....(4.26)$$

***Note :** The gain margin of second order system is infinite.*

4.4 CORRELATION BETWEEN TIME AND FREQUENCY RESPONSE

The correlation between time and frequency response has an explicit form only for first and second order systems. The correlation for second-order system is discussed here.

Consider the standard form of transfer function of second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where, ζ = Damping factor

ω_n = Undamped natural frequency

The sinusoidal transfer function of the system is obtained by letting $s = j\omega$,

$$\frac{C}{R}(j\omega) = T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + j\,2\zeta\,\omega_n\omega + \omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n} + 1\right)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

$$= \frac{1}{(1 - u^2) + j2\zeta u}$$

Where, $u = \left(\frac{\omega}{\omega_n}\right)$ is the normalized frequency.

$$\text{Magnitude of closed loop system, } M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}} \quad .....(4.27)$$

$$\text{Phase of closed loop system, } \alpha = \angle T(j\omega) = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) \quad .....(4.28)$$

The magnitude and phase angle characteristics for normalized frequency u ,for certain values of ζ are shown in fig 4.2 and 4.3. The frequency at which M has a peak value is known as the resonant frequency. The peak value of the magnitude is the resonant peak M_r. At this frequency the slope of the magnitude curve is zero.

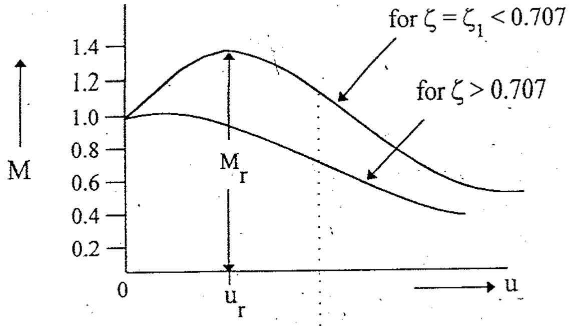


Fig 4.2 : Magnitude, M as a function of u

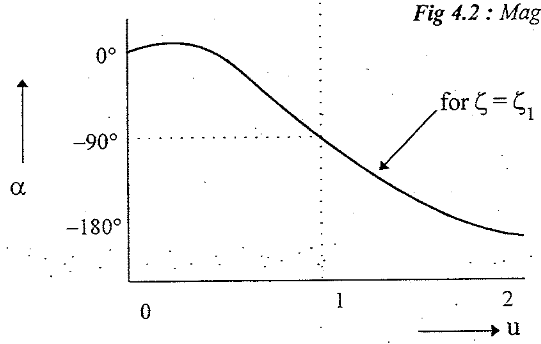


Fig 4.3 : Phase, α as a function of u

Let ω_r be the resonant frequency and $u_r = \left(\frac{\omega_r}{\omega_n}\right)$ be the normalized resonant frequency.

The expression for resonant frequency ω_r can be obtained by differentiating M with respect to ω and equating $\frac{dM}{du}$ to zero.

The M_r and the corresponding phase α_r can be obtained by substituting the expression for ω_r in the equation of M and α.

It can be shown that, $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$; $u_r = \sqrt{1-2\zeta^2}$;

$$\alpha_r = -\tan^{-1}\left[\frac{\sqrt{1-2\zeta^2}}{\zeta}\right] \quad \text{and} \quad \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

When $\zeta = 0$, $\quad \omega_r = \omega_n\sqrt{1-2\zeta^2} = \omega_n$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \infty$$

Hence it is clear that as ζ tends to zero, ω_r approaches ω_n and M_r approaches infinity.

For $0 < \zeta \le 1/\sqrt{2}$, the resonant frequency always has a value less than ω_n and the resonant peak has a value greater than one.

For $\zeta > 1/\sqrt{2}$, the condition $(dM/du) = 0$, will not be satisfied for any real value of ω.

Hence when $\zeta > 1/\sqrt{2}$ the magnitude M decreases monotonically from M = 1 at u = 0 with increasing u. It follows that for $\zeta > 1/\sqrt{2}$ there is no resonant peak and the greatest value of M equals one.

The frequency at which M has a value of $1/\sqrt{2}$ is of special significane and is called the cut-off frequency ω_c. The signal frequencies above cut-off are greatly attenuated on passing through a system.

For feedback control system, the range of frequencies over which $M \ge 1/\sqrt{2}$ is defined as bandwidth ω_b. Control system being low-pass filters (at zero frequency M = 1), the bandwidth ω_b is equal to cut-off frequency ω_c.

In general the bandwidth of a control system indicates the noise-filtering characteristics of the system. Also, bandwidth gives a measure of the transient response.

The normalized bandwidth, $u_b = \left(\frac{\omega_b}{\omega_n}\right)$

$$u_b = \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + \zeta^4}\right]^{\frac{1}{2}}$$

From the equation of u_b it is clear that u_b is a function of ζ alone. The graph between u_b and ζ is shown in fig 4.4.

The expression for the damped frequency of oscillation ω_d and peak overshoot M_p of the step response for $0 \le \zeta \le 1$ are

Damped frequency, $\omega_d = \omega_n \sqrt{1-\zeta^2}$

Peak overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

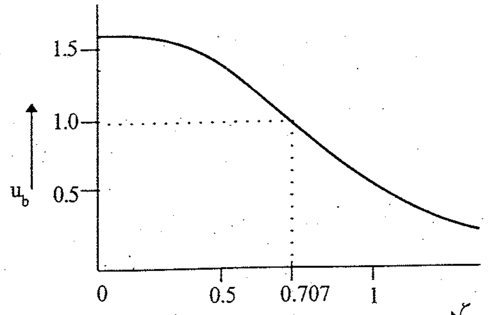


Fig 4.4 : Normalized bandwidth as a function of ζ

Comparison of the equation of M_r and M_p reveals that both are functions of only ζ.

The sketch of M_r and M_p for various value of ζ are shown in fig 4.5. The sketches reveals that a system with a given value of M_r must exhibit a corresponding value of M_p if subjected to a step input. For $\zeta > 1/\sqrt{2}$ the resonant peak M_r does not exist and the correlation breaks down. This is not a serious problem as for this range of ζ, the step response oscillations are well damped and M_p is negligible.

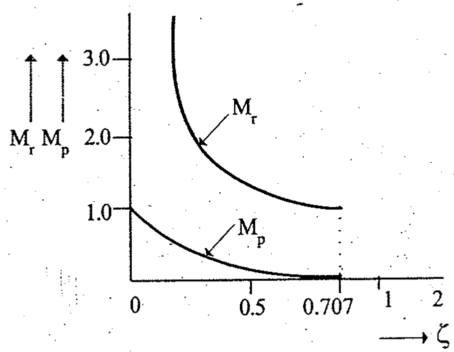


Fig 4.5 : M_r and M_p as a function of ζ

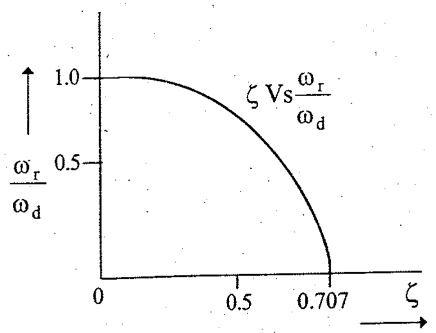


Fig 4.6 : ω_r/ω_d as a function of ζ

The comparison of the equation of ω_r and ω_d reveals that there exists a definite correlation between them. The sketch of ω_r/ω_d with respect to ζ is shown in fig 4.6.

4.5 FREQUENCY RESPONSE PLOTS

Frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are

1. Bode plot
2. Polar plot (or Nyquist plot)
3. Nichols plot
4. M and N circles
5. Nichols chart

The Bode plot, Polar plot and Nichols plot are usually drawn for open loop systems. From the open loop response plot the performance and stability of closed loop system are estimated. The M and N circles and Nichols chart are used to graphically determine the frequency response of unity feedback closed loop system from the knowledge of open loop response.

The frequency response plots are used to determine the frequency domain specifications, to study the stability of the systems and to adjust the gain of the system to satisfy the desired specifications.

4.6 BODE PLOT

The Bode plot is a frequency response plot of the transfer function of a system. A Bode plot consists of two graphs. One is a plot of the magnitude of a sinusoidal transfer function versus log ω. The other is a plot of the phase angle of a sinusoidal transfer function versus log ω.

The Bode plot can be drawn for both open loop and closed loop transfer function. Usually the bode plot is drawn for open loop system. The standard representation of the logarithmic magnitude of open loop transfer function of G(jω) is 20 log |G(jω)| where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated db. The curves are drawn on semilog paper, using the log scale (abcissa) for frequency and the linear scale (ordinate) for either magnitude (in decibels) or phase angle (in degrees).

The main advantage of the bode plot is that multiplication of magnitudes can be converted into addition. Also a simple method for sketching an approximate log-magnitude curve is available.

Consider the open loop transfer function, $G(s) = \dfrac{K\,(1+sT_1)}{s\,(1+sT_2)\,(1+sT_3)}$

$$G(j\omega) = \frac{K\,(1+j\omega T_1)}{j\omega\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

$$= \frac{K\angle 0^\circ \sqrt{1+\omega^2 T_1^2}\ \angle \tan^{-1}\omega T_1}{\omega\angle 90^\circ \sqrt{1+\omega^2 T_2^2}\ \angle \tan^{-1}\omega T_2 \sqrt{1+\omega^2 T_3^2}\ \angle \tan^{-1}\omega T_3}$$

The magnitude of $G(j\omega) = |G(j\omega)| = \dfrac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$

The phase angle of the $G(j\omega) = \angle G(j\omega) = \tan^{-1}\omega T_1 - 90° - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$

The magnitude or $G(j\omega)$ can be expressed in decibels as shown in below.

$$|G(j\omega)| \text{ in db} = 20\log|G(j\omega)|$$

$$= 20\log\left[\frac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\left[\frac{K}{\omega}\times\sqrt{1+\omega^2T_1^2}\times\frac{1}{\sqrt{1+\omega^2T_2^2}}\times\frac{1}{\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\frac{1}{\sqrt{1+\omega^2T_2^2}} + 20\log\frac{1}{\sqrt{1+\omega^2T_3^2}}$$

$$= 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2T_1^2} - 20\log\sqrt{1+\omega^2T_2^2} - 20\log\sqrt{1+\omega^2T_3^2} \quad .....(4.29)$$

From the equation (4.29) it is clear that, when the magnitude is expressed in db, the multiplication is converted to addition. Hence in magnitude plot, the db magnitudes of individual factors of $G(j\omega)$ can be added.

Therefore to sketch the magnitude plot, a knowledge of the magnitude variations of individual factor is essential. The magnitude plot and phase plot of various factors of $G(j\omega)$ are explained in the following section.

BASIC FACTORS OF $G(j\omega)$.

The basic factors that very frequently occur in a typical transfer function $G(j\omega)$ are,

1. Constant gain, K
2. Integral factor, $\dfrac{K}{j\omega}$ or $\dfrac{K}{(j\omega)^n}$
3. Derivative factor, $K(j\omega)$ or $K(j\omega)^n$
4. First order factor in denominator, $\dfrac{1}{1+j\omega T}$ or $\dfrac{1}{(1+j\omega T)^m}$
5. First order factor in numerator, $(1+j\omega T)$ or $(1+j\omega T)^m$.

6. Quadratic factor in denominator, $\left[\dfrac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}\right]$
7. Quadratic factor in numerator, $\left[1+2\zeta\left(\dfrac{j\omega}{\omega_n}\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2\right]$

CONSTANT GAIN, K

Let, $G(s) = K$

$\therefore G(j\omega) = K = K\angle 0°$

$A = |G(j\omega)|$ in db $= 20\log K$

$\phi = \angle G(j\omega) = 0°$

The magnitude plot for a constant gain K is a horizontal straight line at the magnitude of 20 log K db. The phase plot is straight line at 0°

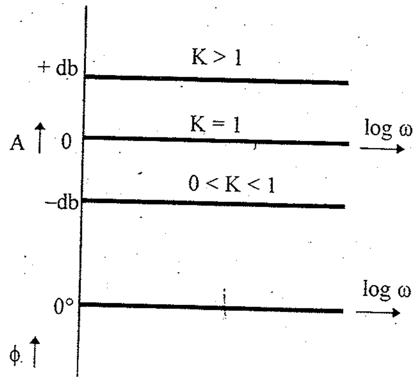


Fig 4.7 : Bode plot of constant gain, K

When $K > 1$, 20 log K is positive

When $0 < K < 1$, 20 log K is negative

When $K = 1$, 20 log K is zero

INTEGRAL FACTOR

Let $G(s) = K/s$

$G(j\omega) = (K/j\omega) = (K/\omega)\angle -90°$

$A = |G(j\omega)|$ in db $= 20\log(K/\omega)$ and

$\phi = \angle G(j\omega) = -90°$

When $\omega = 0.1\,K$, $A = 20\log(1/0.1) = 20$ db

When $\omega = K$, $A = 20\log 1 = 0$ db

When $\omega = 10\,K$, $A = 20\log 1/10 = -20$ db

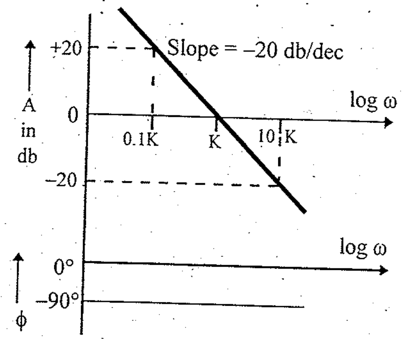


Fig 4.8 : Bode plot of integral factor K/jω

From the above analysis it is evident that the magnitude plot of the factor is a straight line with a slope of –20 db/dec and passing through **zero** db, when $\omega = K$. Since the $\angle G(j\omega)$ is a constant and independent of ω the phase plot is a straight line at –90°

When an integral factor has multiplicity of n, then

$G(s) = K/s^n$

$G(j\omega) = K/(j\omega)^n = K/\omega^n \angle -90n°$

$$A = |G(j\omega)| \text{ in db}$$
$$= 20 \log \frac{K}{\omega^n}$$
$$= 20 \log \left(\frac{K^{\frac{1}{n}}}{\omega} \right)^n = 20\, n \log \left(\frac{K^{\frac{1}{n}}}{\omega} \right)$$
$$\phi = \angle G(j\omega) = -90\, n^\circ$$

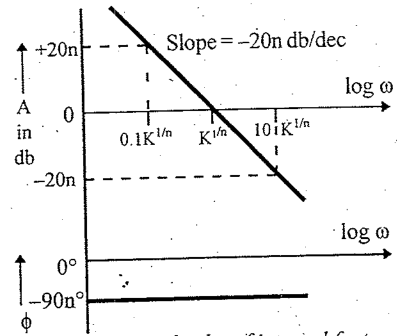


Fig 4.9 : Bode plot of integral factor $K/(j\omega)^n$

Now the magnitude plot of the integral factor is a straight line with a slope of –20 n db/dec and passing through zero db when $\omega = K^{1/n}$. The phase plot is a straight line at –90n°

DERIVATIVE FACTOR

Let $G(s) = Ks$

$G(j\omega) = K(j\omega) = (K\omega) \angle 90^\circ$

$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega)$

and $\phi = \angle G(j\omega) = +90^\circ$

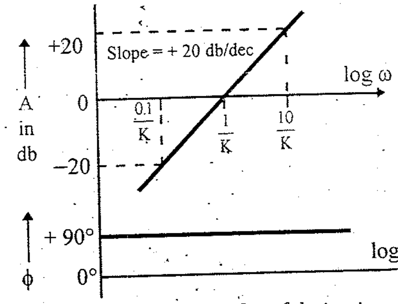


Fig 4.10 : Bode plot of derivative factor $K(j\omega)$

When $\omega = 0.1/K$, $A = 20 \log (0.1) = -20$ db

When $\omega = 1/K$, $A = 20 \log 1 = 0$ db

When $\omega = 10/K$, $A = 20 \log 10 = +20$ db

From the above analysis it is evident that the magnitude plot of the derivative factor is a straight line with a slope of +20n db/dec and passing through zero db when $\omega = 1/K$. Since the $\angle G(j\omega)$ is a constant and independent of ω, the phase plot is a straight line at +90°.

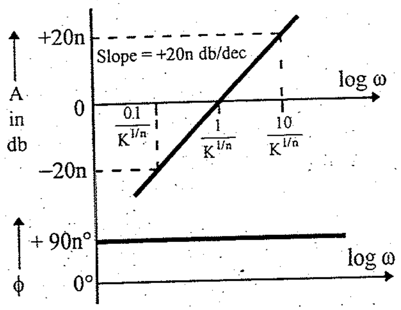


Fig 4.11 : Bode plot of derivative factor $K(j\omega)^n$

When the derivative factor has multiplicity of n then $G(s) = K\, s^n$

$$\therefore G(j\omega) = K(j\omega)^n = K\omega^n \angle 90n^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega^n) = 20 \log (K^{1/n}\, \omega)^n = 20\, n \log (K^{1/n}\, \omega)$$

and $\phi = \angle G(j\omega) = 90n^\circ$

Now the magnitude plot of the derivative factor is a straight line with a slope of +20n db/dec and passing through zero db when $\omega = 1/K^{1/n}$. The phase plot is a straight line at 90n°.

FIRST ORDER FACTOR IN DENOMINATOR

$$G(s) = \frac{1}{1+sT}$$
$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1} \omega T$$
$$A = |G(j\omega)|_{\text{in db}} = 20 \log \frac{1}{\sqrt{1+\omega^2 T^2}} = -20 \log \sqrt{1+\omega^2 T^2}$$

At very low frequencies $\omega T \ll 1$

$$\therefore A = -20 \log \sqrt{1+\omega^2 T^2} \approx -20 \log 1 = 0$$

At very high frequencies $\omega T \gg 1$

$$\therefore A = -20 \log \sqrt{1+\omega^2 T^2} \approx -20 \log \sqrt{\omega^2 T^2} = -20 \log \omega T$$

at $\omega = \frac{1}{T}$, $\quad A = -20 \log 1 = 0$

at $\omega = \frac{10}{T}$, $\quad A = -20 \log 10 = -20$ db

The above analysis shows that the magnitude plot of the factor $1/(1+j\omega T)$ can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < 1/T$ and the other is a straight line with slope –20 db/dec for the frequency range $(1/T < \omega < \infty)$. The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called the corner frequency or break frequency. For the factor $1/(1+j\omega T)$ the frequency, $\omega = 1/T$ is the corner frequency, ω_c. It divides the frequency response curve into two regions, a curve for low frequency region and a curve for high frequency region.

The actual magnitude at the corner frequency, $\omega_c = \frac{1}{T}$ is

$$A = -20\log\sqrt{1+1} = -3\text{ db}.$$

Hence by this approximation the loss in db at the corner frequency is –3 db.

The phase plot is obtained by calculating the phase angle of $G(j\omega)$ for various values of ω

Phase angle, $\phi = \angle G(j\omega) = -\tan^{-1}\omega T$.

At the corner frequency, $\omega = \omega_c = \frac{1}{T}$, $\phi = -\tan^{-1}\omega T = -\tan^{-1}1 = -45°$

As $\omega \to 0$, $\phi \to 0$ and As $\omega \to \infty$, $\phi \to -90°$

The phase angle of the factor $[1/(1+j\omega T)]$ varies from 0° to –90° as ω is varied from zero to infinity. The phase plot is a curve passing through –45° at ω_c.

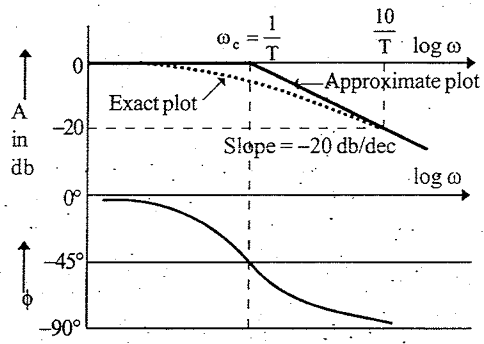


Fig 4.12 : Bode plot of the factor $\frac{1}{1+j\omega T}$

When the first order factor in the denominator has a multiplicity of m then

$$G(s) = \frac{1}{(1+sT)^m} \quad \text{and}$$

$$G(j\omega) = \frac{1}{(1+j\omega T)^m} = \frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T}$$

$$A = |G(j\omega)| \text{ in db} = 20\log\frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m} = -20\,m\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = -m\tan^{-1}\omega T$$

Now the magnitude plot of the factor $1/(1+j\omega T)^m$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0 < \omega < 1/T$ and the other is a straight line with slope –20 m db/sec for the frequency range $1/T < \omega < \infty$. The corner frequency, $\omega_c = 1/T$ and the loss in db at the corner frequency is –3 m db.

The phase angle of the factor $1/(1+j\omega T)^m$ varies from 0° to $-90m°$ as ω is varied from zero to infinity. The phase plot is a curve passing through $-45m°$ at ω_c.

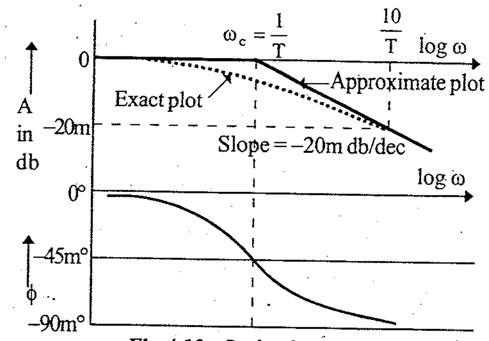


Fig 4.13 : Bode plot of the factor $1/(1+j\omega T)^m$

FIRST ORDER FACTOR IN THE NUMERATOR

$$G(s) = 1+sT$$

$$G(j\omega) = 1+j\omega T = \sqrt{1+\omega^2T^2}\ \angle\tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\sqrt{1+\omega^2T^2} \quad \text{and} \quad \phi = \angle G(j\omega) = \tan^{-1}\omega T$$

By an analysis similar to that of previous section it can be shown that the magnitude plot of the factor $(1+j\omega T)$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0 < \omega < 1/T$ and the other is a straight line with slope +20 db/dec for the frequency range $1/T < \omega < \infty$. The two straight lines are asymptotes of the exact curve.

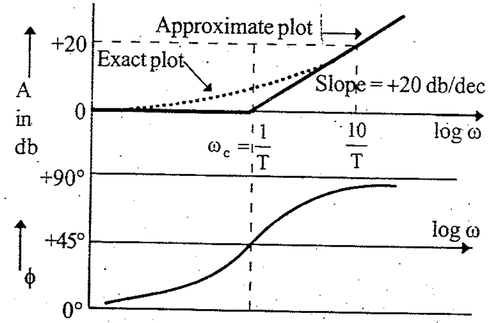


Fig 4.14 : Bode plot of the factor $(1+j\omega T)$

The frequency at which the two asymptotes meet is called the corner frequency or break frequency. For the factor $(1+j\omega T)$, the frequency, $(\omega = 1/T)$ is the corner frequency, ω_c. By this approximation the loss in db at the corner frequency is +3 db. The phase angle of the factor $(1+j\omega T)$ varies from zero to +90° as ω is varied from 0 to ∞. The phase plot is a curve passing through +45° at ω_c.

When the first order factor in the numerator has a multiplicity of m, then

$$G(s) = (1+sT)^m$$

$$G(j\omega) = (1+j\omega T)^m = \left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T$$

$$A = |G(j\omega)|_{\text{in db}} = 20\log\left(\sqrt{1+\omega^2T^2}\right)^m = 20\,m\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = m\tan^{-1}\omega T$$

Now the magnitude plot of the factor $(1 + j\omega T)^m$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0 < \omega < 1/T$ and the other is a straight line with a slope of +20 m db/dec for the frequency range $1/T < \omega < \infty$. The corner frequency, $\omega_c = 1/T$ and the loss in db at this corner frequency is +3 m db.

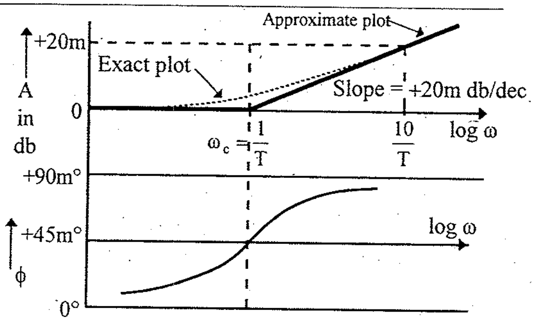


Fig 4.15 : Bode plot of the factor $(1+j\omega T)^m$

The phase angle of the factor $(1 + j\omega T)^m$ varies from zero to $+90m°$ as ω is varied from zero to infinity. The phase plot is a curve passing through $+45\ m°$ at ω_c.

QUADRATIC FACTOR IN THE DENOMINATOR

$$G(s) = \frac{1}{1 + 2\zeta\frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2}$$

$$G(j\omega) = \frac{1}{1 + j\frac{2\zeta\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

$$= \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}} \angle -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

Let $A = |G(j\omega)|$ in db

$$A = 20\log\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}} = -20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$= -20\log\sqrt{1 + \frac{\omega^4}{\omega_n^4} - 2\frac{\omega^2}{\omega_n^2} + 4\zeta^2\frac{\omega^2}{\omega_n^2}} = -20\log\sqrt{1 - \frac{\omega^2}{\omega_n^2}(2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}}$$

At very low frequencies when $\omega << \omega_n$, the magnitude is

$$A = -20\ \log\sqrt{1 - \frac{\omega^2}{\omega_n^2}(2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}} \approx -20\ \log 1 = 0$$

At very high frequencies when $\omega >> \omega_n$, the magnitude is

$$A = -20\ \log\sqrt{1 - \frac{\omega^2}{\omega_n^2}(2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}} \approx -20\ \log\sqrt{\frac{\omega^4}{\omega_n^4}} = -20\ \log\frac{\omega^2}{\omega_n^2}$$

$$\therefore A = -40\ \log\frac{\omega}{\omega_n}$$

At $\omega = \omega_n$, $A = -40\ \log 1 = 0$ db

At $\omega = 10\ \omega_n$, $A = -40\ \log 10 = -40$ db

From the above analysis it is evident that the magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a straight line with slope −40 db/dec for the frequency range $\omega_n < \omega < \infty$. The two straight lines are asymptotes of the exact curve. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency ω_n is the corner frequency, ω_c.

The two asymptotes of the exact curve are independent of the damping ratio, ζ. In the exact magnitude plot resonant peak occurs near the corner frequency and the magnitude of resonant peak depends on ζ. Lower the value of ζ, larger will be the resonant peak. Hence by this approximation the error at the corner frequency depends on damping ratio ζ. The phase plot is obtained by calculating the phase angle of $G(j\omega)$ for various values of ω

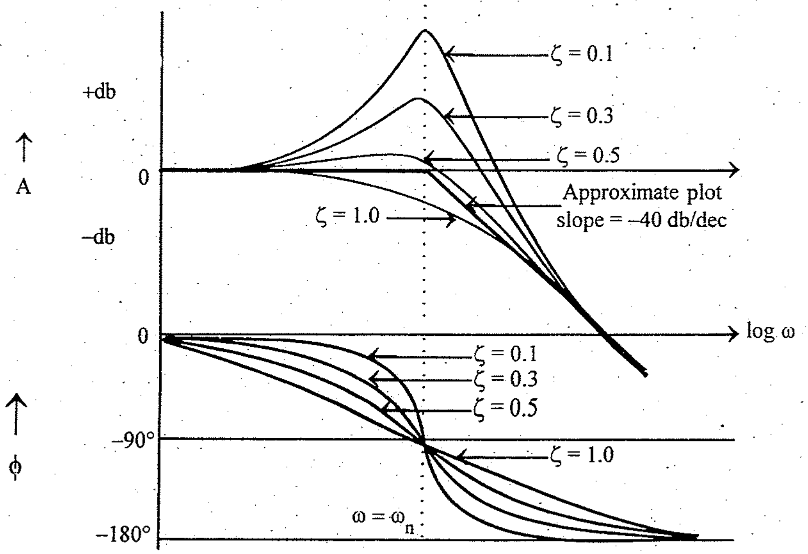


Fig 4.16 : Bode plot of quadratic factor in denominator

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

At $\omega = \omega_n$, $\phi = -\tan^{-1}\frac{2\zeta}{0} = -\tan^{-1}\infty = -90^0$

As $\omega \to 0$, $\phi \to 0$ and As $\omega \to \infty$, $\phi \to -180^0$

The phase angle of the quadratic factor varies from 0 to -180^0 as ω is varied from 0 to ∞. The phase plot is a curve passing through -90^0 at ω_c. At the corner frequency phase angle is -90^0 and independent of ζ, but at all other frequency it depends on ζ.

QUADRATIC FACTOR IN THE NUMERATOR

$$G(s) = 1 + 2\zeta\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2$$

$$G(j\omega) = 1 + j2\zeta\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 = \sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}} \;\angle\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

The magnitude plot of the quadratic factor in the numerator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a

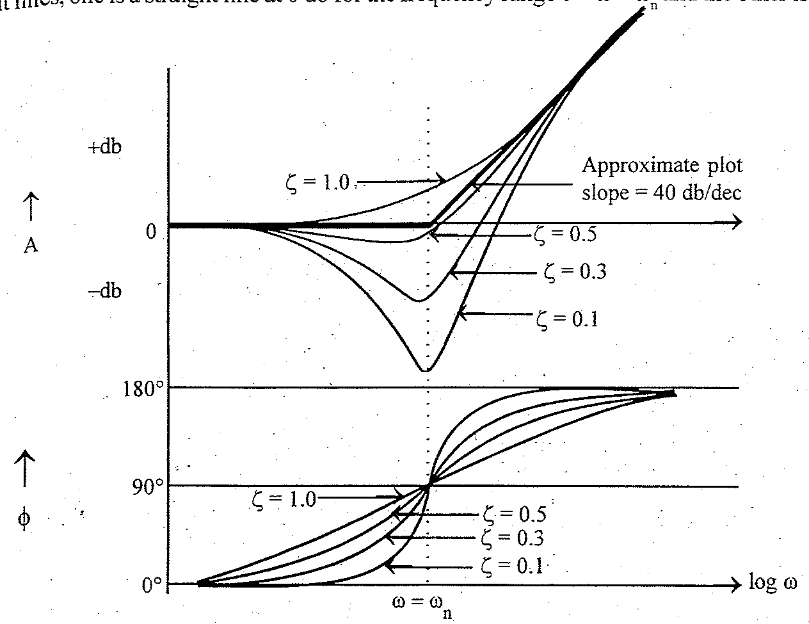


Fig 4.17 : Bode plot of quadratic factor in numerator

straight line with slope +40 db/dec for the frequency range $\omega_n < \omega < \infty$. The corner frequency is ω_n. Due to this approximation the error at the corner frequency depends on ζ.

The phase angle varies from 0 to +180°, as ω is varied from 0 to ∞. At the corner frequency the phase angle is +90° and independent of ζ, but at all other frequency it depends on ζ

PROCEDURE FOR MAGNITUDE PLOT OF BODE PLOT

From the analysis of previous sections the following conclusions can be obtained.

1. The constant gain K, integral and derivative factors contribute gain (magnitude) at all frequencies.
2. In approximate plot the first, quadratic and higher order factors contribute gain (magnitude) only when the frequency is greater than the corner frequency.

Hence the low frequency response upto the lowest corner frequency is decided by K or $K/(j\omega)^n$ or $K(j\omega)^n$ term. Then at every corner frequency the slope of the magnitude plot is altered by the first, quadratic and higher order terms. Therefore the magnitude plot can be started with K or $K/(j\omega)^n$ or $K(j\omega)^n$ term and then the db magnitude of every first and higher order terms are added one by one in the increasing order of the corner frequency.

This is illustrated in the following example.

$$\text{Let, } G(s) = \frac{K(1+sT_1)^2}{s^2(1+sT_2)(1+sT_3)}$$

$$\therefore G(j\omega) = \frac{K(1+j\omega T_1)^2}{(j\omega)^2(1+j\omega T_2)(1+j\omega T_3)}$$

Let $T_2 < T_3 < T_1$

The corner frequencies are $\omega_{c1} = \frac{1}{T_1}$, $\omega_{c2} = \frac{1}{T_2}$, $\omega_{c3} = \frac{1}{T_3}$

Here $\omega_{c1} < \omega_{c3} < \omega_{c2}$

The magnitude plot of the individual terms of $G(j\omega)$ and their combined magnitude plot are shown in fig 4.18.

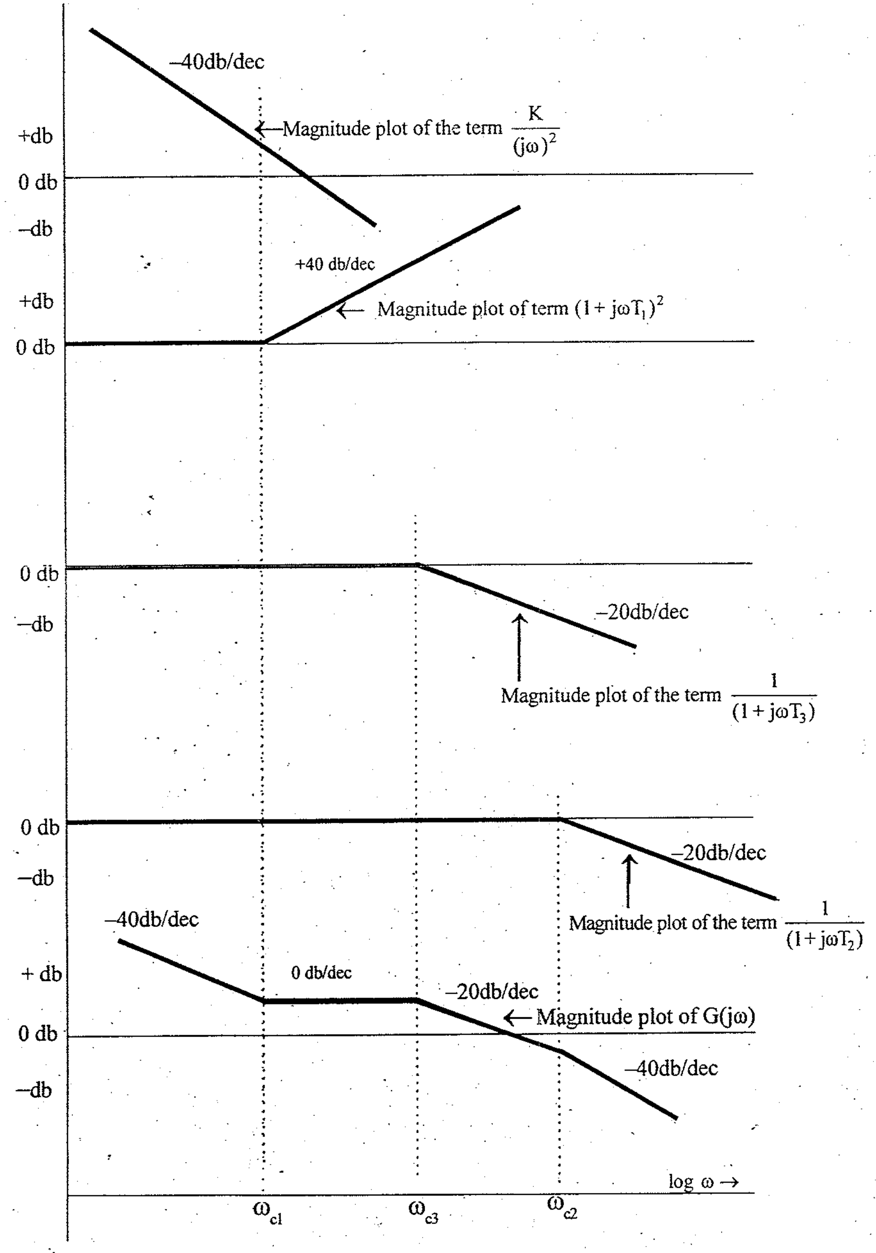


Fig 4.18 : Magnitude plot of bode plot of $G(j\omega) = \dfrac{K(1+j\omega T_1)^2}{(j\omega)^2 (1+j\omega T_2)(1+j\omega T_3)}$

The step by step procedure for plotting the magnitude plot is given below.

Step 1 : Convert the transfer function into Bode form or time constant form. The Bode form of the transfer function is

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)\left(1+\dfrac{s^2}{\omega_n^2}+2\zeta\dfrac{s}{\omega_n}\right)}$$

$$G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)\left(1-\dfrac{\omega^2}{\omega_n^2}+j2\zeta\dfrac{\omega}{\omega_n}\right)}$$

Step 2 : List the corner frequencies in the increasing order and prepare a table as shown below.

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec

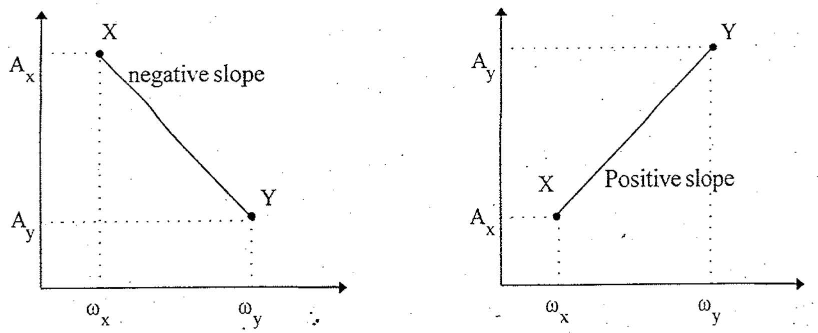
In the above table enter K or $K/(j\omega)^n$ or $K(j\omega)^n$ as the first term and the other terms in the increasing order of corner frequencies. Then enter the corner frequency, slope contributed by each term and change in slope at every corner frequency.

Step 3: Choose an arbitrary frequency ω_l which is lesser than the lowest corner frequency. Calculate the db magnitude of K or $K/(j\omega)^n$ or $K(j\omega)^n$ at ω_l and at the lowest corner frequency.

Step 4 : Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula,

Gain at ω_y = change in gain from ω_x to ω_y + Gain at ω_x

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log\frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$



Step 5 : Choose an arbitrary frequency ω_h which is greater than the highest corner frequency. Calculate the gain at ω_h by using the formula in step 4.

Step 6 : In a semilog graph sheet mark the required range of frequency on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale) after choosing proper scale.

Step 7 : Mark all the points obtained in steps 3, 4, and 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

(Note : The magnitude plot obtained above is an approximate plot. If an exact plot is needed then appropriate corrections should be made at every corner frequencies)

PROCEDURE FOR PHASE PLOT OF BODE PLOT

The phase plot is an exact plot and no approximations are made while drawing the phase plot. Hence the exact phase angles of G(jω) are computed for various values of ω and tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semilog-sheet on a common frequency scale.

Take another y-axis in the graph where the magnitude plot is drawn and in this y-axis mark the desired range of phase angles after choosing proper scale. From the tabulated values of ω and phase angles, mark all the points on the graph. Join the points by a smooth curve.

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM BODE PLOT

The gain margin in db is given by the negative of db magnitude of G(jω) at the phase cross-over frequency, ω_{pc}. The ω_{pc} is the frequency at which phase of G(jω) is –180°. If the db magnitude of G(jω) at ω_{pc} is negative then gain margin is positive and vice versa.

Let ϕ_{gc} be the phase angle of G(jω) at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of G(jω) is zero. Now the phase margin, γ is given by, $\gamma = 180° + \phi_{gc}$. If ϕ_{gc} is less negative than –180° then phase margin is positive and vice versa.

The positive and negative gain margins are illustrated in fig 4.19.

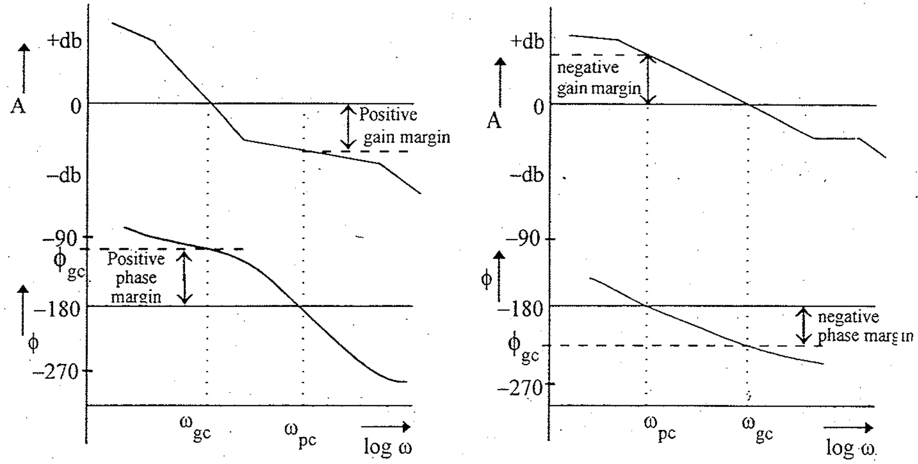


Fig 4.19 : Bode plot showing phase margin (γ) and gain margin (K_g)

GAIN ADJUSTMENT IN BODE PLOT

In the open loop transfer function G(jω) the constant K contributes only magnitude. Hence by changing the value of K the system gain can be adjusted to meet the desired specifications. The desired specifications are gain margin, phase margin, ω_{pc} and ω_{gc}. In a system transfer function if the value of K required to be estimated to satisfy a desired specification then draw the bode plot of the system with K = 1. The constant K can add 20 log K to every point of the magnitude plot and due to this addition the magnitude plot will shift vertically up or down. Hence shift the magnitude plot vertically up or down to meet the desired specification. Equate the vertical distance by which the magnitude plot is shifted to 20 log K and solve for K.

Let x = change in db (x is positive if the plot is shifted up and vice versa).

$$\text{Now } 20 \log K = x \; ; \quad \log K = x/20 \qquad \therefore K = 10^{x/20}$$

Note : *A point in complex plane can be represented by rectangular coordinates or by polar coordinates. Consider a point, z = a+jb in complex plane. Now, $|z| = \sqrt{a^2+b^2}$ and Arg [z]= $\tan^{-1} b/a$. If the point lies in first or fourth quadrant then the argument as calculated by $\tan^{-1} b/a$ will be the correct values. But if it lies either in second or third quadrant then a correction should be made in the calculated values of argument, because the calculator will always give the values of $\tan^{-1} b/a$ either from 0 to + 90° or from 0 to –90° The corrections to be made while converting from rectangular to polar coordinates is shown below.*

A point in I[st] quadrant, $a + jb = \sqrt{a^2+b^2} \angle \tan^{-1} b/a$

A point in II[nd] quadrant, $-a + jb = \sqrt{a^2+b^2} \angle (\pi - \tan^{-1} b/a)$

A point in III[rd] quadrant, $-a - jb = \sqrt{a^2+b^2} \angle (\pi + \tan^{-1} b/a)$

A point in IV[th] quadrant, $a - jb = \sqrt{a^2+b^2} \angle -\tan^{-1} b/a$

EXAMPLE 4.1

Sketch Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec

$$G(s) = \frac{Ks^2}{(1+0.2s)\,(1+0.02s)}$$

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given s-domain transfer function

$$\therefore G(j\omega) = \frac{K(j\omega)^2}{(1+0.2\,j\omega)(1+0.02\,j\omega)}$$

$$\text{Let } K = 1, \ \therefore G(j\omega) = \frac{(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$$

MAGNITUDE PLOT

The corner frequencies are

$$\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec and } \omega_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec}$$

The various terms of G(jω) are listed in Table 1 in the increasing order of their corner frequency. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE 1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$(j\omega)^2$	-	+ 40	
$\frac{1}{1+j0.2\omega}$	$\omega_{c1} = \frac{1}{0.2} = 5$	−20	40-20=20
$\frac{1}{1+j0.02\omega}$	$\omega_{c2} = \frac{1}{0.02} = 50$	−20	20-20=0

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let ω_l=0.5 rad/sec and ω_h=100 rad/sec

Let $A = |G(j\omega)|$ in db

Let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h.

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|(j\omega)^2\right| = 20\log(\omega)^2 = 20\log(0.5)^2 = -12 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\left|(j\omega)^2\right| = 20\log(\omega)^2 = 20\log(5)^2 = 28 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})}$$

$$= 20 \times \log\frac{50}{5} + 28 = 48 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})}$$

$$= 0 \times \log\frac{100}{50} + 48 = 48 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a, b, c and d on the graph. Join the points by straight lines and mark the slope on the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by

$$\phi = \angle G(j\omega) = 180° - \tan^{-1}0.2\omega - \tan^{-1}0.02\omega$$

The phase angle of G(jω)are calculated for various values of ω and listed in table2.

TABLE2

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg
0.5	5.7	0.6	$173.7 \approx 174$
1	11.3	1.1	$167.6 \approx 168$
5	45	5.7	$129.3 \approx 130$
10	63.4	11.3	$105.3 \approx 106$
50	84.3	45	$50.7 \approx 50$
100	87.1	63.4	$29.5 \approx 30$

On the same semilog sheet choose a scale of 1unit = 20°, on the y-axis on the right side of semilog sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

CALCULATION OF K

Given that the gain crossover frequency is 5 rad/sec. At $\omega = 5$ rad/sec the gain is 28 db. If gain crossover frequency is 5 rad/sec then at that frequency the db gain should be zero. Hence to every point of magnitude plot a db gain of –28db should be added. The addition of –28db shifts the plot downwards. The corrected magnitude plot is obtained by shifting the plot with K = 1 by 28db downwards. The magnitude correction is independent of frequency. Hence the magnitude of –28db is contributed by the term K. The value of K is calculated by equating 20 logK to –28 db.

$$\therefore 20 \log K = -28 \text{ db}$$

$$\log K = \frac{-28}{20} \qquad \therefore K = 10^{-\frac{28}{20}} = 0.0398$$

The magnitude plot with K = 1 and 0.0398 and the phase plot are shown in fig 4.1.1.

NOTE

The frequency $\omega = 5$ rad/sec is a corner frequency. Hence in the exact plot the db gain at $\omega = 5$ rad/sec will be 3db less than the approximate plot. Therefore for exact plot the 20logK will contribute a gain of –25db

$$\therefore 20 \log K = -25 \text{ db}$$

$$\log K = \frac{-25}{20} \qquad \therefore K = 10^{\frac{-25}{20}} = 0.0562.$$

CHAPTER 4 - FREQUENCY RESPONSE ANALYSIS

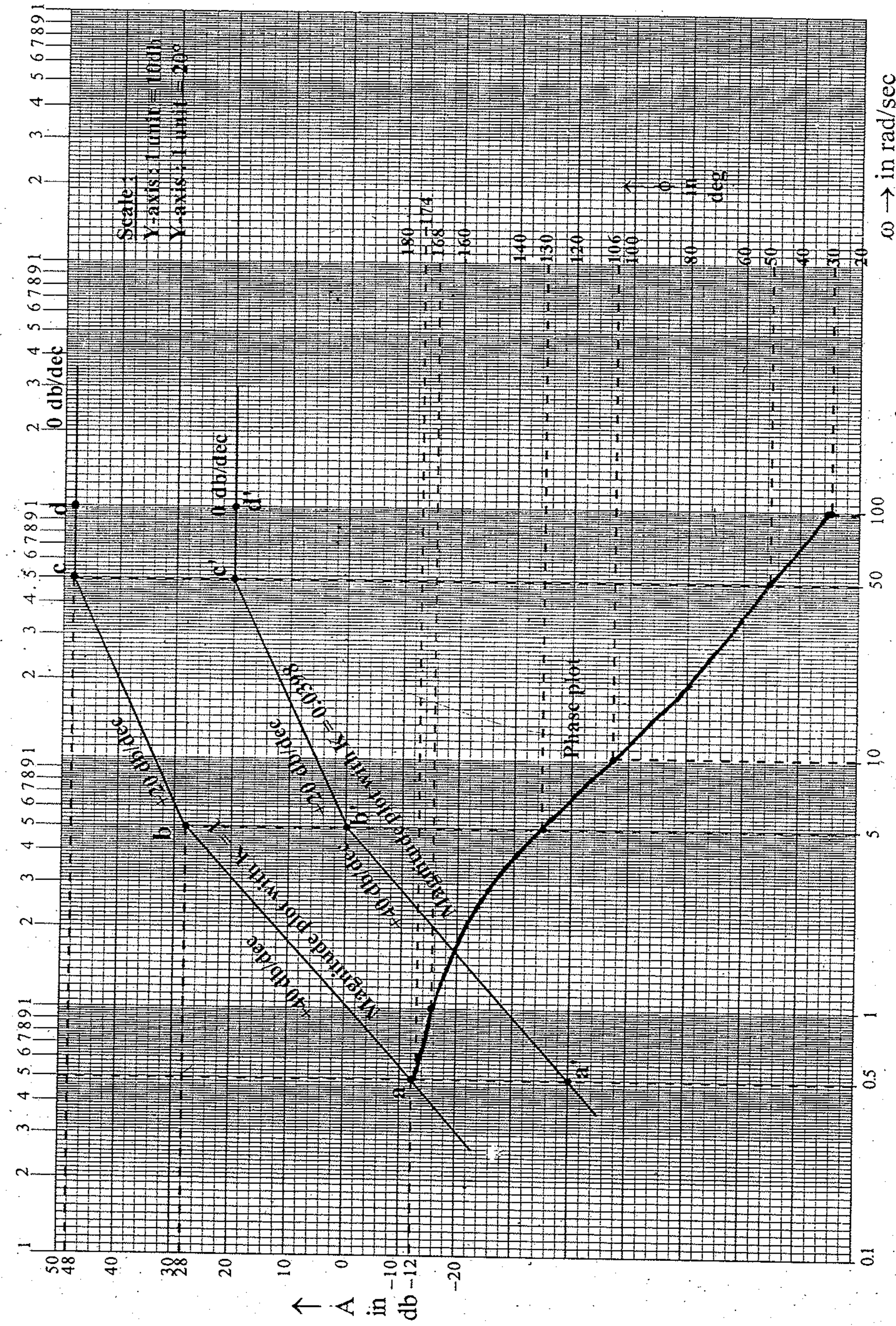


Fig 4.1.1.: Bode plot of transfer function, $G(j\omega) = \frac{K(j\omega)^2}{(1 + j0.2\omega)(1 + j0.02\omega)}$

EXAMPLE 4.2

Sketch the bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s)=\frac{75\,(1+0.2s)}{s\left(s^2+16s+100\right)}$$

SOLUTION

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given s-domain transfer function after converting it to bode form or time constant form.

$$\text{Given that } G(s)=\frac{75(1+0.2s)}{s\left(s^2+16s+100\right)}$$

On comparing the quadratic factor of G(s) with standard form of quadratic factor we can estimate ζ and ω_n

$$\therefore s^2+16s+100=s^2+2\zeta\omega_n s+\omega_n^2$$

On comparing we get,

$\omega_n^2=100$ $\therefore \omega_n=10$	$2\zeta\omega_n=16$ $\therefore \zeta=\frac{16}{2\omega_n}=\frac{16}{2\times 10}=0.8$

$$G(s)=\frac{75\,(1+0.2s)}{s\times 100\left(\frac{s^2}{100}+\frac{16s}{100}+1\right)}=\frac{0.75\,(1+0.2s)}{s\left(1+0.01s^2+0.16s\right)}$$

$$\therefore G(j\omega)=\frac{0.75\,(1+0.2j\omega)}{j\omega\left(1+0.01(j\omega)^2+0.16j\omega\right)}=\frac{0.75\,(1+j0.2\omega)}{j\omega\left(1-0.01\omega^2+j0.16\omega\right)}$$

MAGNITUDE PLOT

The corner frequencies are $\omega_{c1}=\frac{1}{0.2}=5$ rad / sec and $\omega_{c2}=\omega_n=10$ rad / sec

Note : For the quadratic factor the corner frequency is ω_n

The various terms of $G(j\omega)$ are listed in table 1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency

TABLE 1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$	-	-20	
$1+j0.2\omega$	$\omega_{c1}=\frac{1}{0.2}=5$	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2}=\omega_n=10$	-40	$0-40=-40$

Choose a low frequency ω_l such that $\omega_l<\omega_{c1}$ and choose a high frequency ω_h such that $\omega_h>\omega_{c2}$

Let $\omega_l=0.5$ rad/sec and $\omega_h=20$ rad/sec

Let $A=|G(j\omega)|$ in db ; Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At, } \omega=\omega_l,\ A=20\log\left|\frac{0.75}{j\omega}\right|=20\log\frac{0.75}{0.5}=3.5\text{ db}$$

$$\text{At, } \omega=\omega_{c1},\ A=20\log\left|\frac{0.75}{j\omega}\right|=20\log\frac{0.75}{5}=-16.5\text{ db}$$

$$\text{At, } \omega=\omega_{c2},\ A=\left[\text{slope from }\omega_{c1}\text{ to }\omega_{c2}\times\log\frac{\omega_{c2}}{\omega_{c1}}\right]+A_{(\text{at }\omega=\omega_{c1})}$$

$$=0\times\log\frac{10}{5}+(-16.5)=-16.5\text{ db}$$

$$\text{At } \omega=\omega_h,\ A=\left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right]+A_{(\text{at } \omega=\omega_{c2})}$$

$$=-40\times\log\frac{20}{10}+(-16.5)=-28.5\ \text{db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_p, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 5 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a,b,c and d on the graph. Join the points by straight lines and mark the slope on the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by

$$\phi=\angle G(j\omega)=\tan^{-1}0.2\omega-90^\circ-\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}\quad \text{For } \omega\le\omega_n$$

$$\phi=\angle G(j\omega)=\tan^{-1}0.2\omega-90^\circ-\left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}+180^\circ\right)\quad \text{For } \omega>\omega_n$$

The phase angle of G(jω) are calculated for various values of ω and listed in Table2.

TABLE2

ω rad/sec	$\tan^{-1}0.2\,\omega$ deg	$\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi=\angle G(j\omega)$ deg
0.5	5.7	4.6	$-88.9\approx-88$
1	11.3	9.2	$-87.9\approx-88$
5	45	46.8	$-91.8\approx-92$
10	63.4	90	$-116.6\approx-116$
20	75.9	−46.8+180=133.2	$-147.3\approx-148$
50	84.3	−18.4+180=161.6	$-167.3\approx-168$
100	87.1	−9.2+180=170.8	$-173.7\approx-174$

***Note :** In quadratic factors the angle varies from 0° to 180°. But the calculator calculates tan^{-1} only between 0° to 90°. Hence a correction factor of 180° should be added to the phase angle after corner frequency.*

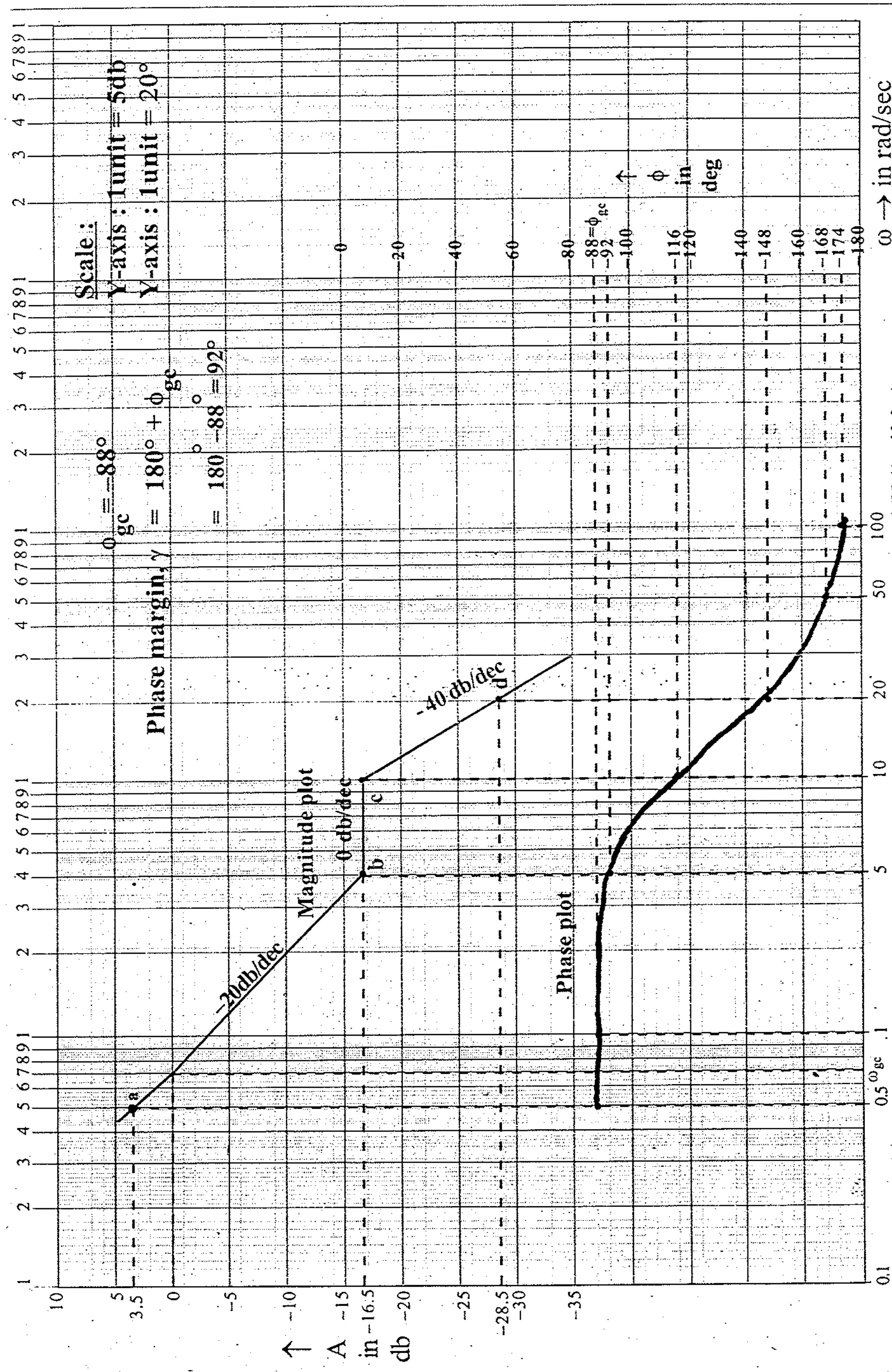


Fig 4.2.1: Bode plot of transfer function, $G(j\omega)=\dfrac{0.75\,(1+j0.2\omega)}{j\omega\,(1-0.01\omega^2+j0.16\omega)}$

On the same semilog sheet choose a scale of 1 unit = 20^0 on the y-axis on the right side of semilog sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

The magnitude plot and the phase plot are shown in fig 4.2.1. From the fig 4.2.1 we find that the phase angle at gain crossover frequency $(\omega_{gc}) = \phi_{gc} = 88°$

$\therefore$ Phase margin, $\gamma = 180 + \phi_{gc} = 180° - 88° = 92°$

Here, Gain margin $= +\infty$.

The phase plot crosses $-180°$ only at infinity. The $|G(j\omega)|$ at infinity is $-\infty$ db. Hence gain margin is $+\infty$.

EXAMPLE 4.3

Given $G(s) = \dfrac{K\,e^{-0.2s}}{s\,(s+2)\,(s+8)}$. Find K so that the system is stable with (a) gain margin equal to 6 db and (b) phase margin equal to 45°.

SOLUTION

$$\text{Given that } G(s) = \frac{K\,e^{-0.2s}}{s\,(s+2)(s+8)} = \frac{K\,e^{-0.2s}}{s \times 2\left(1+\frac{s}{2}\right) \times 8\left(1+\frac{s}{8}\right)}$$

$$= \frac{0.0625\,Ke^{-0.2s}}{s\,(1+0.5s)(1+0.125s)}$$

Let K=1

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given s domain transfer function.

$$G(j\omega) = \frac{0.0625\,e^{-j0.2\omega}}{j\omega\,(1+j0.5\omega)\,(1+j0.125\omega)}$$

Note: $\left|0.0625\,e^{-j0.2\omega}\right| = 0.0625$ *and* $\angle e^{-j0.2\omega} = -0.2\omega$, *radians*

MAGNITUDE PLOT

The corner frequencies are

$$\omega_{c1} = \frac{1}{0.5} = 2 \text{ rad/sec} \quad \text{and } \omega_{c2} = \frac{1}{0.125} = 8 \text{ rad/sec}$$

The various terms of G(jω) are listed in table1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE 1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{0.0625}{j\omega}$	-	-20	
$\dfrac{1}{1+j0.5\omega}$	$\omega_{c1} = \dfrac{1}{0.5} = 2$	-20	$-20-20=-40$
$\dfrac{1}{1+j0.125\omega}$	$\omega_{c2} = \dfrac{1}{0.125} = 8$	-20	$-40-20=-60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 50$ rad/sec

Let $A = |G(j\omega)|$ in db.

Let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h

$$\text{At } \omega = \omega_l,\ A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{0.5} = -18\text{db}$$

$$\text{At } \omega = \omega_{c1},\ A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{2} = -30\text{db}$$

At $\omega = \omega_{c2}$, $A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\dfrac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})}$

$$= -40 \times \log\frac{8}{2} + (-30) = -54\text{db}$$

At $\omega = \omega_h$, $A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\dfrac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})}$

$$= -60 \times \log\frac{50}{8} + (-54) = -102\text{db}$$

Let the points a,b,c,d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10db on y-axis. The frequencies are marked in decades from 0.01 to 100 rad/sec on logarithmic scale in x-axis. Fix the points a, b, c, and d on the graph. Join the points by straight line and mark the slope on the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by

$$\phi = -0.2\omega \times \frac{180}{\pi} - 90 - \tan^{-1} 0.5\omega - \tan^{-1} 0.125\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table 2.

TABLE 2

ω rad/sec	−0.2 ω (180/π) deg	$\tan^{-1}$ 0.5 ω deg	$\tan^{-1}$ 0.125 ω deg	φ = ∠G(jω) deg
0.01	−0.1145	0.2864	0.0716	−90.4 ≈ −90
0.1	−1.145	2.862	0.716	−94.7 ≈ −94
0.5	−5.7	14	3.6	−113.3 ≈ −114
1	−11.4	26	7.12	−134.4 ≈ −134
2	−22.9	45	14	−171.9 ≈ −172
3	−34.37	56.30	20.56	−201.2 ≈ −202
4	−45.84	63.43	26.57	−225.8 ≈ −226

On the same semilog sheet choose a scale of 1unit = 20° on the y-axis on the right side of the semilog sheet. Mark the calculated phase angle on the graph sheet. Join the points by smooth curve

The magnitude and phase plot are shown in fig 4.3.1.

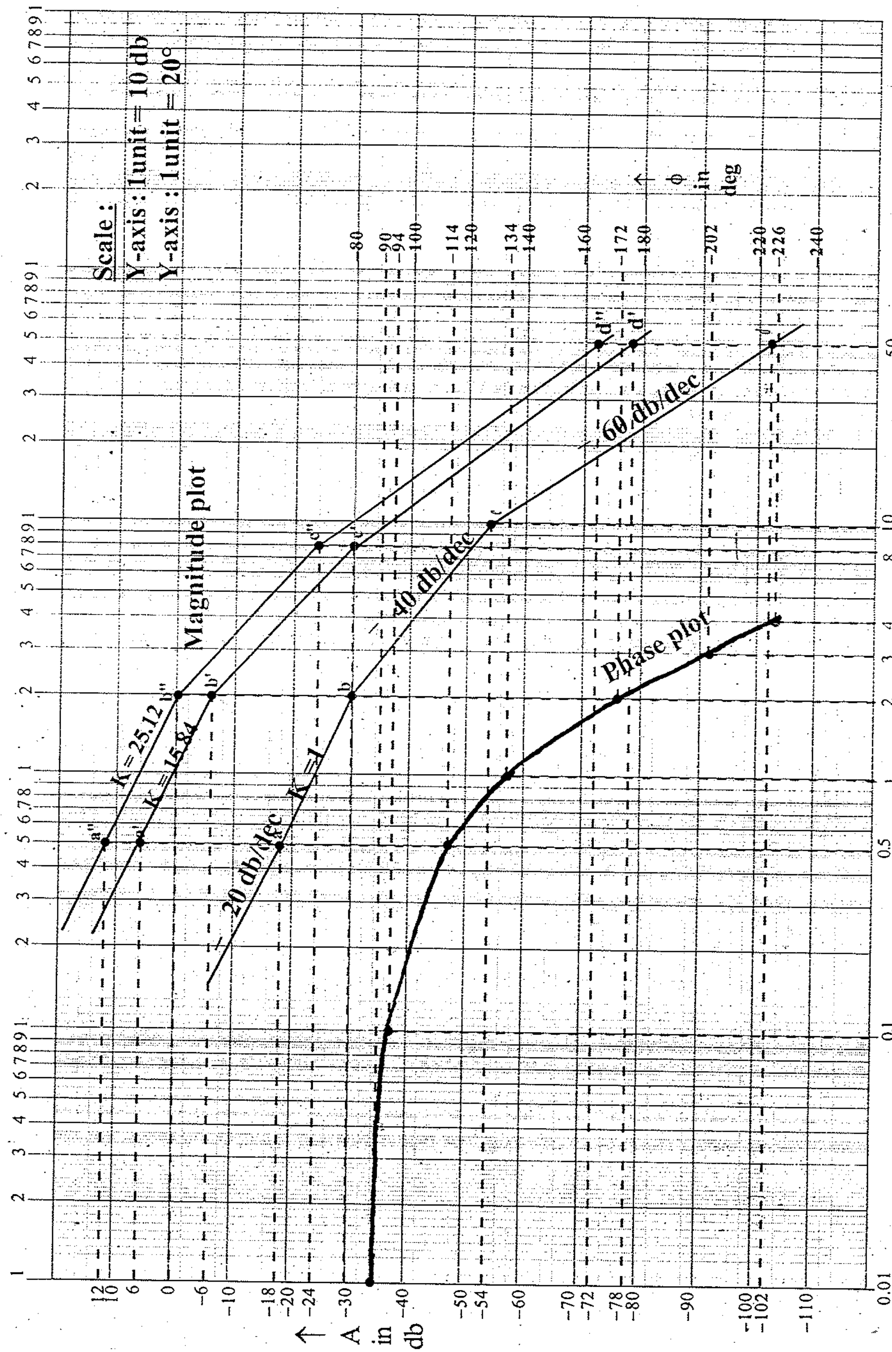


Fig 4.3.1: Bode plot of transfer function, $G(j\omega) = \dfrac{0.0625\, K\, e^{-j0.2\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$

CALCULATION OF K

Phase margin, $\gamma = 180+\phi_{gc}$, where ϕ_{gc} is the phase of $G(j\omega)$ at $\omega = \omega_{gc}$.

When $\gamma = 45°$, $\phi_{gc} = \gamma - 180 = 45 - 180 = -135°$.

With K = 1, the db gain at $\phi = -135°$ is –24 db. This gain should be made zero to have to PM of 45°. Hence to every point of magnitude plot a db gain of 24 db should be added. The corrected magnitude plot is obtained by shifting the plot with K = 1 by 24 db upwards. The magnitude correction is independent of frequency. Hence the magnitude of 24 db is contributed by the term K. The value of K is calculated by equating 20 log K to 24 db.

$$\therefore 20 \log K = 24 \quad ; \quad K = 10^{24/20} \quad ; \quad K = 15.84$$

With K = 1, the gain margin = –(–34) = 34 db. But the required gain margin is 6 db. Hence to every point of magnitude plot a db gain of 28 db should be added. This addition of 28 db shifts the plot upwards. The magnitude correction is independent of frequency. Hence the magnitude of 28 db is contributed by the term K. The value of K is calculated by equating 20 log K to 28 db.

$$\therefore 20 \log K = 28 \quad ; \quad K = 10^{28/20} \quad ; \quad K = 25.12$$

The magnitude plot with K = 15.84 and 25.12 are shown in fig 4.3.1.

EXAMPLE 4.4

Plot the Bode diagram for the following transfer function and obtain the gain and phase cross over frequencies.

SOLUTION

$$\text{Given that } G(s) = \frac{10}{s\,(1+0.4s)\,(1+0.1s)}$$

The sinusoidal transfer function of $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given transfer function,

$$G(j\omega) = \frac{10}{j\omega\,(1+j0.4\omega)\,(1+j0.1\omega)}$$

MAGNITUDE PLOT

The corner frequencies are

$$\omega_{c1} = \frac{1}{0.4} = 2.5 \text{ rad/sec} \quad \text{and} \quad \omega_{c2} = \frac{1}{0.1} = 10 \text{ rad/sec}$$

The various terms of $G(j\omega)$ are listed in table 1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE 1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{10}{j\omega}$	-	– 20	
$\frac{1}{1+j0.4\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	– 20	$-20-20=-40$
$\frac{1}{1+j0.1\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	– 20	$-40-20=-60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.1$ rad/sec, and $\omega_h = 50$ rad/sec.

Let $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h

$$\text{At } \omega = \omega_l,\ A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \frac{10}{0.1} = 40 \text{ db}$$

$$\text{At } \omega = \omega_{c1},\ A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \frac{10}{2.5} = 12 \text{ db}$$

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40 \times \log \frac{10}{2.5} + 12 = -12 \text{ db}$$

$$\text{At } \omega = \omega_h,\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -60 \times \log \frac{50}{10} + (-12) = -54 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by

$$\phi = -90° - \tan^{-1} 0.4\,\omega - \tan^{-1} 0.1\,\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table 2.

TABLE 2

ω rad/sec	$\tan^{-1} 0.4\omega$ deg	$\tan^{-1} 0.1\omega$ deg	$\phi = \angle G(j\omega)$ deg
0.1	2.29	0.57	$-92.86 \approx -92$
1	21.80	5.71	$-117.5 \approx -118$
2.5	45.0	14.0	$-149 \approx -150$
4	57.99	21.8	$-169.79 \approx -170$
10	75.96	45.0	$-210.96 \approx -210$
20	82.87	63.43	$-236.3 \approx -236$

On the same semilog sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

The magnitude and phase plots are shown in fig 4.4.1.

From the graph the gain and phase cross over frequencies are found to be 5 rad/sec.

RESULT

Gain cross-over frequency = 5 rad/sec.

Phase cross-over frequency = 5 rad/sec.

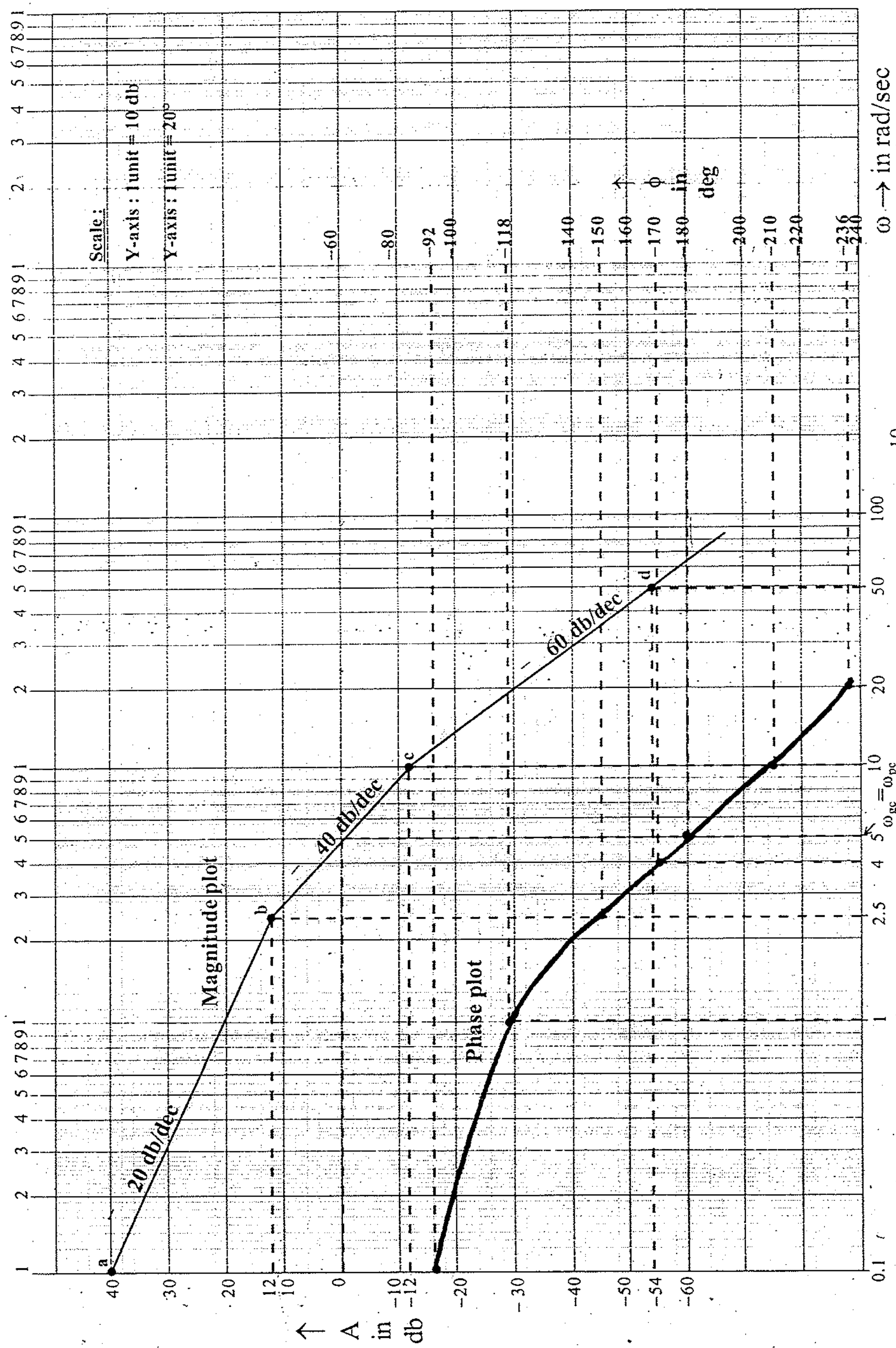


Fig 4.4.1: Bode plot of transfer function, $G(j\omega) = \dfrac{10}{j\omega\,(1 + j0.4\omega)\,(1 + j0.1\omega)}$

EXAMPLE 4.5

For the following transfer function draw bode plot and obtain gain cross-over frequency.

$$G(s) = \frac{20}{s\,(1+3s)\,(1+4s)}$$

SOLUTION

The sinusoidal transfer function of $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given transfer function.

$$G(j\omega) = \frac{20}{j\omega\,(1+j3\omega)\,(1+j4\omega)}$$

MAGNITUDE PLOT

The corner frequencies are $\omega_{c1} = \frac{1}{4} = 0.25$ rad / sec, $\omega_{c2} = \frac{1}{3} = 0.333$ rad / sec.

The various terms of $G(j\omega)$ are listed in table 1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and change in slope at the corner frequency.

TABLE 1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{20}{j\omega}$	-	-20	
$\frac{1}{(1+j4\omega)}$	$\omega_{c1} = \frac{1}{4} = 0.25$	-20	$-20-20=-40$
$\frac{1}{1+j3\omega}$	$\omega_{c2} = \frac{1}{3} = 0.33$	-20	$-40-20=-60$

Choose a frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let $\omega_l = 0.15$ rad/sec and $\omega_h = 1$ rad/sec

Let $A = |G(j\omega)|$ in db

Let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h

$$\text{At } \omega = \omega_l,\ A = |G(j\omega)| = 20\log\left|\frac{20}{0.15}\right| = 42.5 \text{ db}$$

$$\text{At } \omega = \omega_{c1},\ A = |G(j\omega)| = 20\log\left|\frac{20}{0.25}\right| = 38 \text{ db}$$

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40 \times \log\frac{0.33}{0.25} + 38 = 33 \text{ db}$$

$$\text{At } \omega = \omega_h,\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -60 \times \log\frac{1}{0.33} + 33 = 4 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10 db on y-axis. The frequencies are marked in decades from 0.01 to 10 rad/sec on logarithmic scales on x-axis. Fix the points a, b, c and d on the graph sheet. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

The phase angle of $G(j\omega)$, $\phi = -90 - \tan^{-1} 3\omega - \tan^{-1} 4\omega$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table 2.

TABLE 2

ω, rad/sec	$\tan^{-1} 3\omega$, deg	$\tan^{-1} 4\omega$, deg	$\phi = \angle G(j\omega)$, deg
0.15	24.22	30.96	$-145.18 \approx -146$
0.2	30.96	38.66	$-159.61 \approx -160$
0.25	36.86	45.0	$-171.86 \approx -172$
0.33	44.7	52.8	$-187.5 \approx -188$
0.5	56.30	63.43	$-209.73 \approx -210$
0.6	60.14	67.38	$-218.32 \approx -218$
1	71.56	75.96	$-237.56 \approx -238$

On the same semilog sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

The magnitude and phase plots are shown in fig 4.5.1. From the graph the gain cross-over frequency is found to be $\omega_{gc} = 1.1$ rad/sec.

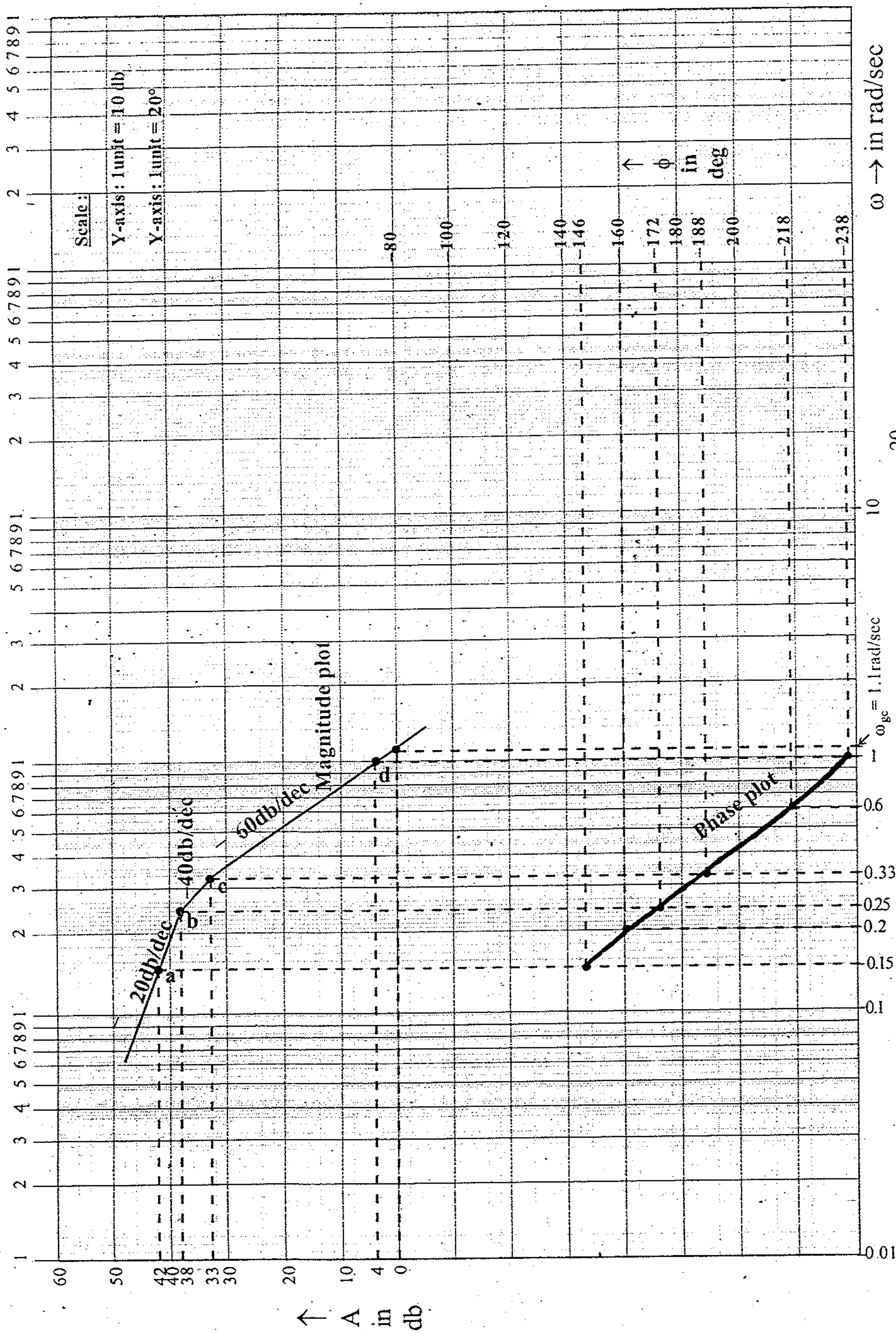


Fig 4.5.1 : Magnitude plot for transfer function, $\frac{20}{j\omega\,(1+j3\omega)\,(1+j4\omega)}$

EXAMPLE 4-6

For the function GH(s) = 5(1+2s)/(1+4s)(1+0.25s), draw the bode plot.

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given s-domain transfer function

$$\therefore G(j\omega) = \frac{5\,(1+j2\omega)}{(1+j4\omega)\,(1+j0.25\omega)}$$

MAGNITUDE PLOT

The corner frequencies are

$$\omega_{c1} = \frac{1}{4} = 0.25 \text{ rad / sec}, \quad \omega_{c2} = \frac{1}{2} = 0.5 \text{rad / sec and } \omega_{c3} = \frac{1}{0.25} = 4 \text{ rad / sec}$$

The various terms of G(jω) are listed in table 1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by the each term and the change in slope at the corner frequency.

TABLE1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/deg
5	-	0	-
$\frac{1}{(1+j4\omega)}$	$\omega_{c1} = \frac{1}{4} = 0.25$	–20	0 –20= –20
(1+j2ω)	$\omega_{c2} = \frac{1}{2} = 0.5$	20	–20+20=0
$\frac{1}{(1+j0.25\omega)}$	$\omega_{c3} = \frac{1}{0.25} = 4$	–20	0–20=–20

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$. Let ω_l = 0.1 rad/sec and ω_h = 10 rad/sec.

Let $A = |G(j\omega)|$ in db and let us calculate A at ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h.

At $\omega = \omega_l$; $A = |G(j\omega)| = 20 \log 5 = +14$ db

At $\omega = \omega_{c1}$; $A = |G(j\omega)| = 20 \log 5 = +14$ db

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})}$$

$$= -20 \times \log\frac{0.5}{0.25} + 14 = +8\text{db}$$

$$\text{At } \omega = \omega_{c3},\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})}$$

$$= 0 \times \log\frac{4}{0.5} + 8 = +8\text{db}$$

$$\text{At } \omega = \omega_{h},\ A = \left[\text{slope from } \omega_{c3} \text{to } \omega_{h} \times \log\frac{\omega_{h}}{\omega_{c3}}\right] + A_{(\text{at } \omega=\omega_{c3})}$$

$$= -20 \log \frac{10}{4} + 8 = 0 \text{ db}$$

Let the points a,b,c, and d be the points correponding to frequencies ω_p, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 5 db on y axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales on x-axis. Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

The phase angle of G(jω), $\phi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$

The phase angle of G(jω) are calculated for various values of ω and listed the table2.

TABLE 2

ω	$\tan^{-1} 2\omega$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1} 0.25\omega$ deg	$\phi = \angle G(j\omega)$
0.1	11.3	21.8	1.43	–12
0.25	26.56	45.0	3.5	–22
0.5	45.0	63.43	7.1	–26
2	75.96	82.87	26.56	–33
4	82.87	86.42	45.0	–49
10	87.13	88.56	68.19	–70
50	89.42	89.71	85.42	–86

On the same semilog graph sheet choose a scale of 1unit = 10° on y-axis on the right side of the semilog sheet. Mark the calculated phase angle on the graph sheet join the points by a smooth curve. The magnitude and phase plots are shown in fig 4.6.1.

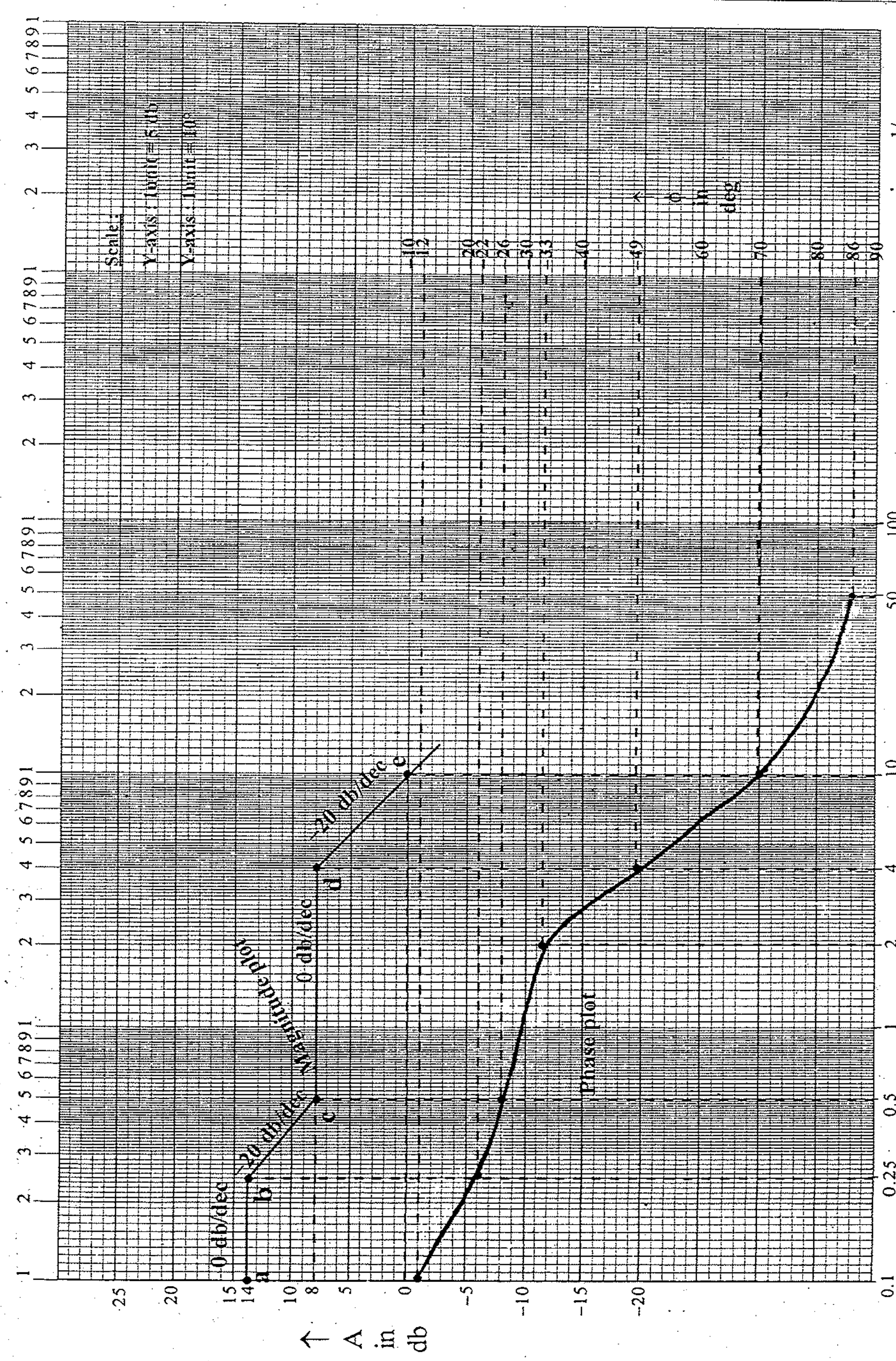


Fig 4.6.1.: Bode plot of transfer function, $G(j\omega) = \dfrac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$

4.7 POLAR PLOT

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity. Thus the polar plot is the locus of vectors |G(jω)| ∠G(jω) as ω is varied from zero to infinity. The polar plot is also called Nyquist plot.

The polar plot is usually plotted on a polar graph sheet. The polar graph sheet has concentric circles and radial lines. The circles represent the magnitude and the radial lines represent the phase angles. Each point on the polar graph has a magnitude and phase angle. The magnitude of a point is given by the value of the circle passing through that point and the phase angle is given by the radial line passing through that point. In polar graph sheet a positive phase angle is measured in anticlockwise from the reference axis (0°) and a negative angle is measured clockwise from the reference axis (0°).

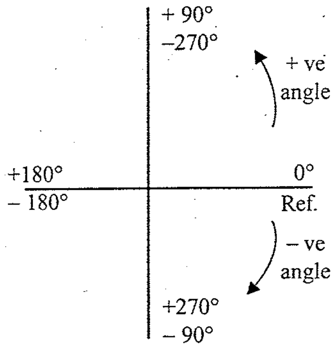


Fig 4.20 : Polar graph

Alternatively, if G(jω) can be expressed in rectangular coordinates as,

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega)$$

where, $G_R(j\omega)$ = Real part of G(jω)

and $G_I(j\omega)$ = Imaginary part of G(jω),

then the polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ as ω is varied from 0 to ∞.

To plot the polar plot, first compute the magnitude and phase of G(jω) for various values of ω and tabulate them. Usually the choice of frequencies are corner frequencies and frequencies around corner frequencies. Choose proper scale for the magnitude circles. Fix all the points on polar graph sheet and join the points by smooth curve. Write the frequency corresponding to each point of the plot.

To plot the polar plot on ordinary graph sheet, compute the magnitude and phase for various values of ω. Then convert the polar coordinates to rectangular coordinates using **P→R** conversion (polar to rectangular conversion) in the calculator. Sketch the polar plot using rectangular coordinates.

For minimum phase transfer function with only poles, the type number of the system determines at what quadrant the polar plot starts and the order of the system determines at what quadrant the polar plot ends.

(Note : The minimum phase systems are systems with all poles and zeros on the left half of s-plane)

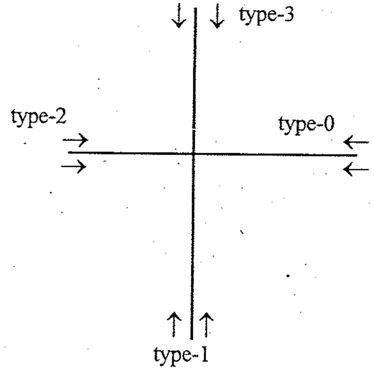


Fig 4.21 : Start of polar plot

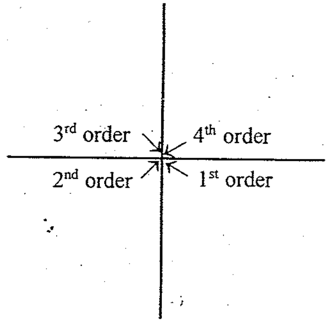


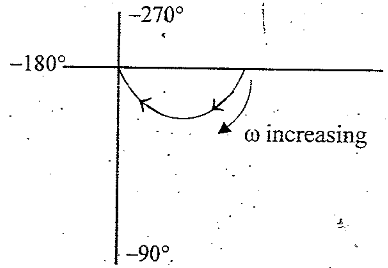
Fig 4.22 : End of polar plot

TYPICAL SKETCHES OF POLAR PLOT

Type : 0, Order : 1

$$G(s) = \frac{1}{1+sT}$$

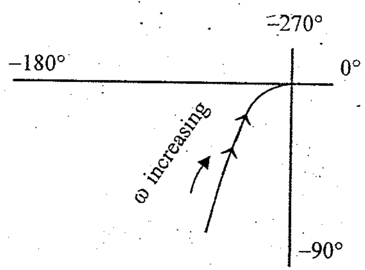
$$G(j\omega) = \frac{1}{1+j\omega T}$$



Type : 1, Order : 2

$$G(s) = \frac{1}{s(1+sT)}$$

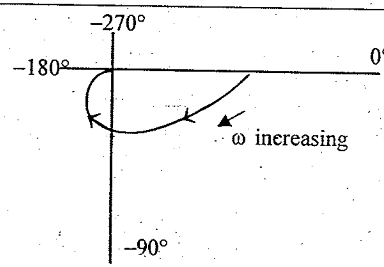
$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$



Type : 0, Order : 2

$$G(s) = \frac{1}{(1+sT_1)\ (1+sT_2)}$$

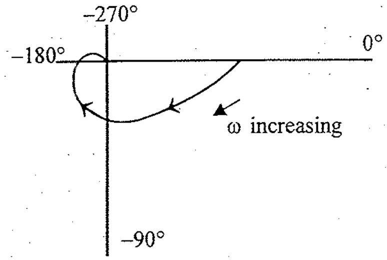
$$G(j\omega) = \frac{1}{(1+j\omega T_1)\ (1+j\omega T_2)}$$



Type : 0, Order : 3

$$G(s) = \frac{1}{(1+sT_1)\,(1+sT_2)\,(1+sT_3)}$$

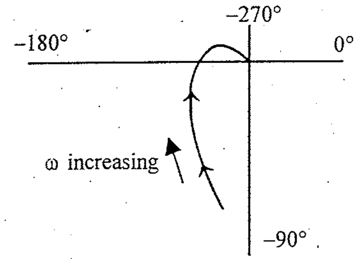
$$G(j\omega) = \frac{1}{(1+j\omega T_1)\,(1+j\omega T_2)\,(1+j\omega T_3)}$$



Type : 1, Order : 3

$$G(s) = \frac{1}{s(1+sT_1)\,(1+sT_2)}$$

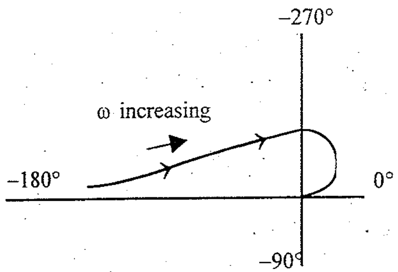
$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)\,(1+j\omega T_2)}$$



Type : 2, Order : 4

$$G(s) = \frac{1}{s^2(1+sT_1)\,(1+sT_2)}$$

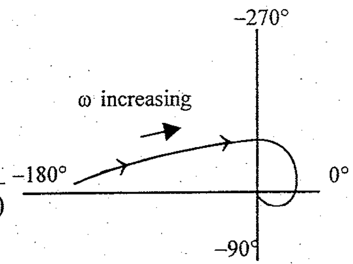
$$G(j\omega) = \frac{1}{(j\omega)^2\,(1+j\omega T_1)\,(1+j\omega T_2)}$$



Type : 2, Order : 5

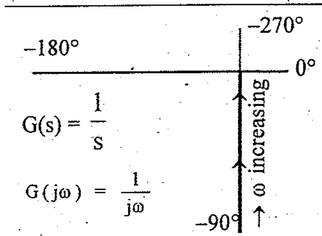
$$G(s) = \frac{1}{s^2(1+sT_1)\,(1+sT_2)\,(1+sT_3)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2\,(1+j\omega T_1)\,(1+j\omega T_2)\,(1+j\omega T_3)}$$



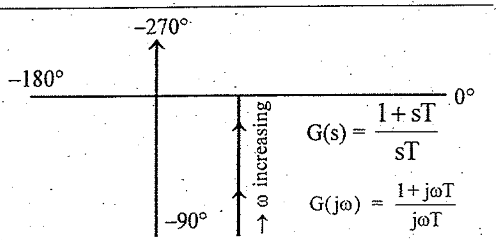
$$G(s) = \frac{1}{s}$$

$$G(j\omega) = \frac{1}{j\omega}$$



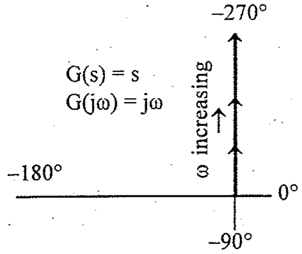
$$G(s) = \frac{1+sT}{sT}$$

$$G(j\omega) = \frac{1+j\omega T}{j\omega T}$$



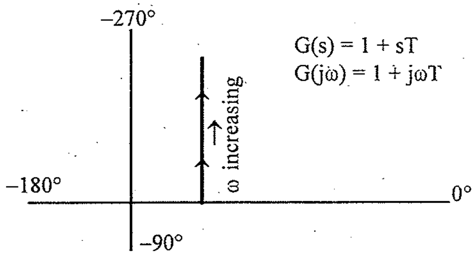
$G(s) = s$

$G(j\omega) = j\omega$



$G(s) = 1 + sT$

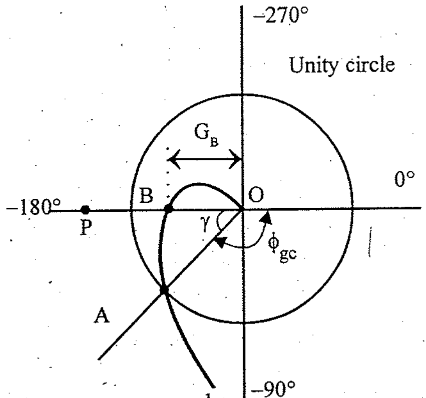
$G(j\omega) = 1 + j\omega T$



DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT

The **gain margin** is defined as the inverse of the magnitude of $G(j\omega)$ at phase crossover frequency. The **phase crossover frequency** is the frequency at which the phase of $G(j\omega)$ is 180°.

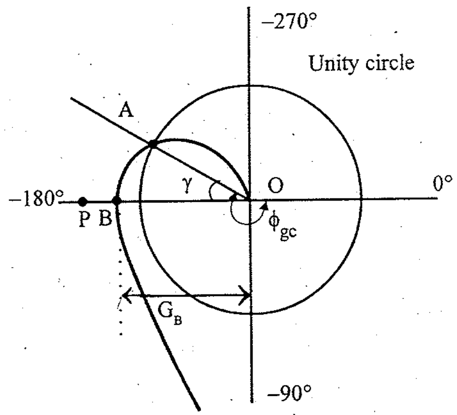
Let the polar plot cut the 180° axis at point **B** and the magnitude circle passing through the point B be G_B. Now the Gain margin, $K_g = 1/G_B$. If the point B lies within unity circle then the Gain margin is positive otherwise negative. (If the polar plot is drawn in ordinary graph sheet using rectangular coordinates then the point B is the cutting point of $G(j\omega)$ locus with negative real axis and $K_g = 1/|G_B|$ where G_B is the magnitude corresponding to point B).



Gain margin, $K_g = \frac{1}{G_B}$

Phase margin, $\gamma = 180° + \phi_{gc}$

Fig 4.23 : Polar plot showing positive gain margin and phase margin



Gain margin, $K_g = 1/G_B$

Phase margin, $\gamma = 180 + \phi_{gc}$

Fig 4.24 : Polar plot showing negative gain margin and phase margin.

The **phase margin** is defined as, phase margin, $\gamma = 180 + \phi_{gc}$ where ϕ_{gc} is the phase angle of G(jω) at gain crossover frequency. The **gain crossover frequency** is the frequency at which the magnitude of G(jω) is unity.

Let the polar plot cut the unity circle at point A as shown in fig (4.23) and (4.24). Now the phase margin, γ is given by ∠AOP, i.e. if ∠AOP is below −180° axis then the phase margin is positive and if it is above −180° axis then the phase margin is negative.

GAIN ADJUSTMENT USING POLAR PLOT

TO DETERMINE K FOR SPECIFIED GM

Draw G(jω) locus with K = 1. Let it cut the −180° axis at point B corresponding to a gain of G_B. Let the specified gain margin be x db. For this gain margin, the G(jω) locus will cut −180° at point A whose magnitude is G_A.

$$\text{Now, } 20\log\frac{1}{G_A} = x$$

$$\log\frac{1}{G_A} = \frac{x}{20}$$

$$\frac{1}{G_A} = 10^{x/20}$$

$$\therefore G_A = \frac{1}{10^{x/20}}$$

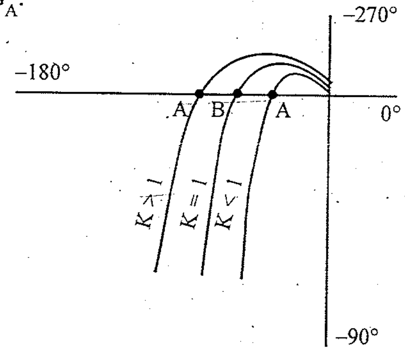


Fig 4.25 : Polar plot for different values of K

Now the value of K is given by, $K = \frac{G_A}{G_B}$.

If, K > 1, then the system gain should be increased.

If, K < 1, then the system gain should be reduced.

TO DETERMINE K FOR SPECIFIED PM

Draw G(jω) locus with K = 1. Let it cut the unity circle at point B. (The gain at point B is G_B and equal to unity). Let the specified phase margin be x°.

For a phase margin of x°, let ϕ_{gcx} be the phase angle of G(jω) at gain crossover frequency.

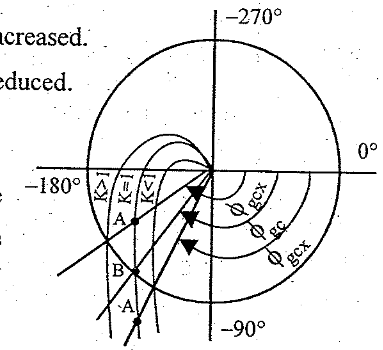


Fig 4.26 : Gain adjustment for required phase margin

$$\therefore x^\circ = 180^\circ + \phi_{gcx} \quad \text{and} \quad \phi_{gcx} = x^\circ - 180^\circ$$

In the polar plot, the radial line corresponding to ϕ_{gcx} will cut the locus of G(jω) with K = 1 at point A and the magnitude corresponding to that point be G_A

$$\text{Now, } K = \frac{G_B}{G_A} = \frac{1}{G_A} \qquad (\because G_B = 1)$$

EXAMPLE 4.7

The open loop transfer function of a unity feedback system is given by G(s) = 1/s(1+s) (1+2s). Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, G(s) = 1/s(1+s) (1 +2s)

Put s = jω

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j\omega)\,(1+j2\omega)}$$

The corner frequencies are $\omega_{c1} = 1/2 = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec. The magnitude and phase angle of G(jω) are calculated for the corner frequencies and for frequencies around corner frequencies and tabulated in table 1. Using polar to rectangular conversion, the polar coordinates listed in table 1 are converted to rectangular coordinates and tabulated in table 2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.7.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.7.2.

$$G(j\omega) = \frac{1}{(j\omega)\,(1+j\omega)\,(1+j2\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}}\angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+4\omega^2+\omega^2+4\omega^4}}$$

$$= \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

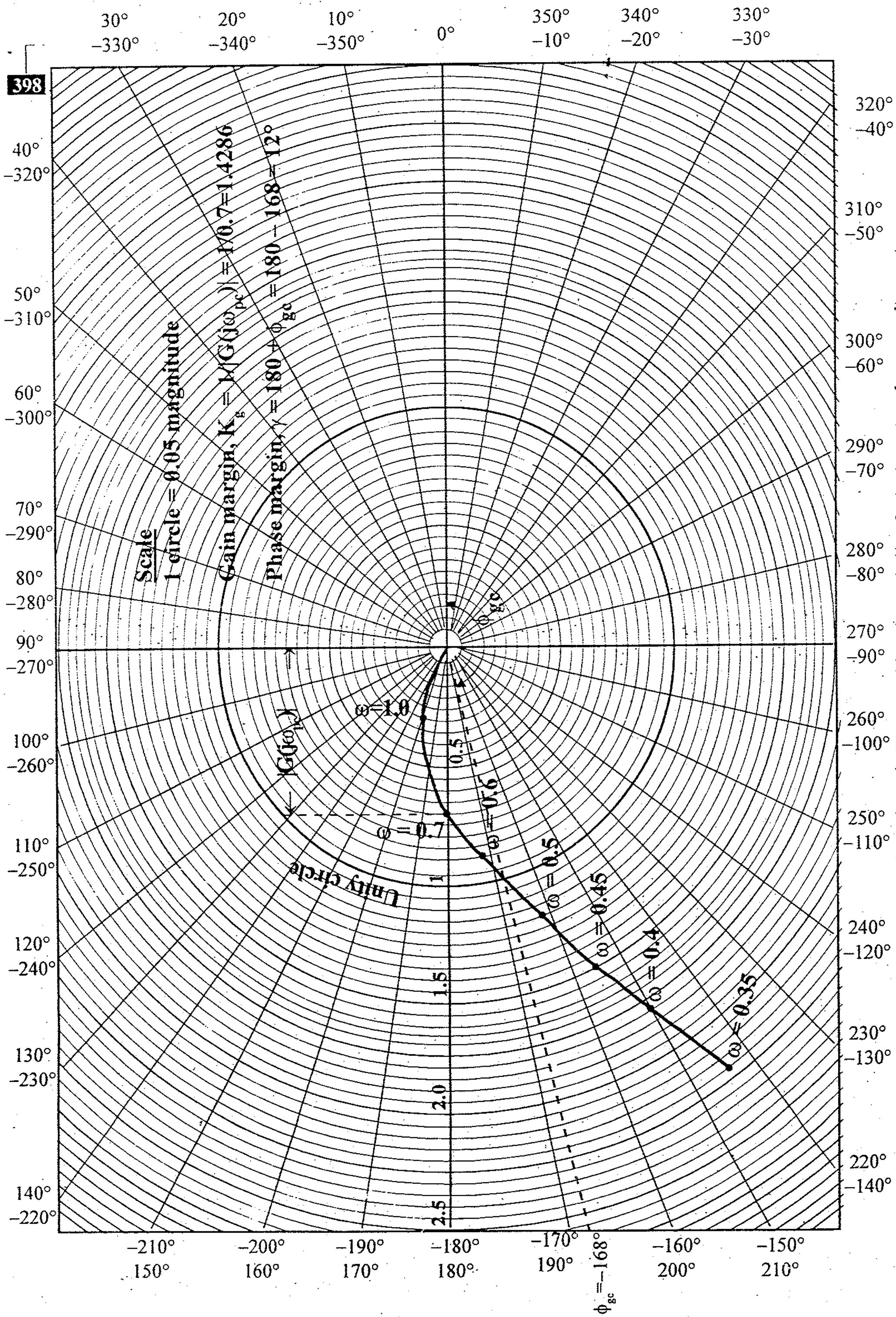


Fig 4.7.1: Polar plot of $G(j\omega) = 1/j\omega\,(1+j\omega)\,(1+j2\omega)$ (using polar coordinates)

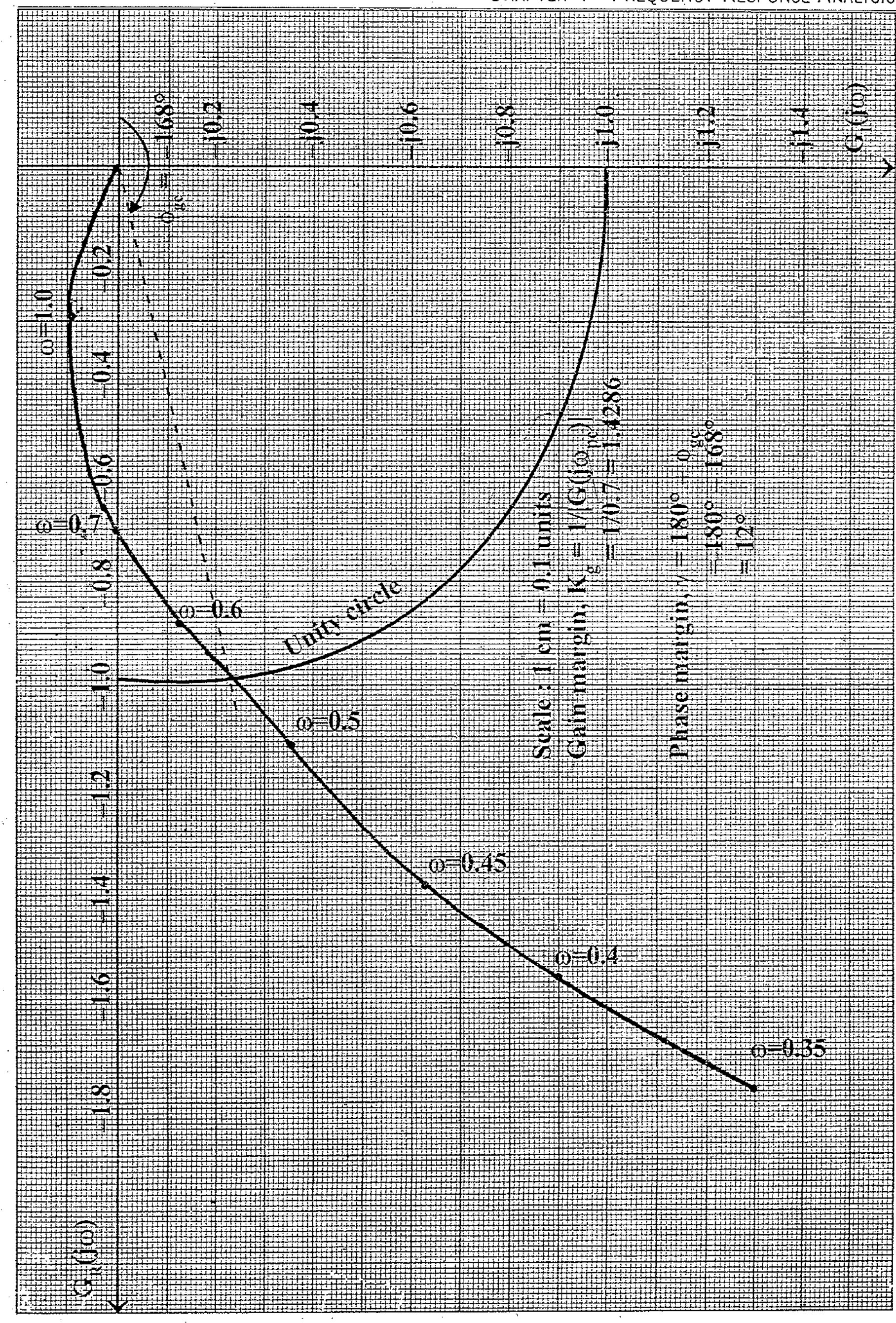


Fig 4.7.2: Polar plot of $G(j\omega) = 1/j\omega\,(1+j\omega)\,(1+j2\omega)$ (using rectangular coordinates)

Table 1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$ deg	−144	−150	−156	−162	−171	−179.5 ≈−180	−198

Table 2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	−1.78	−1.56	−1.37	−1.14	−0.89	−0.7	−0.29
$G_I(j\omega)$	−1.29	−0.9	−0.61	−0.37	−0.14	0	0.09

RESULT

Gain margin, $K_g = 1.4286$

Phase margin, $\gamma = +12^\circ$

EXAMPLE 4.8

The open loop transfer function of a unity feedback system is given by $G(s) = 1/\,s^2(1+s)\,(1+2s)$. Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, $G(s) = 1/\,s^2(1+s)\,(1+2s)$

Put $s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2\,(1+j\omega)\,(1+j2\omega)}$$

The corner frequencies are $\omega_{c1} = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec. The magnitude and phase angle of G(jω) are calculated for the corner frequencies and frequencies around corner frequencies are tabulated in table 1. Using the polar to rectangular conversion, the polar coordinates listed in table 1 are converted to rectangular coordinates and tabulated in table 2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.8.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.8.2.

$$G(j\omega) = \frac{1}{(j\omega)^2\,(1+j\omega)\,(1+j2\omega)}$$

$$= \frac{1}{\omega^2\angle 180^\circ\,\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega\,\sqrt{1+4\omega^2}\,\angle\tan^{-1}2\omega}$$

$$G(j\omega) = \frac{1}{\omega^2\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}}\,\angle(-180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega)$$

$$\therefore\ \lvert G(j\omega)\rvert = \frac{1}{\omega^2\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}} = \frac{1}{\omega^2\sqrt{(1+\omega^2)\,(1+4\omega^2)}}$$

$$= \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

Table 1 : Magnitude and phase plot of G(jω) at various frequencies

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$\lvert G(j\omega)\rvert$	3.3	2.5	1.9	1.5	1.2	0.97 ≈ 1	0.8	0.3
$\angle G(j\omega)$ deg	−246	−251	−256	−261	−265	−269	−273	−288

Table 2 : Real and imaginary part of G(jω)

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$G_R(j\omega)$	−1.34	−0.81	−0.46	−0.23	−0.1	−0.02	0.04	0.09
$G_I(j\omega)$	3.01	2.36	1.84	1.48	1.2	1.0	0.8	0.29

RESULT

Gain margin, $K_g = 0$

Phase margin, $\gamma = -90^\circ$

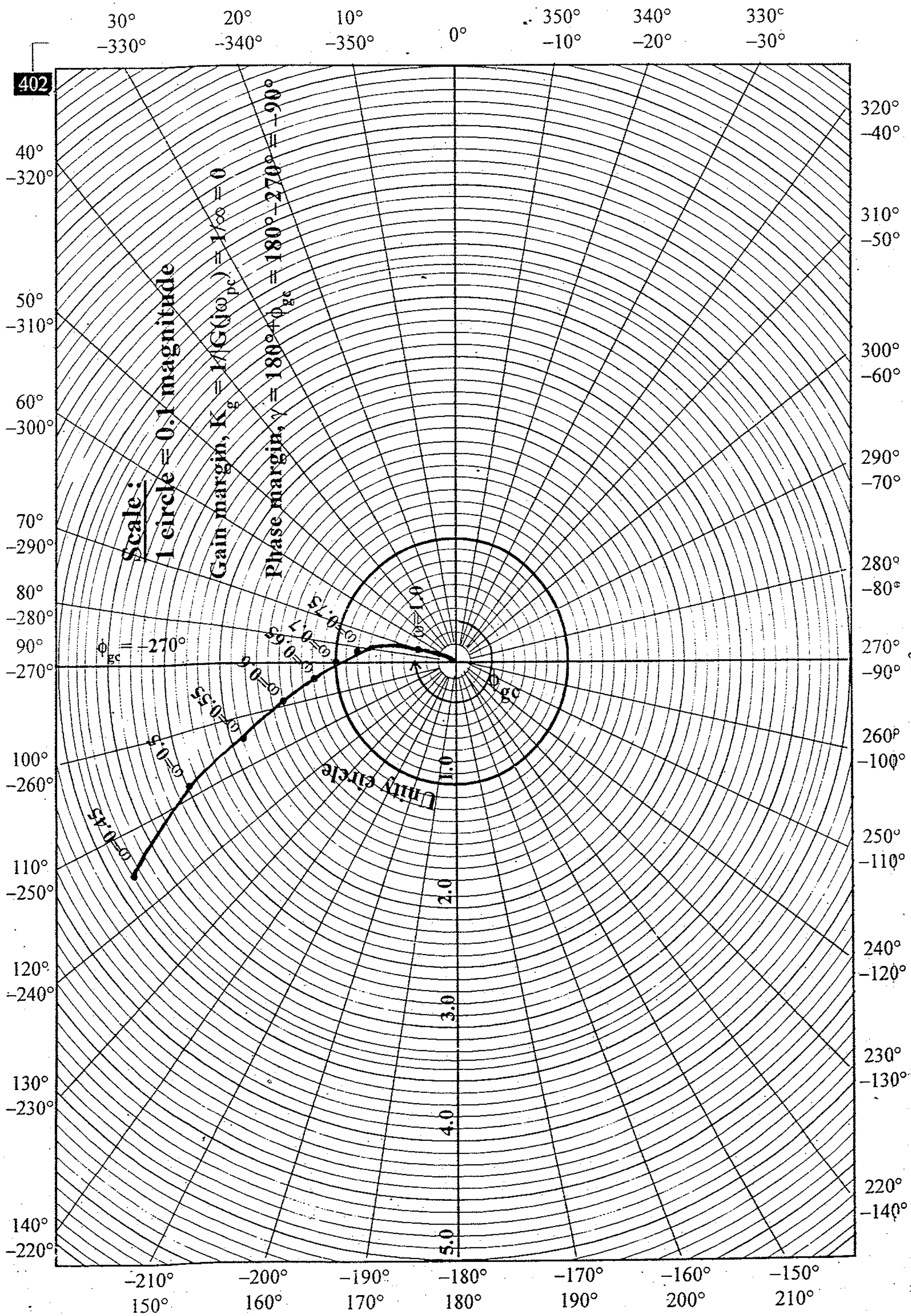


Fig 4.8.1: Polar plot of $G(j\omega) = 1/(j\omega)^2\,(1+j\omega)\,(1+j2\omega)$, *(using polar coordinates)*

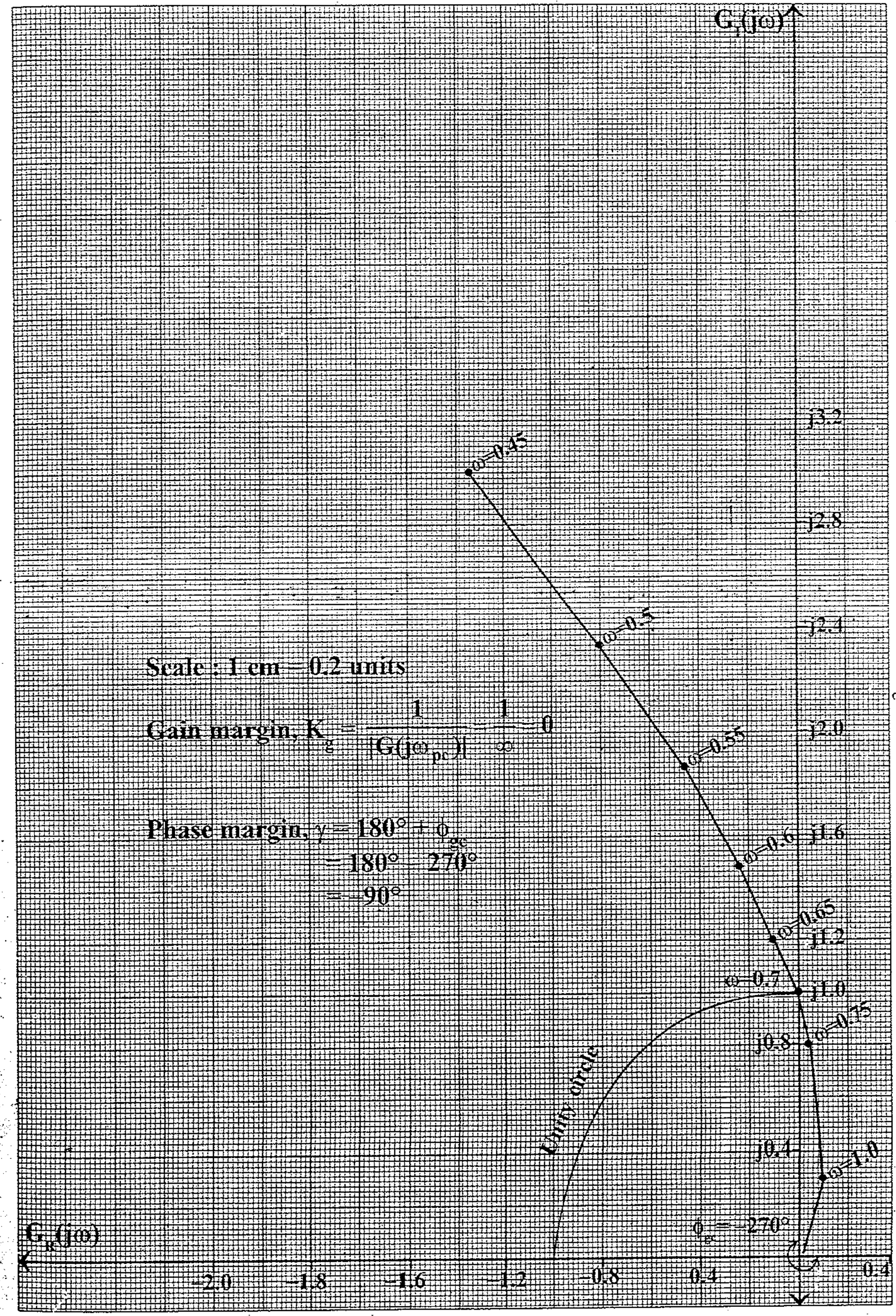


Fig 4.8.2: Polar plot of $G(j\omega) = 1/(j\omega)^2\,(1+j\omega)\,(1+j2\omega)$, *(using rectangular coordinates)*

EXAMPLE 4.9

The open loop transfer function of a unity feedback system is given by

$G(s) = \dfrac{(1+0.2s)\,(1+0.025s)}{s^3(1+0.005s)\,(1+0.001s)}$. Sketch the polar plot and determine the phase margin.

SOLUTION

$$\text{Given that } G(s) = \frac{(1+0.2s)\,(1+0.025s)}{s^3(1+0.005s)\,(1+0.001s)}$$

$$\therefore G(j\omega) = \frac{(1+j0.2\omega)\,(1+j0.025\omega)}{(j\omega)^3\,(1+j0.005\omega)\,(1+j0.001\omega)}$$

$$= \frac{\sqrt{1+(0.2\omega)^2}\;\angle\tan^{-1}0.2\omega\;\sqrt{1+(0.025\omega)^2}\;\angle\tan^{-1}0.025\omega}{\omega^3\;\angle 270^\circ\;\sqrt{1+(0.005\omega)^2}\;\angle\tan^{-1}0.005\omega\;\sqrt{1+(0.001\omega)^2}\;\angle\tan^{-1}0.001\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+(0.2\omega)^2}\;\sqrt{1+(0.025\omega)^2}}{\omega^3\;\sqrt{1+(0.005\omega)^2}\;\sqrt{1+(0.001\omega)^2}}$$

$$\angle G(j\omega) = \tan^{-1}0.2\omega + \tan^{-1}0.025\omega - 270^\circ - \tan^{-1}0.005\omega - \tan^{-1}0.001\omega$$

The magnitude and phase angle of $G(j\omega)$ are calculated for various frequencies and listed in table 1. Using the polar to rectangular conversion, the polar coordinates listed in table 1 are converted to rectangular coordinates and tabulated in table 2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.9.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.9.2.

Table 1 : Magnitude and phase of $G(j\omega)$

ω,rad/sec	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$\lvert G(j\omega)\rvert$	1.4	1.2	1.0	0.8	0.6	0.4	0.2
$\angle G(j\omega)$, deg	−259	−258	−257	−256	−255	−253	−249

Table 2 : Real and imaginary part of $G(j\omega)$

ω, rad/sec	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$G_R(j\omega)$	−0.27	−0.25	−0.22	−0.19	−0.16	−0.12	−0.07
$G_I(j\omega)$	1.37	1.17	0.97	0.78	0.58	0.38	0.19

RESULT

Phase margin, $\gamma = -77^\circ$

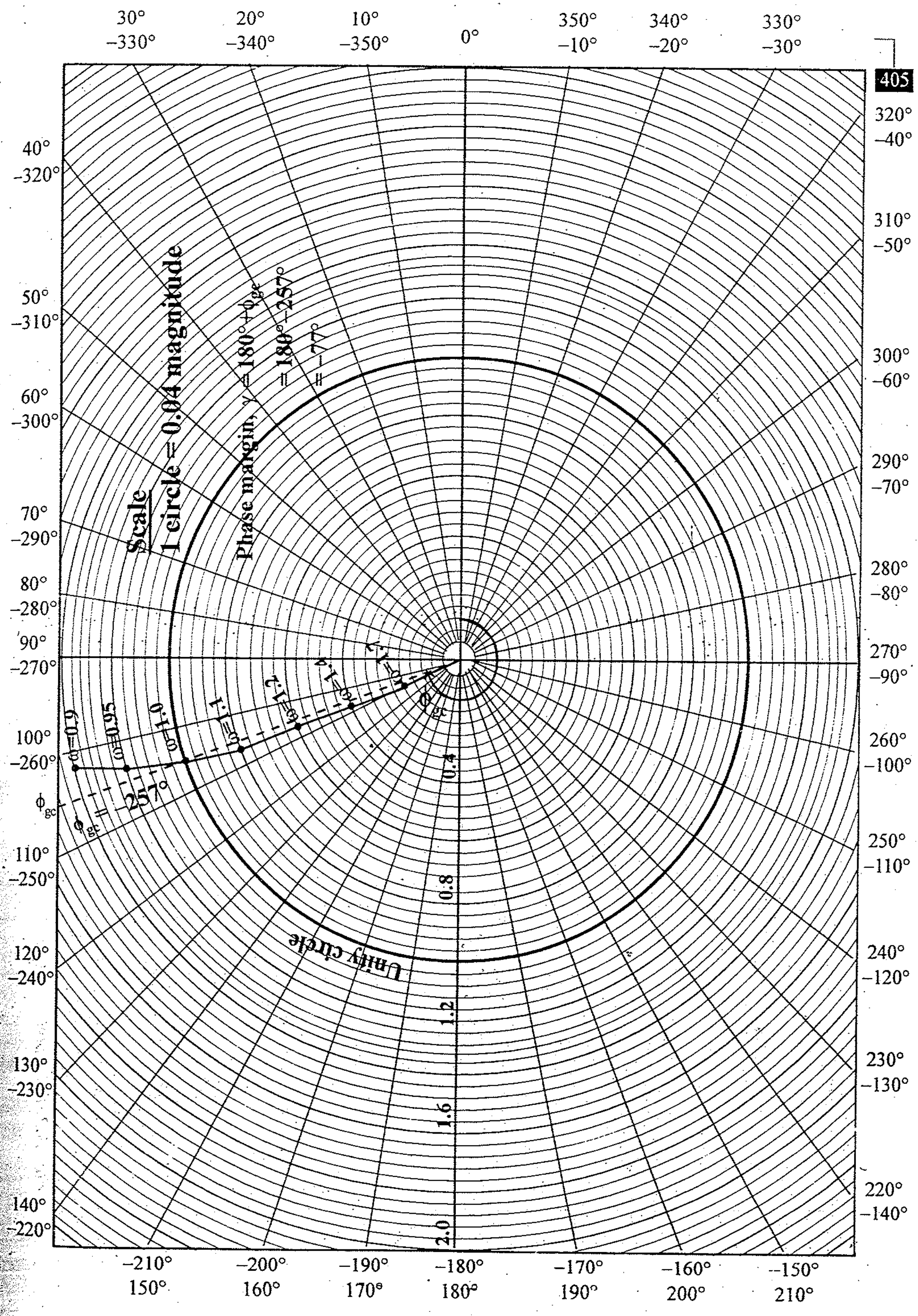


Fig 4.9.1: Polar plot of $G(j\omega) = (1+j0.2\omega)\,(1+j0.025\omega) / (j\omega)^3\,(1+j0.005\omega)\,(1+j0.001\omega)$, *(using polar coordinates)*

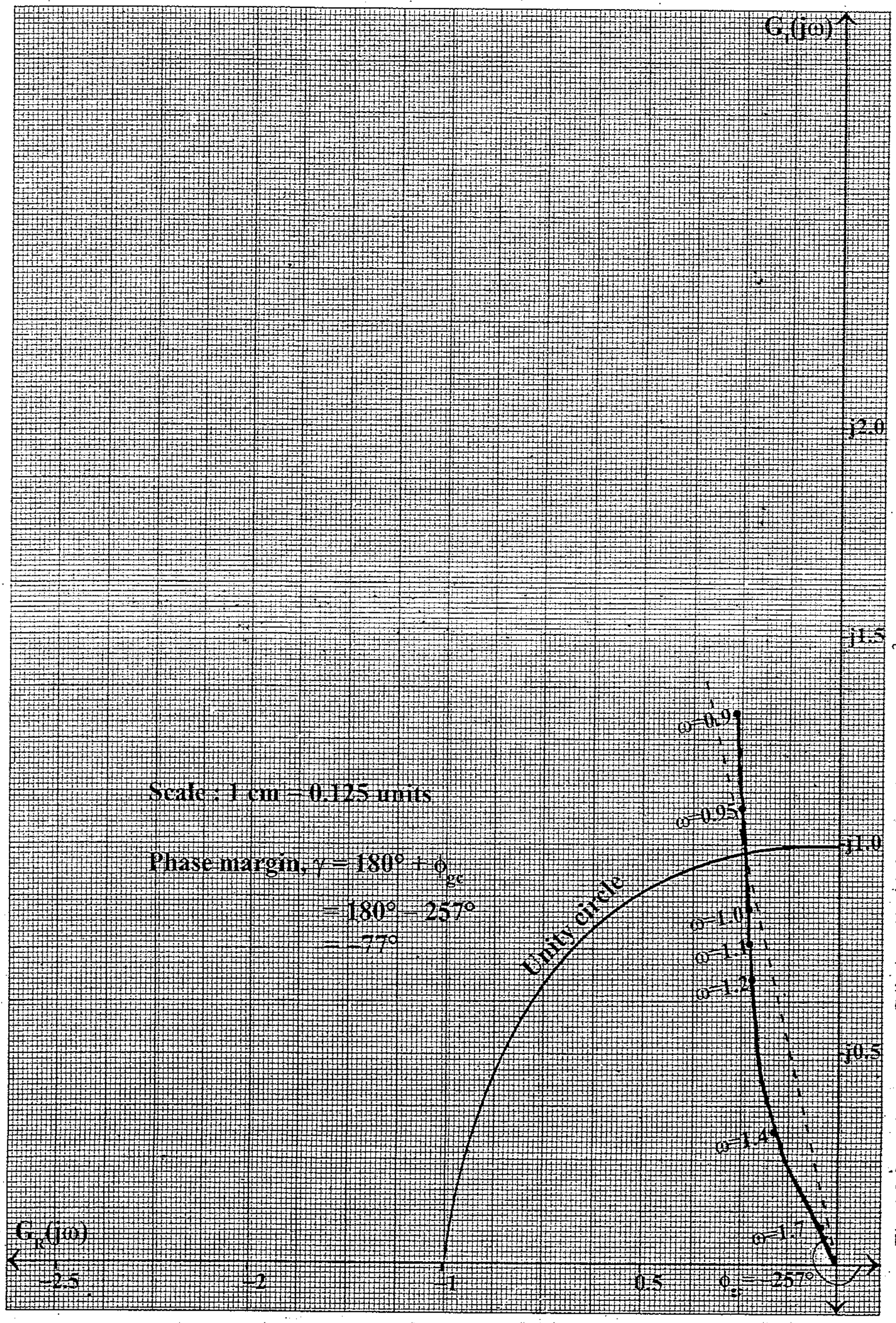


Fig 4.9.2: Polar plot of $G(j\omega) = (1+j0.2\omega)(1+j0.025\omega)/(j\omega)^3(1+j0.005\omega)(1+j0.001\omega)$, *(using rectangular coordinates)*

EXAMPLE 4.10

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+s)^2$. Sketch the polar plot and determine the gain and phase margin.

SOLUTION

Given that, $G(s) = 1/s(1+s)^2$.

Put $s = j\omega$

$$\therefore G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

The corner frequency is $\omega_{c1} = 1$ rad/sec. The magnitude and phase angle of $G(j\omega)$ are calculated for corner frequency and frequencies around corner frequency and tabulated in table 1. Using polar to rectangular conversion the polar coordinates listed in table 1 are converted to rectangular coordinates and tabulated in table 2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.10.1. The polar plot using rectangular coordinates are sketched on an ordinary graph sheet as shown in fig 4.10.2.

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

$$= \frac{1}{\omega\angle 90° \;\sqrt{1+\omega^2}\;\angle\tan^{-1}\omega\;\sqrt{1+\omega^2}\;\angle\tan^{-1}\omega}$$

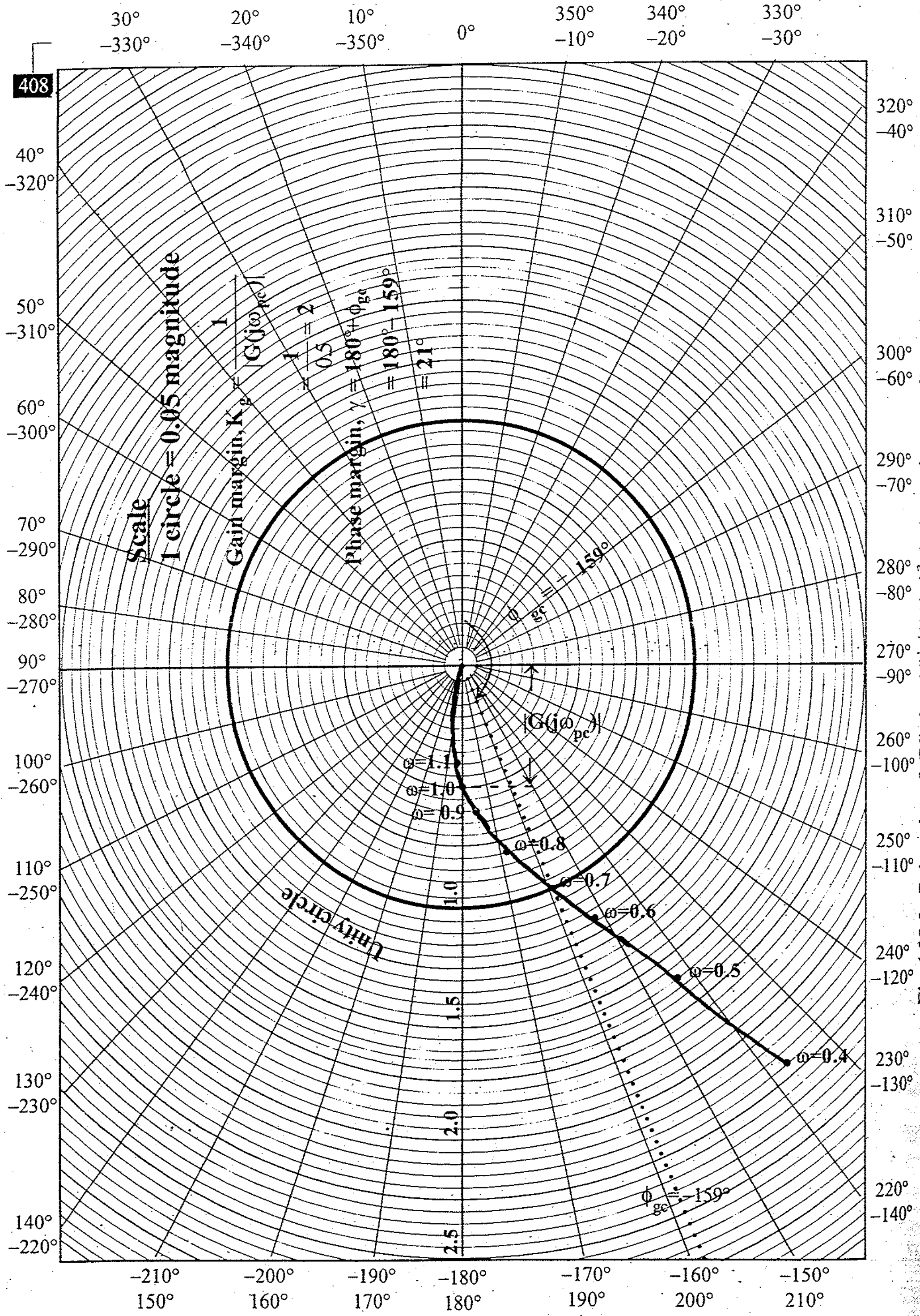
$$= \frac{1}{\omega\left(\sqrt{1+\omega^2}\right)^2} \quad \angle(-90 - 2\tan^{-1}\omega)$$

$$|G(j\omega)| = \frac{1}{\omega(1+\omega^2)} = \frac{1}{\omega+\omega^3}$$

$$\angle G(j\omega) = -90 - 2\tan^{-1}\omega$$

Table 1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
|G(jω)|	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
∠G(jω) deg	−134	−143	−151	−159	−167	−174	−180	−185



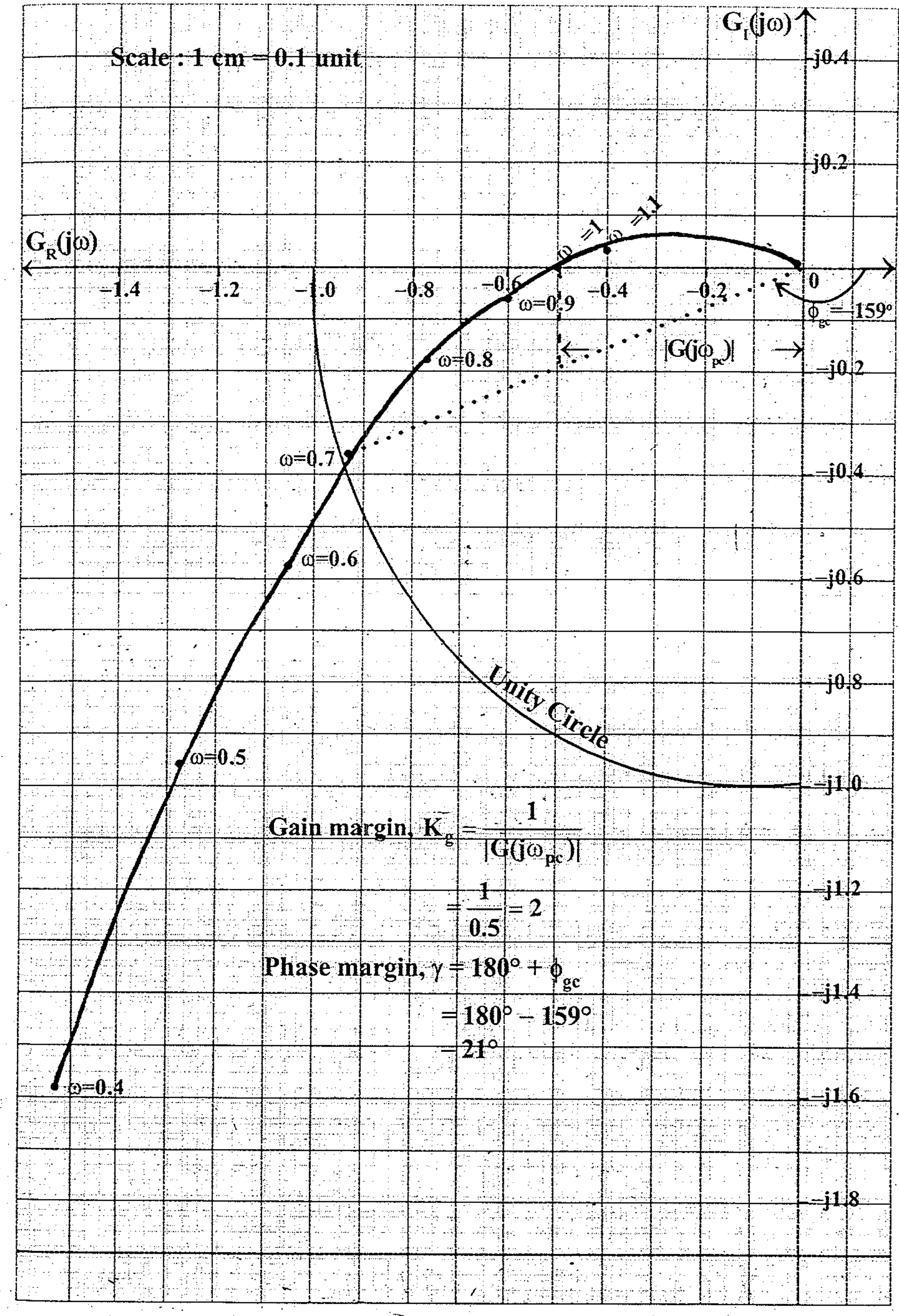


Fig 4.10.2: *Polar plot of* $G(j\omega) = 1/j\omega\,(1+j\omega)^2$ *(using rectangular coordinates)*

Table 2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$G_R(j\omega)$	−1.53	−1.28	−1.05	−0.93	−0.78	−0.6	−0.5	−0.4
$G_I(j\omega)$	−1.58	−0.96	−0.58	−0.36	−0.18	−0.06	0	0.03

RESULT

Gain margin, $K_g = 2$

Phase margin, $\gamma = 21°$

EXAMPLE 4.11

Consider a unity feedback system having an open loop transfer function $G(s) = \dfrac{K}{s\,(1+0.2s)\,(1+0.05s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 18 db (ii) Phase margin is 60°.

SOLUTION

Given that, $G(s) = \dfrac{K}{s\,(1+0.2s)\,(1+0.05s)}$

The polar plot is sketched by taking K = 1.

∴ Put K = 1 and s = jω in G(s)

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)}$$

The corner frequencies are $\omega_{c1} = 1/0.2 = 5$ rad/sec and $\omega_{c2} = 1/0.05 = 20$ rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table 1. Using polar to rectangular conversion the polar coordinates listed in table 1 are converted to rectangular coordinates and tabulated in table 2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.11.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.11.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)}$$

$$= \frac{1}{\omega\angle 90°\ \sqrt{1+(0.2\omega)^2}\ \angle\tan^{-1}0.2\omega\ \sqrt{1+(0.05\omega)^2}\ \angle\tan^{-1}0.05\omega}$$

$$= \frac{1}{\omega\ \sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.05\omega)^2}}\ \angle(-90° - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\ \sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.05\omega)^2}}$$

$$\angle G(j\omega) = -90° - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega$$

Table 1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.6	0.8	1	2	3	4
$\lvert G(j\omega)\rvert$	1.65	1.23	1.0	0.5	0.3	0.2
∠G(jω) deg	−98	−101	−104	−117.5	−129.4	−140

ω rad/sec	5	6	7	9	10	11	14
$\lvert G(j\omega)\rvert$	0.14	0.1	0.07	0.05	0.04	0.03	0.02
∠G(jω) deg	−149	−157	−164	−175	−180	−184	−195

Table 2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.6	0.8	1	2	3	4
$G_R(j\omega)$	−0.23	−0.23	−0.24	−0.23	−0.19	−0.15
$G_I(j\omega)$	−1.63	−1.21	−0.97	−0.44	−0.23	−0.13

ω rad/sec	5	6	7	9	10	11	14
$G_R(j\omega)$	−0.120	−0.092	−0.067	−0.050	−0.04	−0.030	−0.019
$G_I(j\omega)$	−0.072	−0.039	−0.019	−0.0044	0	0.002	0.005

In the polar plot shown in fig 4.11.1 and 4.11.2 there are two curves I and II. These two loci are sketched with different scales to clearly determine the gain margin and phase margin.

From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.04 = 25$

Gain margin in db = 20 log 25 = 28 db

Phase margin, $\gamma = 76°$

Case (i)

With K = 1, let G(jω) cut the –180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.04$. The gain margin of 28 db with K = 1 has to be reduced to 18 db and so K has to be increased to a value greater than one.

Let G_A be the gain at –180° for a gain margin of 18 db.

$$\text{Now, } 20 \log \frac{1}{G_A} = 18$$

$$\log \frac{1}{G_A} = \frac{18}{20}$$

$$\frac{1}{G_A} = 10^{18/20} \qquad \therefore G_A = \frac{1}{10^{18/20}} = 0.125$$

$\therefore$ The value of K is given by $K = \dfrac{G_A}{G_B} = \dfrac{0.125}{0.04} = 3.125$

Case (ii)

With K = 1, the phase margin is 76°. This has to be reduced to 60°. Hence gain has to be increased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 60°

$$\therefore 60° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 60° - 180° = -120°$$

In the polar plot the –120° line cut the locus of G(jω) at point A and cut the unity circle at point B.

Let G_A = Magnitude of G(jω) at point A.

G_B = Magnitude of G(jω) at point B.

From the polar plot, $G_A = 0.425$ and $G_B = 1$.

$$\text{Now, } K = \frac{G_B}{G_A} = \frac{1}{0.425} = 2.353$$

RESULT

(i) When K = 1, Gain margin, $K_g = 25$
Gain margin in db = 28db

(ii) When K = 1, Phase margin, γ = 76°

(iii) For a gain margin of 18 db, K = 3.125

(iv) For a phase margin of 60°, K = 2.353

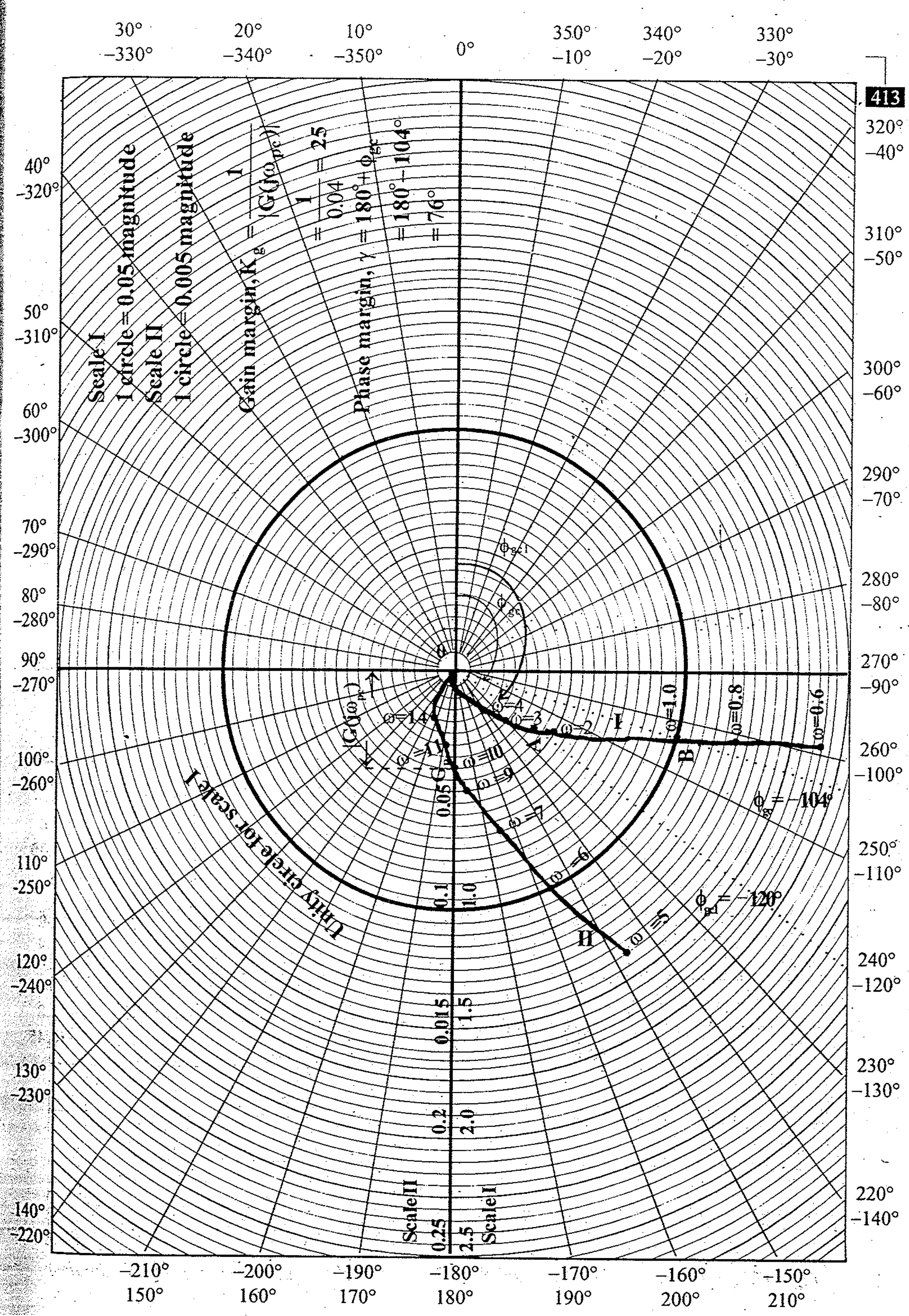


Fig 4.11.1: Polar plot of $G(j\omega) = 1/j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)$, *(using polar coordinates)*

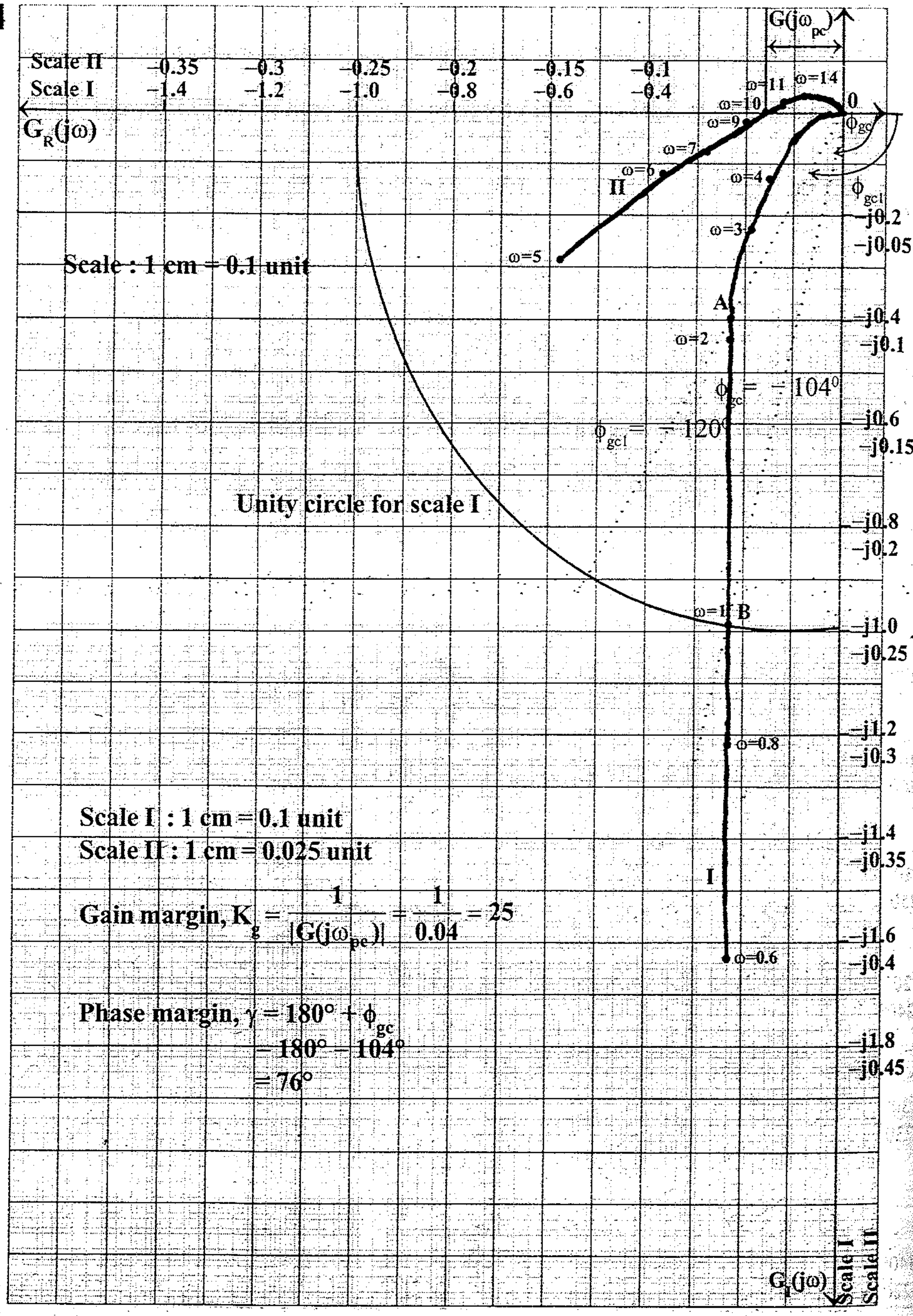


Fig 4.11.2: Polar plot of $G(j\omega) = 1/j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)$, *(using rectangular coordinates)*

EXAMPLE 4.12

Consider a unity feedback system having an open loop transfer function $G(s) = \dfrac{K}{s\,(1+0.5s)\,(1+4s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 20 db and (ii) Phase margin is 30°.

SOLUTION

Given that, $G(s) = K/s\,(1+0.5s)\,(1+4s)$

The polar plot is sketched by taking K = 1

Put K = 1 and s = jω in G(s)

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

The corner frequencies are $\omega_{c1} = 1/4 = 0.25$ rad/sec and $\omega_{c2} = 1/0.5 = 2$ rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table 1. Using polar to rectangular conversion the polar coordinates listed in table 1 are converted to rectangular coordinates and tabulated in table 2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.12.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.12.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+(0.5\omega)^2}\ \angle\tan^{-1}0.5\omega\ \sqrt{1+(4\omega)^2}\ \angle\tan^{-1}4\omega}$$

$$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}\ \angle(-90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\ \sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}0.5\omega - \tan^{-1}4\omega.$$

Table 1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
|G(jω)|	2.11	1.3	0.87	0.61	0.35	0.22	0.15
∠G(jω) deg	−149	−159	−167	−174	−184	−193	−199

Table 2 : Real part and imaginary part of G(jω) at various frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$G_R(j\omega)$	–1.8	–1.21	–0.85	–0.61	–0.35	–0.21	–0.14
$G_I(j\omega)$	–1.09	–0.47	–0.2	–0.06	0.02	0.05	0.05

From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.44 = 2.27$

Gain margin in db = 20 log 2.27 = 7.12 db.

Phase margin, $\gamma = 15°$.

Case (i)

With K = 1, let G(jω) cut the –180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.44$. The gain margin of 7.12 db with K = 1 has to be increased to 20 db and so K has to be decreased to a value less than one.

Let G_A be the gain at –180° for a gain margin of 20 db.

$$\text{Now, } 20\log\frac{1}{G_A} = 20$$

$$\log\frac{1}{G_A} = \frac{20}{20} = 1$$

$$\frac{1}{G_A} = 10^1 = 10$$

$$\therefore G_A = \frac{1}{10} = 0.1$$

$$\text{The value of K is given by } K = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$$

Case (ii)

With K = 1, the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 30°

$$\therefore 30° = 180 + \phi_{gc2}$$

$$\phi_{gc2} = 30° - 180° = -150°$$

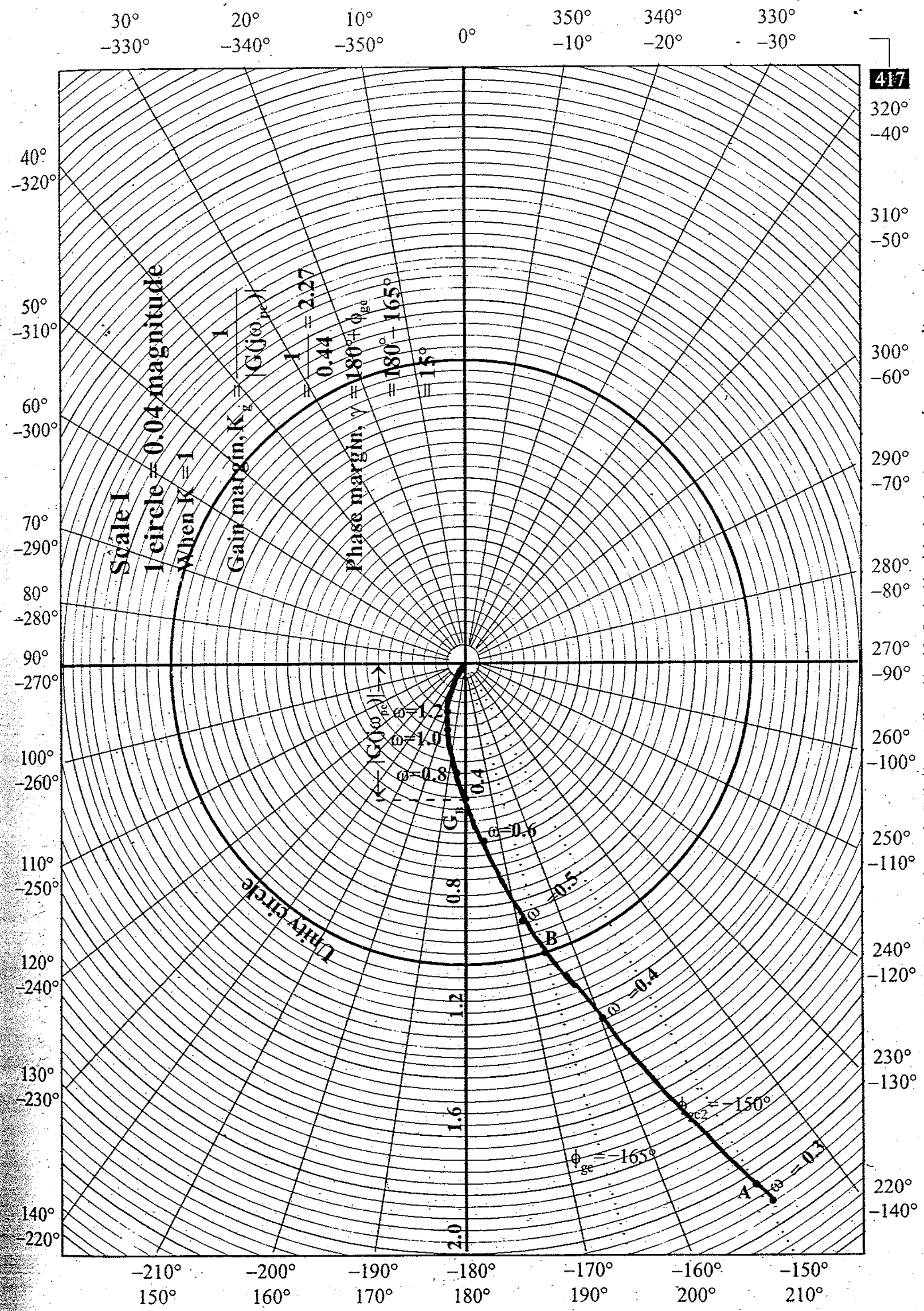


Fig 4.12.1: Polar plot of G(jω) = 1/jω (1+j0.5ω) (1+j4ω), (using polar coordinates)

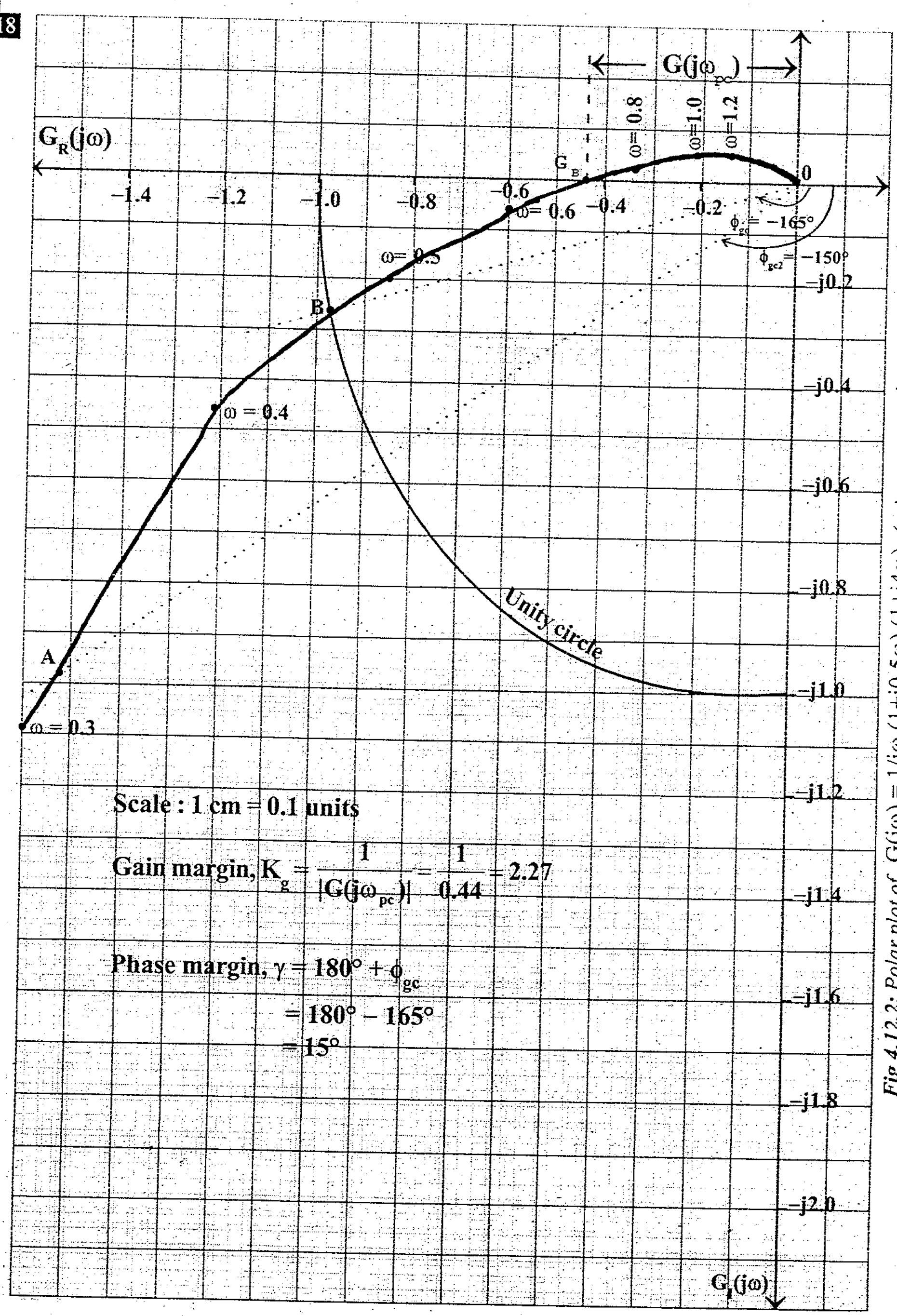


Fig 4.12.2: Polar plot of $G(j\omega) = 1/j\omega\,(1+j0.5\omega)\,(1+j4\omega)$, *(using rectangular coordinates)*

In the polar plot the −150° line cuts the locus of G(jω) at point A and cut the unity circle at point B.

Let G_A = Magnitude of G(jω) at point A.

G_B = Magnitude of G(jω) at point B.

From the polar plot $G_A = 2.04$ and $G_B = 1$

$$\text{Now, } K = \frac{G_B}{G_A} = \frac{1}{2.04} = 0.49$$

RESULT

(i) When K = 1, Gain margin, $K_g = 2.27$

Gain margin in db = 7.12 db

(ii) When K = 1, Phase margin, γ = 15°

(iii) For a gain margin of 20 db, K = 0.227

(iv) For a phase margin of 30°, K = 0.49

4.8 NICHOLS PLOT

The **Nichols plot** is a frequency response plot of the open loop transfer function of a system. The Nichols plot is a graph between magnitude of G(jω) in db and the phase of G(jω) in degree, plotted on a ordinary graph sheet.

To plot the Nichols plot, first compute the magnitude of G(jω) in db and phase of G(jω) in deg for various values of ω and tabulate them. Usually the choice of frequencies are corner frequencies. Choose appropriate scales for magnitude on y–axis and phase on x–axis. Fix all the points on ordinary graph sheet and join the points by smooth curve. Write the frequency corresponding to each point of the plot.

In another method, first the Bode plot of G(jω) is sketched. From the Bode plot the magnitude and phase for various values of frequency, ω are noted and tabulated. Using these values the Nichols plot is sketched as explained earlier.

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM NICHOLS PLOT

The gain margin in db is given by the negative of db magnitude of G(jω) at the phase crossover frequency, ω_{pc}.

The ω_{pc} is the frequency at which phase of G(jω) is −180°. If the db magnitude of G(jω) at ω_{pc} is negative then gain margin is positive and vice versa.

Let ϕ_{gc} be the phase angle of $G(j\omega)$ at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of $G(j\omega)$ is zero. Now the phase margin, γ is given by $\gamma = 180° + \phi_{gc}$. If ϕ_{gc} is less negative than $-180°$ then phase margin is positive and vice versa.

The positive and negative gain margins are illustrated in fig 4.27.

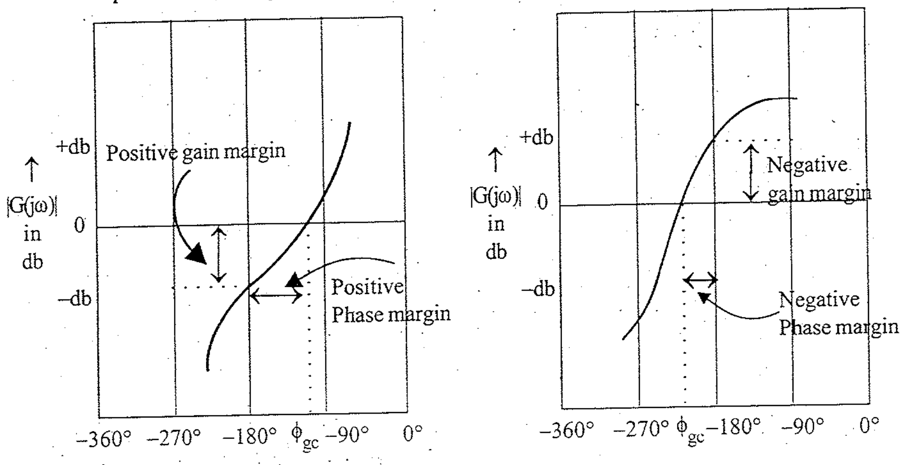


Fig 4.27 : Nichols plot showing phase margin and gain margin

GAIN ADJUSTMENT IN NICHOLS PLOT

In the open loop transfer function, $G(j\omega)$ the constant K contributes only magnitude. Hence by changing the value of K the system gain can be adjusted to meet the desired specifications. The desired specifications are gain margin and phase margin.

In a system transfer function if the value of K required to be estimated to satisfy a desired specification then draw the Nichols plot of the system with K=1. The constant K can add 20logK to every point of the plot. Due to this addition the Nichols plot will shift vertically up or down. Hence shift the plot vertically up or down to meet the desired specification. Equate the vertical distance by which the Nichols plot is shifted to 20 log K and solve for K.

Let x = change in db (x is positive if the plot is shifted up and vice versa)

Now, $20 \log K = x$

$$\log K = \frac{x}{20}$$

$$\therefore K = 10^{\frac{x}{20}}$$

EXAMPLE 4.13

Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K(1+10s)}{s^2(1+s)(1+2s)}$. Sketch the Nichols plot and determine the value of K so that (i) Gain margin is 10db, (ii)Phase margin is 10°

SOLUTION

$$\text{Given that } G(s) = \frac{K(1+10s)}{s^2(1+s)(1+2s)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by letting to $s = j\omega$. Also put K = 1

$$\therefore G(j\omega) = \frac{(1+j10\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

$$= \frac{\sqrt{1+(10\omega)^2}\ \angle\tan^{-1}10\omega}{\omega^2\angle 180^0\sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\sqrt{1+(2\omega)^2}\ \angle\tan^{-1}2\omega}$$

$$\therefore |G(j\omega)| = \frac{\sqrt{1+100\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}$$

$$|G(j\omega)|_{\text{indb}} = 20\log\left[\frac{\sqrt{1+100\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}\right]$$

$$\angle G(j\omega) = \tan^{-1}10\omega - 180^0 - \tan^{-1}\omega - \tan^{-1}2\omega$$

The magnitude of $G(j\omega)$ in db and phase of $G(j\omega)$ in deg are calculated for various values of ω and listed in the following table. The Nichols plot of $G(j\omega)$ with K = 1 is sketched as shown in fig 4.13.1

ω rad/sec	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0
|G(jω)| db	34.1	25.4	19.3	14.3	10	1.4	–5.3	–15.2	–22.5
∠G(jω) deg	–150	–164	–181	–194	–204	–222	–232	–244	–250

From the Nichols plot the gain margin and phase margin of the system when K=1 are

Gain margin = – 19.5 db

Phase margin = – 45°

GAIN ADJUSTMENT FOR REQUIRED GAIN MARGIN

For a gain margin of 10 db, the magnitude of G(jω) should be −10db, when the phase is −180°. When K = 1, the magnitude of G(jω) is +19.5db corresponding to phase angle of −180°. Hence if we add −29.5 db to every point of G(jω) ,then the plot shifts downwards and it will cross −180° axis at a magnitude of −10db. The magnitude correction is independent of frequency and so this gain can be contributed by the term K. Let this value of K be K_1. The value of K_1 is calculated by equating 20 log K_1 to −29.5db.

$$\therefore\ 20\log K_1 = -29.5\text{ db}$$

$$\log K_1 = \frac{-29.5}{20}$$

$$K_1 = 10^{\frac{-29.5}{20}} = 0.0335$$

GAIN ADJUSTMENT FOR REQUIRED PHASE MARGIN

Let ϕ_{gc2} = phase of G(jω) at gain crossover frequency for a phase margin of 10°

$$\therefore \text{Phase margin, } \gamma_2 = 180 + \phi_{gc2}$$

$$\therefore \phi_{gc2} = \gamma_2 - 180° = 10 - 180 = -170°$$

When K = 1, the magnitude of G(jω) is +23 db corresponding to a phase of −170°. But for a phase margin of 10°, this gain should be made zero. Hence if we add −23db to every point of G(jω) locus then the plot shifts downwards and it will cross −170° axis at magnitude of 0 db. The magnitude correction is independent of frequency and so this gain can be contributed by the term K. Let this value of K be K_2. The value of K_2 is calculated by equating 20 logK_2 to −23db.

$$\therefore\ 20\log K_2 = -23$$

$$\log K_2 = -23/\ 20$$

$$K_2 = 10^{\frac{-23}{20}} = 0.07$$

RESULT

When K = 1,

Gain margin	=	−19.5 db
Phase margin	=	−45°
For a gain margin of 10db, K = K_1	=	0.0335
For a phase margin of 10° , K = K_2	=	0.07

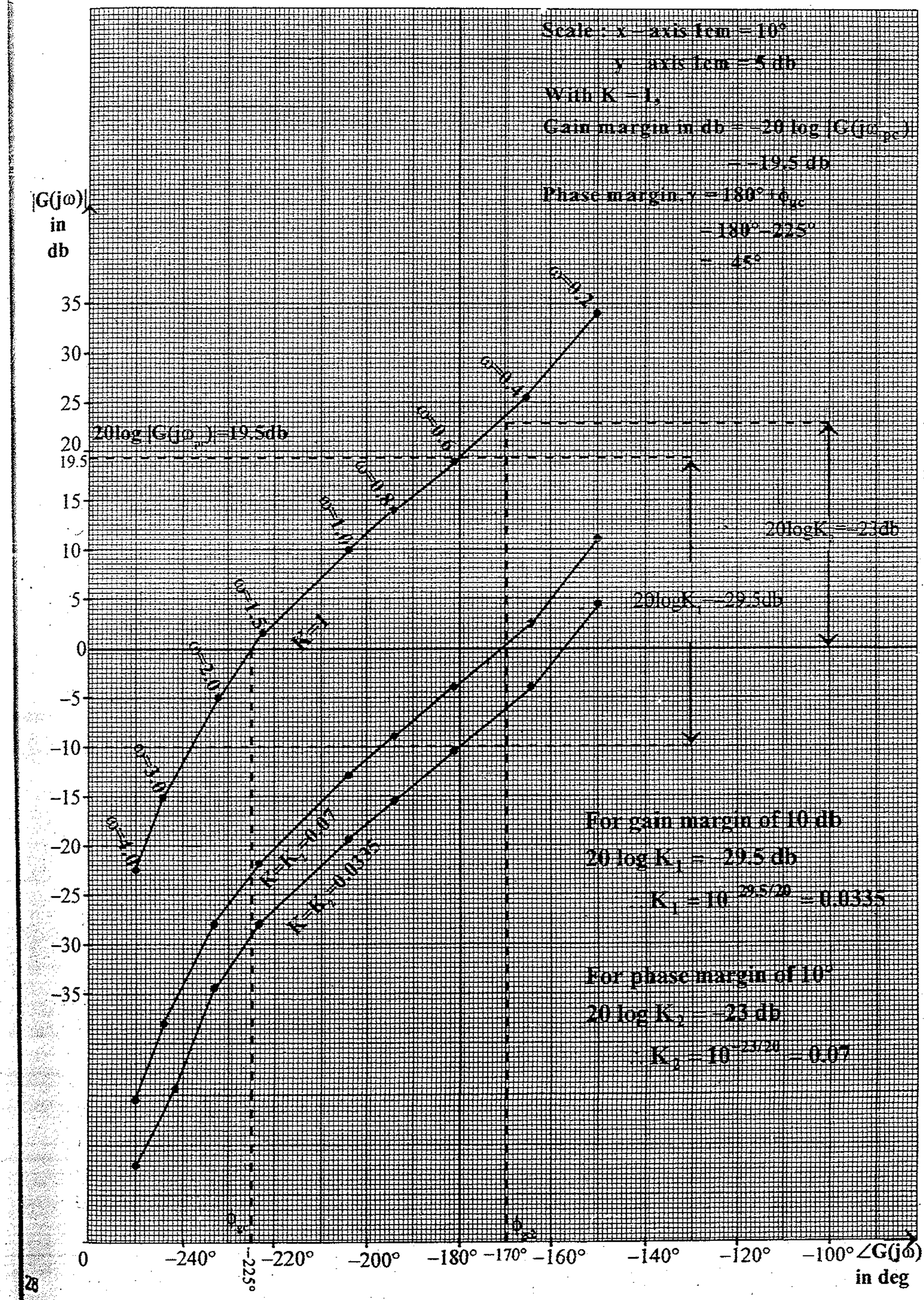


Fig 4.13.1 : Nichols plot of $G(j\omega) = \frac{K(1+j10\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)}$

4.9 CLOSED LOOP RESPONSE FROM OPEN LOOP RESPONSE

The closed loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

The sinusoidal transfer function is obtained by replacing s by jω.

$$\therefore \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)\,H(j\omega)}$$

Let, $$\frac{C(j\omega)}{R(j\omega)} = M(j\omega) = M\angle\alpha$$

Where, M = Magnitude of closed loop transfer function

α = Phase of closed loop transfer function.

The magnitude and phase of closed loop system are functions of frequency, ω. The sketch of magnitude and phase of closed loop system with respect to ω is closed loop frequency response plot. The magnitude and phase of closed loop system for various values of frequency can be evaluated analytically or graphically. The analytical method of determining the frequency response involves tedious calculations. Two graphical methods are available to determine the closed loop frequency response from open loop frequency response. They are M and N circles and Nichols chart.

4.10 M AND N CIRCLES

The magnitude of closed loop transfer function with unity feedback can be shown to be in the form of circle for every value of M. These circles are called **M-circles.**

If the phase of closed loop transfer function with unity feedback is α, then it can be shown that tan α will be in the form of circle for every value of α. These circles are called **N-circles.**

The M and N circles are used to find the closed loop frequency response graphically from the open loop frequency response G(jω) without calculating the magnitude and phase of the closed loop transfer function at each frequency.

The M and N circles are available as standard chart. The chart consists of M and N circles superimposed on ordinary graph sheet. Using ordinary graph the locus of G(jω) (Polar Plot) is sketched. The locus of G(jω) will cut the M-circles and N-circles at various points. The intersection of G(jω) locus with M and N circles gives the magnitude and phase of the closed loop system at frequencies corresponding to the cutting point of G(jω).

The M and α for various values of ω are tabulated. The magnitude and phase response of closed loop system are sketched on semilog graph sheet by taking ω on the logarithmic scale on x-axis. [The closed loop frequency response has two plots. They are M Vs ω and α Vs ω]

M CIRCLES

Consider the closed loop transfer function of unity feedback system,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\text{put } s = j\omega, \quad \therefore \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Let $G(j\omega) = X + jY$

Where X = Real part of G(jω)

and Y = Imaginary part of G(jω).

$$\therefore \frac{C(j\omega)}{R(j\omega)} = \frac{X+jY}{1+X+jY} = \frac{\sqrt{X^2+Y^2}\ \tan^{-1}\frac{Y}{X}}{\sqrt{(1+X)^2+Y^2}\ \tan^{-1}\frac{Y}{1+X}}$$

Let M = Magnitude of $\dfrac{C(j\omega)}{R(j\omega)}$

$$\therefore M = \frac{\sqrt{X^2+Y^2}}{\sqrt{(1+X)^2+Y^2}}$$

On squaring

$$M^2 = \frac{X^2+Y^2}{(1+X)^2+Y^2}$$

$$M^2\left((1+X)^2+Y^2\right) = X^2 + Y^2$$

$$M^2\left(1+X^2+2X+Y^2\right) = X^2 + Y^2$$

$$M^2 + M^2X^2 + M^2 2X + M^2Y^2 - X^2 - Y^2 = 0$$

$$X^2(M^2-1) + M^2 2X + M^2 + Y^2(M^2-1) = 0 \qquad .....(4.30)$$

When M = 1, the equation (4.30) represents a straight line. Consider equation (4.30) when M = 1,

$$X^2(1-1) + 2X + 1 + Y^2(1-1) = 0$$

$$2X + 1 = 0$$

$$X = -1/2$$

Hence when M = 1, the equation represents a straight line passing through X = −1/2 and Y = 0.

When M = 1, the equation (4.30) represents a family of circles. Consider equation (4.30).

$$X^2(M^2-1) + M^2 2X + M^2 + Y^2(M^2-1) = 0$$

Divide throughout by (M^2-1)

$$X^2 + \frac{M^2}{M^2-1}2X + \frac{M^2}{M^2-1} + Y^2 = 0$$

Add $\frac{M^2}{\left(M^2-1\right)^2}$ on both sides,

$$X^2 + \frac{M^2}{M^2-1}2X + \frac{M^2}{M^2-1} + \frac{M^2}{\left(M^2-1\right)^2} + Y^2 = \frac{M^2}{\left(M^2-1\right)^2}$$

$$X^2 + \frac{M^2}{M^2-1}2X + \frac{M^2\left(M^2-1\right)+M^2}{\left(M^2-1\right)^2} + Y^2 = \frac{M^2}{\left(M^2-1\right)^2}$$

$$X^2 + \frac{M^2}{M^2-1}2X + \frac{M^4}{\left(M^2-1\right)^2} + Y^2 = \frac{M^2}{\left(M^2-1\right)^2}$$

$$\left(X + \frac{M^2}{M^2-1}\right)^2 + Y^2 = \frac{M^2}{\left(M^2-1\right)^2} \qquad .....(4.31)$$

The equation of circle with centre at (X_1, Y_1) and radius r is given by

$$(X - X_1)^2 + (Y - Y_1)^2 = r^2 \qquad .....(4.32)$$

On comparing equation (4.31) and equation (4.32), it can be concluded that the equation (4.31) represent a family circles with centre at $(-M^2/M^2-1),0)$ and with radius, $r = M/(M^2-1)$. The circles given by equation (4.31) are called M-circles.

When M = 0

$$X_1 = -\frac{M^2}{M^2-1} = 0$$

$$Y_1 = 0$$

$$r = \frac{M}{M^2-1} = 0$$

Hence when M = 0 the magnitude circle becomes a point at(0,0).

When M = ∞

$$X_1 = \frac{-M^2}{M^2-1} \approx \frac{-M^2}{M^2} = -1$$

$$Y_1 = 0$$

$$r = \frac{M}{M^2-1} \approx \frac{M}{M^2} = \frac{1}{M} = 0$$

Hence when M = ∞, the magnitude circle becomes a point at (−1,0)

From the above analysis it is clear that the magnitude of closed loop transfer function will be in the form of circles when M≠1 and when M = 1, the magnitude is a straight line passing through (−1/2,0).

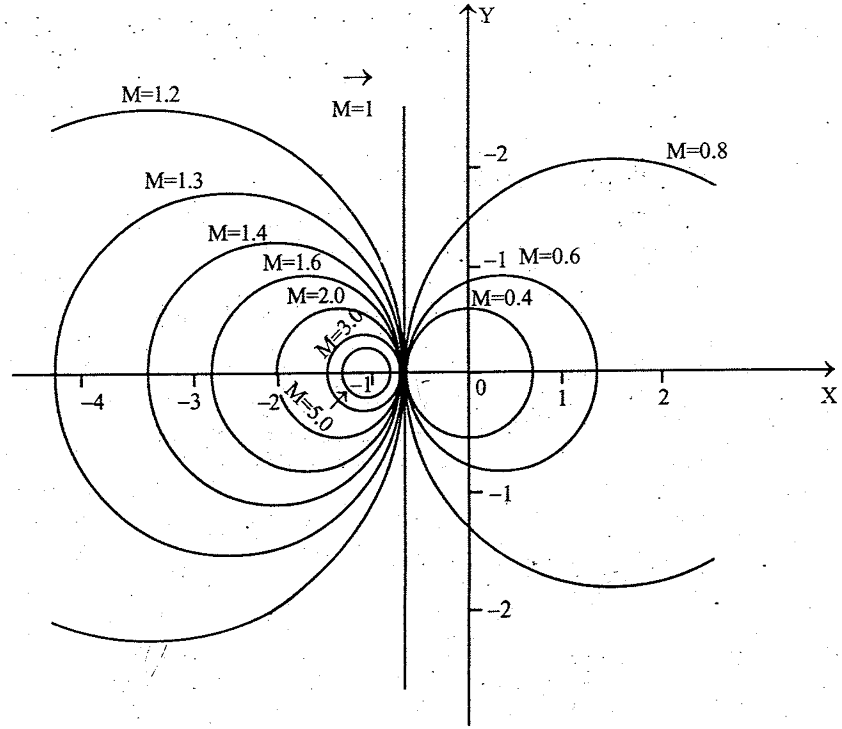


Fig 4.28 : The family of constant M circles.

For values of M less than 1, the magnitude is a circle to the right of the straight line corresponding to M = 1. It is observed that the circles for M<1 passes through (−1/2,0) and (0,0) on the negative real axis. For decreasing values of M, the radius decreases and the circle, becomes a point at (0,0) when M = 0.

For values of M greater than 1, the magnitude is a circle to the left of the straight line corresponding to M = 1. It is observed that circle passes between the points (−1,0) and (−1/2,0) on the negative real axis. For increasing values of M the radius decreases and the circle becomes a point at (−1,0) when $M=\infty$. The family of M-circles are shown in fig 4.28.

N CIRCLES

Consider the closed loop transfer function of unity feedback system.

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}$$

$$\text{Put } s=j\omega, = \frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}$$

Let $G(j\omega)=X+jY$ Where X = Real part of $G(j\omega)$

Y = Imaginary part of $G(j\omega)$.

$$\therefore \frac{C(jw)}{R(jw)}=\frac{X+jY}{1+X+jY}=\frac{\sqrt{X^2+Y^2}\tan^{-1}\frac{Y}{X}}{\sqrt{(1+X)^2+Y^2}\tan^{-1}\frac{Y}{1+X}}$$

$$\text{Let } \alpha=\text{Arg}\left[\frac{C(j\omega)}{R(j\omega)}\right]$$

$$\therefore \alpha=\tan^{-1}\frac{Y}{X}-\tan^{-1}\frac{Y}{1+X}$$

Let $N=\tan\alpha$

$$\therefore N=\tan\left(\tan^{-1}\frac{Y}{X}-\tan^{-1}\frac{Y}{1+X}\right)$$

$$=\frac{\tan\left(\tan^{-1}\frac{Y}{X}\right)-\tan\left(\tan^{-1}\frac{Y}{1+X}\right)}{1+\tan\left(\tan^{-1}\frac{Y}{X}\right)\tan\left(\tan^{-1}\frac{Y}{1+X}\right)}$$

Note :

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\times\tan B}$$

$$=\frac{\frac{Y}{X}-\frac{Y}{1+X}}{1+\frac{Y}{X}\cdot\frac{Y}{1+X}}=\frac{\frac{Y(1+X)-XY}{X(1+X)}}{\frac{X(1+X)+Y^2}{X(1+X)}}=\frac{Y+XY-XY}{X+X^2+Y^2}$$

$$\therefore N=\frac{Y}{X+X^2+Y^2}$$

On rearranging we get,

$$X+X^2+Y^2=\frac{Y}{N}$$

$$X+X^2+Y^2=\frac{Y}{N}=0$$

Add the term $\frac{1}{4}+\left(\frac{1}{2N}\right)^2$ on both sides.

$$X+X^2+Y^2-\frac{Y}{N}+\frac{1}{4}+\left(\frac{1}{2N}\right)^2=\frac{1}{4}+\left(\frac{1}{2N}\right)^2$$

$$\left(X^2+\frac{1}{4}+X\right)+\left(Y^2+\frac{1}{(2N)^2}-\frac{Y}{N}\right)=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(X+\frac{1}{2}\right)^2+\left(Y-\frac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2} \qquad .....(4.33)$$

The equation of circle with centre at (X_1,Y_1) and radius r is

$$(X-X_1)^2+(Y-Y_1)^2=r^2 \qquad .....(4.34)$$

On comparing equation (4.33) and (4.34), it can be concluded that the equation (4.33) represents a family of circle with centre at (−1/2,1/2N) and with radius

radius $\sqrt{\frac{1}{4}+\frac{1}{(2N)^2}}$ The circles given by the equation (4.30) are called N circles.

The equation of N circles is satisfied at two points (0,0) and (−1,0). Hence the N-circles passes through these two points for all values of α. ($N = \tan\alpha$).

Consider the equation of N-circle
When X = 0 and Y = 0

$$\left(X+\frac{1}{2}\right)^2+\left(Y-\frac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\frac{1}{4}+\frac{1}{4N^2}=\frac{1}{4}+\frac{1}{4N^2}$$

Consider the equation of N-circle
when X = −1 and Y = 0

$$\left(X+\frac{1}{2}\right)^2+\left(Y-\frac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(-1+\frac{1}{2}\right)^2+\left(-\frac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(4N)^2}$$

$$\frac{1}{4}+\frac{1}{4N^2}=\frac{1}{4}+\frac{1}{4N^2}$$

The above analysis shows that the equation of N-circle is satisfied at points(0,0) and (−1, 0).

When $\alpha = 180°$ the circle becomes a straight line passing through real axis. It is also observed that the circle for $\alpha = \theta°$ above the real axis will be a part of circle for $\alpha = \theta°$ below the real axis, as shown in fig 4.29. The family of N circles are shown in fig 4.30.

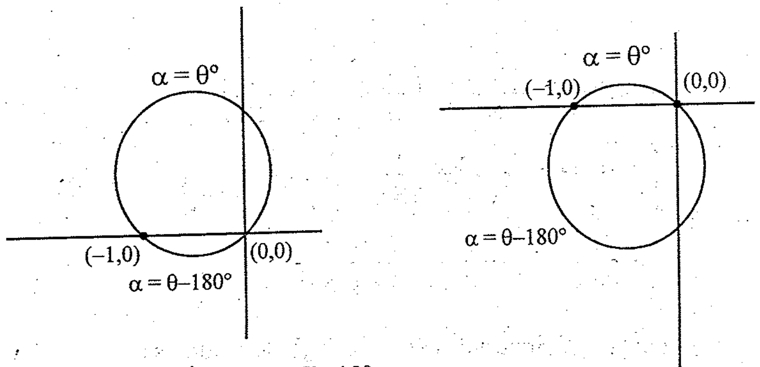


Fig 4.29

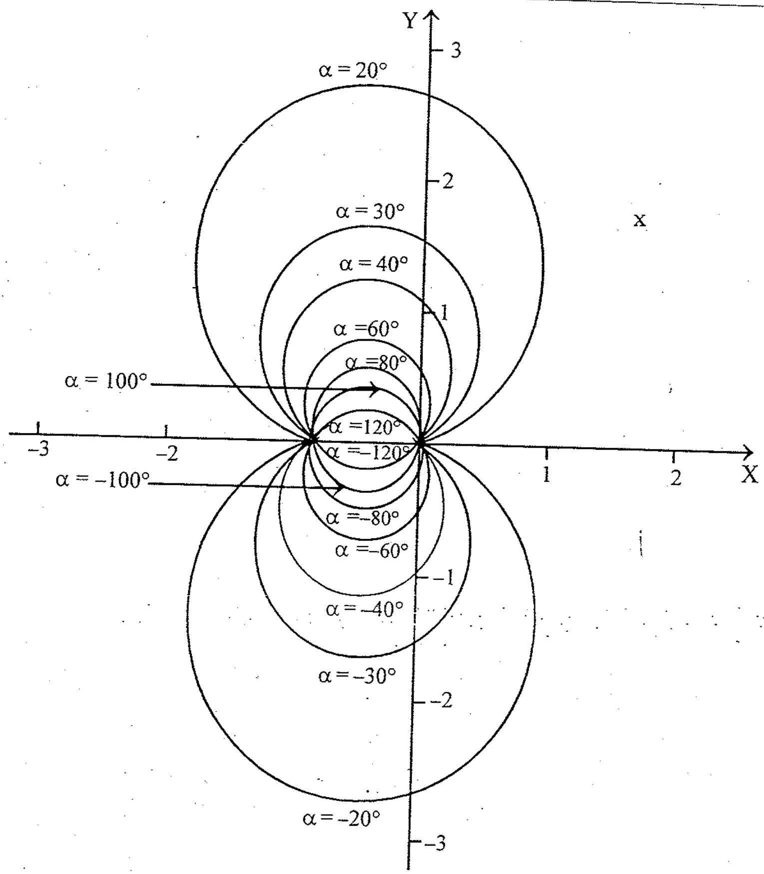


***Fig 4.30. :** The family of constant N circles.*

4.11 NICHOLS CHART

N.B. Nichols transformed the constant M and N circles to log-magnitude and phase angle coordinates and the resulting chart is known as **Nichols chart**.

Nichols chart consist of M and N contours, superimposed on ordinary graph. The M contours are the magnitude of closed loop system in decibels and the N contours are the corresponding phase angle locus of closed loop system. The ordinary graph consist of magnitude in db marked on the Y-axis and the phase in degrees marked on the X-axis.

The Nichols plot of open loop system can be plotted on the ordinary graph. The Nichols plot is a graph between magnitude of $G(j\omega)$ in db and the phase of $G(j\omega)$ in degrees, plotted on an ordinary graph sheet. To draw the Nichols plot the magnitude and phase angle of $G(j\omega)$ are calculated for various values of ω. Alternatively the Bode plot of

$G(j\omega)$ is sketched and from the Bode plot the magnitude and phase of $G(j\omega)$ for any frequency can be obtained.

Using Nichols chart the closed loop frequency response can be determined graphically from the locus of open loop system. When the Nichols plot of $G(j\omega)$ is sketched on Nichols chart, the locus of $G(j\omega)$ will cut the M and N contours at various points. The cutting point of the locus of $G(j\omega)$ with the M-contour gives the magnitude of closed loop system corresponding to a frequency same as that of $G(j\omega)$ at that point. The cutting point of locus of $G(j\omega)$ and N contour gives the phase of closed loop system corresponding to a frequency same as that of $G(j\omega)$ at that point. The magnitude M and phase angle α ($N = \tan\alpha$) of closed loop system are tabulated. The closed loop frequency response can be plotted on a semilog graph sheet using the tabulated values. The closed loop frequency response consists of two plots. They are magnitude, M Vs ω and phase angle α Vs ω.

The frequency domain specifications can be determined from Nichols chart. Fig 4.31, shows various frequency domain specifications of a typical $G(j\omega)$ locus. Also the Nichols plot drawn on a Nichols chart can be used for gain adjustment.

DETERMINATION OF RESONANT PEAK (M_r) AND RESONANT FREQUENCY (ω_r)

The resonant peak is given by the value of M-contour which is tangent to $G(j\omega)$ locus. The resonant frequency is given by the frequency of $G(j\omega)$ at the tangency point.

DETERMINATION OF BANDWIDTH

The Bandwidth is given by frequency corresponding to the intersection point of $G(j\omega)$ and -3 dB M-contour

DETERMINATION OF GAIN MARGIN

The Gain margin is given by negative of magnitude of $G(j\omega)$ in db at phase crossover frequency, ω_{pc}. At phase crossover frequency the phase of $G(j\omega)$ is $-180°$.

$$\text{Gain Margin, } K_g \text{ in db} = -|G(j\omega_{pc})|_{\text{in db}}$$

DETERMINATION OF PHASE MARGIN

The phase margin, γ is given by $\gamma = 180 + \phi_{gc}$ where ϕ_{gc} is the phase of $G(j\omega)$ at gain crossover frequency. At gain crossover frequency the magnitude of $G(j\omega)$ is zero db.

GAIN ADJUSTMENT USING NICHOLS CHART

DETERMINATION OF K FOR SPECIFIED GAIN MARGIN

Draw the $G(j\omega)$ locus with $K = 1$. Determine the amount of gain to be added at $\phi = -180°$, so that db magnitude of $G(j\omega)$ locus at $-180°$ is negative of the specified gain margin. Let the db gain to be added be x db. The gain contribution is independent of

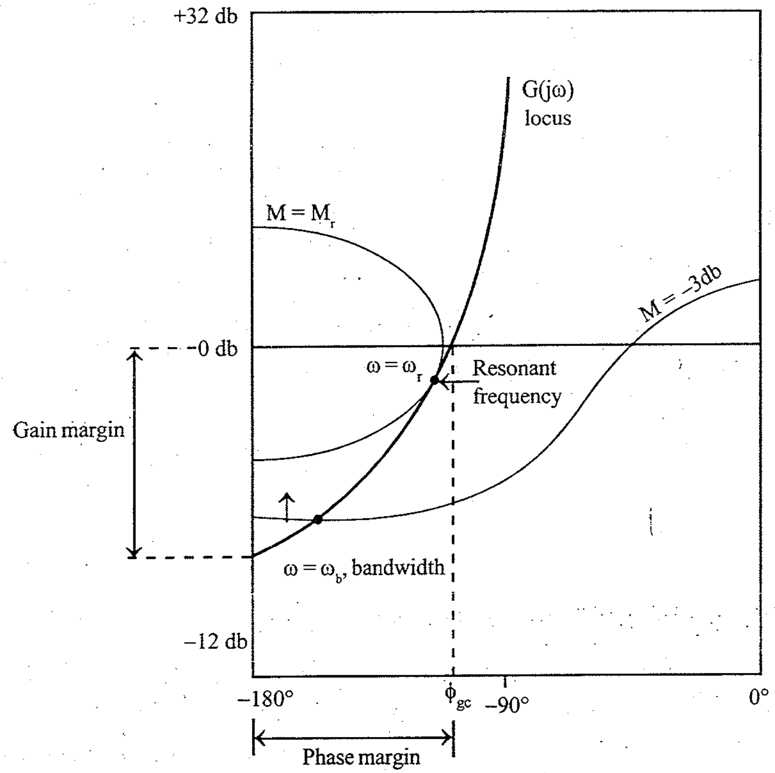


Fig 4.31 : Determination of frequency domain specifications from Nichols Chart

frequency and so it can be achieved by choosing proper value of K. The value of K is obtained by equating 20 log K to x db.

$$\text{Now, } 20 \log K = x$$

$$\therefore K = 10^{\frac{x}{20}}$$

DETERMINATION OF K FOR SPECIFIED PHASE MARGIN

Draw the $G(j\omega)$ locus with K= 1. The phase margin, $\gamma = 180° + \phi_{gc}$ where ϕ_{gc} is phase of $G(j\omega)$ at gain crossover frequency. $\therefore \phi_{gc} = \gamma - 180°$. For specified phase margin, calculate ϕ_{gc} and from the Nichols plot determine the db gain at ϕ_{gc}. Let this gain be y db. For the specified phase margin, this gain should be made zero. Hence $-y$ db should be added to every point of $G(j\omega)$. This is achieved by choosing proper value of K. The value of K is obtained by equating 20 log K to $-y$ db.

Now, $20\log K = -y$

$$\therefore K = 10^{\frac{-y}{20}}$$

DETERMINATION OF K FOR SPECIFIED RESONANT PEAK, M_r

Draw the $G(j\omega)$ locus with $K = 1$. Using a tracing paper, trace the locus of $G(j\omega)$ (A standard tracing paper, called Nichols overlay is available). Then shift the locus vertically up or down, so that $M = M_r$ contour is tangent to $G(j\omega)$ locus. Measure the vertical shift in db. Let the shift be $\pm$ x db. (+ for up and $-$for down)

Now, $20 \log K = \pm x$

$$\therefore K = 10^{\pm\frac{x}{20}}$$

DETERMINATION OF K FOR A SPECIFIED BANDWIDTH

Draw the $G(j\omega)$ locus with $K = 1$. Determine the open loop gain $G(j\omega)$ at $\omega = \omega_b$ where, ω_b is the specified bandwidth. Determine the point of intersection of -3db M-contour and this open loop gain on the Nichols chart. Let this point be point A. Trace the $G(j\omega)$ locus. Shift the $G(j\omega)$ locus vertically up or down, so that it passes through point A. Measure the vertical shift in db. Let the shift be $\pm$ x db (+ for up and $-$ for down).

Now, $20 \log K = \pm x$

$$\therefore K = 10^{\pm\frac{x}{20}}$$

EXAMPLE 4.14

The open loop transfer function of unity feedback system is, $G(s) = Ke^{-0.2s}/s(1+0.25s)(1+0.1s)$. Using Nichols chart, determine the following,

i) The value of K so that the gain margin of the system is 4 db.

ii) The value of K so that the phase margin of the system is 40°

iii) The value of K so that resonant peak M_r of the system is 1db. What are the corresponding values of ω_r and ω_b?

iv) The value of K so that the bandwidth ω_b of the system is 1.5 rad/sec.

SOLUTION

First the actual Bode plot of $G(j\omega)$ with K=1 is plotted on semilog graph sheet. The magnitude of $G(j\omega)$ in db and phase of $G(j\omega)$ for various frequencies are calculated and listed in Table1. The choice of frequencies are chosen such that the magnitude plot extends in the range of 40 db to -14db and the phase plot extends in the range of 0° to -180°.

Given that $\quad G(s) = \dfrac{K\,e^{-0.2s}}{s\,(1+0.25s)\,(1+0.1s)}$

Let $K=1$ and put $s = j\omega$

$$\therefore G(j\omega) = \frac{e^{-j0.2\omega}}{j\omega\,(1+j0.25\omega)\,(1+j0.1\omega)}$$

$$= \frac{1\angle -0.2\omega \times \frac{180}{\pi}}{\omega\angle 90^0\ \sqrt{1+0.0625\,\omega^2}\ \angle\tan^{-1}0.25\omega\sqrt{1+0.01\omega^2}\angle\tan^{-1}0.1\omega}$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+0.0625\omega^2}\ \sqrt{1+0.01\omega^2}}$$

$$|G(j\omega)|_{\text{in db}} = 20\log\left[\frac{1}{\omega\sqrt{1+0.0625\omega^2}\ \sqrt{1+0.01\omega^2}}\right]$$

$$\angle G(j\omega) = -0.2\omega\times\frac{180}{\pi} - 90° - \tan^{-1}0.25\omega - \tan^{-1}0.1\omega$$

Table 1 : Calculated values of $|G(j\omega)|$ and $\angle G(j\omega)$

ω rad/sec	0.01	0.02	0.05	0.1	0.2	0.5	1.0	2.0	4.0
$\lvert G(j\omega)\rvert$ db	40	34	26	20	14	6	0	-7	-16
$\angle G(j\omega)$ deg	-90	-91	-91	-93	-96	-106	-121	-151	-203

The magnitude and phase plot of Bode plot of $G(j\omega)$ are shown in fig 4.14.1. From the bode plot the phase and frequency for various values of magnitudes are noted and tabulated in table-2. (The choice of magnitudes are 20,16,12,.... i.e, in steps of 4 db, which is convenient for Nichols plot on Nichols chart). Using the values listed in table2 the locus of G(jω) on Nichols chart is sketched as shown in fig 4.14.2

Table 2: Values of |G(jω)| and ∠G(jω) noted from bode plot.

ω rad/sec	0.1	0.16	0.25	0.4	0.64	1.0	1.5	2.2	3.0
|G(jω)| db	20	16	12	8	4	0	–4	–8	–12
∠G(jω) deg	–90	–92	–96	–102	–110	–120	–136	–156	–180

GAIN MARGIN AND PHASE MARGIN WHEN K = 1

When K = 1, the G(jω) locus cuts the –180° axis at –12db. Hence the magnitude at phase crossover frequency is –12db.

$$\therefore \text{Gain Margin, } K_g = -|G(j\omega_{pc})|_{in\,db} = -(-12) = +12\text{db}.$$

When K = 1, the phase of G(jω) is -120° corresponding to magnitude of 0 db. Hence the phase at gain crossover frequency is –120°.

$$\therefore \text{Phase margin, } \gamma = 180° + \phi_{gc} = 180° - 120° = 60°$$

i) To find K for a gain margin of 4db.

When gain margin is 4 db, the locus of G(jω) should cross the 180° axis at –4 db. When K = 1, the magnitude of G(jω) is –12db corresponding to a phase of 180°. Hence if we add, –4 – (–12) = 8 db to every point of G(jω) then the plot shifts upwards and crosses –180° axis at –4 db. This magnitude correction is achieved by choosing appropriate value of K. The value of K is obtained by equating 20 log K to 8 db.

$$\therefore 20 \log K = 8\text{db}$$

$$\log K = \frac{8}{20}$$

$$K = 10^{\frac{8}{20}} = 2.5$$

The G(jω), when K = 2.5 is shown in fig 4.14.2.

ii) To find K for phase margin of 40°.

Let ϕ_{gc2} be the phase of G(jω) at gain crossover frequency when the phase margin is 40°

$$\therefore \text{Phase margin, } \gamma_2 = 180° + \phi_{gc2}$$

$$\therefore \phi_{gc2} = \gamma_2 - 180° = 40° - 180° = -140°.$$

From the above calculation it is evident that for a phase margin of 40° the magnitude of G(jω) should be 0 db corresponding to a phase of –140°. When K = 1, the magnitude of G(jω) is –5db corresponding to a phase of –140°. Hence if we add +5 db to every point of G(jω) locus then the plot shifts upwards and crosses –140° axis at 0 db. This magnitude correction is achieved by choosing appropriate values of K. The value of K is obtained by equating 20 logK to 5 db.

$$\therefore 20 \log K = 5$$

$$\log K = \frac{5}{20}$$

$$K = 10^{\frac{5}{20}} = 1.78$$

The G(jω) locus, when K = 1.78 is shown in fig 4.14.2

iii) To find K for a resonant peak of 1 db

The resonant peak, M_r is given by the M-contour which is tangent to G(jω) locus. When K = 1, the G(jω) locus is tangent to M = 0.25 db contour. Hence when K = 1, the resonant peak is 0.25 db.

For a resonant peak of 1 db, the M = 1 db contour should be made tangent to G(jω) locus. For this, G(jω) locus can be shifted vertically up or down so that it becomes tangent to M = 1 db contour. In this problem the G(jω) locus is shifted vertically up to make it tangent to M = 1 db contour. The shifted G(jω) locus is shown in fig 4.14.3.

[Note : Trace the G(jω) locus when K = 1 on a tracing paper and shift the traced locus over the Nichols chart vertically so that it is tangent to required M-contour. By keeping the tracing paper at the shifted position darken the traced locus, so that it makes an impression on nichols chart.]

The vertical shift is equivalent to adding a magnitude of 20 log K to every point of G(jω) locus. From the shifted locus of G(jω) it is observed that +2db is added to every point of G(jω) locus. Hence the value of K is obtained by equating 20 log K to +2 db.

$$20\log K = 2\text{db}$$

$$\log K = \frac{2}{20}$$

$$K = 10^{\frac{2}{20}} = 1.26$$

The resonant frequency, ω_r is given by the frequency of G(jω) at the tangency point. The magnitude of G(jω) is 0 db at the tangency point of M = 1 db contour. The corresponding frequency is noted from the bode plot of G(jω). From the bode plot the frequency at 0 db is 1.0 rad/sec. Hence the resonant frequency, ω_r= 1.0 rad/sec

iv) To find K so that ω_b=1.5 rad/sec

The bandwidth, ω_b is given by the frequency of G(jω) corresponding to the meeting point of G(jω) locus and M = –3db contour. From the bode plot find the magnitude of G(jω) when ω = 1.5 rad/sec. From fig 4.14.1 it is observed that magnitude of G(jω) is – 4db when ω = 1.5 rad/sec.

In the Nichols chart, find the point where the M = –3db contour passes through –4db line. Let this point be P. Now the G(jω) locus with K = 1 is shifted vertically down so that is passes through point P. The shifted G(jω) locus is shown in fig 4.14.3.

*[**Note :** Trace the G(jω) locus when K = 1 on a tracing paper and shift the traced locus over Nichols chart so that it passes through point P. By keeping the tracing paper at the shifted position, darken the traced locus, so that it makes an impression on Nichols chart.]*

The vertical shift is equivalent to adding a magnitude of 20 logK to every point of G(jω) locus. From the shifted locus of G(jω) it is observed that –6db is added to every point of G(jω) locus. Hence the value of K is obtained by equating 20 logK to –6 db.

$$20\log K = -6\text{ db}$$

$$\log K = \frac{-6}{20}$$

$$K = 10^{-6/20} = 0.5$$

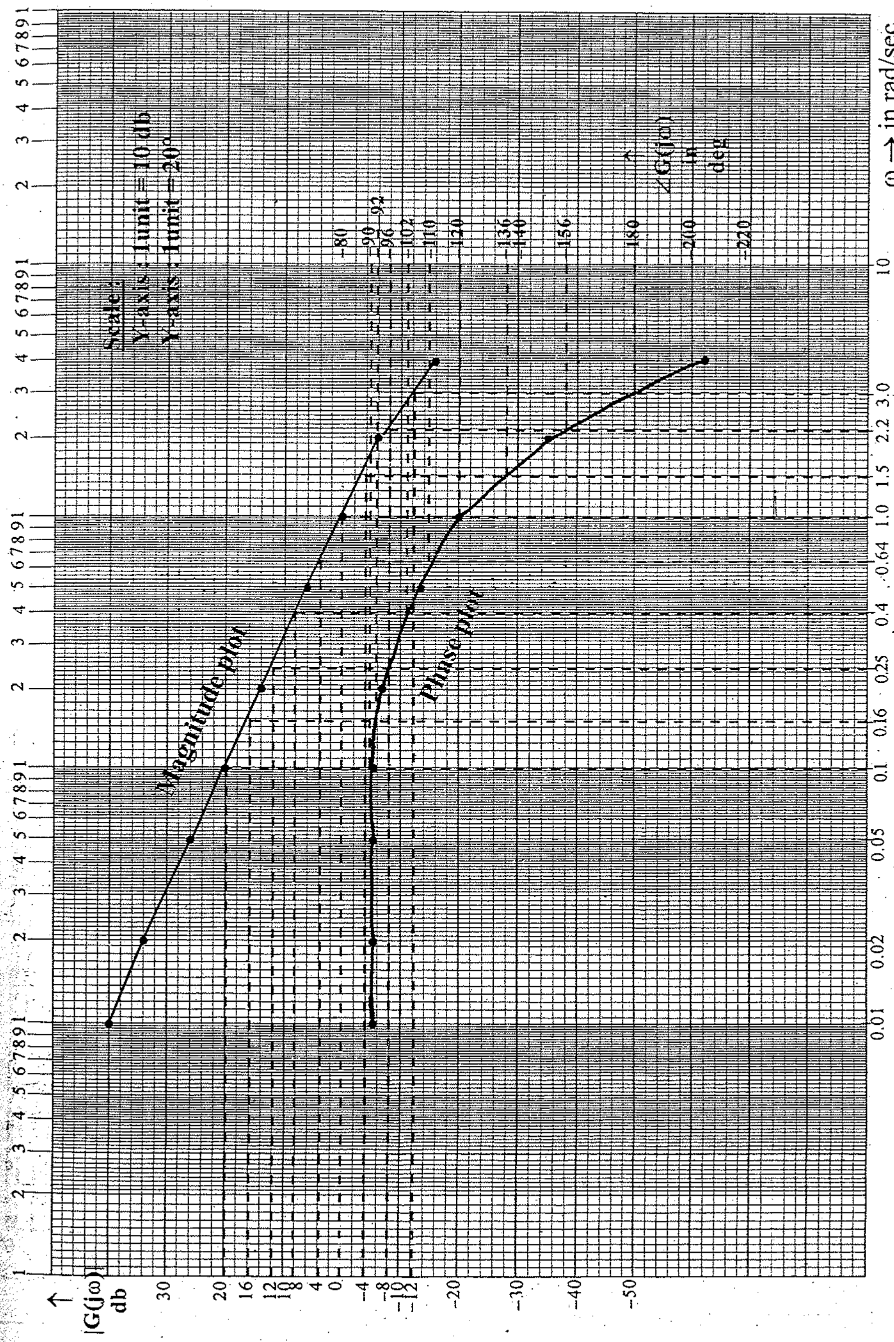


Fig 4.14.1 : Bode plot of $G(j\omega) = \frac{e^{-j0.2\omega}}{j\omega(1+j0.25\omega)(1+j0.1\omega)}$.

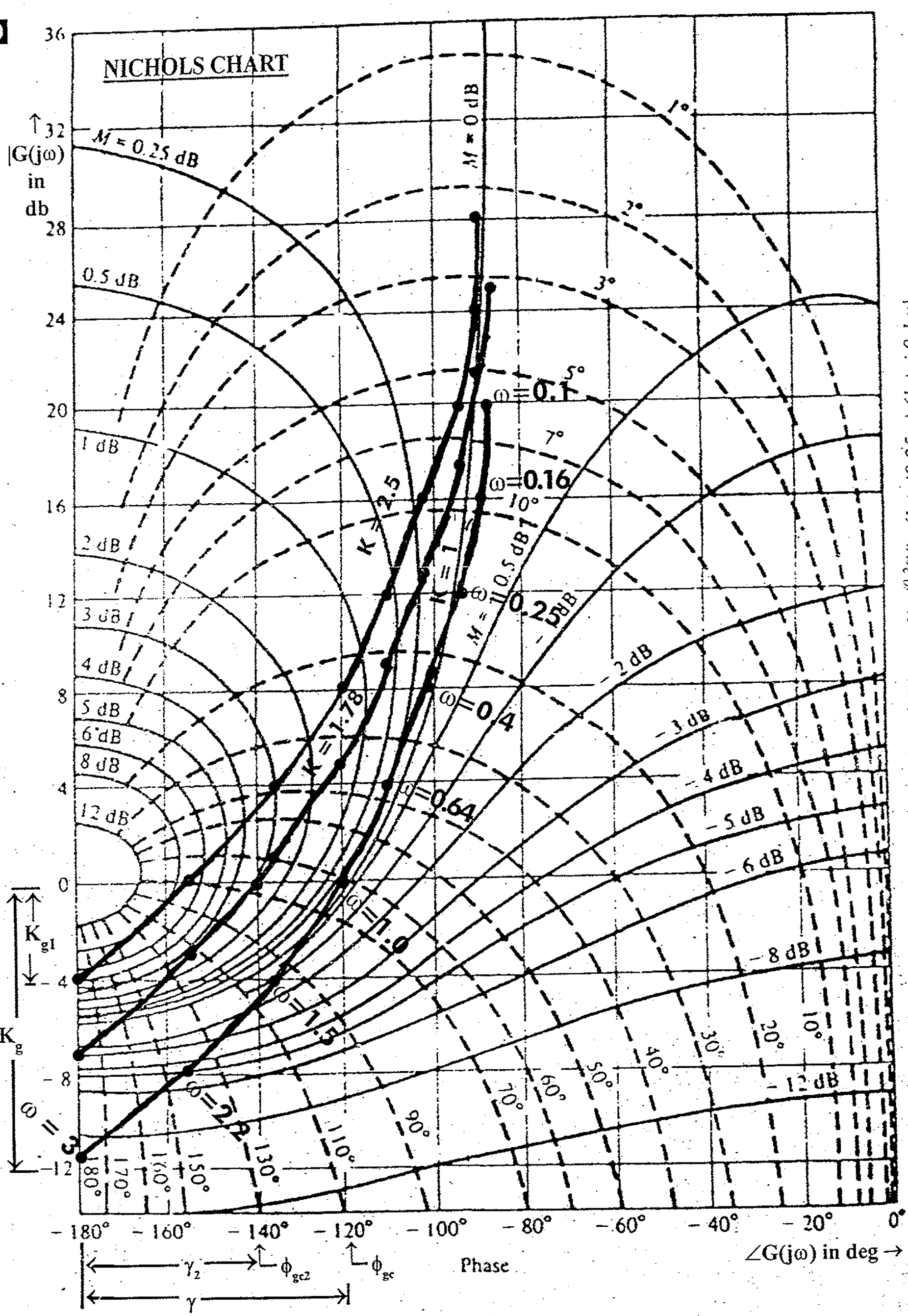


Fig 4.14.2 : Nichols plot of $G(j\omega) = \dfrac{Ke^{-j0.2\omega}}{j\omega(1+j0.25\omega)(1+j0.1\omega)}$

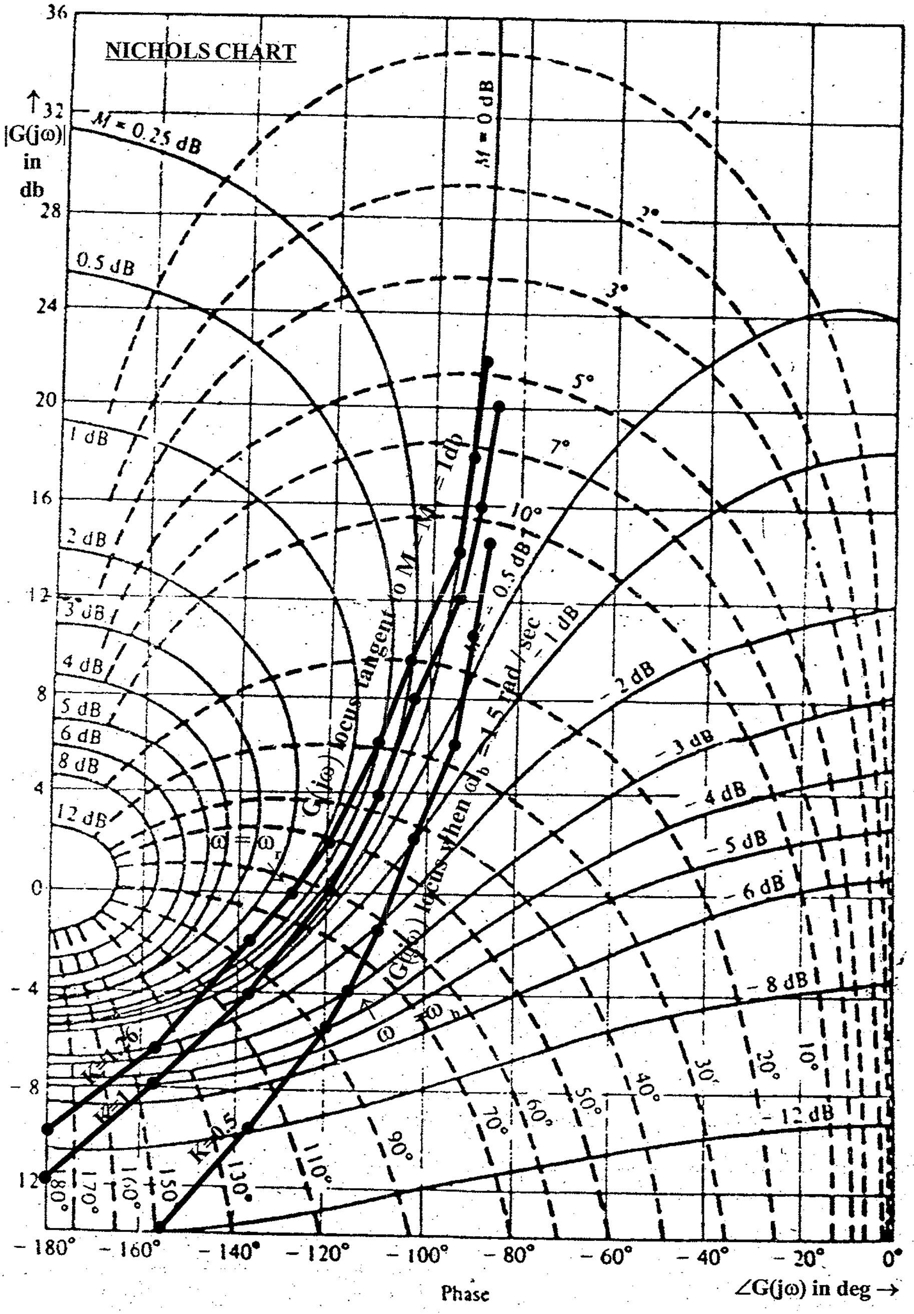


Fig 4.14.3 : Nichols plot of $G(j\omega) = \dfrac{Ke^{-j0.2\omega}}{j\omega(1+j0.25\omega)(1+j0.1\omega)}$

EXAMPLE 4.15

A unity feedback system has open loop transfer function $G(s) = 20/s(s+2)(s+5)$. Using Nichols chart, determine the closed loop frequency response and estimate M_r, ω_r and ω_b.

SOLUTION

Given that, $G(s) = 20/s(s+2)(s+5)$.

The transfer function G(s) is converted to time constant or bode form.

$$G(s) = \frac{20}{s \times 2\left(\frac{s}{2}+1\right) \times 5\left(\frac{s}{5}+1\right)} = \frac{20/(2\times 5)}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)} = \frac{2}{s\,(1+0.5s)\,(1+0.2s)}$$

$$\therefore G(j\omega) = \frac{2}{j\omega\,(1+j0.5\omega)\,(1+j0.2\omega)}$$

$$= \frac{2}{\omega\angle 90^\circ\ \sqrt{1+0.25\omega^2}\ \angle\tan^{-1}0.5\omega\ \sqrt{1+0.04\omega^2}\ \angle\tan^{-1}0.2\omega}$$

$$= \frac{2}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+0.04\omega^2}}\angle(-90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.2\omega)$$

$$|G(j\omega)| = \frac{2}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+0.04\omega^2}}$$

$$|G(j\omega)|_{\text{in db}} = 20\log\left[\frac{2}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+0.04\omega^2}}\right]$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.2\omega$$

First the actual bode plot of $G(j\omega)$ is plotted on semilog graph sheet. The magnitude of $G(j\omega)$ in db and phase of $G(j\omega)$ for various frequencies are calculated and listed in Table 1. The choice of frequencies are chosen such that the magnitude plot extends in the range of 40 db to −14 db and the phase plot extends in the range of 0° to −180°.

Table 1: Calculated values of $|G(j\omega)|$ and $\angle G(j\omega)$.

ω, rad/sec	0.2	0.5	1.0	2.0	3.0	4.0
|G(jω)|, db	20	12	5	−4	−10	−15
∠G(jω), deg	−98	−110	−128	−157	−177	−192

The magnitude and phase plot of bode plot of $G(j\omega)$ are shown in fig 4.15.1 From the bode plot, the phase and frequency for various values of magnitudes are noted and tabulated in table 2. (The choice of magnitudes are 20, 16, 12,..., i.e., in steps of 4 db, which is convenient for Nichols plot on Nichols chart). Using the values listed in table 2, the locus of $G(j\omega)$ is sketched on the Nichols chart as shown in fig 4.15.2.

Table 2: Values of $|G(j\omega)|$ and $\angle G(j\omega)$ noted from bode plot.

ω rad/sec	0.2	0.31	0.1	0.74	1.05	1.5	2.0	2.6	3.2
|G(jω)| db	20	16	12	8	4	0	−4	−8	−11.5
∠G(jω) deg	−98	−102	−110	−118	−130	−146	−156	−168	−180

The locus of $G(j\omega)$ drawn on the Nichols chart cuts the M-contour and N-contour at various points. The meeting points of $G(j\omega)$ locus and various M-contours are noted. The phase α corresponding to the meeting point are noted from N-contours passing through the meeting point. (if the meeting point lies between two N-contours then choose an approximate value of α)

The frequency corresponding to the meeting point are noted from bode plot by transferring the $|G(j\omega)|$ corresponding to meeting point to bode plot. The values of ω, M and α are listed in table 3. The values of M and α are the magnitude and phase of closed loop frequency response of $G(j\omega)$ with unity feedback.

Table 3: Values of M and α from Nichols Chart

ω rad/sec	0.36	0.62	1.0	1.2	1.6	1.8	2.0	2.1	2.3	2.4	2.5	2.8	3.0
M db	0.25	1	2	3	4	3	2	1	0	-2	-3	-6	-8
α deg	12	21	35	50	78	102	120	130	140	151	155	165	175

Using the values listed in table3, the closed loop frequency response plots are sketched as shown in fig 4.15.3. The closed loop frequency response consists of two plots and they are magnitude plot, M Vs ω and phase plot α Vs ω. From the closed loop frequency response the values of M_r, ω_r and ω_b are noted.

Resonant peak, $M_r = +4$ db

Resonant frequency, $\omega_r = 1.6$ rad/sec

Bandwidth, $\omega_b = 2.5$ rad/sec

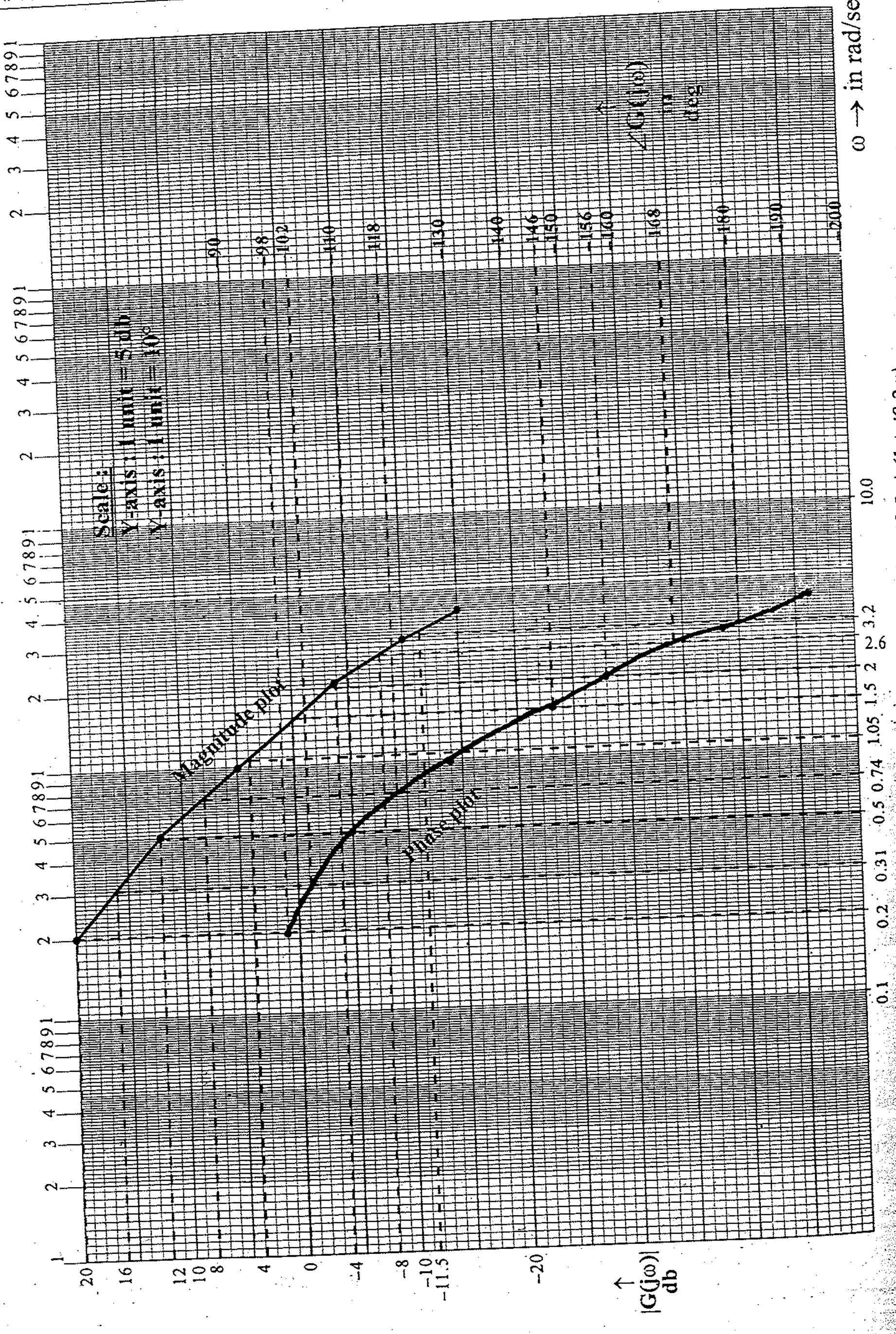


Fig 4.15.1 : Bode plot of $G(j\omega) = 2/j\omega\,(1 + j0.5\omega)\,(1 + j0.2\omega)$

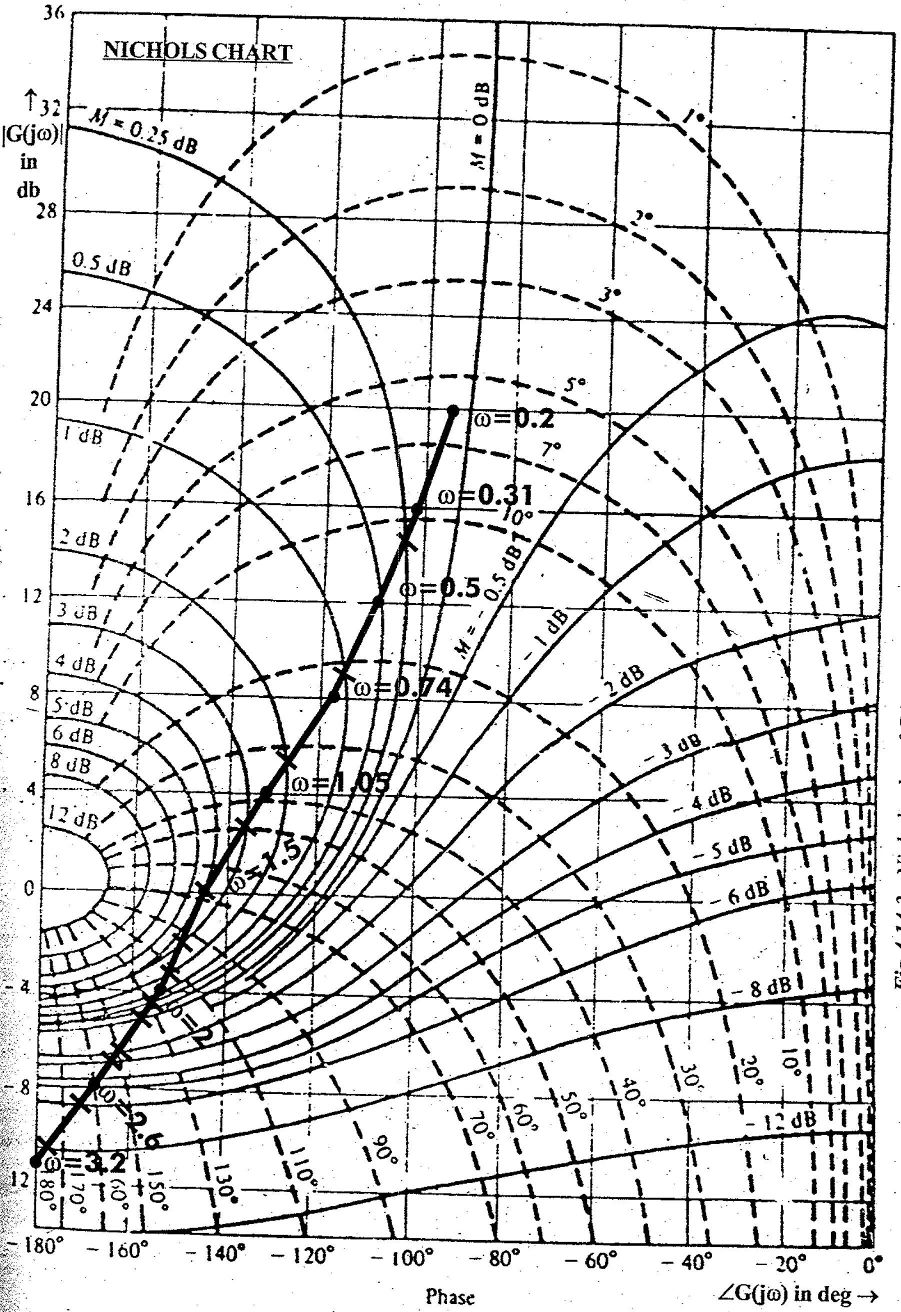


Fig 4.14.3 : Nichols plot of $G(j\omega) = 2/j\omega\,(1 + j0.5\omega)\,(1 + j0.2\omega)$

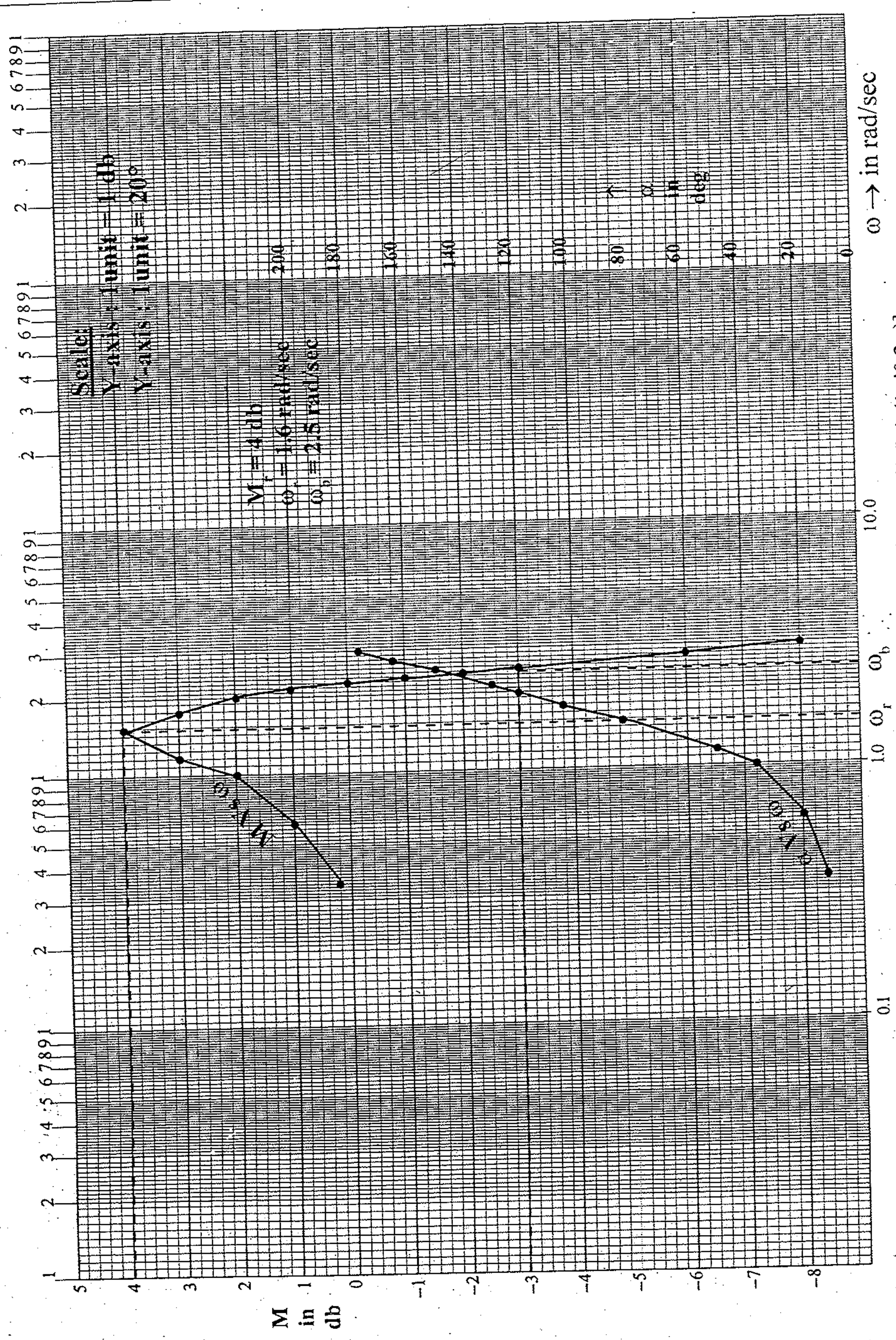


Fig 4.15.3 : Closed loop frequency resposne of $G(j\omega) = 2/[j\omega\ (1+j0.5\omega)\ (1+j0.2\omega)]$

4.12 SHORT QUESTION AND ANSWER

Q4.1 *What is frequency response ?*

The frequency response is a steady-state output of the system, when the input is a sinusoidal signal

Q4.2 *What are advantages of frequency response analysis ?*

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of the open loop frequency response.
2. The practical testing of system can be easily carried with available sinusoidal signal generators and precise measurement equipments.
3. The transfer function of complicated functions can be determined experimentally by frequency response tests.
4. The design and parameter adjustment can be carried more easily.
5. The corrective measure for noise disturbance and parameter variation can be easily carried.
6. It can be extended to certain non-linear systems.

Q4.3 *What are frequency domain specifications?*

The frequency domain specifications indicates the performance of the system in frequency domain, and they are

1. Resonant peak
2. Resonant frequency
3. Bandwidth
4. Cut-off rate
5. Gain margin
6. Phase margin.

Q4.4 *Define Resonant Peak ?*

The maximum value of the magnitude of closed loop transfer function is called Resonant Peak.

Q4.5 *What is Resonant frequency ?*

The frequency at which the resonant peak occurs is called Resonant frequency. The resonant peak is the maximum value of the magnitude of closed loop transfer function.

Q4.6 *Define Bandwidth ?*

The Bandwidth is the range of frequencies for which the system gain is more than –3db.

Q4.7 *What is cut-off rate ?*

The slope of the log-magnitude curve near the cut-off frequency is called cut-off rate.

Q4.8 *Define gain margin ?*

The gain margin, K_g is defined as the reciprocal of the magnitude of open loop transfer function, at phase cross-over frequency, ω_{pc}

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$$

When expressed in decibels, it is given by, the negative of db magnitude of $G(j\omega)$ at phase cross-over frequency.

$$\text{Gain margin in db} = 20\log\frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = -20\log|G(j\omega)|_{\omega=\omega_{pc}}$$

Q4.9 *Define phase margin?*

The phase margin, γ is that amount of additional phase lag at the gain cross-over frequency, ω_{gc} required to bring the system to the verge of instability. It is given by, $180° + \phi_{gc}$, where ϕ_{gc} is the phase of $G(j\omega)$ at the gain cross over frequency.

$$\text{Phase margin, } \gamma = 180° + \phi_{gc}$$

$$\text{Where, } \phi_{gc} = \text{Arg}\,[G(j\omega)]\Big|_{\omega=\omega_{gc}}$$

Q4.10 *What is phase and Gain cross-over frequency ?*

The gain cross-over frequency is the frequency at which the magnitude of the open loop transfer function is unity.

The phase cross-over frequency is the frequency at which the phase of the open loop transfer function is 180°.

Q4.11 *Write the expression for resonant peak and resonant frequency,*

$$\text{Resonant Peak, } M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\text{Resonant frequency, } \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Q4.12 *Write a short note on the correlation between the time and frequency response?*

There exist a correlation between time and frequency response of first or second order systems. The frequency domain specification can be expressed in terms of the time domain parameters ζ and ω_n. For a peak overshoot in time domain there is a corresponding resonant peak in frequency domain.

For higher order systems there is no explicit correlation between time and frequency response. But if there is a pair of dominant complex conjugate poles, then the system can be approximated to second order system and the correlation between time and frequency response can be estimated.

Q4.13 *The damping ratio and natural frequency of oscillation of a second order system is 0.5 and 8 rad/sec respectively. Calculate the resonant peak and resonant frequency?*

$$\text{Resonant peak, } M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2\times(0.5)\sqrt{1-(0.5)^2}} = 1.154$$

$$\text{Resonant frequency, } \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

$$= 8\times\sqrt{1-2\times 0.5^2} = 5.657\ \text{rad/sec}$$

Q4.14 *What is Bode plot ?*

The bode plot is a frequency response plot of the transfer function of a system. It consists of two plots-magnitude plot and phase plot.

The magnitude plot is a graph between magnitude of a system transfer function in db and the frequency ω. The phase plot is a graph between the phase or argument of a system transfer function in degrees and the frequency ω. Usually, both the plots are plotted on a common x-axis in which the frequencies are expressed in logarithmic scale.

Q4.15 *What is approximate bode plot ?*

In approximate bode plot, the magnitude plot of first and second order factors are approximated by two straight lines, which are asymptotes to exact plot. One straight line is at 0db, for the frequency range 0 to ω_c and the other straight line is drawn with a slope of $\pm$20n db/dec for the frequency range ω_c to ∞. Here ω_c is the corner frequency.

Q4.16 *Define corner frequency ?*

The magnitude plot can be approximated by asymptotic straight lines. The frequencies corresponding to the meeting point of asymptotes are called corner frequency. The slope of the magnitude plot changes at every corner frequencies.

Q4.17 *What are the advantages of Bode Plot ?*

1. The magnitudes are expressed in db and so a simple procedure is available to add magnitude of each term one by one.
2. The approximate bode plot can be quickly sketched, and the corrections can be made at corner frequencies to get the exact plot.
3. The frequency domain specifications can be easily determined.
4. The bode plot can be used to analyse both open loop and closed loop system.

Q4.18 *What is the value of error in the approximate magnitude plot of a first order factor at the corner frequency ?*

The error in the approximate magnitude plot of a first order factor at the corner frequency is $\pm$3mdb, where m is multiplicity factor. Positive error for numerator factor and negative error for denominator factor.

Q4.19 *What is the value of error in the approximate magnitude plot of a quadratic factor with ζ=1 at the corner frequency ?*

The error is $\pm$6db, for the quadratic factor with ζ=1. Positive error for numerator factor and negative error for denominator factor.

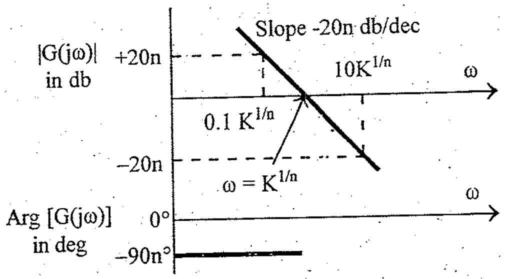
Q4.20 *Draw the bode plot of,* $G(s) = \frac{K}{s^n}$.

Let $s = j\omega$,

$$\therefore G(j\omega) = \frac{K}{(j\omega)^n}$$

The magnitude of G(jω) is unity when $\omega = K^{1/n}$.

The magnitude plot is a straight line with slope of -20n db/dec and passing through $\omega = K^{1/n}$. The phase plot is straight line parallel to x-axis at $-90n°$.

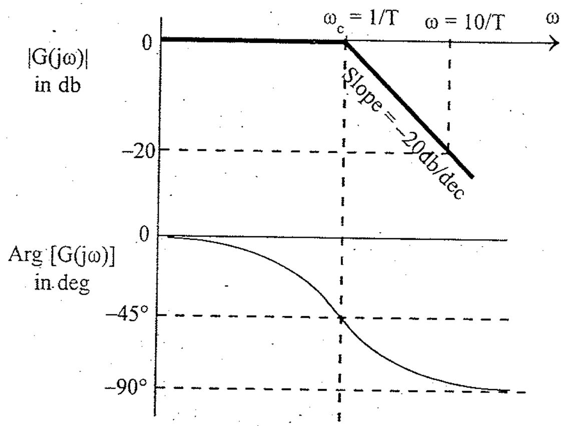


Q4.21 *Sketch the bode plot of G(s) = 1/(1+sT)..*

Let $s = j\omega$, $\therefore G(j\omega) = \frac{1}{1+j\omega T}$; The corner frequency, $\omega_c = \frac{1}{T}$

The magnitude plot is approximated by two straight lines-one straight line at 0db in the frequency range 0 to ω_c and the other straight line with the slope of –20db/dec in the frequency range ω_c to ∞.

The phase of G(jω) varies from 0 to –90° as ω is varied from 0 to ∞. Hence, the phase plot is a curve passing through –45° at the corner frequency.



Q4.22 *What is polar plot ?*

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle/argument of G(jω) on polar or rectangular co-ordinates as ω is varied from zero to infinity.

Q4.23 *What is minimum phase system ?*

The minimum phase systems are systems with minimum phase transfer functions. In minimum phase transfer functions, all poles and zeros will lie on the left half of s-plane.

Q4.24 *What is All-Pass systems ?*

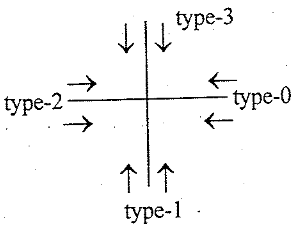
The all pass systems are systems with all pass transfer functions. In all pass transfer functions, the magnitude is unity at all frequencies and the transfer function will have anti-symmetric pole zero pattern (i.e., for every pole in the left half s-plane, there is a zero in the mirror image position with respect to imaginary axis).

Q4.25 *What is non-minimum phase transfer function ?*

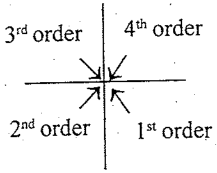
A transfer function which has one or more zeros in the right half s-plane is known as non-minimum phase transfer function.

Q4.26 *In minimum phase system, how the start and end of polar plot are identified ?*

For minimum phase transfer functions, with only poles, the type number of the sytem determines the quadrant in which the polar plot starts, and the order of a system determines the quadrant in which the polar plot ends.



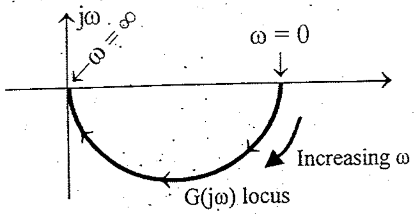
Start of polar plot



End of Polar Plot

Q4.27 *Draw the polar plot of G(s)= 1(1+sT).*

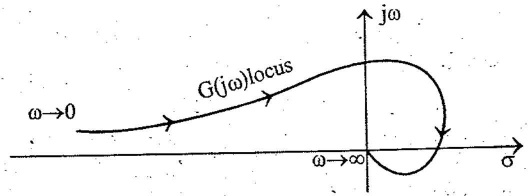
Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{1}{1+j\omega T}$



The given system is of type-0 and order =1. Hence, $G(j\omega)$ locus starts in fourth quadrant at 0° and ends in fourth quadrant along 90° axis.

Q4.28 *Sketch the polar plot of,* $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$.

The given system is all pole minimum phase system. The type number of the system is 2 and the order is 5. Hence, the polar plot starts in second quadrant and ends in fourth quadrant.



Q4.29 *What is a Nichols plot?*

The Nichols plot is a frequency response plot of the open loop transfer function of a system. It is a graph between magnitude of $G(j\omega)$ in db and the phase of $G(j\omega)$ in degree, plotted on a ordinary graph sheet.

Q4.30 *What are M and N circles?*

The magnitude, M of closed loop transfer function with unity feedback will be in the form of circle in complex plane for each constant value of M. The family of these circles are called M circles.

Let $N = \tan \alpha$ where α is the phase of closed loop transfer function with unity feedback. For each constant value of N, a circle can be drawn in the complex plane. The family of these circles are called N circles.

Q4.31 *How closed loop frequency response is determined from open loop frequency response using M and N circles?*

The $G(j\omega)$ locus or the polar plot of open loop system is sketched on the standard M and N circles chart. The meeting point of M circle with $G(j\omega)$ locus gives the magnitude of closed loop system. (the frequency being same as that of open loop system). The meeting point of $G(j\omega)$ locus with N-circle gives the value of phase of closed loop system. (the frequency being same as that of open loop system).

Q4.32 *What is Nichols chart?*

The Nichols chart consists of M and N contours superimposed on ordinary graph. Along each M contour the magnitude of closed loop system, M will be a constant. Along each N contour, the phase α of closed loop system will be constant. The ordinary graph consists of magnitude in db, marked on the y-axis and the phase in degrees marked on x-axis. The Nichols chart is used to find the closed loop frequency response from the open loop frequency response.

Q4.33 *How the closed loop frequency response is determined from the open loop frequency response using Nichols chart?*

The $G(j\omega)$ locus or the Nichols plot is sketched on the standard Nichols chart. The meeting point of M contour with $G(j\omega)$ locus gives the magnitude of closed loop system and the meeting point with N circle gives the argument/phase of the closed loop system.

Q4.34 *What are the advantages of Nichols chart?*

1. It is used to find closed loop frequency response from open loop frequency response.
2. The frequency domain specifications can be determined from Nichols chart.
3. The gain of the system can be adjusted to satisfy the given specification.

4.13 EXERCISES

E4.1 Sketch the bode plot of the following open loop transfer functions and from the plot determine the phase margin and gain margin.

a) $G(s)=100(1+0.1s)/s(1+0.2s)(1+0.5s)$

b) $G(s)=50(1+0.1s)/(1+0.01s)(1+s)$

c) $G(s)=30(1+0.1s)/s(1+0.01s)(1+s)$

d) $G(s)=s^2(s+10)/(s+5)^2(s+0.1)$

e) $G(s)=40(1+s)/(1+5s)(s^2+2s+4)$

f) $G(s)=10(1+s)\,e^{-0.1s}/s(1+0.2s)$

E4.2 The open loop transfer function of a system is given by $G(s)=K/s(1+0.5s)(1+0.2s)$. Using bode plot find the value of K so that (i) The gain margin of the system is 6db and (ii) The phase margin of the system is 25°.

E4.3 Sketch the polar plot of the following transfer functions and from the plot, determine the phase margin and gain margin.

a) $G(s)=10(s+1)/(s+10)^2$

b) $G(s)=200(s+2)/s(s^2+10s+100)$

c) $e^{-0.1s}/s(s+1)(s+5)$

d) $1/s(s+4)(s+8)$

E4.4 The open loop transfer function of a system is given by $G(s)=K/s(s^2+s+4)$. Using polar plot, determine the value of K, so that phase margin is 50°. What is the corresponding value of gain margin?

E4.5 A unity feedback system has $G(s)=K/s(1+0.1s)$. Using Nichols chart find the value of K so that resonant peak, $M_r=1.4$. Find the corresponding value of ω_r.

E4.6 The open loop transfer function of unity feedback system is, $G(s)=K/(1+0.05s)(1+0.1s)(1+0.3s)$.

Using Nichols chart find the value of K so that gain margin of the system is 10db. What is the corresponding value of phase margin.

E4.7 Using Nichols chart determine the closed loop frequency response of the unity feedback system, whose open loop transfer function is, $G(s)=200(s+1)/s(s+10)^2$.

E4.8 A unity feedback system has open loop transfer function $G(s)=54/(1+0.1s)(s^2+8s+25)$.

Using Nichols chart determine the closed loop frequency response. From the closed loop response determine, the resonant peak, resonant frequency and bandwidth.

CHAPTER FIVE

5 CONCEPTS OF STABILITY AND ROOT LOCUS

5.1 DEFINITIONS OF STABILITY

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system the response or output is predictable, finite and stable for a given input (or changes in input or changes in system parameters).

The different definitions of the stability are the following

1. A system is stable if its output is bounded (finite) for any bounded (finite) input.
2. A system is asymptotically stable if in the absence of the input, the output tends towards zero (or to the equilibrium state) irrespective of initial conditions.
3. A system is stable if for a bounded disturbing input signal the output vanishes ultimately as t approaches infinity.
4. A system is unstable if for a bounded disturbing input signal the output is of infinite amplitude or oscillatory.

5. For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable.

6. If a system output is stable for all variations of its parameters then the system is called absolutely stable system.

7. If a system output is stable for a limited range of variations of its parameters, then the system is called conditionally stable system.

RESPONSE OF A SYSTEM

Let the closed loop transfer function (or overall transfer function) $\left.\right\} \frac{C(s)}{R(s)} = M(s)$

$\therefore$ The response or output in s-domain, $C(s) = M(s)\,R(s)$

Let $c(t) = L^{-1}[C(s)]$ (Response in time domain)

$r(t) = L^{-1}[R(s)]$ (Input in time domain)

For an impulse input, $r(t) = \delta(t)$; $\therefore R(s) = L[\delta(t)] = 1$

$$\text{The impulse response of the system} = L^{-1}[C(s)] = L^{-1}[M(s)\,R(s)]$$
$$= L^{-1}[M(s)] = m(t) \quad .....(5.1)$$

Hence the impulse response of a system is the inverse Laplace transform of the system transfer function. The advantage of impulse response is that the knowledge of impulse response can be used to determine the output of a system for any arbitrary input.

The output in s-domain is the product of transfer function and input in s-domain. The convolution theorem states that the inverse Laplace transform of the product of two s-domain functions is equal to the convolution of their time domain functions.

i.e., $L^{-1}[M(s)\,.\,R(s)] = \int_{-\infty}^{+\infty} m(\tau)\, r(t-\tau)\, d\tau$

$\therefore$ The response of a system in time domain $\left.\right\} c(t) = L^{-1}[M(s)\,R(s)] = \int_{-\infty}^{+\infty} m(\tau)\, r(t-\tau)\, d\tau \quad .....(5.2)$

From the equation (5.2) it can be concluded that the response for any arbitrary input is given by convolution of impulse response and the input.

BOUNDED - INPUT BOUNDED - OUTPUT (BIBO) STABILITY

A linear relaxed system is said have BIBO stability if every bounded (finite) input results in a bounded (finite) output. A test for BIBO stability can be obtained from convolution theorem.

The response of a system c(t) for any input is given by convolution of the input and impulse response.

The response in time domain, $c(t) = \int_{0}^{\infty} m(\tau)\, r(t-\tau)\, d\tau \quad .....(5.3)$

Note : A relaxed system is one in which the initial conditions are zero. Hence the limits of integration is from 0 to ∞.

If the input r(t) is bounded then there exist a constant A_1, such that $|r(t)| \le A_1 < \infty$. The condition for bounded output for this bounded input condition can be derived as follows.

On taking the absolute value on both sides of equation (5.3), we get,

$$\therefore |c(t)| = \left| \int_{0}^{\infty} m(\tau)\, r(t-\tau)\, d\tau \right| \quad .....(5.4)$$

Since the absolute value of an integral is not greater than the integral of the absolute value of the integrand the equation (5.4) can be written as,

$$|c(t)| \le \int_{0}^{\infty} |m(\tau)\, r(t-\tau)|\, d\tau$$
$$\le \int_{0}^{\infty} |m(\tau)|\, |r(t-\tau)| d\tau$$
$$\le \int_{0}^{\infty} |m(\tau)|\, A_1 d\tau \qquad (\because \text{For bounded input } |r(t-\tau)| \le A_1)$$
$$\le A_1 \int_{0}^{\infty} |m(\tau)|\, d\tau \quad .....(5.5)$$

If the output c(t) is bounded then there exists a constant A_2 such that $|c(t)| \le A_2 < \infty$.

$$\therefore A_1 \int_{0}^{\infty} |m(\tau)|\, d\tau \le A_2 < \infty \quad .....(5.6)$$

The above condition is satisfied if

$$\int_0^{\infty} |m(\tau)|\, d\tau \quad < \infty$$

Hence for bounded output, $\int_0^{\infty} |m(\tau)|\, d\tau < \infty$

Therefore we can conclude that a system with impulse response $m(\tau)$ is BIBO stable if and only if the impulse response is absolutely integrable (i.e., $\int_0^{\infty} |m(\tau)|\, d\tau$ is finite. This means that area under the absolute value curve of the impulse response $m(\tau)$ evaluated from $t = 0$ to $t = \infty$ must be finite).

5.2 LOCATION OF ROOTS ON THE S-PLANE FOR STABILITY

The closed loop transfer function, M(s) can be expressed as a ratio of two polynomials in s. The denominator polynomial of closed loop transfer function is called characteristic equation. The roots of characteristic equation are poles of closed loop transfer function.

For BIBO stability the integral of impulse response should be finite, which implies that the impulse response should be finite as t tends to infinity. *[The impulse response is the inverse Laplace transform of the transfer function]*. This requirement for stability can be linked to the location of roots of characteristic equation in the s-plane.

The closed loop transfer function M(s) can be expressed as a ratio of two polynomials in s as shown in equation (5.7).

$$M(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots\ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_{n-1} s + a_n} \qquad \ldots..(5.7)$$

$$= \frac{(s+z_1)\,(s+z_2)\,(s+z_3) \ldots\ldots (s+z_m)}{(s+p_1)\,(s+p_2)\,(s+p_3) \ldots\ldots (s+p_n)} \qquad \ldots..(5.8)$$

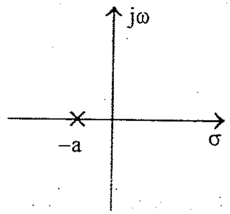
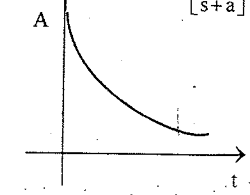
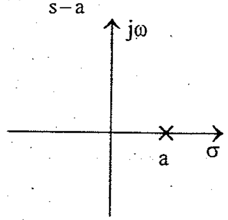
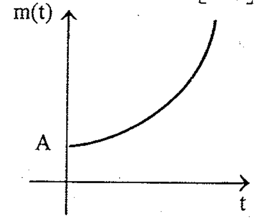
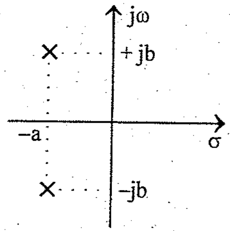
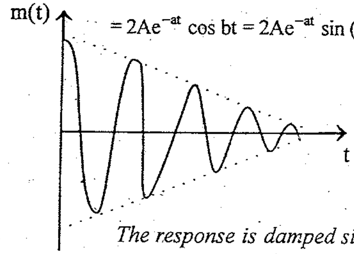
The roots of numerator polynomial $z_1, z_2, \ldots..z_n$ are zeros. The roots of denominator polynomial $p_1, p_2, \ldots p_n$ are poles. The denominator polynomial is the characteristic equation and so the poles are roots of characteristic equation.

By partial fraction expansion we can write,

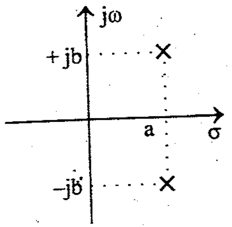
$$M(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + \ldots\ldots + \frac{A_n}{s+p_n} \qquad \ldots..(5.9)$$

The poles $p_1, p_2, p_3, \ldots.. p_n$ may be at origin or lying on imaginary axis or lying on right or left half s-plane. The impulse response is given by inverse Laplace transform of M(s). The inverse Laplace transform of each term of M(s) depends on the location of poles. The impulse response of various terms of M(s) are shown in table 5.1.

TABLE 5.1.

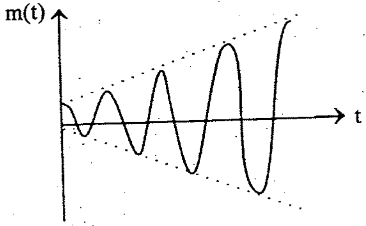
Transfer function and location of roots on s-plane	Impulse response (inverse Laplace transform and the sketch of time domain function)
$M(s) = \frac{A}{s+a}$  *Root on negative real axis*	$m(t) = L^{-1}\left[\frac{A}{s+a}\right] = Ae^{-at}$  *Response is exponentially decaying*
$M(s) = \frac{A}{s-a}$  *Root on positive real axis*	$m(t) = L^{-1}\left[\frac{A}{s-a}\right] = Ae^{+at}$  *Response is exponentially increasing*
$M(s) = \frac{A}{s+a+jb} + \frac{A}{s+a-jb}$  *Complex conjugate roots on left half s-plane*	$m(t) = L^{-1}\left[\frac{A}{s+a+jb} + \frac{A}{s+a-jb}\right]$ $= Ae^{-(a+jb)t} + Ae^{-(a-jb)t}$ $= 2Ae^{-at}\cos bt = 2Ae^{-at}\sin(bt+90°)$  *The response is damped sinusoidal (i.e., damped oscillatory)*

$$M(s) = \frac{A}{s-a+jb} + \frac{A}{s-a-jb}$$



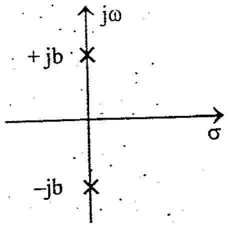
Complex conjugate roots on right half s-plane

$$m(t) = L^{-1}\left[\frac{A}{s-a+jb} + \frac{A}{s-a-jb}\right]$$
$$= Ae^{-(-a+jb)t} + Ae^{-(-a-jb)t}$$
$$= 2Ae^{at}\cos bt = 2Ae^{at}\sin(bt+90°)$$



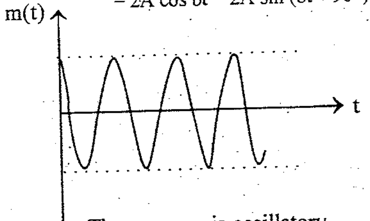
The response is exponentially increasing sinusoidal (i.e., amplitude of oscillations exponentially increases with time)

$$M(s) = \frac{A}{s+jb} + \frac{A}{s-jb}$$



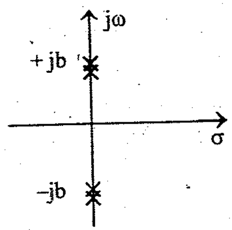
Single pair of roots on imaginary axis

$$m(t) = L^{-1}\left[\frac{A}{s+jb} + \frac{A}{s-jb}\right]$$
$$= Ae^{-jb} + Ae^{+jb}$$
$$= 2A\cos bt = 2A\sin(bt+90°)$$



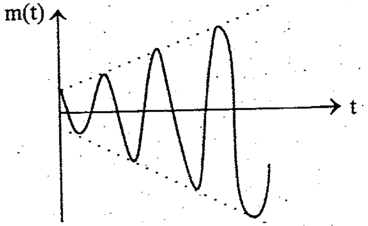
The response is oscillatory

$$M(s) = \frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}$$



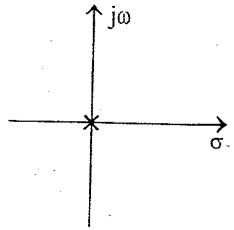
Double pair of roots on imaginary axis

$$m(t) = L^{-1}\left[\frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}\right]$$
$$= At\,e^{-jb} + At\,e^{jb}$$
$$= 2At\cos bt = 2At\sin(bt+90°)$$



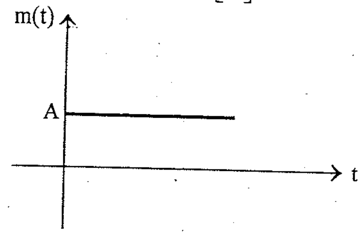
Response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time)

$$M(s) = \frac{A}{s}$$



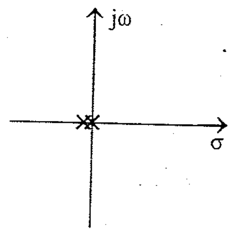
Single root at origin

$$m(t) = L^{-1}\left[\frac{A}{s}\right] = A$$



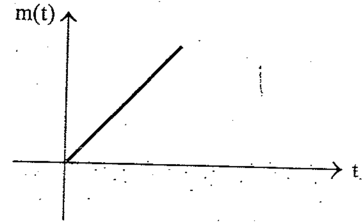
The response is constant

$$M(s) = \frac{A}{s^2}$$



Double root at origin

$$m(t) = L^{-1}\left[\frac{A}{s^2}\right] = At$$



The response linearly increases with time

From table 5.1, the following conclusions are drawn based on the location of roots of characteristic equation.

1. If all the roots of characteristic equation have negative real parts (i.e., lying on left half s-plane) then the impulse response is bounded (i.e., it decreases to zero as t tends to ∞). Hence $\int_0^\infty m(\tau)d\tau$ is finite and the system is bounded-input bounded-output stable.

2. If any root of the characteristic equation has a positive real part (i.e., lying on right half s-plane) then impulse response is unbounded, (i.e., it increases to ∞ as t tends to ∞). Hence $\int_0^\infty m(\tau)d\tau$ is infinite and so system is unstable.

3. If the characteristic equation has repeated roots on the imaginary axis then impulse response is unbounded (i.e., it increases to ∞ as t tends to ∞). Hence $\int_0^\infty m(\tau)d\tau$ is infinite and so the system is unstable.

4. If one or more non repeated roots of the characteristic equation are lying on the imaginary axis, then impulse response is bounded (i.e., it has constant amplitude oscillations) but $\int_0^\infty m(\tau)d\tau$ is infinite and so the system is unstable.

5. If the characteristic equation has single root at origin then the impulse response is bounded (i.e, it has constant amplitude) but $\int_0^\infty m(\tau)d\tau$ is infinite and so the system is unstable.

6. If the characteristic equation has repeated roots at origin then the impulse response is unbounded (i.e., it linearly increases to infinity as t tends to ∞) and so the system is unstable.

7. In system with one or more repeated roots on imaginary axis or with single root at origin, the output is bounded for bounded inputs except for the inputs having poles matching the system poles. These cases may be treated as acceptable or non-acceptable. Hence when the system has non repeated poles on imaginary axis or single pole at origin, it is referred as limitedly or marginally stable system.

In summary the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

(i) If all the roots of characteristic equation has negative real parts, then the system is stable.

(ii) If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.

(iii) If the condition (i) is satisfied except for the presence of one or more non-repeated roots on the imaginary axis, then the system is limitedly or marginally stable.

In order to ascertain the stability of a system, it is necessary to determine if any of the roots of the characteristic equation lie in the right half s-plane. The characteristic equation is given by the denominator polynomial of closed loop transfer function, [equation (5.7)].

Consider the n^{th} order characteristic equation shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots\ldots\ldots a_{n-1} s + a_n = 0 \qquad \ldots..(5.10)$$

Let the roots of n^{th} order characteristic equation [equation (5.10)] be $s = r_1, r_2, \ldots r_n$. These roots are functions of the coefficients $a_0, a_1, a_2, \ldots. a_{n-1}, a_n$.

Consider a second order polynomial,

$$\begin{aligned} a_0 s^2 + a_1 s + a_2 &= a_0 \left(s^2 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} \right) \\ &= a_0 (s - r_1)(s - r_2) \\ &= a_0 s^2 - a_0 (r_1 + r_2) s + a_0 r_1 r_2 \end{aligned} \qquad \ldots..(5.11)$$

Consider a third order polynomial

$$\begin{aligned} a_0 s^3 + a_1 s^2 + a_2 s + a_3 &= a_0 \left(s^3 + \frac{a_1}{a_0} s^2 + \frac{a_2}{a_0} s + \frac{a_3}{a_0} \right) \\ &= a_0 (s - r_1)(s - r_2)(s - r_3) \\ &= a_0 s^3 - a_0 (r_1 + r_2 + r_3) s^2 \\ &\quad + a_0 (r_1 r_2 + r_1 r_3 + r_2 r_3) s - a_0 r_1 r_2 r_3 \end{aligned} \qquad \ldots..(5.12)$$

On extending this expansion to the n^{th} order polynomial, we get.

$$\begin{aligned} a_0 s^n + a_1 s^{n-1} + \ldots..+ a_{n-1} s + a_n &= a_0 s^n - a_0 (\text{sum of all the roots})\, s^{n-1} \\ &\quad + a_0 \left(\begin{matrix} \text{sum of the products of the roots} \\ \text{taken 2 at a time} \end{matrix} \right) s^{n-2} \\ &\quad - a_0 \left(\begin{matrix} \text{sum of the products of the roots} \\ \text{taken 3 at a time} \end{matrix} \right) s^{n-3} \\ &\quad + \ldots\ldots\ldots + a_0 (-1)^n \,(\text{Product of all the n roots}) \end{aligned} \qquad \ldots..(5.13)$$

If all the roots of a polynomial are real and in the left half s-plane, then all r_i in equations (5.11) and (5.12) are real and negative. Therefore all polynomial coefficients are positive. This characteristic also applies to the general case of equation (5.13). If atleast one root is in the right half of s-plane then some of the coefficients will be negative. Also, it can be observed that if all the roots are in the left half plane, no coefficient can be zero.

Since the characteristic polynomial coefficients are real, the complex roots should occur as conjugate pairs. From equation (5.13) it can be inferred that when polynomial coefficients are formed, the imaginary parts of roots/products of roots will cancel. Therefore, if all roots occur in the left half plane, (whether it is complex or real) then all coefficients of the general polynomial of equation (5.13) will be positive. Presence of a negative coefficient implies that there is atleast one root in the right half plane.

A zero coefficient indicates presence of complex-conjugate roots on the imaginary axis and/or one or more roots in the right half plane.

In summary the following conclusions can be made about the coefficients of characteristic polynomial

1. If all the coefficients are positive and if no coefficient is zero then all the roots are in the left half plane.

2. If any coefficient a_i is equal to zero then some of the roots may be on the imaginary axis or on the right half plane.

3. If any coefficient a_i is negative then atleast one root is in the right half plane.

It can be concluded that the absence or negativeness of any of the coefficients of a characteristic polynomial indicates that the system is either unstable or at most marginally stable. Thus the necessary condition for stability of the system is that all the coefficients of its characteristic polynomial be positive. If any coefficient is zero/negative, we can immediately say that the system is unstable.

When all the coefficients are positive, the system is not necessarily stable, because there may be roots in the right half plane and/or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all of the coefficients of characteristic equation be positive. *(For example, consider the characteristic polynomial with all positive coefficients $s^3 + s^2 + 2s + 8 = 0$. The characteristic polynomial can be written as*

$$(s^3+s^2+2s+8)=(s+2)\left(s-\frac{1}{2}-j\frac{\sqrt{15}}{2}\right)\left(s-\frac{1}{2}+j\frac{\sqrt{15}}{2}\right)=0$$

The coefficients of the polynomial are all positive but two roots are on right half plane and so the system is unstable).

5.3 ROUTH HURWITZ CRITERION

The closed loop transfer function of a system is a ratio of two polynomials in s. The denominator polynomial of closed loop transfer function is called the characteristic equation of the system.

The roots of the characteristic equation of a stable system should lie on the left of s-plane. Hence the roots should have negative real parts. The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real parts or not.

The first step in analysing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

When all the coefficients are positive, the system is not necessarily stable. Eventhough the coefficient are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all of the coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability.

A. Hurwitz and E.J. Routh independently published the method of investigating the sufficient conditions of stability of a system. The Hurwitz criterion is in terms of determinants and Routh criterion is in terms of array formulation. The Routh stability criterion is presented here.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation, $a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n = 0$, where $a_0 > 0$ into a schedule, called the Routh array as shown below.

$$\begin{array}{lclllll}
s^n & : & a_0 & a_2 & a_4 & a_6 & a_8 \;\dots\dots \\
s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & a_9 \;\dots\dots \\
s^{n-2} & : & b_0 & b_1 & b_2 & b_3 & b_4 \;\dots\dots \\
s^{n-3} & : & c_0 & c_1 & c_2 & c_3 & c_4 \;\dots\dots \\
\vdots & & & & & & \\
s^1 & : & g_0 & & & & \\
s_0 & : & h_0 & & & &
\end{array}$$

The Routh stability criterion can be stated as follows.

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

***Note:** If the order of sign of first column element is +, +, –, + and +. Then + to – is considered as one sign change and – to + as another sign change.*

CONSTRUCTION OF ROUTH ARRAY

Let the characteristic polynomial be

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \ldots . a_{n-1} s^1 + a_n s^0$$

The coefficients of the polynomial are arranged in two rows as shown below.

$$\begin{array}{llllll} s^n & : & a_0 & a_2 & a_4 & a_6 \ldots\ldots \\ s^{n-1} & : & a_1 & a_3 & a_5 & a_7 \ldots\ldots \end{array}$$

If n is even then s^n row is formed by coefficients of even order terms and s^{n-1} row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s).

If n is odd, then s^n row is formed by coefficients of odd order term and s^{n-1} row is formed by coefficients of even order terms (i.e., coefficients of even powers of s).

The other rows of routh array upto s^0 row can be formed by the following procedure. Each row of Routh array is constructed by using the elements of previous two rows.

Consider two consecutive rows of Routh array as shown below.

$$\begin{array}{lllllll} s^{n-x} & : & x_0 & x_1 & x_2 & x_3 & x_4 \ldots\ldots \\ s^{n-x-1} & : & y_0 & y_1 & y_2 & y_3 & y_4 \ldots\ldots \end{array}$$

Let the next row be,

$$\begin{array}{llllll} s^{n-x-2} & : & z_0 & z_1 & z_2 & z_3 \ldots\ldots \end{array}$$

The elements of s^{n-x-2} row are given by,

$$z_0 = \frac{(-1)\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0 x_1 - y_1 x_0}{y_0}$$

$$z_1 = \frac{(-1)\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0 x_2 - y_2 x_0}{y_0}$$

$$z_2 = \frac{(-1)\begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix}}{y_0} = \frac{y_0 x_3 - y_3 x_0}{y_0}$$

$$z_3 = \frac{(-1)\begin{vmatrix} x_0 & x_4 \\ y_0 & y_4 \end{vmatrix}}{y_0} = \frac{y_0 x_4 - y_4 x_0}{y_0}$$

The elements $z_0, z_1, z_2, z_3, \ldots$ are computed until an element equals to zero or for all possible computations as shown above.

In the process of constructing Routh array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied or divided by a positive constant to simplify the computational work.

In the construction of Routh array one may come across the following three cases.

Case I : Normal Routh array (Non-zero elements in the first column of routh array)

Case II : A row of all zeros

Case III : First element of a row is zero but some or other elements are not zero

CASE I : NORMAL ROUTH ARRAY

In this case there is no difficulty in forming routh array. The routh array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of s-plane and the stability of the system can be estimated.

In this case if there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

CASE II : A ROW OF ALL ZEROS

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is called the auxiliary polynomial. The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrantal symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes. The fig 5.1 shows the roots of an even polynomial.

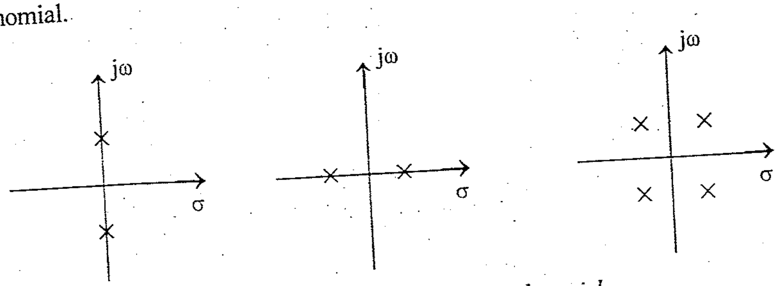


Fig 5.1 : The roots of an even polynomial

The case II polynomial can be analyzed by any one of the following two methods.

METHOD 1

1. Determine the auxiliary polynomial, A(s)
2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds
3. The row of zeros is replaced with coefficients of dA(s)/ds.
4. Continue the construction of the array in the usual manner (as that of case I) and the array is interpreted as follows.
   a. If there are sign changes in the first column of routh array then the system is unstable. The number of roots lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

   b. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

METHOD 2

1. Determine the auxiliary polynomial, A(s).
2. Divide the characteristic equation by auxiliary polynomial.
3. Construct Routh array using the coefficients of quotient polynomial
4. The array is interpretted as follows.
   a. If there are sign changes in the first column of routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of s-plane is given by number of sign changes in first column of routh array.

      The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex.

      The total number of roots on right half of s-plane is given by the sum of number of sign changes and the number of roots of auxiliary polynomial with positive real part. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

   b. If there is no sign change in the first column of routh array of quotient polynomial then the system is limitedly or marginally stable. Since there is no sign change all the roots of quotient polynomial are lying on the left half of s-plane.

      The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex. The number of roots lying on imaginary axis and on the right half of s-plane can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

CASE III : FIRST ELEMENT OF A ROW IS ZERO

While constructing routh array, if a zero is encountered as first element of a row then all the elements of the next row will be infinite. To overcome this problem let $0 \rightarrow \epsilon$ and complete the construction of array in the usual way (as that of case I)

Finally let $\epsilon \rightarrow 0$ and determine the values of the elements of the array which are functions of ϵ. The resultant array is interpreted as follows.

Note : If all the elements of a row are zeros then the solution is attempted by considering the polynomial as case II polynomial. Even if there is a single element zero on s^1 row, it is considered as a row of all zeros.

a. If there is no sign change in first column of routh array and if there is no row with all zeros, then all the roots are lying on left half of s-plane and the system is stable.

b. If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

c. If there is a row of all zeros after letting $\in \to 0$, then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial and divide the characteristic equation by auxiliary polynomial to eliminate the imaginary roots. The routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpreted as explained in method-2 of case-II polynomial

EXAMPLE 5.1

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is $s^4+8s^3+18s^2+16s+5=0$

The given characteristic equation is 4^{th} order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	18	5	.... Row-1
s^3	:	8	16		.... Row-2

The elements of s^3 row can be divided by 8 to simplify the computations.

s^4	:	1	18	5	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	16	5		.... Row-3
s^1	:	1.7			.... Row-4
s^0	:	5			.... Row-5

Column-1

$$s^2 : \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$$
$$s^2 : 16 \quad 5$$
$$s^1 : \frac{16\times2-5\times1}{16}$$
$$s^1 : 1.6875 \approx 1.7$$
$$s^0 : \frac{1.7\times5-0\times16}{1.7}$$
$$s^0 : 5$$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

EXAMPLE 5.2

Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

The characteristic equation of the system is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

The given characteristic polynomial is 6^{th} order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^6	:	1	8	20	16	.... Row-1
s^5	:	2	12	16		.... Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.

s^6	:	1	8	20 16	....Row-1
s^5	:	1	6	8	....Row-2
s^4	:	1	6	8	....Row-3
s^3	:	0	0		....Row-4
s^3	:	1	3		....Row-4
s^2	:	3	8		....Row-5
s^1	:	0.33			....Row-6
s^0	:	8			....Row-7

Column-1

$$s^4 : \frac{1\times8-6\times1}{1} \quad \frac{1\times20-8\times1}{1} \quad \frac{1\times16-0\times1}{1}$$

$$s^4 : \quad 2 \quad 12 \quad 16$$

divide by 2

$$s^4 : \quad 1 \quad 6 \quad 8$$

$$s^3 : \frac{1\times6-6\times1}{1} \quad \frac{1\times8-8\times1}{1}$$

$$s^3 : \quad 0 \quad 0$$

The auxiliary equation is $A = s^4+6s^2+8$. On differentiating A with respect to s we get

$$\frac{dA}{ds} = 4s^3 + 12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row $\quad s^3 : 4 \quad 12$

divide by 4

$s^3 : 1 \quad 3$

$$s^2 : \frac{1\times6-3\times1}{1} \quad \frac{1\times8-0\times1}{1}$$

$$s^2 : \quad 3 \quad 8$$

$$s^1 : \frac{3\times3-8\times1}{3}$$

$$s^1 : 0.33$$

$$s^0 : \frac{0.33\times8-0\times3}{0.33}$$

$$s^0 : 8$$

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is

$$s^4 + 6s^2 + 8 = 0$$

Let $s^2 = x$

$$\therefore x^2 + 6x + 8 = 0$$

The roots of quadratic are, $x = \dfrac{-6 \pm \sqrt{6^2 - 4\times8}}{2}$

$$= -3 \pm 1 = -2 \text{ or } -4$$

The roots of auxiliary polynomial is, $s = \pm\sqrt{x} = \pm\sqrt{-2}$ and $\pm\sqrt{-4}$

$$= +j\sqrt{2},\ -j\sqrt{2},\ +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

EXAMPLE 5.8

Construct Routh array and determine the stability of the system represented by the characteristic equation $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is

$$s^5+s^4+2s^3+2s^2+3s+5=0$$

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2
s^3	:	$\in$	-2		.... Row-3
s^2	:	$\dfrac{2\in+2}{\in}$	5		.... Row-4
s^1	:	$\dfrac{-(5\in^2+4\in+4)}{2\in+2}$			.... Row-5
s^0	:	5			.... Row-6

$$s^3 : \frac{1\times2-2\times1}{1} \quad \frac{1\times3-5\times1}{1}$$

$$s^3 : \quad 0 \quad -2$$

Replace :0 by $\in$

$$s^3 : \ \in \quad -2$$

$$s^2 : \frac{\in\times2-(-2\times1)}{\in} \quad \frac{\in\times5-0\times1}{\in}$$

$$s^2 : \frac{2\in+2}{\in} \quad 5$$

$$s^1 : \frac{\frac{2\in+2}{\in}\times(-2)-(5\times\in)}{\frac{2\in+2}{\in}}$$

$$s^1 : \frac{-(5\in^2+4\in+4)}{2\in+2}$$

$$s^0 : \frac{\frac{-(5\in^2+4\in+4)}{2\in+2}\times5-0\times\frac{2\in+2}{\in}}{\frac{-(5\in^2+4\in+4)}{2\in+2}}$$

$$s^0 : 5$$

On letting $\in \to 0$, we get

s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2
s^3	:	0	-2		.... Row-3
s^2	:	∞	5		.... Row-4
s^1	:	-2			.... Row-5
s^0	:	5			.... Row-6

Column-1

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

RESULT

1. The system is unstable.
2. Two roots are lying on the right half of s-plane and three roots are lying on the left half of s-plane.

EXAMPLE 5.4

By routh stability criterion determine the stability of the system represented by the characteristic equation $9s^5-20s^4+10s^3-s^2-9s-10=0$. Comment on the location of roots of characteristic equation.

SOLUTION

The characteristic polynomial of the system is $9s^5-20s^4+10s^3-s^2-9s-10=0$

On examining the coefficients of the characteristic polynomial, it is found that some of the coefficients are negative and so some roots will lie on the right of s-plane. Hence the system is unstable. The routh array can be constructed to find the number of roots lying on right half of s-plane.

$$9s^5-20s^4+10s^3-s^2-9s-10=0$$

The given characteristic polynomial is 5^{th} order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^5	:	9	10	−9	.... Row-1
s^4	:	−20	−1	−10	.... Row-2
s^3	:	9.55	−13.5		.... Row-3
s^2	:	−29.3	−10		.... Row-4
s^1	:	−16.8			.... Row-5
s^0	:	−10			.... Row-6

Column-1

$s^3: \dfrac{-20\times10-(-1)\times9}{-20}$ $\dfrac{-20\times(-9)-(-10)\times9}{-20}$

s^3: 9.55 −13.5

$s^2: \dfrac{9.55\times(-1)-(-13.5)\times(-20)}{9.55}$ $\dfrac{9.55\times(-10)}{9.55}$

s^2: −29.3 −10

$s^1: \dfrac{-29.3\times(-13.5)-(-10)\times9.55}{-29.3}$

s^1: −16.8

$s^0: \dfrac{-16.8\times(-10)}{-16.8}$

s^0: −10

By examining the elements of I^{st} column of routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is unstable
2. Three roots are lying on the right half and two roots are lying on the left half of s-plane.

EXAMPLE 5.5

The characteristic polynomial of a system is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7^{th} order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations,

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5
s^3	:	2	1			.... Row-5
s^2	:	0.5	1			.... Row-6
s^1	:	−3				.... Row-7
s^0	:	1				.... Row-8

Column-1

$s^5: \dfrac{3\times24-8\times1}{3}$ $\dfrac{3\times24-8\times1}{3}$ $\dfrac{3\times23-5\times1}{3}$

s^5: 21.33 21.33 21.33

Divide by 21.33

s^5: 1 1 1

$s^4: \dfrac{1\times8-1\times3}{1}$ $\dfrac{1\times8-1\times3}{1}$ $\dfrac{1\times5-0\times3}{1}$

s^4: 5 5 5

Divide by 5

s^4: 1 1 1

$s^3: \dfrac{1\times1-1\times1}{1}$ $\dfrac{1\times1-1\times1}{1}$

s^3: 0 0

The auxiliary polynomial is

$$A=s^4+s^2+1$$

Differentiate A with respect to s.

$$\frac{dA}{ds}=4s^3+2s$$

s^3: 4 2

Divide by 2

s^3: 2 1

$s^2: \dfrac{2\times1-1\times1}{2}$ $\dfrac{2\times1-0\times1}{2}$

s^2: 0.5 1

$s^1: \dfrac{0.5\times1-1\times2}{0.5}$

s^1: −3

$s^0: \dfrac{-3\times1}{-3}$

s^0: 1

On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicate the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is $s^4 + s^2 + 1 = 0$

Put $s^2 = x$ in the auxiliary equation,

$$\therefore s^4 + s^2 + 1 = x^2 + x + 1 = 0$$

The roots of quadratic are, $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2}$

$$= 1\angle 120° \text{ or } 1\angle -120°$$

$$\begin{aligned} \text{But } s^2 = x, \quad \therefore s = \pm\sqrt{x} &= \pm\sqrt{1\angle 120°} && \text{or} \quad \pm\sqrt{1\angle -120°} \\ &= \pm\sqrt{1}\angle 120°/2 && \text{or} \quad \pm\sqrt{1}\angle -120°/2 \\ &= \pm 1\angle 60° && \text{or} \quad \pm 1\angle -60° \\ &= \pm(0.5 + j0.866) && \text{or} \quad \pm(0.5 - j0.866) \end{aligned}$$

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on the left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh array. The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

ALTERNATE METHOD

The characteristic equation is $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

$$\begin{array}{lllllll} s^7 & : & 1 & 24 & 24 & 23 & \text{.... Row-1} \\ s^6 & : & 9 & 24 & 24 & 15 & \text{.... Row-2} \end{array}$$

Divide s^6 row by 3 to simplify the computations.

$$\begin{array}{lllllll} s^7 & : & 1 & 24 & 24 & 23 & \text{.... Row-1} \\ s^6 & : & 3 & 8 & 8 & 5 & \text{.... Row-2} \\ s^5 & : & 1 & 1 & 1 & & \text{.... Row-3} \\ s^4 & : & 1 & 1 & 1 & & \text{.... Row-4} \\ s^3 & : & 0 & 0 & & & \text{.... Row-5} \end{array}$$

$$s^5 : \frac{3\times 24 - 8\times 1}{3} \quad \frac{3\times 24 - 8\times 1}{3} \quad \frac{3\times 23 - 5\times 1}{3}$$

$s^5 : 21.33 \quad 21.33 \quad 21.33$

Divide by 21.33

$s^5 : 1 \quad 1 \quad 1$

$$s^4 : \frac{1\times 8 - 1\times 3}{1} \quad \frac{1\times 8 - 1\times 3}{1} \quad \frac{1\times 5 - 0\times 3}{1}$$

$s^4 : 5 \quad 5 \quad 5$

Divide by 5

$s^4 : 1 \quad 1 \quad 1$

$$s^3 : \frac{1\times 1 - 1\times 1}{1} \quad \frac{1\times 1 - 1\times 1}{1}$$

$s^3 : 0 \quad 0$

$$\begin{array}{r|l} & s^3 + 9s^2 + 23s + 15 \\ \hline s^4 + s^2 + 1 & s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 \\ & s^7 + s^5 + s^3 \\ \hline & 9s^6 + 23s^5 + 24s^4 + 23s^3 + 24s^2 + 23s + 15 \\ & 9s^6 + 9s^4 + 9s^2 \\ \hline & 23s^5 + 15s^4 + 23s^3 + 15s^2 + 23s + 15 \\ & 23s^5 + 23s^3 + 23s \\ \hline & 15s^4 + 15s^2 + 15 \\ & 15s^4 + 15s^2 + 15 \\ \hline & 0 \end{array}$$

Since we get a row of zeros, there exists an even polynomial, the even polynomial is nothing but, the auxiliary polynomial.

The auxiliary polynomial is

$$s^4 + s^2 + 1 = 0$$

Divide the characteristic equation by auxiliary polynomial to get the quotient polynomial.

The characteristic polynomial can be expressed as a product of quotient polynomial and auxiliary polynomial.

$$\therefore s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15$$
$$= (s^4 + s^2 + 1)(s^3 + 9s^2 + 23s + 15) = 0$$

The routh array is constructed for quotient polynomial as shown below.

$$\begin{array}{llll} s^3 & : & 1 & 23 \\ s^2 & : & 9 & 15 \end{array}$$

Divide s^2 row by 3,

$$\begin{array}{lcll} s^3 & : & 1 & 23 \\ s^2 & : & 3 & 5 \\ s^1 & : & 21.33 & \\ s^0 & : & 5 & \end{array}$$

Column-1

$$s^1 : \frac{3\times 23-5\times 1}{3}$$
$$s^1 : \ 21.33$$

$$s^0 : \frac{21.33\times 5-0\times 3}{21.33}$$
$$s^0 : \ 5$$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability the roots of auxiliary polynomial should be evaluated.

The auxiliary equation is $s^4+s^2+1=0$.

Put $s^2=x$ in the auxiliary equation,

$$\therefore s^4+s^2+1=x^2+x+1=0$$

The roots of quadradic are, $x=\dfrac{-1\pm\sqrt{1-4}}{2}=-\dfrac{1}{2}\pm j\dfrac{\sqrt{3}}{2}$

$$=1\angle 120^\circ \text{ or } 1\angle -120^\circ$$

But $s^2=x$, $\quad \therefore s=\pm\sqrt{x}=\pm\sqrt{1\angle 120^\circ}$ or $\pm\sqrt{1\angle -120^\circ}$

$$=\pm\sqrt{1}\angle 120^\circ/2 \quad \text{or} \quad \pm\sqrt{1}\angle -120^\circ/2$$
$$=\pm 1\angle 60^\circ \quad \text{or} \quad \pm 1\angle -60^\circ$$
$$=\pm(0.5+j0.866) \quad \text{or} \quad \pm(0.5-j0.866)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by the roots of auxiliary polynomial and the roots of quotient polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so the system is unstable. The remaining five roots are lying on the left half of s-plane.

RESULT

1. The system is unstable
2. Two roots are lying on the right half and five roots are lying on left half of s-plane.

EXAMPLE 5.6

The characteristic polynomial of a system is $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$. Determine the location of roots on the s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row of array using the coefficients of even powers of s as shown below.

$$\begin{array}{lcllll} s^7 & : & 1 & 9 & 4 & 36 & \text{.... Row-1} \\ s^6 & : & 5 & 9 & 20 & 36 & \text{.... Row-2} \end{array}$$

Divide s^6 row by 5 to simplify the computations

$$\begin{array}{lclllll} s^7 & : & 1 & 9 & 4 & 36 & \text{.... Row-1} \\ s^6 & : & 1 & 1.8 & 4 & 7.2 & \text{.... Row-2} \\ s^5 & : & 1 & 0 & 4 & & \text{.... Row-3} \\ s^4 & : & 1 & 0 & 4 & & \text{.... Row-4} \\ s^3 & : & 0 & 0 & & & \text{.... Row-5} \end{array}$$

$$s^5 : \frac{1\times 9-1.8\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 36-7.2\times 1}{1}$$
$$s^5 : 7.2 \quad 0 \quad 28.8$$

Divide by 7.2

$$s^5 : \ 1 \quad 0 \quad 4$$

$$s^4 : \frac{1\times 1.8-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 7.2-0\times 1}{1}$$
$$s^4 : 1.8 \quad 0 \quad 7.2$$

Divide by 1.8

$$s^4 : \ 1 \quad 0 \quad 4$$

$$s^3 : \frac{1\times 0-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1}$$
$$s^3 : \ 0 \quad 0$$

The row of all zeros indicate the existence of even polynomial, which is also the auxiliary polynomial. The auxiliary polynomial is $s^4+4=0$. Divide the characteristic equation by auxiliary equation to get the quotient polynomial.

$$\begin{array}{r|l} & s^3+5s^2+9s+9 \\ \hline s^4+4 & s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36 \\ & s^7 \qquad\qquad\qquad\qquad +4s^3 \\ \hline & 5s^6+9s^5+9s^4 \qquad +20s^2+36s+36 \\ & 5s^6 \qquad\qquad\qquad +20s^2 \\ \hline & 9s^5+9s^4 \qquad +36s+36 \\ & 9s^5 \qquad\qquad +36s \\ \hline & 9s^4 \qquad +36 \\ & 9s^4 \qquad +36 \\ \hline & 0 \end{array}$$

The characteristic equation can be expressed as a product of quotient polynomial and auxiliary equation.

$$\therefore s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36$$
$$=(s^4+4)\,(s^3+5s^2+9s+9)=0$$

The routh array is constructed for quotient polynomial as shown below.

$$\begin{array}{lcll} s^3 & : & \boxed{1} & 9 \\ s^2 & : & 5 & 9 \\ s^1 & : & 7.2 & \\ s^0 & : & 9 & \end{array}$$

Column-1

$$s^1 : \frac{5\times9-9\times1}{5}$$

$$s^1 : 7.2$$

$$s^0 : \frac{7.2\times9-0\times5}{7.2}$$

$$s^0 : 9$$

There is no sign change in the elements of first column of routh array of quotient polynomial. Hence all the roots of quotient polynomial are lying on the left half of s-plane.

To determine the stability the roots of auxiliary polynomial should be evaluated.

The auxiliary polynomial is $s^4 + 4 = 0$.

Put $s^2 = x$ in the auxiliary equation,

$$\therefore s^4 + 4 = x^2 + 4 = 0$$

$$\therefore x^2 = -4$$

$$\text{or } x = \pm\sqrt{-4} = \pm j2 = 2\angle 90^\circ \text{ or } 2\angle -90^\circ$$

$$\text{But } s = \pm\sqrt{x} = \pm\sqrt{2\angle 90^\circ} \text{ or } \pm\sqrt{2\angle -90^\circ}$$

$$= \pm\sqrt{2}\angle 90^\circ/2 \text{ or } \pm\sqrt{2}\angle -90^\circ/2$$

$$= \pm\sqrt{2}\angle 45^\circ \text{ or } \pm\sqrt{2}\angle -45 = \pm(1+j1) \text{ or } \pm(1-j1)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by roots of quotient polynomial and auxiliary polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so the system is unstable. The remaining five roots are lying on the left half of s-plane.

RESULT

1. The system is unstable
2. Two roots are lying on the right half of s-plane and five roots are lying on the left half of s-plane.

EXAMPLE 5.7

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

SOLUTION

The characteristic equation of the system is $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{lcllll} s^5 & : & 1 & 8 & 7 & \text{.... Row-1} \\ s^4 & : & 4 & 8 & 4 & \text{.... Row-2} \end{array}$$

Divide s^4 row by 4 to simplify the calculations

$$\begin{array}{lcllll} s^5 & : & \boxed{1} & 8 & 7 & \text{.... Row-1} \\ s^4 & : & 1 & 2 & 1 & \text{.... Row-2} \\ s^3 & : & 1 & 1 & & \text{.... Row-3} \\ s^2 & : & 1 & 1 & & \text{.... Row-4} \\ s^1 & : & \in & & & \text{.... Row-5} \\ s^0 & : & 1 & & & \text{.... Row-6} \end{array}$$

Column-1

$$s^3 : \frac{1\times8-2\times1}{1} \quad \frac{1\times7-1\times1}{1}$$

$$s^3 : 6 \quad 6$$

Divide by 6

$$s^3 : 1 \quad 1$$

$$s^2 : \frac{1\times2-1\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^2 : 1 \quad 1$$

$$s^1 : \frac{1\times1-1\times1}{1}$$

$$s^1 : 0$$

Let $0 \rightarrow \in$

$$s^1 : \in$$

$$s^0 : \frac{\in\times1-0\times1}{\in}$$

$$s^0 : 1$$

When $\in \rightarrow 0$, there is no sign change in the first column of routh array. But we have a row of all zeros (s' row or row-5) and so there is a possibility of roots on imaginary axis. This can be found from the roots of auxiliary polynomial. Here the auxiliary polynomial is given by s^2 row.

The auxiliary polynomial is $s^2 + 1 = 0$

$$\therefore s^2 = -1$$

$$\text{or } s = \pm\sqrt{-1} = \pm j1$$

The roots of auxiliary polynomial are +j1, and −j1, lying on imaginary axis. The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots of characteristic equation are lying on imaginary axis and so the system is limitedly or marginally stable. The remaining three roots of characteristic equation are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Two roots are lying on imaginary axis and three roots are lying on left half of s-plane.

EXAMPLE 5.8

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

SOLUTION

The characteristic polynomial of the system is $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s as shown below.

$$\begin{array}{llllll}
s^6 & : & 1 & 3 & 3 & 1 & \text{.... Row-1}\\
s^5 & : & 1 & 3 & 2 & & \text{.... Row-2}\\
s^4 & : & \epsilon & 1 & 1 & & \text{.... Row-3}\\
s^3 & : & \dfrac{3\epsilon-1}{\epsilon} & \dfrac{2\epsilon-1}{\epsilon} & & & \text{.... Row-4}\\
s^2 & : & \dfrac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1} & 1 & & & \text{.... Row-5}\\
s^1 & : & \dfrac{4\epsilon^2-\epsilon}{2\epsilon^2-4\epsilon+1} & & & & \text{.... Row-6}\\
s^0 & : & 1 & & & & \text{.... Row-7}
\end{array}$$

$$s^4 : \frac{1\times3-3\times1}{1} \quad \frac{1\times3-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^4 : 0 \quad 1 \quad 1$$

let $0 \to \epsilon$

$$s^4 : \epsilon \quad 1 \quad 1$$

$$s^3 : \frac{\epsilon\times3-1\times1}{\epsilon} \quad \frac{\epsilon\times2-1\times1}{\epsilon}$$

$$s^3 : \frac{3\epsilon-1}{\epsilon} \quad \frac{2\epsilon-1}{\epsilon}$$

$$s^2 : \frac{\frac{3\epsilon-1}{\epsilon}-\frac{2\epsilon-1}{\epsilon}\times\epsilon}{\frac{3\epsilon-1}{\epsilon}} \quad \frac{\frac{3\epsilon-1}{\epsilon}\times1-0\times\epsilon}{\frac{3\epsilon-1}{\epsilon}}$$

$$s^2 : \frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1} \quad 1$$

On letting $\epsilon \to 0$, we get,

$$\begin{array}{llllll}
s^6 & : & 1 & 3 & 3 & 1 & \text{.... Row-1}\\
s^5 & : & 1 & 3 & 2 & & \text{.... Row-2}\\
s^4 & : & 0 & 1 & 1 & & \text{.... Row-3}\\
s^3 & : & -\infty & -\infty & & & \text{.... Row-4}\\
s^2 & : & 1 & 1 & & & \text{.... Row-5}\\
s^1 & : & 0 & & & & \text{.... Row-6}\\
s^0 & : & 1 & & & & \text{.... Row-7}
\end{array}$$

$$s^1 : \frac{\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}\times\frac{2\epsilon-1}{\epsilon}-\frac{3\epsilon-1}{\epsilon}\times1}{\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}}$$

$$s^1 : \frac{(-2\epsilon^2+4\epsilon-1)(2\epsilon-1)-(3\epsilon-1)(3\epsilon-1)}{\epsilon(-2\epsilon^2+4\epsilon-1)}$$

$$s^1 : \frac{-4\epsilon^3+\epsilon^2}{\epsilon(-2\epsilon^2+4\epsilon-1)} = \frac{4\epsilon^2-\epsilon}{2\epsilon^2-4\epsilon+1}$$

$$s^0 : \frac{\frac{4\epsilon^2-\epsilon}{2\epsilon^2-4\epsilon+1}\times1-0\times\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}}{(4\epsilon^2-\epsilon)/(2\epsilon^2-4\epsilon+1)}$$

$$s^0 : 1$$

Since there is a row of all zeros (s^1 row) there is a possibility of roots on imaginary axis. The auxiliary polynomial is $s^2 + 1 = 0$.

The roots of auxiliary polynomial are $\Big\}\ s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots are lying on imaginary axis. Therefore divide the characteristic polynomial by auxiliary equation and construct the routh array for quotient polynomial to find the roots lying on right half of s-plane.

$$\begin{array}{r|l}
 & s^4+s^3+2s^2+2s+1\\
\hline
s^2+1 & s^6+s^5+3s^4+3s^3+3s^2+2s+1\\
 & s^6 \quad +s^4\\
\hline
 & s^5+2s^4+3s^3+3s^2+2s+1\\
 & s^5 \quad +s^3\\
\hline
 & 2s^4+2s^3+3s^2+2s+1\\
 & 2s^4 \quad +2s^2\\
\hline
 & 2s^3+s^2+2s+1\\
 & 2s^3 \quad +2s\\
\hline
 & s^2+1\\
 & s^2+1\\
\hline
 & 0
\end{array}$$

The characteristic polynomial can be expressed as a product of auxiliary polynomial and quotient polynomial.

$$\therefore s^6+s^5+3s^4+3s^3+3s^2+2s+1$$

$$= (s^2+1)\,(s^4+s^3+2s^2+2s+1) = 0$$

The routh array for quotient polynomial is constructed as shown below.

$$\begin{array}{lllll}
s^4 & : & 1 & 2 & 1 & \text{.... Row-1}\\
s^3 & : & 1 & 2 & & \text{.... Row-2}\\
s^2 & : & \epsilon & 1 & & \text{.... Row-3}\\
s^1 & : & \dfrac{2\epsilon-1}{\epsilon} & & & \text{.... Row-4}\\
s^0 & : & 1 & & & \text{.... Row-5}
\end{array}$$

$$s^2 : \frac{1\times2-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^2 : 0 \quad 1$$

let $0 \to \epsilon$

$$s^2 : \epsilon \quad 1$$

$$s^1 : \frac{\epsilon\times2-1\times1}{\epsilon}$$

$$s^1 : \frac{2\epsilon-1}{\epsilon}$$

$$s^0 : \frac{\frac{2\epsilon-1}{\epsilon}\times1-0\times\epsilon}{(2\epsilon-1)/\epsilon}$$

$$s^0 : 1$$

On letting $\in \rightarrow 0$, we get

$$\begin{array}{lllll} s^4 & : & 1 & 2 & 1 & \text{.... Row-1} \\ s^3 & : & 1 & 2 & & \text{.... Row-2} \\ s^2 & : & 0 & 1 & & \text{.... Row-3} \\ s^1 & : & -\infty & & & \text{.... Row-4} \\ s^0 & : & 1 & & & \text{.... Row-5} \end{array}$$

Column-1

On examining the first column of the routh array of quotient polynomial, we found that there are two sign changes. Hence two roots are lying on the right half of s-plane and other two roots of quotient polynomial are lying on the left half of s-plane.

The roots of characteristic equation are given by roots of auxiliary polynomial and quotient polynomial. Hence two roots are lying on imaginary axis, two roots are lying on right half of s-plane and the remaining two roots are lying on left half of s-plane. Hence the system is unstable.

RESULT

1. The system is unstable
2. Two roots are lying on imaginary axis, two roots are lying on right half s-plane and two roots are lying on left half s-plane.

EXAMPLE 5.9

Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s) = \dfrac{K}{s(s+1)(s+2)}$.

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$= \frac{\dfrac{K}{s(s+1)(s+2)}}{1+\dfrac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

The characteristic equation is

$$s(s+1)(s+2)+K=0$$
$$s(s^2+3s+2)+K=0$$
$$s^3+3s^2+2s+K=0$$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{llll} s^3 & : & 1 & 2 \\ s^2 & : & 3 & K \\ s^1 & : & \dfrac{6-K}{3} & \\ s^0 & : & K & \end{array}$$

Column-I

$$s^1 : \frac{3\times 2 - K\times 1}{3}$$
$$s^1 : \frac{6-K}{3}$$
$$s^0 : \frac{\dfrac{6-K}{3}\times K - 0\times 3}{(6-K)/3}$$
$$s^0 : K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K>0$

From s^1 row, for the system to be stable, $\dfrac{6-K}{3}>0$

For $\dfrac{6-K}{3}>0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

RESULT

The value of K is in the range $0<K<6$ for the system to be stable.

EXAMPLE 5.10

The open loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOLUTION

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}$$

$$= \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is

$$(s+2)(s+4)(s^2+6s+25)+K=0$$
$$(s^2+6s+8)(s^2+6s+25)+K=0$$
$$s^4+12s^3+69s^2+198s+200+K=0$$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

$$\begin{array}{llll} s^4 & : & 1 \quad 69 \quad 200+K & \text{..... Row-1} \\ s^3 & : & 12 \quad 198 & \text{..... Row-2} \end{array}$$

Divide s^3 row by 12 to simplify the calculations

$$\begin{array}{lllll} s^4 & : & 1 & 69 \quad 200+K & \text{..... Row-1} \\ s^3 & : & 1 & 16.5 & \text{..... Row-2} \\ s^2 & : & 52.5 & 200+K & \text{..... Row-3} \\ s^1 & : & \frac{666.25-K}{52.5} & & \text{..... Row-4} \\ s^0 & : & 200+K & & \text{..... Row-5} \end{array}$$

Column-1

$$s^2 : \frac{1\times 69-16.5\times 1}{1} \quad \frac{1\times(200+K)}{1}$$
$$s^2 : 52.5 \qquad 200+K$$

$$s^1 : \frac{52.5\times 16.5-(200+K)\times 1}{52.5}$$
$$s^1 : \frac{666.25-K}{52.5}$$

$$s^0 : \frac{\frac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$$
$$s^0 : 200+K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^1 row, for the system to be stable, $(666.25-K)>0$

Since $(666.25-K)>0$, K should be lessthan 666.25.

From s^0 row, for the system to be stable, $(200+K)>0$

Since $(200+K)>0$, K should be greaterthan -200, but practical values of K starts from 0. Hence K should be greater than 0.

$\therefore$ The range of K for the system to be stable is $0<K<666.25$.

When $K=666.25$ the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When $K=666.25$, the coefficients of auxiliary equation are given by the s^2 row.

$\therefore$ The auxiliary equation is $52.5s^2+200+K=0$

$$52.5s^2+200+666.25=0$$
$$s^2=\frac{-200-666.25}{52.5}=-16.5$$
$$s=\pm j4.06$$

When $K=666.25$, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

$\therefore$ The frequency of oscillation, $\omega=4.06$ rad/sec.

RESULT

1. The range of K for stability is $0<K<666.25$
2. The system oscillates when $K=666.25$
3. The frequency of oscillation, $\omega=4.06$ rad/sec (When $K=666.25$).

EXAMPLE 5.11

The open loop transfer function of a unity feedback system is given by $G(s)=\dfrac{K(s+1)}{s^3+as^2+2s+1}$. Determine the value of **K** and **a** so that the system oscillates at a frequency of 2 rad/sec.

SOLUTION

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K(s+1)}{s^3+as^2+2s+1}}{1+\dfrac{K(s+1)}{s^3+as^2+2s+1}}$$

$$= \frac{K(s+1)}{s^3+as^2+2s+1+K(s+1)}$$

$\therefore$ The characteristic equation is

$$s^3+as^2+2s+1+K(s+1)=0$$
$$s^3+as^2+2s+1+Ks+K=0$$
$$s^3+as^2+(2+K)s+1+K=0$$

The routh array of characteristic polynomial is constructed as shown below. The maximum power of s is odd, hence the first row of routh array is formed using coefficients of odd powers of s and the second row of routh array is formed using coefficients of even powers of s.

If the elements of s^1 row are all zeros then there exist an even polynomial (or auxiliary polynomial). If the roots of the auxiliary polynomial are purely imaginary then the roots are lying on imaginary axis and the system oscillates. The frequency of oscillation is the root of auxiliary polynomial.

Routh array

s^3 :	1	2+K
s^2 :	a	1+K
s^1 :	$\frac{a(2+K)-(1+K)}{a}$	
s^0 :	1+K	

From s^1 row

$$\frac{a(2+K)-(1+K)}{a}=0$$
$$a(2+K)-(1+K)=0$$
$$2a+Ka-1-K=0$$
$$2a-1+K(a-1)=0$$

Put K = 4a − 1

$$\therefore 2a-1+(4a-1)(a-1)=0$$
$$2a-1+4a^2-4a-a+1=0$$
$$4a^2-3a=0 \quad \text{(or)} \quad a(4a-3)=0$$

Since $a \neq 0$, $4a-3=0$, $\therefore a=3/4$

When $a=(3/4)$, $K=4a-1=4\times(3/4)-1=2$

The auxiliary polynomial is

$$as^2+(1+K)=0$$
$$as^2=-(1+K)$$
$$s=\pm j\sqrt{\frac{1+K}{a}}$$

Given that $s=\pm j2$

$$\therefore \sqrt{\frac{1+K}{a}}=2$$
$$\frac{1+K}{a}=4$$
$$K=4a-1$$

RESULT

When the system oscillates at a frequency of 2 rad/sec, K = 2 and a = 3/4.

EXAMPLE 5.12

A feedback system has open loop transfer function of $G(s)=\frac{Ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of K for stability of closed loop system.

SOLUTION

Generally control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low frequency ranges the term e^{-s} can be replaced by $(1-s)$ [i.e., $e^{-sT}\approx(1-sT)$]

$$\therefore G(s)=\frac{K\,e^{-s}}{s(s^2+5s+9)} \approx \frac{K(1-s)}{s(s^2+5s+9)}$$

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K(1-s)}{s(s^2+5s+9)}}{1+\dfrac{K(1-s)}{s(s^2+5s+9)}}$$

$$= \frac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

$\therefore$ The characteristic equation is

$$s(s^2+5s+9)+K(1-s)=0$$

$$s(s^2+5s+9)+K(1-s)=s^3+5s^2+9s+K-Ks=0$$

$$\therefore s^3+5s^2+(9-K)s+K=0$$

The routh array of characteristic polynomial is constructed as shown below.

The maximum power of s in the characteristic polynomial is odd, hence form the first row of routh array using coefficients of odd powers of s and second row of routh array using coefficients of even powers of s.

$$\begin{array}{llll} s^3 & : & 1 & 9-K \\ s^2 & : & 5 & K \\ s^1 & : & 9-1.2K & \\ s^0 & : & K & \end{array}$$

$$s^1 : \frac{5\times(9-K)-K\times 1}{5}$$

$$s^1 : \frac{45-5K-K}{5}$$

$$s^1 : \frac{45-6K}{5} \approx 9-1.2K$$

$$s^0 : \frac{(9-1.2K)\times K}{(9-1.2K)}$$

$$s^0 : K$$

From s^1 row, for stability of the system, $(9-1.2K)>0$

If $(9-1.2K)>0$ then $1.2K<9$

$$\therefore K < \frac{9}{1.2} = 7.5$$

From s^0 row, for stability of the system, $K>0$

Finally we can conclude that for stability of the system K is in the range of $0<K<7.5$

RESULT

For stability of the system K is in the range of $0<K<7.5$.

5.4 MATHEMATICAL PRELIMINARIES FOR NYQUIST STABILITY CRITERION

Let F(s) be a function which is expressed as a ratio of two polynomials in s, (the polynomials are expressed in the factored form) as shown in equation (5.14).

$$F(s) = \frac{(s-z_1)\,(s-z_2)\,......\,(s-z_m)}{(s-p_1)\,(s-p_2)\,......\,(s-p_n)} \qquad .....(5.14)$$

The roots of numerator polynomial are zeros and the roots of denominator polynomial are poles. The function has m number of zeros and n number of poles.

Let s be a complex variable represented by $s=\sigma+j\omega$ on the complex s-plane. The function F(s) is also complex and may be defined as $F(s)=u+jv$ and represented on the complex F(s)-plane with coordinates u and v.

The equation (5.14) indicates that for every point s in the s-plane at which F(s) is analytic, there exists a corresponding point F(s) in the F(s)-plane. Hence it can be concluded that the function F(s) maps the points in the s-plane into the F(s)-plane.

Note : *A function is analytic in the s-plane provided the function and all its derivatives exist. The points in the s-plane where the function (or it derivatives) does not exist are called singular points.*

Since any number of points of analycity in the s-plane can be mapped into the F(s)- plane it can be concluded that for a contour in the s-plane which does not go through any singular point, there exists a corresponding contour in the F(s)-plane as shown in fig 5.2.

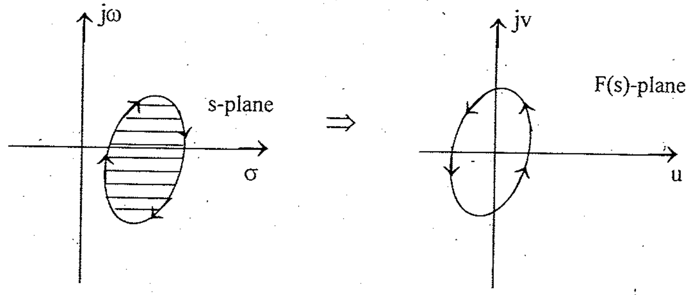


Fig 5.2 : *An arbitrary contour in s-plane and its corresponding contour in F(s)-plane*

Note : *For the development of Nyquist criterion, the exact shape of the contour is not required but only the number of encirclements of the origin of the F(s) - plane is essential.*

CONCEPT OF ENCIRCLED AND ENCLOSED

It is important to distinguish between the concept of encircled and enclosed which are frequently used to apply Nyquist criterion.

Encircled : *A point is said to be encircled by a closed path if it is found inside the path. With reference to fig 1, the point A is encircled in the clockwise direction and the point B is not encircled.*

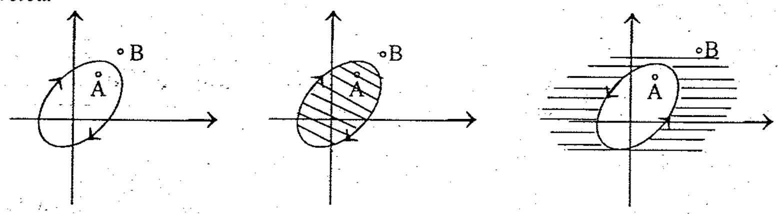


Fig 1 ***Fig 2*** ***Fig3***

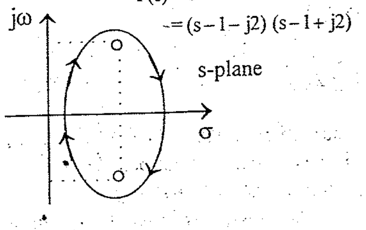
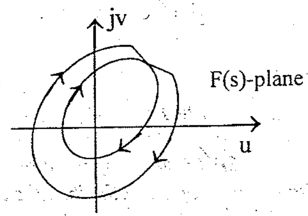
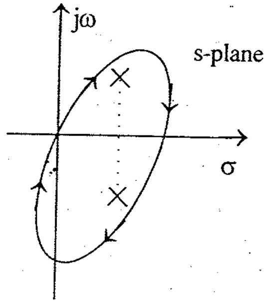
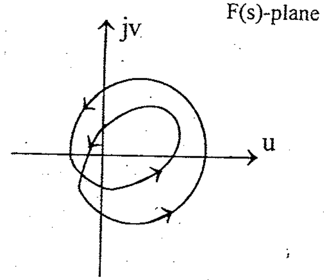
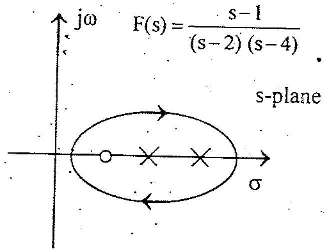
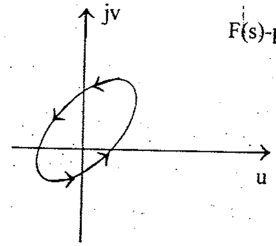
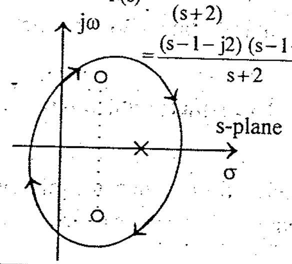
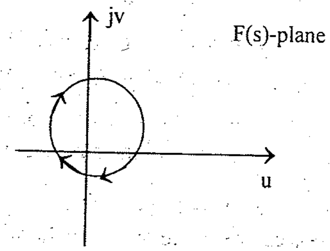
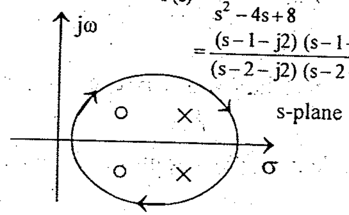
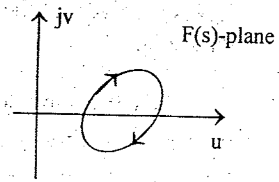
Enclosed : *Any point or region is said to be enclosed by a closed path, if it is found to lie to the right of the path when the path is traversed in the prescribed direction. The shaded regions in fig 2 and 3 are the region enclosed by the closed path. With reference to fig 2, the point A is enclosed by closed path and the point B is not enclosed. With reference to fig 3 the point A is not enclosed by closed path but point B is enclosed.*

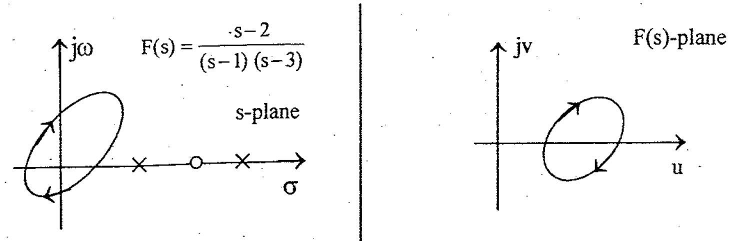
It can be proved that there exists a relationship between the enclosure of poles and zeros by the s-plane closed contour and number of encirclements of the origin of F(s)-plane by the corresponding F(s)-plane contour. The following points is the summary of this relationship.

1. If s-plane closed contour encloses Z number of zeros in the right half s-plane then the corresponding contour in F(s)-plane will encircle, the origin of F(s)-plane Z times in the clockwise direction.
2. If s-plane closed contour encloses P number of poles in the right half s-plane then the corresponding contour in F(s)-plane will encircle the origin of F(s)- plane P times in anticlockwise direction.
3. If the s-plane closed contour encloses Z zeros and P poles in the right half s-plane and if $P > Z$, then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane (P–Z) times in the anti-clockwise direction.
4. If the s-plane closed contour encloses Z zeros and P poles in the right half s-plane and if $P < Z$ then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane (Z–P) times in the clockwise direction.
5. If the s-plane closed contour encloses Z zeros and P poles in the right half s-plane and if $P = Z$, then the corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.
6. If the s-plane closed contour does not enclose any pole or zero then the corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.

The table 5.2 shows examples of arbitrary s-plane contours and their corresponding F(s)-plane contours (exact shape is not shown). Here zeros are marked by small circles (o) and poles by (X).

TABLE 5.2.

The function, F(s) and s-plane contour	F(s)-plane contour
$F(s) = s^2 - 2s + 5 = (s-1-j2)(s-1+j2)$ 	
$F(s) = \frac{1}{s^2 - 4s + 8} = \frac{1}{(s-2-j2)(s-2+j2)}$ 	
$F(s) = \frac{s-1}{(s-2)(s-4)}$ 	
$F(s) = \frac{s^2 - 2s + 5}{(s+2)} = \frac{(s-1-j2)(s-1+j2)}{s+2}$ 	
$F(s) = \frac{s^2 - 2s + 5}{s^2 - 4s + 8} = \frac{(s-1-j2)(s-1+j2)}{(s-2-j2)(s-2+j2)}$ 	



Note : Normally the direction of arbitrary contour in s-plane is chosen as clockwise.

The relation between the enclosure of poles and zeros of F(s) lying on the right half s-plane by the s-plane contour and the encirclements of the origin of F(s)-plane by the corresponding F(s)-plane contour is called principle of argument.

The principle of argument is stated as follows.

Let F(s) is a single valued rational function and is analytic in a given region in the s-plane except at some points. Now, if an arbitrary closed contour is chosen in the s-plane, so that F(s) is analytic at every point on the closed contour in s-plane then the corresponding F(s)-plane contour mapped in the F(s)-plane will encircle the origin N times in anticlockwise direction where N is the difference between the number of poles and number of zeros of F(s) that are encircled by the chosen closed contour in s-plane.

Mathematically, it can be expressed as $N = P - Z$

Where N = Number of encirclement of the origin made by the contour of F(s)-plane.

Z = Number of zeros of F(s) lying on right half s-plane and encircled by the s-plane closed contour.

P = Number of poles of F(s) lying on right half s-plane and encircled by the s-plane closed contour.

The value of N can be positive, zero or negative. Based on the sign of N the following conclusions can be made provided the arbitrary s-plane contour is chosen in the clockwise direction.

1. If N is positive then the encirclement of the origin of the F(s)-plane will be in the anticlockwise direction.

2. If N is zero then the poles and zeros are equal and there will be no encirclement of origin in F(s)-plane.

3. If N is negative then the direction of encirclement of the origin of F(s)-plane will be clockwise.

5.5 NYQUIST STABILITY CRITERION

Consider the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

The characteristic equation of the system is given by the condition, $1+G(s)\,H(s)=0$

Let, $F(s) = 1+G(s)\,H(s)$.

The loop transfer function G(s) H(s) can be expressed as

$$G(s)\,H(s) = \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)} \qquad .....(5.15)$$

Where $m \le n$.

$$\therefore F(s) = 1+G(s)\,H(s) = 1+\frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$= \frac{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)+K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$= \frac{(s+z_1')\,(s+z_2')\,.....\,(s+z_n')}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)} \qquad .....(5.16)$$

Where $z_1', z_2' ..... z_n'$, are zeros of F(s) which are obtained by combining the numerator and denominator polynomial of G(s) H(s).

For the condition $F(s) = 0$, the numerator of F(s) should be equal to zero.

$$\therefore (s+z_1')\,(s+z_2')\,.......\,(s+z_n') = 0 \qquad .....(5.17)$$

We can say that equation (5.17) is the characteristic equation of the system. For the stability of the system the roots of the characteristic equation should not lie on the right half s-plane. The roots of characteristic equation are zeros of F(s) and also they are poles of closed loop transfer function.

Hence we can conclude that for the stability of closed loop system the zeros of F(s) should not lie on the right half s-plane.

Note : For a unity feedback system.

$$G(s)\,H(s) = G(s) = \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = \frac{\dfrac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}}{1+\dfrac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}}$$

$$= \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)+K(s+z_1)\,(s+z_2)\,.....(s+z_m)}$$

$$= \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+z_1')\,(s+z_2')\,.....\,(s+z_n')}$$

From the above equation we can say that the poles of closed loop transfer function are z_1', z_2', ... z_n'.

Let us choose an arbitrary contour in the s-plane which encircles the right half zeros and poles of F(s) (equation (5.16)). The principle of argument (explained in section 5.4) states that the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane, N times in the anticlockwise direction.

Where $\quad N = P - Z$ .....(5.18)

Here P = Number of poles of F(s) lying on right half s-plane

Z = Number of zeros of F(s) lying on right half s-plane

For the stability of the system the roots of characteristic equation and so the zeros of F(s) should not lie on the right half of s-plane. Hence for a stable system $Z = 0$. Hence from equation (5.18) when $Z = 0$, we get

When $Z = 0,\ N = P$ .....(5.19)

When $Z \neq 0,\ N \neq P$ .....(5.20)

From equation (5.15) and (5.16) we can say that the poles of F(s) are also poles of loop transfer function. Hence for the stability of the system, (with reference to equation (5.19) and equation (5.20)) number of poles of loop transfer function lying on right of s-plane should be equal to counter clockwise encirclement of the origin of F(s)-plane. If this condition is not met the system is unstable.

The loop transfer function, G(s)H(s) can be expressed as

$G(s)\,H(s) = [1+G(s)\,H(s)]-1 = F(s) - 1$ .....(5.21)

From equation (5.21) it can be concluded that the F(s)-plane contour drawn with respect to origin is same as the F(s)-plane contour of F(s)–1 drawn with respect to (–1+j0) as shown in fig 5.3.

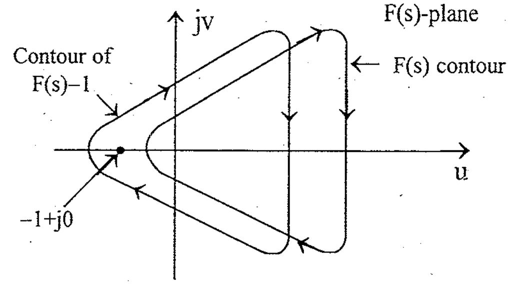


Fig 5.3

Thus the encirclement of the origin of F(s)-plane by the F(s) contour is equivalent to the encirclement of the point $(-1+j0)$ by the G(s) H(s) contour. The Nyquist stability criterion have been proposed based on this concept.

In order to investigate the presence of poles of G(s) H(s) on the right half s-plane a contour, C is chosen such that it encloses the entire right half s-plane as shown in fig 5.4, such a contour C is called Nyquist contour.

The Nyquist contour is directed clockwise and comprises of two segments

(i) An infinite line segment C_1 along the imaginary axis.

(ii) An arc, C_2 of infinite radius.

Along C_1, $s = j\omega$ with s varying from $-j\infty$ to $+j\infty$.

Along C_2, $s = \underset{R\to\infty}{\mathrm{Lt}}\ R\,e^{j\theta}$ with θ varying from $+\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

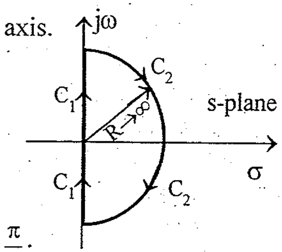


***Fig 5.4 :** Nyquist contour*

***Note :** The s-plane is a complex plane. Any point on a complex plane can be expressed by the complex number in polar form, $Re^{j\theta}$, where R is the magnitude and θ is the argument (or phase).*

The Nyquist stability criterion can be stated as follows. "If the G(s) H(s) contour in the G(s) H(s) plane corresponding to Nyquist contour in the s-plane encircles the point (–1 + j0) in the anticlockwise direction as many times as the number of right half s-plane poles of G(s) H(s), then the closed loop system is stable".

In examining the stability of linear control systems using the Nyquist stability criterion, we come across the following three situations.

(i) There is no encirclement of $-1 + j0$ point. This implies that the system is stable if there are no poles of G(s) H(s) in the right half s-plane. If there are poles on right half s-plane then the system is unstable.

(ii) An anticlockwise encirclement or (or encirclements) of $-1 + j0$ point. In this case the system is stable if the number of anticlockwise encirclements is same as the number of poles of G(s) H(s) in the right half s-plane. If the number of encirclements is not equal to number of poles on right half s-plane then the system is unstable.

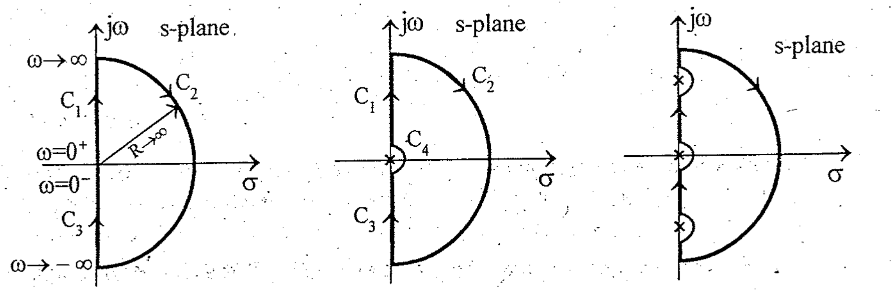
(iii) There is a clockwise encirclement (or encirclements) of the $-1 + j0$ point. In this case the system is always unstable.

PROCEDURE FOR INVESTIGATING THE STABILITY USING NYQUIST CRITERION

The following procedure can be followed to investigate the stability of closed loop system from the knowledge of open loop system, using Nyquist stability criterion.

1. Choose a Nyquist contour as shown in fig 5.5, which encloses the entire right half s-plane except the singular points. The Nyquist contour encloses all the right half s-plane poles and zeros of G(s) H(s). [The poles on imaginary axis are singular points and so they are avoided by taking a detour around it as shown in fig 5.5 b and c].

Note : For mapping a contour from s-plane to G(s) H(s) plane the Nyquist contour in s-plane should be analytic at every point. At singular points it is not analytic.



a. Nyquist contour when there is no pole on imaginary axis

b. Nyquist contour when there are poles at origin

c. Nyquist contour when there are poles on imaginary axis and at origin

***Fig 5.5 :** Nyquist contour*

2. The Nyquist contour should be mapped in the G(s) H(s) plane using the function G(s) H(s) to determine the encirclement $-1+ j0$ point in the G(s) H(s) plane. The Nyquist contour of fig 5.5b can be divided into four sections C_1, C_2, C_3 and C_4. The mapping of the four sections in the G(s) H(s) plane can be carried sectionwise and then combined together to get entire G(s) H(s) contour.

3. In section C_1 the value of ω varies from 0 to $+\infty$. The mapping of section C_1 is obtained by letting $s = j\omega$ in G(s) H(s) and varying ω from 0 to $+\infty$,

$$\text{i.e. } G(s)\,H(s)\Big|_{\substack{s=j\omega \\ \omega=0 \text{ to } \infty}} = G(j\omega)\,H(j\omega)\Big|_{\omega=0 \text{ to } \infty}$$

The locus of $G(j\omega)\,H(j\omega)$ as ω is varied from 0 to $+\infty$ will be the G(s) H(s) contour in G(s) H(s) plane corresponding to section C_1 in s-plane. This locus is the polar plot of $G(j\omega)\,H(j\omega)$. There are three ways of mapping this section of G(s) H(s) contour, they are,

(i) Calculate the values of $G(j\omega)\,H(j\omega)$ for various values of ω and sketch the actual locus of $G(j\omega)\,H(j\omega)$.

(or)

(ii) Separate the real part and imaginary part of $G(j\omega)\,H(j\omega)$. Equate the imaginary part to zero, to find the frequency at which the $G(j\omega)\,H(j\omega)$ locus crosses real axis (to find phase crossover frequency). Substitute this frequency on real part and find the crossing point of the locus on real axis. Sketch the approximate locus of $G(j\omega)\,H(j\omega)$ from the knowledge of type number and order of the system or from the value of $G(j\omega)\,H(j\omega)$ at $\omega = 0$ and $\omega = \infty$.

(iii) Separate the magnitude and phase of $G(j\omega)\,H(j\omega)$. Equate the phase of $G(j\omega)\,H(j\omega)$ to -180 and solve for ω. This value of ω is the phase crossover frequency and the magnitude at this frequency is the crossing point on real axis. Sketch the approximate root locus as mentioned in method (ii).

4. The section C_2 of Nyquist contour has a semicircle of infinite radius. Therefore, every point on section C_2 has infinite magnitude but the argument varies from $+\pi/2$ to $-\pi/2$. Hence the mapping of section C_2 from s-plane to G(s) H(s) plane can be obtained by letting $s = \underset{R\to\infty}{\mathrm{Lt}}\; Re^{j\theta}$ in G(s) H(s) and varying θ from $+\pi/2$ to $-\pi/2$.

Consider the loop transfer function in time constant form and with y number of poles at origin, as shown here,

$$G(s)\,H(s) = \frac{K\,(1+sT_1)\,(1+sT_2)\,(1+sT_3)\ldots\ldots}{s^y\,(1+sT_a)\,(1+sT_b)\,(1+sT_c)\ldots\ldots}$$

Let G(s) H(s) has m zeros and n poles including poles at origin. For practical systems, m > n.

Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the term $(1+sT)$ can be approximated to sT [i.e., $(1+sT) \approx sT$]

$$\therefore G(s)\,H(s) \approx K\frac{sT_1.sT_2.sT_3\ldots..}{s^y.sT_a.sT_b.sT_c\ldots..} = K_1\frac{s^m}{s^n} = \frac{K_1}{s^{n-m}}$$

On letting, $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ we get

$$G(s)\,H(s)\Big|_{s=\underset{R\to\infty}{Lt}Re^{j\theta}} = \frac{K_1}{\underset{R\to\infty}{Lt}\ (Re^{j\theta})^{n-m}} = 0e^{-j\theta(n-m)}$$

When $\theta = \frac{\pi}{2}$, $G(s)\,H(s) = 0e^{-j\frac{\pi}{2}(n-m)}$

When $\theta = -\frac{\pi}{2}$, $G(s)\,H(s) = 0e^{+j\frac{\pi}{2}(n-m)}$

From the above two equations we can conclude that the section C_2 of Nyquist contour in s-plane is mapped as circles/circular arc around origin with radius tending to zero in the G(s) H(s) plane.

5. In section C_3 the value of ω varies from $-\infty$ to 0. The mapping of section C_3 is obtained by letting $s = +j\omega$ in G(s) H(s) and varying ω from $-\infty$ to 0.

$$\text{i.e., } G(s)\,H(s)\Big|_{\substack{s=+j\omega \\ \omega=-\infty \text{ to } 0}} = G(j\omega)\,H(j\omega)\Big|_{\omega=-\infty \text{ to } 0.}$$

The locus of $G(j\omega)\,H(j\omega)$ as ω is varied from $-\infty$ to 0 will be the G(s) H(s) contour in G(s) H(s) plane corresponding to section C_3 in s-plane. This locus is the inverse polar plot of $G(j\omega)\,H(j\omega)$. The inverse polar plot is given by the mirror image of polar plot with respect to real axis.

6. The section C_4 of Nyquist contour has a semicircle of zero radius. Therefore every point on semicircle has zero magnitude but the argument varies from $-\pi/2$ to $+\pi/2$. Hence the mapping of section C_4 from s-plane to G(s) H(s) plane can be obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{-j\theta}$ in G(s) H(s) and varying θ from $-\pi/2$ to $+\pi/2$.

Consider the loop transfer function in time constant form and with y number of poles at origin as shown below

$$G(s)\,H(s) = \frac{K\,(1+sT_1)\,(1+sT_2)\,(1+sT_3)\ldots..}{s^y\,(1+sT_a)\,(1+sT_b)\,(1+sT_c)\ldots..}$$

Let G(s) H(s) has m zeros and n poles including poles at origin. For practical systems, m > n.

Since $s \to R\,e^{j\theta}$ and $R \to 0$, the term $1+sT$ can be approximated to 1 [i.e. $(1+sT) \approx 1$]

$$\therefore G(s)\,H(s) \approx K\frac{1}{s^y}$$

On letting $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$ we get

$$G(s)\,H(s)\Big|_{s=\underset{R\to 0}{Lt}Re^{j\theta}} = \frac{K}{\underset{R\to 0}{Lt}\ (Re^{j\theta})^y} = \infty e^{-j\theta y}$$

When $\theta = -\frac{\pi}{2}$, $G(s)\,H(s) = \infty e^{j\frac{\pi}{2}y}$

When $\theta = \frac{\pi}{2}$, $G(s)\,H(s) = \infty e^{-j\frac{\pi}{2}y}$

From the above two equations we can conclude that the section C_4 of Nyquist contour in s-plane is mapped as circles/circular arc in G(s) H(s) plane with origin as centre and infinite radius.

Note :

1. *If there are no poles on the origin then the section C_4 of Nyquist contour will be absent.*

2. *If there are poles on imaginary axis as shown below then the Nyquist contour is divided into the following 8 sections and the mapping is performed sectionwise.*

Section C_1 : From $+j0^+$ to $+j\omega_1^-$

Section C_2 : From $+j\omega_1^-$ to $+j\omega_1^+$

Section C_3 : From $+j\omega_1^+$ to $+j\infty$

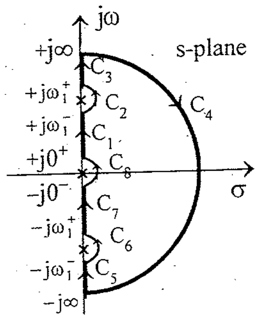
Section C_4 : From $+j\infty$ to $-j\infty$

Section C_5 : From $-j\infty$ to $-j\omega_1^-$

Section C_6 : From $-j\omega_1^-$ to $-j\omega_1^+$

Section C_7 : From $-j\omega_1^+$ to $-j0^-$

Section C_8 : From $-j0^-$ to $+j0^+$



EXAMPLE 5.13

Draw the Nyquist plot for the system whose open loop transfer function is $G(s)\,H(s) = \dfrac{K}{s\,(s+2)\,(s+10)}$. Determine the range of K for which closed loop system is stable.

SOLUTION

$$\text{Given that, } G(s)\,H(s) = \frac{K}{s\,(s+2)\,(s+10)} = \frac{K}{s \times 2\left(\dfrac{s}{2}+1\right) \times 10\left(\dfrac{s}{10}+1\right)}$$

$$= \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)}$$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig 5.13.1.

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

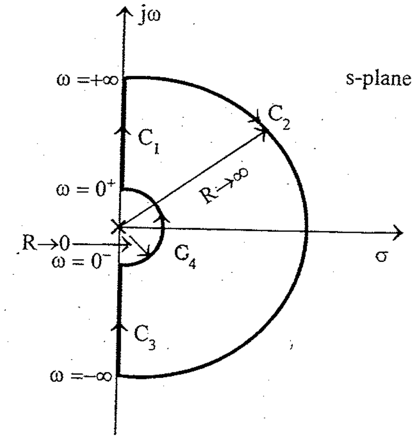


Fig 5.13.1 : Nyquist contour in s-plane

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)\,H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)\,H(j\omega)$.

$$G(s)\,H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)}$$

Let $s = j\omega$

$$\therefore G(j\omega)\,H(j\omega) = \frac{0.05K}{j\omega\,(1+j0.5\omega)\,(1+j0.1\omega)}$$

$$= \frac{0.05K}{j\omega\,(1+j0.6\omega-0.05\omega^2)}$$

$$= \frac{0.05K}{-0.6\omega^2 + j\omega\,(1-0.05\omega^2)}$$

When the locus of $G(j\omega)\,H(j\omega)$ crosses real axis the imaginary term will be zero and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc}, \quad \omega_{pc}(1-0.05\omega_{pc}^2) = 0$$

$$\therefore 1-0.05\omega_{pc}^2 = 0$$

$$\omega_{pc} = \sqrt{\frac{1}{0.05}} = 4.472 \text{ rad / sec}$$

At $\omega = \omega_{pc} = 4.472$ rad / sec,

$$G(j\omega)\,H(j\omega) = \frac{0.05K}{-0.6\omega^2} = -\frac{0.05K}{0.6 \times (4.472)^2}$$

$$= -0.00417K$$

The open loop system is type-1 and third order system. Also it is a minimum phase system with all poles. Hence the polar plot of G(jω) H(jω) starts at –90° axis at infinity, crosses real axis at –0.00417K and ends at origin in second quadrant. The section C_1 and its mapping are shown in fig 5.13.2. and 5.13.3.

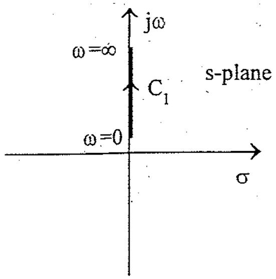


Fig 5.13.2 : Section C_1 in s-plane

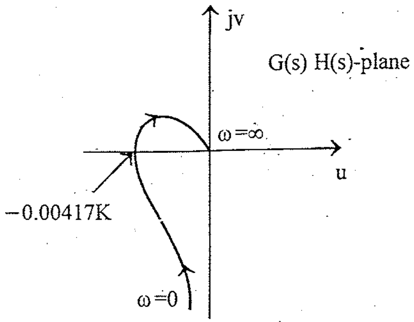


Fig 5.13.3 : Mapping of section C_1 in G(s) H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from +π/2 to –π/2. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the G(s) H(s) can be approximated as shown below [i.e., (1+sT) ≈ sT].

$$G(s)\,H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)} \approx \frac{0.05K}{s \times 0.5s \times 0.1s} = \frac{K}{s^3}$$

Let, $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{s^3}\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{\underset{R\to\infty}{Lt}\ (Re^{j\theta})^3} = 0e^{-j3\theta}$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)\,H(s) = 0e^{-j3\frac{\pi}{2}} \quad .....(5.13.1)$$

$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)\,H(s) = 0e^{+j3\frac{\pi}{2}} \quad .....(5.13.2)$$

From the equations (5.13.1) and (5.13.2) we can say that section C_2 in s-plane (fig 5.13.4.) is mapped as circular arc of zero radius around origin in G(s) H(s)-plane with argument (phase) varying from –3π/2 to +3π/2 as shown in fig 5.13.5.

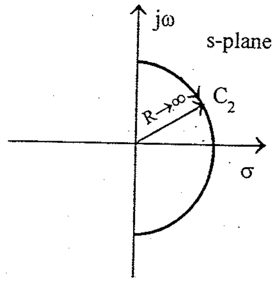


Fig 5.13.4 : Section C_2 in s-plane

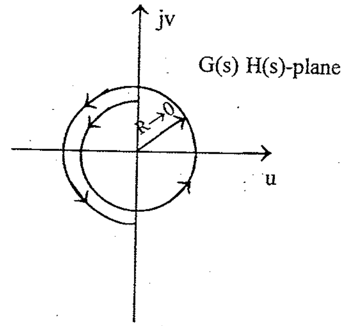


Fig 5.13.5 : Mapping of section C_2 in G(s) H(s)-plane

MAPPING OF SECTION C_3

In section C_3, ω varies from –∞ to 0. The mapping of section C_3 is given by the locus of G(jω) H(jω) as ω is varied from –∞ to 0. This locus is the inverse polar plot of G(jω) H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s) H(s) plane are shown in fig 5.13.6 and fig 5.13.7.

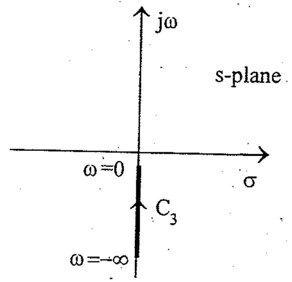


Fig 5.13.6 : Section C_3 in s-plane

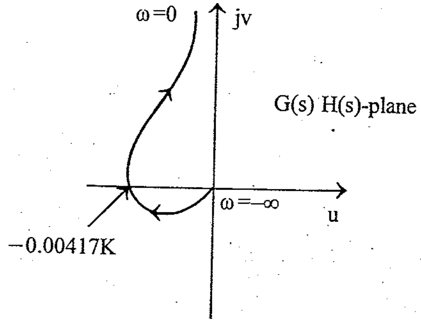


Fig 5.13.7 : Mapping of section C_3 in G(s) H(s)-plane

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from –π/2 to +π/2. Since $s \to R\,e^{j\theta}$ and $R \to 0$, the G(s) H(s) can be approximated as shown below [i.e., (1+sT) ≈ 1].

$$G(s)\,H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)} \approx \frac{0.05K}{s \times 1 \times 1} = \frac{0.05K}{s}$$

Let $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{0.05K}{s}\Big|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{0.05K}{\underset{R\to 0}{Lt}\,(Re^{j\theta})} = \infty e^{-j\theta}$$

$$\text{When } \theta = -\frac{\pi}{2},\quad G(s)\,H(s) = \infty e^{+j\frac{\pi}{2}} \qquad .....(5.13.3)$$

$$\text{When } \theta = \frac{\pi}{2},\quad G(s)\,H(s) = \infty e^{-j\frac{\pi}{2}} \qquad .....(5.13.4)$$

From the equations (5.13.3) and (5.13.4) we can say that section C_4 in s-plane (fig 5.13.8.) is mapped as a circular arc of infinite radius with argument (phase) varying from $+\pi/2$ to $-\pi/2$ as shown in fig 5.13.9.

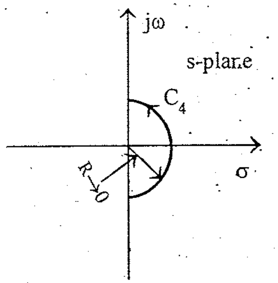


Fig 5.13.8 : Section C_4 in s-plane

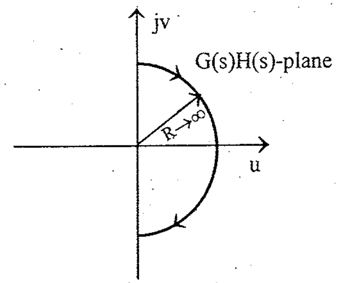


Fig 5.13.9 : Mapping of section C_4 in G(s)H(s)-plane

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s) H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 5.13.10.

STABILITY ANALYSIS

When, $-0.00417K = -1$, the contour passes through $(-1+j0)$ point and corresponding value of K is the limiting value of K for stability.

$$\therefore \text{ Limiting value of K} = \frac{1}{0.00417} = 240$$

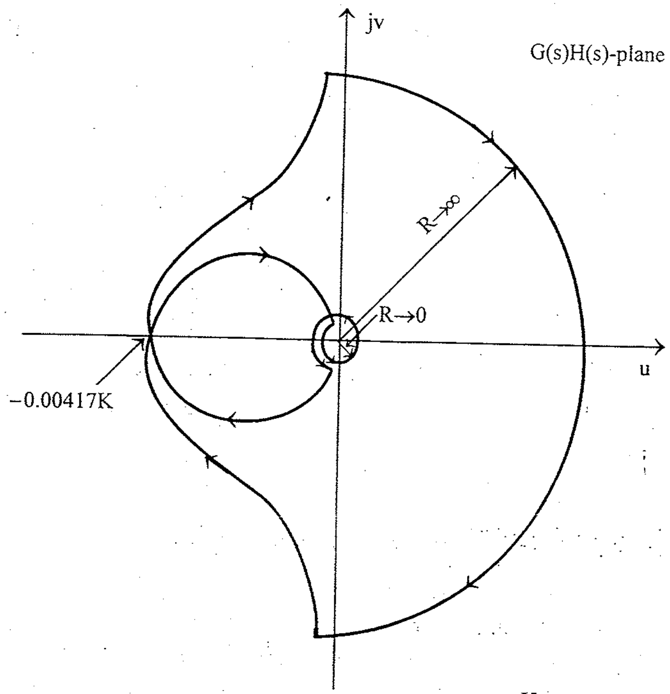


Fig 5.13.10 : Nyquist plot of $G(s)\,H(s) = \dfrac{K}{s(s+2)\,(s+10)}$

When K < 240

When K is lessthan 240, the contour crosses real axis at a point between 0 and –1. On travelling through Nyquist plot along the indicated direction it is found that the point $-1+j0$ is not encircled. Also the open loop transfer function has no poles on the right half of s-plane. Therefore the closed loop system is stable.

When K > 240

When K is greater than 240, the contour crosses real axis at a point between –1 and $-\infty$. On travelling through Nyquist plot along the indicated direction it is found that the point $-1+j0$ is encircled in clockwise direction two times. Therefore the closed loop system is unstable.

RESULT

The value of K for stability is $0 < K < 240$

EXAMPLE 5.14

Construct the Nyquist plot for a system whose open loop transfer function is given by $G(s)\,H(s) = \dfrac{K\,(1+s)^2}{s^3}$. Find the range of K for stability.

SOLUTION

Given that, $G(s)\,H(s) = \dfrac{K\,(1+s)^2}{s^3}$

The open loop transfer function has three poles at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig 5.14.1.

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

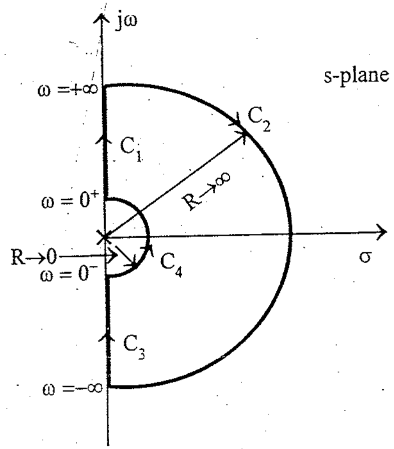


Fig 5.14.1 : Nyquist contour in s-plane

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to +∞. The mapping of section C_1 is given by the locus of G(jω) H(jω) as ω is varied from 0 to ∞. This locus is the polar plot of G(jω) H(jω).

$$G(s)\,H(s) = \frac{K\,(1+s)^2}{s^3}$$

Let s = jω

$$\therefore G(j\omega)\,H(j\omega) = \frac{K\,(1+j\omega)^2}{(j\omega)^3} = \frac{K(1-\omega^2+2j\omega)}{-j\omega^3}$$

$$= \frac{K\,(1-\omega^2)}{-j\omega^3} + \frac{K\,2j\omega}{-j\omega^3}$$

$$= -\frac{2K}{\omega^2} + j\frac{K\,(1-\omega^2)}{\omega^3}$$

When the G(jω) H(jω) locus crosses real axis the imaginary term will be zero and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc}, \quad K(1-\omega_{pc}^2) = 0$$

$$\therefore 1-\omega_{pc}^2 = 0$$

$$\omega_{pc} = 1 \text{ rad / sec}$$

At $\omega = \omega_{pc} = 1$ rad / sec,

$$G(j\omega)\,H(j\omega) = -\frac{2K}{\omega^2} = -\frac{2K}{1^2} = -2K \qquad .....(5.14.1)$$

$$G(j\omega)\,H(j\omega) = \frac{K(1+j\omega)^2}{(j\omega)^3}$$

$$= \frac{K\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega\,\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega}{\omega^3\angle 270°}$$

$$= \frac{K\,(1+\omega^2)}{\omega^3}\angle(2\tan^{-1}\omega - 270°)$$

As $\omega \to 0$, $\quad G(j\omega)\,H(j\omega) \to \infty\angle -270°$ .....(5.14.2)

As $\omega \to \infty$, $\quad G(j\omega)\,H(j\omega) \to 0\angle -90°$ .....(5.14.3)

From equations (5.14.1), (5.14.2) and (5.14.3) we can say that the polar plot starts at −270° axis at infinity, crosses real axis at −2K and ends at origin in third quadrant. The section C_1 and its mapping are shown in fig 5.14.2. and 5.14.3.

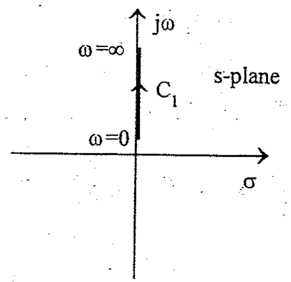


Fig 5.14.2 : Section C_1 in s-plane

⇒

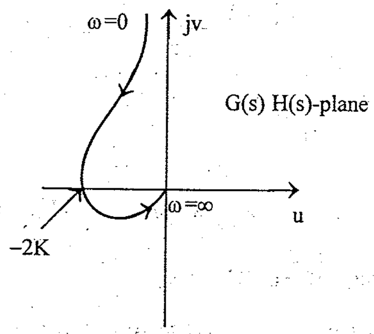


Fig 5.14.3 : Mapping of section C_1 in G(s) H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the G(s) H(s) can be approximated as shown below [i.e., $(1+sT) \approx sT$].

$$G(s)\,H(s) = \frac{K\,(1+s)^2}{s^3} \approx \frac{K\,s^2}{s^3} = \frac{K}{s}$$

Let $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{s}\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{\underset{R\to\infty}{Lt}\ Re^{j\theta}} = 0e^{-j\theta}$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)\,H(s) = 0e^{-j\frac{\pi}{2}} \qquad .....(5.14.4)$$

$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)\,H(s) = 0e^{j\frac{\pi}{2}} \qquad .....(5.14.5)$$

From the equations (5.14.4) and (5.14.5) we can say that section C_2 in s-plane (fig 5.14.4.) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument (phase) varying from $-\pi/2$ to $+\pi/2$ as shown in fig 5.14.5.

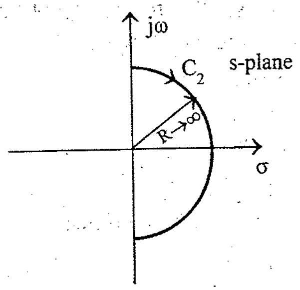


Fig 5.14.4 : Section C_2 in s-plane

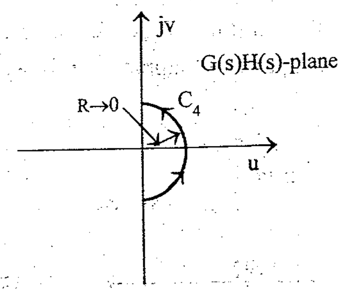


Fig 5.14.5 : Mapping of section C_2 in G(s)H(s)-plane

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of G(jω) H(jω) as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of G(jω) H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 5.14.6 and fig 5.14.7.

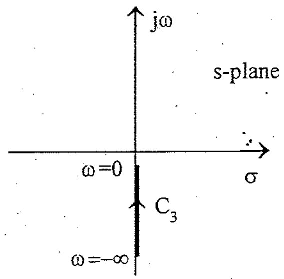


Fig 5.14.6 : Section C_3 in s-plane

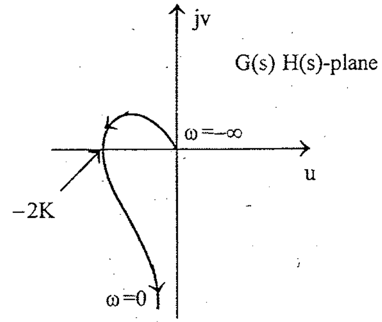


Fig 5.14.7 : Mapping of section C_3 in G(s) H(s)-plane

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, the G(s) H(s) can be approximated as shown below [i.e., $(1+sT) \approx 1$].

$$G(s)\,H(s) = \frac{K\,(1+s)^2}{s^3} \approx \frac{K\times 1}{s^3} = \frac{K}{s^3}$$

Let $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to 0}{Lt} Re^{+j\theta}} = \frac{K}{s^3}\Big|_{s=\underset{R\to 0}{Lt} Re^{+j\theta}} = \frac{K}{\underset{R\to 0}{Lt}\ (Re^{+j\theta})^3} = \infty e^{-j3\theta}$$

$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)\,H(s) = \infty e^{+j3\frac{\pi}{2}} \qquad .....(5.14.6)$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)\,H(s) = \infty e^{-j3\frac{\pi}{2}} \qquad .....(5.14.7)$$

From the equations (5.14.6) and (5.14.7) we can say that section C_4 in s-plane (fig 5.14.8.) is mapped as a circular arc of infinite radius with argument (phase) varying from $+3\pi/2$ to $-3\pi/2$ as shown in fig 5.14.9.

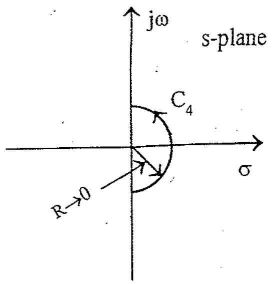


Fig 5.14.8 : Section C_4 in s-plane

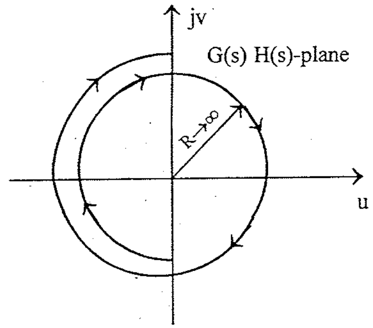


Fig 5.14.9 : Mapping of section C_4 in G(s) H(s)-plane

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 5.14.10.

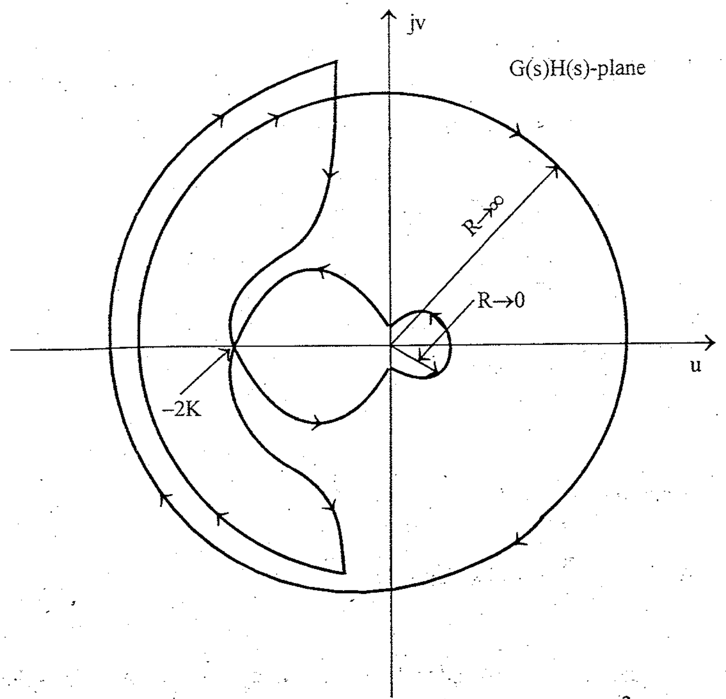


Fig 5.14.10 : Nyquist plot of $G(s)H(s) = \frac{K\,(1+s)^2}{s^3}$

STABILITY ANALYSIS

When, $-2K = -1$, the contour passes through $(-1+j0)$ point and corresponding value of K is the limiting value of K for stability.

$$\therefore \text{Limiting value of } K = \frac{1}{2} = 0.5$$

When K < 0.5

When K is lessthan 0.5, the contour crosses real axis at a point between 0 and −1. On travelling through Nyquist plot along the indicated direction it is observed that the −1+j0 point is encircled in clockwise direction one time. Therefore the system is unstable.

When K > 0.5

When K is greater than 0.5, the contour crosses real axis at a point between −1 and −∞. On travelling through Nyquist plot along the indicated direction it is observed that (−1+j0) point is encircled in both clockwise and anticlockwise direction one time. Hence net encirclement is zero. Also the open loop system has no poles at the right half of s-plane. Therefore the closed loop system is stable.

RESULT

The system is stable when K > 0.5.

EXAMPLE 5.15

The open loop transfer function of a system is $G(s)\,H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)}$.

Determine the stability of closed loop system. If the closed loop system is not stable then find the number of closed-loop poles lying on the right half of s-plane.

SOLUTION

Given that, $G(s)\,H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)}$

The open loop transfer function has two poles at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig 5.15.1.

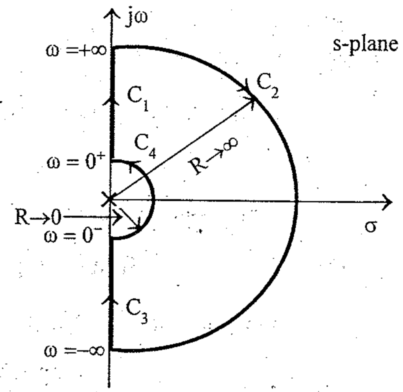


Fig 5.15.1 : Nyquist contour in s-plane

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)\,H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)\,H(j\omega)$.

$$G(s)\,H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)}$$

Let $s = j\omega$

$$\therefore G(j\omega)\,H(j\omega) = \frac{(1+j4\omega)}{(j\omega)^2\,(1+j\omega)\,(1+j2\omega)}$$

$$= \frac{\sqrt{1+16\omega^2}\ \angle\tan^{-1}4\omega}{\omega^2\ \angle 180^\circ\sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$= \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}\ \angle(\tan^{-1}4\omega - 180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega)$$

$$\therefore |G(j\omega)\,H(j\omega)| = \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}$$

$$\angle G(j\omega)\,H(j\omega) = \tan^{-1}4\omega - 180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

When the $G(j\omega)\,H(j\omega)$ locus crosses real axis, the phase will be -180° and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc},\quad \angle G(j\omega)\,H(j\omega) = -180^\circ$$

$$\therefore \tan^{-1}4\omega_{pc} - 180^\circ - \tan^{-1}\omega_{pc} - \tan^{-1}2\omega_{pc} = -180^\circ$$

$$\tan^{-1}4\omega_{pc} = \tan^{-1}\omega_{pc} + \tan^{-1}2\omega_{pc}$$

On taking tan on both sides we get,

$$\tan\,[\tan^{-1}4\omega_{pc}] = \tan\,[\tan^{-1}\omega_{pc} + \tan^{-1}2\omega_{pc}]$$

$$4\omega_{pc} = \frac{\tan\,\tan^{-1}\omega_{pc} + \tan\,\tan^{-1}2\omega_{pc}}{1 - \tan\,\tan^{-1}\omega_{pc}\ .\ \tan\,\tan^{-1}2\omega_{pc}}$$

$$4\omega_{pc} = \frac{\omega_{pc} + 2\omega_{pc}}{1 - 2\omega_{pc}^2}$$

$$\therefore 1 - 2\omega_{pc}^2 = \frac{3\omega_{pc}}{4\omega_{pc}}$$

Note : $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\ .\ \tan B}$

$$-2\omega_{pc}^2 = \frac{3}{4} - 1$$

$$\omega_{pc} = \sqrt{\frac{-0.25}{-2}} = 0.354\ \text{rad/sec}$$

At $\omega = \omega_{pc} = 0.354$ rad / sec,

$$|G(j\omega)\,H(j\omega)| = \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}} = \frac{\sqrt{1+16\times 0.354^2}}{(0.354)^2\ \sqrt{1+0.354^2}\ \sqrt{1+4\times 0.354^2}}$$

$$= 10.64 \qquad .....(5.15.1)$$

Hence $G(j\omega)\,H(j\omega)$ locus crosses the real axis at -10.64.

$$\text{At } \omega \to 0,\ G(j\omega)\,H(j\omega) \to \infty\angle -180^\circ \qquad .....(5.15.2)$$

$$\text{At } \omega \to \infty,\ G(j\omega)\,H(j\omega) \to 0\angle -270^\circ \qquad .....(5.15.3)$$

From equations (5.15.1), (5.15.2) and (5.15.3) we can say that the polar plot starts at -180° axis at infinity, travels in third quadrant and crosses real axis at -10.64 to enter second quadrant and then ends at origin in second quadrant. The section C_1 and its mapping are shown in fig 5.15.2. and 5.15.3.

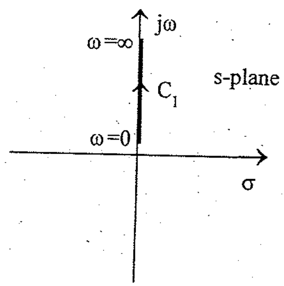


Fig 5.15.2 : Section C_1 in s-plane

$\Rightarrow$

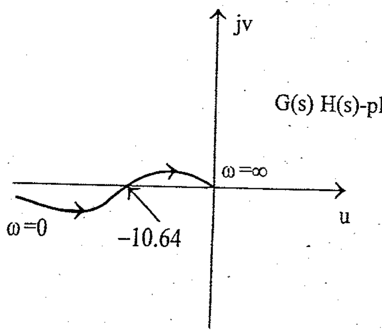


Fig 5.15.3 : Mapping of section C_1 in G(s) H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \lim_{R\to\infty} R e^{j\theta}$ in G(s) H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R e^{j\theta}$ and $R \to \infty$, the G(s) H(s) can be approximated as shown below [i.e., $(1+sT) \approx sT$].

$$G(s)\,H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)} \approx \frac{4s}{s^2 . s . 2s} = \frac{2}{s^3}$$

Let, $s = \underset{R\to\infty}{\mathrm{Lt}}\ Re^{j\theta}$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to\infty}{\mathrm{Lt}} Re^{j\theta}} = \frac{2}{s^3}\Big|_{s=\underset{R\to\infty}{\mathrm{Lt}} Re^{j\theta}} = \frac{2}{\underset{R\to\infty}{\mathrm{Lt}}\ (Re^{j\theta})^3} = 0e^{-j3\theta}$$

$$\text{When } \theta = \frac{\pi}{2}, \qquad G(s)\,H(s) = 0e^{-j3\frac{\pi}{2}} \qquad .....(5.15.4)$$

$$\text{When } \theta = -\frac{\pi}{2}, \qquad G(s)\,H(s) = 0e^{j3\frac{\pi}{2}} \qquad .....(5.15.5)$$

From the equations (5.15.4) and (5.15.5) we can say that section C_2 in s-plane (fig 5.15.4.) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument (phase) varying from $-3\pi/2$ to $+3\pi/2$ as shown in fig 5.15.5.

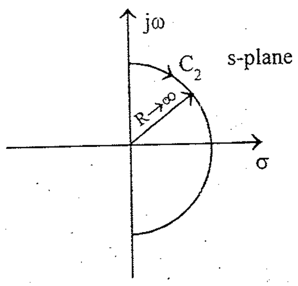


Fig 5.15.4 : Section C_2 in s-plane

⇒

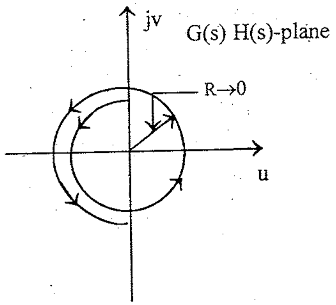


Fig 5.15.5 : Mapping of section C_2 in G(s) H(s)-plane

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)\,H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)\,H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 5.15.6 and fig 5.15.7.

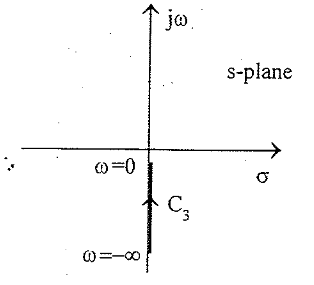


Fig 5.15.6 : Section C_3 in s-plane

⇒

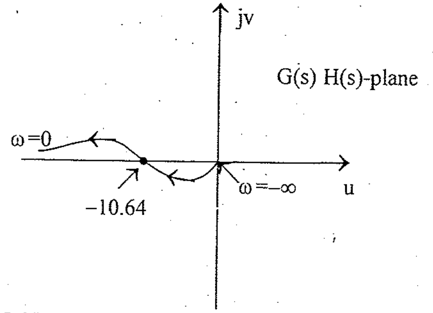


Fig 5.15.7 : Mapping of section C_3 in G(s) H(s)-plane

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to 0}{\mathrm{Lt}}\ R e^{j\theta}$ in G(s) H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R e^{j\theta}$ and $R \to 0$, the G(s) H(s) can be approximated as shown below [i.e., $(1+sT) \approx 1$].

$$G(s)\,H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)} \approx \frac{1}{s^2 \times 1 \times 1} = \frac{1}{s^2}$$

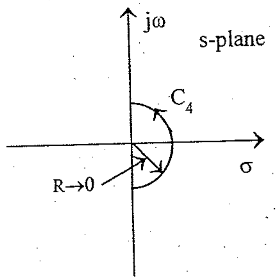
Let $s = \underset{R\to 0}{\mathrm{Lt}}\ Re^{j\theta}$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to 0}{\mathrm{Lt}} Re^{j\theta}} = \frac{1}{s^2}\Big|_{s=\underset{R\to 0}{\mathrm{Lt}} Re^{j\theta}} = \frac{1}{\underset{R\to 0}{\mathrm{Lt}}\ (Re^{j\theta})^2} = \infty e^{-j2\theta}$$

$$\text{When } \theta = -\frac{\pi}{2}, \qquad G(s)\,H(s) = \infty e^{+j\pi} \qquad .....(5.15.6)$$

$$\text{When } \theta = \frac{\pi}{2}, \qquad G(s)\,H(s) = \infty e^{-j\pi} \qquad .....(5.15.7)$$

From the equations (5.15.6) and (5.15.7) we can say that section C_4 in s-plane (fig 5.15.8.) is mapped as a circle of infinite radius with argument (phase) varying from $+\pi$ to $-\pi$ as shown in fig 5.15.9.



⇒

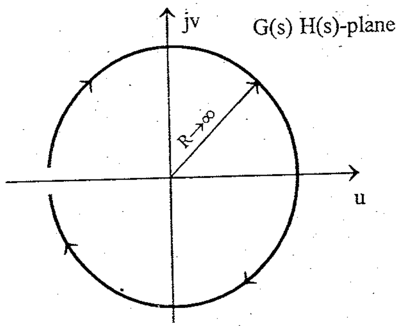


Fig 5.15.8 : Section C_4 in s-plane

Fig 5.15.9 : Mapping of section C_4 in G(s) H(s)-plane

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 5.15.10.

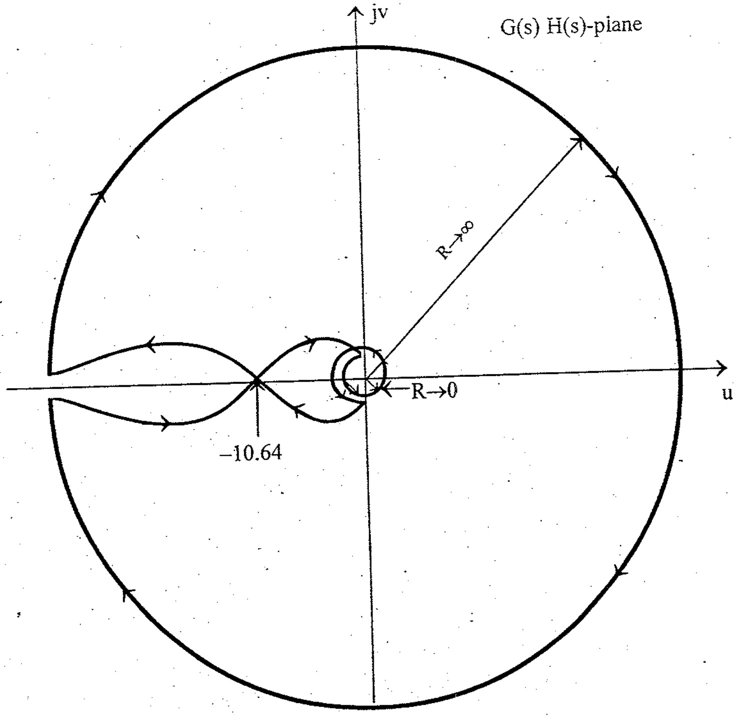


Fig 5.15.10 : Nyquist plot of $G(s)H(s) = \dfrac{1+4s}{s^2(1+s)(1+2s)}$

STABILITY ANALYSIS

On travelling through Nyquist contour in G(s)H(s)-plane it is observed that (−1+j0) point is encircled in anticlockwise direction two times. Therefore the closed loop system is unstable.

Since the (−1+j0) is encircled twice, two poles of closed loop system are lying on the right half s-plane.

RESULT

1. Closed loop system is unstable.
2. Two poles of closed loop system are lying on the right half s-plane.

EXAMPLE 5.16

Sketch the Nyquist plot for a system with the open loop transfer function $G(s)\,H(s) = \dfrac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$. Determine the range of values of K for which the system is stable.

SOLUTION

$$\text{Given that, } G(s)\,H(s) = \frac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$$

The open loop transfer function does not have a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane as shown in fig 5.16.1.

The Nyquist contour has three sections C_1, C_2 and C_3. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

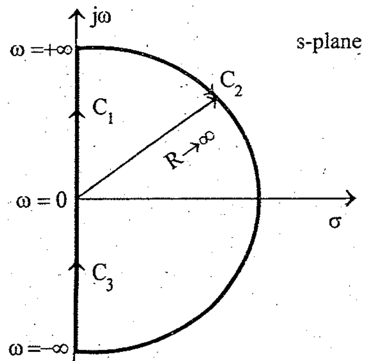


Fig 5.16.1 : Nyquist contour in s-plane

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to +∞. The mapping of section C_1 is given by the locus of G(jω) H(jω) as ω is varied from 0 to ∞. This locus is the polar plot of G(jω) H(jω).

$$G(s)\,H(s) = \frac{K\,(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$$

Let $s = j\omega$

$$\therefore G(j\omega)\,H(j\omega) = \frac{K\,(1+j0.5\omega)\,(1+j\omega)}{(1+j10\omega)\,(-1+j\omega)}$$

$$= \frac{K(1+j1.5\omega-0.5\omega^2)}{-1-j9\omega-10\omega^2} = \frac{K(1-0.5\omega^2)+j1.5\omega K}{-(1+10\omega^2)-j9\omega}$$

On multiplying the numerator and denominator by the complex conjugate of denominator we get,

$$G(j\omega)\,H(j\omega) = \frac{K(1-0.5\omega^2)+j1.5\omega K}{-(1+10\omega^2)-j9\omega} \times \frac{-(1+10\omega^2)+j9\omega}{-(1+10\omega^2)+j9\omega}$$

$$= \frac{-K(1-0.5\omega^2)\,(1+10\omega^2)-13.5\omega^2K + j\,[9\omega K(1-0.5\omega^2)-1.5\omega K(1+10\omega^2)]}{(1+10\omega^2)^2+(9\omega)^2}$$

When the $G(j\omega)\,H(j\omega)$ locus crosses real axis the imaginary term is zero and the corresponding frequency is the phase crossover frequency.

$$\therefore \text{At } \omega=\omega_{pc},\quad 9\omega_{pc}\,K(1-0.5\omega_{pc}^2)-1.5\omega_{pc}\,K(1+10\omega_{pc}^2)=0$$

$$\therefore 9\omega_{pc}\,K(1-0.5\omega_{pc}^2) = 1.5\omega_{pc}\,K(1+10\omega_{pc}^2)$$

$$1-0.5\omega_{pc}^2 = \frac{1.5}{9}(1+10\omega_{pc}^2)$$

$$1-0.5\omega_{pc}^2 = 0.167+1.67\omega_{pc}^2$$

$$2.17\omega_{pc}^2 = 0.833$$

$$\omega_{pc} = \sqrt{\frac{0.833}{2.17}} = 0.62 \text{ rad / sec}$$

At $\omega=\omega_{pc} = 0.62$ rad/sec

$$G(j\omega)\,H(j\omega) = \frac{-K(1-0.5\omega^2)\,(1+10\omega^2)-13.5\omega^2K}{(1+10\omega^2)^2+(9\omega)^2}$$

$$= -K\left[\frac{(1-0.5\times0.62^2)\,(1+10\times0.62^2)+13.5\times0.62^2}{(1+10\times0.62^2)^2+(9\times0.62)^2}\right]$$

$$= -K\left[\frac{3.913+5.189}{23.464+31.136}\right] = -0.1667K$$

The $G(j\omega)H(j\omega)$ locus crosses real axis at a point $-0.1667K$.

The exact shape of $G(j\omega)H(j\omega)$ locus is determined by calculating the magnitude and phase of $G(j\omega)H(j\omega)$ for various values of ω

Note : $(-1+j\omega)$ represents a point in second quadrant

$$G(j\omega)H(j\omega) = K\frac{(1+j0.5\omega)\,(1+j\omega)}{(1+j10\omega)\,(-1+j\omega)}$$

$$= K\frac{\sqrt{1+(0.5\omega)^2}\,\angle\tan^{-1}0.5\omega \,.\, \sqrt{1+\omega^2}\,\angle\tan^{-1}\omega}{\sqrt{1+(10\omega)^2}\,\angle\tan^{-1}10\omega\,\sqrt{1+\omega^2}\,\angle(180^\circ-\tan^{-1}\omega)}$$

$$= K\frac{\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}\,\angle(\tan^{-1}0.5\omega+2\tan^{-1}\omega-\tan^{-1}10\omega-180^\circ)$$

$$|G(j\omega)H(j\omega)| = K\frac{\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}0.5\omega+2\tan^{-1}\omega-\tan^{-1}10\omega-180^\circ$$

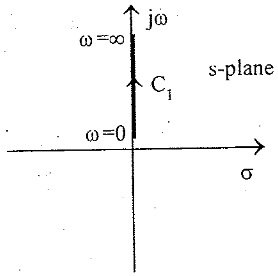
ω rad/sec	0	0.1	0.5	1.5	2.0	5.0	∞
|G(jω) H(jω)|	K	0.707K	0.202K	0.083K	0.07K	0.054K	0
∠G(jω)H(jω) deg	−180	−210	−191	−116	−95	−43	0

From the above analysis, the following conclusions are made,

1. The locus of $G(j\omega)H(j\omega)$ starts at $K\angle-180^\circ$ when $\omega=0$ and travels in second quadrant.
2. The locus crosses real axis at $-0.1667K$ and enters third quadrant.
3. Then the locus crosses negative imaginary axis and enters fourth quadrant.
4. Finally the locus ends at origin when $\omega=\infty$.

(Note : The exact plot can also be sketch on polar graph sheet).

The section C_1 in s-plane and its corresponding mapping in $G(s)H(s)$ plane are shown in fig 5.16.2. and 5.16.3.



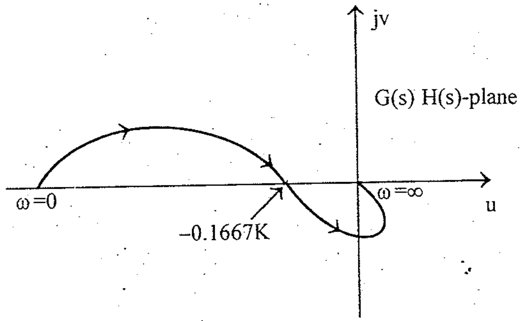


Fig 5.16.2 : Section C_1 in s-plane

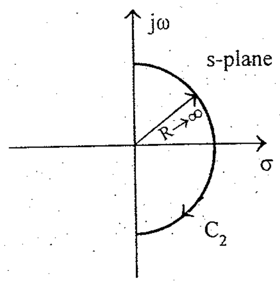
Fig 5.16.3 : Mapping of section C_1 in G(s) H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt} R\,e^{j\theta}$ in G(s) H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, G(s) H(s) can be approximated as shown below [i.e., $(1+sT) \approx sT$; Here $(s-1) \approx s$].

$$G(s)\,H(s) = \frac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)} \approx \frac{K\,0.5s \times s}{10s \times (s)} = 0.05K$$

The approximate G(s)H(s) is independent of s and so the contour of section C_2 in s-plane is mapped as a point at 0.05K in G(s)H(s) plane.



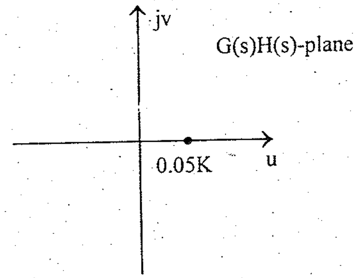


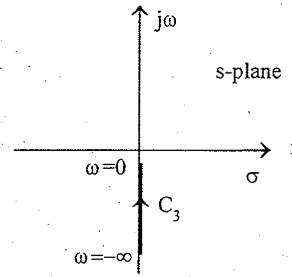
Fig 5.16.4 : Section C_2 in s-plane

Fig 5.16.5 : Mapping of section C_2 in G(s)H(s)-plane

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of G(jω) H(jω) as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of G(jω) H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 5.16.6 and fig 5.16.7.



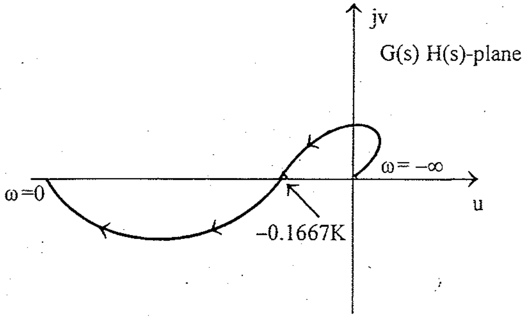


Fig 5.16.6 : Section C_3 in s-plane

Fig 5.16.7 : Mapping of section C_3 in s-plane

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 5.16.8.

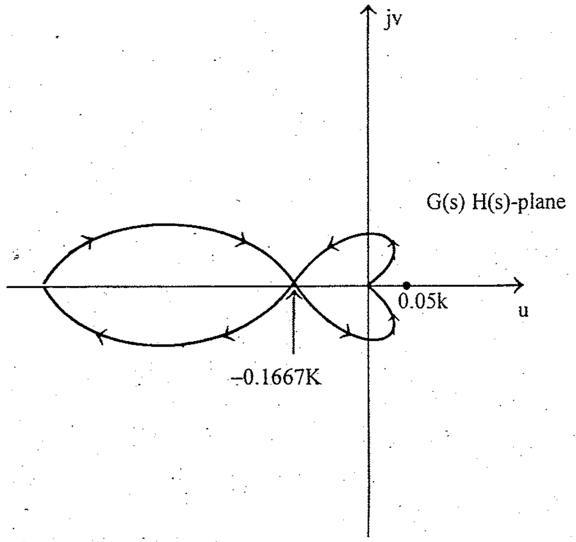


Fig 5.16.8 : The Nyquist plot of $G(s) = \dfrac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$

When $-0.1667K = -1$, the contour passes through $-1+j0$ point and this value of K is the limiting value for stability.

$$\text{The limiting value of } K = \frac{1}{0.1667} = 6$$

When K<6

When K<6, the locus crosses real axis between 0 and -1. On travelling through the locus it is observed that the $-1+j0$ point is encircled clockwise and so the closed loop system is unstable.

When K>6

When K>6, the locus crosses real axis between -1 and $-\infty$. On travelling through the locus it is observed that the $-1+j0$ point is encircled anticlockwise one time. Also the open loop system has one pole at the right half s-plane. Hence the system is stable.

RESULT

1. The open loop system is unstable.
2. For stability of the closed loop system, K>6.

EXAMPLE 5.17

Construct Nyquist plot for a feedback control system whose open loop transfer function is given by, $G(s)\,H(s) = \dfrac{5}{s(1-s)}$. Comment on the stability of open-loop and closed loop system.

SOLUTION

Given that, $G(s)\,H(s) = \dfrac{5}{s(1-s)}$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half s-plane except the origin as shown in fig 5.17.1.

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

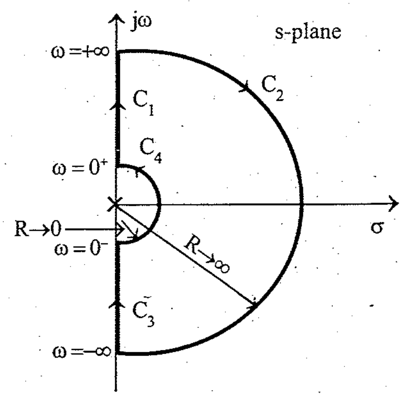


Fig 5.17.1 : Nyquist contour in s-plane

MAPPING OR SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)\,H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)\,H(j\omega)$.

$$G(s)\,H(s) = \frac{5}{s(1-s)}$$

Note : $(1-j\omega)$ represents a point in fourth quadrant

Let $s = j\omega$

$$\therefore G(j\omega)\,H(j\omega) = \frac{5}{j\omega(1-j\omega)} = \frac{5}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle -\tan^{-1}\omega}$$

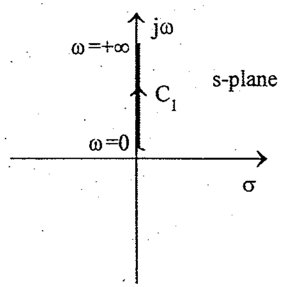
$$= \frac{5}{\omega\sqrt{1+\omega^2}}\angle(-90+\tan^{-1}\omega)$$

$$\therefore |G(j\omega)H(j\omega)| = \frac{5}{\omega\sqrt{1+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = -90^\circ + \tan^{-1}\omega$$

The exact shape of $G(j\omega)H(j\omega)$ locus is determined by calculating the magnitude and phase of $G(j\omega)H(j\omega)$ for various values of ω.

ω rad/sec	0	0.6	1.0	2.0	10.0	∞
$\lvert G(j\omega)\,H(j\omega)\rvert$	∞	7.15	3.53	1.12	0.05	0
$\angle G(j\omega)H(j\omega)$ deg	−90	−59	−45	−26	−5	0



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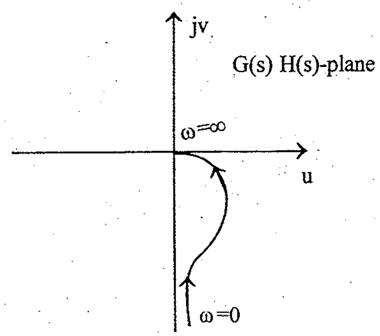


Fig 5.17.2 : Section C_1 in s-plane

Fig 5.17.3 : Mapping of section C_1 in G(s)H(s)-plane

From the above analysis, we can conclude that $G(j\omega)H(j\omega)$ locus starts at $-90°$ axis at infinity for $\omega = 0$ and meets the origin along $0°$ axis when $\omega = \infty$.

The section C_1 in s-plane and its corresponding mapping in G(s)H(s)-plane are shown in fig 5.17.2. and 5.17.3.

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the G(s) H(s) can be approximated as shown below. [i.e., $(1-s) \approx -s$)]

$$G(s)\,H(s) = \frac{5}{s(1-s)} \approx \frac{5}{s\,(-s)} = \frac{5}{s^2\,e^{j\pi}}$$

$\boxed{\textit{Note}: -1 = e^{j\pi}}$

$$\text{Let } s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$$

$$\therefore\ G(s)H(s)\Big|_{s=\underset{R\to\infty}{Lt} R e^{j\theta}} = \frac{5}{\underset{R\to\infty}{Lt}\ (R\,e^{j\theta})^2\,e^{j\pi}} = 0e^{-j(2\theta+\pi)}$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)H(s) = 0e^{-j2\pi} \qquad \text{.....(5.17.1)}$$

$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)H(s) = 0e^{j0} \qquad \text{.....(5.17.2)}$$

From the equation (5.17.1) and (5.17.2) we can say that section C_2 in s-plane (fig 5.17.4) is mapped as circular arc of zero radius around origin in G(s)H(s) plane with argument varying from -2π to $+0$ as shown in fig 5.17.5.

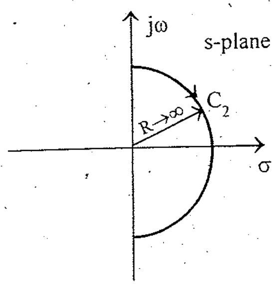


Fig 5.17.4 : Section C_2 in s-plane

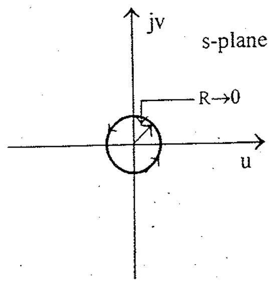


Fig 5.17.5 : Mapping of section C_2 in G(s)H(s)-plane

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 5.17.6 and fig 5.17.7.

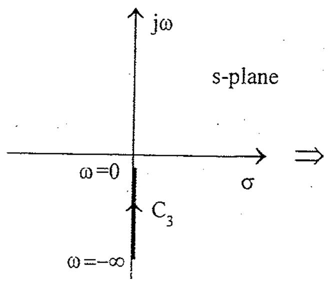


Fig 5.17.6 : Section C_3 in s-plane

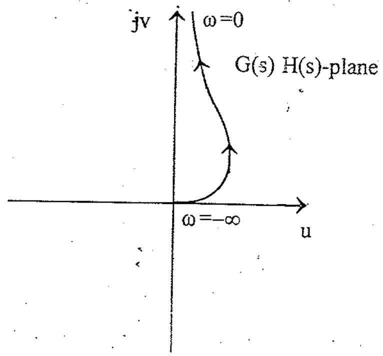


Fig 5.17.7 : Mapping of section C_3 in G(s)H(s)-plane

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, G(s) H(s) can be approximated as shown below. [i.e., $(1-s) \approx 1$]

$$G(s)\,H(s) = \frac{5}{s\,(1-s)} \approx \frac{5}{s \times 1} = \frac{5}{s}$$

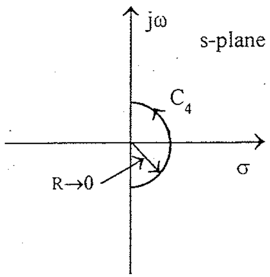
$$\text{Let } s = \underset{R\to 0}{Lt}\ Re^{j\theta}$$

$$\therefore\ G(s)\,H(s)\Big|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{5}{\underset{R\to 0}{Lt}\ Re^{j\theta}} = \infty e^{-j\theta}$$

$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)\,H(s) = \infty e^{j\frac{\pi}{2}} \qquad \text{.....(5.17.3)}$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)\,H(s) = \infty e^{-j\frac{\pi}{2}} \qquad \text{.....(5.17.4)}$$

From the equations (5.17.3) and (5.17.4) we can say that section C_4 in s-plane (fig 5.17.8.) is mapped as a circular arc of infinite radius with argument varying from $\pi/2$ to $-\pi/2$ as shown in fig 5.17.9.



⇒

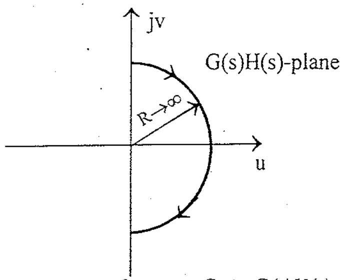


Fig 5.17.8 : Section C_4 in s-plane *Fig 5.17.9 : Mapping of section C_4 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 5.17.10.

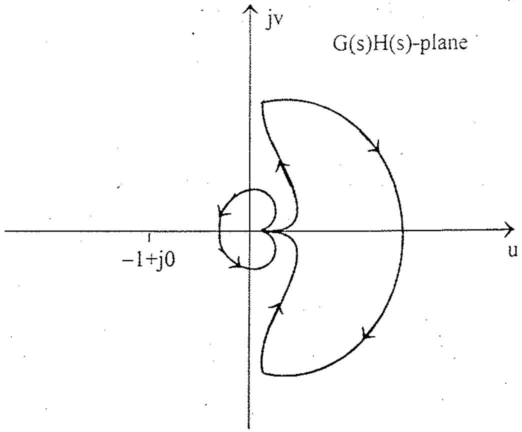


Fig 5.17.10 : Nyquist plot of $G(s)=\dfrac{5}{s(1-s)}$

STABILITY ANALYSIS

The Nyquist contour in G(s)H(s)-plane does not encircle the point (−1+j0) but the open loop transfer function has one pole on the right half s-plane. Therefore the system is unstable.

RESULT

Both open loop and closed loop systems are unstable.

EXAMPLE 5.18

By Nyquist stability criterion determine the stability of closed loop system, whose open loop transfer function is given by, $G(s)\,H(s)=\dfrac{(s+2)}{(s+1)\,(s-1)}$. Comment on the stability of open-loop and closed loop system.

SOLUTION

Given that, $G(s)\,H(s)=\dfrac{(s+2)}{(s+1)\,(s-1)}$

The open loop transfer function does not have a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane as shown in fig 5.18.1.

The Nyquist contour has three sections C_1, C_2 and C_3. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

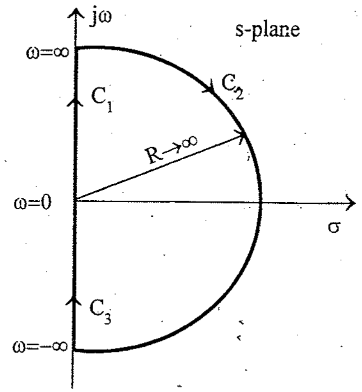


Fig 5.18.1 : Nyquist contour in s-plane

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to +∞. The mapping of section C_1 is given by the locus of G(jω)H(jω) as ω is varied from 0 to ∞. This locus is the polar plot of G(jω) H(jω).

$$G(s)\,H(s)=\frac{s+2}{(s+1)\,(s-1)}=\frac{2(1+0.5s)}{(1+s)\,(-1+s)}$$

> *Note : (−1+jω) represents a point in second quadrant*

Let s = jω

$$\therefore G(j\omega)\,H(j\omega)=\frac{2(1+j0.5\omega)}{(1+j\omega)\,(-1+j\omega)}=\frac{2\sqrt{1+0.25\omega^2}\;\angle\tan^{-1}0.5\omega}{\sqrt{1+\omega^2}\;\angle\tan^{-1}\omega\;\sqrt{1+\omega^2}\;\angle(180°-\tan^{-1}\omega)}$$

$$=\frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}\angle(-180+\tan^{-1}0.5\omega)$$

$$\therefore |G(j\omega)H(j\omega)|=\frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}$$

$$\angle G(j\omega)H(j\omega)=-180°+\tan^{-1}0.5\omega$$

The exact shape of $G(j\omega)H(j\omega)$ locus is determined by calculating the magnitude and phase of $G(j\omega)H(j\omega)$ for various values of ω.

ω rad/sec	0	0.4	1.0	2.0	10.0	∞
$\lvert G(j\omega)H(j\omega)\rvert$	2	1.76	1.12	0.57	0.1	0
$\angle G(j\omega)H(j\omega)$ deg	−180	−168	−153	−135	−101	−90

From the above analysis, we can conclude that $G(j\omega)H(j\omega)$ locus starts at $-180°$ axis at a magnitude of -2 for $\omega = 0$ and meets the origin along $-90°$ axis when $\omega = \infty$.

The section C_1 in s-plane and its corresponding mapping in G(s)H(s)-plane are shown in fig 5.18.2. and 5.18.3.

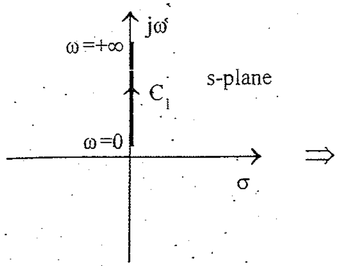


Fig 5.18.2 : Section C_1 in s-plane

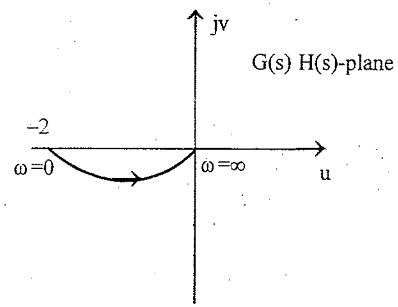


Fig 5.18.3 : Mapping of section C_1 in G(s)H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$ in G(s) H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, G(s) H(s) can be approximated as shown below [i.e., $(1+sT) \approx sT$]

$$G(s)\,H(s) = \frac{2(1+0.5s)}{(1+s)\,(-1+s)} \approx \frac{2\times 0.5s}{s\times s} = \frac{1}{s}$$

Let, $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$

$$\therefore G(s)H(s)\Big|_{s=\underset{R\to\infty}{Lt} R e^{j\theta}} = \frac{1}{\underset{R\to\infty}{Lt}\ R\,e^{j\theta}} = 0e^{-j\theta}$$

When $\theta = \frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{-j\frac{\pi}{2}}$ .....(5.18.1)

When $\theta = -\frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{j\frac{\pi}{2}}$ .....(5.18.2)

From the equations (5.18.1) and (5.18.2) we can say that section C_2 in s-plane (fig 5.18.4) is mapped as circular arc of zero radius around origin in G(s)H(s) plane with argument varying from $-\pi/2$ to $+\pi/2$ as shown in fig 5.18.5.

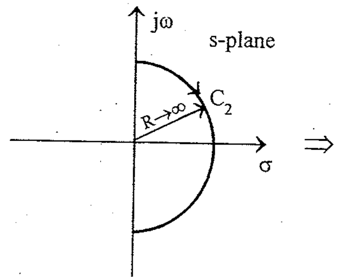


Fig 5.18.4 : Section C_2 in s-plane

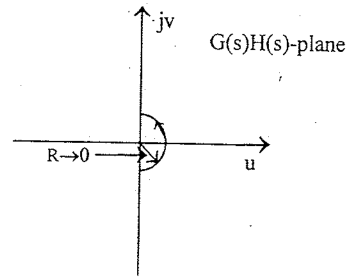


Fig 5.18.5 : Mapping of section C_2 in G(s)H(s)-plane

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 5.18.6 and fig 5.18.7.

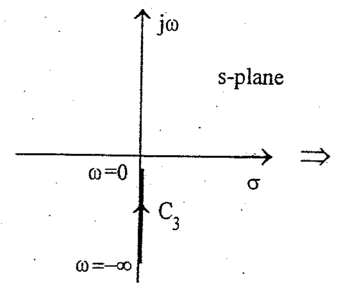


Fig 5.18.6 : Section C_3 in s-plane

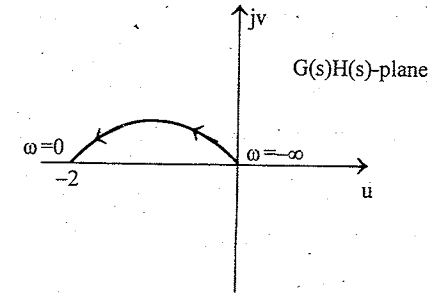


Fig 5.18.7 : Mapping of section C_3 in G(s)H(s)-plane

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 5.18.8.

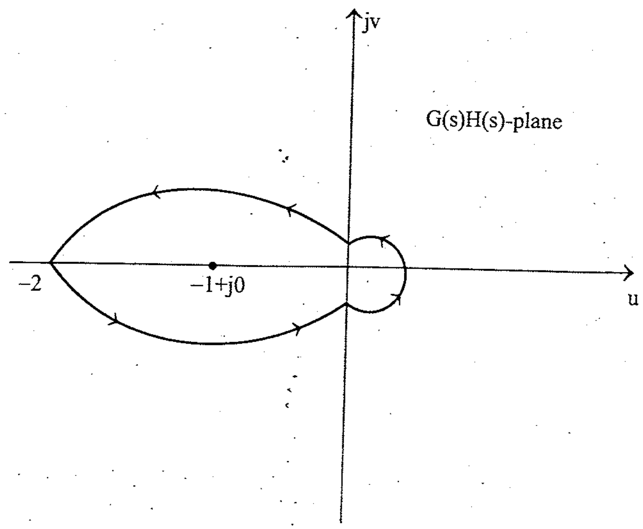


Fig 5.18.8 : Nyquist plot of $G(s)\,H(s) = \dfrac{(s+2)}{(s+1)\,(s-1)}$

STABILITY ANALYSIS

On travelling through Nyquist contour it is observed that −1+j0 point is encircled in anticlockwise direction one time. Also the open loop transfer function has one pole at right half s-plane. Since the number of anticlockwise encirclement is equal to number of open loop poles on right half s-plane, the closed loop system is stable.

RESULT

1. Open loop system is unstable
2. Closed loop system is stable.

5.6 RELATIVE STABILITY

The Relative stability indicates the closeness of the system to stable region. It is an indication of the strength or degree of stability.

In time domain, the relative stability may be measured by relative settling times of each root or pair of roots. The settling time is inversely proportional to the location of roots of characteristic equation. If the root is located far away from the imaginary axis, then the transients dies out faster and so the relative stability of system will improve. The transient response and so the relative stability for various location of roots in s-plane are shown in fig 5.6.

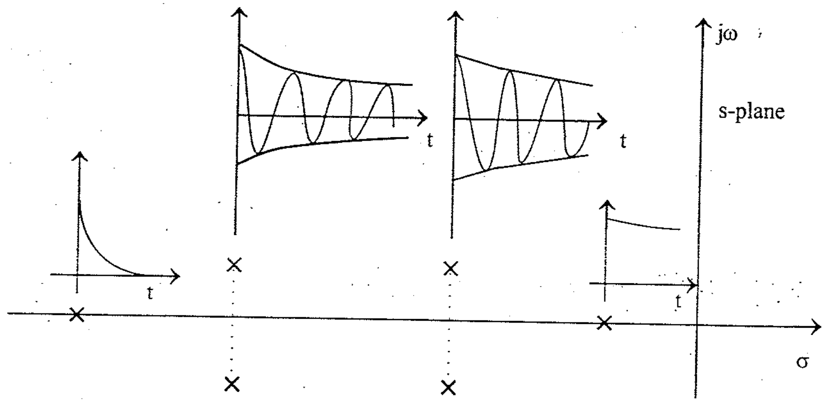


Fig 5.6 : Transient response and relative stability for various locations of roots on s-plane

In frequency domain the relative stability of a system can be studied from Nyquist plot. The relative stability of the system is given by closeness of polar plot to −1+j0 point. As the polar plot gets closer to −1+j0 point the system move towards instability.

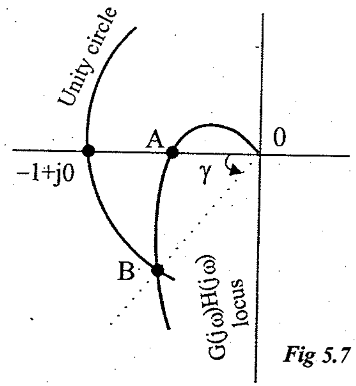


Fig 5.7

The relative stability in frequency domain are quantitatively measured in terms of phase margin and gain margin. Consider a G(jω)H(jω) locus as shown in fig 5.7. Let this locus cross the real axis at **A** and a unit circle drawn with origin as centre cuts this locus at point-B. Let γ be the angle between negative real axis and line OB.

If the gain of the system is increased then the locus will shift upwards and it may cross real axis at −1+j0 point. When the locus passes through −1+j0 point, A→1 and γ→0. Hence the closeness of G(jω) H(jω) locus to the critical point −1+j0 can be measured in terms of intercept A and angle γ. The value of A and γ are quantitative indications of relative stability. These values are used to define gain margin and phase margin as practical measures of relative stability. The concepts of gain margin and phase margin are defined for open loop systems but from the values of gain margin and phase margin the stability of closed loop system can be judged.

5.7 GAIN MARGIN AND PHASE MARGIN

Gain margin is a factor by which the system gain can be increased to drive the system to the verge of instability. With reference to fig 5-7 the magnitude of $G(j\omega)H(j\omega)$ is A when it crosses real axis and the phase corresponding to that point is -180^0. The frequency corresponding to that point be ω_{pc}. If the gain of the system is increased by a factor 1/A then the magnitude at the frequency ω_{pc} will be $A(1/A) = 1$. Now the $G(j\omega)H(j\omega)$ locus will pass through $-1+j0$ point driving the system to the verge of instability. Hence the gain margin, K_g of the system may be defined as the reciprocal of the gain at which the phase angle is $-180°$. The frequency at which the phase angle is $-180°$ is called phase crossover frequency.

$$\text{Gain margin, } K_g = \left.\frac{1}{|G(j\omega)H(j\omega)|}\right|_{\omega=\omega_{pc}}$$

$$\text{Gain margin in db} = 20\log\frac{1}{|G(j\omega)H(j\omega)|} = -20\log|G(j\omega)H(j\omega)|\Big|_{\omega=\omega_{pc}}$$

Note : *Gain Margin in decibels is given by negative of db magnitude of $G(j\omega)$ at phase crossover frequency. Hence at ω_{pc}, if db magnitude is negative, then gain margin is positive and vice versa.*

The phase margin is defined as the amount of additional phase lag at gain crossover frequency required to bring the system to verge of instability. The frequency at which the magnitudes of $G(j\omega)H(j\omega)$ equals unity is called the gain crossover frequency, ω_{gc}. With reference to fig 5.7 the phase angle corresponding to the meeting point of unity circle and $G(j\omega)H(j\omega)$ locus is $-180° + \gamma$. Now with magnitude remaining unity if an additional phase lag equal to γ is introduced then the net phase angle becomes $-180°$ and $G(j\omega)H(j\omega)$ locus will pass through $-1+j0$ point driving the system to the verge of instability. This additional phase lag γ is known as phase margin.

$$\text{Let, } \phi_{gc} = \angle G(j\omega)H(j\omega)\big|_{\omega=\omega_{gc}} \; ; \; \text{Now } -180°+\gamma = \phi_{gc}$$

$$\therefore \text{Phase margin, } \gamma = 180° + \phi_{gc}$$

For stability of closed loop system the gain margin of open loop system should be greaterthan 1 or if it is expressed in db it should be positive and phase margin of open loop system should be positive.

EXAMPLE 5.19

The open loop transfer function of a unity feedback system is given by $G(s) = \dfrac{K}{s(1+sT_1)(1+sT_2)}$. Derive an expression for gain K in terms of T_1, T_2 and specified gain margin, K_g.

SOLUTION

$$\text{Given that, } G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

Let $s = j\omega$

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)} = \frac{K}{j\omega(1+j\omega T_2+j\omega T_1-\omega^2 T_1 T_2)}$$

$$= \frac{K}{j\omega\left[1+j\omega(T_1+T_2)-\omega^2 T_1 T_2\right]}$$

$$= \frac{K}{j\omega-\omega^2(T_1+T_2)-j\omega^3 T_1 T_2}$$

$$= \frac{K}{-\omega^2(T_1+T_2)+j\omega(1-\omega^2 T_1 T_2)}$$

The gain margin, K_g is defined as the reciprocal of the magnitude of $G(j\omega)$ at phase crossover frequency. At phase crossover frequency the magnitude is purely real. Hence at phase crossover fréquency, ω_{pc}, the imaginary part of $G(j\omega)$ is zero.

$$\therefore \text{At } \omega = \omega_{pc}, \quad \omega_{pc}\left(1-\omega_{pc}^2 T_1 T_2\right) = 0$$

$$\therefore 1-\omega_{pc}^2 T_1 T_2 = 0$$

$$-\omega_{pc}^2 T_1 T_2 = -1$$

$$\omega_{pc} = \frac{1}{\sqrt{T_1 T_2}}$$

At $\omega = \omega_{pc}$, the imaginary part is zero,

$$\therefore |G(j\omega)| = \left|\frac{K}{-\omega^2(T_1+T_2)}\right| = \frac{K}{\omega^2(T_1+T_2)}$$

$$\therefore \text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{K/\omega_{pc}^2(T_1+T_2)} = \frac{\omega_{pc}^2(T_1+T_2)}{K}$$

Put $\omega_{pc}^2 = \dfrac{1}{T_1T_2}$

$$\therefore K_g = \frac{\left(\frac{1}{T_1T_2}\right)(T_1+T_2)}{K}$$

$$K = \frac{1}{K_g}\frac{T_1+T_2}{T_1T_2} = \frac{1}{K_g}\left(\frac{T_1}{T_1T_2}+\frac{T_2}{T_1T_2}\right)$$

$$K = \frac{1}{K_g}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)$$

The expression for gain K in terms of K_g, T_1 and T_2 is, $K = \dfrac{1}{K_g}\left(\dfrac{1}{T_1}+\dfrac{1}{T_2}\right)$

EXAMPLE 5.20

Determine the Gain crossover frequency, phase crossover frequency, Gain margin and phase margin of a system with open loop transfer function, $G(s) = \dfrac{1}{s(1+2s)(1+s)}$

SOLUTION

To find phase crossover frequency and gain margin

Given that $G(s) = \dfrac{1}{s\,(1+2s)\,(1+s)}$

Let $s = j\omega$

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j2\omega)\,(1+j\omega)} = \frac{1}{j\omega\left(1+j\omega+j2\omega-2\omega^2\right)}$$

$$= \frac{1}{j\omega\,(j3\omega+1-2\omega^2)} = \frac{1}{-3\omega^2+j\omega\,(1-2\omega^2)}$$

At phase crossover frequency the imaginary part of $G(j\omega)$ is zero. Hence put $\omega = \omega_{pc}$ in imaginary part and equate to zero to solve for ω_{pc}.

$$\therefore \omega_{pc}\left(1-2\omega_{pc}^2\right) = 0$$

since $\omega_{pc} \neq 0$, $1-2\omega_{pc}^2 = 0$

$$-2\omega_{pc}^2 = -1$$

$$\omega_{pc}^2 = \frac{1}{2}$$

$$\omega_{pc} = \frac{1}{\sqrt{2}} = 0.707 \text{ rad/sec}$$

The gain margin, K_g is defined as the reciprocal of the magnitude of $G(j\omega)$ at phase cross over frequency.

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{|1/-3\omega^2|_{\omega=\omega_{pc}}}$$

$$= 3\omega_{pc}^2 = 3\times 0.707^2 = 1.5$$

Gain margin in db $= 20 \log K_g = 20 \log 1.5 = 3.5$ db

To find gain crossover frequency and phase margin

Given that $G(s) = \dfrac{1}{s\,(1+2s)\,(1+s)}$

Let $s = j\omega$

$$\therefore G(j\omega) = \frac{1}{j\omega(1+j2\omega)(1+j\omega)} = \frac{1}{\omega\angle 90^\circ\sqrt{1+4\omega^2}\angle\tan^{-1}2\omega\sqrt{1+\omega^2}\angle\tan^{-1}\omega}$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}2\omega - \tan^{-1}\omega$$

At gain crossover frequency, ω_{gc} the magnitude of G(jω) is unity.

$$\therefore \text{At } \omega = \omega_{gc}, \qquad |G(j\omega)| = \frac{1}{\omega_{gc}\sqrt{1+4\omega_{gc}^2}\,\sqrt{1+\omega_{gc}^2}} = 1$$

Solving the above equation for ω_{gc} will be tedious. Hence by trial and error find the root of the above equation.

$$\text{When } \omega = 1, \quad |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\,\sqrt{1+\omega^2}} = \frac{1}{1\sqrt{1+4}\,\sqrt{1+1}} = 0.3$$

$$\text{When } \omega = 0.5, \quad |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\,\sqrt{1+\omega^2}} = \frac{1}{0.5\sqrt{1+4\times 0.5^2}\,\sqrt{1+0.5^2}} = 1.26$$

From the two calculations shown above, we can conclude that the unity magnitude will occur for a frequency between 0.5 and 1.0.

$$\text{When } \omega = 0.6, \quad |G(j\omega)| = \frac{1}{0.6\sqrt{1+4\times 0.6^2}\,\sqrt{1+0.6^2}} = 0.915$$

$$\text{When } \omega = 0.57, \quad |G(j\omega)| = \frac{1}{0.57\sqrt{1+4\times 0.57^2}\,\sqrt{1+0.57^2}} = 1.005$$

Let ω = 0.57 be the gain crossover frequency, since for this value of ω the magnitude of G(jω) is approximately equal to one.

$\therefore$ Gain crossover frequency, $\omega_{gc} = 0.57$ rad/sec.

Let the phase of G(jω) at ω_{gc} be ϕ_{gc}

$$\text{At } \omega = \omega_{gc} = 0.57, \qquad \phi_{gc} = -90 - \tan^{-1}2\omega - \tan^{-1}\omega$$
$$= -90 - \tan^{-1}(2\times 0.57) - \tan^{-1}0.57$$
$$= -168°$$

$$\therefore \text{Phase margin, } \gamma = 180 + \phi_{gc} = 180 - 168 = 12°$$

RESULT

1. The phase crossover frequency, $\omega_{pc} = 0.707$ rad/sec
2. The gain crossover frequency, $\omega_{gc} = 0.57$ rad/sec
3. The gain margin, $K_g = 1.5$
   The gain margin in db = 3.5 db
4. The phase margin, $\gamma = 12°$

EXAMPLE 5.2[illegible]

The open loop transfer function of a system is $G(s) = \dfrac{K}{s\,(1+0.1s)\,(1+s)}$.

(i) Determine the value of K so that gain margin is 6 db.

(ii) Determine the value of K so that phase margin is 40°.

SOLUTION

(i) To find K for specified gain margin

$$\text{Given that, } G(s) = \frac{K}{s\,(1+0.1s)\,(1+s)}$$

Let s = jω

$$\therefore G(j\omega) = \frac{K}{j\omega\,(1+j0.1\omega)\,(1+j\omega)} = \frac{K}{j\omega\,(1+j1.1\omega - 0.1\omega^2)}$$
$$= \frac{K}{-1.1\omega^2 + j\omega\,(1-0.1\omega^2)}$$

At phase crossover frequency ω_{pc}, the G(jω) is real and so equate the imaginary part to zero to solve for ω_{pc}.

$$\text{At } \omega = \omega_{pc}, \qquad \omega_{pc}(1 - 0.1\omega_{pc}^2) = 0$$
$$\therefore 1 - 0.1\omega_{pc}^2 = 0$$
$$-0.1\omega_{pc}^2 = -1$$
$$\therefore \omega_{pc} = \frac{1}{\sqrt{0.1}} = 3.162 \text{ rad / sec}$$

$$|G(j\omega)|_{\omega=\omega_{pc}} = \left|\frac{K}{-1.1\omega^2}\right|_{\omega=\omega_{pc}} = \frac{K}{1.1\times 3.162^2} = 0.0909K$$

Given that gain margin = 6db

$$\therefore 20\log K_g = 6$$

$$\log K_g = \frac{6}{20}$$

$$\therefore \text{Gain margin, } K_g = 10^{\frac{6}{20}} = 1.9953$$

By definition of gain margin,

Gain margin, $K_g = \dfrac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$

$$\therefore 1.9953 = \frac{1}{0.0909K} \quad ; \quad \therefore K = \frac{1}{0.0909 \times 1.9953} = 5.5135$$

(ii) To find K for specified phase margin.

Given that $G(s) = \dfrac{K}{s\,(1+0.1s)\,(1+s)}$

Let $s = j\omega$

$$\therefore G(j\omega) = \frac{K}{j\omega\,(1+j0.1\omega)\,(1+j\omega)}$$

$$= \frac{K}{\omega\angle 90^\circ \sqrt{1+(0.1\omega)^2}\,\angle\tan^{-1}0.1\omega\,\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega}$$

$$|G(j\omega)| = \frac{K}{\omega\sqrt{1+0.01\omega^2}\,\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}0.1\omega - \tan^{-1}\omega$$

Let ω_{gc} = Gain crossover frequency

and $\phi_{gc} = \angle G(j\omega)$ at $\omega = \omega_{gc}$.

$$\text{At } \omega = \omega_{gc}, \qquad \phi_{gc} = \angle G(j\omega)|_{\omega=\omega_{gc}} = -90 - \tan^{-1}0.1\,\omega_{gc} - \tan^{-1}\omega_{gc}$$

By definition of phase margin,

Phase margin, $\gamma = 180 + \phi_{gc}$

The required phase margin is 40°, $\therefore \gamma = 40^\circ$

$$40 = 180 - 90 - \tan^{-1}0.1\omega_{gc} - \tan^{-1}\omega_{gc}$$

$$\tan^{-1}0.1\,\omega_{gc} + \tan^{-1}\omega_{gc} = 180 - 90 - 40$$

$$\tan^{-1}0.1\,\omega_{gc} + \tan^{-1}\omega_{gc} = 50^\circ$$

On taking tan on either side we get

$$\tan\left[\tan^{-1}0.1\omega_{gc} + \tan^{-1}\omega_{gc}\right] = \tan 50^\circ$$

$$\frac{\tan\,\tan^{-1}0.1\omega_{gc} + \tan\,\tan^{-1}\omega_{gc}}{1-\tan\,\tan^{-1}0.1\omega_{gc}\,.\,\tan\,\tan^{-1}\omega_{gc}} = \tan 50^\circ$$

Note : $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\frac{0.1\omega_{gc} + \omega_{gc}}{1 - 0.1\omega_{gc} \times \omega_{gc}} = 1.192$$

$$1.1\,\omega_{gc} = 1.192\,(1 - 0.1\omega_{gc}^2)$$

$$0.1192\omega_{gc}^2 + 1.1\omega_{gc} - 1.192 = 0$$

$$\omega_{gc}^2 + \frac{1.1}{0.1192}\,\omega_{gc} - \frac{1.192}{0.1192} = 0$$

$$\omega_{gc}^2 + 9.228\,\omega_{gc} - 10 = 0$$

$$\therefore \omega_{gc} = \frac{-9.228 \pm \sqrt{9.228^2 + 4\times 10}}{2} = \frac{-9.228 \pm 11.1873}{2}$$

On taking positive value we get

$$\omega_{gc} = \frac{-9.228 + 11.1873}{2} = 0.98 \text{ rad/sec}$$

$$\text{At } \omega = \omega_{gc}, \qquad |G(j\omega)| = 1$$

$$|G(j\omega)|_{\omega=\omega_{gc}} = \frac{K}{\omega\sqrt{1+0.01\omega^2}\,\sqrt{1+\omega^2}} = 1$$

$$K = \omega\sqrt{1+0.01\omega^2}\,\sqrt{1+\omega^2}\Big|_{\omega=\omega_{gc}}$$

$$= 0.98\sqrt{1+0.01\times 0.98^2}\,\sqrt{1+0.98^2}$$

$$= 1.3787$$

RESULT

For a gain margin of 6 db, K = 5.5135

For a phase margin of 40°, K = 1.3787

5.8 ROOT LOCUS

The root locus technique was introduced by **W.R.Evans** in 1948 for the analysis of control systems. The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

Consider the open loop transfer function of system, $G(s) = \dfrac{K}{s\,(s+p_1)\,(s+p_2)}$

The closed loop transfer function of the system with unity feedback is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s+p_1)\,(s+p_2)}}{1+\dfrac{K}{s\,(s+p_1)\,(s+p_2)}}$$

$$= \frac{K}{s\,(s+p_1)\,(s+p_2)+K}$$

The denominator polynomial of C(s)/R(s) is the characteristic equation of the system. The characteristic equation is given by

$$s\,(s+p_1)\,(s+p_2)+K=0$$

The roots of characteristic equation is a function of open loop gain K. [In other words the roots of characteristic equation depend on open loop gain K]. When the gain K is varied from 0 to ∞, the roots of characteristic equation will take different values. When $K=0$, the roots are given by open loop poles. When $K \to \infty$, the roots will take the value of open loop zeros.

The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called root loci (or the path taken by a root of characteristic equation when open loop gain K is varied from 0 to ∞ is called root locus).

***Note :** In general the roots of characteristic equation can be varied by varying any other system parameter other than gain.*

In general the closed loop transfer function of system with multiple loops is obtained from the signal flow graph of the system using Mason's gain formula.

$$\frac{C(s)}{R(s)} = T(s) = \frac{1}{\Delta}\sum_k P_k\Delta_k$$ *(Refer chapter 1 section 1.12)*

The determinant, Δ is the denominator polynominal of C(s)/R(s). The characteristic equation of the system is given by , $\Delta = 0$

For the single loop system shown in fig 5.8

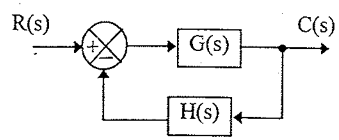


Fig 5.8

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The Characteristic equation, is $1+G(s)H(s)=0$

Let, $D(s) = G(s)H(s)$

$\therefore 1 + G(s)\,H(s) = 1+D(s) = 0$

(or) $D(s) = -1$ .....(5.22)

From equation (5.22) it can be concluded that the roots of the characteristic equation occur only for those values of s for which, $D(s) = -1$

The equation (5.22) can be converted to two Evans conditions given below,

$|D(s)| = 1$ ..... (5.23)

$\angle D(s) = \pm 180^\circ (2q+1)$ ..... (5.24)

Where $q = 0,1,2,3,$ .....

The equation (5.23) is called magnitude criterion and equation (5.24) is called angle criterion. The magnitude criterion states that $s = s_a$ will be a point on root locus if for that value of s, $|D(s)| = |G(s)\,H(s)| = 1$. The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s, $\angle D(s) = \angle G(s)\,H(s)$ is equal to an odd multiple of 180°.

The function D(s) can be expressed as a ratio of two polynomials in s as shown below.

$$D(s) = G(s)H(s) = K\,\frac{(s+z_1)\,(s+z_2)\,(s+z_3).....}{(s+p_1)\,(s+p_2)\,(s+p_3).....} \quad .....(5.25)$$

$$\therefore |D(s)| = K\,\frac{|s+z_1|.|s+z_2|.|s+z_3|.....}{|s+p_1|.|s+p_2|.|s+p_3|.....}$$

$$= K\,\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|}$$

Where, m = Number of zeros

n = Number of poles

The magnitude criterion states that $|D(s)| = 1$

$$\therefore K\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|} = 1 \quad \text{or} \quad K = \frac{\prod_{i=1}^{n}|s+p_i|}{\prod_{i=1}^{m}|s+z_i|} \qquad .....(5.26)$$

The open-loop gain K corresponding to a point $s = s_a$ on root locus can be calculated using equation (5.26). It can be shown that $|s + p_i|$ is equal to the length of vector drawn from $s = p_i$ to $s = s_a$ and $|s + z_i|$ is equal to the length of vector drawn from $s = z_i$ to $s = s_a$. Hence the equation for K can be written as

$$K = \frac{\text{Product of length of vector from open loop poles to the point } s = s_a}{\text{Product of length of vectors from open loop zeros to the point } s = s_a}$$

From equation (5.25)

$$\angle D(s) = \angle(s+z_1) + \angle(s+z_2) + \angle(s+z_3) + ... - \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)....$$

$$= \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$$

Where, m = Number of zeros

n = Number of poles

The angle criterion states that $\angle D(s) = \pm 180°\ (2q + 1)$

$$\therefore \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = \pm 180°(2q+1) \qquad .....(5.27)$$

The equations (5.27) can be used to check whether a point $s = s_a$ is a point on root locus or not. It can be shown that $\angle(s+p_i)$ is equal to the angle of vector drawn from $s = p_i$ to $s = s_a$ and $\angle(s+z_i)$ is equal to the angle of vector drawn from $s = z_i$ to $s = s_a$. Hence equation (5.27) can be written as

$$\begin{pmatrix} \text{Sum of angles of vectors} \\ \text{from open loop zeros} \\ \text{to the point } s = s_a \end{pmatrix} - \begin{pmatrix} \text{Sum of angles of vector} \\ \text{from open loop poles} \\ \text{to the point } s = s_a \end{pmatrix} = \pm 180°\ (2q+1)$$

CONSTRUCTION OF ROOT LOCUS

The exact root locus is sketched by trial and error procedure. In this method, the poles and zeros of G(s) H(s) are located on the s-plane on a graph sheet and a trial point $s = s_a$ is selected. Determine the angles of vectors drawn from poles and zeros to the trial point. From the angle criterion, determine the angle to be contributed by these vectors to make the trial point as a point on root locus. Shift the trial point suitably so that the angle criterion is satisfied.

A number of points are determined using the above procedure. Join the points by a smooth curve which is the root locus. The value of K for a particular root can be obtained from the magnitude criterion.

The trial and error procedure for sketching root locus is tedious. A set of rules have been developed to reduce the task involved in sketching root locus and to develop a quick approximate sketch. From the approximate sketch, a more accurate root locus can be obtained by a few trials.

RULES FOR CONSTRUCTION OF ROOT LOCUS

RULE 1 : The root locus is symmetrical about the real axis.

RULE 2 : Each branch of the root locus originates from an open-loop pole corresponding to $K = 0$ and terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to $K = \infty$. The number of branches of the root locus terminating on infinity is equal to $(n-m)$, i.e., the number of open loop poles minus the number of finite zeros.

RULE 3 : Segments of the real axis having an odd number of real axis open-loop poles plus zeros to their right are parts of the root locus.

RUEL 4 : The $(n-m)$ root locus branches that tend to infinity, do so along straight line asymptotes making angles with the real axis given by

$$\phi_A = \frac{\pm 180°[2q+1]}{(n-m)} \quad ; \quad q = 0,1,2,....,(n-m).$$

RULE 5 : The point of intersection of the asymptotes with the real axis is at $s = \sigma_A$ where

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

RULE 6 : The breakaway and breakin points of the root locus are determined from the roots of the equation $dK/ds = 0$. If r numbers of branches of root locus meet at a point, then they break away at an angle of $\pm 180°/r$.

RULE 7 : The angle of departure from a complex open-loop pole is given by

$$\phi_p = \pm 180^\circ (2q+1) + \phi \; ; q = 0, 1, 2, ...$$

Where ϕ is the net angle contribution at the pole by all other open loop poles and zeros. Similarly the angle of arrival at a complex open loop zero is given by

$$\phi_z = \pm 180^\circ (2q+1) + \phi \;\; ; q = 0, 1, 2, ...$$

Where ϕ is the net angle contribution at the zero by all other open-loop poles and zeros.

RULE 8 : The intersection of root locus branches with the imaginary axis can be determined by use of the Routh criterion, or by letting $s = j\omega$ in the characteristic equation and equating the real part and imaginary part to zero, to solve for ω and K. The value of ω is the intersection point on imaginary axis and K is the value of gain at the intersection point.

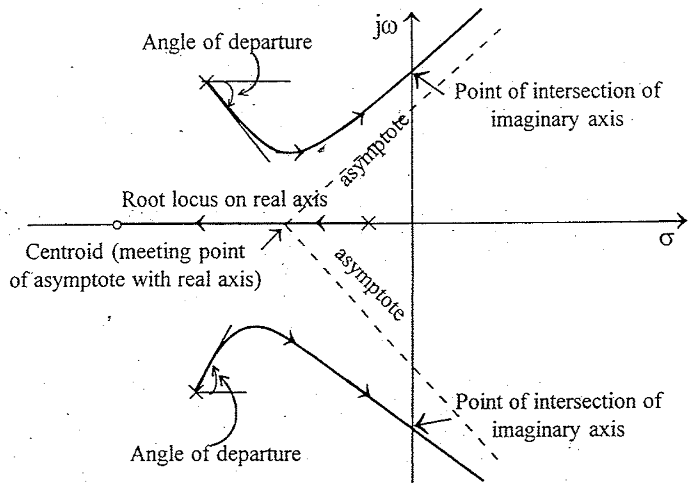
RULE 9 : The open-loop gain K at any point $s = s_a$ on the root locus is given by

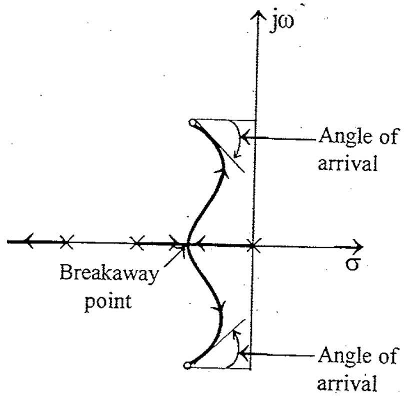
$$K = \frac{\prod_{i=1}^{n} |s_a + p_i|}{\prod_{i=1}^{m} |s_a + z_i|}$$

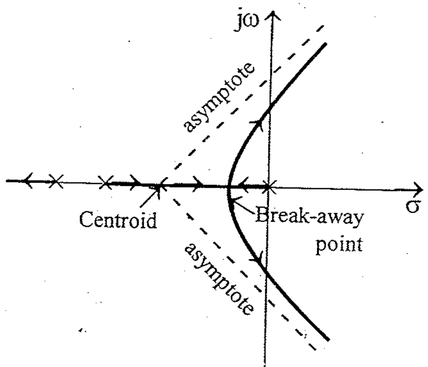
$$= \frac{\text{Product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeros to the point } s_a}$$

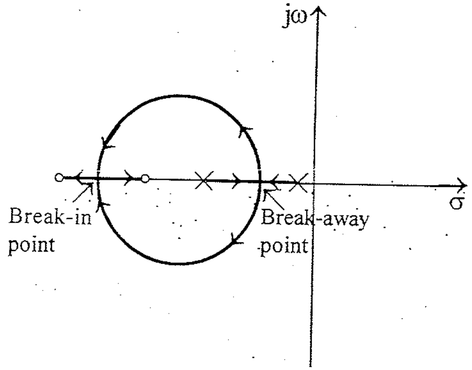
Note : *The length of vector should be measured to scale. If there is no finite zero then the product of vector lengths from zeros is equal to 1.*

TYPICAL SKETCHES OF ROOT LOCUS PLOTS









PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Locate the poles and zeros of G(s). H(s) on the s-plane. The root locus branch start from open loop poles and terminate at zeros.

Step 2 : Determine the root locus on real axis.

Step 3 : Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step 4 : Find the breakaway and breakin points.

Step 5 : If there is a complex pole then determine the angle of departure from the complex pole. If there is a complex zero then determine the angle of arrival at the complex zero.

Step 6 : Find the points where the root loci may cross the imaginary axis.

Step 7 : Take a series of test point in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test point by smooth curve.

Step 8 : The value of gain K at any point on the locus can be determined from magnitude condition.

The magnitude condition is given by

$$\left.\begin{array}{c}\text{Gain K at a point}\\ (s=s_a)\end{array}\right\} = \frac{\text{product of length of vectors from poles to the point } (s=s_a)}{\text{product of length of vectors from finite zeros to the point } (s=s_a)}$$

Note : When there is no finite zero, the denominator is taken as unity. The length of vectors should be measured to scale.

EXPLANATION FOR THE VARIOUS STEPS IN THE PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Location of poles and zeros

Draw the real and imaginary axis on an ordinary graph sheet and choose same scales both on real and imaginary axis.

The poles are marked by cross **"X"** and zeros are marked by small circle **"o"**. The number of root locus branches is equal to number of poles of open loop transfer function. The origin of a root locus is at a pole and the end is at a zero.

If n = number of poles and m = number of finite zeros,

then **m** root locus branches ends at finite zeros. The remaining **(n–m)** root locus branches will end at zeros at infinity.

Step 2 : Root locus on real axis

To decide the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Step 3 : Angles of asymptotes and centroid

If n = number of poles and m = number of zeros,

then $(n-m)$ root locus branches will terminate at zeros at infinity. These root locus branches will go along an asymptotic path and meets the asymptotes at infinity.

Hence number of asymptotes is equal to number of root locus branches going to infinity. The angles of asymptotes and the centroid are given by the following formulae

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

Where, $q = 0, 1, 2, 3, ...(n-m)$

$$\text{Centroid (meeting point of asymptote with real axis)} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Step 4 : Breakaway and Breakin points

The breakaway or breakin points either lie on real axis or exist as complex conjugate pairs. If there is a root locus on real axis between 2 poles then there exist a breakaway point. If there is a root locus on real axis between 2 zeros then there exist a breakin point. If there is a root locus on real axis between pole and zero then there may be or may not be breakaway or breakin point.

Let the characteristic equation be in the form

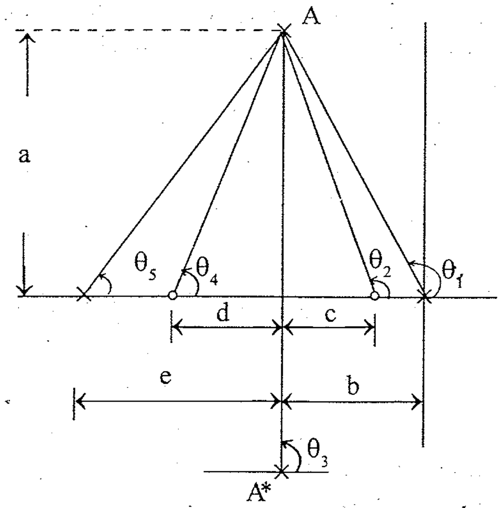
$$B(s) + K\,A(s) = 0$$

$$\therefore K = \frac{-B(s)}{A(s)}$$

The breakaway and breakin point is given by roots of the equation $dK/ds = 0$. The roots of $dK/ds = 0$ are actual breakaway or breakin point provided for this value of root the gain K should be positive and real.

Step 5 : Angle of Departure and angle of arrival

$$\left.\begin{array}{l}\text{Angle of Departure}\\ \text{(from a complex pole A)}\end{array}\right\} = 180^\circ - \left(\begin{array}{l}\text{Sum of angles of vector to the}\\ \text{complex pole A from other poles}\end{array}\right) + \left(\begin{array}{l}\text{Sum of angles of vectors to the}\\ \text{complex pole A from zeros}\end{array}\right)$$



$$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90^\circ$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig 5.9 : Calculation of angle of departure

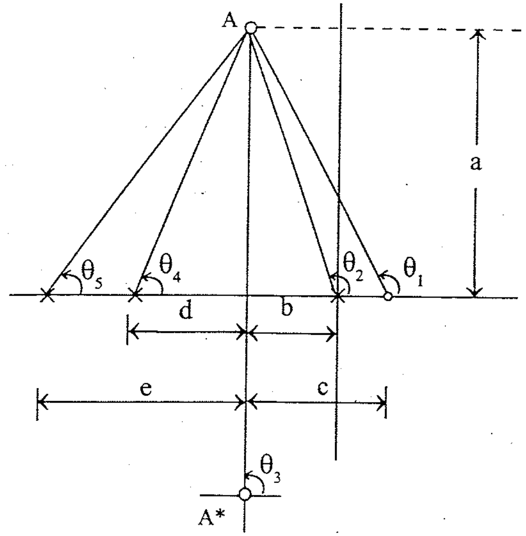
If poles are complex then they exist only as conjugate pairs. Consider the two complex conjugate poles A and A* shown in fig 5.9.

$$\left.\begin{matrix}\text{Angle of departure}\\ \text{at pole A}\end{matrix}\right\} = 180^\circ - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$$

$$\left.\begin{matrix}\text{Angle of departure}\\ \text{at pole A*}\end{matrix}\right\} = -[\text{Angle of departure at pole A}]$$

Note : *The angles can be calculated as shown in fig 5.9. or they can be measured using protractor.*

$$\left.\begin{matrix}\text{Angle of arrival at a}\\ \text{complex zero A}\end{matrix}\right\} = 180^\circ - \begin{pmatrix}\text{Sum of angles of vectors to the}\\ \text{complex zero A from all other zeros}\end{pmatrix} + \begin{pmatrix}\text{Sum of angles of vectors to the}\\ \text{complex zero A from poles}\end{pmatrix}$$



$$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{c}$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{b}$$

$$\theta_3 = 90^\circ$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig 5.10 : Calculation of angle of arrival

If zeros are complex then they exist only as conjugate pairs. Consider the two complex conjugate zeros A and A* as shown in fig 5.10.

$$\left.\begin{matrix}\text{Angle of arrival}\\ \text{at zero A}\end{matrix}\right\} = 180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$$

$$\left.\begin{matrix}\text{Angle of arrival}\\ \text{at zero A*}\end{matrix}\right\} = -[\text{Angle of arrival at zero A}]$$

Note : *The angles can be calculated as shown in fig 5.10 or they can be measured using protractor.*

Step 6 : Point of intersection of root locus with imaginary axis

The point where the root loci intersects the imaginary axis can be found by following three methods.

1. By Routh Hurwitz array
2. By trial and error approach.
3. Letting $s = j\omega$ in the characteristic equation and separate the real part and imaginary part. Two equations are obtained-one by equating real part to zero and the other by equating imaginary part to zero. Solve the two equations for ω and K. The value of ω gives the point where the root locus crosses imaginary axis. The value of K gives the value of gain K at this crossing point. Also this value of K is the limiting value of K for stability of the system.

Step 7 : Test points and root locus

Choose a test point. Using a protractor roughly estimate the angles of vectors drawn to this point and adjust the point to satisfy angle criterion. Repeat the procedure for few more test points. Sketch the root locus from the knowledge of typical sketches and the informations obtained in steps 1 through 6.

Note : In practice the approximate root locus can be sketched from the informations obtained in steps 1 through 6 and from the knowledge of typical sketches of root locus.

DETERMINATION OF OPEN LOOP GAIN FOR A SPECIFIED DAMPING OF THE DOMINANT ROOTS

The dominant pole is a pair of complex conjugate pole which decides the transient response of the system. In higher order systems the dominant poles are given by the poles which are very close to origin, provided all other poles are lying faraway from the dominant poles. The poles which are faraway from the origin will have less effect on the transient response of the system.

The transfer function of higher order systems can be approximated to a second order transfer function, whose standard form of closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The dominant poles (s_d and s_d^*) are given by the roots of quadratic factor, $(s^2+2\zeta\omega_n s+\omega_n^2) = 0$.

$$\therefore s_d = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The dominant pole can be plotted on the s-plane as shown in fig 5.11.

In the right angle triangle OAP,

$$\cos\alpha = \frac{\zeta\omega_n}{\omega_n} = \zeta \qquad \therefore \alpha = \cos^{-1}\zeta$$

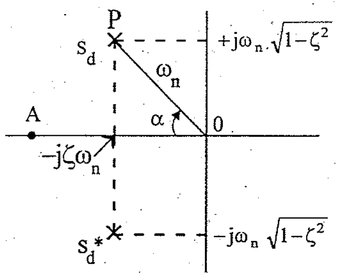


Fig 5.11 : Dominant pole, s_d

To fix a dominant pole on root locus, draw a line at an angle of $\cos^{-1}\zeta$ with respect to negative real axis. The meeting point of this line with root locus will give the location of dominant pole. The value of K corresponding to dominant pole can be obtained from magnitude condition.

$$\left.\begin{array}{r}\text{The gain K corresponding}\\ \text{to dominant pole, } s_d\end{array}\right\} = \frac{\begin{array}{c}\text{Product of length of vectors from}\\ \text{open loop poles to dominant pole}\end{array}}{\begin{array}{c}\text{Product of length of vectors from}\\ \text{open loop zeros to dominant pole}\end{array}}$$

IMPORTANCE OF ROOT LOCUS

The root locus technique is an important tool in designing control systems with desired performance characteristics. The desired performance of the system can be achieved by adjusting the location of its closed-loop poles in the s-plane by varying one or more system parameters.

The root locus can be plotted in the s-plane by varying a system parameter (usually gain, K) over the complete range of values. The roots corresponding to a particular value of the system parameter can then be located on the locus or the value of the parameter for a desired root location can be determined from the locus.

The root locus technique is also used for stability analysis. Using root locus the range of values of K, for a stable system can be determined. It is also easier to study the relative stability of the system from the knowledge of location of closed loop poles. The dominant roots are used to estimate the damping ratio and natural frequency of oscillation of the system. From ζ and ω_n the time domain specifications can be calculated.

EXAMPLE 5.22

A unity feedback control system has an open loop transfer function

$G(s) = \dfrac{K}{s\,(s^2+4s+13)}$. Sketch the root locus.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s\,(s^2+4s+13)=0$.

The roots of the quadratic are, $s = \dfrac{-4 \pm \sqrt{4^2 - 4\times 13}}{2} = -2 \pm j3$

$\therefore$ The poles are 0, $-2+j3$ and $-2-j3$.

The poles are marked by X (cross) as shown in fig 5.22.1

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig 5.22.1.

Note : For the given transfer function one root locus branch will start at the pole at the origin and travel through the negative real axis to meet the zero at infinity.

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180° \ (2q+1)}{n-m}$$

Here n = 3, and m = 0.

$$\text{If } q=0, \ \text{Angles} = \pm\frac{180°}{3} = \pm 60°$$

$$\text{If } q=1, \ \text{Angles} = \pm\frac{180°\times 3}{3} = \pm 180°$$

$$\text{If } q=2, \ \text{Angles} = \pm\frac{180°\times 5}{3} = \pm 300° = \mp 60°$$

$$\text{If } q=3, \ \text{Angles} = \pm\frac{180°\times 7}{3} = \pm 420° = \pm 60°$$

Note : It is enough if you calculate the required number of angles. Here it is given by first three values of angles. The remaining values will be repetitions of the previous values.

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 5.22.1.

Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{K}{s\,(s^2+4s+13)}}{1+\dfrac{K}{s\,(s^2+4s+13)}} = \frac{K}{s\,(s^2+4s+13)+K}$$

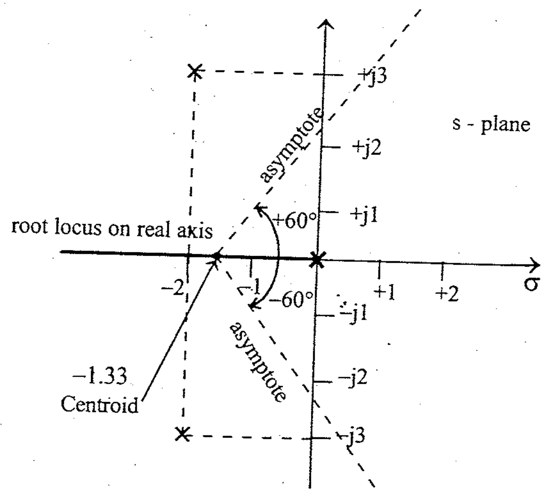


Fig : 5.22.1 : Figure showing the asymptotes, root locus on real axis and location of poles and centroid

The characteristic equation is $s(s^2+4s+13)+K=0$

$$s^3+4s^2+13s+K=0$$

$$\therefore K = -s^3-4s^2-13s$$

On differentiating the equation of K with respect to s we get.

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

$$\text{Put } \frac{dK}{ds} = 0$$

$$\therefore -(3s^2+8s+13) = 0$$

$$(3s^2+8s+13) = 0$$

$$\therefore s = \frac{-8\pm\sqrt{8^2-4\times 13\times 3}}{2\times 3}$$

$$= -1.33 \pm j1.6$$

Check for K

When $s = (-1.33 + j1.6)$, the value of K is given by

$$K = -(s^3 + 4s^2 + 13s)$$

$$= -[(-1.33 + j1.6)^3 + 4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)]$$

$\neq$ positive and real.

Also when $s = -1.33 - j1.6$ the value of K is not equal to real and positive.

Since the values of K for $s = (-1.33 \pm j1.6)$ are not real and positive, the points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step 5 : To find the angle of departure

Let us consider the complex pole A shown in fig 5.22.2. Draw vectors from all other poles to the pole A as shown in fig 5.22.2. Let the angles of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(3/2) = 123.7°$

$\theta_2 = 90°$

Angle of departure from the complex pole $A = 180° - (\theta_1 + \theta_2)$

$$= 180° - (123.7° + 90°) = -33.7°$$

The angle of departure at complex pole A* is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole A* = +33.7°

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0$$

$$-j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

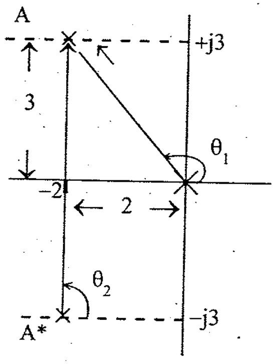


Fig 5.22.2.

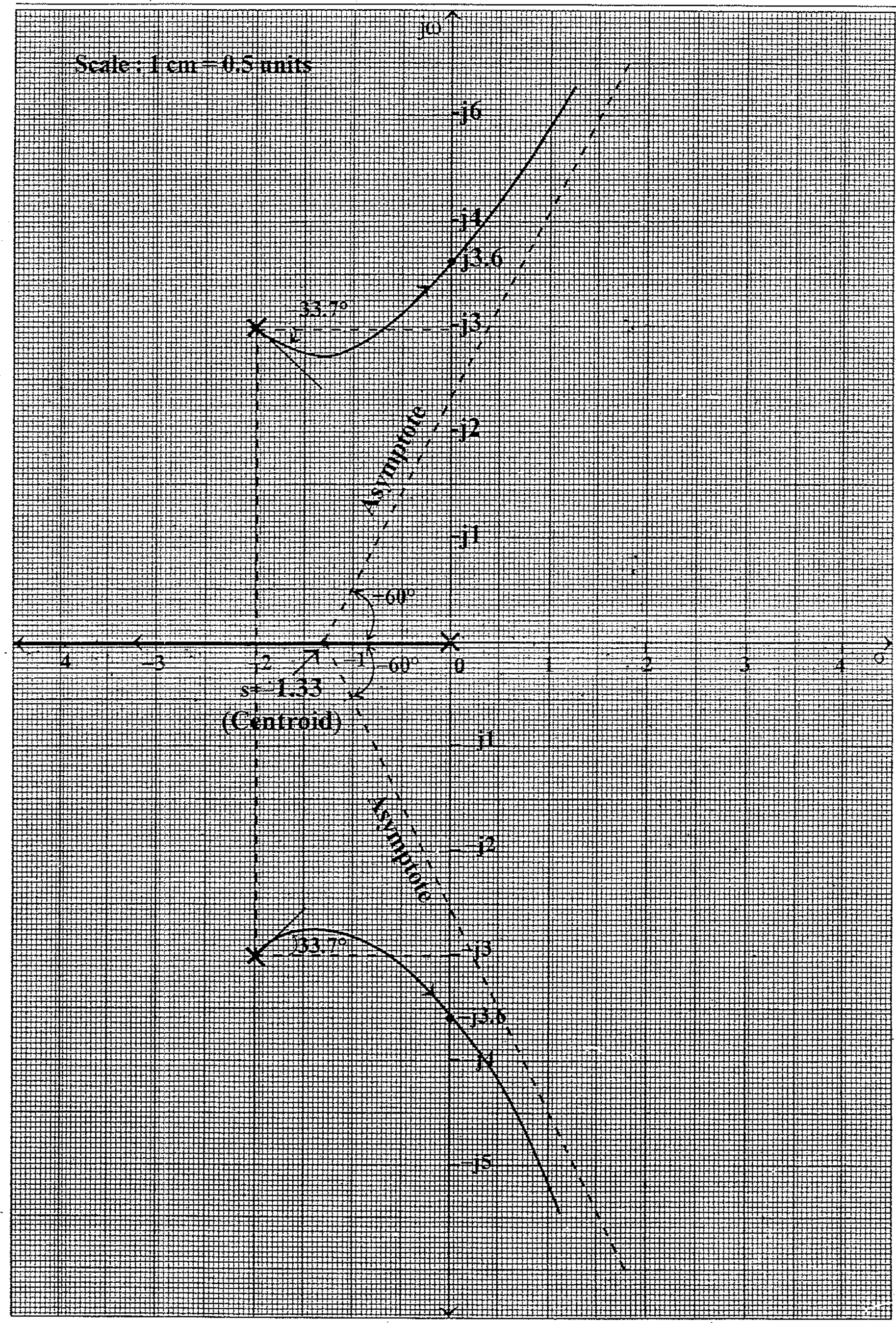


Fig 5.22.3. : Root locus sketch of, $1 + G(s) = 1 + \dfrac{K}{s(s^2 + 4s + 13)}$

On equating imaginary part to zero, we get

$$-\omega^3 + 13\omega = 0$$
$$-\omega^3 = -13\omega$$
$$\omega^2 = 13$$
$$\omega = \pm\sqrt{13}$$
$$= \pm 3.6$$

On equating real part to zero, we get

$$-4\omega^2 + K = 0$$
$$K = 4\omega^2$$
$$= 4(13) = 52.$$

The crossing point of root locus is $\pm j3.6$. The value of K at this crossing point is K = 52. (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig 5.22.3. The root locus has three branches one branch starts at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles (along the angle of departure), crosses the imaginary axis at $\pm j3.6$ and travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 5.23

Sketch the root locus of the system whose open loop transfer function is $G(s) = \frac{K}{s\,(s+2)\,(s+4)}$. Find the value of K so that the damping ratio of the closed loop system is 0.5.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+2)(s+4) = 0$.

$\therefore$ The poles are, $s = 0, -2, -4$.

The poles are marked by X(cross) as shown in fig 5.23.1.

Step 2 : To find the root locus on real axis

There are three poles on the real axis. Choose a test point on real axis between $s = 0$ and $s = -2$. To the right of this point the total number of real poles and zeros is one, which is an odd number. Hence the real axis between $s = 0$ and $s = -2$ will be a part of root locus.

Choose a test point on real axis between $s = -2$ and $s = -4$. To the right of this point, the total number of real poles and zeros is two which is an even number. Hence the real axis between $s = -2$ and $s = -4$ will not be a part of root locus.

Choose a test point on real axis to the left of $s = -4$. To the right of this point, the total number of real poles and zeros is three, which is an odd number. Hence the entire negative real axis from $s = -4$ to $-\infty$ will be a part of root locus.

Step 3 : To find asymptotes and centroid

Since there are three poles the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

Here n = 3 and m = 0.

$$\text{If } q = 0,\ \text{Angles} = \pm\frac{180}{3} = \pm 60°$$

$$\text{If } q = 1,\ \text{Angles} = \pm\frac{180\times 3}{3} = \pm 180°$$

$$\text{If } q = 2,\ \text{Angles} = \pm\frac{180\times 5}{3} = \pm 300° = \mp 60°$$

Note : It is enough if you calculate the required number of angles. Here it is given by first three values of angles. The remaining values will be repetitions of the previous values.

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$
$$= \frac{0-2-4-0}{3} = -2$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 5.23.1.

Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

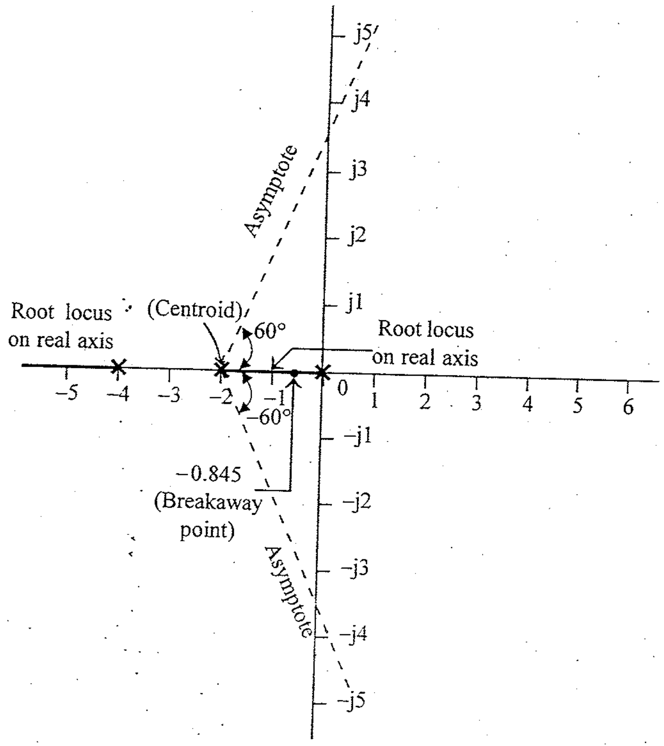


Fig 5.23.1: Figure showing the asymptotes, root locus on real axis and location of poles, centroid and breakaway points

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+2)(s+4)}}{1+\dfrac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is given by

$$s(s+2)(s+4)+K=0$$

$$s(s^2+6s+8)+K=0$$

$$s^3+6s^2+8s+K=0 \qquad \therefore K=-s^3-6s^2-8s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -(3s^2+12s+8)$$

$$\text{Put } \frac{dK}{ds}=0$$

$$\therefore -(3s^2+12s+8)=0$$

$$(3s^2+12s+8)=0$$

$$s=\frac{-12\pm\sqrt{12^2-4\times3\times8}}{2\times3}=-0.845 \quad \text{or} \quad -3.154$$

Check for K

When $s=-0.845$, the value of K is given by

$$K=-[(-0.845)^3+6(-0.845)^2+8(-0.845)]=3.08$$

Since K is positive and real for, $s=-0.845$, this point is actual breakaway point.

When $s=-3.154$, the value of K is given by

$$K=-[(-3.154)^3+6(-3.154)^2+8(-3.154)]=-3.08$$

Since K is negative for, $s=-3.154$, this is not a actual breakaway point. The breakaway point is marked on the negative real axis as shown in fig 5.23.1.

Step5 : To find angle of departure

Since there are no complex pole or zero, we need not find angle of departure or arrival.

Step 6 : To find the crossing point of imaginary axis

The characteristic equation is given by

$$s^3+6s^2+8s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+8(j\omega)+K=0$$

$$-j\omega^3-6\omega^2+j8\omega+K=0$$

Equating imaginary part to zero	Equating real part to zero
$-j\omega^3+j8\omega=0$	
$-j\omega^3=-j8\omega$	$-6\omega^2+K=0$
$\omega^2=8$	$K=6\omega^2=6\times8=48$
$\omega=\pm\sqrt{8}=\pm2.8$	

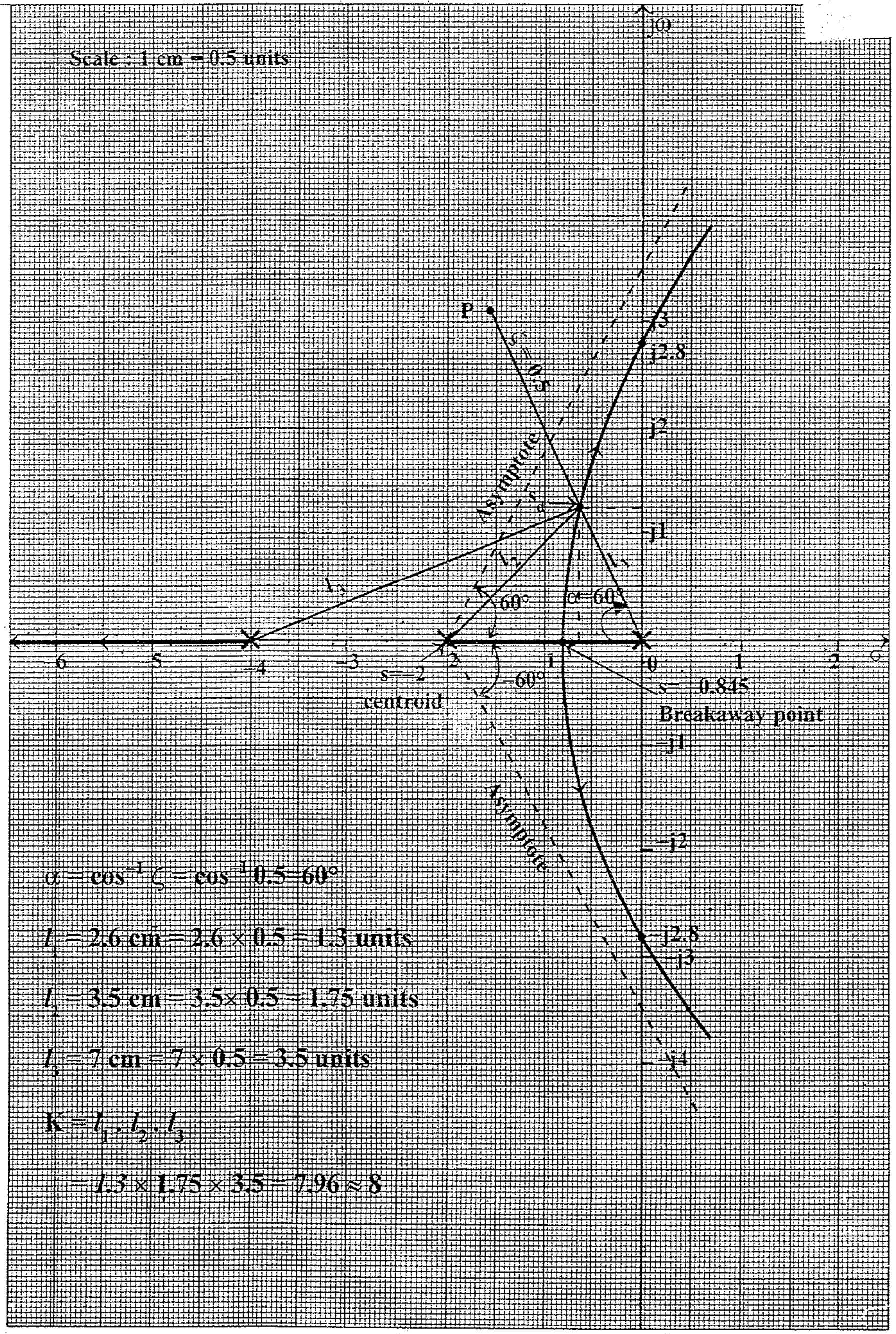


Fig 5.23.2. : Root locus sketch of, $1+G(s) = 1+\dfrac{1}{s\,(s+2)\,(s+4)}$

The crossing point of root locus is $\pm j2.8$. The value of K corresponding to this point is K = 48. (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig 5.23.2. The root locus has three branches. One branch starts at the pole at s = − 4 and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at s = 0 and s = − 2 and travel through negative real axis, breakaway from real axis at s = −0.845, then crosses imaginary axis at s = $\pm j2.8$ and travel parallel to asymptotes to meet the zeros at infinity

TO FIND THE VALUE OF K CORRESPONDING TO $\zeta = 0.5$

Given that $\zeta = 0.5$

Let $\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60°$

Draw a line OP, such that the angle between line OP and negative real axis is 60° ($\alpha = 60°$) as shown in fig 5.23.2. The meeting point of the line OP and root locus gives the dominant pole, s_d.

$$\left.\begin{array}{l}\text{The value of K corresponding}\\ \text{to the point, } s = s_d\end{array}\right\} = \frac{\begin{array}{c}\text{Product of length of vector from}\\ \text{all poles to the point, } s = s_d\end{array}}{\begin{array}{c}\text{Product of length of vectors from}\\ \text{all zeros to the point, } s = s_d\end{array}}$$

$$= \frac{l_1 . l_2 . l_3}{1} = 1.3 \times 1.75 \times 3.5 = 7.96 \approx 8$$

Note : The length of vectors are measured to scale.

EXAMPLE 5.24

The open loop transfer function of a unity feedback system is given by $G(s) = \dfrac{K\,(s+9)}{s\,(s^2+4s+11)}$. Sketch the root locus of the system.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation

$$s\,(s^2 + 4s + 11) = 0$$

The roots of the quadratic are, $s = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 11}}{2} = -2 \pm j2.64$

$\therefore$ The poles are lying at $s = 0, -2 + j2.64, -2 - j2.64$

The zeros are lying at $s = -9$ and infinity

The poles are marked by X(cross) and zeros by **"o"** (circle) as shown in fig 5.24.1.

Step 2 : To find the root locus on real axis.

One pole and one zero lies on real axis. Choose a test point to the left of $s = 0$, then to the right of this point, the total number of poles and zeros is one which is an odd number. Hence the portion of real axis from $s = 0$ to $s = -9$ will be a part of root locus. If we choose a test point to the left of $s = -9$ then to the right of this point, the total number of pole and zeros is two, which is an even number. Hence the real axis from $s = -9$ to $-\infty$ will not be a part of root locus. The root locus on real axis is shown as a bold line in fig 5.24.1.

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. One root locus branch start at the pole at origin and travel along negative real axis to meet the zero at $s = -9$. The other two root locus branches meet the zeros at infinity. The number of asymptotes required are two.

$$\text{Angle of asymptotes} = \pm\frac{180\,(2q+1)}{n-m}$$

Here $n = 3$ and $m = 1$.

If $q = 0$, $\quad \text{Angles} = \dfrac{\pm 180}{2} = \pm 90°$

If $q = 1$, $\quad \text{Angles} = \dfrac{\pm 180 \times 3}{2} = \pm 270° = \mp 90°$

If $q = 2$, $\quad \text{Angles} = \dfrac{\pm 180 \times 5}{2} = \pm 450° = \pm 90°$

***Note :** It is enough if you calculate the required number of angles. Here it is given by first two values of angles. The remaining values will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0 - 2 + j2.64 - 2 - j2.64 - (-9)}{2} = 2.5$$

The centroid is marked and from the centroid, the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown 5.24.1.

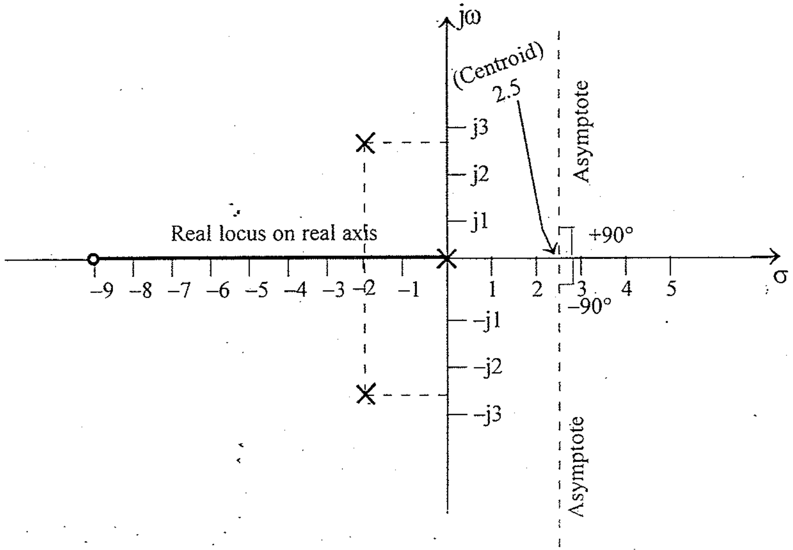


Fig 5.24.1 : Figure showing the asymptotes root locus on real axis and location of poles, zero and centroid.

Step 4 : To find the breakaway and breakin points.

From the location of poles and zero and from the knowledge of typical sketches of root locus, it can be concluded that there is no possibility of breakaway or breakin points.

Step 5 : To find the angle of departure

Let us consider the complex pole A as shown in fig 5.24.2. Draw vectors from all other poles and zero to the pole A as shown in fig 5.24.2. Let the angles of these vectors be θ_1, θ_2 and θ_3.

$$\text{Here } \theta_1 = 180° - \tan^{-1}\frac{2.64}{2} = 127.1°$$

$$\theta_2 = 90°$$

$$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7°$$

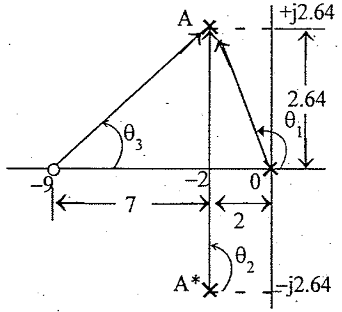


Fig 5.24.2

$$\left.\begin{array}{l}\text{Angle of departure from}\\ \text{the complex pole A}\end{array}\right\} = 180 - (\theta_1 + \theta_2) + \theta_3$$

$$= 180 - (127.1 + 90) + 20.7 = -16.4°$$

The angle of departure at the complex pole A* is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole A* = −(−16.4) = +16.4°

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point of imaginary axis

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K(s+9)}{s\,(s^2+4s+11)}}{1+\dfrac{K\,(s+9)}{s\,(s^2+4s+11)}}$$

$$= \frac{K(s+9)}{s\,(s^2+4s+11)+K(s+9)}$$

The characteristic equation is the denominator polynomial of C(s)/R(s).

$\therefore$ The characteristic equation is,

$$s\,(s^2+4s+11)+K\,(s+9)=0$$

$$(s^3+4s^2+11s)+Ks+9K=0$$

put $s = j\omega$

$$(j\omega)^3+4(j\omega)^2+11(j\omega)+K(j\omega)+9K=0$$

$$-j\omega^3-4\omega^2+j11\omega+jK\omega+9K=0$$

On equating imaginary part to zero,

$$-j\omega^3+j11\omega+jK\omega=0$$

$$-j\omega^3=-j11\omega-jK\omega$$

$$\omega^2=11+K$$

Put $K = 8.8$

$$\therefore \omega^2=11+8.8=19.8$$

$$\omega=\pm\sqrt{19.8}$$

$$=\pm 4.4$$

On equating real part to zero,

$$-4\omega^2+9K=0$$

$$9K=4\omega^2$$

Put $\omega^2=11+K$

$$\therefore 9K=4(11+K)$$

$$9K=44+4K$$

$$9K-4K=44$$

$$5K=44 \qquad \therefore K=\frac{44}{5}=8.8$$

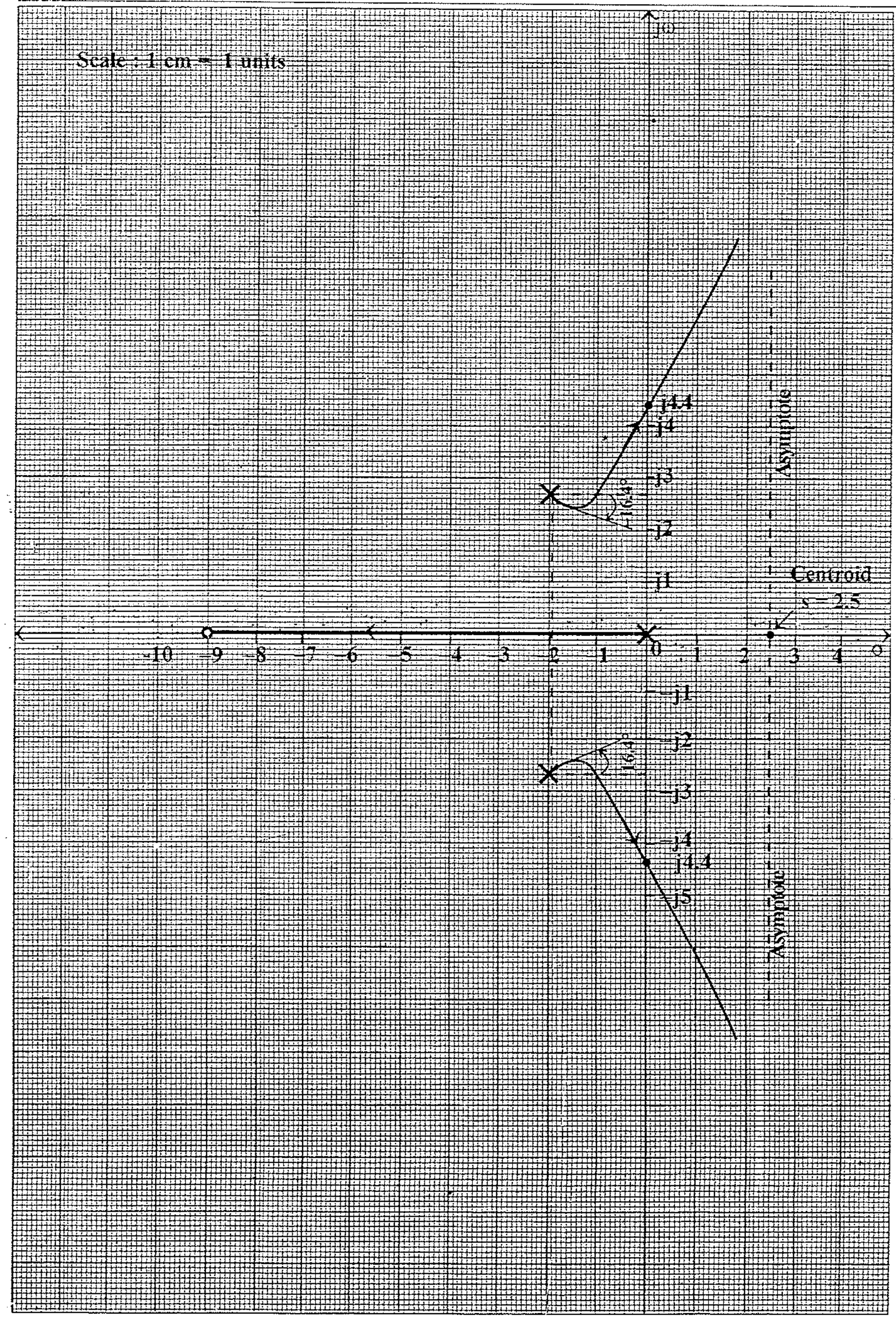


Fig 5.24.3. : Root locus sketch for, $1+G(s)=1+\dfrac{K\,(s+9)}{s\,(s^2+4s+11)}$

The crossing point of root locus is $\pm j4.4$. The value of K at this crossing point $K = 8.8$ (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig 5.24.3. The root locus has three branches. One branch starts at pole at origin and travel through negative real axis to meet the zero at $s = -9$.

The other two root locus branches starts at complex poles (along the angle of departure) crosses the imaginary axis at $\pm j4.4$ and travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 5.25

Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s)\,H(s) = \dfrac{K}{s\,(s+4)\,(s^2+4s+20)}$.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s\,(s+4)\,(s^2 + 4s + 20) = 0$.

The roots of the quadratic are, $s = \dfrac{-4 \pm \sqrt{4^2 - 4\times 1\times 20}}{2}$

$= -2 \pm j4$

$\therefore$ The poles are lying at $s = 0, -4, -2 + j4$ and $-2 - j4$

The zeros are lying at infinity

The poles are marked by X (cross) as shown in fig 5.25.1.

Step 2 : To find root locus on real axis

There are two poles on the real axis. Choose a test point on real axis between $s = 0$ and $s = -4$. To the right of this point, the total number of real poles is one which is an odd number. Hence the real axis between $s = 0$ and $s = -4$ will be a part of root locus. Choose a test point to the left of $s = -4$, now to the right of this test point the total number of poles and zeros is two which is even number. Hence the real axis from $s = -4$ to $s = -\infty$ will not be a part of root locus. The root locus on real axis is shown as a bold line in fig 5.25.1.

Step 3 : To find angles of asymptotes and centroid

Since there are four poles, the number of root locus branches are four. There is no finite zero. Hence all the four root locus branches ends at zeros at infinity. Hence the number of asymptotes required is four.

$$\text{Angle of asymptotes} = \pm\frac{180(2q+1)}{n-m}$$

Here $n = 4$ and $m = 0$.

If $q = 0$, Angles $= \dfrac{\pm 180}{4} = \pm 45°$

If $q = 1$, Angles $= \dfrac{\pm 180\times 3}{4} = \pm 135°$

If $q = 2$, Angles $= \dfrac{\pm 180\times 5}{4} = \pm 225° = \mp 135°$

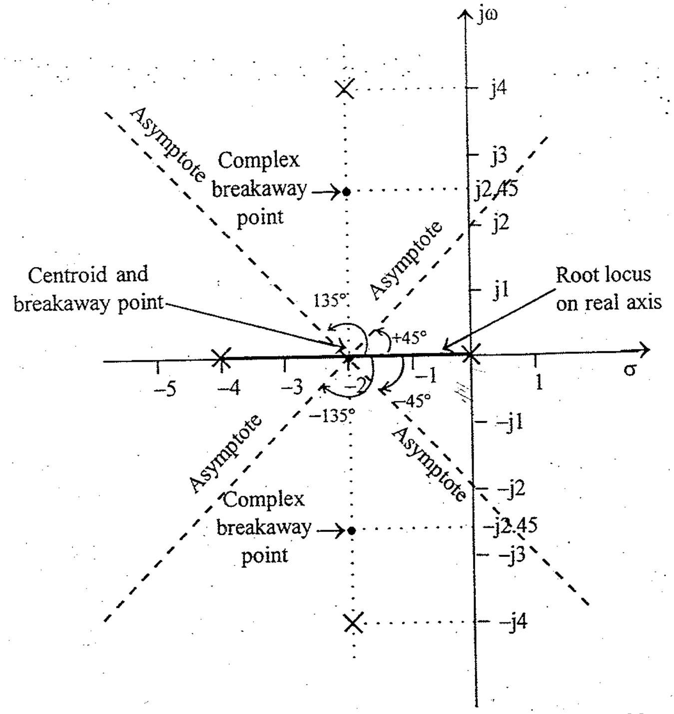


Fig 5.25.1 : Figure showing the asymptotes, root locus on real axis and location of poles, centroid and breakaway points

If q = 3, $\quad \text{Angles} = \dfrac{\pm 180° \times 7}{4} = \pm 315° = \mp 45°$

Note : *It is enough if you calculate the required number of angles. Here it is given by first four values of angles. The remaining will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n - m}$$

$$= \frac{0 - 4 - 2 + j4 - 2 - j4 - 0}{4 - 0} = \frac{-8}{4} = -2.$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 5.25.1.

Step 4 : To find the breakaway and breakin point

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+4)(s^2+4s+20)}}{1 + \dfrac{K}{s(s+4)(s^2+4s+20)}}$$

$$= \frac{K}{s(s+4)(s^2+4s+20) + K}$$

The characteristic equation is, $s(s+4)(s^2+4s+20) + K = 0$

$$\therefore K = -s(s+4)(s^2+4s+20) = -(s^2+4s)(s^2+4s+20)$$

$$K = -(s^4 + 8s^3 + 36s^2 + 80s)$$

On differentiating the equation of K with respect to s we get

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80)$$

put $\dfrac{dK}{ds} = 0$

$$\therefore -(4s^3 + 24s^2 + 72s + 80) = 0$$

$$4s^3 + 24s^2 + 72s + 80 = 0.$$

On dividing by 4 we get,

$$s^3 + 6s^2 + 18s + 20 = 0.$$

The equation $s^3 + 6s^2 + 18s + 20 = 0$ will have atleast one real root. By trial and error, the real root is found to be $s = -2$. *(Refer Appendix II for Lin's method.)*

To find the real root of $\dfrac{dK}{ds} = 0$ by Lin's method.

The first trial divisor is chosen as the last two terms of the polynomial

I^st trial

Trial divisor $= 18s + 20$

$$= s + \frac{20}{18} = s + 1.11$$

$$\begin{array}{r|l} & s^2 + 4.89s + 12.57 \\ \hline s + 1.11 & s^3 + 6s^2 + 18s + 20 \\ & s^3 + 1.11s^2 \\ \hline & 4.89s^2 + 18s \\ & 4.89s^2 + 5.43s \\ \hline & 12.57s + 20 \\ & 12.57s + 13.95 \\ \hline & 6.05 \end{array}$$

Next trial divisor $\rightarrow 12.57s + 20$

II^nd trial

Trial divisor $= 12.57s + 20$

$$= s + \frac{20}{12.57} = s + 1.59$$

$$\begin{array}{r|l} & s^2 + 4.41s + 11 \\ \hline s + 1.59 & s^3 + 6s^2 + 18s + 20 \\ & s^3 + 1.59s^2 \\ \hline & 4.41s^2 + 18s \\ & 4.41s^2 + 7s \\ \hline & 11s + 20 \\ & 11s + 17.49 \\ \hline & 2.51 \end{array}$$

Next trial divisor $\rightarrow 11s + 20$

III^rd trial

Trial divisor $= 11s + 20$

$$= s + \frac{20}{11} = s + 1.82$$

$$\begin{array}{r|l} & s^2 + 4.18s + 10.4 \\ \hline s + 1.82 & s^3 + 6s^2 + 18s + 20 \\ & s^3 + 1.82s^2 \\ \hline & 4.18s^2 + 18s + 20 \\ & 4.18s^2 + 7.6s \\ \hline & 10.4s + 20 \\ & 10.4s + 18.9 \\ \hline & 1.1 \end{array}$$

Since the remainder converge for every trial, let us approximate the root to $s = -2$. On dividing the polynomial by s + 2, we found that (s+2) is a divisor of the polynomial.

$$\begin{array}{r|l} & s^2 + 4s + 10 \\ \hline s + 2 & s^3 + 6s^2 + 18s + 20 \\ & s^3 + 2s^2 \\ \hline & 4s^2 + 18s \\ & 4s^2 + 8s \\ \hline & 10s + 20 \\ & 10s + 20 \\ \hline & 0 \end{array}$$

The polynomial $(s^3 + 6s^2 + 18s + 20) = 0$ can be expressed as

$$s^3 + 6s^2 + 18s + 20 = (s+2)(s^2 + 4s + 10) = 0$$

The root of the quadratic $(s^2 + 4s + 10)$ are given by,

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 10}}{2} = -2 \pm j2.45$$

Check for K

When $s = -2$, $K = -(s^4 + 8s^3 + 36s^2 + 80s)$

$$= -\left[(-2)^4 + 8(-2)^3 + 36(-2)^2 + 80(-2)\right]$$
$$= -[-64] = 64$$

When $s = -2 \pm j2.45$, $= 3.16\angle \pm 129°$ $\quad K = -\left[s^4 + 8s^3 + 36s^2 + 80s\right]$

$$= -[(3.16\angle \pm 129°)^4 + 8(3.16\angle \pm 129°)^3 + 36(3.16\angle \pm 129°)^2 + 80(3.16\angle \pm 129°)]$$
$$= -[99.7\angle \pm 156° + 252.4\angle \pm 27° + 359.5\angle \pm 258° + 252.8\angle \pm 129°]$$

For positive values of angles,

$$K = -[-91 + j40 + 225 + j115 - 75 - j351 - 159 + j196]$$
$$= -[-100] = 100$$

For negative values of angles,

$$K = -[-91 - j40 + 225 - j115 - 75 + j351 - 159 - j196]$$
$$= -[-100] = 100$$

For all the roots of the equation $dK/ds = 0$, the value of K is positive and real. Hence all the three roots are actual breakaway points. The breakaway points are shown in fig 5.25.1.

Step 5 : To find angle of departure

Let us consider the complex pole A shown in fig 5.25.2. Draw vectors from all other poles to the pole A as shown in fig 5.25.2. Let angles of these vectors be θ_1, θ_2 and θ_3.

Here,

$$\theta_1 = 180° - \tan^{-1}\frac{4}{2} = 117°$$

$$\theta_2 = 90°$$
$$\theta_3 = \tan^{-1}\frac{4}{2} = 63°$$

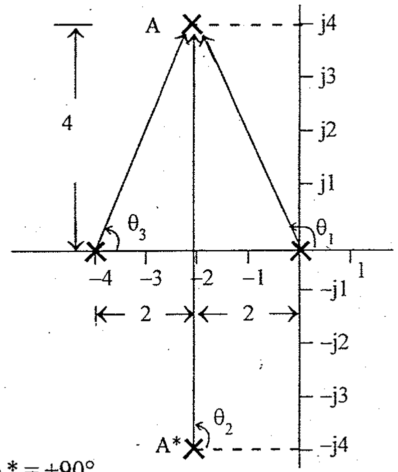


Fig 5.25.2.

$$\left.\begin{array}{l}\text{Angle of departure}\\ \text{from complex pole A}\end{array}\right\} = 180 - (\theta_1 + \theta_2 + \theta_3)$$
$$= 180 - (117 + 90 + 63)$$
$$= -90°$$

The angle of departure at complex pole A* is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure from complex pole A* = +90°

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by,

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0.$$

put $s = j\omega$,

$$(j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + K = 0.$$
$$\omega^4 - j8\omega^3 - 36\omega^2 + j80\omega + K = 0.$$

On equating imaginary part to zero,

$$-j8\omega^3 + j80\omega = 0$$
$$-j8\omega^3 = -j80\omega$$
$$\omega^2 = 10$$
$$\omega = \pm\sqrt{10} = \pm 3.2$$

On equating real part to zero,

$$\omega^4 - 36\omega^2 + K = 0$$
$$K = -\omega^4 + 36\omega^2$$
$$\text{Put } \omega^2 = 10$$
$$\therefore K = -(10)^2 + (36 \times 10) = 260.$$

The crossing point of root locus is ±j3.2. The value of K at this crossing point is K = 260. (This is the limiting value of K for stability).

The complete root locus is sketched as shown in fig 5.25.3. The root locus has four branches. All the root locus branches goes to infinity along the asymptotic lines to meet the zeros at infinity.

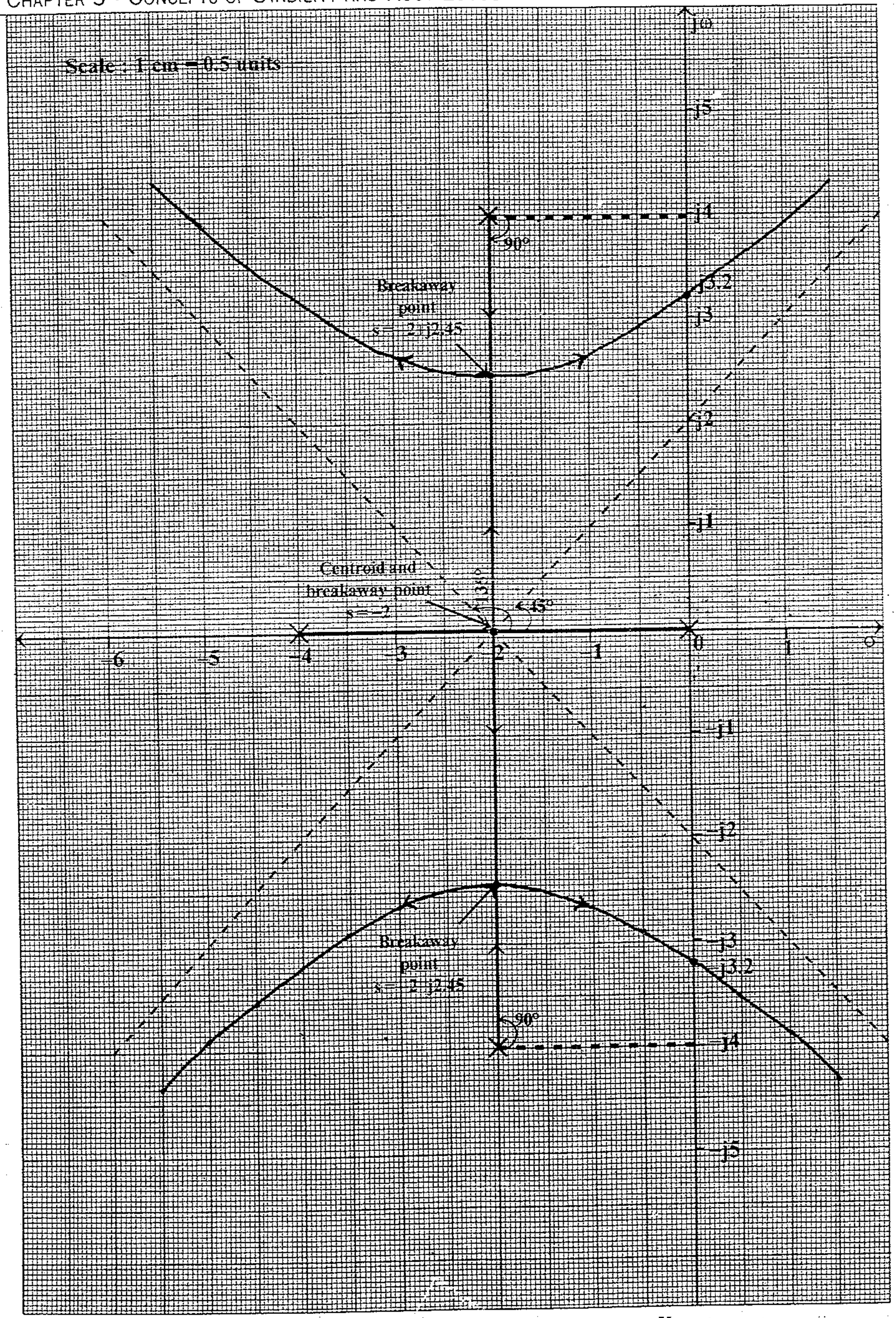


Fig 5.25.3. : Root locus sketch of, $1+G(s) = 1+\dfrac{K}{s\,(s+4)\,(s^2+4s+20)}$

EXAMPLE 5.26

Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s)\,H(s) = \dfrac{K\,(s+1.5)}{s\,(s+1)\,(s+5)}$.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+1)(s+5)=0$ and the zeros are the roots of the equation, $(s+1.5)=0$.

The poles are lying at $s = 0, -1, -5$.

The zeros are lying at $s = -1.5$ and infinity.

The poles are marked by X(cross) and zeros by **"o"** (circle) as shown in fig 5.26.1.

Step 2 : To find root locus on real axis

The segment of real axis between $s = 0$ and $s = -1$ and the segment of real axis between $s = -1.5$ and $s = -5$ will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros. The root locus on real axis are shown as bold lines in fig 5.26.1.

Step 3 : To find angles of asymptotes and centroid

Since there are three poles, the number of root locus branches are three. There is one finite zero, so one root locus branch will end at finite zero. The other two branches will meet the zeros at infinity. Hence the number of asymptotes required is two.

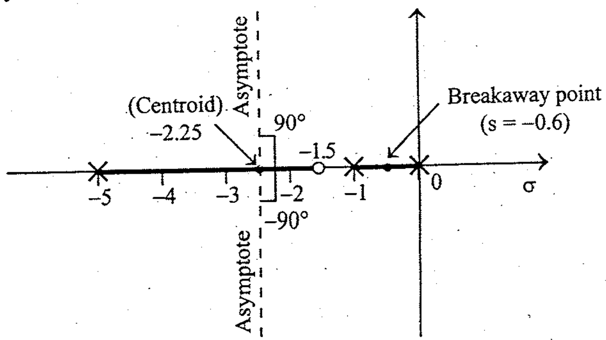


Fig 5.26.1: Figure showing the asymptote, root locus on real axis and location of poles, zero, centroid and breakaway point.

$$\text{Angle of asymptotes} = \pm\frac{180(2q+1)}{n-m}$$

Here n = 3 and m = 1.

If q = 0, $\quad \text{Angles} = \frac{\pm 180}{2} = \pm 90°$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-1-5-(-1.5)}{2} = -2.25$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 5.26.1.

Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K\,(s+1.5)}{s\,(s+1)\,(s+5)}}{1+\dfrac{K\,(s+1.5)}{s\,(s+1)\,(s+5)}}$$

$$= \frac{K\,(s+1.5)}{s\,(s+1)\,(s+5)+K(s+1.5)}$$

The characteristic equation is, s (s + 1) (s + 5) + K (s +1.5) = 0

$$\therefore K = \frac{-s\,(s+1)\,(s+5)}{s+1.5} = \frac{-s\,(s^2+6s+5)}{s+1.5}$$

$$= \frac{-(s^3+6s^2+5s)}{s+1.5}$$

On differentiating K with respect to s we get,

$$\frac{dK}{ds} = \frac{-(3s^2+12s+5)\,(s+1.5)-\left[-(s^3+6s^2+5s)\right](1)}{(s+1.5)^2}$$

$$= \frac{-3s^3-4.5s^2-12s^2-18s-5s-7.5+s^3+6s^2+5s}{(s+1.5)^2}$$

$$= \frac{-2s^3-10.5s^2-18s-7.5}{(s+1.5)^2} = \frac{-2(s^3+5.25s^2+9s+3.75)}{(s+1.5)^2}$$

For $\frac{dK}{ds} = 0$, the numerator should be zero.

$$\therefore s^3+5.25s^2+9s+3.75=0$$

The third order polynomial will have one real root. The real root of the above polynomial can be determined by Lin's method. *(Refer Appendix II).*

To find the real root of $s^3 + 5.25s^2 + 9s + 3.75 = 0$ by Lin's method

The last two terms of the polynomial are chosen as I[st] trial divisor,

I[st] trial

$$\text{I}^{st}\ \text{Trial divisor} = 9s+3.75 = s+\frac{3.75}{9} = s+0.42$$

$$\begin{array}{r|l} & s^2+4.83s+6.97 \\ \hline s+0.42 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.42s^2 \\ \hline & 4.83s^2+9s \\ & 4.83s^2+2.03s \\ \hline \text{II}^{nd}\ \text{trial divisor} \rightarrow & 6.97s+3.75 \\ & 6.97s+2.93 \\ \hline & 0.82 \end{array}$$

II[nd] trial

$$\text{II}^{nd}\ \text{Trial divisor} = 6.97s+3.75 = s+\frac{3.75}{6.97} = s+0.54$$

$$\begin{array}{r|l} & s^2+4.71s+6.46 \\ \hline s+0.54 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.54s^2 \\ \hline & 4.71s^2+9s \\ & 4.71s^2+2.54s \\ \hline \text{II}^{nd}\ \text{trial divisor} \rightarrow & 6.46s+3.75 \\ & 6.46s+3.49 \\ \hline & 0.26 \end{array}$$

III[rd] trial

$$\text{III}^{rd}\ \text{Trial divisor} = 6.46s+3.75 = s+\frac{3.75}{6.46} = s+0.58$$

$$\begin{array}{r|l} & s^2+4.67s+6.3 \\ \hline s+0.58 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.58s^2 \\ \hline & 4.67s^2+9s \\ & 4.67s^2+2.7s \\ \hline \text{IV}^{th}\ \text{trial divisor} \rightarrow & 6.3s+3.75 \\ & 6.3s+3.65 \\ \hline & 0.1 \end{array}$$

IV[th] trial

$$\text{IV}^{th}\ \text{Trial divisor} = 6.3s+3.75 = s+\frac{3.75}{6.3} = s+0.6$$

$$\begin{array}{r|l} & s^2+4.65s+6.2 \\ \hline s+0.6 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.6s^2 \\ \hline & 4.65s^2+9s \\ & 4.65s^2+2.8s \\ \hline & 6.2s+3.75 \\ & 6.2s+3.72 \\ \hline & 0.03 \end{array}$$

On neglecting the small value of 0.03, one of the root of the polynomial is, s = –0.6.

The polynomial can be expressed $s^3+5.25s^2+9s+3.75=(s+0.6)(s^2+4.65s+6.2)=0$

The roots of the quadratic $(s^2+4.65s+6.2)$ are

$$s=\frac{-4.65\pm\sqrt{4.65^2-4\times 6.2}}{2}=-2.3\pm j0.89$$

Check for K

$$\text{When } s=-0.6,\ K=\frac{-(s^3+6s^2+5s)}{s+1.5}$$

$$=\frac{-\left[(-0.6)^3+6(-0.6)^2+5(-0.6)\right]}{-0.6+1.5}=1.17$$

For $s=-0.6$, the value of K is positive and real and so it is actual breakaway point. It can be shown that for $s=-2.3\pm j0.86$ the value of K is not positive and real and so they cannot be breakaway points. The actual breakaway point is shown in fig 5.26.1.

Step 5 : To find angle of departure

Since there are no complex pole or zero we need not find angle of departure or arrival.

Step 6 : To find crossing point of imaginary axis.

The characteristic equation is

$$s(s+1)(s+5)+K(s+1.5)=0$$

$$s(s^2+6s+5)+Ks+1.5K=0$$

$$s^3+6s^2+5s+Ks+1.5K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+5(j\omega)+K(j\omega)+1.5K=0$$

$$-j\omega^3-6\omega^2+j5\omega+jK\omega+1.5K=0$$

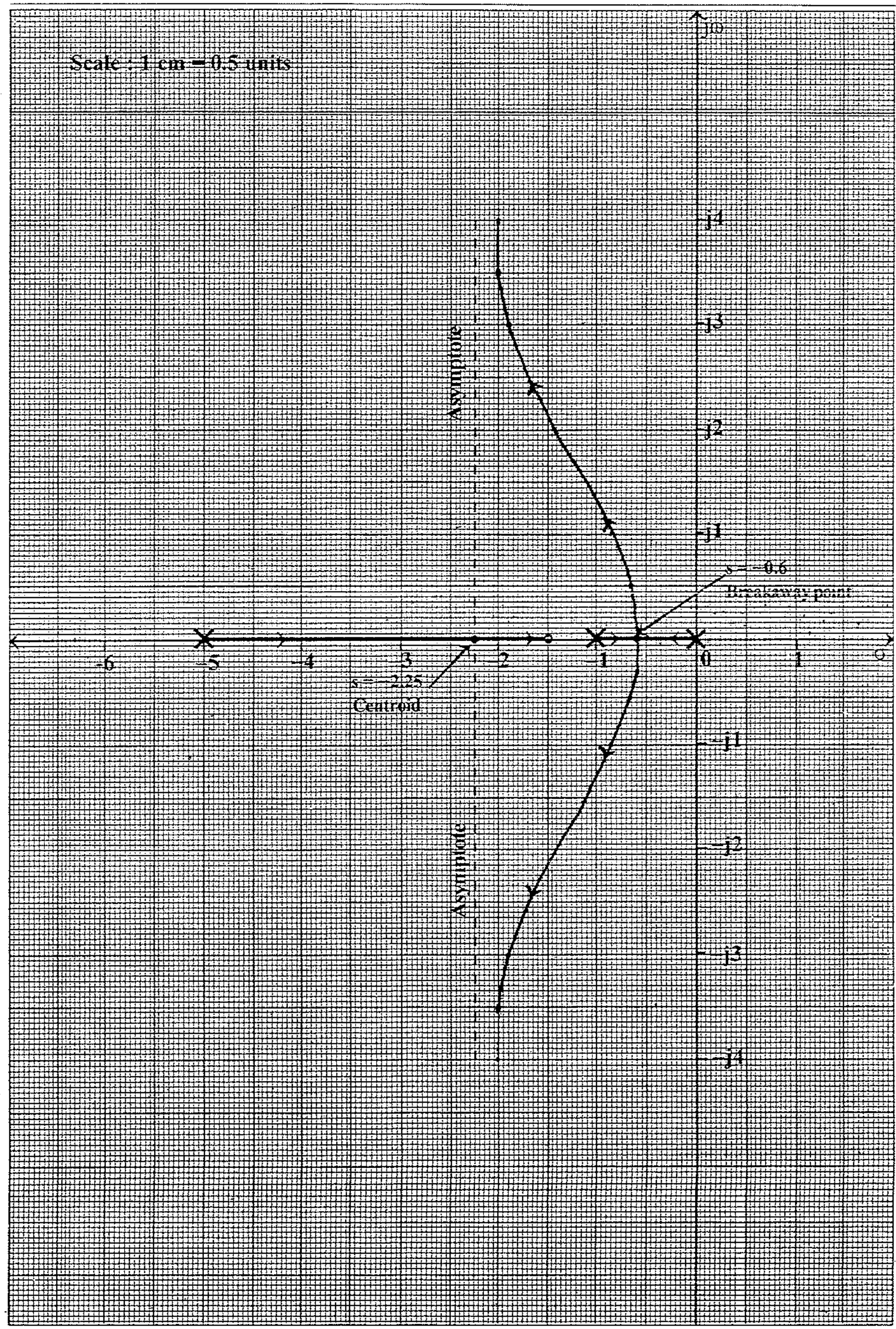


Fig 5.26.2. : Root locus sketch of, $1+G(s)=1+\dfrac{K(s+1.5)}{s(s+1)(s+5)}$

On equating imaginary part to zero, we get,

$$-j\omega^3 + j5\omega + jK\omega = 0$$

$$-j\omega^3 = -j5\omega - jK\omega$$

$$\omega^2 = 5 + K$$

On equating real part to zero, we get,

$$-6\omega^2 + 1.5K = 0$$

put $\omega^2 = 5 + K$

$$-6(5+K) + 1.5K = 0$$

$$-30 - 4.5K = 0$$

$$-4.5K = 30$$

$$K = -\frac{30}{4.5} = -6.67$$

Since the value of K is negative, there is no crossing point on imaginary axis, or for any positive values of K the root locus will not cross imaginary axis.

The complete root locus sketch is shown in figure 5.26.2. The root locus has three branches. One branch starts at $s = -5$ and ends at finite zero at $s = -1.5$. The other two root locus starts at $s = 0$ and $s = -1$ and breakaway from real axis at $s = -0.6$, then travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 5.27

Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s) = \dfrac{K\,(s^2 + 6s + 25)}{s\,(s+1)\;(s+2)}$

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s+1)(s+2) = 0$ and the zeros are the roots of the equation $(s^2 + 6s + 25) = 0$.

The roots of quadratic are, $s = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 25}}{2}$

$$= -3 \pm j4$$

The poles are lying at $s = 0, -1, -2$

The zeros are lying at $s = -3 + j4, -3 - j4$.

The poles are marked by X (cross) and zeros by **"o"** (circle) as shown in fig 5.27.1.

Step 2 : To find root locus on real axis

The segment of real axis between $s = 0$ and $s = -1$ and the entire negative real axis from $s = -2$ will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros. The root locus on real axis are shown as a bold line in fig 5.27.1.

Step 3 : To find angles of asymptotes and centroid

Since there are three poles the number of root locus branches are three. There are two finite zeros, so two root locus branch will end at finite zeros. The third root locus will meet the zero at infinity by travelling through negative real axis. Here the number of asymptote is one and the angle of asymptote is $\pm 180°$.

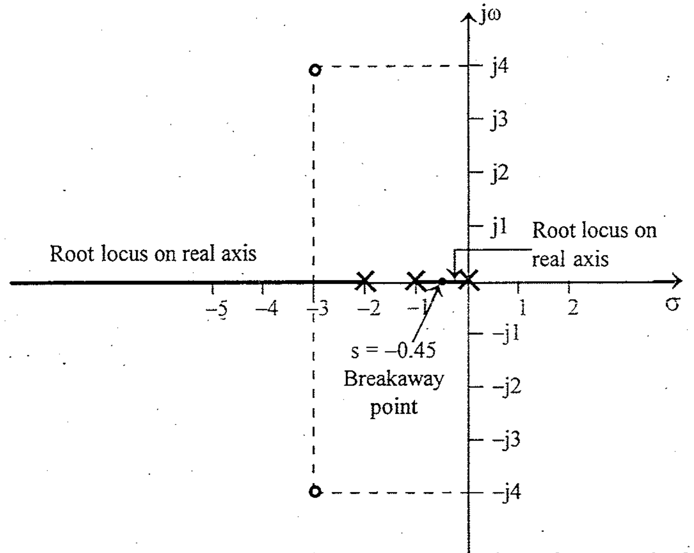


Fig 5.27.1 : Figure showing the root locus on real axis, location of poles, zeros and breakaway point.

Step 4 : To find the breakaway and breakin points

The closed loop transfer function $\Big\}$ $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K\,(s^2+6s+25)}{s\,(s+1)\,(s+2)}}{1 + \dfrac{K\,(s^2+6s+25)}{s\,(s+1)\,(s+2)}}$

$$= \frac{K\,(s^2+6s+25)}{s\,(s+1)\,(s+2) + K(s^2+6s+25)}$$

The characteristic equation is, $s\,(s+1)\,(s+2) + K\,(s^2 + 6s + 25) = 0$

$$\therefore K = \frac{-s\,(s+1)\,(s+2)}{s^2+6s+25} = \frac{-s\,(s^2+3s+2)}{s^2+6s+25}$$

$$= \frac{-s^3-3s^2-2s}{s^2+6s+25}$$

On differentiating K with respect to s we get,

$$\frac{dK}{ds} = \frac{(-3s^2-6s-2)\,(s^2+6s+25)-(-s^3-3s^2-2s)\,(2s+6)}{(s^2+6s+25)^2}$$

$$= \frac{-(s^4+12s^3+91s^2+150s+50)}{(s^2+6s+25)^2}$$

For $\frac{dK}{ds} = 0$, the numerator should be zero.

$$\therefore s^4+12s^3+91s^2+150s+50 = 0$$

The fourth order polynomial can be split into two quadratic equations. The two quadratic factors can be obtained by Lin's method. *(Refer AppendixII).*

To find quadratic factors by Lin's Method.

The first trial divisor be the last three terms

I^{st} trial

I^{st} Trial divisor $= 91s^2+150s+50$

$$= s^2+\frac{150}{91}s+\frac{50}{91} = s^2+1.65s+0.55$$

$$\begin{array}{r|l}
 & s^2+10.35s+73.37 \\
\hline
s^2+1.65s+0.55 & s^4+12s^3+91s^2+150s+50 \\
 & s^4+1.65s^3+0.55s^2 \\
\hline
 & 10.35s^3+90.45s^2+150s \\
 & 10.35s^3+17.08s^2+5.7s \\
\hline
II^{nd}\text{ trial divisor} \rightarrow & 73.37s^2+144.3s+50 \\
 & 73.37s^2+121.1s+40.35 \\
\hline
 & 23.2s+9.65
\end{array}$$

II^{nd} trial

II^{nd} Trial divisor $= 73.37s^2+144.3s+50$

$$= s^2+\frac{144.3}{73.37}s+\frac{50}{73.37} = s^2+2s+0.7$$

$$\begin{array}{r|l}
 & s^2+10s+70.3 \\
\hline
s^2+2s+0.7 & s^4+12s^3+91s^2+150s+50 \\
 & s^4+2s^3+0.7s^2 \\
\hline
 & 10s^3+90.3s^2+150s \\
 & 10s^3+20s^2+7s \\
\hline
 & 70.3s^2+143s+50 \\
 & 70.3s^2+140.6s+49.2 \\
\hline
 & 2.4s+0.8
\end{array}$$

On neglecting the small remainder we can write,

$$s^4+12s^3+91s^2+150s+50 \approx (s^2+2s+0.7)\,(s^2+10s+70.3)$$

The roots of the quadratic $(s^2+2s+0.7)$ are

$$s = \frac{-2\pm\sqrt{2^2-4\times0.7}}{2} = -0.45,\ -1.55$$

The roots of the quadratic $(s^2+10s+70.3)$ are

$$s = \frac{-10\pm\sqrt{10^2-4\times70.3}}{2} = -5\pm j6.73$$

Here, $s=-1.55$ is not a point on root locus, hence it cannot be a breakaway point.

Check the other three values for actual breakaway point.

$$\text{When } s=-0.45,\ K = \frac{-s^3-3s^2-2s}{s^2+6s+25}$$

$$= \frac{-(-0.45)^3-3(-0.45)^2-2(-0.45)}{(-0.45)^2+6(-0.45)+25} = 0.017$$

For $s=-0.45$, the value of K is positive and real and so it is actual breakaway point. It can be shown that for $s=-5\pm j6.73$ the value of K is not positive and real and so they cannot be breakaway points. The actual breakaway point is shown in fig 5.27.1.

Step 5 : To find angle of arrival

Let us consider the complex zero A shown in fig 5.27.2. Draw vectors from all other poles to the pole A as shown in fig 5.27.2. Let the angles of these vectors be θ_1, θ_2, θ_3 and θ_4.

$$\text{Here, } \theta_1 = 180° - \tan^{-1}\frac{4}{3} = 126.9°$$

$$\theta_2 = 180° - \tan^{-1}\frac{4}{2} = 116.6°$$

$$\theta_3 = 180° - \tan^{-1}\frac{4}{1} = 104°$$

$$\theta_4 = 90°$$

$$\left.\begin{array}{l}\text{Angle of arrival at}\\ \text{complex zero A}\end{array}\right\} = 180° - (\theta_4) + (\theta_1 + \theta_2 + \theta_3)$$

$$= 180° - 90° + 126.9° + 116.6° + 104°$$

$$= 437.5° = 77.5°.$$

Angle of arrival at complex zero A* is negative of the angle of arrival at complex zero A.

$$\therefore \left.\begin{array}{l}\text{Angle of arrival at}\\ \text{complex zero A*}\end{array}\right\} = -77.5°$$

Mark the angles of arrival at complex zeros using protractor.

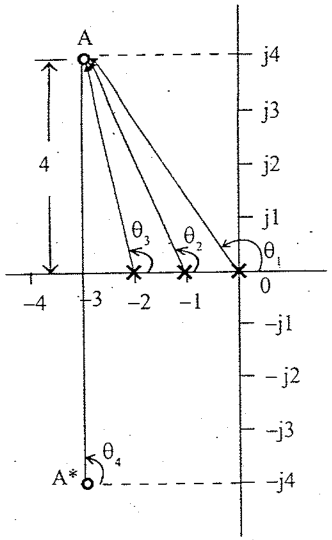


Fig 5.27.2.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is

$$s(s+1)(s+2) + K(s^2 + 6s + 25) = 0$$

$$s(s^2 + 3s + 2) + Ks^2 + 6Ks + 25K = 0$$

$$s^3 + 3s^2 + 2s + Ks^2 + 6Ks + 25K = 0$$

$$s^3 + (3+K)s^2 + (2+6K)s + 25K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + (3+K)(j\omega)^2 + (2+6K)(j\omega) + 25K = 0$$

$$-j\omega^3 - (3+K)\omega^2 + j(2+6K)\omega + 25K = 0$$

On equating imaginary part to zero

$$-j\omega^3 + j(2+6K)\omega = 0$$

$$-j\omega^3 = -j(2+6K)\omega$$

$$\omega^2 = (2 + 6K)$$

On equating real part to zero

$$-(3+K)\omega^2 + 25K = 0$$

Put $\omega^2 = 2 + 6K$

$$-(3+K)(2+6K) + 25K = 0$$

$$-(6 + 18K + 2K + 6K^2) + 25K = 0$$

$$-6K^2 + 5K - 6 = 0$$

$$K = \frac{-5 \pm \sqrt{5^2 - 4\times(-6)(-6)}}{2\times(-6)} = 0.4 \pm j0.9$$

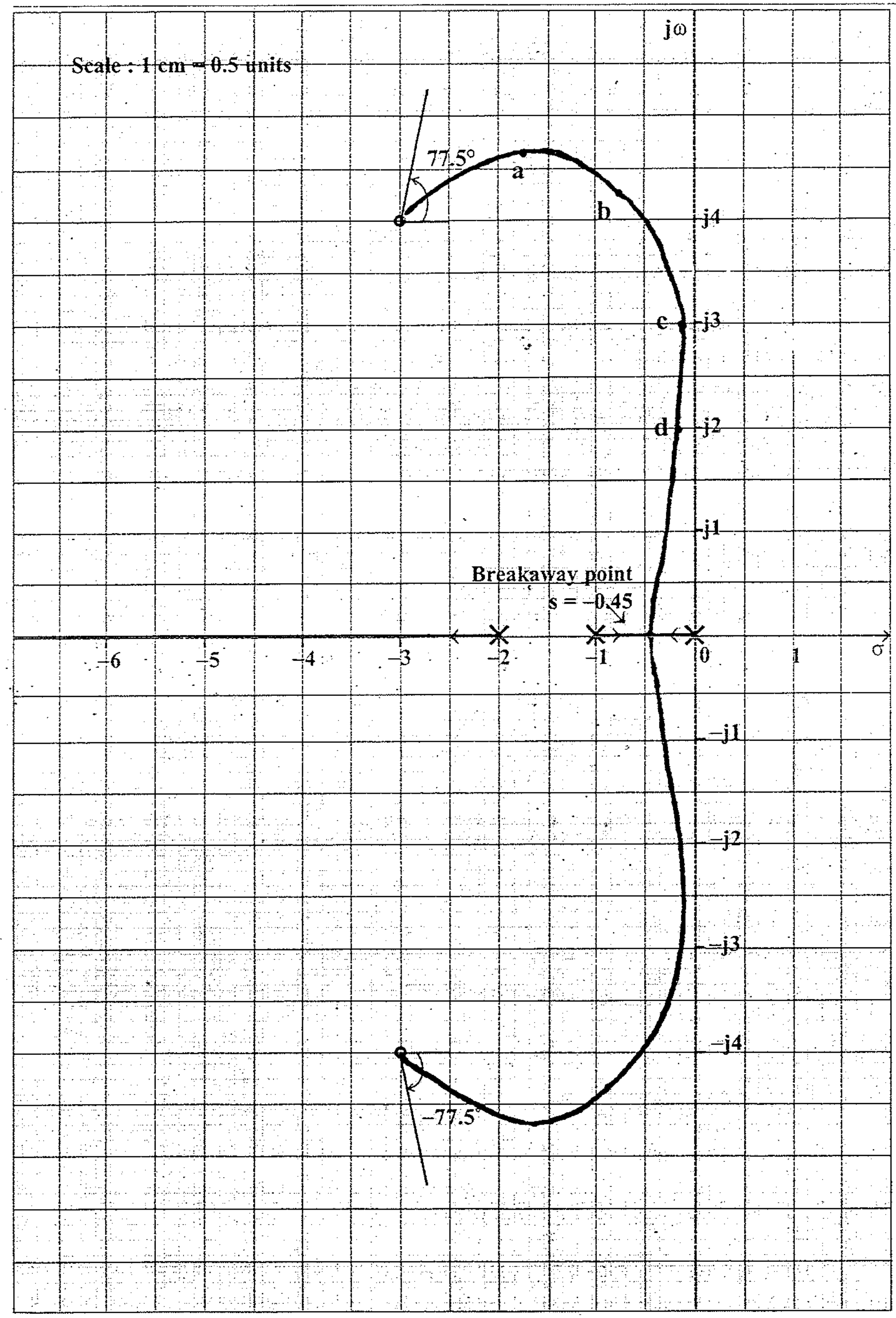


Fig 5.27.3. : Root locus sketch of, $1 + G(s) = 1 + \dfrac{K(s^2 + 6s + 25)}{s(s+1)(s+2)}$

Since the value of K is not real and positive, there is no crossing point on imaginary axis, or for any positive values of K the root locus will not cross imaginary axis.

Step 7 : To find points on root locus

Choose test points a, b, c, d on the s-plane and adjust the test points to satisfy angle criterion. The test points are shown in fig 5.27.3.

On the upper half of s-plane the root locus is sketched through the test points a, b, c and d. The root locus on the lower half of s-plane is the mirror image of the root locus on the upper half of s-plane.

The complete root locus sketch is shown in fig 5.27.3. The root locus has three branches. One branch starts at $s = -2$ and goes to infinity along negative real axis. The other two root locus branches starts at $s = 0$ and $s = -1$ and breaks from real axis at $s = -0.45$, then meets the complex zeros.

EXAMPLE 5.28

Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s) = \dfrac{K}{s(s^2+6s+10)}$.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s^2 + 6s + 10) = 0$

$$\text{The roots of the quadratic are, } s = \frac{-6 \pm \sqrt{6^2 - 4\times 10}}{2}$$

$$= -3 \pm j1$$

The poles are 0, $-3+j1$ and $-3-j1$

The poles are marked by X(cross) as shown in fig 5.28.1

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as bold line in fig 5.28.1.

Note: For the given transfer function one root locus branch will start at the pole at the origin and meet the zero at infinity through the negative real axis.

Step 3: To find angles of asymptotes and centroid.

Since there are 3 poles the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180(2q+1)}{n-m}$$

Here $n = 3$ and $m = 0$,

$$\text{If } q = 0,\ \text{Angles} = \pm\frac{180}{3} = \pm 60^\circ$$

$$\text{If } q = 1,\ \text{Angles} = \pm\frac{180\times 3}{3} = \pm 180^\circ$$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{-3+j1-3-j1}{3} = -2$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 5.28.1.

Step 4: To find the breakaway and breakin points.

$$\text{The closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s^2+6s+10)}}{1+\dfrac{K}{s(s^2+6s+10)}} = \frac{K}{s(s^2+6s+10)+K}$$

The characteristic equation is $s(s^2+6s+10)+K=0$

$$\therefore K = -s(s^2+6s+10) = -s^3-6s^2-10s$$

On differentiating the equation of K with respect to s we get

$$\frac{dK}{ds} = -3s^2-12s-10$$

$$\text{Put } \frac{dK}{ds} = 0$$

$$\therefore -3s^2-12s-10=0$$

$$3s^2+12s+10=0 \qquad \therefore s = \frac{-12 \pm \sqrt{12^2-4\times3\times10}}{2\times3} = -1.18 \text{ or } -2.82$$

Check for K

When, $s=-1.18$, $K=-s^3-6s^2-10s$

$$= -(-1.18)^3-6(-1.18)^2-10(-1.18) = 5.09$$

When, $s=-2.82$, $K=-s^3-6s^2-10s$

$$= -(-2.82)^3-6(-2.82)^2-10(-2.82) = 2.91$$

Since the values of K for $s=-1.18$ and -2.82 are positive and real, both the points are actual breakaway or breakin points. It can be proved that $s=-2.82$ is a breakin point and $s=-1.18$ is a breakaway point. The breakin and breakaway points are shown in fig 5.28.1.

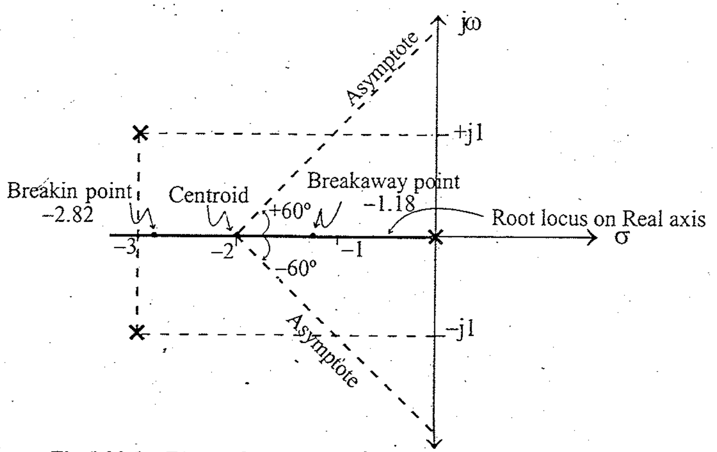


Fig 5.28.1 : Figure showing the asymptote, root locus on real axis and location of poles, centroid, breakin and breakaway points.

Step 5: To find the angle of departure

Consider the complex pole A shown in fig 5.28.2. Draw vectors from all other poles to the pole A as shown in fig 5.28.2. Let the angle of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(1/3) = 161.6°$; $\theta_2 = 90°$

Angle of departure from the complex pole A $= 180° - (\theta_1+\theta_2)$

$$= 180° - (161.6°+90°) = -71.6° \approx -72°$$

The angle of departure at complex pole A* is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole A* = +72

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point on imaginary axis.

The characteristic equation is given by

$$s(s^2+6s+10)+K=0$$

$$s^3+6s^2+10s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+10(j\omega)+K=0$$

$$-j\omega^3-6\omega^2+j10\omega+K=0$$

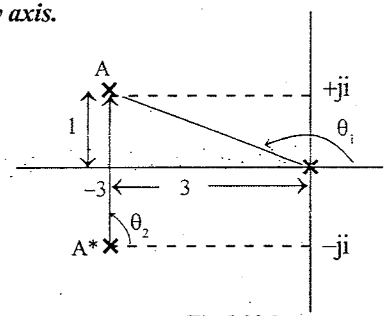


Fig 5.28.2

On equating imaginary part to zero we get,

$$-\omega^3+10\omega=0$$
$$\omega^3=10\omega$$
$$\omega^2=10$$
$$\omega=\pm\sqrt{10}=\pm3.16\approx\pm3.2$$

On equating real part to zero we get

$$-6\omega^2+K=0$$
$$K=6\omega^2$$
$$=6\times10=60$$

The root locus crosses imaginary axis at $\pm j3.2$ and the gain K corresponding to this point is 60. This is the limiting value of K for the stability of the system.

The complete root locus sketch is shown in fig 5.28.3. The root locus has three branches. One branch starts at $s=0$ and goes to infinity along negative real axis. The other two root locus branches starts at $s=-3\pm j1$ and enter the real axis at $s=-2.82$ and then breakaway from real axis at $s=-1.18$. Finally they travel parallel to asymptotes to meet the zeros at infinity.