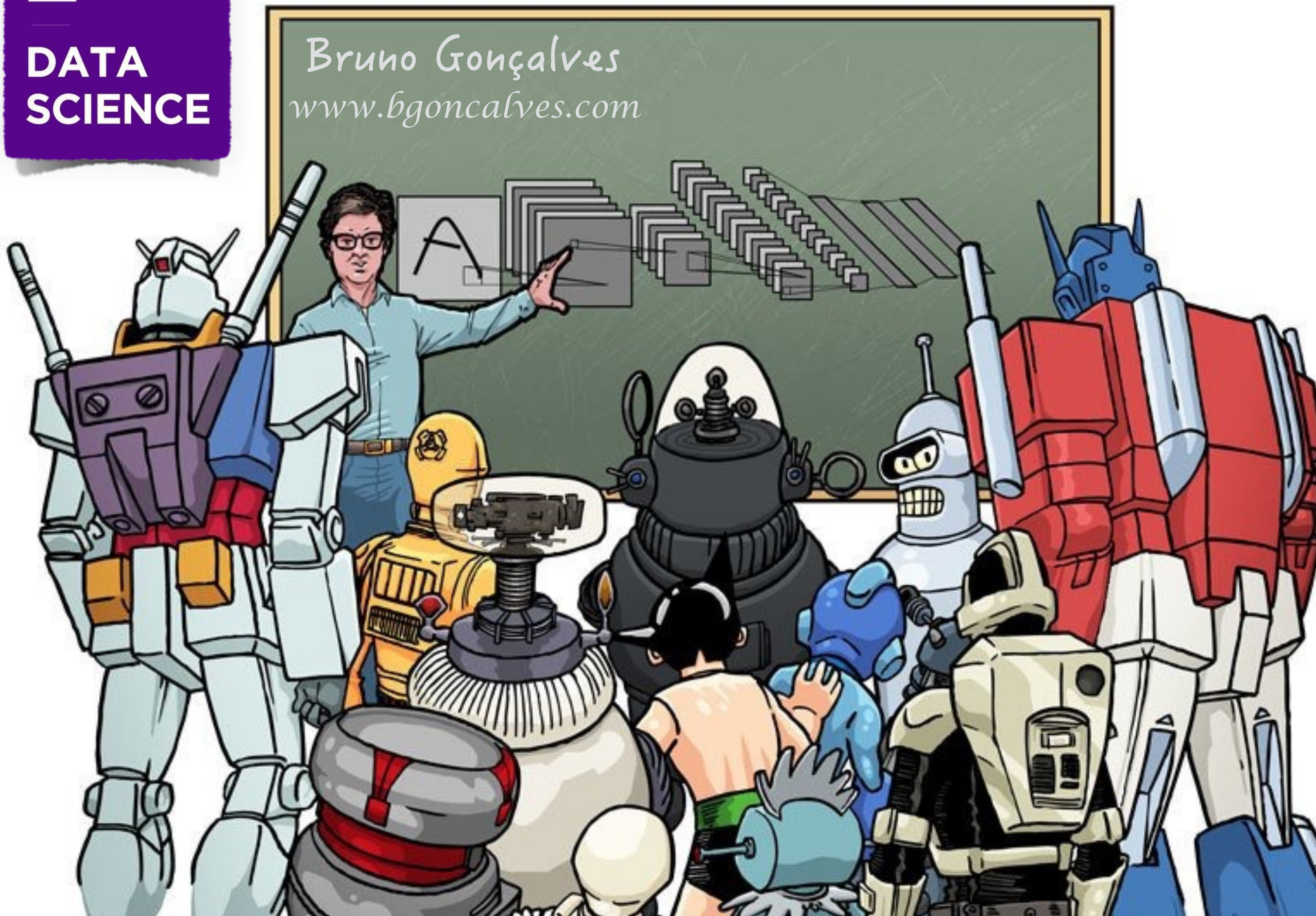
NYU

DATA SCIENCE

Machine(s) Learning and Data Science

Bruno Gonçalves
www.bgoncalves.com



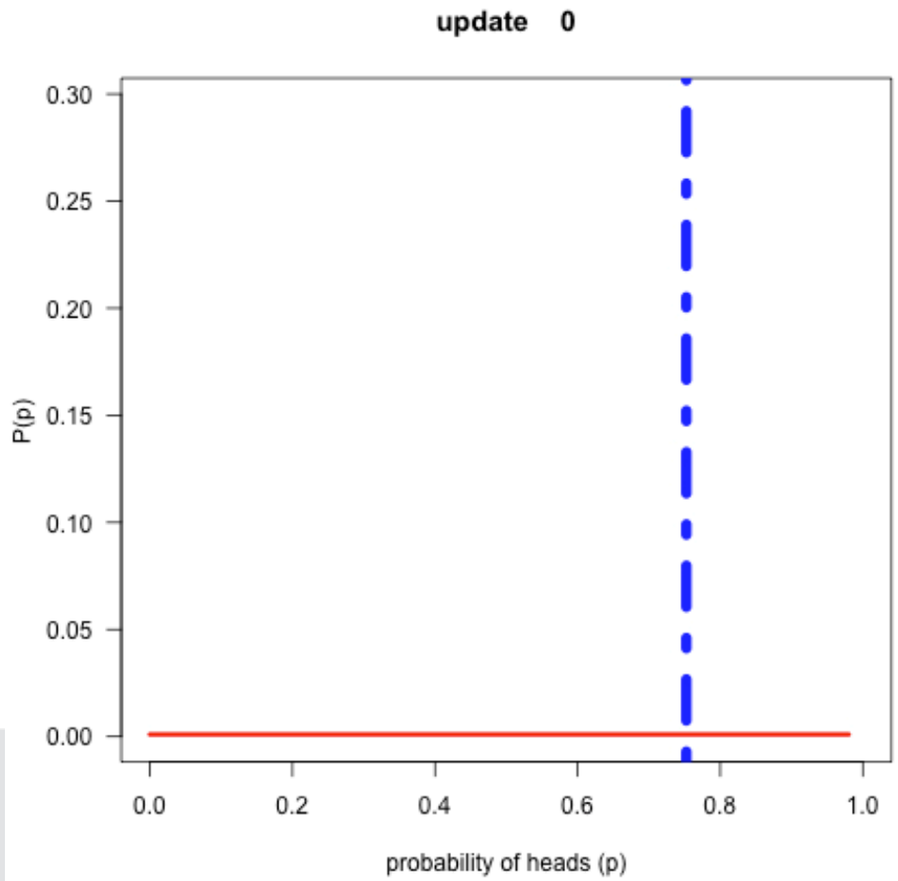
Bayesian Coin Flips

http://youtu.be/GTx0D8VY0CY

- Biased coin with unknown probability of heads (p)
- Perform N flips and update our belief after each flip using Bayes Theorem

$$P\left(p|heads\right) = \frac{P\left(heads|p\right)P\left(p\right)}{P\left(heads\right)}$$

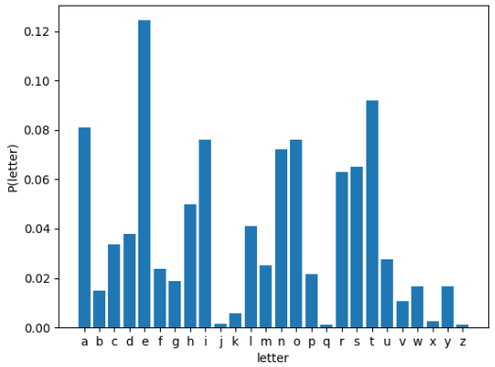
$$P\left(p|tails\right) = \frac{P\left(tails|p\right)P\left(p\right)}{P\left(tails\right)}$$




```
# Uninformative prior
prior = np.ones(bins, dtype='float')/bins
likelihood_heads = np.arange(bins)/float(bins)
likelihood_tails = 1-likelihood_heads
flips = np.random.choice(a=[True, False], size=flips, p=[0.75, 0.25])

for coin in flips:
    if coin:  # Heads
        posterior = prior * likelihood_heads
    else:  # Tails
        posterior = prior * likelihood_tails

    # Normalize
    posterior /= np.sum(posterior)

    # The posterior is now the new prior
    prior = posterior
```

coins.py

Language Detection



- Previously, we measured the probability distribution of letters in the english language. In effect, we calculated:

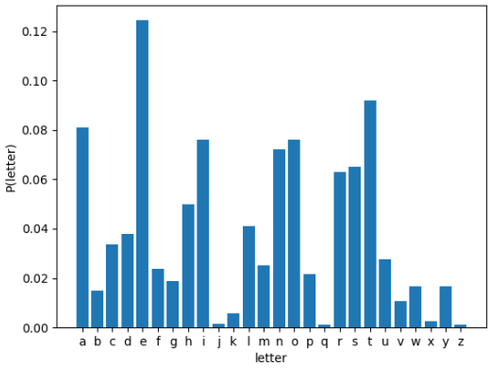
$$P\left(letter|english\right)$$

- The probability of seeing a specific letter given that the text is in English. If we do this for a few other languages we can have a table of the form:

$$P\left(letter|language\right)$$

- In the repository you'll find table_langs.dat where this conditional probability is calculated for 5 different languages: English, French, German, Italian and Spanish.

Language Detection



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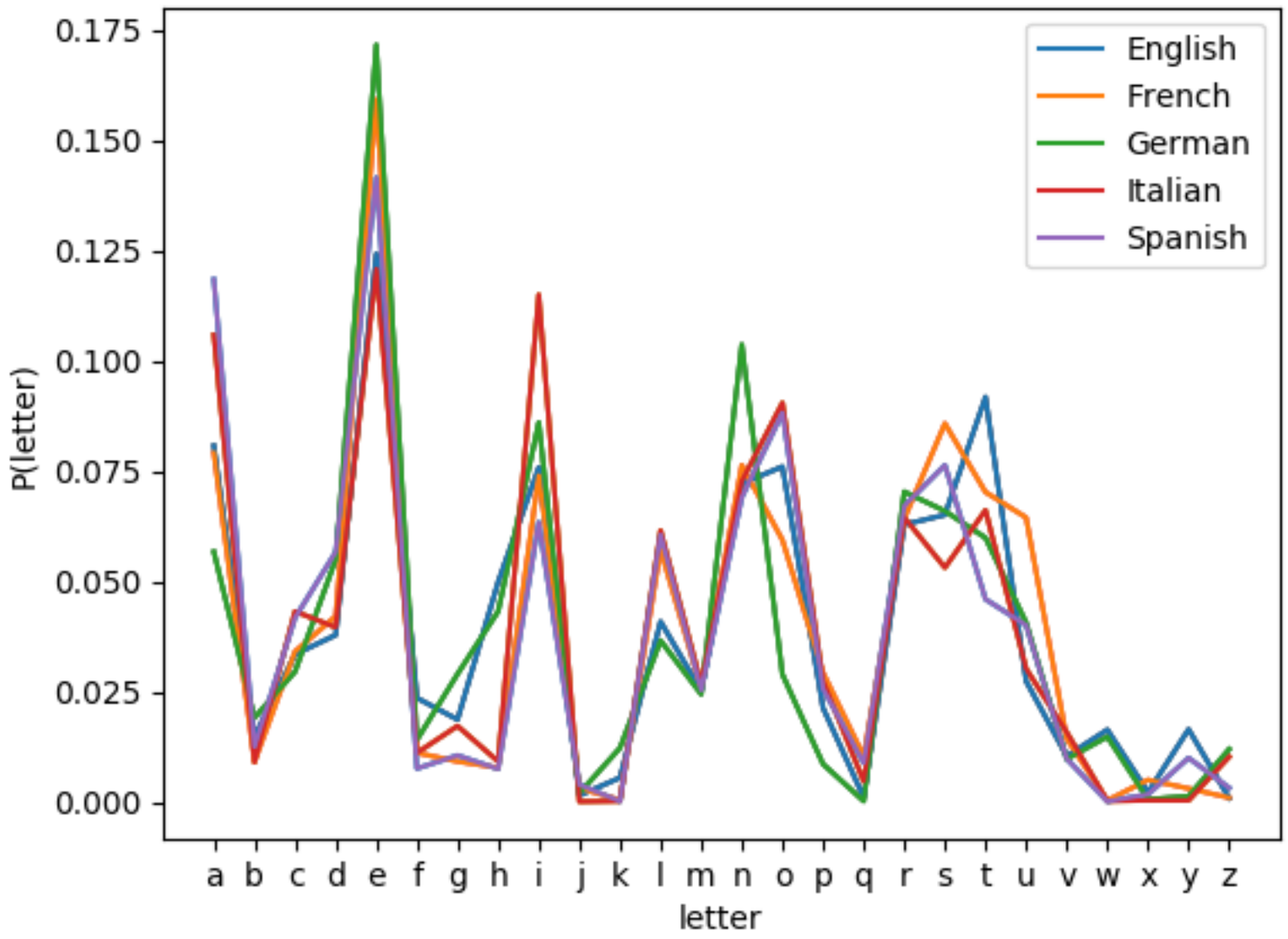
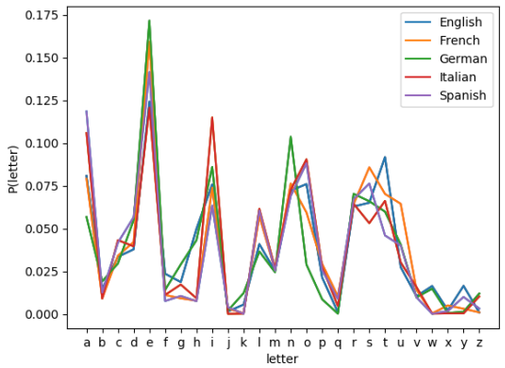


plot_langs.py

Language Detection



- A few minutes ago we measured the probability distribution of letters in the english language. In effect, we calculated

$$P\left(letter|english\right)$$

- The probability of seeing a specific letter given that the text is in English. If we do this for a few other languages we can have a table of the form:

$$P\left(letter|language\right)$$

- In the repository you'll find table_langs.dat where this conditional probability is calculated for 5 different languages: English, French, German, Italian and Spanish.

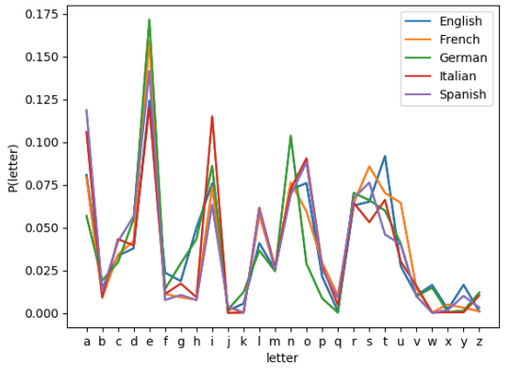
- Using these conditional probabilities, and Bayes Theorem, we can easily build a language detector. For that we just need to calculate:

$$P\left(language|text\right)$$

- We can rewrite it as a:

$$P\left(language|letter_1, letter_2, \cdots, letter_n\right)$$

Naive Bayes Classifier



$$P\left(language|letter_1, letter_2, \cdots, letter_n\right)$$

- If we treat each letter independently, we obtain:

$$P\left(language|letter_1, letter_2, \cdots, letter_n\right) = \prod_i P\left(language|letter_i\right)$$

- This is known as the Naive Bayes Approach and is an obvious oversimplification as it completely ignores correlations present in the sequence of letters.

- All we have to do is apply Bayes Theorem to our original table:

$$P\left(language|letter\right) = \frac{P\left(letter|language\right)P\left(language\right)}{P\left(letter\right)}$$

- And if we assume that all languages are equally probable (non-informative prior)

$$P\left(letter\right) = \frac{1}{N_{langs}}$$

Naive Bayes Classifier - Numerical Considerations

- Naive Bayes approaches (and many others) use terms of the form:

$$\prod_i P\left(A|B_i\right)$$

- which implies multiplying many small numbers. To avoid numerical complications, it is best to use, instead:

$$\sum_i \log P\left(A|B_i\right)$$

- Which is commonly referred to as the "Log-Likelihood". Our expression then becomes:

$$\mathcal{L}\left(language|letter_1, letter_2, \cdots, letter_n\right) = \sum_i \log\left[\frac{P\left(letter_i|language\right)P\left(language\right)}{P\left(letter_i\right)}\right]$$

- Or more simply:

$$\mathcal{L}\left(language|text\right) = \sum_i \log\left[\frac{P\left(letter_i|language\right)P\left(language\right)}{P\left(letter_i\right)}\right]$$

$$\mathcal{L}\left(language|text\right) = \sum_i \mathcal{L}\left(language|letter_i\right)$$

Naive Bayes Classifier - Normalization

- The final question to answer is, how can we compare the different results?

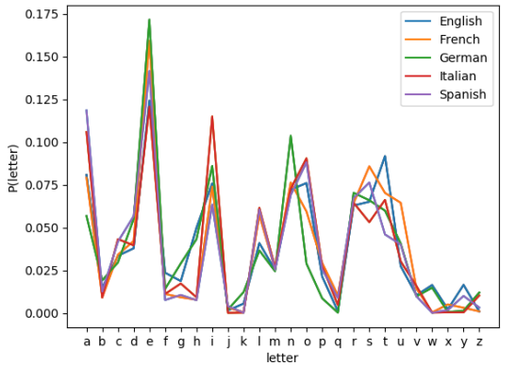
- Intuitively, we expect that the language with the highest likelihood is, well, the most likely one.

- Can we quantify how certain we are that we have the correct answer?

- We just need to remember that likelihoods actually represent probabilities and renormalize them!

$$P\left(language|text\right) = \frac{\exp\left[\mathcal{L}\left(language|text\right)\right]}{\sum_{language}\exp\left[\mathcal{L}\left(language|text\right)\right]}$$

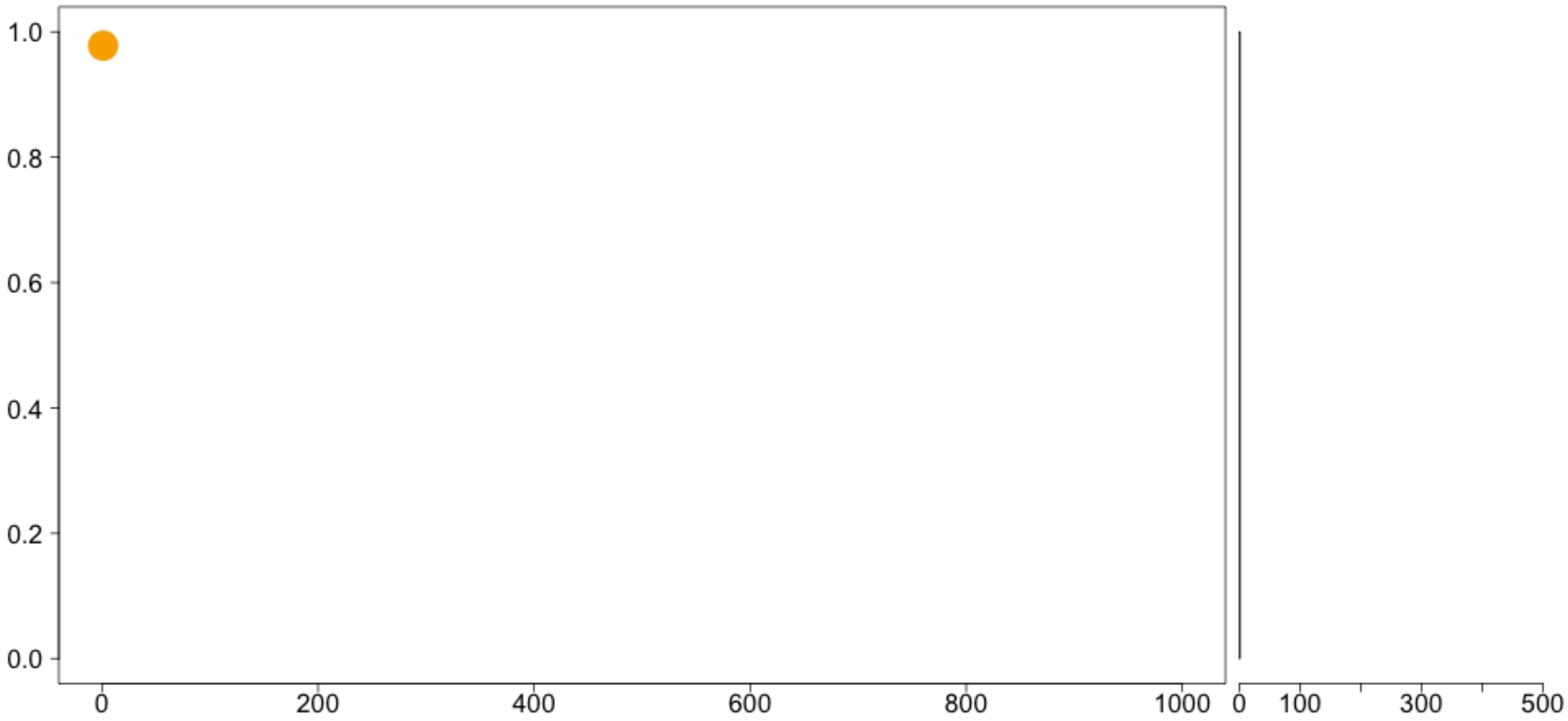
Language Detection




```python
import pandas as pd
import numpy as np
from collections import Counter

def load_data():
    P_letter_lang = pd.read_csv('table_langs.dat', sep=' ', header=0, index_col = 0)

    langs = list(P_letter_lang.columns)

    P_letter = P_letter_lang.mean(axis=1)
    P_letter /= P_letter.sum()

    P_lang_letter = np.array(P_letter_lang)/(P_letter_lang.shape[1]*P_letter.T[:,None])

    L_lang_letter = np.log(P_lang_letter.T)

    return langs, P_letter, L_lang_letter

def detect_lang(langs, P_letter, L_lang_letter, text):
    counts = np.zeros(26, dtype='int')
    pos = dict(zip(P_letter.index, range(26)))

    text_counts =  Counter(text).items()

    for letter, count in text_counts:
        if letter in pos:
            counts[pos[letter]] += count

    L_text = np.dot(L_lang_letter, counts)
    index = np.argmax(L_text)
    lang_text = langs[index]
    prob = np.exp(L_text[index])/np.sum(np.exp(L_text))*100

    return lang_text, prob, L_text
```

detect.py

Law of Large Numbers



Central Limit Theorem

- As $n \to \infty$ the random variables:

$$\sqrt{n}\left(S_n - \mu\right)$$

- with:

$$S_n = \frac{1}{n}\sum_i x_i$$

- converge to a normal distribution:

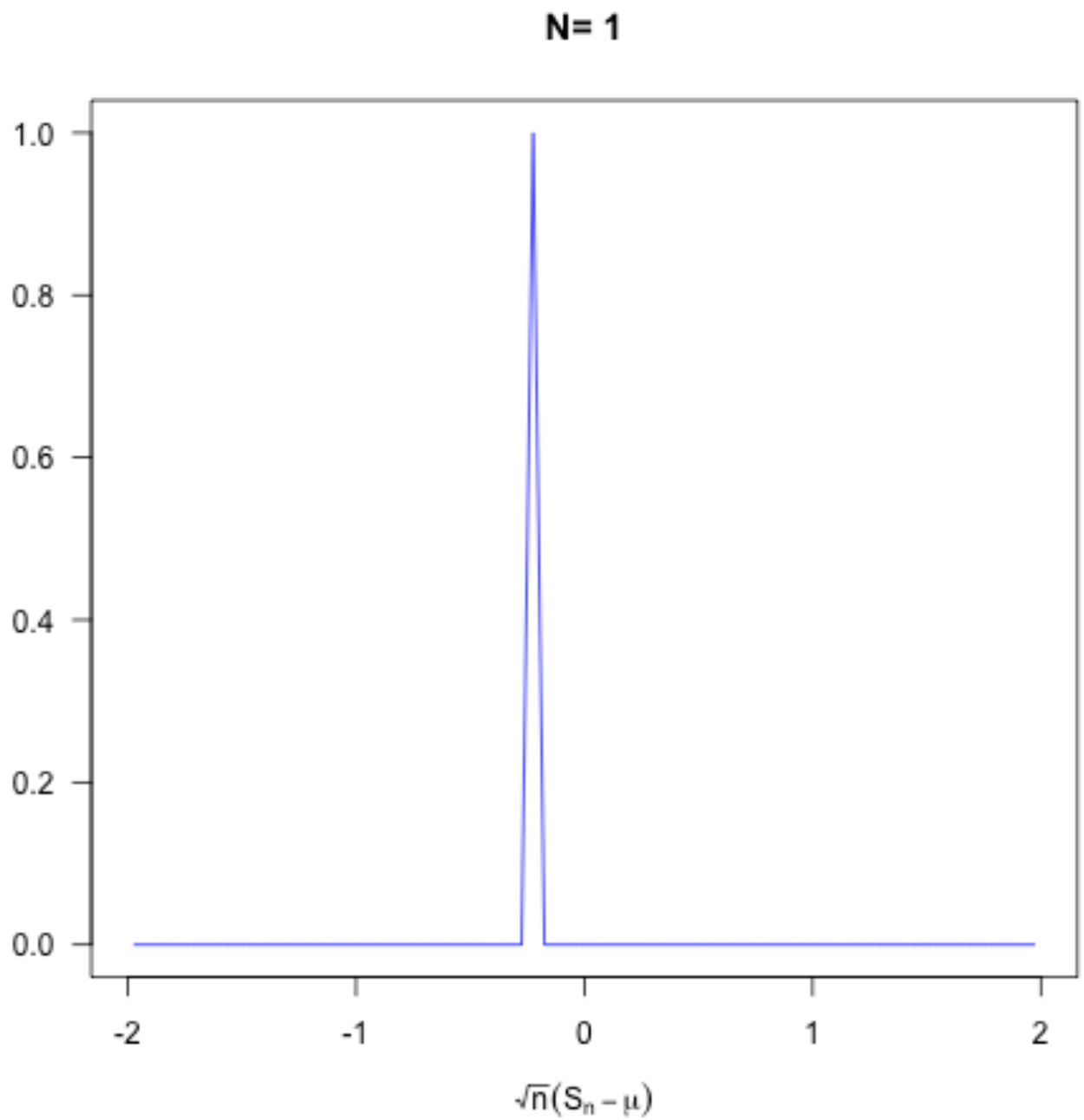
$$\mathcal{N}\left(0, \sigma^2\right)$$

- after some manipulations, we find:

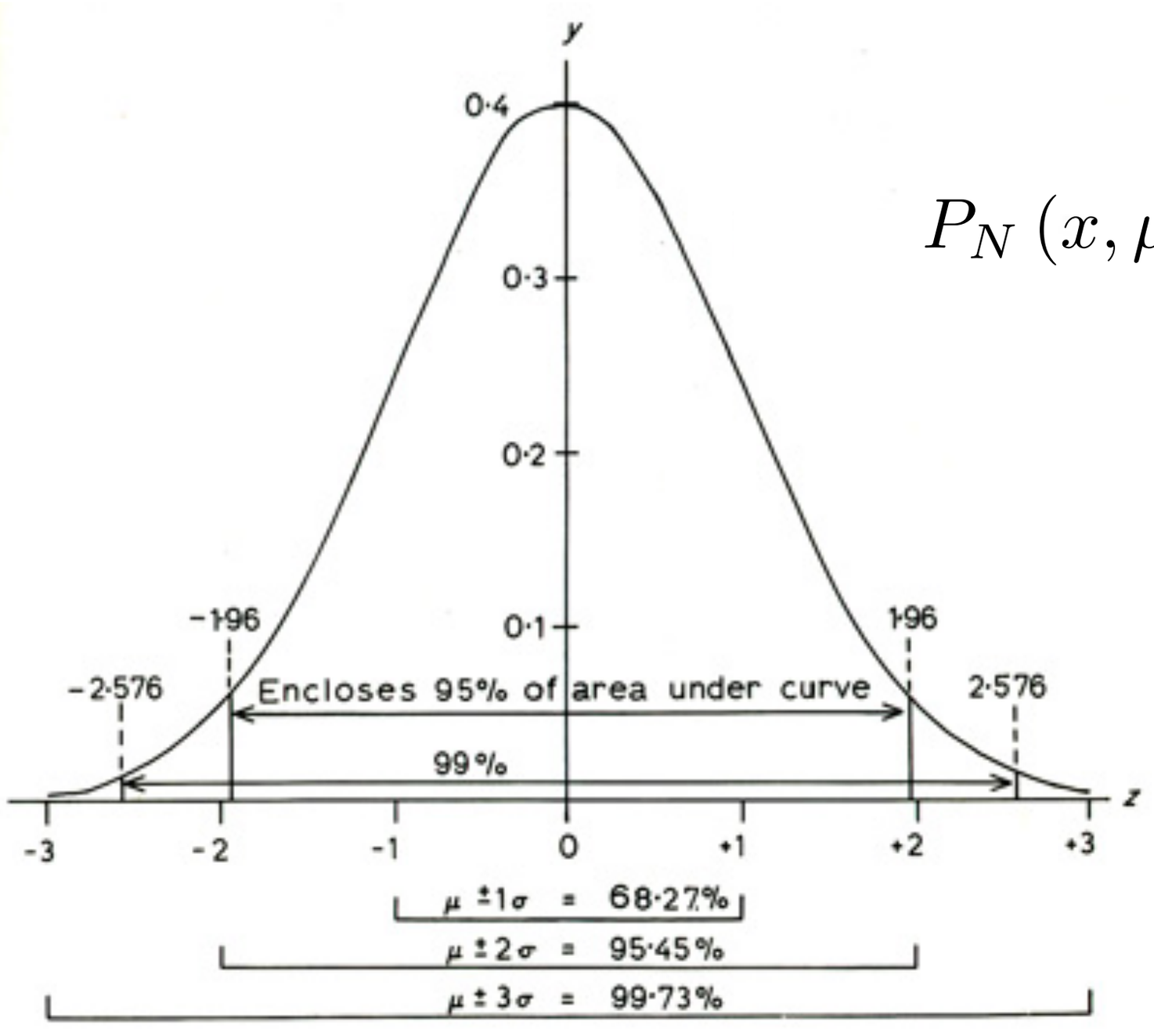
$$S_n \sim \mu + \frac{\mathcal{N}\left(0, \sigma^2\right)}{\sqrt{n}} \to SE = \frac{\sigma}{\sqrt{n}}$$

The estimation of the mean converges to the true mean with the square root of the number of samples

Central Limit Theorem



Gaussian Distribution



$$P_N\left(x,\mu,\sigma\right)=\frac{1}{\sqrt{2}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Experimental Measurements

- Experimental errors commonly assumed gaussian distributed

- Many experimental measurements are actually averages:

    - Instruments have a finite response time and the quantity of interest varies quickly over time

    - Stochastic Environmental factors

    - Etc

MLE - Fitting a theoretical function to experimental data

- In an experimental measurement, we expect (CLT) the experimental values to be normally distributed around the theoretical value with a certain variance. Mathematically, this means:

$$y - f\left(x\right) \approx \frac{1}{\sqrt{2\sigma^2}} \exp\left[-\frac{\left(y - f\left(x\right)\right)^2}{2\sigma^2}\right]$$

- where y are the experimental values and $f\left(x\right)$ the theoretical ones. The likelihood is then:

$$\mathcal{L} = -\frac{N}{2}\log\left[2\sigma^2\right] - \sum_i \left[\frac{\left(y_i - f\left(x_i\right)\right)^2}{2\sigma^2}\right]$$

- Where we see that to maximize the likelihood we must minimize the sum of squares

Least Squares Fitting

MLE - Linear Regression

- Let's say we want to fit a straight line to a set of points:

$$y = w \cdot x + b$$

- The Likelihood function then becomes:

$$\mathcal{L} = -\frac{N}{2}\log\left[2\sigma^2\right] - \sum_i \left[\frac{\left(y_i - w \cdot x_i - b\right)^2}{2\sigma^2}\right]$$

- With partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial w} = \sum_i \left[2x_i\left(y_i - w \cdot x_i - b\right)\right]$$

$$\frac{\partial \mathcal{L}}{\partial b} = \sum_i \left[\left(y_i - w \cdot x_i - b\right)\right]$$

- Setting to zero and solving for $\hat{w}$ and $\hat{b}$:

$$\hat{w} = \frac{\sum_i \left(x_i - \langle x \rangle\right)\left(y_i - \langle y \rangle\right)}{\sum_i \left(x_i - \langle x \rangle\right)^2}$$

$$\hat{b} = \langle y \rangle - \hat{w}\langle x \rangle$$

MLE for Linear Regression

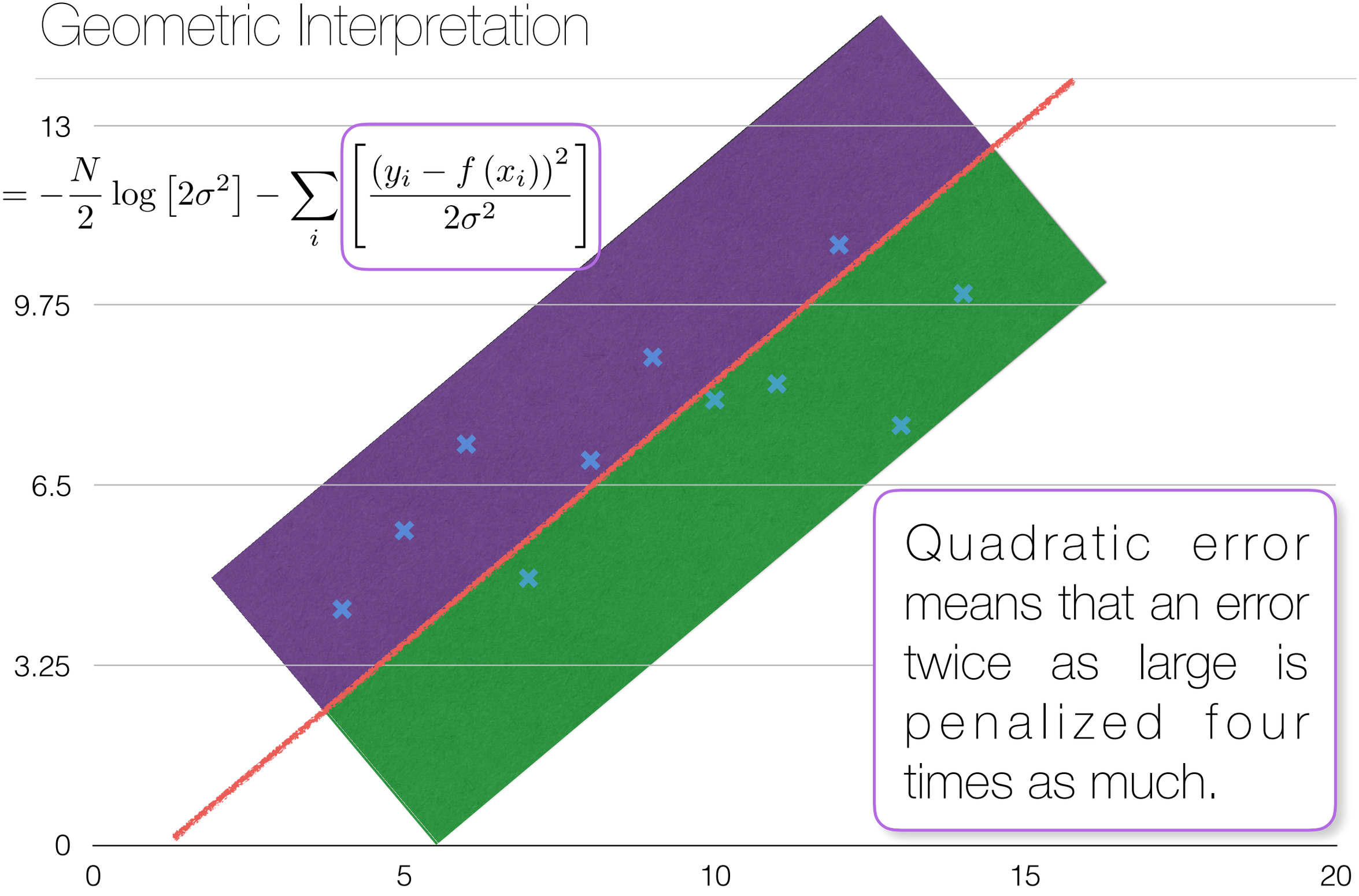
```python
from __future__ import print_function
import sys
import numpy as np
from scipy import optimize

data = np.loadtxt(sys.argv[1])

x = data.T[0]
y = data.T[1]

meanx = np.mean(x)
meany = np.mean(y)

w = np.sum((x-meanx)*(y-meany))/np.sum((x-meanx)**2)
b = meany-w*meanx

print(w, b)

#We can also optimize the Likelihood expression directly
def likelihood(w):
    global x, y
    sigma = 1.0
    w, b = w

    return np.sum((y-w*x-b)**2)/(2*sigma)

w, b = optimize.fmin_bfgs(likelihood, [1.0, 1.0])

print(w, b)
```

MLElinear.py

@bgoncalves

Geometric Interpretation

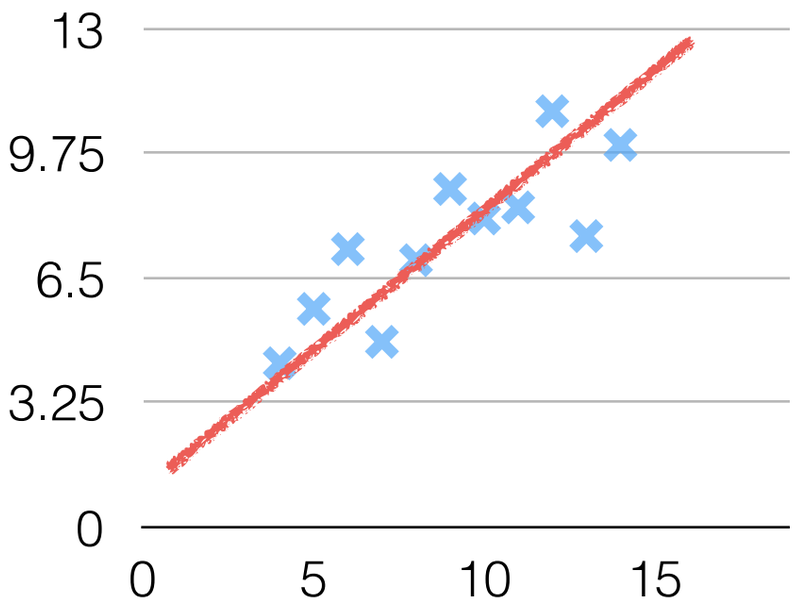
$$\mathcal{L} = -\frac{N}{2}\log\left[2\sigma^2\right] - \sum_i \left[\frac{\left(y_i - f\left(x_i\right)\right)^2}{2\sigma^2}\right]$$



Quadratic error means that an error twice as large is penalized four times as much.

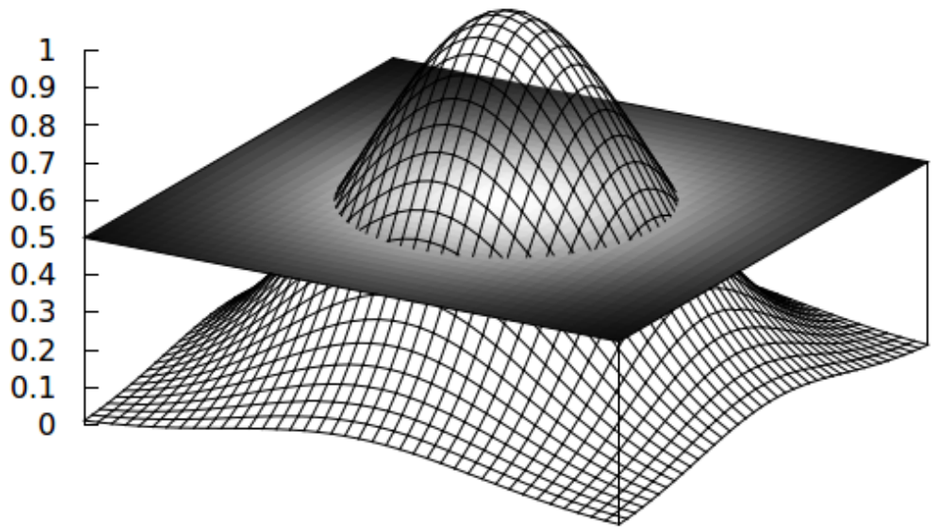
Geometric Interpretation

2D



$$y = \omega_0 + \omega_1 x_1$$

3D

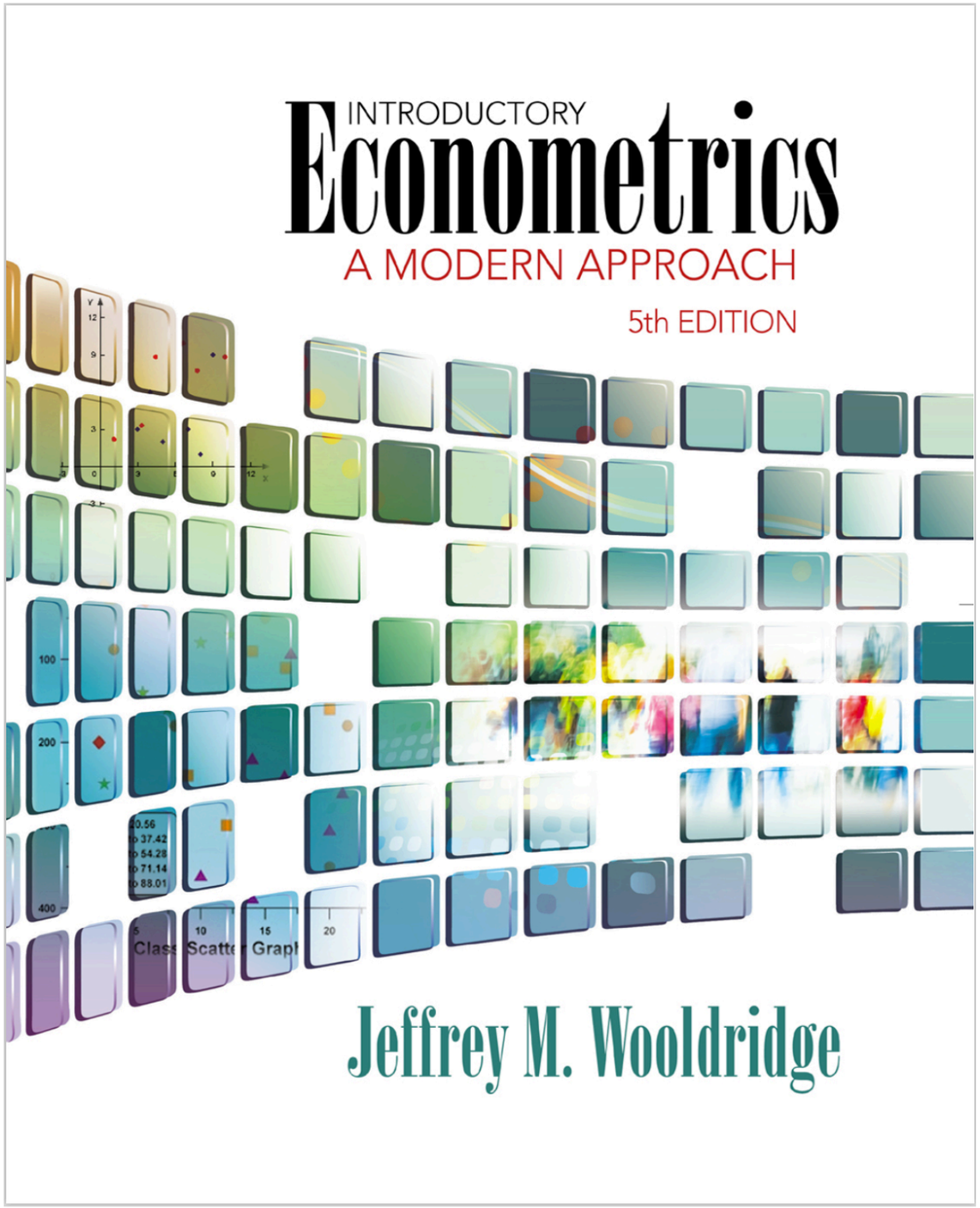


$$y = \omega_0 + \omega_1 x_1 + \omega_2 x_2$$

nD

$$y = \vec{\omega}^T \vec{x}$$

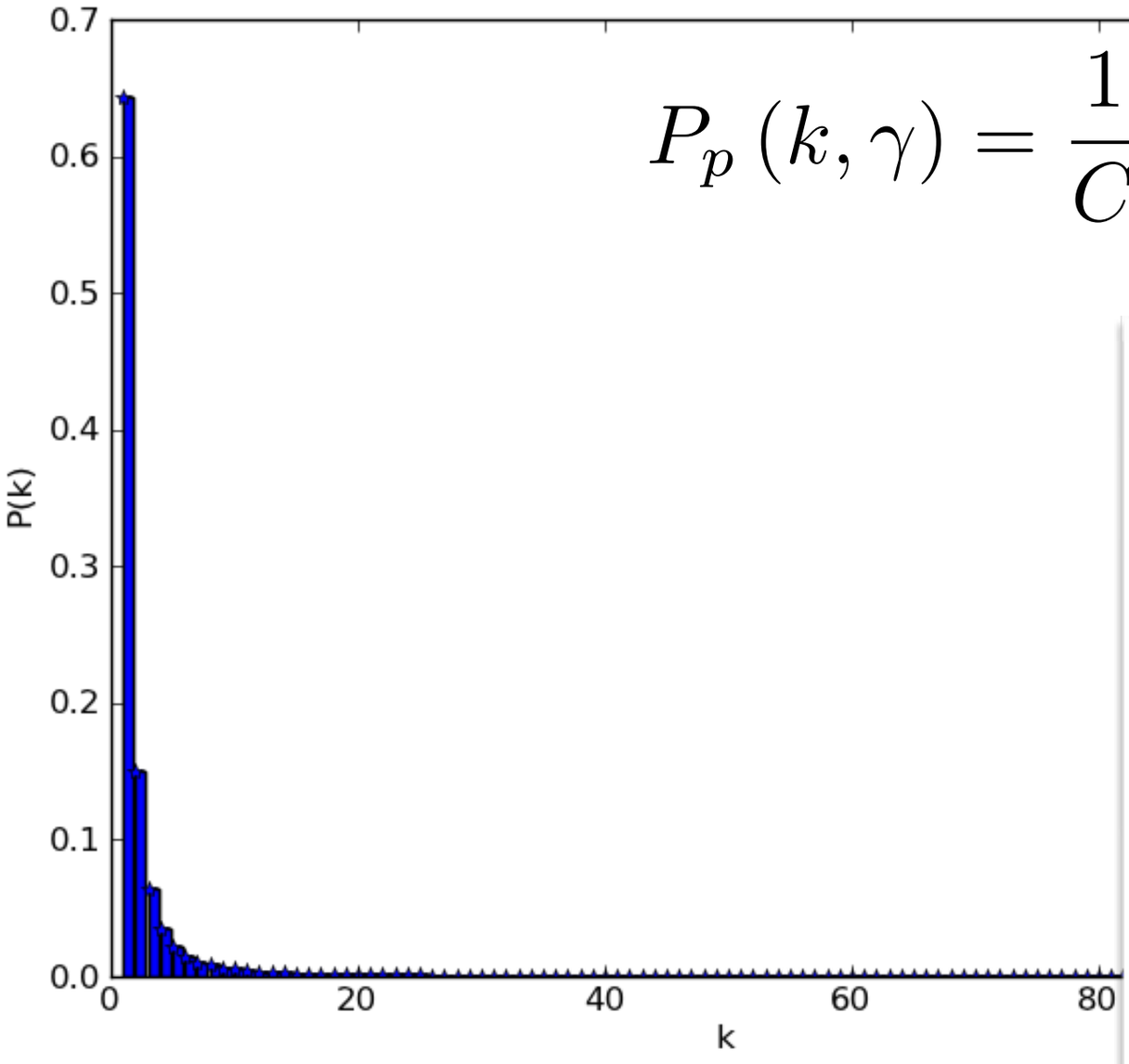
Multiple Regression...

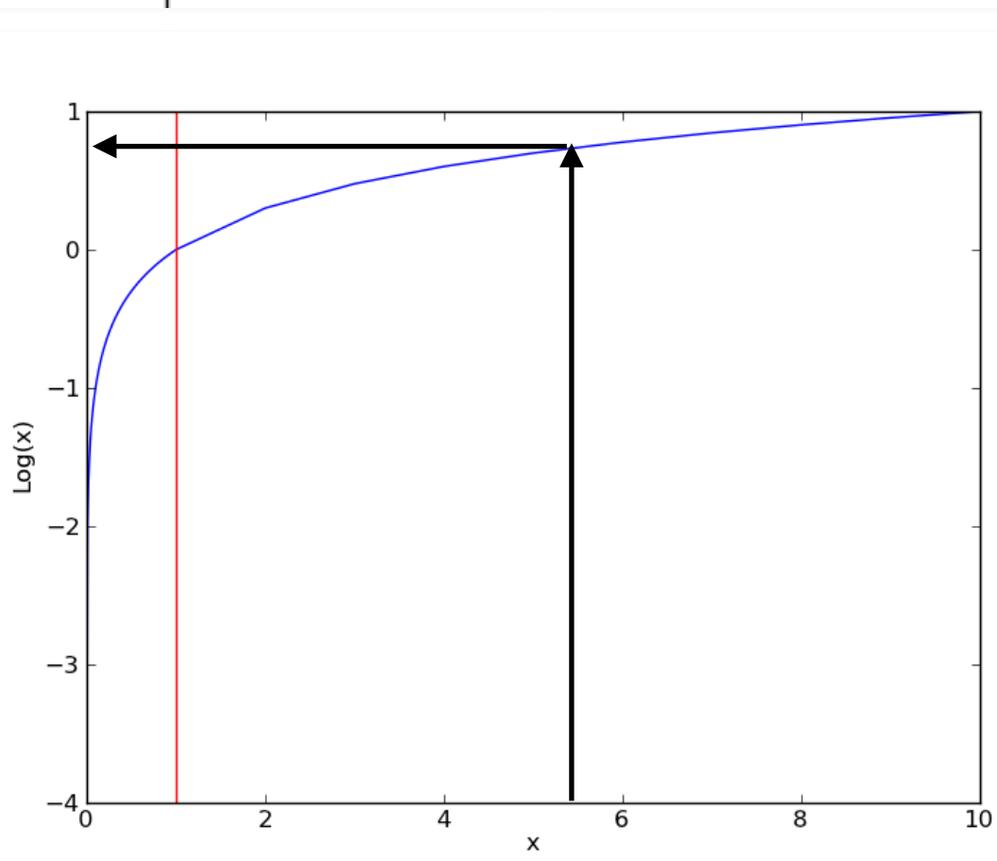


BRIEF CONTENTS

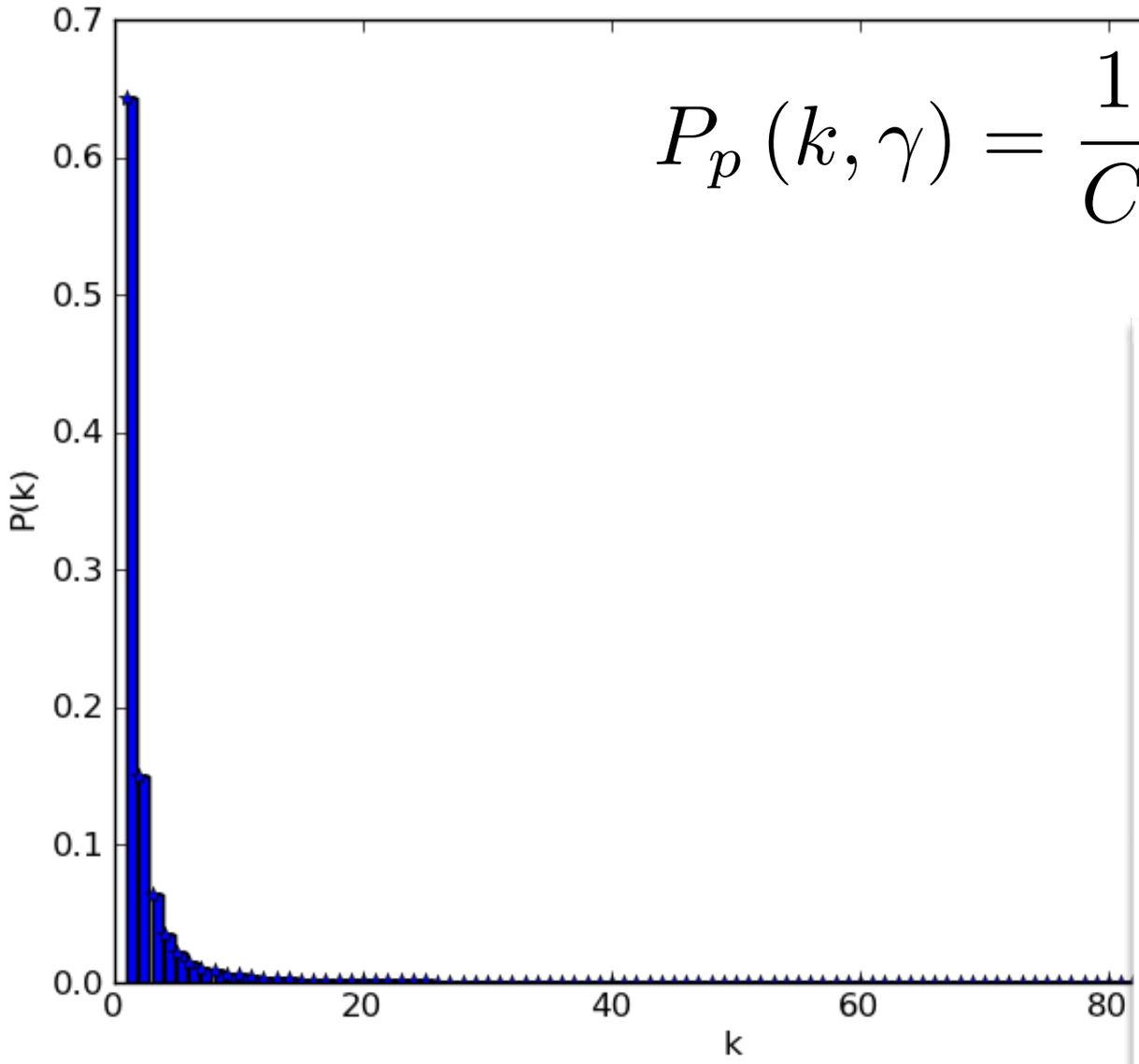
Broad-tailed distributions

$$P_p\left(k,\gamma\right) = \frac{1}{C}k^{-\gamma}$$

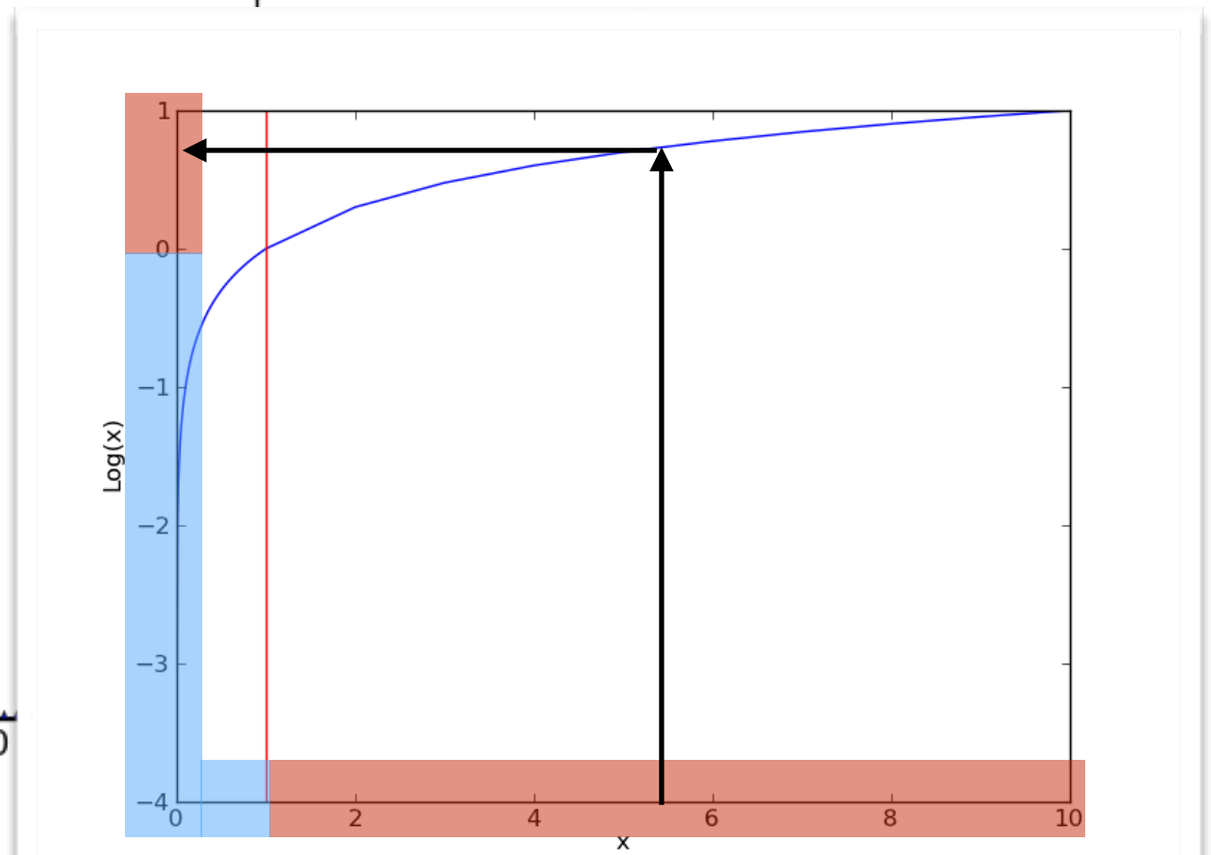




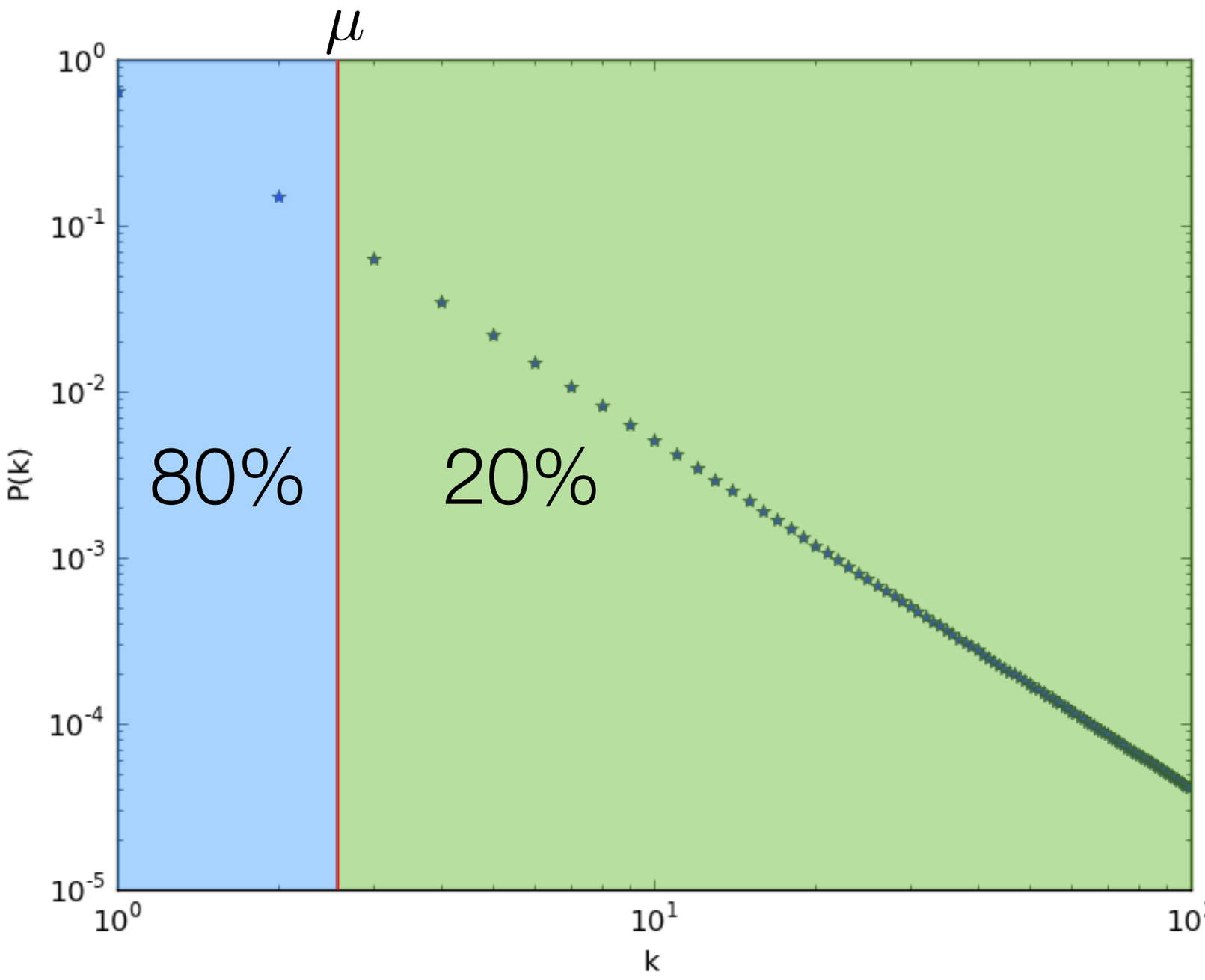
Broad-tailed distributions

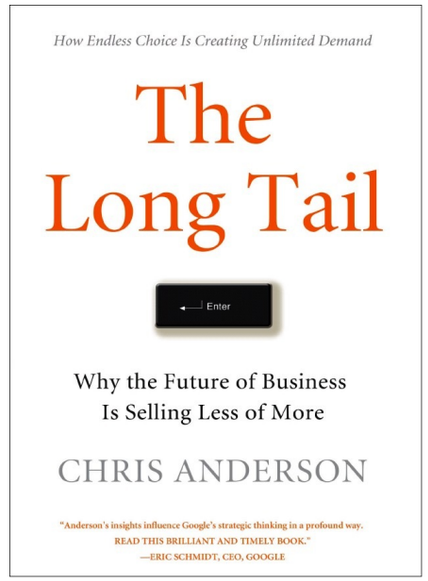


$$P_p\left(k,\gamma\right) = \frac{1}{C}k^{-\gamma}$$



(Almost) Everyone is below average!





MLE - Fitting a power-law to experimental data

SIAM Rev. 51, 661 (2009)

- We often find what look like power-law distributions in empirical data:

$$P(k) = \frac{\gamma - 1}{k_{min}} \left(\frac{k}{k_{min}} \right)^{-\gamma}$$

and we would like to find the right parameter values.

- The likelihood of any set of points is:

$$\mathcal{L} = \sum_i \log \left[\frac{\gamma - 1}{k_{min}} \left(\frac{k_i}{k_{min}} \right)^{-\gamma} \right]$$

- And maximizing, we find:

$$\gamma = 1 + n \left[\sum_i \log \left(\frac{k_i}{k_{min}} \right) \right]^{-1}$$

- with a standard error of:

$$SE = \frac{\gamma - 1}{\sqrt{n}}$$

MLE Coin Flips

- Biased coin with unknown probability of heads (p)
- In a sequence of N flips, the likelihood of N_h heads and N_t=N-N tails is:

$$\mathcal{L} = \log\left[p^{N_h}\left(1-p\right)^{N-N_h}\right]$$

Ignoring the combinatorial factor!

- or simply:

$$\mathcal{L} = N_h \log\left[p\right] + \left(N-N_h\right)\log\left[1-p\right]$$

- Taking the derivative:

$$\frac{\partial \mathcal{L}}{\partial p} = \frac{N_h}{p} - \frac{N-N_h}{1-p}$$

- Setting to zero and solving for p:

$$p = \frac{N_h}{N}$$

Binomial Distribution

- The probability of getting k successes with n trials of probability p (k heads in n coin flips):

$$P_B\left(k,n,p\right) = \frac{n!}{k!\left(n-k\right)!}p^k\left(1-p\right)^{n-k}$$

- The mean value is:

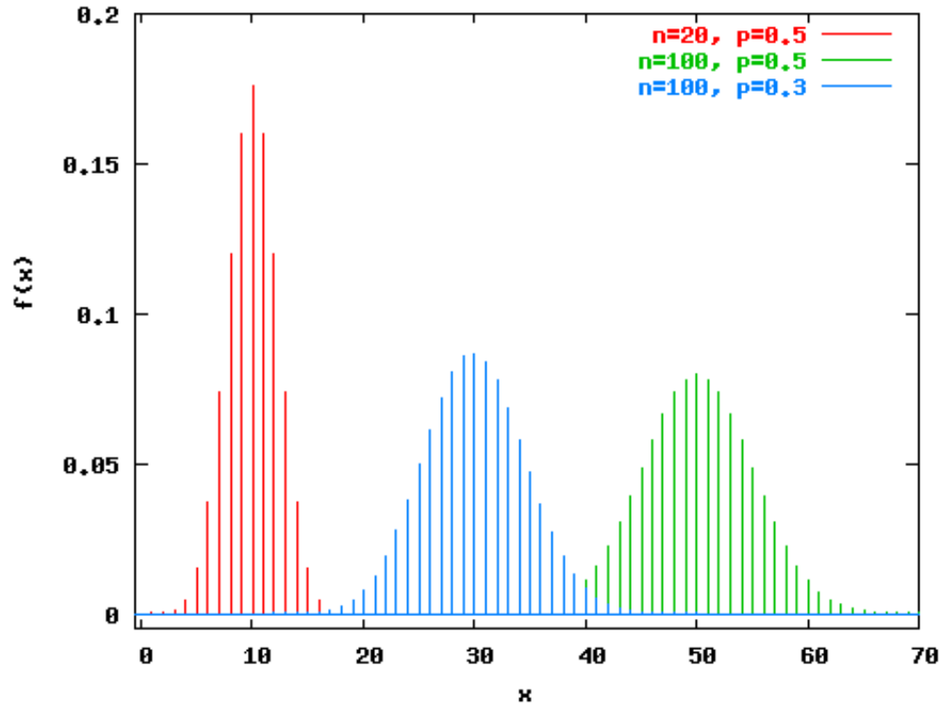
$$\mu = np$$

- and the variance:

$$\sigma = np\left(1-p\right)$$

- and for sufficiently large n:

$$P_B\left(k,n,p\right) \sim P_N\left(np, np\left(1-p\right)\right)$$



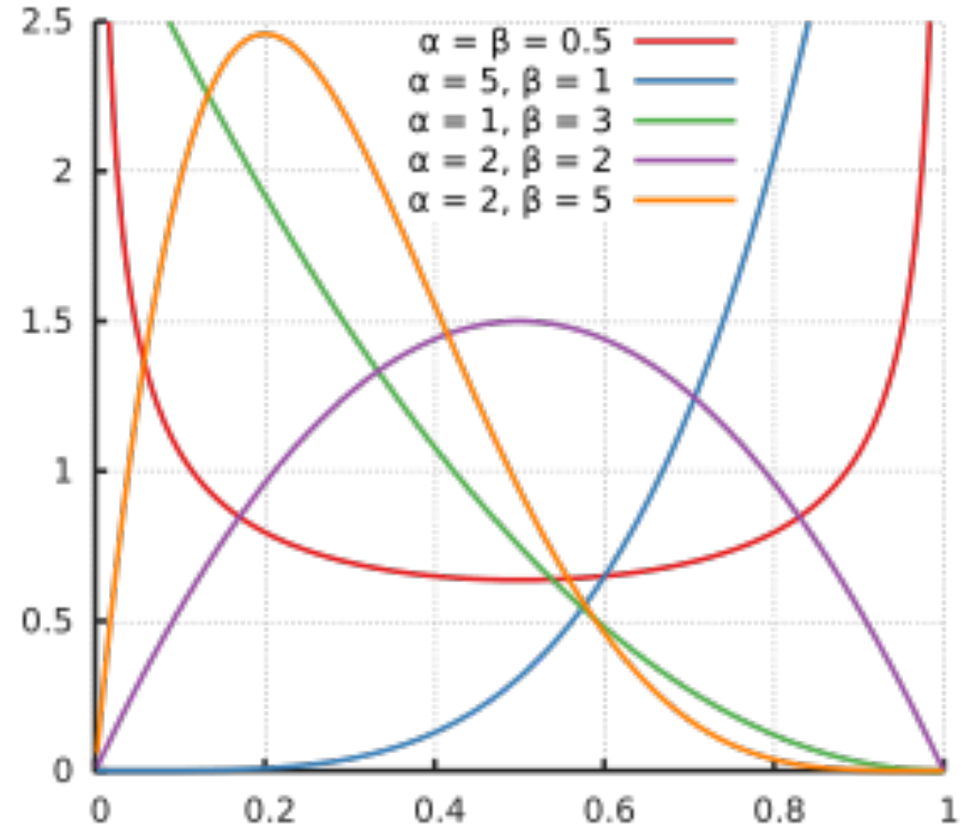
(Beta Distribution)

$$P_B\left(k,n,p\right)=\frac{n!}{k!\left(n-k\right)!}p^k\left(1-p\right)^{n-k}$$

- Related to Binomial and has a very similar form:

$$P_\beta\left(x,\alpha,\beta\right)=\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1}$$

- with $x\in[0,1]$ and $\alpha,\beta>0$.



- $\Gamma\left(a\right)$ is the continuous extension to $a!$

- The mean is:

$$\mu=\frac{\alpha}{\alpha+\beta}$$

- And the variance:

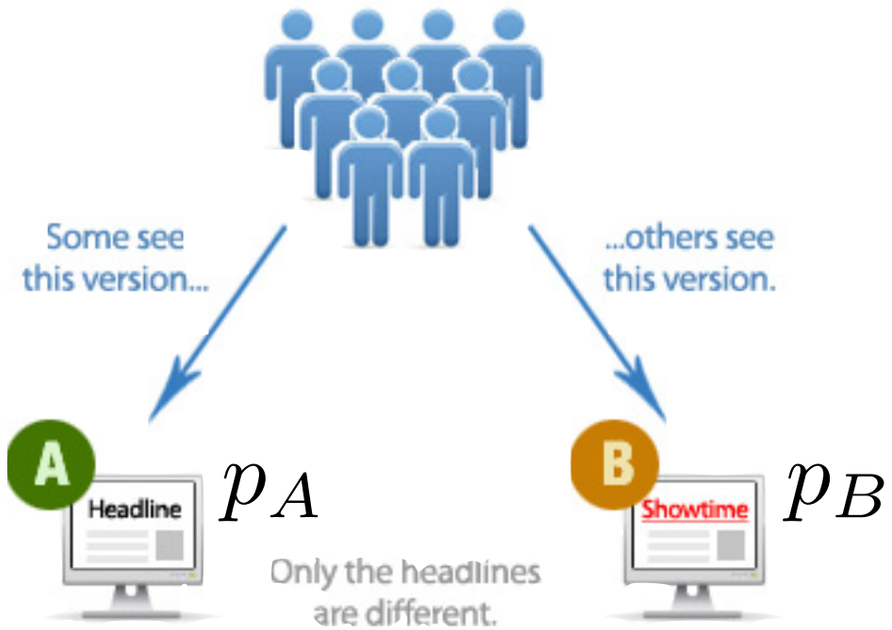
$$\sigma=\frac{\alpha\beta}{\left(\alpha+\beta\right)^2\left(\alpha+\beta+1\right)}$$

A/B Testing

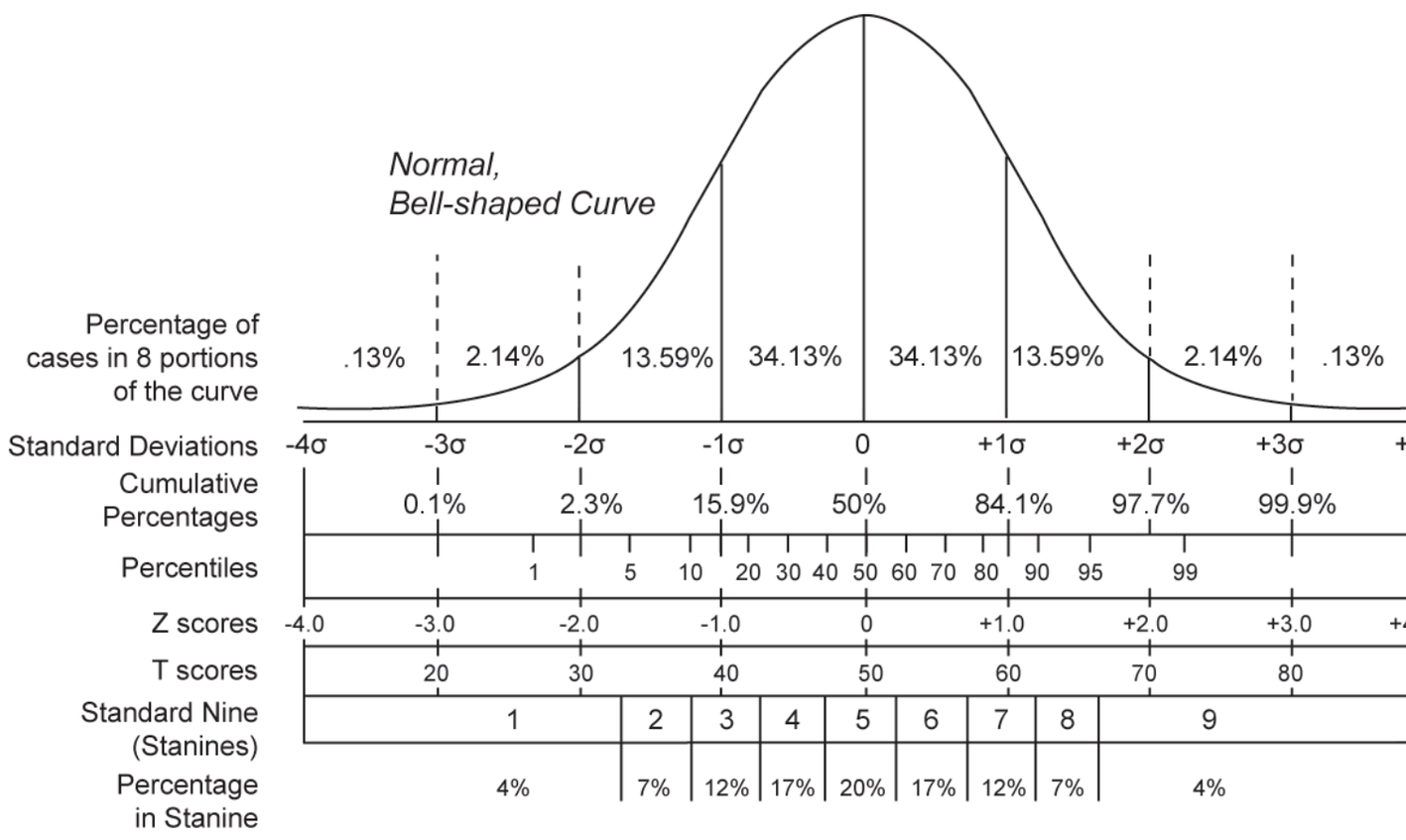
- Divide users into two groups A and B
- Measure some metric for each group (conversion probability, for example)

$$p_A \qquad p_B$$

A/B Testing



p_A p_B



- If conversion is a binomial process, then:
- Standard Error

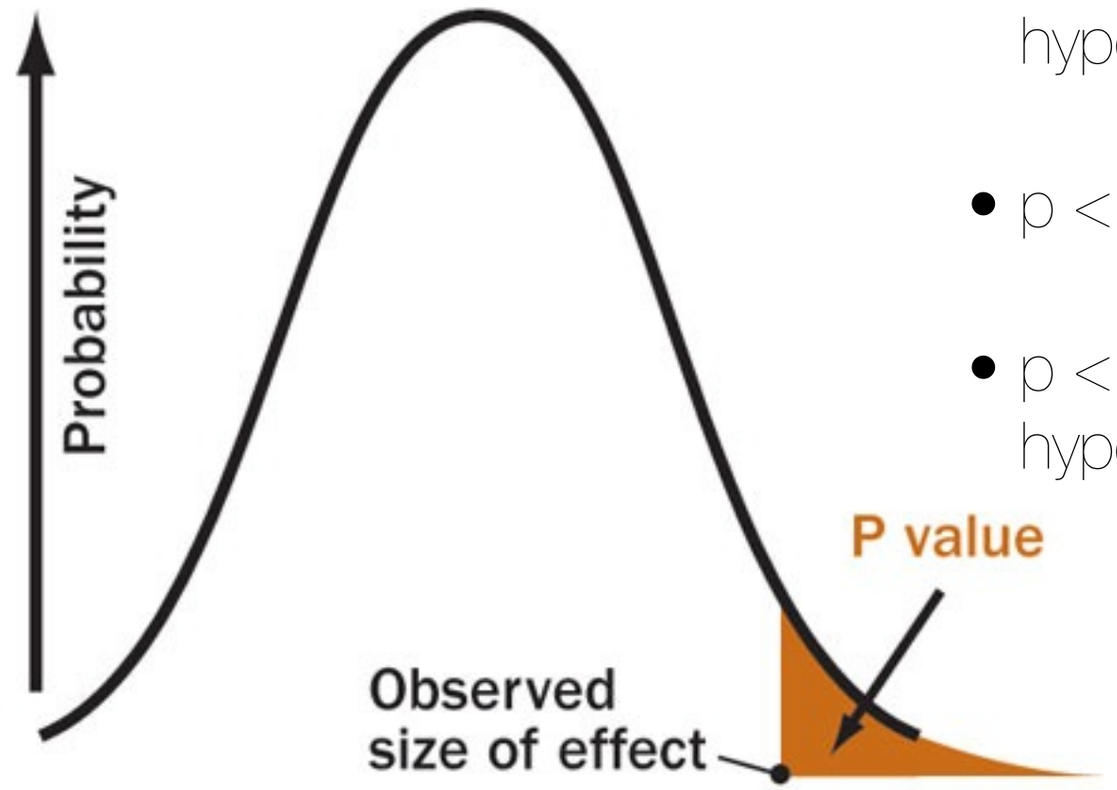
$$SE = \sqrt{\frac{p\left(1-p\right)}{N}}$$

- Z score

$$Z = \frac{p_A - p_B}{\sqrt{SE_A^2 + SE_B^2}}$$

p-value

- Calculate the probability of an event more extreme that the observation under the "null hypothesis"



- $p < 0.05$ Moderate evidence agains null-hypothesis
- $p < 0.01$ Strong evidence against null-hypothesis
- $p < 0.001$ Very strong evidence agains the null-hypothesis

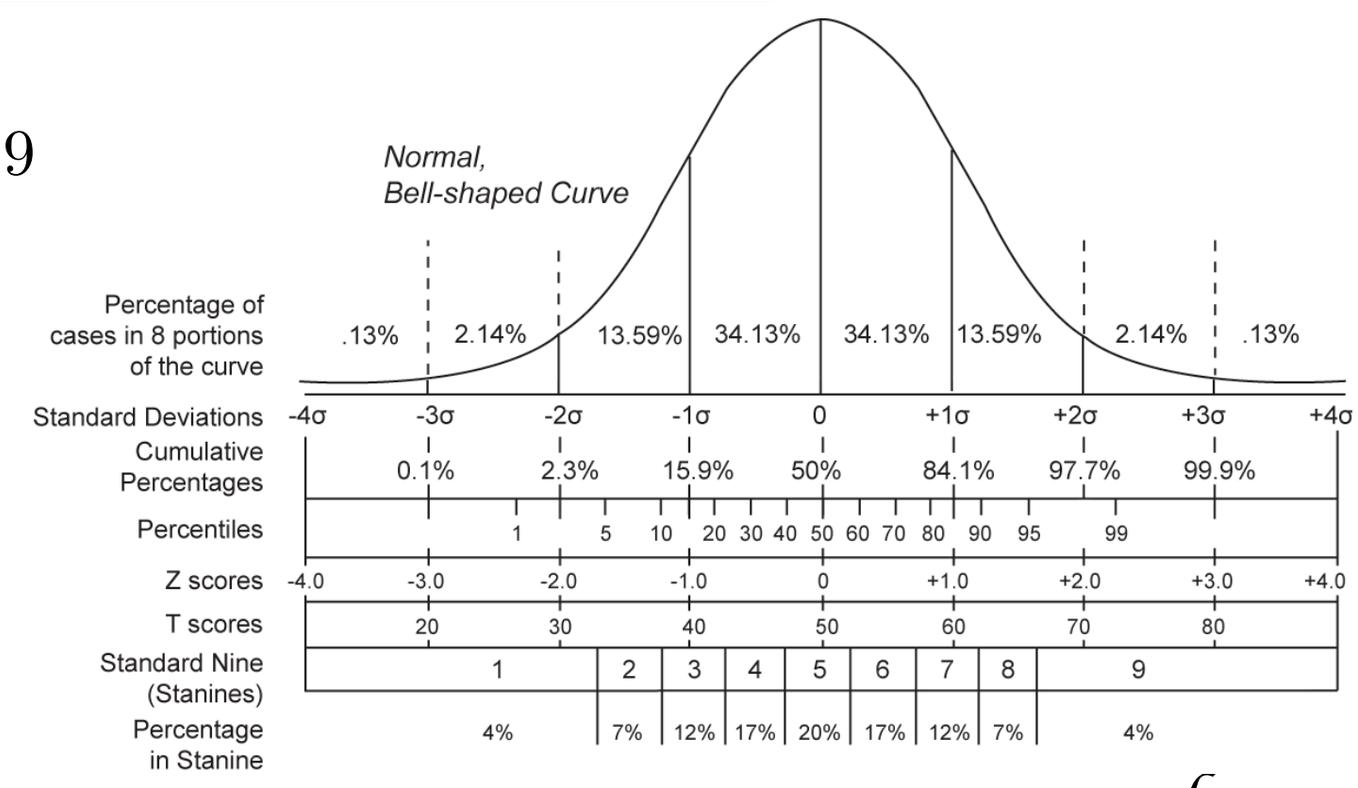
- The smaller the p-value the better.

Berkeley Discrimination Case Part I

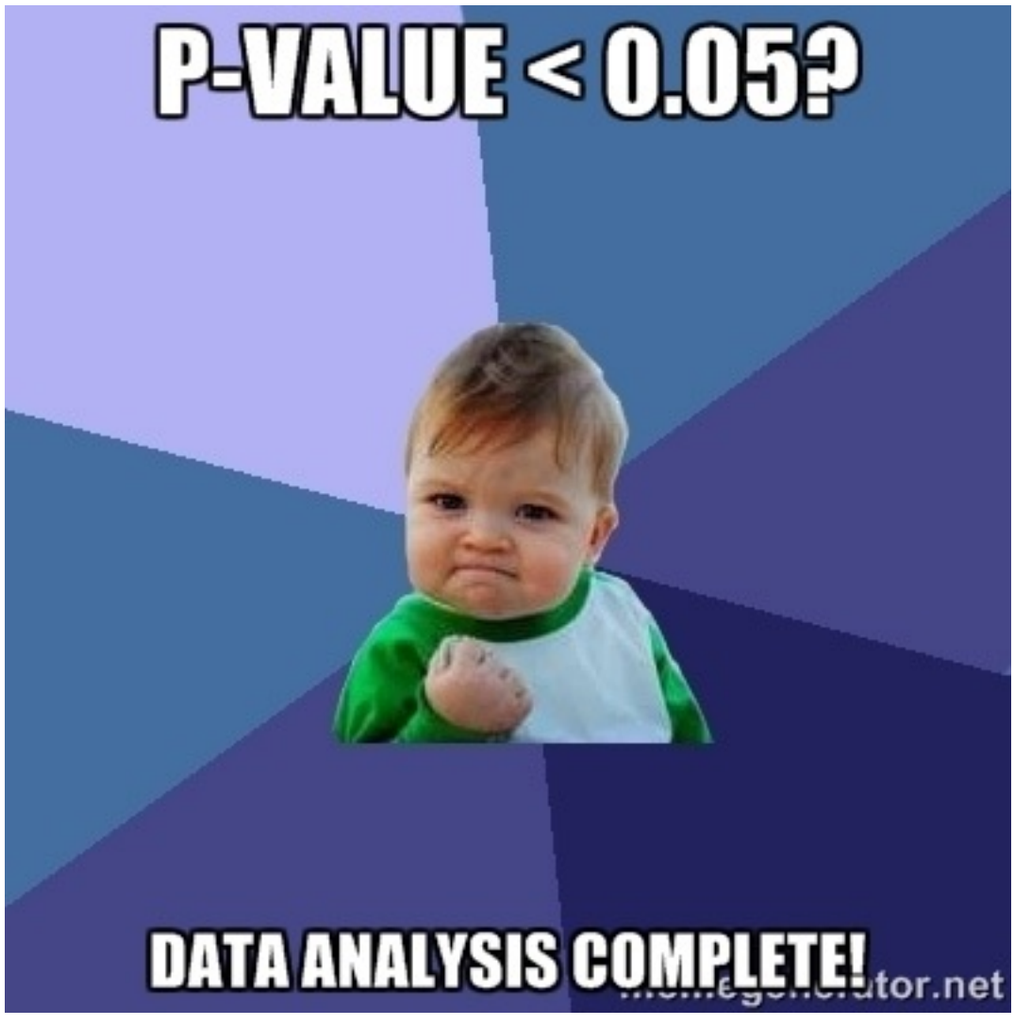
	Candidates	Acceptance Rate	SE
Men	8442	0.44	5.4×10^{-3}
Women	4321	0.35	7.2×10^{-3}

$$Z = \frac{p_A - p_B}{\sqrt{SE_A^2 + SE_B^2}} = 9.9$$

$$p \approx 10^{-23}$$

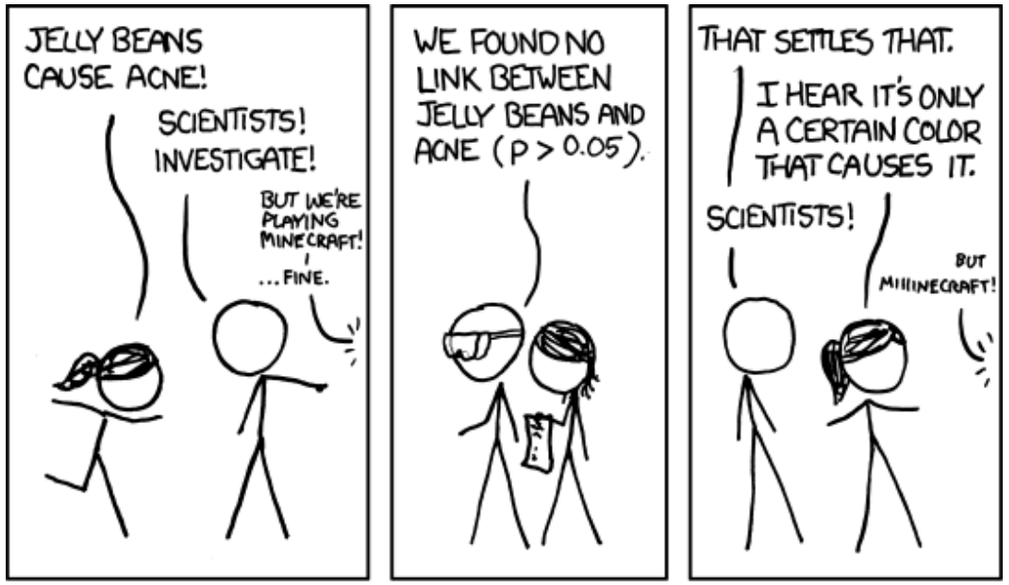


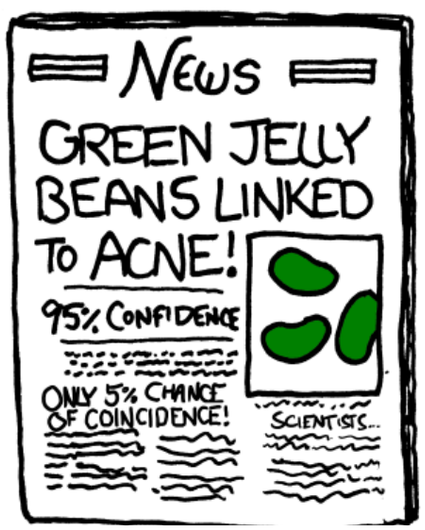
p-value

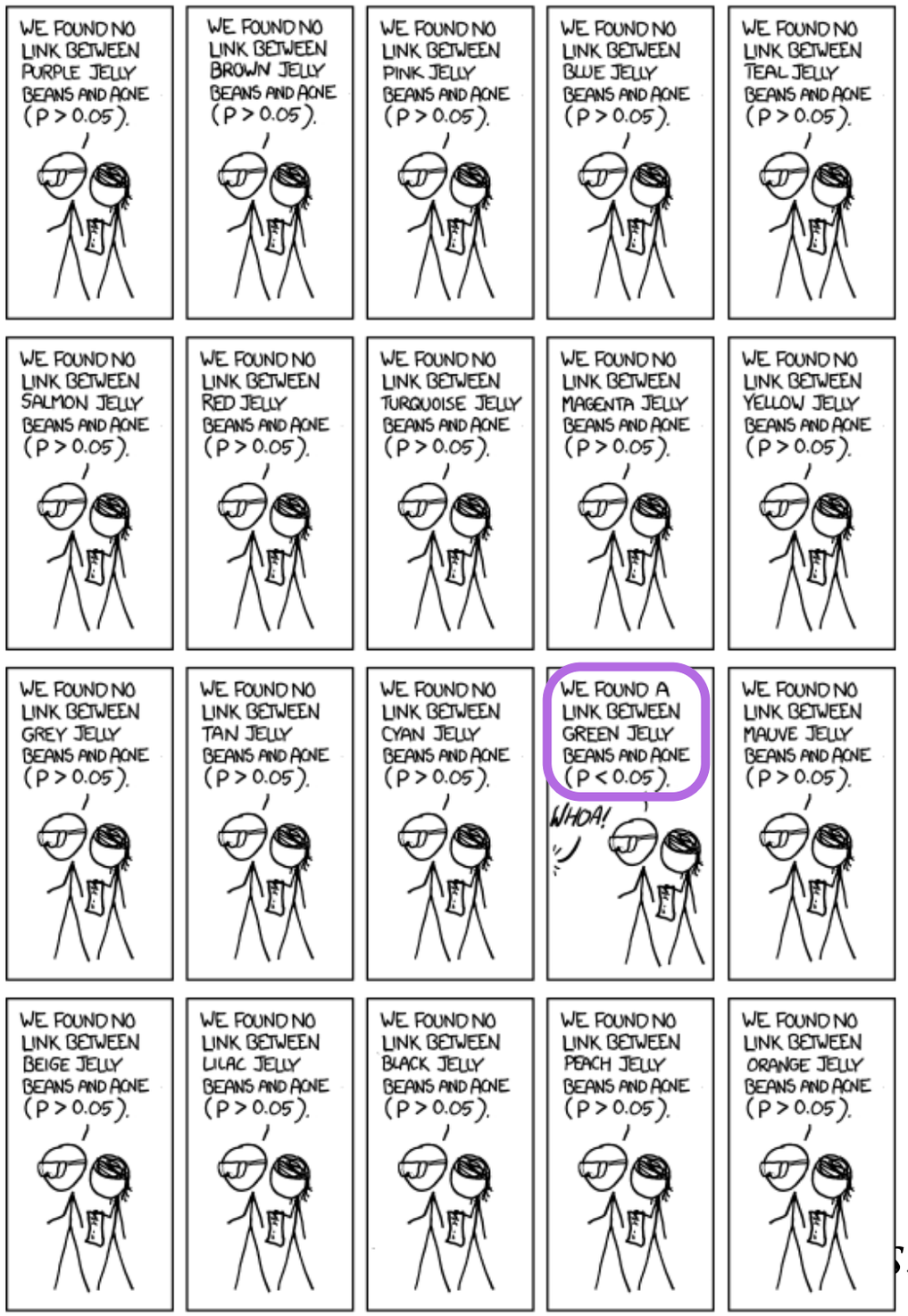


"Statistical significance does not imply scientific significance"

(Bonferoni Correction)







(Bonferoni Correction)

- You can think of p as the probability of observing a result as extreme by chance. With n comparisons, this probability becomes:

$$p_n = 1 - (1 - p)^n$$

  which quickly goes to 1 as n increases.

- However, by replacing p by p/n for each individual comparison, we obtain:

$$p_n = 1 - \left(1 - \frac{p}{n}\right)^n$$

- and for sufficiently large n:

$$p_n \approx 1 - e^{-p} \approx p$$

- allowing us to keep the probability of false results arbitrarily low even with arbitrarily large numbers of comparisons.

The Simpsons

@bgoncalves

Simpson's Paradox

Science 187, 398 (1975)

	Candidates	Acceptance Rate
Men	8442	0.44
Women	4321	0.35

Berkeley Discrimination Case Part II:
The statisticians strike back.

Simpson's Paradox

Science 187, 398 (1975)

	Candidates	Acceptance Rate
Men	8442	0.44
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"aggregated data can appear
to reverse important trends
in the numbers being combined"
WSJ, Dec 2, 2009

Dept	Men		Women	
	Candidates	Acceptance	Candidates	Acceptance
A	825	0.62	108	**0.82**
B	560	0.63	25	**0.68**
C	325	0.37	594	0.34
D	417	0.33	375	0.35
E	191	**0.28**	393	0.24
F	272	**0.06**	341	0.07
	2590	**0.46**	**1835**	**0.30**