Coordinate Geometry and Lines

1. **Distance Formula**
   The distance between point $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

2. The **slope** of non-vertical line that passes through points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is
   $k = \dfrac{y_2-y_1}{x_2-x_1}$. The slope of vertical line is not defined.

3. **Point-slope form of line equation**
   The equation of line passing through the point $P_1(x_1, y_1)$ and having slope k is $y-y_1 = k\cdot(x-x_1)$.

4. Slope-intercept form of a line equation
   The line equation with slope k and y-intercept b is $y=k\cdot x+b$

5. Alternate form of line equation
   $A\cdot x+B\cdot y+C=0$

6. Parallel and Perpendicular Lines

   a) Two nonvertical lines are **parallel** iff they have **same slope**.

   b) Two lines with slopes k_1 and k_2 are **perpendicular** iff $k_1\cdot k_2=-1 \Rightarrow k_2 = -\dfrac{1}{k_1}$

7. Alternative forms of line that passes through points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$k = \frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1} = \frac{y-y_2}{x-x_2} \Rightarrow \frac{x-x_1}{x-x_2} = \frac{y-y_1}{y-y_2}$$

In determinant form:

$$\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ y_2 & y_2 & 1 \end{bmatrix} = 0$$

8. **Line Equation in Segments**
   Given 2 points: A(0, b) and B(a, 0) liying at y-axis and x-axis correspondingly.
   $\dfrac{x}{a}+\dfrac{y}{b}=1$

Analysis of Line Equation

$A \cdot x + B \cdot y + C = 0$

If 1 or 2 coefficients are 0, this equation is incomplete. Here are possible cases:

1. C=0; $\Rightarrow$ $A \cdot x + B \cdot y = 0$ - line through origin (0,0)

2. B=0 (A≠0) => $A \cdot x + C = 0$ $<\equiv>$ $x = -\frac{C}{A} = a$; line ⊥ x-axis, can be written as x=a.
3. B=0, C=0 (A≠0) <≡> x=0 - y-axis itself

4. A=0 (B≠0) $\Rightarrow$ $B \cdot y + C = 0$ <≡> y= $-\frac{C}{B} = b$; line ⊥y-axis, can be writeen as y=b.

5. A=0, C=0 (B≠0) $\Rightarrow$ $B \cdot y = 0$ $<\equiv>$ y=0 - x-axis itself

6. If A≠0 and B≠0 and C≠0, line equation can be written in this form:

$\frac{x}{a} + \frac{y}{b} = 1$ This is line equation in segments.

where a= $-\frac{C}{A}$, $b = -\frac{C}{B}$

Analysis of Lines Intersection

If 2 lines have the following equations $A_1 \cdot x + B_1 \cdot y + C_1 = 0$ and $A_2 \cdot x + B_2 \cdot y + C_2 = 0$
4 cases are possible:

a. $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$ $\Rightarrow$ 2 lines have 1 common point

b. $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$ $\Rightarrow$lines are parallel (||)

c. $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$ $\Rightarrow$lines are the same, i.e. 2 equation define the same line.

d. $\frac{A_1}{B_2} = -\frac{B_1}{A_2}$ $<\equiv>$ $A_1 \cdot A_2 + B_1 \cdot B_2 = 0$ $\Rightarrow$ lines are perpendicular (⊥)

To see if 3 lines with given equations of $A \cdot x + B \cdot y + C = 0$, $A_1 \cdot x + B_1 \cdot y + C_1 = 0$ and $A_2 \cdot x + B_2 \cdot y + C_2 = 0$

intersect, check if d= $\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ y_2 & y_2 & 1 \end{bmatrix} = 0$. If d=0, these 3 lines intersect.

Alternative view at the same question is to solve the system below for x, y, and z:

$$\begin{cases} Ax+By+C=0 \\ A_1x+B_1y+C_1=0 \\ A_2x+B_2y+C_2=0 \end{cases}$$