# AE 4733 — Midterm Examination

**Name:** Bailey Waterman **Score:** ___ / 80

— Take-home exam. **Due:** No later than Saturday, 23-Sep-2017 at 6:00 PM EDT
— Submit one PDF document through Canvas. Do not submit code.
— If you wish to show efforts that did not result in correctly functioning code, include commented code in the PDF and describe your efforts at making the code work.
— **Collaboration in any form, including verbal discussion, with any other individual is strictly disallowed.** Students found in violation of this policy will forfeit all points for this assignment, and will be immediately reported for academic dishonesty to the AE Program Chair. Consult WPI's academic integrity guide.

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**Problem 1.** (10 points)

1. The power flux density of GPS signals available to users on the surface of Earth is approximately $2 \times 10^{-32}$ Watts/m$^2$.
2. The CTRS is Earth-fixed. ☒ True ☐ False
3. The tangent axes system is Earth-fixed. ☒ True ☐ False
4. The tangent axes system is inertial. ☐ True ☒ False
5. The CIRS is Earth-fixed. ☐ True ☒ False
6. The angular velocity of the tangent axes system relative to CTRS is $(\,0\,, \,0\,, \,0\,)$ rad/s.
7. The angular velocity of CTRS relative to CIRS, with coordinates in CTRS, is $(\,0\,, \,0\,, \,2\pi/86400\,)$ rad/s.
8. According to WGS 84, the length of the semi-*minor* axis of the global datum ellipsoid (Earth's approximate shape) is 6356.752 km.
9. An accelerometer on a gimbaled platform measures:
   ☒ Inertial acceleration in an inertial coordinate system
   ☐ Inertial acceleration in a body-fixed coordinate system
   ☐ Proper acceleration in an inertial coordinate system
   ☐ Proper acceleration in a body-fixed coordinate system
10. An accelerometer in free fall measures $(\,0\,, \,0\,, \,0\,)$ m/s$^2$.

**Problem 2.** The $1-2-3$ Euler angle sequence, consists of three successive planar rotations through angles $\psi$, $\theta$ and $\phi$ as follows:

$$\mathcal{I} \xrightarrow{\psi,\text{ about } x\text{-axis}} \mathcal{A} \xrightarrow{\theta,\text{ about } y\text{-axis}} \mathcal{B} \xrightarrow{\phi,\text{ about } z\text{-axis}} \mathcal{E}.$$

1. (3 points)

$$R_{\mathrm{i}}^{\mathrm{a}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & S\psi \\ 0 & -S\psi & C\psi \end{bmatrix}, \quad R_{\mathrm{a}}^{\mathrm{b}} = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix},$$

$$R_{\mathrm{b}}^{\mathrm{e}} = \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. (1 point)
$R_{\mathrm{i}}^{\mathrm{e}} =$ $\underline{R_b^e R_a^b R_i^a}$ (symbols only, no need to multiply matrices).

3. (2 points)
$\omega_{\mathrm{ie}}^{\mathrm{e}} =$ $\underline{R_b^e R_a^b}$ $\omega_{\mathrm{ia}}^{\mathrm{a}} +$ $\underline{R_b^e}$ $\omega_{\mathrm{ab}}^{\mathrm{b}} +$ $\underline{R_e^e}$ $\omega_{\mathrm{be}}^{\mathrm{e}}$.

4. (3 points)

$$\omega_{\mathrm{ia}}^{\mathrm{a}} = \begin{bmatrix} \dot\psi \\ 0 \\ 0 \end{bmatrix}, \quad \omega_{\mathrm{ab}}^{\mathrm{b}} = \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix}, \quad \omega_{\mathrm{be}}^{\mathrm{e}} = \begin{bmatrix} 0 \\ 0 \\ \dot\phi \end{bmatrix}.$$

5. (3 points)

$$\omega_{\mathrm{ie}}^{\mathrm{e}} = \begin{bmatrix} C\phi C\theta & S\phi & 0 \\ -C\theta S\phi & C\phi & 0 \\ S\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} d\psi/dt \\ d\theta/dt \\ d\phi/dt \end{bmatrix}.$$

6. (3 points)

$$\begin{bmatrix} d\psi/dt \\ d\theta/dt \\ d\phi/dt \end{bmatrix} = \frac{1}{C\theta(C\phi^2+S\phi^2)} \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi C\theta & C\phi C\theta & 0 \\ -C\phi S\theta & S\phi S\theta & 1 \end{bmatrix} \omega_{ie}^{e}.$$

7. (5 points)
At time $t = 0$, the three angles are $\psi = 5°$, $\theta = 10°$, and $\phi = 15°$. A 3-axis rate gyro in a strapdown inertial measurement unit measures the angular rates of axes $\mathcal{E}$ relative to $\mathcal{I}$ over the interval of $t = 0$ through $t = 20$ s as a *constant* $(0, 0.25, 0.75)$ deg/s.

- The notation for the quantity measured by the rate gyro is $\omega_{ie}^{e}$.
- The values of the three angles at time $t = 20$ s are:

$\psi(20) =$ 21.06 deg, $\theta(20) =$ 10.55 deg, $\phi(20) =$ 17.88 deg.

8. (5 points)
Meanwhile in a parallel universe, angular rates recorded by a 3-axis rate gyro in a strapdown inertial measurement unit over the interval of $t = 0$ through $t = 20$ s are time-varying, as provided in the attached .mat file.

- Plot the three angles $\psi, \theta, \phi$ over this time interval.
- The values of the three angles at time $t = 20$ s are: (3 points)

$\psi(20) =$ 26.33 deg, $\theta(20) =$ 50.88 deg, $\phi(20) =$ 17.57 deg.

**Problem 3.** (15 points)

The direction of the velocity vector of an aircraft is described by two successive rotations: a $z$-axis rotation through angle $\psi$, which is known as the *heading angle*, followed by a $y$-axis rotation through angle $\gamma$, which is known as the *angle of climb*. These are similar to the yaw and pitch rotations; different names are used because the aircraft velocity vector can be pointed in a direction different from the aircraft's orientation (e.g., crosswinds can cause sideslip).

At a specific instant of time $t_0$, an aircraft is recorded to have heading angle $\psi(t_0) = 45°$ and angle of climb $\gamma(t_0) = 2°$. At this very instant, the rates of change of these angles are also known:

$$\left.\frac{d\psi}{dt}\right|_{t=t_0} = 0 \text{ deg/s} \qquad \left.\frac{d\gamma}{dt}\right|_{t=t_0} = 1.5 \text{ deg/s}.$$

At this instant of time, the aircraft's orientation is (by sheer coincidence) perfectly aligned with its velocity vector, i.e., the velocity vector is parallel to the aircraft body-fixed $x$-axis. A pitot tube (also aligned with the aircraft body-fixed $x$-axis) measures airspeed as 75 m/s at this time instant.

What will be the reading of a strapdown (i.e., aircraft body-fixed) 3-axis accelerometer at this time instant? Ignore all sensor noise (including that of the pitot tube). Ignore wind.

Hint: This problem is similar to Problem 3 in HW 2, except this is in 3D.

Fun fact: In telescope pointing systems, this two-angle sequence of describing the telescope's direction is known as the *azimuth-elevation* system, sometimes also called the "alt-az" system.

$\underline{v}^e = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$, $\dot{\underline{v}}^e = \begin{bmatrix} \dot{V} \\ 0 \\ 0 \end{bmatrix}$, $\underline{\omega}_{ie}^e = \begin{bmatrix} 0 \\ \dot{\gamma} \\ \dot{\psi} \end{bmatrix}$

$\underbrace{R_i^e \dot{v}^i}_{\text{accelerometer data}} = \dot{v}^e + \underline{\omega}_{ie}^e \times \underline{v}^e$

$\underline{\omega}_{ie}^e \times \underline{v}^e = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \dot{\gamma} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ V\dot{\psi} \\ V\dot{\gamma} \end{bmatrix}$

$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \dot{V} \\ V\dot{\psi} \\ V\dot{\gamma} \end{bmatrix}$

$V(t_0) = 75 \text{ m/s} \rightarrow \dot{V}(t_0) = 0$

$\psi(t_0) = 45°$

$\gamma(t_0) = 2°$

$\dot{\psi}(t_0) = 0$

$\dot{\gamma}(t_0) = 1.5 \text{ °/s} = 0.02618 \text{ rad/s}$

$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ 75(0) \\ 75(0.02618) \end{bmatrix}$

$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.9635 \end{bmatrix} \text{m/s}^2$

**Problem 4.** (20 points)

In a time interval of $t = 0$ through $t = 60$ s, an aircraft's orientation remains (again, by sheer coincidence) perfectly aligned with its velocity vector. In this time interval, a strapdown 3-axis accelerometer measures the aircraft's acceleration. After correcting for gravity, these measurements are provided in the attached .mat file, i.e. the measurements provided are of inertial acceleration, not proper acceleration.

At time $t = 0$, the position coordinates of the aircraft's center of mass (which is where the accelerometers are attached) in a tangent axis system (North-East-Down (NED) system) $\mathcal{T}$ are:

$$\mathbf{p}^t(0) = \begin{bmatrix} 1 \\ -2.5 \\ -1 \end{bmatrix} \text{ km.}$$

For this problem, the NED system is assumed to be inertial. At time $t = 0$, the airspeed is 75 m/s. The heading angle and the angle of climb are:

$$\psi(0) = 45°, \qquad \gamma(0) = 2°.$$

$\dot{V}^i$

For the given time interval, plot the following: the airspeed, the heading angle, the angle of climb, and the three position coordinates. Write down the position coordinates of the aircraft's c.m. at time $t = 60$ s.

Plot the two angles in degrees, not radians. Plot the position coordinates as a 3D trajectory using, say, the `plot3` command in MATLAB®.

<u>Hint:</u> This problem is also similar to Problem 3 in HW 2, except this is in 3D.

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \dot{V} \\ V\dot{\psi} \\ V\dot{\gamma} \end{bmatrix}$$

$$\dot{V}^i = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{V} \\ V\dot{\psi} \\ V\dot{\gamma} \end{bmatrix}$$

$$\dot{V}^i = \begin{bmatrix} \dot{V}\cos\psi - V\dot{\psi}\sin\psi \\ \dot{V}\sin\psi + V\dot{\psi}\cos\psi \\ V\dot{\gamma} \end{bmatrix}$$

**Problem 5.** (10 points)

Consider a thin disc. A coordinate axes system $\mathcal{I}$ is fixed in space. Initially, the thin disc has an orientation such that the $\hat{\mathbf{k}}_\mathrm{i}$ basis vector (i.e., $z$-axis of the $\mathcal{I}$ system) is perpendicular to the plane of the disc, and its origin is at the center of the disc. Now consider the following *two different pairs* of successive planar rotations of the disc:

$$P^i = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathcal{I} \xrightarrow{\psi,\ x} \mathcal{A} \xrightarrow{\theta,\ y} \mathcal{B},$$

$$\mathcal{I} \xrightarrow{\alpha,\ z} \mathcal{C} \xrightarrow{\beta,\ x} \mathcal{D}.$$

Find out the relationships between the pair $(\psi, \theta)$ and the pair $(\alpha, \beta)$ such that the basis vectors $\hat{\mathbf{k}}_\mathrm{b}$ and $\hat{\mathbf{k}}_\mathrm{d}$ are parallel, i.e., the plane of the disc has the same orientation in space at the end of each pair of rotations.

$$R_i^a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & S\psi \\ 0 & -S\psi & C\psi \end{bmatrix} \qquad R_a^b = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix}$$

$$R_i^b = R_a^b R_i^a = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & S\psi \\ 0 & -S\psi & C\psi \end{bmatrix} = \begin{bmatrix} C\theta & 0 & 0 \\ 0 & C\psi & 0 \\ 0 & 0 & C\psi C\theta \end{bmatrix}$$

$$R_i^c = \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_c^d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\beta & S\beta \\ 0 & -S\beta & C\beta \end{bmatrix}$$

$$R_i^d = R_c^d R_i^c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\beta & S\beta \\ 0 & -S\beta & C\beta \end{bmatrix} \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\alpha & 0 & 0 \\ 0 & C\alpha C\beta & 0 \\ 0 & 0 & C\beta \end{bmatrix}$$

$$P^b = \begin{bmatrix} C\theta & 0 & 0 \\ 0 & C\psi & 0 \\ 0 & 0 & C\psi C\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C\theta x \\ C\psi y \\ C\psi C\theta z \end{bmatrix} \qquad P^d = \begin{bmatrix} C\alpha & 0 & 0 \\ 0 & C\alpha C\beta & 0 \\ 0 & 0 & C\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C\alpha x \\ C\alpha C\beta y \\ C\beta z \end{bmatrix}$$

$$C\theta = C\alpha \longrightarrow \theta = \alpha \qquad (\psi, 0),\ (0, \beta)$$

$$\left.\begin{aligned} C\psi &= C\alpha C\beta \\ C\psi C\theta &= C\beta \end{aligned}\right\} \begin{aligned} &\alpha = 0,\ \theta = 0 \\ &\psi = \beta \end{aligned}$$

## Contents



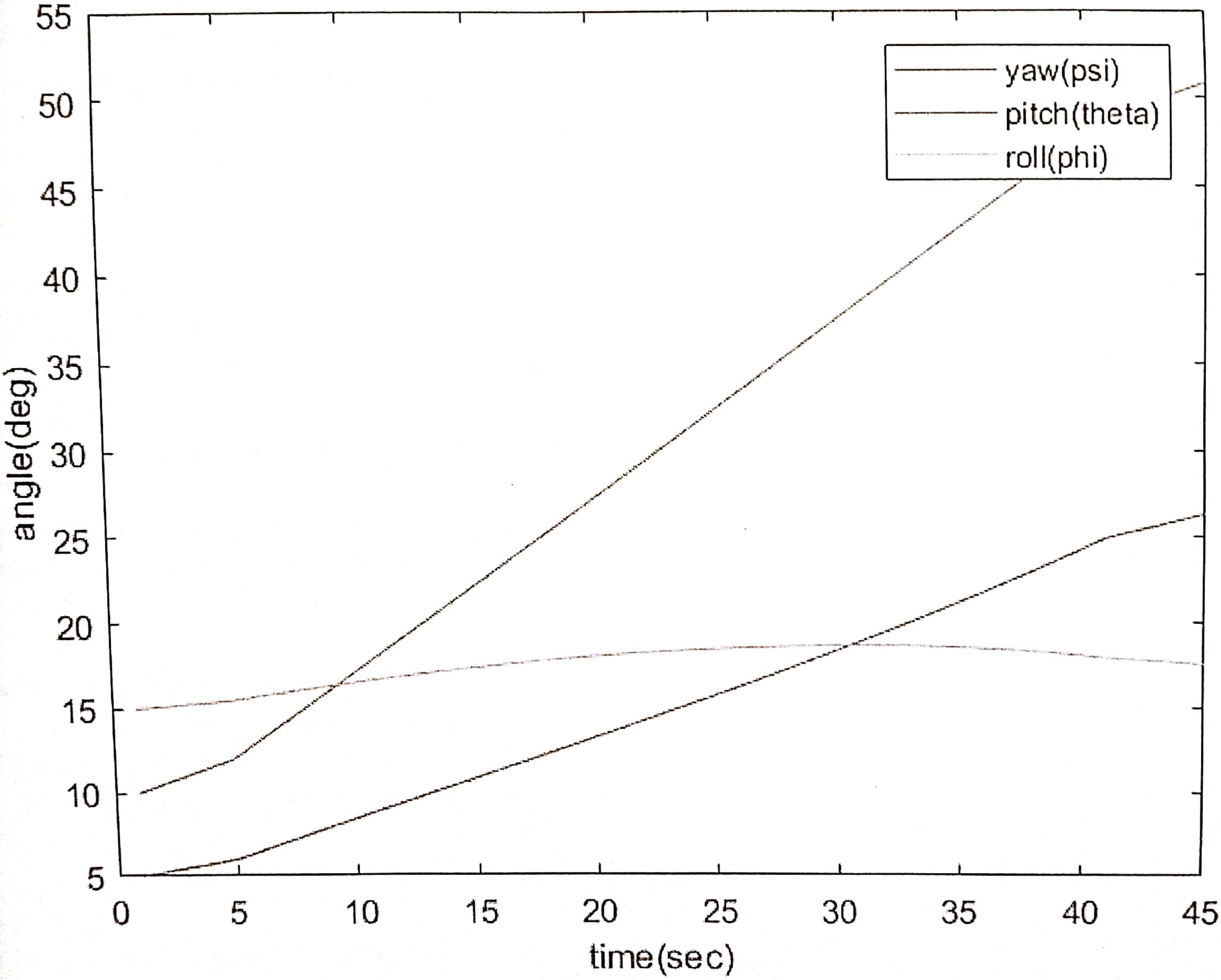
```
clear all;
close all;
clc;
```

## Problem 2 Question 7

```
%initial conditions
euler321_angles_initial = [5; 10; 15] * pi/180; %radians
time_span = [0 20]; %seconds
omega_iee_1 = [0; 0.25; 0.75;] *pi/180; %radians/second

%solve ODE
[t_sim, euler321_angles_sim] = ...
    ode45(@(t, y)midterm_kinematics(t, y, omega_iee_1), time_span, euler321_angles_initial);

euler321_angles_degrees = euler321_angles_sim * 180/pi;
```

## Problem 2 Question 8

```
load('gyrodata_p2_8_2017.mat')

%initial conditions
euler321_angles_initial = [5; 10; 15] * pi/180; %radians
time_span = [0 20]; %seconds
omega_iee = [omega_iee(1);omega_iee(2);omega_iee(3);];; %radians/second

%solve ODE
[t_sim, euler321_angles_sim] = ...
ode45(@(t, y)midterm_kinematics(t, y, omega_iee), time_span, euler321_angles_initial);

euler321_angles_degrees = euler321_angles_sim * 180/pi;

plot(euler321_angles_degrees)

xlabel('time(sec)')
ylabel ('angle(deg)')
legend ('yaw(psi)','pitch(theta)','roll(phi)')
```




**Problem 4**

```
clear variables; close all; clc;
load('accel_readings.mat');
load('time_pts.mat');
```

```
ax =  accel_readings(1);
ay =  accel_readings(2);
az =  accel_readings(3);



v_dot_i = @(t) [midterm_v_dot(t)*cos(midterm_psii(t)) - midterm_v(t)*midterm_psii_dot(t)*sin(
midterm_psii(t));
            midterm_v_dot(t)*sin(midterm_psii(t)) + midterm_psii_dot(t)*midterm_v(t)*cos(midt
erm_psii(t));
            midterm_v(t)*midterm_gamma_dot(t)];

% in odefun and initial conditon y0,
% y(1:3, :) = p_i (position in m)
% y(4:6, :) = v_i (velocity in m/s)

odefun = @(t,y) [y(4:6, :); v_dot_i(t)];
y0 = [1000 -2500 -1000 75 0 0]';
warning('off','all')
[t_sim, y_sim] = ode45(odefun, [0 60], y0);
warning('on','all')

figure();
plot(t_sim, y_sim(:, 1:3));
xlabel('time(s)')
ylabel('position(m)')
legend({'$\hat{i}$', '$\hat{j}$', '$\hat{k}$'},'Interpreter','latex')
figure();
plot(t_sim, y_sim(:, 4:6));
xlabel('time(s)')
ylabel('velocity(m/s)')
legend({'$\hat{i}$', '$\hat{j}$', '$\hat{k}$'},'Interpreter','latex')
% at t=40 in I frame,
position = y_sim(end, 1:3)' % m
velocity = y_sim(end, 4:6)' % m/s

%Tried running script but kept getting error "Subscript indices must be
%positive integers or logicals" which I can't seem to resolve
```