24.22 $y' - 5y = e^{5x}$ $y(0) = 2$

$sY - 2 - 5Y = \frac{1}{s-5}$

$Y(s-5) = \frac{1}{s-5} + 2$

$Y = \frac{1}{(s-5)^2} + \frac{2}{s-5}$

$y = xe^{5x} + 2e^{5x}$

24.23 $y' + y = xe^{-x}$ $y(0) = -2$

$sY + 2 + Y = \frac{1}{(s+1)^2}$

$Y(s+1) = \frac{1}{(s+1)^2} - 2$

$Y = \frac{1 \cdot 2}{(s+1)^3 \cdot 2} - \frac{2}{s+1}$

$\frac{2}{s^3} \to x^2$

$y = \frac{1}{2}x^2e^{-x} - 2e^{-x}$

24.24 $y' + y = \sin x$ $y(0)$ not given.

$sY - y(0) + Y = \frac{1}{s^2+1}$

$(s+1)Y = y(0) + \frac{1}{s^2+1}$

$Y = \frac{y(0)}{s+1} + \frac{1}{(s+1)(s^2+1)}$

$y(0)e^{-x}$

par frac. decomp

$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$

$1 = A(s^2+1) + (Bs+C)(s+1)$

$0s^2 + 0s + 1 = s^2(A+B) + s(B+C) + (A+C)$

so $0 = A+B \rightarrow B = -A$ so $A = C$

$0 = B+C \rightarrow B = -C$

$1 = A+C$

$A = \frac{1}{2} = C$

$B = -\frac{1}{2}$

$Y = \frac{y(0)}{s+1} + \frac{1/2}{s+1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1}$

$y = y(0)e^{-x} + \frac{1}{2}e^{-x} - \frac{1}{2}\cos(x) + \frac{1}{2}\sin x$

24.25

$y' + 20y = 6\sin 2x \quad y(0) = 6$

$sY - 6 + 20Y = \dfrac{6\cdot 2}{s^2+4} \nearrow Y = \dfrac{6}{s+20} + \dfrac{12}{(s+20)(s^2+4)}$

$\dfrac{12}{(s+20)(s^2+4)} = \dfrac{A}{s+20} + \dfrac{Bs+C}{s^2+4}$

$12 = A(s^2+4) + (Bs+C)(s+20)$

$12 = s^2(A+B) + s(20B+C) + (4A+20C)$

$0s^2 + 0s \quad 12 = s^2(A+B) + s(20B+C) + (4A+20C)$

$0 = A+B$

$0 = 20B + C$

$\cancel{12 = 4A + 20C}$

$3 = A + 5C$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 20 & 1 & 0 \\ 1 & 0 & 5 & 3 \end{bmatrix} R_3 - R_1$ $\quad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 20 & 1 & 0 \\ 0 & -1 & 5 & 3 \end{matrix} \quad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 5 & 3 \end{matrix}$

more simple, $A = -B$.

$20B + C = 0$

$-B + 5C = 3$

$\left[\begin{array}{cc|c} 20 & 1 & 0 \\ -1 & 5 & 3 \end{array}\right]$

$A = 3/101$

$B = \dfrac{\begin{vmatrix} 0 & 1 \\ 3 & 5 \end{vmatrix}}{101} = \dfrac{-3}{101}$

$C = \dfrac{\begin{matrix} 20 & 0 \\ -1 & 3 \end{matrix}}{101} = \dfrac{60}{101}$

$Y = \dfrac{6}{s+20} + \dfrac{3/101}{s+20} + \dfrac{-3/101\, s}{s^2+4} + \dfrac{\overset{30}{\cancel{60}}/101 \cdot 2}{(s^2+4)\cdot \cancel{2}}$

$y = 6e^{-20x} + \dfrac{3}{101}e^{-20x} - \dfrac{3}{101}\cos(2x) + \dfrac{30}{101}\sin(2x)$

24.26 $\quad y'' - y = 0 \quad y(0) = 1 \quad y'(0) = 1$

$s^2Y - s(1) - (1) \quad -Y = 0$

$Y(s^2-1) = s+1$

$\rightarrow Y = \dfrac{s+1}{s^2-1} = \dfrac{\cancel{s+1}}{\cancel{(s+1)}(s-1)}$

$= \dfrac{1}{s-1}$

$y = e^x$

24.28

$y'' - y = e^x \quad y(0) = 1 \quad y'(0) = 0$

$$\underbrace{s^2Y - s(1) - 0} - \underbrace{Y} = \frac{1}{s-1}$$

$$(s^2-1)(Y) = s + \frac{1}{s-1} = \frac{s^2 - s + 1}{s-1}$$

$(s-1)(s+1)$

$$Y = \frac{s^2-s+1}{(s-1)^2(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1}$$

So $\boxed{y(x) = Ae^x + Bxe^x + Ce^{-x}}$

I just need A, B, C.

$$s^2 - s + 1 = A(s-1)(s+1) + B(s+1) + C(s-1)^2$$

$$\underbrace{1}s^2 \underbrace{-}s + \underbrace{1} = A(s^2-1) + B(s+1) + C(s^2 - 2s + 1)$$

$1 = A + C$ (collect s^2 coefficients)

$-1 = B - 2C$ (collect s coefficients)

$1 = -A + B + C$

alternate.

$s = 1 \Rightarrow 1 = 2B$

$s = -1 \Rightarrow 3 = 4C$

$s = 0 \Rightarrow 1 = -A + B + C$

So $A = B + C - 1$

$= \frac{1}{2} + \frac{3}{4} - 1$

$= \frac{1}{4}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ -1 & 1 & 1 & 1 \end{array}\right] R_1 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & 2 & 2 \end{array}\right] R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 4 & 3 \end{array}\right] \begin{array}{l} 4R_1 - R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 4 & 3 \end{array}\right]$$

$A = 1/4$

$B = 1/2$

$C = 3/4$

So $\boxed{y(x) = \frac{1}{4}e^x + \frac{1}{2}xe^x + \frac{3}{4}e^{-x}}$

24.29 $y'' + 2y' - 3y = \sin 2x \quad y(0) = y'(0) = 0$

$$s^2 Y - s(0) - 0 + 2(sY - 0) - 3Y = \frac{2}{s^2+4}$$

$$\underbrace{Y(s^2+2s-3)}_{(s+3)(s-1)} = \frac{2}{s^2+4} \Rightarrow Y = \frac{2}{(s-1)(s+3)(s^2+4)}$$

$$\frac{2}{(s-1)(s+3)(s^2+4)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4}$$

$y = Ae^{x} + Be^{-3x} + C\cos 2x + \frac{D}{2}\sin 2x$

where

$$2 = A(s+3)(s^2+4) + B(s-1)(s^2+4) + (Cs+D)(s-1)(s+3)$$

let $s = +1$. Then $2 = A(4)(5) + 0 + 0$ means $2 = 20A$, $A = \frac{1}{10}$

let $s = -3$ Then $2 = 0 + B(-4)(13) + 0$ mean $2 = B(-4)(13)$ $B = \frac{-1}{26}$

To find C and D will not be as simple. I need to solve a system of equations. (I would not expect you to do this much by hand).

$$2 = A(s^3+3s^2+4s+12) + B(s^3-s^2+4s-4) + (Cs+D)(s^2+2s-3)$$

$$0s^3+0s^2+0s+2 = s^3(A+B+C) + s^2(3A-B+2C+D) + s(4A+4B-3C+2D) + (12A-4B-3D)$$

So we need to solve

$$0 = A+B+C$$
$$0 = 3A-B+2C+D$$
$$0 = 4A+4B-3C+2D$$
$$2 = 12A-4B-3D$$

or rref $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 1 & 0 \\ 4 & 4 & -3 & 2 & 0 \\ 12 & -4 & 0 & -3 & 2 \end{bmatrix}$

Since I already know $A = \frac{1}{10}$ and $B = \frac{-1}{26}$,

then row 1 says $C = -A - B = -\frac{1}{10} + \frac{1}{26} = \frac{-26+10}{260} = \frac{-16}{260}$

$= \frac{-8}{130}$

$C = \frac{-4}{65}$

Then $D = -3A + B - 2C = \frac{-3}{10} - \frac{1}{26} + \frac{32}{260} = \frac{-78-10+32}{260}$

$= \frac{-56}{260} = \frac{-14}{65} = D$

Plug in above

24.31 $y''+y'+y=0 \quad y(0)=4 \quad y'(0)=-3$

$$(s^2Y-4s+3)+(sY-4)+Y=0$$

$$Y(s^2+s+1)=4s-3+4=4s+1$$

$$Y=\frac{4s+1}{s^2+s+1}=\frac{4s+1}{s^2+s+\frac{1}{4}-\frac{1}{4}+1}=\frac{4s+1}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

$$=\frac{4(s+\frac{1}{2}-\frac{1}{2})+1}{(s+\frac{1}{2})^2+3/4}=\frac{4(s+\frac{1}{2})-2+1}{(s+\frac{1}{2})^2+3/4}$$

$$=\frac{4(s+\frac{1}{2})}{(s+\frac{1}{2})^2+3/4}+\frac{-1}{(s+\frac{1}{2})^2+3/4}\cdot\frac{\sqrt{3}/2}{\sqrt{3}/2}$$

$$\frac{4u}{u^2+3/4}$$

$$\downarrow$$

$$4\cos\left(\frac{\sqrt{3}}{2}x\right)e^{-\frac{1}{2}x}-\frac{1}{\sqrt{3}/2}\sin\left(\frac{\sqrt{3}}{2}x\right)e^{-\frac{1}{2}x}$$

24.33

$$y'' + 5y' - 3y = u(x-4) \qquad y(0)=0 \quad y'(0)=0$$

$$(s^2Y - 0 - 0) + 5(sY - 0) - 3Y = \frac{1}{s}e^{-4s}$$

$$Y = \frac{1}{s(s^2+5s-3)}e^{-4s} \qquad \frac{1}{s(s^2+5s-3)} = \frac{A}{s} + \frac{Bs+C}{s^2+5s-3}$$

$$1 = A(s^2+5s-3) + (Bs+C)s$$

$$0s^2 + 0s + 1 = s^2(A+B) + s(5A+C) + (-3A)$$

$$0 = A + B \qquad B = 1/3$$

$$0 = 5A + C \qquad C = 5/3$$

$$1 = -3A \qquad A = -1/3$$

$$Y = \left(\frac{-1/3}{s} + \frac{1/3\, s + 5/3}{s^2+5s-3}\right)e^{-4s} \qquad s^2 + 5s + \frac{25}{4} - \frac{25}{4} - 3 = (s+5/2)^2 - 37/4$$

$$= \left(\frac{-1/3}{s} + \frac{1/3(s + 5/2 - 5/2) + 5/3}{(s+\frac{5}{2})^2 + 37/4}\right)e^{-4s} \qquad -5/6 + 5/3 = 5/6$$

$$= \left(\frac{-1/3}{s} + \frac{1/3(s+5/2)}{(s+\frac{5}{2})^2 + 37/4} + \frac{5/6}{(s+\frac{5}{2})^2 + 37/4}\cdot\frac{\cdot\frac{\sqrt{37}}{2}}{\cdot\frac{\sqrt{37}}{2}}\right)e^{-4s}$$

without replacing t

$$-1/3 + \frac{1}{3}\cosh\left(\frac{\sqrt{37}}{2}t\right)e^{-\frac{5}{2}t} + \frac{5}{\cancel{6}}\cdot\frac{\cancel{2}}{\sqrt{37}}\sinh\left(\frac{\sqrt{37}}{2}t\right)e^{-\frac{5}{2}t}$$

$$\boxed{y = \left(-\frac{1}{3} + \frac{1}{3}\cosh\left(\frac{\sqrt{37}}{2}(t-4)\right)e^{-\frac{5}{2}(t-4)} + \frac{5}{3\sqrt{37}}\sinh\left(\frac{\sqrt{37}}{2}(t-4)\right)e^{-\frac{5}{2}(t-4)}\right)u(t-4)}$$