1) Solve

$$y' = ay + f$$

$$\underbrace{(-ay - f)}_{M}\,dt + \underbrace{1}_{N}\,dy = 0$$

$$M_y = -a \qquad N_t = 0$$

$$\frac{M_y - N_t}{N} = -a \qquad e^{\int -a\,dt} = e^{-at}$$ Integrating factor.

multiply by e^{-at}

$$(-ae^{-at}y - fe^{-at})\,dt + e^{-at}\,dy = 0$$

$$e^{-at}y - \int e^{-at} f(t)\,dt = c$$

$$e^{-at}y = c + \int e^{-at} f(t)\,dt$$

$$y = e^{at}c + e^{at}\int e^{-at} f(t)\,dt$$

2)

a) $y_1' = 2y_1 + 4y_2$ $\quad y_1(0) = 1$

$y_2' = 4y_1 + 2y_2$ $\quad y_2(0) = 4$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

evals

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 16 = 4 - 4\lambda + \lambda^2 - 16 = \lambda^2 - 4\lambda - 12$$

$(\lambda-6)(\lambda+2)$

$\lambda = 6, -2$

If $\lambda = 6$, $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If $\lambda = -2$ $\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (either works)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{+6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

@ $t = (0)$ we have $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ so

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}$$

so $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + \frac{3}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$

or
$$\boxed{\begin{aligned} y_1 &= 5/2\, e^{6t} - 3/2\, e^{-2t} \\ y_2 &= 5/2\, e^{6t} + 3/2\, e^{-2t} \end{aligned}}$$

Now you can check your answer by computing

$y_1' = 15e^{6t} + 3e^{-2t}$

$y_2' = 15e^{6t} - 3e^{-2t}$

$2y_1 + 4y_2 = (5e^{6t} - 3e^{-2t}) + (10e^{6t} + 6e^{-2t}) = 15e^{6t} + 3e^{-2t} = y_1'$ ✓ yep

and $4y_1 + 2y_2 = (10e^{6t} - 6e^{-2t}) + (5e^{6t} + 3e^{-2t}) = 15e^{6t} - 3e^{-2t} = y_2'$ ✓ yep

2) b $y_1' = y_1 + 2y_2$ $\quad y_1(0) = 6 \quad y_2(0) = 0$

$y_2' = 3y_1$ $\quad \vec{y}(0) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$\vec{y}' = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}}_{A=} \vec{y}$

eigvals $\begin{vmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6$

$(\lambda-3)(\lambda+2) = 0$

$\boxed{\lambda = 3, -2}$

$\lambda = 3 \quad \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2 \quad \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$ I'll use $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2t}$

Now use $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^0 = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \left(\frac{1}{5}\right) \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 18 \\ -6 \end{bmatrix} = \begin{bmatrix} 18/5 \\ -6/5 \end{bmatrix}$$

So
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{18}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - \frac{6}{5} \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2t}$$

$y_1 = \frac{18}{5} e^{3t} + \frac{12}{5} e^{-2t}$

$y_2 = \frac{18}{5} e^{3t} - \frac{18}{5} e^{-2t}$

Check $y_1' = \frac{54}{5} e^{3t} - \frac{24}{5} e^{-2t}$

$y_2' = \frac{54}{5} e^{3t} + \frac{36}{5} e^{-2t}$

$y_1 + 2y_2 = \frac{54}{5} e^{3t} - \frac{24}{5} e^{-2t}$ ✓

$3y_1 = \frac{54}{5} e^{3t} + \frac{36}{5} e^{-2t}$ ✓

yep. It's correct

2c) $y_1' = y_1 + 4y_2$
$y_2' = 3y_1 + 2y_2$ $= \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 12$
$= \lambda^2 - 3\lambda + 2 - 12$
$= \lambda^2 - 3\lambda - 10$
$(\lambda - 5)(\lambda + 2)$
$\lambda = 5, -2$

$\lambda = 5$ $\left[\begin{array}{cc:c} -4 & 4 & 0 \\ 3 & -3 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2$ $\left[\begin{array}{cc:c} 3 & 4 & 0 \\ 3 & 4 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} e^{-2t}}$$

Using the conditions $y_1(0) = 0$, $y_2(0) = 1$
we have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 1/7 \end{bmatrix}$

Our solution is:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{4}{7}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \frac{1}{7}\begin{bmatrix} -4 \\ 3 \end{bmatrix} e^{-2t}$$

2d) $y_1' = y_2$ $\quad y_1(0) = 1$

$y_2' = -3y_1 - 4y_2$ $\quad y_2(0) = 2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{vmatrix} 0-\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = 4\lambda + \lambda^2 + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\boxed{\lambda = -3, -1}$$

If $\lambda = -3$ $\left(\begin{array}{cc|c} 3 & 1 & 0 \\ -3 & -1 & 0 \end{array}\right) \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\lambda = -1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Now @ $t=0$ we have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

So $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{pmatrix} -3/2 \\ 5/2 \end{pmatrix}$

So $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t} + \frac{5}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

$y_1 = -\frac{3}{2} e^{-3t} + \frac{5}{2} e^{-t}$

$y_2 = +\frac{9}{2} e^{-3t} - \frac{5}{2} e^{-t}$

check $y_1' = \frac{9}{2} e^{-3t} - \frac{5}{2} e^{-t} = y_2$ ✓

$-3y_1 - 4y_2$

$y_2' = \frac{-27}{2} e^{-3t} + \frac{5}{2} e^{-t} = \left(\frac{9}{2} e^{-3t} - \frac{15}{2} e^{-t}\right) + \left(\frac{-36}{2} e^{-3t} + \frac{20}{2} e^{-t}\right)$

$-36 + 9 = -27$ ✓

$-15 + 20 = 5$ ✓

yep ~~It's~~ correct.

3) a $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\lambda = 1, 3$ (its triangular so use diagonals).

$$\lambda = 1 \quad \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = 3 \quad \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

check

$$Q^{-1}AQ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \checkmark$$

3b) error. I meant ~~I meant~~ $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. Oh well, we'll do it both ways

$$\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix} = \lambda + \lambda^2 + 2$$
$$\lambda^2 + \lambda + 2$$

$$\frac{-1 \pm \sqrt{1-4(1)(2)}}{2(1)} = \frac{-1 \pm i\sqrt{3}}{2}$$ ugly

Try $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ instead.

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} = 2\lambda + \lambda^2 + 1 = \lambda^2 + 2\lambda + 1 = \underline{(\lambda+1)^2} \quad \lambda = -1, -1$$

$\lambda = -1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightarrow \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ There is only 1 eigenvector.

To find Generalized eigenvector, augment by eigenvector instead of by zero.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \rightarrow \text{rref} \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \text{ so } x+y=1$$

both $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are solutions. I'll pick $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = V_2$

Then $Q = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

and
$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = J$$

3C) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\lambda = 1, 1, 1$ (a triple eigenvalue so algebraic multiplicity is 3.)

$A - 1I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ eigenvalue is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

geometric multiplicity is 1.

I need 2 generalized eigenvectors. (one at rank 2, one at rank 3)

Solve $\left[\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{matrix} 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$

x = arbitrary. let $= 0$

$y = 1/2$

$z = 0$

$\begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} = \vec{v}_2$

Solve $\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x = arbitrary let $= 0$

$y = -1/4$

$z = 1/4$

$v_3 = \begin{bmatrix} 0 \\ -1/4 \\ 1/4 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$ $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

3 d) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \lambda = 1, 1, 1$ (algebraic multiplicity is 1)

2 free variables

Evecs $\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref is}} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \vec{X}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(geom mult is 2)

Now we need 1 more. Solve.

$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\qquad \left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$

There is a solution, $\qquad$ no solution

either $\begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$. You can pick either.

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad J = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

2 vectors make a 2 by 2 block

check $Q^{-1} A Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$= \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = J$ ✓

2e) $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $\begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}$: $\lambda^2+1=0$

$\underline{\lambda=\pm i}$

$\lambda=i$ $\left[\begin{array}{cc|c}-i&1&0\\-1&-i&0\end{array}\right]\to\vec{x}=\begin{bmatrix}1\\i\end{bmatrix}$

$\lambda=-i$ $\left[\begin{array}{cc|c}i&1&0\\ \cancel{-1}&\cancel{i}&\cancel{0}\end{array}\right]\to\vec{x}=\begin{bmatrix}1\\-i\end{bmatrix}$

$Q=\begin{bmatrix}1&1\\i&-i\end{bmatrix}$ $J=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$ $Q^{-1}=\frac{1}{-2i}\begin{bmatrix}-i&-1\\-i&1\end{bmatrix}$

$$Q^{-1}AQ=\frac{1}{-2i}\begin{bmatrix}-i&-1\\-i&1\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}1&1\\i&-i\end{bmatrix}$$

$$=\frac{1}{2i}\begin{bmatrix}-i&-1\\-i&1\end{bmatrix}\begin{bmatrix}i&-i\\-1&-1\end{bmatrix}$$

$$=\frac{1}{2i}\begin{bmatrix}-i^2+1&i^2+1\\-i^2-1&i^2-1\end{bmatrix}\qquad i^2=-1$$

$$=\frac{1}{2i}\begin{bmatrix}1+1&-1+1\\+1-1&-1-1\end{bmatrix}$$

$$=\frac{1}{-2i}\begin{bmatrix}2&0\\0&-2\end{bmatrix}=\begin{bmatrix}-1/i&0\\0&+1/i\end{bmatrix}=\begin{bmatrix}i&0\\0&-i\end{bmatrix}\checkmark$$

note

$\frac{1}{i}=\frac{i^3}{i^4}=\frac{i^3}{1}=-i$

4) a) $\begin{bmatrix} e^{2} & 0 \\ 0 & e^{3} \end{bmatrix}$ b) $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ c) $\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$

d) $\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$ e) $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}$

g) $\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ h) $\begin{bmatrix} e^{5t} & te^{5t} & \frac{t^2}{2}e^{5t} \\ 0 & e^{5t} & te^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$

i) $e^{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & t & t^2/2 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

j) $e^{\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}} = \begin{bmatrix} e^{3t} & te^{3t} & \frac{t^2}{2}e^{3t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & 0 & 0 & e^{-2t} \end{bmatrix}$

3) a) $\begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}$ $\det\begin{vmatrix}-\lambda & 1\\ -3 & 4-\lambda\end{vmatrix} = -4\lambda+\lambda^2+3$

5

$(\lambda-3)(\lambda-1)=0$

$\lambda=3 \quad \lambda=1$

$\lambda=3$

$\left[\begin{array}{cc|c}-3 & 1 & 0\\ -3 & 1 & 0\end{array}\right] \quad \vec{x}=\begin{bmatrix}1\\ +3\end{bmatrix}$

$\lambda=1$

$\left[\begin{array}{cc|c}-1 & 1 & 0\\ -3 & 3 & 0\end{array}\right] \quad \vec{x}=\begin{bmatrix}1\\ +1\end{bmatrix}$

$Q=\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix} \quad J=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix} \quad Q^{-1}=\frac{1}{1-3}\begin{bmatrix}1 & -1\\ -3 & 1\end{bmatrix}$

$e^J=\begin{bmatrix}e^3 & 0\\ 0 & e^1\end{bmatrix}$

$$e^A = Qe^JQ^{-1}$$

$$= \begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}e^3 & 0\\ 0 & e^1\end{bmatrix}\begin{bmatrix}1 & -1\\ -3 & 1\end{bmatrix}\left(-\frac{1}{2}\right)$$

$$= \begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}\left[\begin{array}{c|c}e^3 & -e^3\\ \hline -3e^1 & +e^1\end{array}\right]\left(-\frac{1}{2}\right)$$

$$= \left[\begin{array}{c|c}e^3-3e^1 & -e^3+e^1\\ \hline 3e^3-3e^1 & -3e^3+e^1\end{array}\right]\cdot\left(-\frac{1}{2}\right)$$

$$= \begin{bmatrix}\frac{3e-e^3}{2} & \frac{e-e^3}{2}\\ \frac{3e-3e^3}{3} & \frac{e-3e^3}{2}\end{bmatrix}$$

~~3~~ 5 b) $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 5\lambda + \lambda^2 + 6$

$(\lambda+2)(\lambda+3) = 0$

$\lambda = -2, -3$

$\lambda = -2$

$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = -3$

$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ -6 & -2 & 0 \end{array}\right)$ $\vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$ $J = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ $Q^{-1} = \dfrac{1}{-3+2}\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$

$e^{Jt} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$ (we'll get the answer if $t=1$).

$$e^{A} = Qe^{Jt}Q^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 3e^{-2t} & e^{-2t} \\ -2e^{-3t} & -e^{-3t} \end{bmatrix}$$

$$= \left[\begin{array}{c|c} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{array}\right]$$

let $t=1$ to get answer.

3c) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 2\lambda + \lambda^2 + 1 = (\lambda+1)^2$ $\lambda = -1, -1.$

$\lambda = -1$

to get generalized we solve

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$ $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right]$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ $Q^{-1} = \frac{1}{0-(-1)} \begin{bmatrix} 0 & -1 \\ +1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

$$e^{Jt} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$e^{At} = Q e^{Jt} Q^{-1} -$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} te^{-t} & -e^{-t} + te^{-t} \\ e^{-t} & e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} te^{-t} + e^{-t} & -e^{-t} + te^{-t} + e^{-t} \\ -te^{-t} & +e^{-t} - te^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} te^{-t} + e^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

let $t = 1$ to get answer.

5 3d) $\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$ $\left|\begin{bmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{bmatrix}\right| = 4\lambda + \lambda^2 + 4 = (\lambda+2)^2 = 0$

$\lambda = -2$

$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$ $x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right]$ $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ $Q^{-1} = \frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$e^{Jt} = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$e^{A} = Qe^{Jt}Q^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t} + e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} e^{-2t} + 2te^{-2t} & te^{-2t} \\ -2e^{-2t} - 4te^{-2t} + 2e^{-2t} & -2te^{-2t} + e^{-2t} \end{array}\right]$$

let $\underline{t=1}$.

$\cancel{3}$ 5 e) $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $\begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}=\lambda^2+1=0$ $\lambda=\pm i$

$\lambda=i$ $\qquad\qquad$ $\lambda=-i$

$\left[\begin{array}{cc|c}-i&1&0\\-1&-i&0\end{array}\right]\vec{x}=\begin{bmatrix}1\\i\end{bmatrix}$ $\qquad$ $\left[\begin{array}{cc|c}i&1&0\\-1&i&0\end{array}\right]\vec{x}=\begin{bmatrix}1\\-i\end{bmatrix}$

$Q=\begin{bmatrix}1&1\\i&-i\end{bmatrix}$ $\quad J=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$ $\quad Q^{-1}=\frac{1}{-2i}\begin{bmatrix}-i&-1\\-i&1\end{bmatrix}$

$$e^{Jt}=\begin{bmatrix}e^{it}&0\\0&e^{-it}\end{bmatrix}$$

$$e^{At}=\overset{Q}{\begin{bmatrix}1&1\\i&-i\end{bmatrix}}\overset{e^{Jt}}{\begin{bmatrix}e^{it}&0\\0&e^{-it}\end{bmatrix}}\overset{Q^{-1}}{\begin{bmatrix}-i&-1\\-i&1\end{bmatrix}}\cdot\left(\frac{1}{-2i}\right)$$

$$=\begin{bmatrix}1&1\\i&-i\end{bmatrix}\begin{bmatrix}-ie^{it}&-e^{it}\\-ie^{-it}&e^{-it}\end{bmatrix}\left(\frac{1}{-2i}\right)$$

$$=\begin{bmatrix}-ie^{+it}-ie^{-it}&-e^{it}+e^{-it}\\-i^2e^{it}+i^2e^{-it}&-ie^{it}-ie^{-it}\end{bmatrix}\left(\frac{1}{-2i}\right)$$

We now use $\cosh x$ and $\sinh x$

$$=\begin{bmatrix}-i2\left(\frac{e^{it}+e^{-it}}{2}\right)&-2\left(\frac{e^{it}-e^{-it}}{2}\right)\\2\left(\frac{e^{it}-e^{-it}}{2}\right)&-i2\left(\frac{e^{it}-e^{-it}}{2}\right)\end{bmatrix}\left(\frac{1}{-2i}\right)$$

$$=\frac{1}{-2i}\begin{bmatrix}-2i\cosh(it)&-2\sinh(it)\\2\sinh(it)&-2i\cosh(it)\end{bmatrix}=\frac{1}{-2i}\begin{bmatrix}-2i\cos(t)&-2i\sin t\\2i\sin t&-2i\cos t\end{bmatrix}$$

$$=\begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}$$ Let $t=1$

3f) $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 \quad \lambda = \pm 2i$

$\lambda = 2i$

$$\left[\begin{array}{cc|c} -2i & 1 & 0 \\ -4 & -2i & 0 \end{array}\right] \quad \vec{x} = \begin{bmatrix} 1 \\ +2i \end{bmatrix}$$

$\lambda = -2i$

$$\left[\begin{array}{cc|c} 2i & 1 & 0 \\ 4 & 2i & 0 \end{array}\right] \quad \vec{x} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \quad J = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} -2i & -1 \\ -2i & 1 \end{bmatrix} \cdot \frac{1}{-4i}$$

$$e^{Jt} = \begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix}$$

$$e^{At} = Q e^{Jt} Q^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix} \begin{bmatrix} -2i & -1 \\ -2i & 1 \end{bmatrix} \left(\frac{1}{-4i}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} -2ie^{2it} & -e^{2it} \\ -2ie^{-2it} & e^{-2it} \end{bmatrix} \left(\frac{1}{-4i}\right)$$

$$= \begin{bmatrix} -2ie^{2it} - 2ie^{-2it} & -e^{2it} + e^{-2it} \\ -4i^2 e^{2it} + 4i^2 e^{-2it} & -2ie^{2it} - 2ie^{-2it} \end{bmatrix} \left(\frac{1}{-4i}\right)$$

$$= \begin{bmatrix} -2i \cdot 2\left(\frac{e^{2it} + e^{-2it}}{2}\right) & -2\left(\frac{e^{2it} - e^{-2it}}{2}\right) \\ 4 \cdot 2\left(\frac{e^{2it} - e^{-2it}}{2}\right) & -2i \cdot 2\left(\frac{e^{2it} + e^{-2it}}{2}\right) \end{bmatrix} \left(\frac{1}{-4i}\right)$$

$$= \frac{1}{-4i} \begin{bmatrix} -4i \cosh(2it) & -2 \sinh(2it) \\ 8 \sinh(2it) & -4i \cosh(2it) \end{bmatrix}$$

$$= \frac{1}{-4i} \begin{bmatrix} -4i \cos(2t) & -2i \sin(2t) \\ 8i \sin(2t) & -4i \cos(2t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2t) & +\frac{1}{2} \sin(2t) \\ -2 \sin(2t) & \cos(2t) \end{bmatrix}$$

3g
5

$$\begin{bmatrix}0&1\\0&3\end{bmatrix} \qquad \begin{vmatrix}-\lambda & 1\\ 0 & 3-\lambda\end{vmatrix} = \lambda^2-3\lambda=0$$
$$\lambda(\lambda-3) \qquad \lambda=0,\ \lambda=3$$

$\lambda = 0$

$$\left[\begin{array}{cc|c}0&1&0\\0&3&0\end{array}\right] \quad x=\begin{bmatrix}1\\0\end{bmatrix}$$

$\lambda = 3$

$$\left[\begin{array}{cc|c}-3&1&0\\0&0&0\end{array}\right] \quad \begin{bmatrix}1\\3\end{bmatrix}$$

$$Q=\begin{bmatrix}1&1\\0&3\end{bmatrix} \qquad J=\begin{bmatrix}0&0\\0&3\end{bmatrix} \qquad Q^{-1}=\frac{1}{3}\begin{bmatrix}3&-1\\0&1\end{bmatrix}$$

$$e^{Jt}=\begin{bmatrix}1&0\\0&e^{3t}\end{bmatrix}$$

$$\begin{aligned}
e^{At} &= Qe^{Jt}Q^{-1}\\
&= \begin{bmatrix}1&1\\0&3\end{bmatrix}\begin{bmatrix}1&0\\0&e^{3t}\end{bmatrix}\begin{bmatrix}3&-1\\0&1\end{bmatrix}\frac{1}{3}\\
&= \begin{bmatrix}1&1\\0&3\end{bmatrix}\begin{bmatrix}3&-1\\0&e^{3t}\end{bmatrix}\frac{1}{3}\\
&= \begin{bmatrix}3&-1+e^{3t}\\0&3e^{3t}\end{bmatrix}\frac{1}{3}\\
&= \begin{bmatrix}1&\frac{-1+e^{3t}}{3}\\0&e^{3t}\end{bmatrix}
\end{aligned}$$

3h) 5

$$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \quad \left| \begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \right| = (2-\lambda)^2 - 16 = 0$$

$$(2-\lambda)^2 = 16$$

$$2-\lambda = \pm 4 \qquad \lambda = 2 \pm 4 = -2, 6$$

$\lambda = -2$

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right] \bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 6$

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right] \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad J = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2}$$

$$e^{Jt} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{6t} \end{bmatrix}$$

$$e^{At} = Q e^{Jt} Q^{-1}$$

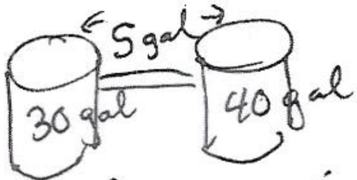
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{6t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & -e^{-2t} \\ e^{6t} & e^{6t} \end{bmatrix} \frac{1}{2}$$

$$= \begin{bmatrix} e^{-2t} + e^{6t} & -e^{-2t} + e^{6t} \\ -e^{-2t} + e^{6t} & e^{2t} + e^{6t} \end{bmatrix} \frac{1}{2}$$

$$= \begin{bmatrix} \dfrac{e^{-2t} + e^{6t}}{2} & \dfrac{-e^{-2t} + e^{6t}}{2} \\ \dfrac{-e^{-2t} + e^{6t}}{2} & \dfrac{e^{2t} + e^{6t}}{2} \end{bmatrix}$$

6 a)

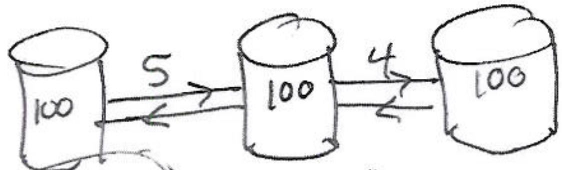


$y_1(0) = 20$
$y_2(0) = 120$

$$y_1' = \left(\frac{5}{40} y_2\right) - \left(\frac{5}{30} y_1\right)$$
$$y_2' = \left(\frac{5}{30} y_1\right) - \left(\frac{5}{40} y_2\right)$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -5/30 & 5/40 \\ 5/30 & -5/40 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

b)



Remember

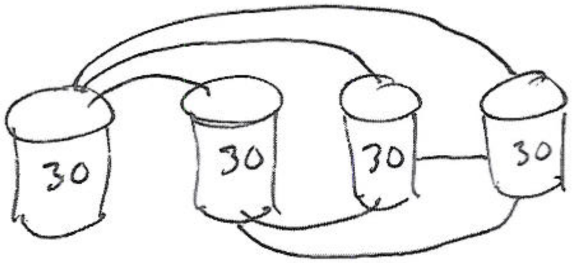
(Inflow) - (outflow

$y_1(0) = 400$
$y_2(0) = 0$
$y_3(0) = 0$

$$y_1' = \left(\frac{5}{100} y_2\right) - \left(\frac{5}{100} y_1\right)$$
$$y_2' = \left(\frac{5}{100} y_1 + \frac{4}{100} y_3\right) - \left(\frac{9}{100} y_2\right)$$
$$y_3' = \left(\frac{4}{100} y_2\right) - \left(\frac{4}{100} y_3\right)$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -5/100 & 5/100 & 0 \\ 5/100 & -9/100 & 4/100 \\ 0 & 4/100 & -4/100 \end{bmatrix}$$

c)



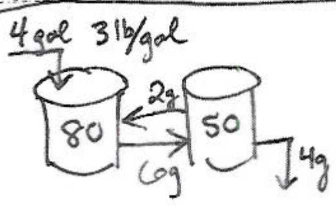
$y_1(0) = 50$
$y_2(0) = 80$
$y_3(0) = 10$
$y_4(0) = 0$

$$y_1' = \left(\frac{1}{30} y_2 + \frac{1}{30} y_3 + \frac{1}{30} y_4\right) - \left(\frac{3}{30} y_1\right)$$
$$y_2' = \left(\frac{1}{30} y_1 + \frac{1}{30} y_3 + \frac{1}{30} y_4\right) - \left(\frac{3}{30} y_2\right)$$
$$y_3' = \left(\frac{1}{30} y_1 + \frac{1}{30} y_2 + \frac{1}{30} y_4\right) - \left(\frac{3}{30} y_3\right)$$
$$y_4' = \left(\frac{1}{30} y_1 + \frac{1}{30} y_2 + \frac{1}{30} y_3\right) - \left(\frac{3}{30} y_4\right)$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} -3/30 & 1/30 & 1/30 & 1/30 \\ 1/30 & -3/30 & 1/30 & 1/30 \\ 1/30 & 1/30 & -3/30 & 1/30 \\ 1/30 & 1/30 & 1/30 & -3/30 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

---

d)



$y_1(0) = 0$
$y_2(0) = 0$

$$y_1' = \left(4 \cdot 3 + \frac{2}{50} y_2\right) - \left(\frac{6}{80} y_1\right)$$
$$y_2' = \frac{6}{80} y_1 - \frac{6}{50} y_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -\frac{6}{80} & \frac{2}{50} \\ \frac{6}{80} & -\frac{6}{50} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix}$$

Nonhomogeneous

8/7

a) $y'' + 4y' + 3y = 0$

$y_1 = y$

$\boxed{y_2 = y_1'}$

$y_2' + 4y_2 + 3y_1 = 0$

$y_2' = -3y_1 - 4y_2$

$y_1' = y_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Homogeneous

---

b) same as a, except. $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4t \end{bmatrix}$

not zero

Non homogeneous

---

c) $y'' + ty' - 2y = 0$

$y_1 = y$

$y_2 = y_1'$

$y_2' = y'' = 2y - ty' = 2y_1 - ty_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -t \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Homogeneous

---

d) Same as C, except Nonhomogeneous and

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -t \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

---

e) $y''' + 3y'' + 3y' + y = 0$ Homogeneous

$$\boxed{\begin{aligned} y_1 &= y \\ y_2 &= y_1' \\ y_3 &= y_2' \end{aligned}}$$

$= y'$

$= y''$

$y_3' = y''' = -y - 3y' - 3y'' = -y_1 - 3y_2 - 3y_3$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

---

f) $$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & +4 & -6 & +4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}$$

Non homogeneous

~~6~~)a $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ Its homogeneous

8

so

$A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}$ $\vec{y} = e^{At}c$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5e^{2t} \\ 6e^{4t} \end{bmatrix}$$

So $y_1 = 5e^{2t}$ $y_2 = 6e^{4t}$.

6b) $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $e^{At} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$ so $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$y_1 = -e^{2t} + 3te^{2t}$ $y_2 = 3e^{2t}$

6c) $y'' + 4y' + 3y = 0$ $y(0) = 0$ $y'(0) = 1$

$y_1 = y$
$y_2 = y_1'$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$

$(\lambda+3)(\lambda+1) = 0$

$\lambda = -3, -1.$

$\lambda = -3$ $\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -3 & -1 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\lambda = -1$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right)$ $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$ $J = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ $Q^{-1} = \frac{1}{-1+3}\begin{bmatrix} -1 & - \\ 3 & 1 \end{bmatrix}$

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} -e^{-3t} & -e^{-3t} \\ 3e^{-t} & e^{-t} \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} -e^{-3t} + 3e^{-t} & -e^{-3t} + e^{-t} \\ 3e^{-3t} - 3e^{-t} & 3e^{-3t} - e^{-t} \end{bmatrix}\left(\frac{1}{2}\right).$$

Hence $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (-e^{-3t} + e^{-t})/2 \\ (3e^{-3t} - e^{-t})/2 \end{bmatrix}$

Since $y_1 = y$, we have

$$y = \frac{(-e^{-3t} + e^{-t})}{2}$$

as our solution

~~6~~d) $y''+2y'+y=0 \quad y(0)=2 \quad y'(0)=0$

8

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$ on 3c we showed $e^{At} = \begin{bmatrix} te^{-t}+e^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{bmatrix}$

So the solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{At}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2te^{-t}+2e^{-t} \\ -2te^{-t} \end{bmatrix}$

So $y_1(t) = \boxed{y(t) = 2te^{-t}+2e^{-t}}$

6e $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2-1$

$2-\lambda = \pm 1$

$\lambda = 2 \pm 1$

$= 1, 3$

$\lambda = 1$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 3$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad Q^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & -e^t \\ e^{3t} & e^{3t} \end{bmatrix}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\begin{bmatrix} e^t+e^{3t} & -e^t+e^{3t} \\ -e^t+e^{3t} & e^t+e^{3t} \end{bmatrix}$$

$$\text{so } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{At}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2e^t+2e^{3t} - e^t+e^{3t} \\ -2e^t+2e^{3t}+e^t+e^{3t} \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} e^t+3e^{3t} \\ -e^t+3e^{3t} \end{bmatrix}$$

so $y_1 = \frac{1}{2}(e^t+3e^{3t})$ and $y_2 = \frac{1}{2}(-e^t+3e^{3t})$

6f $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

In 3e we found $e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

$$\text{so } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos t + 2\sin t \\ -\sin t + 2\cos t \end{bmatrix}$$

$y_1 = \cos t + 2\sin t \qquad y_2 = -\sin t + 2\cos t$