1)

m	$f^m(x)$	@0	$\to /m!$	$\to x^m$
0	e^{4x}	1	1	$1\cdot x^0$
1	$4e^{4x}$	4	$4/1$	$4\cdot x$
2	$4^2 e^{4x}$	4^2	$4^2/2!$	$\frac{4^2}{2!}x^2$
3	$4^3 e^{4x}$	4^3	$4^3/3!$	$\frac{4^3}{3!}x^3$

$$P_3(x) = 1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3$$

---

2) $\cos x \quad n=4 \quad c=\pi$

m	$f^m(x)$	@π	$\to /m!$	times $(x-\pi)^m$
0	$\cos x$	-1	-1	-1
1	$-\sin x$	0	0	0
2	$-\cos x$	1	$1/2!$	$\frac{1}{2!}(x-\pi)^2$
3	$\sin x$	0	0	0
4	$\cos x$	-1	$-1/4!$	$-\frac{1}{4!}(x-\pi)^4$

$$P_4(x) = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4$$

---

3) $\cos(2x) \quad n=4 \quad c=0$

m	$f^m(x)$	@0
0	$\cos 2x$	1
1	$-2\sin 2x$	0
2	$-2^2\cos 2x$	-2^2
3	$+2^3\sin 2x$	0
4	$+2^4\cos 2x$	$+2^4$

$$P_4(x) = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4$$

---

4) $\sin(\frac{1}{2}x) \quad n=5 \quad c=0$

m		@0
0	$\sin(\frac{1}{2}x)$	0
1	$\frac{1}{2}\cos\frac{1}{2}x$	$\frac{1}{2}$
2	$-(\frac{1}{2})^2\sin(\frac{1}{2}x)$	0
3	$-(\frac{1}{2})^3\cos(\frac{1}{2}x)$	$-(\frac{1}{2})^3$
4	$+(\frac{1}{2})^4\sin(\frac{1}{2}x)$	0
5	$+(\frac{1}{2})^5\cos(\frac{1}{2}x)$	$+(\frac{1}{2})^5$

$$P_5(x) = \frac{1}{2}x^1 - \frac{(\frac{1}{2})^3}{3}x^3 + \frac{(\frac{1}{2})^5}{5!}x^5$$

5) $\frac{1}{x} = x^{-1}$ $C=1$ $n=3$

m	$f^{m}(x)$	@1	divid by $/m!$
0	x^{-1}	1	$1 = 1$
1	$-1x^{-2}$	-1	$-1/1 = -1$
2	$(-1)(-2)x^{-3}$	$2!$	$2!/2! = 1$
3	$(-1)(-2)(-3)x^{-4}$	$-3!$	$-3!/3! = -1$

$$\boxed{P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3}$$

6) $\ln x$ $C=1$ $n=3$

		@1	$/m!$
0	$\ln x$	0	0
1	x^{-1}	1	$1/1!$
2	$-1x^{-2}$	-1	$-1/2!$
3	$(-1)(-2)x^{-3}$	$2!$	$2/3!$

$$P_3(x) = 0 + 1x - \frac{1}{2!}x^2 + \frac{2!}{3!}x^3$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

7) $\ln(1-x)$ $n=4$ $c=0$

m		@0
0	$\ln(1-x)$	0
1	$(1-x)^{-1}\cdot(-1)$	-1
2	$(-1)(1-x)^{-2}(-1)^2$	-1
3	$(-1)(-2)(1-x)^{-3}(-1)^3$	$-2!$
4	$(-1)(-2)(-3)(1-x)^{-4}(-1)^4$	$-3!$

$$P_4(x) = 0 - \frac{1}{1!}x^1 - \frac{1}{2!}x^2 - \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4$$

$$= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$$

8) $\ln(1+x)$ $n=4$ $c=0$

m	$f^{m}(x)$	@0
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$(-1)(1+x)^{-2}$	-1
3	$(-1)(-2)(1+x)^{-3}$	$2!$
4	$(-1)(-2)(-3)(1+x)^{-4}$	$-3!$

$$P_4(x) = 1x - \frac{1}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

9) $a_n = \frac{1}{3^n} x^n$ $\qquad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} \Big| \frac{3^n}{3^{n+1}} = \frac{\cancel{x^n} x}{\cancel{x^n}} \Big| \frac{\cancel{3^n}}{\cancel{3^n} 3} = \frac{x}{3}$

$\lim_{n\to\infty} \left|\frac{x}{3}\right| = \frac{|x|}{3} < 1$ when $|x| < 3$ so $\boxed{R=3}$

---

10) $\frac{(-1)^{n+1}}{(-1)^n} \Big| \frac{4^{n+1}}{4^{n+2}} \Big| \frac{x^{n+1}}{x^n} = \frac{-1}{1} \cdot \frac{1}{4} \cdot \frac{x}{1}$ $\qquad \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \frac{|x|}{4} < 1$

when $|x| < 4$ so $\boxed{R=4}$

---

11) $\frac{n+1}{n} \Big| \frac{\cancel{2^n}}{\cancel{2^{n}}2} \Big| \frac{x^{3(n+1)}}{\cancel{x^{3n}}} \quad x^{3(n+1)} = \cancel{x^{3n}} x^3$ $= \frac{n+1}{n} \Big| \frac{1}{2} \Big| \frac{x^3}{1} = \frac{n+1}{n} \; \frac{x^3}{2}$

$\lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)\left(\frac{x^3}{2}\right) \right| = (1)\frac{|x|^3}{2} < 1$ when $|x|^3 < 2$

or $|x| < \boxed{\sqrt[3]{2}}$ $\; = R$

---

12) $\lim_{n\to\infty} \left| \frac{3(n+1)+1}{3n+1} \Big| \frac{n^2+4}{(n+1)^2+4} \Big| \frac{x^{n+1}}{x^n} \right|$

gives $\left|(1)\cdot(1)\cdot(x)\right| = |x| < 1$ when $|x| < 1$ so $R=1$

---

13) $\frac{(-4)^{n+1}}{(-4)^n} \Big| \frac{n+1}{n} \Big| \frac{n^2+1}{(n+1)^2+1} \Big| \frac{x^{2(n+1)}}{\cancel{x^{2n}}} \quad x^{2(n+1)} = \cancel{x^{2n+2}}$

$\left|(-4)(1)(1)(x^2)\right| = 4|x|^2 < 1$ when $|x|^2 < \frac{1}{4}$ $\; |x| < \frac{1}{2} = R$

~~14~~ 15) $\frac{(-1)^{n+1}}{(-1)^n} \Big| \frac{n!}{(n+1)!} \Big| \frac{x^{n+1}}{x^n} = \frac{(-1)}{1} \Big| \frac{\cancel{1\cdot 2\cdot 3\cdots n}}{\cancel{1\cdot 2\cdot 3\cdots n}\cdot(n+1)} \Big| \frac{x}{1} = \frac{-x}{n+1}$

$\lim_{n\to\infty} \left|\frac{-x}{n+1}\right| = 0 < 1$ for all x so

14 on next page

$R = \infty$

(15 on last page)

14) $\frac{n+1}{n} \bigg| \frac{2^n}{2^{n+1}} \bigg| \frac{x^{2(n+1)}}{x^{2n}} = \frac{n+1}{n} \frac{1}{2} \frac{x^2}{1} \xrightarrow[n\to\infty]{} (1)(\tfrac{1}{2})(x^2) < 1$

when $|x|^2 < 2$ or $|x| < \sqrt{2}$ R

16) $\frac{(n+1)!}{n!} \bigg| \frac{10^n}{10^{n+1}} \bigg| \frac{x^{2(n+1)}}{x^{2n}} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}\cdot(n+1)}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}} \bigg| \frac{1}{10} \bigg| \frac{x^2}{1} = \frac{(n+1)x^2}{10}$

$\lim_{n\to\infty} \frac{(n+1)x^2}{10} = \infty < 1$ never, so $\boxed{R=0}$

17)

m	e^x	@0
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1

$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$

$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Radius $\frac{n!}{(n+1)!} \bigg| \frac{x^{n+1}}{x^n} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}(n+1)} \bigg| \frac{x}{1} = \frac{x}{n+1} \to 0$ as $n\to\infty$

$0 < 1$ always so $R = \infty$

18)

m	$f^n(x)$	@0
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
⋮		⋮

$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \ldots$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Radius

$\left| \frac{(2n)!}{(2(n+1))!} \bigg| \frac{(-1)^{n+1}}{(-1)^n} \bigg| \frac{x^{2(n+1)}}{x^{2n}} \right| = \left| \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{(2n)}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{(2n)}(2n+1)(2n+2)} \bigg| \frac{-1}{1} \bigg| \frac{x^2}{1} \right|$

$= \frac{+x^2}{(2n+1)(2n+2)} \to 0$ <1 always as $n\to\infty$ so $R = \infty$

19)

n	$f^{n}(x)$	@0
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
	$\vdots$	1
		0
		-1

$$1x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

radius

$$\left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(2n+1)!}{\underset{(2n+3)!}{(2(n+1)+1)!}} \cdot \frac{x^{\overset{2n+3}{2(n+1)+1}}}{x^{2n+1}} \right| = \left| \frac{-\cancel{1}}{1} \cdot \frac{\cancel{1 \cdot 2 \cdots (2n+1)}}{\cancel{1 \cdot 2 \cdots (2n+1)}(2n+2)(2n+3)} \cdot \frac{x^2}{1} \right|$$

$$= \frac{x^2}{(2n+2)(2n+3)} \xrightarrow{\text{as } n\to\infty} 0 < 1 \text{ always so } R = \infty$$

20

$$\frac{1}{1-x} = (1-x)^{-1}$$

n		@0	$/n!$
0	$(1-x)^{-1}$	1	1
1	$(-1)(1-x)^{-2}(-1)$	1	1
2	$(-1)(-2)(1-x)^{-3}(-1)^2$	$2!$	1
3	$(-1)(-2)(-3)(1-x)^{-3}(-1)^3$	$3!$	1
4		$4!$	1
5		$5!$	1

$$1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1 \quad \boxed{R = 1}$$

21)

		@0	/m!
0	$(1+x)^{-1}$	1	1
1	$(-1)(1+x)^{-2}$	-1	-1
2	$(-1)(-2)(1+x)^{-3}$	2!	1
3	$-3!\,(1+x)^{-4}$	-3!	-1
4	$+4!\,(1+x)^{-5}$	4!	1
	$\vdots$	-5!	-1
		6!	1
		$\vdots$	

$$1-x+x^2-x^3+x^4-x^5+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\,\frac{x^{n+1}}{x^n}\right|=|(-1)x|=|x|<1$$

$$R=1$$

22

m		@0
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
		$\vdots$

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}$$

$$R=\infty$$

similar to 18

23)

m		@0
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	1
		$\vdots$

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}$$

$$R=\infty$$

similar to 19

24)

$$e^{ix}=\sum_{n=0}^{\infty}\frac{1}{n!}(ix)^n=1+(ix)+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\cdots$$

$$=1+ix-\frac{1}{2}x^2-i\frac{1}{3!}x^3+\frac{1}{4!}x^4+i\frac{1}{5!}x^5$$

factor by imaginary or not.

$$=\left(1-\frac{1}{2}x^2+\frac{1}{4}x^4+\cdots\right)+i\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$=\cos x+i\sin x$$

25) $\cosh(ix) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(ix)^{2n} \underset{\text{by \#22}}{=} \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n)!}x^{2n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = \cos x$$

---

26) $\cos(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(ix)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n (i^2)^n}{(2n)!}x^{2n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{(2n)!}x^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} = \cosh(x)$$

---

27) $\sinh(ix) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(ix)^{2n+1}$

$$= (ix) + \frac{1}{3!}(ix)^3 + \frac{1}{5!}(ix)^5 + \frac{1}{7!}(ix)^7 + \cdots$$

$$= i\left(x - \frac{1}{3!}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right) = i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$= i \sin x$$

---

28) $\sin(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(ix)^{2n+1}$

$$= (ix) - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + \cdots$$

$$= i\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right)$$

$$= i\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} = i\sinh(x)$$

29) $x^2 e^{3x}$

we know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

so $e^{3x} = \sum_{n=0}^{\infty} \frac{1}{n!} (3x)^n$

so $x^2 (e^{3x}) = x^2 \sum_{n=0}^{\infty} \frac{1}{n!} (3x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} 3^n x^{n+2}$

check

$R = \frac{1}{3}$

---

30) $\frac{x^2}{e^{3x}} = x^2 e^{-3x}$

$R=1$ $\quad R = \frac{1}{3}$

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ so $e^{-3x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-3x)^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n$

so $x^2 e^{-3x} = x^2 \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^{n+2}$

check $R = \frac{1}{3}$

---

31) $\cos 4x$.

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ so $\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (4x)^{2n}$

or $\sum_{n=0}^{\infty} \frac{(-1)^n 16^n}{(2n)!} x^{2n}$ $\qquad R = \infty$

---

32) $x \sin(2x)$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ so $\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1}$

multiply by x to obtain $x \sin(2x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+1)!} x^{2n+1}$

$R = \infty$ $\qquad = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n \cdot 2}{(2n+1)!} x^{2n+2}$

33) $\frac{x}{1+x} = x\left(\frac{1}{1+x}\right) = x\left(\frac{1}{1-(-x)}\right)$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad$ $R=1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ $\quad$ $R=1$

$\frac{x}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+1}$ $\quad$ $R=1$

---

34) $\frac{1}{1+x^2} = \frac{1}{1+(x^2)} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad$ $R=1$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, R=1$

replace x with $-x^2$

---

35) $\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $\quad$ $R=1$

---

36) replace x with $3x$ $\quad$ $R=\frac{1}{3}$

$\sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}$

---

37) $\sum_{n=3}^{6} n+2 = \sum_{s=0}^{3} (s+3)+2$

let $s=n-3$

$n=s+3$

---

38) $\sum_{n=2}^{8} n^2 = \sum_{n=0}^{6} (s+2)^2$

let $s=n-2$

$n=s+2$

39) $\sum_{n=4}^{\infty} 2^n = \sum_{s=0}^{\infty} 2^{s+4}$

$s = n-4$
$n = s+4$

---

40) $\sum_{n=2}^{\infty} x^n \qquad \sum_{s=0}^{\infty} x^{s+2}$

$s = n-2$
$n = s+2$

---

41) $\sum_{n=1}^{\infty} n a_n x^n = \sum_{s=0}^{\infty} (s+1) a_{s+1} x^{s+1}$

$s = n-1$
$n = s+1$

---

42) $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s$

$s = n-2$
$n = s+2$