21.5

$$\mathcal{L}(x^2) = \int_0^\infty e^{-sx} x^2 \, dx$$ use Int by parts

$$= x^2 \frac{e^{-sx}}{-s} - 2x \frac{e^{-sx}}{+s^2} + \frac{2e^{-sx}}{-s^3} \Big|_0^\infty$$

	D	I
+	x^2	e^{-sx}
−	$2x$	$e^{-sx}/-s$
+	2	e^{-sx}/s^2
	0	$e^{-sx}/-s^3$

when you put in ∞, you have to ask if the limit exists or not.

$$\lim_{x\to\infty} x^2 \frac{e^{-sx}}{-s} = \lim_{x\to\infty} \frac{x^2}{-s\, e^{+sx}} \; \frac{\infty}{\infty}$$ If $s>0$ then this limit is zero.

(use L'Hopital's rule twice)

$$\frac{2x}{-s^2 e^{sx}} \quad \frac{\infty}{\infty}$$

$$\frac{2}{-s^3 e^{sx}} \quad \frac{2}{\infty} = 0$$

Similarly, you get 0 with all 3 limits, provided $s>0$.

Plug in 0 for x and you get $\left(0 - 0 + \frac{2}{-s^3}\right)$

So we have $(0) - \left(0 - 0 + \frac{2}{-s^3}\right) = \boxed{\frac{2}{s^3}}$

21.29 $\quad f(x) = e^{2x}.$

$$\mathcal{L}(e^{2x}) = \int_0^\infty e^{-sx} e^{2x}\,dx = \underbrace{\int_0^\infty e^{(-sx+2x)}\,dx}_{\text{add exponents}}$$

$$= \int_0^\infty e^{(-s+2)x}\,dx$$

$$= \left.\frac{e^{(-s+2)x}}{-s+2}\right|_0^\infty$$

To get a limit, we need $-s+2<0$ or $2<s$ $\boxed{s>2}$

$$= (0) - \left(\frac{1}{-s+2}\right)$$

$$= \boxed{\frac{1}{s-2}}$$ provided $\underline{s>2}$

4.2

$$y' = y^2 x^3$$

we first seperate variables

$$\frac{dy}{dx} = y^2 x^3 \quad \text{or} \quad \underbrace{\frac{1}{y^2}\,dy}_{y\text{'s on left}} = \underbrace{x^3\,dx}_{x\text{'s on right}}$$

Now integrate both sides

$$\int \frac{1}{y^2}\,dy = \int x^3\,dx$$

$$\frac{y^{-1}}{-1} + \cancel{C} = \frac{x^4}{4} + C$$

only need 1 arbitrary constant.

Solve for y.

$$-\frac{1}{y} = \frac{x^4}{4} + C \qquad \boxed{y = \frac{-1}{\frac{x^4}{4} + C}}$$

OK here.

In schaums they write $k = 4C$ and simplify more. Here's why.

If we replace C with $\frac{K}{4}$, then

$$y = \frac{-1}{\frac{x^4}{4} + \frac{K}{4}} = \frac{-4}{x^4 + K}.$$

It doesn't matter which you use.

4.8

Solve $e^x dx - y\,dy = 0$, $y(0) = 1$.

well, schaum's just integrates each part. I'm going to advocate "Separate the variables by getting all x's on one side and all y's on the other." This gives

$e^x dx = y\,dy$. Now integrate

$$\int e^x dx = \int y\,dy$$

$$e^x + c = \frac{y^2}{2}.$$

So $2e^x + \boxed{2c} = y^2$

This is still an arbitrary constant, so lets just call it K.

$$2e^x + K = y^2.$$

or $y = \pm\sqrt{2e^x + K}$,

We need $y(0) = 1$, so when $x = 0$, $y = 1$.

$$1 = \pm\sqrt{2e^0 + K}$$

choose +

$$1 = \sqrt{2 + K}$$

$$1 = 2 + K$$

$$\boxed{-1 = K}$$

so $y = \sqrt{2e^x - 1}$

1.2 (Schaum's solution is sufficient)

1.7 To determine if it is a solution, we just have to differentiate and check the initial conditions.

ODE $y'' + 4y = 0$ IC $y(0) = 0$ $y'(0) = 1$.

a) $y_1 = \sin 2x$ $y_1' = 2\cos 2x$
$y_1'' = -4\sin 2x$

$y_1'' + 4y = -4\sin 2x + 4(\sin 2x) = 0$ ✓ It is a solution to the ODE.

However, $y_1(0) = 0$ ✓
$y_1'(0) = 2 \neq 1$ ⇐ Problem
Not a solution

b) $y_2 = x$
$y_2' = 1$
$y_2'' = 0$

$y_2'' + 4y_2 = 0 + 4x \neq 0$ Not a solution

c) $y_3 = \frac{1}{2}\sin 2x$
$y_3' = \cos 2x$
$y_3'' = -2\sin 2x$

$y_3'' + 4y_3 = -2\sin 2x + 4\left(\frac{1}{2}\sin 2x\right) = 0$ ✓
satisfies ODE.

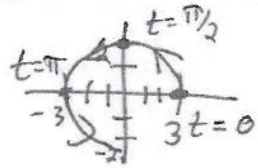
$y_3(0) = 0$ ✓
$y_3'(0) = \cos 0 = 1$ ✓

Yes it is a solution.

Review Problems Math 316 ~~problems 1-7~~ I renumbered them

1a $\vec{r}(t) = \langle 3\cos t, 2\sin t \rangle \quad 0 \le t \le 3\pi/2$.

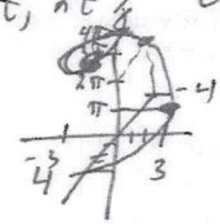
$D\vec{r} = \begin{bmatrix} -3\sin t \\ 2\cos t \end{bmatrix}$



3/4 of a circle.

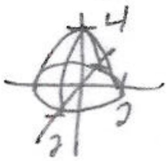
1b $\vec{r}(t) = \langle 4\cos t, 3\sin t, 2t \rangle \quad 0 \le t \le 2\pi$.

$D\vec{r}(t) = \begin{bmatrix} -4\sin t \\ 3\cos t \\ 2 \end{bmatrix}$

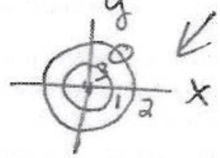


once around an elliptical helix

1c $f(x,y) = 4 - x^2 - y^2 \quad x^2 + y^2 \le 2$



paraboloid

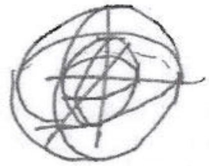


level curve

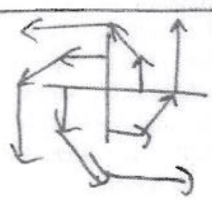
$Df(x,y) = [-2x \;\; -2y]$

1d $f(x,y,z) = x^2 + y^2 + z^2$ a level surface is $\left.\begin{array}{l} 1 = x^2+y^2+z^2 \\ 4 = x^2+y^2+z^2 \end{array}\right\}$ spheres.

$Df = [2x \;\; 2y \;\; 2z]$



1e $F(x,y) = \langle -y, x \rangle$



a spin field

$D\vec{F}(x,y) = \begin{array}{c c} x & y \end{array}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

1f $F(x,y,z) = \langle -x, -y, -z \rangle$

$DF(x,y,z) = \begin{array}{ccc} x & y & z \end{array}$
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

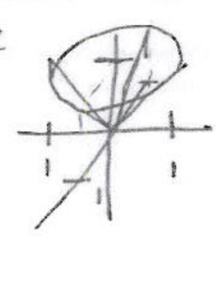
a radial field

every vector points to origin.

1g $\vec{r}(u,v) = \langle u\cos v, u\sin v, u \rangle$

parametric surface

$Dr(u,v) = \begin{array}{cc} u & v \end{array}$
$\begin{bmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 1 & 0 \end{bmatrix}$



a cone

(II)

2a

$f(x,y) = x^2 y \qquad x = \cos t,\ y = \sin t \qquad r(t) = \langle \cos t, \sin t \rangle.$

$$Df = \begin{bmatrix} 2xy & x^2 \end{bmatrix} \qquad Dr = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

$$Df \cdot Dr = \left[2xy(-\sin t) + x^2(\cos t) \right]$$

$$= \boxed{2(\cos t)(\sin t)(-\sin t) + (\cos t)^2 \cos t} = \frac{df}{dt}.$$

2b

$f = 3x - 4y \qquad r(t) = \langle 2u - v,\ 6uv \rangle$

$$Df = \begin{bmatrix} 3 & -4 \end{bmatrix} \qquad D\vec{r} = \begin{array}{cc} u & v \\ \left[\begin{matrix} 2 & -1 \\ 6v & 6u \end{matrix}\right. & \left.\right] \end{array} \qquad Df \cdot Dr = \begin{array}{cc} u & v \\ \left[3 \cdot 2 - 4 \cdot 6v \right. & \left. 3(-1) + (-4)(6u) \right] \end{array}$$

$$\text{so } f_u = 3 \cdot 2 + (-4)(6v) \qquad f_v = (3)(-1) + (-4)(6u)$$

2c

$f = 9 - x^2 - y^2 \qquad \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta \rangle \leftarrow$ polar coordinates

$$Df = \begin{array}{cc} x & y \\ \left[-2x \right. & \left. -2y \right] \end{array} \qquad D\vec{r} = \begin{array}{cc} r & \theta \\ \left[\begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix}\right] \end{array}$$

$$Df \cdot D\vec{r} = \begin{array}{cc} r & \theta \\ \left[(-2x)(\cos\theta) + (-2y)(\sin\theta) \right. & \left. (-2x)(-r\sin\theta) + (-2y)(r\cos\theta) \right] \end{array}$$

$$\frac{\partial f}{\partial r} = f_r = (-2(\overset{x}{r\cos\theta}))(\cos\theta) + (-2(\overset{y}{r\sin\theta}))(\sin\theta)$$

$$\frac{\partial f}{\partial \theta} = f_\theta = (-2(\overset{x}{r\cos\theta}))(-r\sin\theta) + (-2(\overset{y}{r\sin\theta}))(r\cos\theta)$$

2d

$\vec{r}(x,y) = \langle x, y, x^2 - y \rangle \qquad g(u,v) = \langle x, y \rangle = \langle 2u - v, 6uv \rangle$

$$D\vec{r}(x,y) = \begin{array}{cc} x & y \\ \left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 2x & -1 \end{matrix}\right] \end{array} \qquad Dg(u,v) = \begin{array}{cc} u & v \\ \left[\begin{matrix} 2 & -1 \\ 6v & 6u \end{matrix}\right] \end{array} \qquad D(r \circ g) = Dr \cdot Dg$$

$$\vec{r}_u = \begin{pmatrix} 2 \\ 6v \\ 4(2u-v) - 6v \end{pmatrix} \qquad \vec{r}_v = \begin{pmatrix} -1 \\ 6u \\ -2(2u-v) - 6u \end{pmatrix}$$

$$Dr \cdot Dg = \begin{bmatrix} (1)(2) + 0(6v) & (1)(-1) + (0)(6u) \\ (0)(2) + 1(6v) & (0)(-1) + (1)(6u) \\ (2x)(2) + (-1)(6v) & (2x)(-1) + (-1)(6u) \end{bmatrix} = \begin{array}{cc} u & v \\ \left[\begin{matrix} 2 & -1 \\ 6v & 6u \\ 4x - 6v & -2x - 6u \end{matrix}\right] \\ \vec{r}_u & \vec{r}_v \end{array}$$

Review 316 pg 3

2e $\vec{F} = \langle -y, x\rangle \qquad \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$

$$D\vec{F} = \begin{array}{cc} x & y \\ \end{array}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad D\vec{r} = \begin{array}{cc} r & \theta \end{array}\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

oops

$\cancel{D\vec{F}\cdot D\vec{r}} = \cancel{\begin{bmatrix} (0)(\cos\theta) + (-1)(\sin\theta) \\ (0)(-r\sin\theta) + (-1)(r\cos\theta) \end{bmatrix}}$

I do make mistakes occasionally. If you find some, please email me

$$D\vec{F}\cdot D\vec{r} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} 0\cdot\cos\theta - 1\sin\theta & 0(-r\sin\theta) - 1r\cos\theta \\ 1\cos\theta + 0\sin\theta & (-r\sin\theta) + 0\,r\cos\theta \end{bmatrix}$$

$$= \begin{array}{cc} r & \theta \end{array}\begin{bmatrix} -\sin\theta & -r\cos\theta \\ \cos\theta & -r\sin\theta \end{bmatrix}$$

So $\vec{F}_r = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \qquad \vec{F}_\theta = \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \end{bmatrix}$

2f $f = x^2 + 4yz \qquad T(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$

$$Df = \begin{bmatrix} 2x & 4z & 4y \end{bmatrix} \qquad DT = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Df\cdot DT = \begin{bmatrix} 2x\cos\theta + 4z\sin\theta & 2x(-r\sin\theta) + 4z(r\cos\theta) & 4y \end{bmatrix}$$

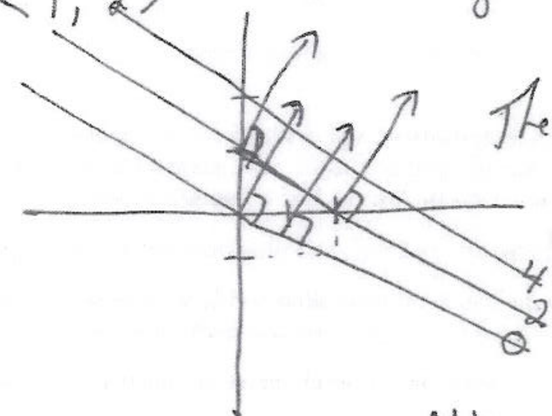
$$\begin{array}{ccc} \| & \| & \| \\ f_r & f_\theta & f_z \\ \| & \| & \| \\ 2(r\cos\theta)\cos\theta + 4z\sin\theta & 2(r\cos\theta)(-r\sin\theta) + 4zr\cos\theta & 4(r\sin\theta) \end{array}$$

3a $f = x + 2y$.

$\nabla f = \langle 1, 2 \rangle$

$0 = x + 2y$ means $y = -\frac{1}{2}x$

$2 = x + 2y$ means $y = 1 - \frac{1}{2}x$



The gradient is $\perp$ to level curves which are parallel lines for level curves

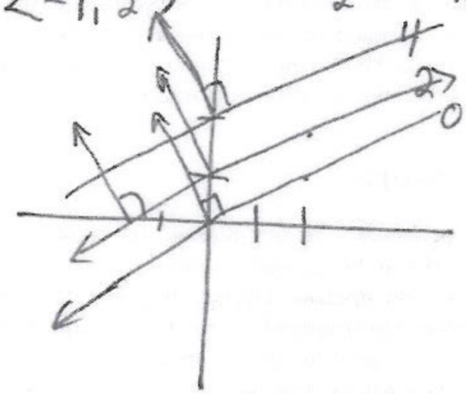
Notice gradient points in direction of increase

3b $f = -x + 2y$

$\nabla f = \langle -1, 2 \rangle$

$0 = -x + 2y \Rightarrow y = \frac{1}{2}x$

$2 = -x + 2y \Rightarrow y = 1 + \frac{1}{2}x$



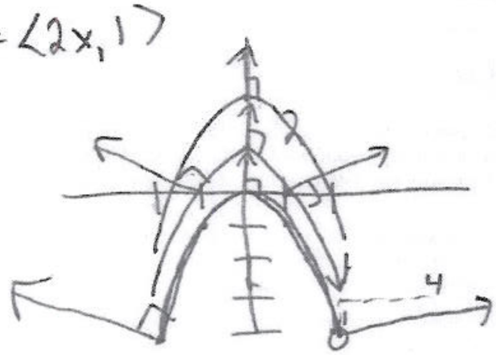
3c $f = x^2 + y$

$0 = x^2 + y$ mean $y = -x^2$

$1 = x^2 + y$ mean $y = 1 - x^2$

parabolas opening down

$\nabla f = \langle 2x, 1 \rangle$

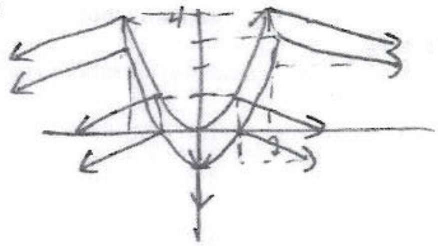


[redacted] 3d $f = x^2 - y$

$0 = x^2 - y \Rightarrow y = x^2$

$1 = x^2 - y \Rightarrow y = x^2 - 1$

$\nabla f = \langle 2x, -1 \rangle$

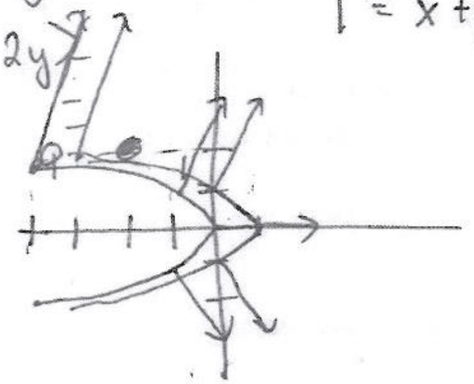


3e [redacted] $f = x + y^2$

$0 = x + y^2 \Rightarrow x = -y^2$

$1 = x + y^2 \Rightarrow x = 1 - y^2$ parabolas opening left.

$\nabla f = \langle 1, 2y \rangle$

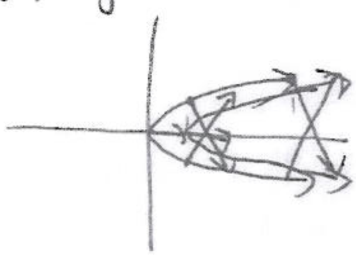


[redacted] 3f $f = x - y^2$

$\nabla f = \langle 1, -2y \rangle$

$0 = x - y^2 \Rightarrow x = y^2$

[redacted] $= x - y^2 \Rightarrow x = 1 + y^2$



4a $\langle 4x+5y,\ 5x+6y\rangle$

$M_y = 5 \quad N_x = 5$ ✓ They are equal

$\int 4x+5y\,dx \quad | \quad \int 5x+6y\,dy$

$\frac{4x^2}{2} + 5xy \quad | \quad 5xy + \frac{6y^2}{2}$

same

$f = \frac{4x^2}{2} + 5xy + \frac{6y^2}{2}$

4b ~~2xy~~ $\langle 2x-y,\ x+2y\rangle$ Not equal.

$M_y = -1 \quad N_x = 1$ No potential

4c $F = \langle \underset{M=}{e^{3x}+e^{2y}},\ \underset{N=}{2xe^{2y} - \frac{1}{1+y^2}}\rangle$

$M_y = 2e^{2y} \quad N_x = 2e^{2y}$ ✓ They are equal

$\int M\,dx = \frac{e^{3x}}{3} + xe^{2y} \quad | \quad \int N\,dy = xe^{2y} - \tan^{-1}(y)$.

equal

$f = \frac{e^{3x}}{3} + xe^{2y} - \tan^{-1}(y)$

4d $\langle 4x+5y,\ 5x+6y\rangle$ Same as 2a. Oops.

4e $\langle x+y+z,\ x+y+z,\ x+y+z\rangle$ ✓ equal pairs

$M_y = 1 \quad N_x = 1 \quad P_x = 1$

$M_z = 1 \quad N_z = 1 \quad P_y = 1$

$\int M\,dx = \frac{x^2}{2} + xy + xz \quad | \quad \int N\,dy = xy + \frac{y^2}{2} + yz \quad | \quad \int P\,dz = xz + yz + \frac{z^2}{2}$

$f = \frac{x^2}{2} + xy + xz + \frac{y^2}{2} + yz + \frac{z^2}{2}$

~~18~~ 4f $\langle 3y + yz,\ 3x + xz + 2y + 5z,\ zy + 5y \rangle$

$M_y = 3+z \quad N_x = 3+z \quad P_{\cancel{x}} = 0$

$M_z = y \quad N_z = x+5 \quad P_{\cancel{y}} = z+5$

problem problem

No potential. I meant xy instead of zy.

In this case

$\langle 3y + yz,\ 3x+xz+2y+5z,\ xy+5y \rangle$

$M_y = 3+z \quad N_x = 3+z \quad P_y = y$

$M_z = y \quad N_z = x+5 \quad P_y = x+5$

pairs are equal.

$\int M dx = 3xy + xyz$

$\int N dy = 3xy + xyz + \frac{2y^2}{2} + 5yz$

$\int P dz = xyz + 5xy$

(ignore all duplicates)

$f = 3xy + xyz + \frac{2y^2}{2} + 5yz$

5a-e repeats 4a-e just with differential notation.

~~26-30 repeat 20-24~~

change to x

~~31~~ 5f $(x+3y+yz)dx + (3x+xz+2y+5z)dy + (xy+5y+4z)dz.$

integrate. $\int dx$ | $\int dy$ | $\int dz$

$\frac{x^2}{2} + 3xy + xyz$ | $\cancel{3xy} + \cancel{xyz} + y^2 + 5yz$ | $\cancel{xyz} + \cancel{5yz} + \frac{4z^2}{2}$

$f = \frac{x^2}{2} + 3xy + xyz + y^2 + 5yz + \frac{4z^2}{2}$