28.4

$$8x^2y'' + 10xy' + (x-1)y = 0$$

divide by $8x^2$

$$y'' + \frac{10x}{8x^2}y' + \frac{x-1}{8x^2}y = 0$$

$$P(x) = \frac{10}{8x} \qquad Q(x) = \frac{x-1}{8x^2}$$

$xP = \frac{10}{8}$ ← analytic $\qquad x^2Q = \frac{x-1}{8}$ ← anaylitic

so 0 is a <u>regular singular point</u>

28.5 $\quad 8x^2y'' + 10xy' + (x-1)y = 0$

$$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\lambda}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+\lambda)x^{n+\lambda-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+\lambda)(n+\lambda-1)x^{n+\lambda-2}$$

$$8x^2 \xrightarrow[\text{2 to } n+\lambda-2]{\text{add}} \qquad + 10x \qquad + \underset{\text{2 series}}{(x-1)} \qquad = 0$$

$$\sum_{n=0}^{\infty} 8a_n(n+\lambda)(n+\lambda-1)x^{n+\lambda} + \sum_{n=0}^{\infty} 10a_n(n+\lambda)x^{n+\lambda} + \sum_{n=0}^{\infty} a_n x^{n+\lambda+1} - \sum_{n=0}^{\infty} a_n x^{n+\lambda}$$

$s+\lambda = n+\lambda+1$

$s-1 = n$

all have $n+\lambda$ but the third, so shift it. $\quad \sum_{s=1}^{\infty} a_{s-1}x^{s+1}$

so if $s=0$, skip the 3^{rd}

$s=0 \quad 8a_0(0+\lambda)(0+\lambda-1) + 10a_0(\lambda) + \text{skip} - a_0 = 0$

$a_0(8\lambda^2 - 8\lambda + 10\lambda - 1) = a_0\,\underline{\underline{(8\lambda^2 + 2\lambda - 1)}} = 0$ ← here it is.

(28.6) From 28.5 we have $8\lambda^2+2\lambda-1=0$ or $(4\lambda-1)(2\lambda+1)=0$ $\lambda=1/4, -1/2$

$y=\sum_{n=0}^{\infty} a_n x^{n+\lambda}$ $\quad$ $y'=\sum_{n=0}^{\infty} a_n (n+\lambda) x^{n+\lambda-1}$ $\quad$ $y''=\sum_{n=0}^{\infty} a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda-2}$

times $(x-1)$ $\quad$ times $10x$ $\quad$ times $8x^2$

$$\sum_{n=0}^{\infty} a_n x^{n+\lambda+1} - \sum_{n=0}^{\infty} a_n x^{n+\lambda} + \sum_{n=0}^{\infty} 10 a_n (n+\lambda) x^{n+\lambda} + \sum_{n=0}^{\infty} 8 a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda}$$

change.

$s=n+1$

$$\sum_{n=1}^{\infty} a_{s-1} x^{n+\lambda}$$

for $n\geq 1$

So

$$a_{n-1} - a_n + 10a_n(n+\lambda) + 8a_n(n+\lambda)(n+\lambda-1) = 0$$

$$a_n = \frac{-1\, a_{n-1}}{10(n+\lambda)+8(n+\lambda)(n+\lambda-1)-1}$$

move a_{n-1} to other side + divide by coefficients of a_n.

$$= \frac{-a_{n-1}}{8(n+\lambda)^2 - 8(n+\lambda) + 10(n+\lambda) - 1}$$

$+2(n+\lambda)$.

If $u=n+\lambda$

$8u^2+2u-1$

$(4u-1)(2u+1)$

$$= \frac{-a_{n-1}}{(4(n+\lambda)-1)(2(n+\lambda)+1)}$$

let $\lambda = 1/4$ $\quad$ let $\lambda=-1/2$

n=1 gives $a_1 = \dfrac{-a_0}{(4(1+\lambda)-1)(2(1+\lambda)+1)}$ $\quad$ $\dfrac{-a_0}{(5-1)(7/2)} = \dfrac{-a_0}{14}$ $\quad$ $\dfrac{-a_0}{(1)(2)}$

$2\cdot 7$

n=2 gives $a_2 = \dfrac{-a_1}{(4(2+\lambda)-1)(2(2+\lambda)+1)}$ $\quad$ $\dfrac{-a_1}{(8)(11/2)} = \dfrac{a_0}{2\cdot7\cdot4\cdot11}$ $\quad$ $\dfrac{-a_1}{(5)(4)} = \dfrac{a_0}{1\cdot2\cdot5\cdot4}$

$\dfrac{-a_2}{(12)(15/2)} = \dfrac{-a_0}{2\cdot7\cdot4\cdot11\cdot6\cdot15}$ $\quad$ $\dfrac{-a_0}{1\cdot2\cdot5\cdot4\cdot9\cdot6}$

$\dfrac{-a_3}{(16)(19/2)} = \dfrac{+a_0}{2\cdot7\cdot4\cdot11\cdot6\cdot15\cdot8\cdot19}$

$$y_1 = x^{1/4} a_0\left(1 - \frac{1}{2\cdot7}x + \frac{1}{2\cdot7\cdot4\cdot11}x^2 - \frac{1}{2\cdot7\cdot4\cdot11\cdot6\cdot15}x^3 + \cdots\right)$$

$$y_2 = x^{-1/2} a_0\left(1 - \frac{1}{1\cdot2}x + \frac{1}{1\cdot2\cdot5\cdot4}x^2 - \frac{1}{1\cdot2\cdot5\cdot4\cdot9\cdot6}x^3 + \cdots\right)$$

$$y = c_1 y_1 + c_2 y_2$$

28.9 $3x^2y'' - xy' + y = 0$

$$y'' - \frac{x}{3x^2}y' + \frac{1}{3x^2}y = 0$$

$$x \cdot \left(\frac{1}{3x}\right) = \frac{1}{3} \quad x^2 \cdot \left(\frac{1}{3x^2}\right) = \frac{1}{3}$$ both analytic so use Frobenius.

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \qquad y' = \sum_{n=0}^{\infty} a_n (n+\lambda) x^{n+\lambda-1} \qquad y'' = \sum_{n=0}^{\infty} a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda-2}$$

$\cdot (1)$ $\quad \cdot(-x)$ $\quad \cdot 3x^2$

$$\sum_{n=0}^{\infty} a_n x^{n+\lambda} - \sum_{n=0}^{\infty} a_n (n+\lambda) x^{n+\lambda} + \sum_{n=0}^{\infty} 3(n+\lambda)(n+\lambda-1) x^{n+\lambda}$$

We know for $n \geq 0$

$$a_n - a_n(n+\lambda) + 3(n+\lambda)(n+\lambda-1)a_n = 0$$

so $a_n\left(1-(n+\lambda)+3(n+\lambda)(n+\lambda-1)\right) = 0$ for all n.

This means $a_n = 0$ if $n > 0$

when $n = 0$ $\quad a_n(1 - \lambda + 3(\lambda)(\lambda-1)) = 0$

so since $a_0 \neq 0$, $\quad 1 - \lambda + 3\lambda^2 - 3\lambda = 0$

$$3\lambda^2 - 4\lambda + 1 = (3\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = 1 \text{ or } 1/3$$

Hence $a_n = 0$ for $n > 0$ and $\lambda = 1$ or $1/3$.

$y = x^1$ or $y = x^{1/3}$ are solutions

So $y = c_1 x + c_2 x^{1/3}$ is the general solution

27.12 $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$

It's ordinary @ 0 so use power series method. use m in power series instead of n

$y = \sum_{m=0}^{\infty} a_n x^m$ $\quad y' = \sum_{m=0}^{\infty} a_m m x^{m-1}$ $\quad y'' = \sum_{m=0}^{\infty} a_m m(m-1) x^{m-2}$

$(n(n+1))$ $\quad (-2x)$ $\quad (1-x^2)$

$$\sum_{m=0}^{\infty} n(n+1) a_m x^m + \sum -2a_m m x^m + \sum_{m=0,1,2}^{\infty} a_m m(m-1) x^{m-2} - \sum_{m=0}^{\infty} a_m m(m-1) x^m$$

$s = m-2$

$s+2 = m$

$$\sum_{s=0}^{\infty} n(n+1) a_s x^s + \sum_{s=0}^{\infty} -2a_s s x^s + \sum_{s=0}^{\infty} a_{s+2}(s+2)(s+1) x^s + \sum -a_s s(s-1) x^s$$

so $n(n+1)a_s - 2a_s s + a_{s+2}(s+2)(s+1) - a_s s(s-1)$

$$\boxed{a_{s+2} = \frac{2a_s s + a_s s(s-1) - n(n+1)a_s}{(s+2)(s+1)}}$$

ok here.

$$\text{or} = a_s \frac{(s^2 - s + 2s - n(n+1))}{(s+2)(s+1)}$$

$$= a_s \frac{s^2 + s - n(n+1)}{(s+2)(s+1)}$$

$$= a_s \frac{(s+(n+1))(s-n)}{(s+2)(s+1)}$$

notice that when $s = n$.

$$\boxed{a_{s+2} = 0}$$

29.4 If $n=4$. the ODE is $(1-x^2)y''-2xy'+20y=0$.

$P_4 = \frac{1}{8}(35x^4-30x^2+3)$ | (20)

To verify I just have to take 2 derivatives, plug them in, and see if I get 0.

Times by

$P_4' = \frac{1}{8}(35\cdot 4x^3 - 30\cdot 2x)$ | $(-2x)$

$P_4'' = \frac{1}{8}(35\cdot 4\cdot 3x^2 - 30\cdot 2)$ | $(1-x^2)$

$$20P_4 - 2xP_4' + (1-x^2)P_4''$$

$$= \frac{20}{8}(35x^4-30x^2+3) - \frac{2x}{8}(35\cdot 4x^3 - 30\cdot 2x) + \frac{(1-x^2)}{8}(35\cdot 4\cdot 3x^2 - 30\cdot 2)$$

$$= \frac{1}{8}\Big(x^4\big(\underbrace{\cancel{20\cdot 35 - 2\cdot 35\cdot 4 - 35\cdot 4\cdot 3}}_{35(20-8-12)=0}\big) + x^3\big(\cancel{0+0+0}\big) + x^2\big(20\cdot(-30)+2\cdot 30\cdot 2 + 35\cdot 4\cdot 3 + 30\cdot 2\big)$$

$$+ x\big(\cancel{0+0+0}\big) + \big(\underbrace{\cancel{20\cdot 3 - 0 + (-30\cdot 2)}}_{0}\big)\Big)$$

$$30(-20+4+2) + 35(4\cdot 3)$$

$$= (30)(-14) + 7\cdot 5\cdot 4\cdot 3$$

$$= -\underbrace{6\cdot 5\cdot 7\cdot 2}_{12\cdot 35} + \underbrace{7\cdot 5}_{35}\cdot\underbrace{4\cdot 3}_{12} = 0$$

all are zero

So yes its a solution

28.10 $x^2y'' + xy' + x^2y = 0$ check $1y'' + \frac{x}{x^2}y' + \frac{x^2}{x^2}y = 0$

singular

$x\frac{1}{x} = 1$ $x^2 \cdot 1 = x^3$

both analytic

regular singular

$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n+\lambda) x^{n+\lambda-1}$ $y'' = \sum a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda-2}$

(x^2) (x) (x^2)

$$\sum_{n=0}^{\infty} a_n x^{n+\lambda+2} + \sum_{n=0}^{\infty} a_n (n+\lambda) x^{n+\lambda} + \sum_{n=0}^{\infty} a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda}$$

let $s = n+2$

$s-2 = n$

$$\sum_{s=2}^{\infty} a_{s-2} x^{s+\lambda}$$

so

$s=0$ skip $+ a_0(\lambda) + a_0(\lambda)(\lambda-1) = 0$

$a_0(\lambda + \lambda^2 - \lambda) = 0$ $\lambda = 0, 0$

a double root

$s=1$ skip $+ a_1\underbrace{(1+\cancel{\lambda})}_{\text{ignore } \lambda \text{ now}} + a_0(0)(-1) = 0$

$\boxed{a_1 = 0}$

$s \geq 2$ $a_{s-2} + a_s(s+\cancel{\lambda}) + a_s(s+\cancel{\lambda})(s+\cancel{\lambda}-1) = 0$

$$a_s = \frac{-a_{s-2}}{s + s(s-1)} = \frac{-a_{s-2}}{s^2}$$

$s=2$ $a_2 = \frac{-a_0}{4} = \frac{-a_0}{2^2}$

$s=3$ $a_3 = \frac{-a_1}{9} = 0$

$s=4$ $a_4 = \frac{-a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}$

$a_5 = 0$

$a_6 = \frac{-a_4}{6^2} = \frac{-a_0}{2^2 \cdot 4^2 \cdot 6^2}$

so $y = a_0\left(1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 - \frac{1}{2^2 4^2 6^2}x^6 + \frac{1}{2^2 4^2 6^2 8^2}x^8 - \cdots\right)$

28.14

$$x^2 y'' + (x^2 - 2x) y' + 2y = 0$$

$$1y'' + \frac{x^2-2x}{x^2} y' + \frac{2}{x^2} y = 0.$$

singular

$x\left(1-\frac{2}{x}\right) = x-2 \qquad x^2 \frac{2}{x^2} = 2$

analytic

So regular singula

$y = \sum a_n x^{n+\lambda}$ $\qquad y' = \sum a_n (n+\lambda) x^{n+\lambda-1}$ $\qquad y'' = \sum a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda-2}$

$(2) \qquad (x^2-2x) \qquad (x^2)$

$$\sum_{n=0}^{\infty} 2a_n x^{n+\lambda} + \sum_{n=0}^{\infty} a_n (n+\lambda) x^{n+\lambda+1} - \sum_{n=0}^{\infty} 2a_n (n+\lambda) x^{n+\lambda} + \sum_{n=0}^{\infty} a_n (n+\lambda)(n+\lambda-1) x^{n+\lambda}$$

$s = n+1 \quad n = s-1$

$$\sum_{s=1}^{\infty} a_{s-1} (s+\lambda-1) x^{s+\lambda}$$

$s=0 \qquad 2a_0 + \text{skip} - 2a_0(\lambda) + a_0(\lambda)(\lambda-1) = 0$

$$2 - 2\lambda + \lambda^2 - \lambda = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$$

$2, 1$

use $\lambda = 2$. the larger

$s \geq 1 \qquad 2a_s + a_{s-1}(s+2-1) - 2a_s(s+2) + a_s(s+2)(s+1) = 0$

$$a_s = \frac{-a_{s-1}(s+1)}{2 - 2(s+2) + (s+2)(s+1)} = \frac{-a_{s-1}(s+1)}{\cancel{2} - 2s \cancel{-4} + s^2 + 3s \cancel{+2}} = \frac{-a_{s-1}\cancel{(s+1)}}{s\cancel{(s+1)}} = \frac{-a_{s-1}}{s}$$

$s^2 + s = s(s+1)$

So $s=1 \quad a_1 = \frac{-a_0}{1}$

$s=2 \quad a_2 = \frac{-a_1}{2} = \frac{a_0}{2!}$

$s=3 \quad a_3 = \frac{-a_2}{3} = \frac{-a_0}{3!}$

So $\boxed{y = a_0 x^2 \left(1 - \frac{1}{1} x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots\right)}$

pull a_0 out.

OK Here

$$= a_0 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = a_0 x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$= \boxed{a_0 x^2 e^{-x}}$$

30.4

Prove $\Gamma(p+1) = p\Gamma(p) \quad p>0$

$$\Gamma(p+1) = \int_0^\infty x^{p+1-1} e^{-x}\,dx = \int_0^\infty x^p e^{-x}.$$

$$= x^p(-e^{-x})\Big|_0^\infty + \int_0^\infty p x^{p-1} e^{-x}\,dx$$

D	I
$+\ x^p$	e^{-x}
$-\ p x^{p-1}$	$-e^{-x}$

remember $\frac{x^p}{e^x} \to 0$ no matter what $p>0$ is. e^x grows faster than x^p.

$$\frac{x^p}{-e^{+x}}\Big|_0^\infty = 0-0$$

$$= \int_0^\infty p x^{p-1} e^{-x}\,dx = p\int_0^\infty x^{p-1} e^{-x}\,dx = p\Gamma(p)$$

30.6

Prove $\Gamma(n+1) = n!$ if n is a positive integer

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = \frac{e^{-x}}{-1}\Big|_0^\infty = \frac{e^{-\infty}}{-1} - \left(\frac{1}{-1}\right) = 0+1 = 1$$

so $\Gamma(0+1) = 1 = 0!$

Now use 30.4, the fact that $\Gamma(p+1) = p\Gamma(p)$.

You can do it without induction.

$$\text{so } \Gamma(1+1) = 1\Gamma(1) = 1\cdot 1 = 1!$$

$$\Gamma(2+1) = 2\Gamma(2) = 2\cdot 1! = 2!$$

$$\Gamma(3+1) = 3\Gamma(3) = 3\cdot 2! = 3!$$

$$\Gamma(4+1) = 4\Gamma(4) = 4\cdot 3! = 4!$$

$$\Gamma(n+1) = n\Gamma(n-1+1) = n(n-1)\Gamma(n-2+1) = n(n-1)(n-2)\Gamma(n-3+1)$$

$$= \cdots = n(n-1)(n-2)\cdots(3)(2)1\Gamma(1)$$

$$= n!\cdot 1 = n!$$

30.8

$$\int_0^\infty e^{-x^2}dx$$

let $x^2 = u$ $\quad x = \pm\sqrt{u}$ since $x > 0$ we know $x = +\sqrt{u}$

$2x\,dx = du$

$dx = \frac{1}{2x}du$

$dx = \frac{1}{2\sqrt{u}}du$

$x = 0$ means $u = 0$

$x = \infty$ mean $u = \infty$

$$\int_{0=u}^{\infty=u} e^{-u}\frac{1}{2\sqrt{u}}du = \int_0^\infty \frac{1}{2}u^{-1/2}e^{-u}du$$

$$= \frac{1}{2}\int_0^\infty u^{-1/2}e^{-u}du$$

$$= \frac{1}{2}\int_0^\infty u^{1/2-1}e^{-u}du = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \boxed{\frac{1}{2}\sqrt{\pi}}$$

Here's why.

<u>Extra:</u>

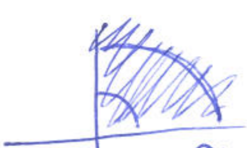
As a side note: How do you compute $\int_0^\infty e^{-x^2}dx$? The Idea is as follows:

Let $I = \int_0^\infty e^{-x^2}dx$. So $I^2 = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-x^2}dx\right)$

change dummy variable to y.

$$= \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-x^2}\cdot e^{-y^2}dy\,dx$$

$$= \int_0^\infty\int_0^\infty e^{-x^2-y^2}dy\,dx$$ Now switch to polar coordinates.



$0 \le \theta \le \pi/2$

$0 \le r \le \infty$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta$$

$x^2 + y^2 = r^2$

$dy\,dx = r\,dr\,d\theta$

$$= \frac{\pi}{2}\left(-\frac{1}{2}e^{-r^2}\Big|_0^\infty\right) = \frac{\pi}{2}\left(0 + \frac{1}{2}\right) = \pi/4$$

So $I^2 = \pi/4$ or $I = \sqrt{\pi}/2$. Hence $\frac{\sqrt{\pi}}{2} = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ or $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

30.9 $x^2y'' + xy' + (x^2-p^2)y = 0$ $\qquad y'' + \frac{x}{x^2}y' + \frac{x^2-p^2}{x^2}y = 0$

$x\cdot\frac{1}{x} = 1 \qquad x^2\left(\frac{x^2-p^2}{x^2}\right) = x^2-p^2$

analytic so

regular singular

$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \qquad y' = \sum_{n=0}^{\infty} a_n(n+\lambda)x^{n+\lambda-1} \qquad y'' = \sum a_n(n+\lambda)(n+\lambda-1)x^{n+\lambda-2}$

$(x^2-p^2) \qquad (x) \qquad (x^2)$

$$\sum_{n=0} a_n x^{n+\lambda+2} - \sum_{n=0} p^2 a_n x^{n+\lambda} + \sum_{n=0}^{\infty} a_n(n+\lambda)x^{n+\lambda} + \sum_{n=0}^{\infty} a_n(n+\lambda)(n+\lambda-1)x^{n+\lambda}$$

shift

$s = n+2$

$s-2 = n$

$\sum_{s=2}^{\infty} a_{s-2}x^{s}$

$s=0$ skip $-p^2a_0 + a_0\lambda + a_0\lambda(\lambda-1) = 0$

$-p^2 + \cancel{\lambda} + \lambda^2 - \cancel{\lambda} = 0$

$(\lambda-p)(\lambda+p) = 0 \quad \lambda = p, -p$

use $\boxed{\lambda = p}$

$s=1$ skip $-p^2a_1 + a_1(1+p) + a_1(1+p)(1+p-1) = 0$

$a_1(\cancel{-p^2} \cancel{+1} \cancel{+p} + 1^2 + 1p \cancel{-1} + p\cdot 1 \cancel{+p^2} \cancel{-p}) = 0$

$a_1(1^2+2p) = 0 \Rightarrow \boxed{a_1 = 0} \Rightarrow a_3 = 0 = a_5 = a_7 = a_9 = a_{11}$

$s \geq 2 \qquad a_{s-2} - p^2a_s + a_s(s+p) + a_s(s+p)(s+p-1) = 0$

$$a_s = \frac{-a_{s-2}}{(s+p)(s+p-1)+(s+p)-p^2} = \frac{-a_{s-2}}{(s+p)(s+p-1+1)-p^2}$$

$$= \frac{-a_{s-2}}{(s+p)^2-p^2} = \frac{-a_{s-2}}{(s+p+p)(s+p-p)} = \boxed{\frac{-a_{s-2}}{(s+2p)s} = a_s}$$

$s=2 \quad a_2 = \frac{-a_0}{(2+2p)(2)} = \frac{-a_0}{2\cdot 2(1+p)}$

$s=4 \quad a_4 = \frac{-a_2}{(4+2p)4} = \frac{a_0}{2(2+p)\cdot 4\cdot 2\cdot 2(1+p)}$

$s=6 \quad a_6 = \frac{-a_4}{(6+2p)6} = \frac{-a_0}{2(2+p)\underbrace{4}_{2\cdot 2}\cdot 2\cdot 2(1+p)\cdot 2(3+p)\cdot \underbrace{6}_{2\cdot 3}}$

$$y = a_0\left(1 - \frac{1}{2^2(1+p)}x^2 + \frac{1}{2^4\cdot 2(1+p)(2+p)}x^4 - \frac{1}{2^6 3!(1+p)(2+p)(3+p)}x^6 + \cdots\right)$$