Book 1-3

① $\frac{8(s+3)}{(s+3)^2+4} - \frac{4\cdot 5}{(s-4)^2+25} + \frac{3\cdot 5!}{(s-7)^6}$

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② $L(x\,u(x-4) + \delta(x-6)) = \boxed{\left(\frac{1}{s^2}+\frac{4}{s}\right)e^{-4s} + e^{-6s}}$

$\delta(x-6) \xrightarrow{L} e^{-6s}$

$x\,u(x-4) \rightarrow (\underline{x-4}+4)u(\underline{x-4})$

$L(t+4) = \frac{1}{s^2} + \frac{4}{s}$

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③ $L(e^{3x}u(x-2) + 7\delta(x-4)) = \boxed{\frac{e^6}{s-3}e^{-2s} + 7e^{-4s}}$

$7\delta(x-4) \xrightarrow{L} 7e^{-4s}$

$e^{3x}u(x-2) \rightarrow e^{3(\underline{x-2}+2)}u(\underline{x-2})$

$e^{3(t+2)} = e^{3t}e^{6}$

$\xrightarrow{L} \frac{e^6}{s-3}$

$4\frac{s+3-3}{(s+3)^2+25} + \frac{2}{(s-2)^2}e^{-3s} = \frac{s+3}{(s+3)^2+25} - \frac{3\cdot 5}{((s+3)^2+25)\cdot 5} + \left(\frac{2}{(s-2)^2}\right)e^{-3s}$

$\frac{2}{(s-2)^2} \rightarrow 2te^{2t}$

$\boxed{\cos(5t)e^{-3t} - \frac{3}{5}\sin(5t)e^{-3t} + 2(t-3)e^{2(t-3)}u(t-3)}$

$5\left(\frac{s}{s^2+4s+13}\right)e^{-4s} = \left(\frac{s+2-2}{(s+2)^2+9}\right)e^{-4s} = \left(\frac{s+2}{(s+2)^2+9} - \frac{2\cdot 3}{((s+2)^2+9)\cdot 3}\right)e^{-4s}$

$s^2+4s+13 \quad +4-4$

$\cos(3t)e^{-2t} - \frac{2}{3}\sin 3t e^{-2t}$ replace t with $t-4$

$\boxed{\left(\cos(3(t-4))e^{-2(t-4)} - \frac{2}{3}\sin(3(t-4))e^{-2(t-4)}\right)u(t-4)}$

Book 6

$$\frac{1}{s(s^2+1)}e^{-5s}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$$0s^2+0s+1 = s^2(A+B) + s(C) + A$$

$0 = A+B$

$0 = C$

$1 = A$

$B=-1$

$$\left(\frac{1}{s} + \frac{-s}{s^2+1}\right)e^{-5s}$$

$1 - \cos(t)$ replace t with $t-5$ so

$$\boxed{(1-\cos(t-5))\,u(t-5)}$$

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7) $$\frac{1}{s^2(s^2+1)}e^{-3s}$$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$0s^3+0s^2+0s+1 = s^3(A+C) + s^2(B+D) + s(A) + B$$

$0=A+C$ $C=0$

$0=B+D$ $D=-1$

$0=A$

$1=B$

$$\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-3s}$$ replace t with $t-3$

$t - \sin(t)$

$$\boxed{((t-3) - \sin(t-3))\,u(t-3)}$$

Book #8

$$\frac{2s+1}{(s-1)^2(s+1)}e^{-7s} \quad \Bigg| \quad \frac{2s+1}{(s-1)^2(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1}$$

$$2s+1 = A(s-1)(s+1) + B(s+1) + C(s-1)^2$$

let $s=1$ $\quad 3 = 0 + 2B + 0 \quad$ so $B = 3/2$

let $s=-1$ $\quad -1 = 0 + 0 + 4C \quad$ so $C = -1/4$

let $s=0$ $\quad 1 = -A + B + C \quad$ so $A = B + C - 1$

$$= 3/2 - 1/4 - 1$$

$$= \frac{6-1-4}{4} = 1/4$$

$$\left(\frac{1/4}{s-1} + \frac{3/2}{(s-1)^2} - \frac{1/4}{s+1}\right)e^{-7s}$$

$$\frac{1}{4}e^{t} + \frac{3}{2}te^{t} - \frac{1}{4}e^{-t}$$

replace t with $t-7$

$$\boxed{\left(\frac{1}{4}e^{t-7} + \frac{3}{2}(t-7)e^{t-7} - \frac{1}{4}e^{-(t-7)}\right)u(t-4)}$$

Book #9

$$\frac{1}{(s-1)(s+2)(s-3)}e^{-4s} \qquad \frac{1}{(s-1)(s+2)(s-3)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$1 = A(s+2)(s-3) + B(s-1)(s-3) + C(s-1)(s+2)$$

let $s=1$ $\quad 1 = A(3)(-2) + 0 + 0 \Rightarrow A = -1/6$

let $s=-2$ $\quad 1 = 0 + B(-3)(-5) + 0 \Rightarrow B = 1/15$

let $s=3$ $\quad 1 = 0 + 0 + C(2)(5) \Rightarrow C = 1/10$

$$\left(\frac{-1/6}{s-1} + \frac{1/15}{s+2} + \frac{1/10}{s-3}\right)e^{-4s}$$

$$-1/6\, e^{+t} + 1/15\, e^{-2t} + \frac{1}{10}e^{3t}$$

replace t with $t-4$

$$\left(-1/6\, e^{t-4} + \frac{1}{15}e^{-2(t-4)} + \frac{1}{10}e^{3(t-4)}\right)u(t-4)$$

Book 10

$m=1 \quad c=0 \quad k=4 \quad r=u(t-1) \quad y(0)=1 \quad y'(0)=0$

$$y''+4y=u(t-1)$$

$$s^2Y-s-0+4Y=\frac{1}{s}e^{-s}$$

$$Y(s^2+4)=s+\frac{1}{s}e^{-s} \Rightarrow Y=\frac{s}{s^2+4}+\frac{1}{s(s^2+4)}e^{-s}$$

use convolution.

$$y=\cos(2t)+\left(1*\frac{1}{2}\sin(2t)\right)u(t-1)$$

replace t with t-4

$$\frac{1}{2}\sin(2t)*1=\int_0^t\frac{1}{2}\sin(2p)\cdot 1\,dp$$

$$=-\frac{1}{4}\cos(2p)\Big|_0^t$$

$$=-\frac{1}{4}\cos(2t)+\frac{1}{4}.$$

so $y=\cos 2t+\left(\frac{1}{4}-\frac{1}{4}\cos(2(t-1))\right)u(t-1)$

You could instead use a partial fraction decomposition.

$$Y=\frac{s}{s^2+4}+\frac{1}{s(s^2+4)}e^{-s} \qquad \frac{1}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}$$

$$1=A(s^2+4)+(Bs+C)s$$

$$1=s^2(A+B)+s(C)+4A$$

$0=A+B \qquad B=-\frac{1}{4}$

$0=C$

$1=4A \qquad A=\frac{1}{4}$

$$=\frac{s}{s^2+4}+\left(\frac{1/4}{s}-\frac{1/4\,s}{s^2+4}\right)e^{-s}$$

$$\boxed{\cos(2t)+\left(\frac{1}{4}-\frac{1}{4}\cos(2(t-1))\right)u(t-1)}$$

Either way works well.
Would you rather do an integral (convolution) or a partial fraction decomposition

Book #11

$$y'' + 4y = \delta(t-3) \quad y(0) = 1 \quad y'(0) = 0$$

$$s^2 Y - s + 4Y = e^{-3s}$$

$$Y = \frac{s + e^{-3s}}{s^2+4} = \frac{s}{s^2+4} + \frac{2 \cdot 1}{2(s^2+4)} e^{-3s}$$ replace t with $t-3$

$$y = \cos(2t) + \frac{1}{2}\sin(2(t-3))\,u(t-3)$$

Book #12

$$y'' + 4y = 7u(t-3) \quad y(0) = 1 \quad y'(0) = 0$$

$$s^2 Y - s + 4Y = \frac{7}{s} e^{-3s}$$

$$Y = \frac{s}{s^2+4} + 7\underbrace{\left(\frac{1}{s(s^2+4)}\right)}_{\text{from \#10 we know } \mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right) = \frac{1}{4} - \frac{1}{4}\cos(2t)} e^{-3s}$$

$$y = \cos(2t) + 7\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-3))\right)u(t-3)$$

#13

$$y'' + 4y = 7u(t-3) + 11\delta(t-5) \quad y(0) = 1 \quad y'(0) = 0$$

$$s^2 Y - s + 4Y = \frac{7}{s}e^{-3s} + 11e^{-5s}$$

$$Y = \frac{s}{s^2+4} + 7\left(\frac{1}{s(s^2+4)}\right)e^{-3s} + \frac{11e^{-5s}}{s^2+4}\cdot\frac{2}{2}$$

$$y = \cos(2t) + \underbrace{7\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-3))\right)u(t-3)}_{\text{from \#12}} + \frac{11}{2}\sin(2(t-5))\,u(t-5)$$

Book #14

$$y'' + 4y = \underline{7t\,u(t-3)} \quad y(0) = 1 \quad y'(0) = 0$$

$$7(\underline{t-3} + 3)\,u(\underline{t-3})$$

$$L(7(x+3)) = 7\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$s^2Y - s + 4Y = 7\left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-3s}$$

$$Y = \underbrace{\frac{s}{s^2+4} + \frac{7}{s(s^2+4)}e^{-3s}}_{\text{known from \#12}} + \underbrace{\frac{7}{s^2(s^2+4)}e^{-3s}}_{\text{need to find.}}$$

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$1 = A(s)(s^2+4) + B(s^2+4) + (Cs+D)s^2$$

$$1 = s^3(A+C) + s^2(B+D) + s(4A) + 4B$$

$$0 = A + C \quad C = 0$$

$$0 = B + D \quad D = -1/4$$

$$0 = 4A \quad A = 0$$

$$1 = 4B \quad B = 1/4$$

$$Y = \frac{s}{s^2+4} + \frac{7}{s(s^2+4)}e^{-3s} + 7\left(\frac{1/4}{s^2} - \frac{1/4 \cdot 2}{s^2+4 \cdot 2}\right)e^{-3s}$$

$$\boxed{y = \cos(2t) + 7\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-3)) + \frac{1}{4}(t-3) - \frac{1}{8}\sin(2(t-3))\right)u(t-3)}$$

The computer will simplify this

to $\frac{1}{4} + \frac{1}{4}t - \frac{3}{4} = \frac{1}{4}t - \frac{1}{2} = \frac{1}{4}(t-2)$

Book #15

$y'' + 4y = 7 \quad y(0) = 1 \quad y'(0) = 0$

$s^2 Y - s + 4Y = \frac{7}{s}$

$Y = \frac{s}{s^2+4} + \frac{7}{s(s^2+4)}$ ↙ from #12

$y = \cos(2t) + 7\left(\frac{1}{4} - \frac{1}{4}\cos(2t)\right)$ no need to replace t with anything

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#16 $y'' + 4y = 7 \quad y(\pi) = 1 \quad y'(\pi) = 0$

Replace in $y(t)$ each t with $u = t - \pi$.

So $\hat{y}(u)$ is now our function.

$\hat{y}(0)$ means $0 = t - \pi$ or $t = \pi$.

so $\hat{y}(0) = 1 \quad \hat{y}'(0) = 0$.

$\hat{y}'' + 4\hat{y} = 7$

$s^2 \hat{Y} - s + 4\hat{Y} = 7/s$

$\hat{Y} = \frac{s}{s^2+4} + \frac{7}{s(s^2+4)}$

$\hat{y}(u) = \cos(2u) + 7\left(\frac{1}{4} - \frac{1}{4}\cos(2u)\right)$ replace <u>u with $t-\pi$</u>

$$\boxed{y(t) = \cos(2(t-\pi)) + 7\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-\pi))\right)}$$

Mainly its like # 15, but each t was replaced with $t-\pi$.

20) $y'' + 3y' + 2y = 4u(t-1) + 10\delta(t-2)$ $\quad y(0)=0 \quad y'(0)=0$

$$s^2 Y + 3sY + 2Y = \frac{4e^{-s}}{s} + 10e^{-2s}$$

$s^2+3s+2 = (s+2)(s+1)$

$$Y = \frac{4}{s(s+2)(s+1)}e^{-s} + \frac{10}{(s+2)(s+1)}e^{-2s}$$

$$\frac{4}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$\frac{10}{(s+2)(s+1)} = \frac{D}{s+2} + \frac{E}{s+1}$$

$$4 = A(s+2)(s+1) + B(s)(s+1) + C(s)(s+2)$$

$10 = D(s+1) + E(s+2)$

$s=-1 \quad 10 = 0 + E(1)$

$s=-2 \quad 10 = D(-1) + 0$

$E = 10$

$D = -10$

let $s=0 \quad 4 = A(2)(1) + 0 + 0 \quad A = 2$

$s=-2 \quad 4 = 0 + B(-2)(-1) + 0 \quad B = 2$

$s=-1 \quad 4 = 0 + 0 + C(-1)(+1) \quad C = 4$

$$Y = \left(\frac{2}{s} + \frac{2}{s+2} + \frac{4}{s+1}\right)e^{-s} + \left(\frac{-10}{s+2} + \frac{10}{s+1}\right)e^{-2s}$$

$$y(t) = \left(2 + 2e^{-2(t-1)} + 4e^{-(t-1)}\right)u(t-1) + \left(-10e^{-2(t-2)} + 10e^{-(t-1)}\right)u(t-2)$$

That's it notice there are 2 Heavisides

22) $L = 0 \quad R = 2 \quad C = 1/5. \quad E = 12u(t-2) \quad I(0) = 0$

$E' = 12\delta(t-2)$

$\quad ' + RI' + \frac{1}{c} I = E'$

no inductor.

$$2I' + 5I = 12\delta(t-2).$$

$$2(sY - 0) + 5Y = 12e^{-2s}$$

$$Y = \frac{12e^{-2s}}{2s+5} = \frac{\overset{6}{\cancel{12}}\, e^{-2s}}{\cancel{2}(s + 5/2)}$$

$$y = 6e^{-5/2(t-2)}\, u(t-2)$$

Ignore the capacitor.

23] $L=1,\ R=2,\ C=0$ $\quad E(t)=12u(t-2)\quad I(0)=0.$

Use the equation $LI'+RI=E(t)$. (don't take derivatives)

$$1I'+2I=12u(t-2)$$

$$sY-0+2Y=12\frac{e^{-2s}}{s}$$

$$Y=\frac{12e^{-2s}}{s(s+2)}=\left(\frac{A}{s}+\frac{B}{s+2}\right)e^{-2s}$$

$$=\left(\frac{6}{s}-\frac{6}{s+2}\right)e^{-2s}$$

$$\frac{12}{s(s+2)}=\frac{A}{s}+\frac{B}{s+2}$$

$$12=A(s+2)+Bs$$

Let $s=0$ $\quad 12=2A\quad A=6$

Let $s=-2$ $\quad 12=0-2B\quad B=-6$

invert

$$\boxed{I(t)=\left(6-6e^{-2(t-2)}\right)u(t-2)}$$

25) $L=1$ $R=2$ $C=1/5$ $E=12u(t-2)$ $I(0)=0$

$Q(0)=0.$

$$1y' + 2y + 5\int y\,dt = 12u(t-2)$$

@ 0 $y'(0)+0+0=0$ so $\underline{y'(0)=0.}$

now differentiate

$$y'' + 2y' + 5y = 12\delta(t-2)$$

$$s^2Y + 2sY + 5Y = 12e^{-2s}$$

Complete the square.

$\underline{s^2+2s+1}-1+5$

$$Y = \frac{12e^{-2s}}{s^2+2s+5} = \frac{12}{(s+1)^2+4}e^{-2s}$$

$$\frac{12}{(s+1)^2+4}\cdot\frac{2}{2}$$

$$y(t) = \frac{12}{2}\sin(2(t-2))e^{-1(t-2)}u(t-2)$$

27) This is the hardest one of all. $\quad y = I$

Find $y'(0)$ first

$$y' + 2y + 5\int y\,dt = 4\cos 3t$$

$y(0)=0$

$\int y\,dt @ 0 = 0$

$$y'(0) + 0 + 0 = 4 \quad \text{so } \underline{y'(0) = 4} \text{ Important}$$

Now differentiate

$$y'' + 2y' + 5y = -12\sin 3t$$

$$(s^2 Y - s0 - 4) + 2sY + 5Y = \frac{-12 \cdot 3}{s^2+9}$$

$$Y(s^2+2s+5) = \frac{-36}{s^2+9} + 4$$

Remember to complete the square.

$$Y = \frac{-36}{(s^2+2s+5)(s^2+9)} + \frac{4}{s^2+2s+5} \text{ OK.}$$

s^2+2s+5
$= (s+1)^2+4$

We need a partial fraction decomposition.

$$L^{-1}\left(\frac{4}{(s+1)^2+4} \cdot \frac{2}{2}\right) = \frac{4}{2}\sin(2t)$$

$$\frac{-36}{\text{stuff}} = \frac{As + B}{(s+1)^2+4} + \frac{Cs+D}{s^2+9}$$

$$-36 = (As+B)(s^2+9) + (Cs+D)(s^2+2s+5)$$

use a calculator now.

$$-36 = As^3 + 9As + Bs^2 + 9B + Cs^3 + 2s^2C + 5Cs + Ds^2 + 2Ds + 5D$$

s^3	s^2	s	1
A	B	9A	9B
C	2C	5C	
	D	2D	5D
0	0	0	-36

$A + C = 0 \quad \boxed{A = -C}$

$B + 2C + D = 0$

$9A + 5C + 2D = 0 \qquad -4C + 2D = 0 \text{ so } \boxed{D = 2C}$

$9B + 5D = -36$

$B + 2C + 2C = 0$ or $B + 4C = 0$

$9B + 10C = -36$

$$B = \frac{\begin{vmatrix} 0 & 4 \\ -36 & 10 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ 9 & 10 \end{vmatrix}} = \frac{36 \cdot 4}{-26}$$

$$C = \frac{\begin{vmatrix} 1 & 0 \\ 9 & -36 \end{vmatrix}}{-26} = \frac{-36}{-26}$$

Next page

$C = \frac{+36}{+26}$ $B = \frac{144}{-26}$ $A = -\frac{36}{26}$ $D = \frac{36}{13}$

$C = \frac{18}{13}$ $B = -\frac{72}{13}$ $A = \frac{-18}{13}$ $D = \frac{36}{13}$

$4 \cdot 13 = 52$
$52 - 18 = 34$
$52 - 72 = -20$

$$\text{So } Y = \frac{-18/13\, s + \frac{-72}{13}}{(s+1)^2+4} + \frac{18/13\, s + \frac{36}{13}}{s^2+9} + \frac{4}{(s+1)^2+4}$$

$$= \frac{(\cancel{34}^{-18}/13)s - \frac{\cancel{72}^{-20}}{13}}{(s+1)^2+4} + \frac{18/13\, s + \frac{36}{13}}{s^2+9}$$

Now make s be an s+1.

$+18 - 20 = -2$

$$\frac{\cancel{34}^{-18}}{13}(s+1-1) - \frac{\cancel{72}^{-20}}{13} = \frac{\cancel{34}^{-18}}{13}(s+1) + \frac{\cancel{34}^{18}}{13} - \frac{\cancel{72}^{20}}{13} = \frac{-2}{13}$$

$$\text{So } Y = \frac{\frac{\cancel{34}^{-18}}{13}(s+1) - \frac{2}{13}}{(s+1)^2+4} + \frac{\left(\frac{18}{13}\right)s + \frac{36}{13}}{s^2+9}$$

$$y(t) = \frac{\cancel{34}^{18}}{13}\cos(2t)e^{-t} + \left(\frac{-2}{13}\right)\left(\frac{1}{2}\right)e^{-t}\sin 2t + \frac{18}{13}\cos(3t) + \frac{36}{13}\left(\frac{1}{3}\right)\sin(3t)$$

$$y = \boxed{e^{-t}\left(\frac{18}{13}\cos 2t - \frac{1}{13}\sin 2t\right) + \frac{18}{13}\cos 3t + \frac{12}{13}\sin 3t}$$

This is the ugliest. If you can do this, you have it mastered.