5.5 $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$

$\int dx$	$\int dy$
$\frac{x^2}{2} + x\sin y$	$x\sin y - y^2$

$$f = \boxed{\frac{x^2}{2} + x\sin y - y^2 = C}$$

~~[illegible]~~ Do lots of these type till you can do them <u>quickly</u>.

5.21 $y' = 2xy - x$ I'm going to solve it

$$dy = (2xy - x)dx$$

$$\underbrace{(-(2xy - x))}_{M}dx + dy = 0$$

$M_y = -2x$ $N_x = 0$

$$\frac{M_y - N_x}{N}$$

$$\frac{(-2x)-(0)}{1} = -2x$$

$$e^{\int -2x\,dx} = e^{-x^2} = F$$ ← our integrating factor.

multiply by F

$$\left(-e^{x^2} 2xy + e^{-x^2} x\right)dx + e^{x^2} dy = 0$$

$\int dy = e^{u} y$

To get solution $\int -e^{-x^2} 2xy + e^{-x^2} x\,dx$

$u = -x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$= \int \frac{-e^{u} y}{-1} + \frac{e^{u}}{-2}\,du$

$= +e^{u} y - \frac{1}{2} e^{u}$

$$f = \boxed{e^{-x^2} y - \frac{1}{2} e^{-x^2} = C}$$ ← solution.

(This is almost just like #5,21)

6.4

$$y' - 2xy = x$$

rewrite in differential form

$$(-x - 2xy)\,dx + dy = 0$$

$$M_y = -2x \qquad N_x = 0$$

$$\frac{-2x - 0}{1} = -2x$$

$$e^{\int -2x\,dx} = e^{-x^2} = F$$

$$(-xe^{-x^2} - 2xye^{-x^2})\,dx + e^{-x^2}\,dy = 0$$

$\int dx$ $\qquad$ $\int dy = ye^{-x^2}$

use u-substitution

$u = -x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$$= \frac{1}{2}e^{-x^2} + ye^{-x^2}$$

same

$$f = \boxed{\frac{1}{2}e^{-x^2} + ye^{-x^2} = C}$$

or $\quad ye^{-x^2} = C - \frac{1}{2}e^{-x^2}$

$$\boxed{y = e^{x^2}(C) - \frac{1}{2}}$$

7.4 $y' = ky$ (grow @ rate prop to itself)

or $N'(t) = kN$ (using book notation).

$\frac{dN}{N} = k\,dt$

$\int \frac{1}{N} dN = \int k\,dt$

$e^{\ln|N|} = e^{kt+C}$

$$\boxed{N = e^{kt} c}$$

@ $t=1$ $N=1000$ $1000 = e^{k} \cdot c \Rightarrow c = \frac{1000}{e^k}$

@ $t=4$ $N=3000$ $3000 = e^{4k} \cdot c$

$$\frac{3000}{1000} = \frac{e^{4k} \frac{1000}{e^k}}{1000} = \frac{e^{3k} \cdot 1000}{1000}$$

so $3 = e^{3k}$

$\ln 3 = 3k$

$\ln 3/3 = k$

$C = \frac{1000}{e^{\ln 3/3}} = \frac{1000}{3^{1/3}}$

a) $N = e^{t \ln 3} \cdot \frac{1000}{\sqrt[3]{3}}$

b) $N(0) = \frac{1000}{\sqrt[3]{3}} \approx 693.361 \ldots$

7.10 $A = 30$

$T(10) = 0°$
$T(20) = 15°$

Note

(This is used by criminal investigators in courts of law)

$$\frac{dT}{dt} = k(T-A) \quad k<0$$

$$T' = k(T-30)$$

Separate $\longrightarrow$

$$\frac{1}{T-30}\,dT = k\,dt$$

Integrate $\longrightarrow$

$$\ln|T-30| = kt + C$$

exponentiate $\longrightarrow$ $e \qquad e$

$$T-30 = e^{kt}\cdot C \qquad T = 30 + ce^{kt}$$

$$T(10) = 0 \Rightarrow 0 = 30 + ce^{10k} \Rightarrow c = \frac{-30}{e^{10k}}$$

$$T(20) = 15 \Rightarrow 15 = 30 + ce^{20k}$$

$$-15 = \frac{-30}{e^{10k}}e^{20k}$$

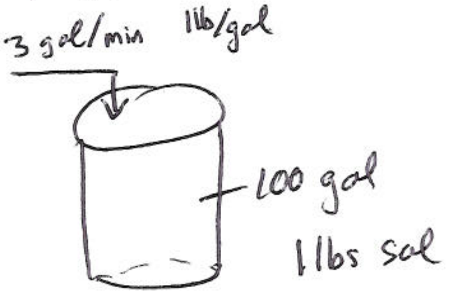
$$\frac{1}{2} = e^{10k} \qquad \ln\frac{1}{2} = 10k \qquad k = \frac{1}{10}\ln\frac{1}{2} \qquad c = \frac{-30}{e^{10(\frac{1}{10}\ln\frac{1}{2})}} \qquad c = -60$$

$$T = 30 - 60e^{(\frac{1}{10}\ln\frac{1}{2})t}$$

$$\boxed{T(0) = 30 - 60 = -30°F}$$

~~T(20) =~~

7.17



Q = Quantity of salt

Q' = inflow − outflow

$$Q' = \left(3\,\frac{gal}{min}\right)\left(1\,\frac{lb}{gal}\right) - \left(\frac{3\,gal/min}{100\,gal}\right)(Q)^{lbs}$$

lbs/min   lbs/min

differential form

$$Q' = 3 - \frac{3}{100}Q$$

$$\left(\frac{3}{100}Q - 3\right)dt + 1\,dQ = 0$$

$M_Q = \frac{3}{100}$ $N_t = 0$

$$F = e^{\int \frac{\frac{3}{100} - 0}{1} dt} = e^{\frac{3}{100}t}$$

Multiply by F.

$$\left(\frac{3}{100}e^{3/100\,t}Q - 3e^{3/100\,t}\right)dt + e^{3/100\,t}\,dQ = 0$$

Integrate

$$e^{3/100\,t}Q - 100e^{3/100\,t} \quad \Big| \quad e^{3/100\,t}Q$$

Solution is $\boxed{e^{3/100\,t}Q - 100e^{3/100\,t} = C}$ or $Q = \dfrac{C + 100e^{3/100\,t}}{e^{3/100\,t}}$

$$= ce^{-3/100\,t} + 100$$

If $t=0$, $Q=1$, so $1(1) - 100(1) = C = -99$

---

b) to get t when $Q = 2$ we have.

$$2 = -99e^{-3/100\,t} + 100 \quad \text{or} \quad e^{-3/100\,t} = \frac{98}{100} \qquad -\frac{3}{100}t = \ln\left(\frac{98}{100}\right)$$

$$\boxed{t = -\frac{100}{3}\ln\left(\frac{98}{100}\right)}$$

4.11 Solve $y' = \frac{y+x}{x}$

Note that if we replace $x + y$ with $tx + ty$,

then $\frac{ty+tx}{tx} = \frac{y+x}{x}$ (so its homogeneous).

we let $u = \frac{y}{x}$, ~~and then compute~~

or $y = ux$

so $\frac{dy}{dx} = \frac{du}{dx} \cdot x + u \cdot \frac{dx}{dx} = u'x + u$

$\frac{y}{x} + 1 = \frac{y+x}{x} = u' \cdot x + u$

This is $(u+1)$

so $(u+1) = u'x + u$

or $1 = u'x$

$\frac{1}{x} = u'$

$\frac{1}{x}dx = du$

$\ln|x| = u + C$

$\ln x = \frac{y}{x} + C$

$y = x(\ln x - c)$

$= x(\ln x + \ln k)$

$= x \ln(kx)$

6.1.16

$$y' + xy = xy^2$$

Bernoulli $n = 2$

let $u = y^{1-2} = y^{-1}$.

or $y = \frac{1}{u}$

$$y' = -\frac{1}{u^2}\frac{du}{dx} = -\frac{1}{u^2}u'$$

Replace y' and y's

$$-\frac{1}{u^2}u' + x\frac{1}{u} = x\frac{1}{u^2}$$

multiply by u^2

$$-u' + xu = x.$$

now differential form

$$(xu - x)dx - du = 0$$

$M_u = x$ $\quad N_x = 0$ $\quad N = -1$

$$\frac{x - 0}{-1} = -x \qquad e^{\int -x\,dx} = e^{-x^2/2} = F$$

$$\left(xe^{-x^2/2}u - xe^{-x^2/2}\right)dx - \left(e^{-x^2/2}\right)du = 0$$

$\int dx$ $\qquad\qquad$ $\int du$

$$-e^{-x^2/2}u + e^{-x^2/2} \quad\Big|\quad -e^{-x^2/2}u$$

solution: $\boxed{-e^{-x^2/2}\left(\frac{1}{y}\right) + e^{-x^2/2} = C}$

or $u = ce^{x^2/2} + 1$

$$y = \frac{1}{ce^{x^2/2} + 1}$$

22.1-3

① $L^{-1}\left(\frac{1}{s}\right) = 1$ (Recall $L(1) = \frac{1}{s}$)

② $L^{-1}\left(\frac{1}{s-8}\right) = e^{8x}$ (recall $L(e^{8x}) = \frac{1}{s-8}$)

③ $L^{-1}\left(\frac{s}{s^2+6}\right) = \cos(\sqrt{6}x)$.

$a^2 = 6$
$a = \sqrt{6}$

$L(\cos(ax)) = \frac{s}{s^2+a^2}$

so to undo it.

24.1

$y' - 5y = 0 \quad y(0) = 2$

Laplace trans form both sides

$(sL(y) - y(0)) - 5L(y) = 0$ (we call $L(y) = Y$)

$sY - 2 - 5Y = 0$

$Y(s-5) = 2$

$Y = \frac{2}{s-5}$

Inverse transform

$$\boxed{L^{-1}(Y) = 2e^{5x}}$$

$$\boxed{y(x) = 2e^{5x}}$$

done

Notice we didn't have to use integration at all.

24.14

$$\frac{dN}{dt} = .05N \qquad N(0) = 20{,}000$$

Laplace both sides $\quad sN_{(s)} - N(0) = .05\,N_{(s)}$

$$sN - 20000 = .05N$$

$$N(s - .05) = 20{,}000$$

$$N = \frac{20{,}000}{s - .05}$$

Now invert $\quad L^{-1}(N_{(s)}) = \boxed{N(t) = 20{,}000\,e^{.05t}}$

Fast

7.7 logistic equation

Pop max = 500 $N(t) = \#$ infected $N(0) = 5$.

$$\frac{dN}{dt} = k(N)(500 - N)$$

rate of change) proportional number infected number not infected

I'm going to use a Bernoulli. The book uses a partial fraction approach. Both are valid. Try both (count it as 2 HW problems)

$$N' = kN \cdot 500 - kN^2 \qquad n=2 \qquad \text{let } u = N^{1-2} = N^{-1} = \frac{1}{N}$$

$$N = \frac{1}{u} \qquad N' = \frac{-1}{u^2}u'$$

$$\frac{-1}{u^2}u' = 500k\frac{1}{u} - k\frac{1}{u^2}$$

mult by u^2 $\quad -u' = 500ku - k \qquad$ or $0 = (500ku - k) + u'$.

so $(500ku - k)dt + du = 0$

$M_u = 500k \qquad N_t = 0$

$$e^{\int \frac{500k - 0}{1}dt} = e^{500kt}$$

$$(500ke^{500kt}u - ke^{500kt})dt + e^{500kt}du = 0$$

$$e^{500kt}u - \frac{1}{500}e^{500kt} = C$$

times by 500, divide by e^{500kt}

$$500u = Ce^{-500kt} + 1$$

$$500\frac{1}{y} = Ce^{-500kt} + 1.$$

$$N = y = \frac{500}{Ce^{-500kt} + 1}$$

$$= \frac{500}{99e^{-500kt} + 1}.$$

Now $N(0) = 5$, so

$$5 = \frac{500}{C+1}$$

$$C + 1 = 100$$

$$C = 99.$$

To get $N(t) = 250$, I need.

$$250 = \frac{500}{99e^{-500kt} + 1}$$

or $99e^{-500kt} + 1 = 2$

$$e^{-500kt} = \frac{1}{99}$$

$$-500kt = \ln\frac{1}{99}$$

$$t = \frac{1}{-500k}\ln\left|\frac{1}{99}\right|$$