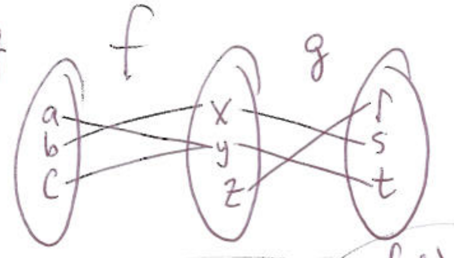
Mappings

8.4



$f(a)=y$ $f(b)=x$ $f(c)=y$

$g(x)=s$ $g(y)=t$ $g(z)=r$

a) $g(f(a)) = g(y) = t$
$g(f(b)) = g(x) = s$
$g(f(c)) = g(y) = t$

b) image of f is $\{x,y\}$
z is not the image of any of $\{a,b,c\}$
image of g is $\{r,s,t\}$ (g is onto)
image of $g\circ f$ is $\{s,t\}$

8.14 $F(x,y,z) = (x+y+z,\ 2x-3y+4z)$ The easiest way is to show F equals a matrix times $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\vec{x}.$$ Hence

① $F(\vec{x}_1+\vec{x}_2) = A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = F(\vec{x}_1)+F(\vec{x}_2)$
(so F preserves vector addition)

② $F(c\vec{x}_1) = A(c\vec{x}_1) = cA\vec{x}_1 = cF(\vec{x}_1)$
(so F preserves scalar multiplication

8.17

$F(1,2) = (2,3)$ and $F(0,1) = (1,4)$.

We know $F = A\vec{x}$ for some A.

so $\begin{bmatrix}2\\3\end{bmatrix} = A\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\4\end{bmatrix} = A\begin{bmatrix}0\\1\end{bmatrix}$

or we could write

$$A\left[\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right] = \left[\begin{pmatrix}2\\3\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix}\right]$$

which means

$$A\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}$$

multiply on the right by $\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}$

Gives

$$A = \begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\-5 & 4\end{bmatrix}$$

So $F = \begin{bmatrix}0 & 1\\-5 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = (0x+y, -5x+4y)$

check $F(1,2) = (0+2, -5+8) = (2,3)$ ✓
and $F(0,1) = (0+1, 0+4) = (1,4)$ ✓

(Draw This in Maple)

9.7

a) $L(1,0) = (2,4)$ $\quad$ $L(0,1) = (5,8)$

$$A = \left[\begin{pmatrix}2\\4\end{pmatrix} \begin{pmatrix}5\\8\end{pmatrix}\right] \quad \text{or} \quad A\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 5\\ 4 & 8\end{bmatrix}$$

$$\text{so } F = \begin{bmatrix}2 & 5\\ 4 & 8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = (2x+5y,\ 4x+8y)$$

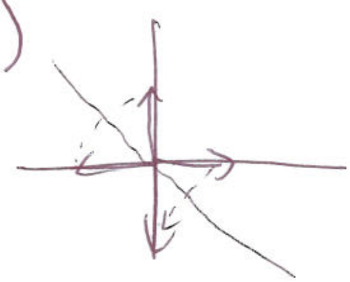
b) 

$F(1,0) = (0,1)$

$F(0,1) = (-1,0).$

$$\text{so } \boxed{A = \left[\begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}-1\\0\end{pmatrix}\right]}$$

Just keep track of where (1,0) and (0,1) go, and then you're done.

c) 

$F(1,0) = (0,-1)$

$F(0,1) = (-1,0)$

$$A = \left[\begin{pmatrix}0\\-1\end{pmatrix} \begin{pmatrix}-1\\0\end{pmatrix}\right]$$

Always think of matrices as column vectors.

Draw all of these in Maple

8.21

$$F(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$$

$F(1,0,0,0) = (1,1,1)$
$F(0,1,0,0) = (-1,0,1)$
$F(0,0,1,0) = (1,2,3)$
$F(0,0,0,1) = (1,-1,-3)$

(The book does this by reducing A^T. It can all be done by reducing A).

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) The image basis is the same as the column space basis. Just use the pivot columns

$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ dim $= 2 = \dim(\operatorname{Im} F) = 2$

" <u>rank</u> $= 2$ "

b) solving $F(v)=0$ means solving $[A \vdots \vec{0}]$ or $\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

x_3 & x_4 are free.

$x_1 + 2x_3 - x_4 = 0$
$x_2 + x_3 - 2x_4 = 0$
$x_3 = x_3$
$x_4 = x_4$

Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_4$

basis $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(\ker) = 2$

8.29

a) $F = (x-y, x-2y)$ $A = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ $\det A = -2+1 = -1 \neq 0$

$F(1,0) = (1,1)$

$F(0,1) = (-1,-2)$

A has an inverse, so F is nonsingular.

b) $G(x,y) = (2x-4y,\ 3x-6y)$ $A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$ $\det A = -12+12 = 0$

no inverse

lets reduce $\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 3 & -6 & 0 \end{array}\right]$ to find a nonzero solution.

rref $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ here is a solution

check

$G(2,1) = (2(2)-4(1),\ 3(2)-6(1))$

$= (0,0)$ ✓

Draw both in Maple. Notice how G collapses the 2D image onto a line.

9.4

a) $F = (x+2y-3z,\ 4x-5y-6z,\ 7x+8y+9z)$

$F(1,0,0) = (1,4,7)$
$F(0,1,0) = (2,-5,8)$
$F(0,0,1) = (-3,-6,9)$

$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \end{bmatrix}$

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 5 & 5 \end{pmatrix}$ ← The matrix is the one given

c)
$$\begin{array}{ccc} F(e_1) & F(e_2) & F(e_3) \end{array}$$
$$\begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 7 \\ 5 & 6 & 7 \end{bmatrix}$$

Draw all 3 in maple.

8.38

$T(x,y) = (2x+4y,\ 3x+6y)$ $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ $\det A = 12 - 12 = 0.$

a) T^{-1} does not exist (det is zero).

b) $T^{-1}(8,12)$ = all pts whose image under T is $(8,12)$

we solve $A\vec{x} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & 6 & 12 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$x + 2y = 4$ $\quad x = -2y + 4$

$y = y.$ $\quad y = y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} y + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

These are the solutions. either.

c) $T^{-1}(1,2)$ reduce $\left[\begin{array}{cc|c} 2 & 4 & 1 \\ 3 & 6 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$ $0 = 1$ no solution.

so $T^{-1}(1,2) = \emptyset = \{\}$ the empty set

d) T is <u>not</u> onto because $(1,2)$ is not in the image of T.

<u>Draw in Maple</u> Notice it collapses to a line

5.24

a) $\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -1 & 3 & 0 \\ 1 & 2 & 3 & 1 & 1 & 0 \\ 3 & 6 & 8 & 1 & 5 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -5 & 7 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

free variables contribute the basis vectors

Basis is $\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

Just change the signs of the nonzero terms in non pivot columns, and put in 0's + 1's in the free variable spot.

dimension of solution space is $\underline{\underline{3}}$.

3.15

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \xrightarrow[R_2 + 2R_1]{E_1} \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix} \xrightarrow[R_2(-\frac{1}{2})]{E_2} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 + 3R_2]{E_3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = I \quad \text{so} \quad A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$\begin{array}{c} R_2 + 2R_1 \\ E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} R_2(-\frac{1}{2}) \\ E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \end{array} \qquad \begin{array}{c} R_1 + 3R_2 \\ E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} R_2 - 2R_1 \\ E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} R_2(-2) \\ E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \end{array} \qquad \begin{array}{c} R_1 - 3R_2 \\ E_3^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \checkmark$$

yep

Handout 14

$$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 2 & 1 & 1 & -2 \\ 0 & 3 & -1 & 2 \\ -1 & 2 & 0 & 4 \end{bmatrix} \xrightarrow[\text{times by } (-1)]{R_1 \leftrightarrow R_4} \begin{bmatrix} -1 & 2 & 0 & 4 \\ 2 & 1 & 1 & -2 \\ 0 & 3 & -1 & 2 \\ 2 & -1 & 0 & 3 \end{bmatrix} \xrightarrow[R_4+2R_1]{R_2+2R_1} \begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & 5 & 1 & 6 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & 0 & 11 \end{bmatrix}$$

no change

$$\xrightarrow[\text{(times by } -1)]{R_2 \leftrightarrow R_4} \begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 11 \\ 0 & 3 & -1 & 2 \\ 0 & 5 & 1 & 6 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 11 \\ 0 & 0 & -1 & -9 \\ 0 & 5 & 1 & 6 \end{bmatrix} \xrightarrow[\text{times by } (\frac{1}{3})]{R_4 \cdot 3} \begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 11 \\ 0 & 0 & -1 & -9 \\ 0 & 15 & 3 & 18 \end{bmatrix} \xrightarrow{R_4-5R_2}$$

$$\begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 11 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & 3 & -37 \end{bmatrix} \xrightarrow{R_4+3R_3} \begin{bmatrix} -1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 11 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & 0 & -64 \end{bmatrix}$$

$18-55 = -37$ $\quad -37-27 = -64$

det of is $(-1)(3)(-1)(-64) = 3(-64)$

$$\det(A) = (-1)(-1)\left(\tfrac{1}{3}\right)\left(3(-64)\right) = -64$$

or expand along 3rd column.

$$0\,|\;| - (1)\begin{vmatrix} 2 & -1 & 3 \\ 0 & 3 & 2 \\ -1 & 2 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 & 3 \\ 2 & 1 & -2 \\ -1 & 2 & 4 \end{vmatrix} + 0\,|\;|$$

$$-1\left(-0\,|\;| + 3\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}\right) - 1\left(2\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}\right)$$

2nd row $\qquad$ 1st row

$$-1\left(3(11) - 2(3)\right) - 1\left(2(8) + 1(6) + 3(5)\right)$$

$$-27 - 1(16+15+6) = -64$$

$27+37=64$