1f) Eigenvalues & Eigenvectors.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{x}_4 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

eigenvector

$\lambda = 1$

$$A\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \\ 3 \end{bmatrix}$$ $\neq$ not a multiple of $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ so <u>Not</u> an eigenvector.

$$A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ — eigenvector.

$\lambda = 3$

$$A\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1$ eigenvector.

We know $\lambda = 1$ and 3 are eigenvalues

I'll let you create your own example.

2a) $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \\ 1 & 2 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \\ -1 & 0 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix}$

rrefs are $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

A & C are row equivalent because they are both row equivalent to the same matrix in rref.

<u>Schaums 4.22.</u>

To see if column spaces are same, switch to rows.

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & -2 & 7 \\ 2 & -3 & -4 \\ 3 & 12 & 17 \end{bmatrix}$$

The rref of A^T gives a basis for column space of A.
The rref of B^T gives a basis for " " " B.
If these rrefs are the same, the spaces are the same.

$$RREF(A^T) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad RREF(B^T) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So they both <u>have the same column space</u> as they both have a basis $\{(1,0,3), (0,1,-2)\}$. $\blacksquare$

2f) $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 3 & -1 & 0 \\ 3 & 5 & -1 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & -12 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\uparrow\ \uparrow$ pivot columns.

Thm 3.15. • Rank of $A = 2$

Thm 3.16 • Since not every column is a pivot column, the columns are <u>dependent</u>

3.20 • Basis is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \right\}$

coordinates are

vector	coordinates
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$(1,0)$
$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$	$(0,1)$
$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$	$(-2,1)$
$\begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$	$(-12,8)$

3.20 Basis for row space is $\{(1,0,-2,-12), (0,1,1,8)\}$

vector	coordinates
$(1\ 2\ 0\ 4)$	$(1\ 2)$
$(2\ 3\ -1\ 0)$	$(2\ 3)$
$(3\ 5\ -1\ 4)$	$(3\ 5)$

3.19 Dimensions of both row + column space are 2.

You can now repeat with another matrix.

2i) $A = \begin{bmatrix} 1 & 8 & -12 & 0 \\ 0 & 2 & 5 & 16 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

cofactor along column 1

$$1 \begin{vmatrix} 2 & 5 & 16 \\ 0 & -3 & -4 \\ 0 & 0 & 4 \end{vmatrix} = (1)(2)\begin{vmatrix} -3 & -4 \\ 0 & 4 \end{vmatrix}$$

$$= (1)(2)(-3)(4)$$

product of diagonal entries

Thm 3.28

$$= \boxed{-24}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 8 & -12 & 0 \\ 0 & 2-\lambda & 5 & 16 \\ 0 & 0 & -3-\lambda & -4 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 5 & 16 \\ 0 & -3-\lambda & -4 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)\begin{vmatrix} -3-\lambda & -4 \\ 0 & 4-\lambda \end{vmatrix}$$

Characteristic polynomial $\Rightarrow (1-\lambda)(2-\lambda)(-3-\lambda)(4-\lambda)$

eigenvalues $\rightarrow \boxed{\lambda = 1, 2, -3, 4}$

The #s on diagonals.

Thm 3.36

I'll let you repeat with another matrix

2j) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -3 & 1 \end{bmatrix}$ $\quad |A| = 1\begin{vmatrix} 3 & 4 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix}$

$$= 1\big((3)(1) - (-3)(4)\big) - 2\big((-1)(1) - (2)(4)\big)$$

notice transpose

$$= 1(15) - 2(-9) = \boxed{33}$$

$$|A^T| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 0 & 4 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & -3 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 3 & -3 \end{vmatrix}$$

$$= 1\big((3)(1) - (4)(-3)\big) - 2\big((-1)(1) - (4)(2)\big)$$

$$= 1(15) - 2(-9) = \boxed{33}$$

Notice how the determinant of the transpose just involves lots of smaller determinants of transposes. on 2 by 2's, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ so we use the fact that on 2 by 2's $|A^T| = |A|$ to show $|A^T| = |A|$ for all A.

Eigen values of A

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 3-\lambda & 4 \\ 2 & -3 & 1-\lambda \end{vmatrix} = (1-\lambda)\big((3-\lambda)(1-\lambda) - (-12)\big) - 2\big((-1)(1-\lambda) - 2(4)\big)$$

same

same.

of A^T

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & 3-\lambda & -3 \\ 0 & 4 & 1-\lambda \end{vmatrix} = (1-\lambda)\big((3-\lambda)(1-\lambda) - (-12)\big) - 2\big((-1)(1-\lambda) - (4)(2)\big)$$

Since characteristic polynomials are the same, eigen values are the same.

page 71 - I'm looking @ the negative of each line.

4b) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

(1) We solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

There are ∞ many solutions

(2) $A\vec{x} = \vec{b}$ may have ∞ many solutions or none at all if rref is $\left[\begin{array}{cc|c} 2 & 1 & a \\ 4 & 2 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 1 \end{array}\right]$

(3) $rref(A) \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it equals $\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$

(4) column 2 is not a pivot column

(5) The columns are dependent & dimension is 1 (which is less than 2)

(6) $rank(A) = 1 \neq 2$

(7) rows are dependent: a basis for row space is $\{(1, 1/2)\}$

(8) $rref \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 0 \end{array}\right] \to \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in column space of A.

(neither is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.)

(9) No inverse since $\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/2 & \text{stuff} \\ 0 & 0 & \end{array}\right]$

(10) $|A| = 2 \cdot 2 - 4 \cdot 1 = 0$ It is zero.

(11) $\begin{vmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = \lambda^2 - 4\lambda + 4 - 4 = \lambda^2 - 4\lambda = (\lambda-4)(\lambda)$

$\lambda = 4, \boxed{0}$

Zero is an eigen value.

4.6 We are just defining new ways to add + scale vectors and then showing those new ways do not define vector spaces. The point is to be able to notice when something is not a vector space. You'll want to pay attention to the axioms on page 73 (def 3.17).

a) usual addition: change scalar mult to $k(a,b) = (ka, b)$.

The problem is with scalar mult. so let's check the axioms.

M_1 $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$.

holds

$k((a,b)+(c,d))$
$= k(a+c, b+d)$
$= (k(a+c), b+d)$

$k(a,b) + k(c,d)$
$= (ka, b) + (kc, d)$
$= (k(a+c), b+d)$

same.

M_2 $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$(k_1+k_2)(a,b)$
$= ((k_1+k_2)a, b)$

$k_1(a,b) + k_2(a,b)$
$= (k_1 a, b) + (k_2 a, b)$
$= ((k_1+k_2)a, 2b)$

$b \neq 2b$
so M_2 fails

---

b) $(a,b) + (c,d) = (a,b)$ usual multiplication.

ignore 2nd.

we'll show addition is not commutative. $(\vec{u}+\vec{v} \neq \vec{v}+\vec{u})$

$(c,d) + (a,b) = (c,d)$ (a,b) does not have to equal (c,d).

---

c) usual addition $k(a,b) = (k^2 a, k^2 b)$

other option.

$(1-1)(a,b) \overset{?}{=} 1(a,b) + (-1)(a,b)$
$= (a,b) + (a,b)$
$= (2a, 2b)$

$(0,0)$ not equal.

Does $(k_1+k_2)(a,b) \overset{?}{=} k_1(a,b) + k_2(a,b)$

$(1+2)(3,4)$
$= (9\cdot3, 9\cdot4)$
$= (27, 36)$

$\overset{?}{=}$

$1(3,4) + 2(3,4)$
$= (3,4) + (12,16) = (15,20)$

not the same.

4:13 To show something is a subspace, we must show

1. $\vec{0}$ belongs to the space
2. $\vec{u}+\vec{v}$ belongs to the space whenever $\vec{u}$ and $\vec{v}$ do
3. $c\vec{u} \in V$ whenever $c \in \mathbb{R}$ and $\vec{u} \in V$.

a) $W = \{(a,b,c) \mid a=b=c\} = \{(a,a,a) \mid a \in \mathbb{R}\}$

$= \boxed{\text{span}\{(1,1,1)\}}$

<u>Yes</u>

If $a=b=c=0$, then

1. $(0,0,0) \in W$

2. $\underset{\vec{u}}{(a,a,a)} \underset{+}{+} \underset{\vec{v}}{(b,b,b)} = (a+b, a+b, a+b)$ equal ✓

so $\vec{u}+\vec{v} \in W$

3. $c(a,a,a) = (ca, ca, ca)$ all are equal ✓

---

b) $W = \{(a,b,c) \mid a+b+c=0\} = \{(a,b,-a-b) \mid a,b \in \mathbb{R}\}$

$c=-a-b$

$= \boxed{\text{span}\{(1,0,-1),(0,1,-1)\}}$

1. $(0,0,0) \in W$ because $0+0+0=0$ ✓

2. If $\overset{\vec{u}}{(a_1, b_1, c_1)} \in W$ and $\overset{\vec{v}}{(a_2, b_2, c_2)} \in W$

then $a_1+b_1+c_1 = 0$ and $a_2+b_2+c_2=0$.

The sum $\vec{u}+\vec{v} = (a_1+a_2, b_1+b_2, c_1+c_2)$ has sum

$(a_1+a_2)+(b_1+b_2)+(c_1+c_2)$

$= (a_1+b_1+c_1)+(a_2+b_2+c_2)$

$= 0 + 0 = 0$ so $\vec{u}+\vec{v} \in W$.

3. $k(a_1, b_1, c_1) = (ka_1, kb_1, kc_1)$ and $k(a_1+b_1+c_1) = k \cdot 0 = 0$

so $k\vec{u} \in W$.

Alternately, if you can show W is the span of a set of vectors, then it automatically is a subspace. The simplest way to show something is a subspace is to find a spanning set.

4.22) see 2a) solution.

5,18) Schaum's solution is great. No need to rref any matrices because there are only 2 vectors.

for c) and d)

c) $(1, 2, -3, 4)$ and $(2, 4, -6, 8)$ are multiples so dependent

d) $(1, -2, 0, 3)$ and $(0, 1, -5, 3)$ are not multiples so independent.

6.6 first we ~~redo~~ rewrite each matrix in the basis as ~~its~~ its coordinates relative to $\left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$

$$\begin{bmatrix}1&1\\1&1\end{bmatrix} \quad \begin{bmatrix}0&-1\\1&0\end{bmatrix} \quad \begin{bmatrix}1&-1\\0&0\end{bmatrix} \quad \begin{bmatrix}1&0\\0&0\end{bmatrix} \quad \begin{bmatrix}2&3\\4&-7\end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$[1,1,1,1] \quad (0,-1,1,0) \quad [1,-1,0,0] \quad (1,0,0,0) \quad (2,3,4,-7)$$

we now put each coordinate vector in a column + then reduce.

$$\left[\begin{array}{cccc|c} 1&0&1&1&2\\ 1&-1&-1&0&3\\ 1&1&0&0&4\\ 1&0&0&0&-7 \end{array}\right] \to \left[\begin{array}{c|c} & -7\\ I & 11\\ & -21\\ & 30 \end{array}\right]$$

so relative to new basis given, we have

$\begin{bmatrix}2&3\\4&-7\end{bmatrix}$ with coordinates $[-7, 11, -21, 30]_B$.

which means $\begin{bmatrix}2&3\\4&-7\end{bmatrix} = -7\begin{bmatrix}1&1\\1&1\end{bmatrix} + 11\begin{bmatrix}0&-1\\1&0\end{bmatrix} - 21\begin{bmatrix}1&-1\\0&0\end{bmatrix} + 30\begin{bmatrix}1&0\\0&0\end{bmatrix}$

(6.9)

If we use the basis $B = \{t^3, t^2, t, 1\}$ then the coodinates are

$$[u_1]_B = (1, 3, -2, 4), \quad (u_2) = [2, 7, -2, 5], \quad (u_3) = (1, 5, 2, -2)$$

Schaum's places these in rows + reduces, finding the rank is 2.

Alternatively, place them in columns

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ -2 & -2 & 2 \\ 4 & 5 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

dependent

column 3 is not a pivot column.

This again shows the rank is 2, and the third vector is a linear combination of the first 2.

Notice that because of our column vectors

we know $u_3 = -3u_1 + 2u_2$, whereas using the row approach in Schaum's does not help us see how the 3 vectors are related.

The row approach only shows us that the vectors are dependent, <u>not</u> how they depend on each other.