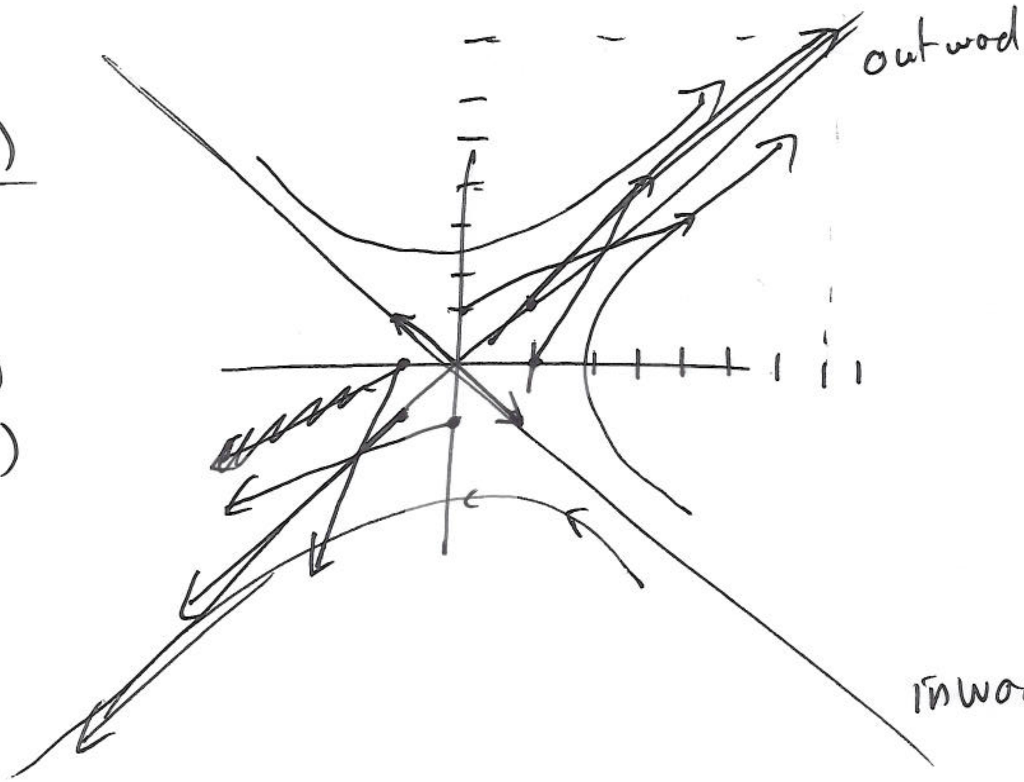
1b) $F = (2x+4y,\ 4x+2y)$ $\quad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 16 = 4 - 4\lambda + \lambda^2 - 16$$
$$= \lambda^2 - 4\lambda - 12$$
$$= (\lambda-6)(\lambda+2)$$
$$\boxed{\lambda = 6, -2}$$

$\lambda = 6$ $\quad \left[\begin{array}{cc|c} -4 & 4 & 0 \\ \cancel{4} & \cancel{-4} & \cancel{0} \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \begin{matrix} x = y \\ y = y \end{matrix}$ $\quad \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ when $\lambda = 6$

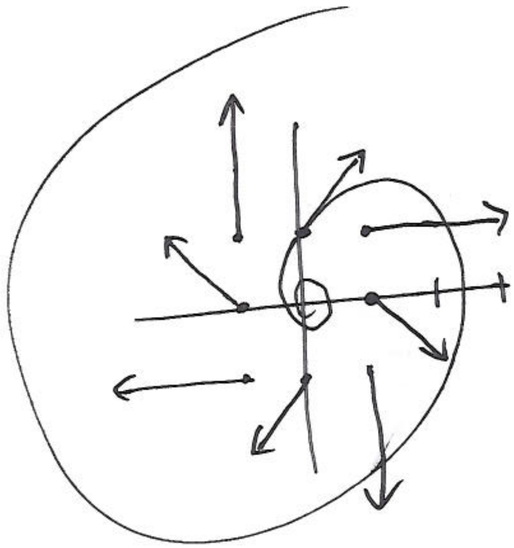
$\lambda = -2$ $\quad \left[\begin{array}{cc|c} 4 & 4 & 0 \\ \cancel{4} & \cancel{4} & \cancel{0} \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \begin{matrix} x = -y \\ y = 1 \end{matrix}$ $\quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ when $x = -2$.

| $(x,y)$ | $F(x,y)$ |
|---|---|
| $(1,0)$ | $(2,4)$ |
| $(1,1)$ | $(6,6)$ |
| $(0,1)$ | $(4,2)$ |
| $(-1,+1)$ | $(2,-2)$ |



1 g) $F = (x+y, -x+y)$ $\quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

| $(x,y)$ | $F(x,y)$ |
|---|---|
| $(1,0)$ | $(1,-1)$ |
| $(1,1)$ | $(2,0)$ |
| $(0,1)$ | $(1,1)$ |
| $(-1,1)$ | $(0,2)$ |
| $(-1,0)$ | $(-1,1)$ |



rotational
outward
spiral

Eigenvalues should have a positive real part.
lets find them

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+1 = \cancel{\lambda^2-2\lambda+1+1} = 0$$

ignore.

$$(1-\lambda)^2 = -1$$

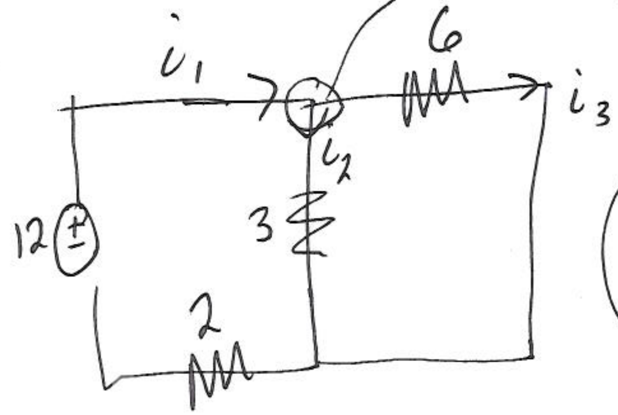
$$(1-\lambda) = \pm\sqrt{-1}$$

$$1-\lambda = \pm i$$

$$\boxed{1 \pm i = \lambda}$$

positive real part is why <u>outward</u> spiral.

2c




① $i_1 = i_2 + i_3$

flow in = flow out.

at the top node

Volts in = Volts out

left loop: $12 = 2i_1 + 3i_2$

in out

right loop $0 = 6i_3 - 3i_2$

no battery

opposite direction

3 eqns - 3 unknowns

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ 2i_1 + 3i_2 \quad &= 12 \\ -3i_2 + 6i_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 12 \\ 0 & -3 & 6 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{i_1 = 3 \quad i_2 = 2 \quad i_3 = 1}$$

3b) $x + y = 2$

$x - y = 3$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 3 \end{array}\right]$$

$$D = \left|\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right| = -1-1 = -2$$

$$D_1 = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -2-3 = -5$$ replace col 1 with $\binom{2}{3}$

$$D_2 = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3-2 = 1$$ replace col 2 with $\binom{2}{3}$

$$x_1 = x = \frac{D_1}{D} = \frac{-5}{-2} = 5/2$$

$$x_2 = y = \frac{D_2}{D} = \frac{1}{-2} = -1/2$$

4b) $(0,1), (2,3), (-1,4)$

$$y = a_0 + a_1 x + a_2 x^2$$

$$1 = a_0 + 0 + 0$$

$$3 = a_0 + a_1 \cdot 2 + a_2 \cdot 4$$

$$4 = a_0 + a_1(-1) + a_2(1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & -1 & 1 & 4 \end{array}\right]$$

I'll use cramer's rule to solve. Gaussian elim may be faster here, but I want you to see the practice with Cramer's rule.

$$D = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} = 1(2+4) = 6$$

$$D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \\ 4 & -1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} = 1(2+4) = 6$$

$$D_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 3 & 4 \\ 1 & 4 & 1 \end{vmatrix} = 1\begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix} - 1\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} = 1(3-16) - 1(1-4) = -13+3 = -10$$

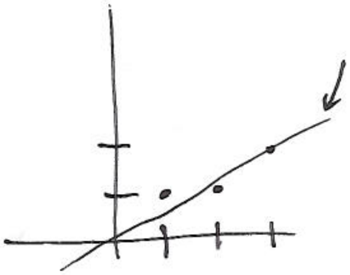
$$D_3 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 4 \end{vmatrix} = 1\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} + 1\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = 1(11) + 1(-3) = 8$$

$$a_0 = \frac{D_1}{D} = \frac{6}{6} = 1 \qquad a_1 = \frac{D_2}{D} = \frac{-10}{6} = -5/3 \qquad a_2 = \frac{D_3}{D} = \frac{8}{6} = \frac{4}{3}$$

$$y = 1 - \tfrac{5}{3}x + \tfrac{4}{3}x^2$$

you can now plug in the 3 points to check.

5b) (1,1) (2,1) (3,2)



I want this line.

$$a_0 + a_1 x = y$$
$$a_0 + a_1 1 = 1$$
$$a_0 + a_1 2 = 1$$
$$a_0 + a_1 3 = 2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

there is no solution.

multiply both sides by $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

using inverse. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

$$\frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 56-54 \\ -24+27 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$$

$$\boxed{y = \tfrac{1}{3} + \tfrac{1}{2}x}$$

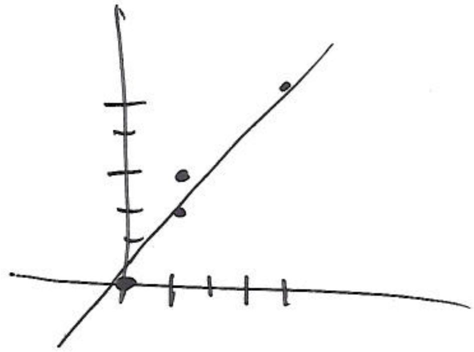
using cramer's rule.

$$D = \begin{vmatrix} 3 & 6 \\ 6 & 14 \end{vmatrix} = 42-36 = 6$$
$$D_1 = \begin{vmatrix} 4 & 6 \\ 9 & 14 \end{vmatrix} = 56-54 = 2$$
$$D_2 = \begin{vmatrix} 3 & 4 \\ 6 & 9 \end{vmatrix} = 27-24 = 3$$
$$a_0 = \frac{2}{6} = \frac{1}{3} \quad a_1 = \frac{3}{6} = \frac{1}{2}$$
$$y = \tfrac{1}{3} + \tfrac{1}{2}x$$

Both require about the same work. $A^{-1}$ maybe a little shorter.

5d) $(0,0), (1,3), (1,2), (4,5)$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 25 \end{bmatrix}$$

using inverse

$$\frac{1}{72-36} \begin{bmatrix} 18 & -6 \\ -6 & 4 \end{bmatrix} \begin{pmatrix} 10 \\ 25 \end{pmatrix} = \frac{1}{36} \begin{bmatrix} 180 - 150 \\ -60 + 100 \end{bmatrix} = \begin{pmatrix} 30 \\ 40 \end{pmatrix} / 36$$

$$\begin{bmatrix} 30/36 \\ 40/36 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 10/9 \end{bmatrix} \quad \boxed{y = 5/6 + \frac{10}{9} x}$$

6b) $f(x,y) = x^2 + 4xy + y^2$

$f_x = 2x + 4y + 0$ (think $y$ is constant, so $(y^2)' = 0$)

$f_y = 0 + 4x + 2y$

find critical values

$Df = [2x+4y,\ 4x+2y] = [0, 0]$

solve $\begin{aligned} 2x+4y &= 0 \\ 4x+2y &= 0 \end{aligned}$ $\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 4 & 2 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ $\boxed{\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}}$ location of max or min.

$D^2 f = \left[\begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}\right]$

find eigenvalues. $\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 16$

$= 4 - 4\lambda + \lambda^2 - 16$

$= \lambda^2 - 4\lambda - 12$

$= (\lambda - 6)(\lambda + 2)$

$\lambda = 6, \ -2$

concave up    concave down

when $\lambda = 6$

$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} y$

Eigen vectors

when $\lambda = -2$

$\left(\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right) \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} y$

saddle

6e) $f = x^2 - 2x + xy + y^2$

find critical pt

$$Df = [f_x \ \ f_y] = [2x - 2 + y, \ x + 2y] = [0, 0]$$

$$\begin{aligned} 2x - 2 + y &= 0 \\ x + 2y &= 0 \end{aligned} \Rightarrow \begin{aligned} 2x + y &= 2 \\ x + 2y &= 0 \end{aligned}$$

$\left|\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & 2 & 0 \end{array}\right|$ cramer's rule

$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad x = 4/3$

$\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$ $\quad y = -2/3$

$\begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = -2$

or

Inverse $\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -2/3 \end{pmatrix}$

$\boxed{\text{There is a critical pt @ } (4/3, -2/3)}$

Now find $D^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ eigenvalues are

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$$

$$4 - 4\lambda + \lambda^2 - 1$$

$$\lambda^2 - 4\lambda + 3$$

$$(\lambda - 3)(\lambda - 1)$$

$$\lambda = 3, 1$$

Both are positive so concave up in every direction

$\boxed{\text{minimum @ } (4/3, -2/3)}$

Eigen vectors are $\lambda = 3 \ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 1 \ \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

7a)

$$\begin{array}{cc} & \begin{array}{ccc} R & C & I \end{array} \\ \begin{array}{c} \text{to } R \\ \text{to } C \\ \text{to } I \end{array} & \begin{bmatrix} .85 & .10 & .15 \\ .10 & .70 & .25 \\ .05 & .20 & .60 \end{bmatrix} \end{array} = A.$$

Transition matrix.

Make sure your order on Rows & colums remains the same.

Current state is $\vec{X}_0 = \begin{bmatrix} 40 \\ 30 \\ 30 \end{bmatrix}$.

after 5 years $\vec{X}_1 = A\vec{X} = \begin{bmatrix} .85 & .10 & .15 \\ .10 & .70 & .25 \\ .05 & .20 & .60 \end{bmatrix} \begin{bmatrix} 40 \\ 30 \\ 30 \end{bmatrix} = \begin{pmatrix} 41.5 \\ 32.5 \\ 26 \end{pmatrix}$

after 10 years $\vec{X}_2 = A^2 x = A\vec{X}_1 = \begin{bmatrix} A \end{bmatrix} \begin{pmatrix} 41.5 \\ 32.5 \\ 26 \end{pmatrix} = \begin{pmatrix} 42.425 \\ 33.4 \\ 24.175 \end{pmatrix}$

Please use a calculator here

Try to get the eigenvector without a calc.

$$\begin{bmatrix} .85-1 & .10 & .15 & 0 \\ .10 & .70-1 & .25 & 0 \\ .05 & .20 & .60-1 & 0 \end{bmatrix} = \begin{bmatrix} -.15 & .10 & .15 & 0 \\ .10 & -.3 & .25 & 0 \\ .05 & .2 & -.4 & 0 \end{bmatrix} \begin{array}{l} \times 100 \\ \times 100 \\ \times 100 \end{array}$$

$$\begin{bmatrix} -15 & 10 & 15 & 0 \\ 10 & -30 & 25 & 0 \\ 5 & 20 & -40 & 0 \end{bmatrix} \begin{array}{l} /5 \\ /5 \\ /5 \end{array} \to \begin{bmatrix} -3 & 2 & 3 & 0 \\ 2 & -6 & 5 & 0 \\ 1 & 4 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & -6 & 5 & 0 \\ -3 & 2 & 3 & 0 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{array}$$

$$\to \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -14 & 21 & 0 \\ \cancel{0} & \cancel{14} & \cancel{-21} & \cancel{0} \end{bmatrix} \begin{array}{l} \\ R_2/-7 \\ R_3 + R_2 \end{array} \quad \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2 \quad \begin{array}{cccc} 1 & 0 & -2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$x = 2z$
$y = 3z/2$
$z = z$

$\begin{bmatrix} 2 \\ 3/2 \\ 1 \end{bmatrix}$

or times by 2 to get $\underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{t=9}$ $\begin{pmatrix} 4/9 \\ 3/9 \\ 2/9 \end{pmatrix} \begin{array}{l} -R \\ -C \\ -I \end{array}$

7b)

currently

$$\begin{array}{cc} & \begin{array}{cc} R & IF \end{array} \\ \begin{array}{c}\text{to } R\\ \text{to } IF\end{array} & \begin{bmatrix} .60 & .20 \\ .40 & .80 \end{bmatrix} \end{array} = A \qquad \vec{X}_0 = \begin{bmatrix} 60 \\ 140 \end{bmatrix}$$

next week $\vec{X}_1 = A\vec{X}_0 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}\begin{bmatrix} 60 \\ 140 \end{bmatrix} = \begin{bmatrix} 64 \\ 136 \end{bmatrix}$

use calculator

2 weeks $\vec{X}_2 = A\vec{x}_1 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}\begin{bmatrix} 64 \\ 136 \end{bmatrix} = \begin{bmatrix} 66.24 \\ 133.76 \end{bmatrix}$ so about $\begin{pmatrix} 66 \\ 134 \end{pmatrix}$

Steady state (do by hand).

$$\left[\begin{array}{cc|c} .6-1 & .2 & 0 \\ .4 & .8-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -.4 & .2 & 0 \\ .4 & -.2 & 0 \end{array}\right] \begin{array}{l} R_1 \cdot 5 \\ R_2 + R_1 \end{array} \Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-2x + y = 0$

$x = 1$
$y = 2$ solves

so eigenvector are multiples of: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

33%
66%

33% of 200 cars is 66.7 cars, so 66 to 67 will be in Rex, and 134 to 133 in IF.

8b)

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

col space basis : $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$

row space basis : $\{ (1, 2, 0), (0,0,1) \}$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 1 & 3 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

col space basis : $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

row space basis : $\{ (2,4,1) \; (1,2,3)$

8c)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -3 & -2 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/4 \\ 0 & 0 & 0 \end{bmatrix}$$

col $\left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\}$

row $\left\{ (1, 0, \tfrac{1}{2}), (0, 1, \tfrac{5}{4}) \right\}$

$$A^T = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -2 \\ 3 & 1 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

col $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

row $\left\{ (1, 2, 3), (2, 0, 1) \right\}$

9b $v = (6,3)$

$w = (-1,3)$

from 8b we have col space basis $\{(2,1),(1,3)\}$

$$\left[\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}6\\3\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix}\right] \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 3 & -6/5\\ 0 & 1 & 0 & 7/5\end{bmatrix}$$

So the coordinates of $\vec{v}$ relative to the basis are

3 and 0

meaning $\begin{pmatrix}6\\3\end{pmatrix} = 3\begin{pmatrix}2\\1\end{pmatrix} + 0\begin{pmatrix}1\\3\end{pmatrix}$. ✓

The coordinates of $\vec{w}$ relative to the basis are

$-6/5$ and $7/5$, meaning

$$\begin{pmatrix}-1\\3\end{pmatrix} = -\tfrac{6}{5}\begin{pmatrix}2\\1\end{pmatrix} + \tfrac{7}{5}\begin{pmatrix}1\\3\end{pmatrix}$$ ✓

RREF tells you the <u>coordinates of vectors relative to the pivot columns.</u>

This problem is really just a vocab problem.

10 b)

$$\left[\begin{array}{cccc|c} 2 & 4 & -4 & -7 & 0 \\ 3 & 6 & 1 & 0 & 0 \\ -1 & -2 & 1 & 2 & 0 \\ 0 & 0 & 8 & 12 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2$ & $x_4$ are free.

$x_1 + 2x_2 - \frac{1}{2}x_4 = 0 \qquad x_1 = -2x_2 + \frac{1}{2}x_4$

$x_2 = x_2$

$x_3 + \frac{3}{2}x_4 = 0 \qquad x_3 = -\frac{3}{2}x_4$

$x_4 = x_4$

Try to get the answer without the middle work

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/2 \\ 0 \\ -3/2 \\ 1 \end{pmatrix}$$

Basis is $\{(-2,1,0,0), (\frac{1}{2}, 0, -\frac{3}{2}, 1)\}$

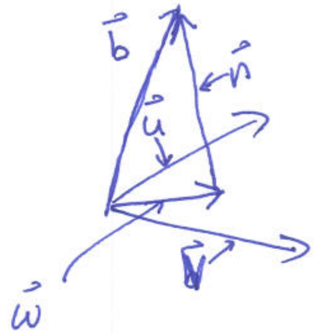
or an alternate basis is (clearing fractions)

$$\boxed{\{(-2,1,0,0), (1,0,-3,2)\}}$$

2 dimensional nullspace.

11 b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\vec{u} = (1,1,1)$
$\vec{v} = (1,2,3)$




$A\vec{x} = \vec{w}$
$= \vec{b} - \vec{n}$

$A^TA\vec{x} = A^Tb - \underbrace{\cancel{A^Tn}}_{0}$

$\boxed{\vec{x} = (A^TA)^{-1}A^Tb}$

$\boxed{\text{Compare to 5b)}}$

It's the same thing with new vocabulary

$$\overset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \qquad \overset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \overset{b}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

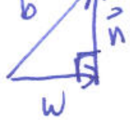
$$A^TA^{-1} = \frac{1}{42-36}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

$$\vec{x} = (A^TA)^{-1}A\vec{b} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 56-54 \\ -24+27 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$$

So the coordinates of $\vec{w}$ relative to $\vec{u}$ + $\vec{v}$ are $\boxed{(1/3, 1/2)}$.

or $\boxed{\vec{w} = (1/3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 4/3 \\ 11/6 \end{pmatrix}}$

The normal vector is $\vec{n} = \vec{b} - \vec{w}$ or $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5/6 \\ 4/3 \\ 11/6 \end{pmatrix} = \begin{pmatrix} 1/6 \\ -1/3 \\ 1/6 \end{pmatrix}$




and $\vec{n} \cdot \vec{u} = (1/6, -1/3, 1/6) \cdot (1,1,1) = 1/6 - 1/3 + 1/6 = 0$ ✓

$\vec{n} \cdot \vec{v} = (1/6, -1/3, 1/6) \cdot (1,2,3) = 1/6 - 2/3 + 3/6 = 0$ ✓

$\boxed{A^T\vec{n} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{pmatrix} 1/6 \\ -1/3 \\ 1/6 \end{pmatrix} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

Remember, the dot product $=0$ means vectors are $\perp$.