1) a) $y' = .03y \quad y(0) = 2000$

$y = e^{.03t} \cdot C$ @ 0 we have $2000 = e^{.03 \cdot 0} \cdot C$
$= C$

so $y = 2000e^{.03t}$ $\quad y(30) = 2000\, e^{.03 \cdot 30}$
$= \cancel{(2000)(e}$
$= 4919.21$

You invest \$2000 @ 3% (a money market account) interest for 30 years
at the end of 30 years your account has \$4919.21

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b) $y' = .012\, y \quad y(0) = 5000$

$y = e^{.012t} C$ $\quad 5000 = e^{.012(0)} \cdot C$ so $C = 5000$

(I meant to put a .12 not .012. oops

$y = 5000\, e^{.012t}$ $\quad$ @ $t = 20$ we have
$y(20) = 6356.25$

Invest 5000 @ 1.2% interest (an interest bearing checking account)
and @ the end of 20 years you'll have 6356.25

If it had been 12% instead of 1.2, then

$y = 5000\, e^{.12(20)} = 55115.88$ → a huge growth

---

c) $y' = -.05y \quad y(0) = 10{,}000 \qquad y = e^{-.05t} \cdot 10{,}000 \qquad y(2) = 10{,}000\, e^{-.1}$
$= 9048.37$

If the stock market drops an average of 5% per year for 2 years, a \$10,000 investment will drop to 9048.37

2)

a) $y_1' = 2y_1 + 4y_2$ $\quad y_1(0) = 1$

$y_2' = 4y_1 + 2y_2$ $\quad y_2(0) = 4$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

evals

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 16$$
$$= 4 - 4\lambda + \lambda^2 - 16$$
$$= \lambda^2 - 4\lambda - 12$$
$$(\lambda-6)(\lambda+2)$$
$$\lambda = 6, -2$$

If $\lambda = 6$, $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If $\lambda = -2$ $\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (either works)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{+6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

@ $t=(0)$ we have $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ so

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}$$

so $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + \frac{3}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

or $$\boxed{\begin{aligned} y_1 &= 5/2\, e^{6t} - 3/2\, e^{-2t} \\ y_2 &= 5/2\, e^{6t} + 3/2\, e^{-2t} \end{aligned}}$$

Now you can check your answer by computing

$y_1' = 15e^{6t} + 3e^{-2t}$

$y_2' = 15e^{6t} - 3e^{-2t}$

$2y_1 + 4y_2 = (5e^{6t} - 3e^{-2t}) + (10e^{6t} + 6e^{-2t})$

$= 15e^{6t} + 3e^{-2t} = y_1'$ ✓ yep

and $4y_1 + 2y_2 = (10e^{6t} - 6e^{-2t}) + (5e^{6t} + 3e^{-2t})$

$= 15e^{6t} - 3e^{-2t} = y_2'$ ✓ yep

2) b $y_1' = y_1 + 2y_2$ $\quad y_1(0) = 6 \quad y_2(0) = 0$

$y_2' = 3y_1 \quad \vec{y}(0) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$\vec{y}' = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}}_{A=} \vec{y}$ $\quad$ eigvals $\begin{vmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6$

$(\lambda-3)(\lambda+2) = 0$

$\boxed{\lambda = 3, -2}$

$\lambda = 3 \quad \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2 \quad \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$ I'll use $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2t}$

Now use $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^0 = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \left(\frac{1}{5}\right) \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 18 \\ -6 \end{bmatrix} = \begin{bmatrix} 18/5 \\ -6/5 \end{bmatrix}$$

So $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{18}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - \frac{6}{5} \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2t}$

$y_1 = \frac{18}{5} e^{3t} + \frac{12}{5} e^{-2t}$

$y_2 = \frac{18}{5} e^{3t} - \frac{18}{5} e^{-2t}$

Check

$y_1' = \frac{54}{5} e^{3t} - \frac{24}{5} e^{-2t}$

$y_2' = \frac{54}{5} e^{3t} + \frac{36}{5} e^{-2t}$

$y_1 + 2y_2 = \frac{54}{5} e^{3t} - \frac{24}{5} e^{-2t}$ ✓

$3y_1 = \frac{54}{5} e^{3t} + \frac{36}{5} e^{-2t}$ ✓

yep. It's correct

2c) $y_1' = y_1 + 4y_2$
$y_2' = 3y_1 + 2y_2$ $= \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 12$$
$$= \lambda^2 - 3\lambda + 2 - 12$$
$$= \lambda^2 - 3\lambda - 10$$
$$(\lambda-5)(\lambda+2)$$
$$\underline{\lambda = 5, -2}$$

$\lambda = 5$ $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 3 & -3 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2$ $\left[\begin{array}{cc|c} 3 & 4 & 0 \\ 3 & 4 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} e^{-2t}}$$

Using the conditions $y_1(0) = 0$, $y_2(0) = 1$
we have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 1/7 \end{bmatrix}$

Our solution is.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{4}{7}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + \frac{1}{7}\begin{bmatrix} -4 \\ 3 \end{bmatrix} e^{-2t}$$

2d) $y_1' = y_2$ $\quad y_1(0) = 1$

$y_2' = -3y_1 - 4y_2 \quad y_2(0) = 2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{vmatrix} 0-\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = 4\lambda + \lambda^2 + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\boxed{\lambda = -3, -1}$$

If $\lambda = -3$ $\left(\begin{array}{cc|c} 3 & 1 & 0 \\ -3 & -1 & 0 \end{array}\right) \rightarrow \vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\lambda = -1$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right) \rightarrow \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Now @ $t=0$ we have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{pmatrix} -3/2 \\ 5/2 \end{pmatrix}$

So $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -3/2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t} + 5/2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

$y_1 = -3/2\, e^{-3t} + 5/2\, e^{-t}$

$y_2 = +9/2\, e^{-3t} - 5/2\, e^{-t}$

check $y_1' = \frac{9}{2} e^{-3t} - 5/2\, e^{-t} = y_2$ ✓

$-3y_1 - 4y_2$

$y_2' = \frac{-27}{2} e^{-3t} + \frac{5}{2} e^{-t} = \left(\frac{9}{2} e^{-3t} - \frac{15}{2} e^{-t}\right) + \left(\frac{-36}{2} e^{-3t} + \frac{20}{2} e^{-t}\right)$

$-36 + 9 = -27$ ✓

$-15 + 20 = 5$ ✓

yep ~~It's~~ correct.

3) a $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\lambda = 1, 3$ (its triangular so use diagonals).

$\lambda = 1$ $\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3$ $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

check

$$Q^{-1}AQ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \checkmark$$

3b) error. I meant $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. OH well, well do it both ways

$$\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix} = \lambda + \lambda^2 + 2$$
$$\lambda^2 + \lambda + 2$$

$$\frac{-1 \pm \sqrt{1-4(1)(2)}}{2(1)} = \frac{-1 \pm i\sqrt{3}}{2}$$ ugly

Try $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ instead.

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} = 2\lambda + \lambda^2 + 1 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \quad \lambda = -1, -1$$

$\lambda = -1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightarrow \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ There is only 1 eigenvector.

To find Generalized eigenvector, augment by eigenvector instead of by zero.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \rightarrow \text{rref} \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \text{ so } x+y=1$$

both $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are solutions. I'll pick $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = V_2$

Then $Q = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

and $$Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = J$$

3 C) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\lambda = 1, 1, 1$ (a triple eigenvalue so algebraic multiplicity is 3.)

$A - 1I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ eigenvalue is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

geometric multiplicity is 1.

I need 2 generalized eigenvectors. (one at rank 2, one at rank 3)

Solve $\left[\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{matrix} 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$

x = arbitrary. let $= 0$
$y = 1/2$
$z = 0$

$\begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} = \vec{v}_2$

Solve $\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x = arbitrary let $= 0$
$y = -1/4$
$z = 1/4$

$v_3 = \begin{bmatrix} 0 \\ -1/4 \\ 1/4 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$ $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

3 d) $A=\begin{bmatrix}1&2&2\\0&1&0\\0&0&1\end{bmatrix}$ $\lambda = 1,1,1$ (algebraic multiplicity is 1)

2 free variables

Evecs $\left[\begin{array}{ccc|c}0&2&2&0\\0&0&0&0\\0&0&0&0\end{array}\right] \xrightarrow{\text{rref is}} \left[\begin{array}{ccc|c}0&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$ $\vec{X}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $X_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$

(geom mult is 2)

Now we need 1 more. Solve.

$\left[\begin{array}{ccc|c}0&2&2&1\\0&0&0&0\\0&0&0&0\end{array}\right]$ $\left[\begin{array}{ccc|c}0&2&2&0\\0&0&0&1\\0&0&0&-1\end{array}\right]$

no solution

There is a solution, either $\begin{bmatrix}0\\1/2\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1/2\end{bmatrix}$. You can pick either.

$Q = \begin{bmatrix}1&0&0\\0&1/2&1\\0&0&-1\end{bmatrix}$ $J = \left[\begin{array}{cc|c}1&1&0\\0&1&0\\\hline 0&0&1\end{array}\right]$ $Q^{-1} = \begin{bmatrix}1&0&0\\0&2&2\\0&0&-1\end{bmatrix}$

2 vectors make a 2 by 2 block

check $Q^{-1}AQ = \begin{bmatrix}1&0&0\\0&2&2\\0&0&-1\end{bmatrix}\begin{bmatrix}1&2&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/2&1\\0&0&-1\end{bmatrix}$

$= \begin{bmatrix}1&0&0\\0&2&2\\0&0&-1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1/2&1\\0&0&-1\end{bmatrix}$

$= \left[\begin{array}{cc|c}1&1&0\\0&1&0\\\hline 0&0&1\end{array}\right] = J$ ✓

4) a) $\begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}$ b) $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ c) $\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$

use the computer

Do you see a pattern?

d) $\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$ e) $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}$

g) $\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ h) $\begin{bmatrix} e^{5t} & te^{5t} & \frac{t^2}{2}e^{5t} \\ 0 & e^{5t} & te^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$

i) $e^{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & t & t^2/2 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

j) $e^{\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}} = \begin{bmatrix} e^{3t} & te^{3t} & \frac{t^2}{2}e^{3t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & 0 & 0 & e^{-2t} \end{bmatrix}$

~~3~~ 5) a) $\begin{bmatrix}0 & 1\\ -3 & 4\end{bmatrix}$ $\quad \det\begin{vmatrix}-\lambda & 1\\ -3 & 4-\lambda\end{vmatrix} = -4\lambda+\lambda^2+3$

$$(\lambda-3)(\lambda-1)=0$$

$$\lambda=3 \quad \lambda=1$$

$\lambda=3$

$$\left[\begin{array}{cc|c}-3 & 1 & 0\\ -3 & 1 & 0\end{array}\right] \quad \vec{x}=\begin{bmatrix}1\\ +3\end{bmatrix}$$

$\lambda=1$

$$\left[\begin{array}{cc|c}-1 & 1 & 0\\ -3 & 3 & 0\end{array}\right] \quad \vec{x}=\begin{bmatrix}1\\ +1\end{bmatrix}$$

$$Q=\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix} \quad J=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix} \quad Q^{-1}=\frac{1}{1-3}\begin{bmatrix}1 & -1\\ -3 & 1\end{bmatrix}$$

$$e^J=\begin{bmatrix}e^3 & 0\\ 0 & e^1\end{bmatrix}$$

use computer

Can you do this by hand? This part is just a review of similar matrices

$$e^A = Q e^J Q^{-1}$$

$$= \begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}e^3 & 0\\ 0 & e^1\end{bmatrix}\begin{bmatrix}1 & -1\\ -3 & 1\end{bmatrix}\left(-\frac{1}{2}\right)$$

$$= \begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}e^3 & -e^3\\ -3e^1 & +e^1\end{bmatrix}\left(-\frac{1}{2}\right)$$

$$= \begin{bmatrix}e^3-3e^1 & -e^3+e^1\\ 3e^3-3e^1 & -3e^3+e^1\end{bmatrix}\cdot\left(-\frac{1}{2}\right)$$

$$= \begin{bmatrix}\frac{3e-e^3}{2} & \frac{e-e^3}{2}\\ \frac{3e-3e^3}{3} & \frac{e-3e^3}{2}\end{bmatrix}$$

use the computer

~~3~~ 5 b) $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 5\lambda+\lambda^2+6$

$(\lambda+2)(\lambda+3)=0$

$\lambda=-2,-3$

$\lambda=-2$

$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right]$ $\vec{x}=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda=-3$

$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -6 & -2 & 0 \end{array}\right]$ $\vec{x}=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

can you do by hand?

$$\boxed{Q=\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \quad J=\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \quad Q^{-1}=\frac{1}{-3+2}\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}}$$

$$e^{Jt}=\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$ (we'll get the answer if $t=1$).

$$e^{A}=Qe^{Jt}Q^{-1}$$

$$=\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 3e^{-2t} & e^{-2t} \\ -2e^{-3t} & -e^{-3t} \end{bmatrix}$$

$$=\left[\begin{array}{c|c} 3e^{-2t}-2e^{-3t} & e^{-2t}-e^{-3t} \\ -6e^{-2t}+6e^{-3t} & -2e^{-2t}+3e^{-3t} \end{array}\right]$$

let $t=1$ to get answer.

use the computer

5 c) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 2\lambda + \lambda^2 + 1 = (\lambda+1)^2$ $\lambda = -1, -1.$

$\lambda = -1$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$ $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

to get generalized we solve

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right]$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{Q = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad Q^{-1} = \frac{1}{0-(-1)} \begin{bmatrix} 0 & -1 \\ +1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}}$

can you do by hand?

$$e^{Jt} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$e^{At} = Qe^{Jt}Q^{-1} -$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} te^{-t} & -e^{-t} + te^{-t} \\ e^{-t} & e^{-t} \end{bmatrix}$$

Use the computer

$$= \begin{bmatrix} te^{-t} + e^{-t} & -e^{-t} + te^{-t} + e^{-t} \\ -te^{-t} & +e^{-t} - te^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} te^{-t} + e^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

let $t = 1$ to get answer.

3d) 5

$\begin{bmatrix}0 & 1\\ -4 & -4\end{bmatrix}$ $\qquad \left|\begin{bmatrix}-\lambda & 1\\ -4 & -4-\lambda\end{bmatrix}\right| = 4\lambda+\lambda^2+4 = (\lambda+2)^2 = 0$

$\lambda = -2$

$\left[\begin{array}{cc|c}2 & 1 & 0\\ -4 & -2 & 0\end{array}\right] \quad x_1 = \begin{bmatrix}1\\ -2\end{bmatrix} \qquad \left[\begin{array}{cc|c}2 & 1 & 1\\ -4 & -2 & -2\end{array}\right] \quad x_2 = \begin{bmatrix}0\\ 1\end{bmatrix}$

$Q = \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix} \quad J = \begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix} \quad Q^{-1} = \frac{1}{1-0}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$

can you do by hand?

$$e^{Jt} = \begin{bmatrix}e^{-2t} & te^{-2t}\\ 0 & e^{-2t}\end{bmatrix}$$

$$e^{A} = Qe^{Jt}Q^{-1}$$

$$= \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} & te^{-2t}\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}e^{-2t} & te^{-2t}\\ -2e^{-2t} & -2te^{-2t}+e^{-2t}\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

use the computer.

$$= \left[\begin{array}{c|c}e^{-2t}+2te^{-2t} & te^{-2t}\\ \hline -2e^{-2t}-4te^{-2t}+2e^{-2t} & -2te^{-2t}+e^{-2t}\end{array}\right]$$

let $t=1$.

Se $\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda = 0$ $\quad \lambda(\lambda-3) \quad \lambda = 0, \lambda = 3$

$\lambda = 0$
$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right] \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3$
$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \quad Q^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$

Can you do by hand?

$e^{Jt} = \begin{bmatrix} 1 & 0 \\ 0 & e^{3t} \end{bmatrix}$

$$
\begin{aligned}
e^{At} &= Q e^{Jt} Q^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 0 & e^{3t} \end{bmatrix} \frac{1}{3} \\
&= \begin{bmatrix} 3 & -1+e^{3t} \\ 0 & 3e^{3t} \end{bmatrix} \frac{1}{3} \\
&= \begin{bmatrix} 1 & \frac{-1+e^{3t}}{3} \\ 0 & e^{3t} \end{bmatrix}
\end{aligned}
$$

Use the computer.

if $t=1$ then

$$\begin{bmatrix} 1 & \frac{-1+e^{3}}{3} \\ 0 & e^{3} \end{bmatrix}$$

5f) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ $\left| \begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \right| = (2-\lambda)^2 - 16 = 0$

$(2-\lambda)^2 = 16$

$2-\lambda = \pm 4 \quad \lambda = 2 \pm 4$

$= -2, 6$

$\lambda = -2$ $\begin{bmatrix} 4 & 4 & \vdots & 0 \\ 4 & 4 & \vdots & 0 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda = 6$ $\begin{bmatrix} -4 & 4 & \vdots & 0 \\ 4 & -4 & \vdots & 0 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad J = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2}$

can you do by hand

$e^{Jt} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{6t} \end{bmatrix}$

$$e^{At} = Q e^{Jt} Q^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{6t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & -e^{-2t} \\ e^{6t} & e^{6t} \end{bmatrix} \frac{1}{2}$$

use the computer.

$$= \begin{bmatrix} e^{-2t} + e^{6t} & -e^{-2t} + e^{6t} \\ -e^{-2t} + e^{6t} & e^{2t} + e^{6t} \end{bmatrix} \frac{1}{2}$$

$$= \begin{bmatrix} \frac{e^{-2t} + e^{6t}}{2} & \frac{-e^{-2t} + e^{6t}}{2} \\ \frac{-e^{-2t} + e^{6t}}{2} & \frac{e^{2t} + e^{6t}}{2} \end{bmatrix}$$

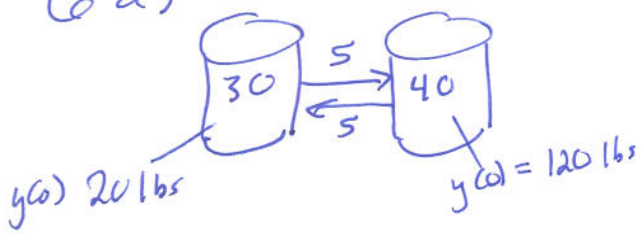
let $t=1$

$\frac{e^{-2} + e^{6}}{2} \quad \frac{-e^{-2} + e^{6}}{2}$

$\frac{-e^{-2} + e^{6}}{2} \quad \frac{e^{2} + e^{6}}{2}$

Jordan form

(6 a)



inflow − out flow

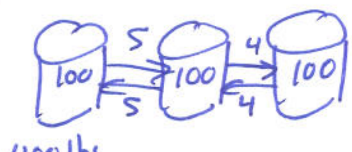
$$y_1' = \frac{5}{40} y_2 - \frac{5}{30} y_1$$

$$y_2' = \frac{5}{30} y_1 - \frac{5}{40} y_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -5/30 & 5/40 \\ 5/30 & -5/40 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{array}{l} y_1(0) = 20 \\ y_2(0) = 120 \end{array}$$

Check online for solution + graph.

(6 b)



inflow − out flow

$$y_1' = \left(\frac{5}{100} y_2\right) - \left(\frac{5}{100} y_1\right)$$

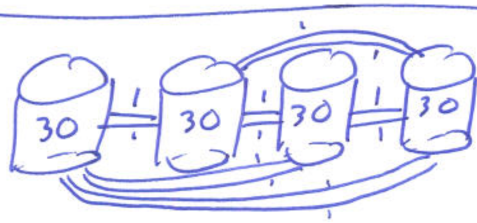
$$y_2' = \left(\frac{5}{100} y_1 + \frac{5}{100} y_3\right) - \left(\frac{10}{100} y_2\right)$$

5+5 = 10 gallon total.

$$y_3' = \left(\frac{5}{100} y_2\right) - \left(\frac{5}{100} y_3\right)$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -5/100 & +5/100 & 0 \\ 5/100 & -10/100 & 5/100 \\ 0 & 5/100 & -5/100 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 400 \\ 0 \\ 0 \end{bmatrix}$$

(6c)



Inflow − Out flow

$$y_1' = \left(\frac{1}{30} y_2 + \frac{1}{30} y_3 + \frac{1}{30} y_4\right) - \left(\frac{3}{30} y_1\right)$$

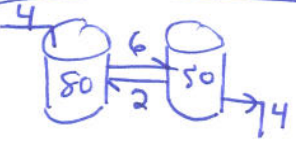
$$y_2' = \left(\frac{1}{30} y_1 + \frac{1}{30} y_3 + \frac{1}{30} y_4\right) - \left(\frac{3}{30} y_2\right)$$

etc.

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} -3/30 & 1/30 & 1/30 & 1/30 \\ 1/30 & -3/30 & 1/30 & 1/30 \\ 1/30 & 1/30 & -3/30 & 1/30 \\ 1/30 & 1/30 & 1/30 & -3/30 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$y_1(0) = 50$
$y_2(0) = 80$
$y_3(0) = 10$
$y_4(0) = 0$

(6d)



Since $y_1(0) = 0$ and $y_2(0) = 0$, the solution is ~~[0,0]~~ always zero

otherwise we write

$$y_1' = \frac{2}{50} y_2 - \frac{6}{80} y_1 = \begin{bmatrix} -6/80 & 2/50 \\ 6/80 & -6/50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_2' = \frac{6}{80} y_2 - \frac{6}{50} y_2$$

~~6~~) a #7

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Its homogeneous so

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \quad \vec{y} = e^{At} c$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5e^{2t} \\ 6e^{4t} \end{bmatrix}$$

So $y_1 = 5e^{2t}$ $\quad y_2 = 6e^{4t}$.

~~6~~ #7 b)

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad e^{At} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$y_1 = -e^{2t} + 3te^{2t} \qquad y_2 = 3e^{2t}$$

~~6c~~) ~~$y'' + 4y' + 3y = 0$~~ $\quad y(0) = 0$ ($y_1(0)$) $\quad y'(0) = 1$ ($y_2(0)$)

#7c

~~$y_1 = y$~~

~~$y_2 = y'$~~

$$\boxed{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \quad A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \quad \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\lambda = -3, -1.$$

$\lambda = -3$
$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ -3 & -1 & 0 \end{array}\right) \quad \vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = -1$
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right) \quad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{Q = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix} \quad J = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \quad Q^{-1} = \frac{1}{-1+3}\begin{bmatrix} -1 & - \\ 3 & 1 \end{bmatrix}}$$

by hand?

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} -e^{-3t} & -e^{-3t} \\ 3e^{-t} & e^{-t} \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} -e^{-3t} + 3e^{-t} & -e^{-3t} + e^{-t} \\ 3e^{-3t} - 3e^{-t} & 3e^{-3t} - e^{-t} \end{bmatrix}\left(\frac{1}{2}\right).$$

~~Since $y_1 = y$ we have $y = \dfrac{(-e^{-3t} + e^{-t})}{2}$ as our solution~~

$$\text{Hence} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} (-e^{-3t} + e^{-t})/2 \\ (3e^{-3t} - e^{-t})/2 \end{bmatrix}}$$

use computer

7d) ~~$y''+2y'+y=0$~~ $y_1(0)=2$ $y_2(0)=0$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$ on 5c we showed $e^{At} = \begin{bmatrix} te^{-t}+e^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{bmatrix}$

So the solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{At}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2te^{-t}+2e^{-t} \\ -2te^{-t} \end{bmatrix}$

So $y_1(t) = \boxed{y(t) = 2te^{-t}+2e^{-t}}$

7e $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2-1$

$2-\lambda = \pm 1$

$\lambda = 2 \pm 1$

$= 1, 3$

$\lambda = 1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda = 3$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $Q^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

by hand.

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & -e^t \\ e^{3t} & e^{3t} \end{bmatrix}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\begin{bmatrix} e^t+e^{3t} & -e^t+e^{3t} \\ -e^t+e^{3t} & e^t+e^{3t} \end{bmatrix}$$

$$\text{so } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{At}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2e^t+2e^{3t} & -e^t+e^{3t} \\ -2e^t+2e^{3t} & +e^t+e^{3t} \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} e^t+3e^{3t} \\ -e^t+3e^{3t} \end{bmatrix}$$

so $y_1 = \frac{1}{2}(e^t+3e^{3t})$ and $y_2 = \frac{1}{2}(-e^t+3e^{3t})$

7f $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

In 3e we found $e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

Use a computer for the whole problem.

$$\text{So } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos t+2\sin t \\ -\sin t+2\cos t \end{bmatrix}$$

(the eigenvalues are complex)

$y_1 = \cos t + 2\sin t$ $\quad y_2 = -\sin t + 2\cos t$