1c) $\vec{u} = (1,2,2)$ $\quad |\vec{u}| = \sqrt{1+4+4} = 3$

$\vec{v} = (3,0,-4)$ $\quad |\vec{v}| = \sqrt{9+0+16} = 5$

$\vec{u}\cdot\vec{v} = 3+0-8 = -5$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-5}{3\cdot 5} = -\frac{1}{3}$$

Not orthogonal.
dot product is <u>not</u> zero
so angle $\neq 90°$.

2h)

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 4 \end{bmatrix}$$

$$CC^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 4 \end{bmatrix} = \left[1\begin{pmatrix}1\\2\end{pmatrix} + 0\begin{pmatrix}0\\3\end{pmatrix} - 1\begin{pmatrix}-1\\4\end{pmatrix} \;\middle|\; 2\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}0\\3\end{pmatrix} + 4\begin{pmatrix}-1\\4\end{pmatrix}\right]$$

$$= \left(\begin{matrix} 1+0+1 \\ 2+0-4 \end{matrix} \;\middle|\; \begin{matrix} 2+0-4 \\ 4+9+16 \end{matrix}\right) = \begin{bmatrix} 2 & -2 \\ -2 & 29 \end{bmatrix}$$

$$C^TC = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix} = \left[1\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}2\\3\\4\end{pmatrix} \;\middle|\; 0\begin{pmatrix}1\\0\\-1\end{pmatrix} + 3\begin{pmatrix}2\\3\\4\end{pmatrix} \;\middle|\; -1\begin{pmatrix}1\\0\\-1\end{pmatrix} + 4\begin{pmatrix}2\\3\\4\end{pmatrix}\right]$$

$$= \begin{bmatrix} 5 & 6 & 7 \\ 6 & 9 & 12 \\ 7 & 12 & 17 \end{bmatrix}$$

2o

next page

20) $C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 4 \\ -1 & 1 & 2 \end{bmatrix}$ (see page 7) for exaples.

Lin comb of columns

$$\left[0\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}0\\3\end{pmatrix} - 1\begin{pmatrix}-1\\4\end{pmatrix} \;\middle|\; 2\begin{pmatrix}1\\2\end{pmatrix} + 0\begin{pmatrix}0\\3\end{pmatrix} + 1\begin{pmatrix}-1\\4\end{pmatrix} \;\middle|\; 1\begin{pmatrix}1\\2\end{pmatrix} + 4\begin{pmatrix}0\\3\end{pmatrix} + 2\begin{pmatrix}-1\\4\end{pmatrix} \right]$$

$$\left[\begin{matrix} 0 + 0 + 1 \\ 0 + 9 - 4 \end{matrix} \;\middle|\; \begin{matrix} 2 + 0 - 1 \\ 4 + 0 + 4 \end{matrix} \;\middle|\; \begin{matrix} 1 + 0 - 2 \\ 2 + 12 \; 8 \end{matrix} \right]$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 5 & 8 & 22 \end{bmatrix}$$

Dot Product.

$$\begin{bmatrix} \underset{0\,0\,1}{[1\,0\,-1]}\begin{bmatrix}0\\3\\-1\end{bmatrix} & \underset{2+0-1}{[1\,0\,-1]}\begin{bmatrix}2\\0\\1\end{bmatrix} & \underset{1+0-2}{[1\,0\,-1]}\begin{bmatrix}1\\4\\2\end{bmatrix} \\ \underset{0+9-4}{[2\,3\,4]}\begin{bmatrix}0\\3\\-1\end{bmatrix} & \underset{4+0+4}{[2\,3\,4]}\begin{bmatrix}2\\0\\1\end{bmatrix} & \underset{2+12+8}{[2\,3\,4]}\begin{bmatrix}1\\4\\2\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 5 & 8 & 22 \end{bmatrix}$$

Lin comb of rows.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 4 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1[0\,2\,1] + 0(3\,0\,4) - 1[-1\,1\,2] \\ 2[0\,2\,1] + 3[3\,0\,4] + 4[-1\,1\,2] \end{bmatrix}$$

$$= \begin{bmatrix} [0\;2\;1] + [0\;0\;0] + [1\;-1\;-2] \\ [0\;4\;2] + [9\;0\;12] + [-4\;4\;8] \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 5 & 8 & 22 \end{bmatrix}$$

3b) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ -3 & -6 & 0 & 12 \end{array}\right] \xRightarrow{R_3+3R_1} \boxed{\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]}$

$$x + 2y = -4$$
$$z = 3$$

Since the middle column does not have a leading 1
let $y = y$. (y is a free variable).
Then ~~w~~write all the others in terms of y.

$$x = -4 - 2y \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$y = y$$
$$z = 3$$

This is the solution.

Sometimes people like to use a variable such as "t" to write the solution in terms of,

$$x = -4 - 2t \qquad t \text{ can be any variable.}$$
$$y = t$$
$$z = 3$$

$$\boxed{(x, y, z) = (-4-2t, t, 3)}$$

4c)

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & -5 \\ 2 & -1 & 3 & 4 \\ 4 & 1 & 4 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -5 \\ 4 & 1 & 4 & 5 \end{bmatrix} R_3 - 2R_1$$

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -5 \\ 0 & 3 & -2 & -3 \end{bmatrix} R_3 - 3R_2 \longrightarrow \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 4 & 12 \end{bmatrix} R_3/4$$

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{R_1 - 3R_3 \\ R_2 + 2R_3}} \begin{bmatrix} 2 & -1 & 0 & -5 \\ 0 & 1 & 0 & +1 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_1 + R_2$$

$$\begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_1/2 \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= -2 \\ x_2 &= 1 \\ x_3 &= 3 \end{aligned}}$$

Solution

4e) $\left[\begin{array}{ccc|c} 2 & 1 & 4 & -1 \\ -1 & 3 & 5 & 2 \\ 0 & 1 & 2 & -2 \end{array}\right]$ swap to simplify later $\left[\begin{array}{cccc} -1 & 3 & 5 & 2 \\ 2 & 1 & 4 & -1 \\ 0 & 1 & 2 & -2 \end{array}\right]$ R_2+2R_1

$\left[\begin{array}{cccc} -1 & 3 & 5 & 2 \\ 0 & 7 & 14 & -3 \\ 0 & 1 & 2 & -2 \end{array}\right]$ $R_1(-1)$ swap $\longrightarrow$ $\left[\begin{array}{cccc} 1 & -3 & -5 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 7 & 14 & -3 \end{array}\right]$ R_3-7R_2

$\left[\begin{array}{cccc} 1 & -3 & -5 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 11 \end{array}\right]$ — problem.

$0 \neq 11$

No solution.

You didn't have to do all the swapping I did. I did it because it placed 1's ~~[illegible]~~ where I needed 1's.

4f) $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 3 & 8 \\ 2 & -4 & 1 & 5 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & -1 & -3 \end{bmatrix} \begin{array}{l} R_2/3 \\ R_3/-1 \end{array}$

$\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_3 - R_2 \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - R_2$

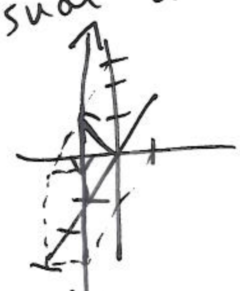
$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 - 2x_2 = 1 \\ x_2 = x_2 \\ x_3 = 3 \end{array}$

x_2 is free variable. Write others in terms of x_2

$$\begin{aligned} x_1 &= 1 + 2x_2 \\ x_2 &= x_2 \\ x_3 &= 3 \end{aligned} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

or $(x, y, z) = (1+2t, t, 3)$

Here's a 3D visual of the line of solutions.



5a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 1 \\ 3 & 0 & 4 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -10 & -11 \\ 0 & -6 & -5 \end{bmatrix} \; R_3(5) - R_2(3)$

$\begin{matrix} 1 & 2 & 3 \\ 0 & -10 & -11 \\ 0 & 0 & \text{not zero} \end{matrix}$

$\boxed{\text{Rank} = 3}$

every column contains a pivot, so the columns are $\boxed{\text{Linearly independent.}}$

No column is a linear combination of any of the others.

5c) $\begin{bmatrix} 1 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \\ 2 & 1 & 3 & -2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{bmatrix} 1 & 3 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & -5 & 5 & -20 \end{bmatrix} R_3+5R_2$

$\begin{bmatrix} 1 & 3 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank = 2 because 2 pivots.

Now to write nonpivot columns as linear combinations of pivot columns.

$$2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$-3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ -2 \end{pmatrix}$$

6a) $\left[\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}\right]$ swap R_1 + R_2 to simplify $\longrightarrow \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 3 & 4 \end{bmatrix} R_2 + 2R_1$

$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 3 & 4 \end{bmatrix} R_3 - 3R_2 \longrightarrow \begin{matrix} -1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & \text{not zero} \end{matrix}$

3 pivots, Rank = 3, 3 columns

$\boxed{\text{Linearly Independent.}}$

6 b) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & -2 \\ -3 & -1 & -3 \\ 4 & -2 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \\ R_4 - 4R_1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & -8 & -12 \\ 0 & 8 & 12 \\ 0 & -14 & -21 \end{bmatrix} \begin{matrix} \\ R_2/-8 \\ R_3/8 \\ R_4/-14 \end{matrix}$

$\begin{matrix} 1 & 3 & 5 \\ 0 & 1 & 3/2 \\ 0 & 1 & 3/2 \\ 0 & 1 & 3/2 \end{matrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 - 3R_2 \qquad 5 - 9/2 = 1/2$

$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Rank = 2 < # of columns

Linearly dependent.

Column 3 $= \left(\frac{1}{2}\right)$(column 1) $+ \left(\frac{3}{2}\right)$(column 2)

$$\boxed{\begin{pmatrix} 5 \\ -2 \\ -3 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ -1 \\ -2 \end{pmatrix}}$$

7d) $\left|\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \\ 2 & 2 & 1 \end{bmatrix}\right|$ You need to do it 2 ways.

use row 1

$+3\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$

$3(-1-8) \quad -1(\underbrace{1-8}_{-7}) \quad -2(2+2)$

$-27 \quad +7 \quad -8 \quad = \quad -28$

use column 2

$\begin{bmatrix} + & - & \\ & + & \\ & - & \end{bmatrix}$

minus plus minus.

$-(1)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} - (2)\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}$

$-1(1-8) - 1(3+4) - 2(12+2)$

$7 \quad -7 \quad -28 \quad = \quad -28$

7h) $\begin{vmatrix} 3 & 2 & 5 & -1 \\ 0 & 8 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & -5 & 3 & -1 \end{vmatrix}$ use column 1 because all entries are zero but 1.

$= +3 \begin{vmatrix} 8 & 4 & 2 \\ -1 & 0 & 0 \\ -5 & 3 & -1 \end{vmatrix}$ now use ~~column~~ row 2.

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$= 3\left(-(-1)\begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix}\right)$

$= (3)(1)(-4-6) = 3(-10) = -30$

---

alternate. Use row 3

$\begin{bmatrix} + \\ - \\ + & - \end{bmatrix}$

$\begin{vmatrix} 3 & 2 & 5 & -1 \\ 0 & 8 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & -5 & 3 & -1 \end{vmatrix} = -(-1) \begin{vmatrix} 3 & 5 & -1 \\ 0 & 4 & 2 \\ 0 & 3 & -1 \end{vmatrix}$

$= 1\left(3 \begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix}\right) = 1 \cdot 3(-4-6)$

$= -30$

7l) $\begin{vmatrix} 1 & -1 & -1 \\ -2 & 3 & 5 \\ 3 & 1 & 9 \end{vmatrix}$

$= +1\begin{vmatrix} 3 & 5 \\ 1 & 9 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 5 \\ 3 & 9 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix}$

$(27-5) + (-18-15) - (-2-9)$

$22 - 33 - (-11) = \boxed{0}$

Since det $= 0$, the columns are linearly <u>dependent</u>

8b)

$$\left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 3 & -5 & 0 & 1 \end{array}\right] R_2(2) + R_1(3)$$

$$\left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 0 & 2 & 3 & 2 \end{array}\right] R_1 - 2R_2$$

$$\left[\begin{array}{cc|cc} -2 & 0 & -5 & -4 \\ 0 & 2 & 3 & 2 \end{array}\right] \begin{array}{l} R_1/-2 \\ R_2/2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5/2 & 4/2 \\ 0 & 1 & 3/2 & 2/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 5/2 & 2 \\ 3/2 & 1 \end{bmatrix}$$

Check $AA^{-1} = \begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 5/2 & 2 \\ 3/2 & 1 \end{bmatrix}$

$$= \left[\begin{array}{c|c} \frac{5}{2}\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 4 \\ -5 \end{pmatrix} & (2)\begin{pmatrix} -2 \\ 3 \end{pmatrix} + (1)\begin{pmatrix} 4 \\ -5 \end{pmatrix} \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

8j)

$$\left[\begin{array}{ccc|ccc} -2 & 0 & 5 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \text{ swap.}$$

$$\left[\begin{array}{cccccc} -1 & 0 & 3 & 0 & 1 & 0 \\ -2 & 0 & 5 & 1 & 0 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+4R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 11 & 0 & 4 & 1 \end{array}\right] \begin{array}{l} R_1(-1) \\ \text{swap.} \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & -3 & 0 & -1 & 0 \\ 0 & 1 & 11 & 0 & 4 & 1 \\ 0 & 0 & -1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_2+11R_3]{R_1-3R_3} \left[\begin{array}{cccccc} 1 & 0 & 0 & -3 & 5 & 0 \\ 0 & 1 & 0 & 11 & -18 & 1 \\ 0 & 0 & -1 & 1 & -2 & 0 \end{array}\right] R_3(-1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5 & 0 \\ 0 & 1 & 0 & 11 & -18 & 1 \\ 0 & 0 & 1 & -1 & +2 & 0 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -3 & 5 & 0 \\ 11 & -18 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

9b) $\begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = \boxed{A^{-1}\vec{b} = x}$

$\downarrow$

Inverse is $\begin{bmatrix} 5/2 & 2 \\ 3/2 & 1 \end{bmatrix}$ from 8b.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5/2 & 2 \\ 3/2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$= 4\begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 6 \\ 4 \end{pmatrix}}$$

$x = 6$

$y = 4$

10 e) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$\left|\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}\right| = (1-\lambda)(3-\lambda)-8$$
$$= 3-4\lambda+\lambda^2-8$$
$$= \lambda^2-4\lambda-5$$
$$= (\lambda-5)(\lambda+1)$$

eigenvalues

$$\boxed{\lambda=5 \quad \lambda=-1}$$

For $\lambda=-1$, $\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow[R_2-R_1]{} \left(\begin{array}{ccc} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x+2y=0$
$y=y$

$\begin{pmatrix} x \\ y \end{pmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

nonzero lin. comb of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

---

for $\lambda=5$ $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $x-y=0$ $x=y$
$y=y$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} y$

Nonzero Linear comb of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

10g) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

use column 1

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)((1-\lambda)(1-\lambda)-0)$$

$$= (1-\lambda)^3$$

$\underline{\underline{\lambda = 1}}$ is a $\underline{\text{triple root}}$.

Eigenvectors

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1/2,\ R_2/2} \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \; R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 is free

$x_1 = x_1$
$x_2 = 0$
$x_3 = 0$

so $\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1}$ $x_1 \neq 0$

non zero lin. comb. of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

10 h) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

use column 1

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$= (1-\lambda)^3$ $\lambda = 1$ is a triple root.

Eigen vectors

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

X_1 + X_3 are free.

$X_1 = X_1$ $\quad$ $X_1 = X_1$

$X_2 + X_3 = 0$ $\quad$ $X_2 = -X_3$

$X_3 = X_3$ $\quad$ $X_3 = X_3$

So

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = X_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvectors are all nonzero lin. comb. of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$