Differential Equations with Linear Algebra

Ben Woodruff[1]

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[1]Mathematics Faculty at Brigham Young University–Idaho, woodruffb@byui.edu

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Chapter 1

Review

This chapter covers the following ideas.

1. Graph basic functions by hand. Compute derivatives and integrals, in particular using the product rule, quotient rule, chain rule, integration by u-substitution, and integration by parts (the tabular method is useful for simplifying notation). Explain how to find a Laplace transform.

2. Explain how to verify a function is a solution to an ODE, and illustrate how to solve separable ODEs.

3. Explain how to use the language of functions in high dimensions and how to compute derivatives using a matrix. Illustrate the chain rule in high dimensions with matrix multiplication.

4. Graph the gradient of a function together with several level curves to illustrate that the gradient is normal to level curves.

5. Explain how to test if a differential form is exact (a vector field is conservative) and how to find a potential.

1.1 Basics

We need to review our ability to graph functions with multiple inputs and/or outputs. The next few problems ask you to practice some skills that will be crucial as the course progresses.

Problem 1.1 Construct graphs of the following functions. Explain how to obtain each graph by transforming and rescaling the first. Then state the amplitude and period of the function.

1. $y = \sin(x)$

2. $y = 5\sin(x) + 1$

3. $y = 4\sin(3(x - \pi)) + 2$

4. $y = 4\sin(3x - \pi) + 2$

Problem 1.2 Consider the function $f(x) = e^{-x}$.

1. Construct graphs of $y = f(x)$ and $y = 2f(-(x + 3)) - 1$.

2. State $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ from your graph.
3. Compute $\lim_{x\to\infty} xf(x)$ and $\lim_{x\to\infty} x^2 f(x)$. [Hint: L'Hopital's rule will help.]

As the semester progresses, we'll need the functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are the hyperbolic trig functions, and we say the hyperbolic sine of x when we write $\sinh x$. These functions are very similar to sine and cosine functions, and have very similarly properties.

Problem 1.3 Three useful facts about the trig functions are (1) $\frac{d}{dx}\sin x = \cos x$, (2) $\frac{d}{dx}\cos x = -\sin x$, and (3) $\cos^2 x + \sin^2 x = 1$. Use the definitions above to show the following:

1. $\frac{d}{dx}\sinh x = \cosh x$,
2. $\frac{d}{dx}\cosh x = \sinh x$, and
3. $\cosh^2 x - \sinh^2 x = 1$.

[Hint: Start by replacing the hyperbolic function with its definition in terms of exponentials. Then perform the computations.]

Problem 1.4 The three facts from the previous problem are crucial tools need to prove that $\frac{d}{dx}\tan x = \sec^2 x$.

1. Use the quotient rule to give a formula for $\frac{d}{dx}\tanh x$ in terms of hyperbolic trig functions.
2. Similarly obtain a formula for the derivative of $\operatorname{sech} x = \frac{1}{\cosh x}$.
3. What is $\frac{d}{dx}\operatorname{csch} x$?

You might ask why these function are called the hyperbolic trig functions. What does a hyperbola have to do with anything?

Problem 1.5 Each pair of parametric equations traces out a curve in the xy plane. Given a Cartesian equation of the curve by eliminating the parameter t, and then graph the curve.

1. $x = \cos t$, $y = \sin t$, $-2\pi < t < 2\pi$.
2. $x = \cosh t$, $y = \sinh t$, $-\infty < t < \infty$.

Give a reason as to why do we call cosh the hyperbolic cosine.

Problem 1.6 Use implicit differentiation to find the derivative of $y = sinh^{-1}x$. Your answer should not involve any hyperbolic trig functions, and should be in terms of x. [Hint: First write $x = sinh(y)$, and then implicitly differentiate both sides. You'll need the key identity from a few problems above to help you finish.]

The problems above asked you to review your differentiation skills. You'll want to make sure you can use the basic rules of differentiation (such as the power, product, quotient, and chain rules). The next few problems will help you review your integration techniques, and you will apply them to two new ideas.

Problem 1.7 Compute the three integrals

$$\int xe^{-x^2}dx \qquad \text{and} \qquad \int_0^1 xe^{-x^2}dx \qquad \text{and} \qquad \int_0^\infty xe^{-x^2}dx.$$

If you have never used the tabular method to perform integration-by-parts, I strongly suggest that you open the online text and read a few examples (see the bottom of page 2).

Problem 1.8 Compute $\displaystyle\int x\sin(5x)dx$ and $\displaystyle\int x^2\sin(5x)dx$.

Problem 1.9 Compute $\displaystyle\int \tanh^{-1} x dx$. The derivative of $\tanh^{-1} x$ is $\dfrac{1}{1-x^2}$.

1.2 Laplace Transforms

Definition 1.1: The Laplace Transform. Let $f(t)$ be a function that is defined for all $t \geq 0$. Using the function $f(t)$, we define the Laplace transform of f to be a few function F where for each s we obtain the value by computing the integral

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt.$$

The domain of F is the set of all s such that the improper integral above converges. The function $f(t)$ is called the inverse Laplace transform of $F(s)$, and we write $f(t) = \mathscr{L}^{-1}(F(s))$.

Note that the Laplace transform of a function with independent variable t is another function with a different independent variable s. After integration, all t's will be removed from $F(s)$. You can of course use any other letters besides t and s.

We will use the Laplace transform throughout the semester to help us solve many problems related to mechanical systems, electrical networks, and more. The mechanical and electrical engineers in this course will use Laplace transforms in many future courses. Our goal in the problems that follow is to practice integration-by-parts. As an extra bonus, we'll learn the Laplace transforms of some basic functions.

Problem 1.10 Compute the integral $\displaystyle\int_0^\infty e^{-st}dt$, and state for which s the integral converges. What is the Laplace transform of $f(t) = 1$? (If the last question seems redundant, then horray.)

Problem 1.11 Compute the Laplace transform of $f(t) = e^{2t}$, and state the domain. Then compute the Laplace transform of $f(t) = e^{3t}$ and state the domain. Finally, compute the Laplace transform of $f(t) = e^{at}$ for any a, and state the domain.

Problem 1.12 Suppose $s > 0$ and n is a positive integer. Explain why

$$\lim_{t\to\infty} \frac{t^n}{e^{st}} = 0.$$

Use this fact to prove that the Laplace transform of t^2 is

$$\mathscr{L}\{t^2\}\frac{2}{s^3}.$$

[You'll need to do integration-by-parts twice, try the tabular method.]

Problem 1.13 In the previous problems, you showed that

$$\mathscr{L}\{t^0\} = \frac{1}{s^1} \qquad \text{and} \qquad \mathscr{L}\{t^2\} = \frac{2}{s^3}.$$

Show that the Laplace transform of t is $\mathscr{L}\{t^1\} = \frac{1}{s^2}$. Then compute the Laplace transforms of t^3, t^4, and so on until you see a pattern. Use this pattern to state the Laplace transform of t^10 and t^n, provided n is a positive integer. [Hint: Try the tabular method of integration-by-parts. After evaluating at 0 and ∞, all terms but 1 will be zero.]

Theorem 1.2. *Since integration can be done term-by-term, and constants can be pulled out of the integral, we have the crucial fact that*

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$$

for functions f, g and constants a, b.

Problem 1.14 Without integrating, rather using the results above, compute the Laplace transform $L(3 + 5t^2 - 6e^{8t})$, and state the domain.

Problem 1.15 Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}$ and $\sinh t = \dfrac{e^t - e^{-t}}{2}$. Use this to prove that

$$\mathscr{L}\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad \text{and} \qquad \mathscr{L}\{\sinh at\} = \frac{a}{s^2 - a^2}.$$

1.3 Ordinary Differential Equations

A differential equation is an equation which involves derivatives (of any order) of some function. For example, the equation $y'' + xy' + \sin(xy) = xy^2$ is a differential equation. An **ordinary differential equation (ODE)** is a differential equation involving an unknown function y which depends on only one independent variable (often x or t). A partial differential equation involves an unknown function y that depends on more than one variable (such as $y(x, t)$). The order of an ODE is the order of the highest derivative in the ODE. A solution to an ODE on an interval (a, b) is a function $y(x)$ which satisfies the ODE on (a, b).

Example 1.3. The first order ODE $y'(x) = 2x$, or just $y' = 2x$, has unknown function y with independent variable x. A solution on $(-\infty, \infty)$ is the function $y = x^2 + C$ for any constant C. We obtain this solution by simply integrating both sides. Notice that there are infinitely many solutions to this ODE.

Typically a solution to an ODE involves an arbitrary constant C. There is often an entire family of curves which satisfy a differential equation, and the constant C just tells us which curve to pick. A **general solution** of an ODE is an infinite class of solutions of the ODE. A **particular solution** is one of the infinitely many solutions of an ODE.

Often an ODE comes with an **initial condition** $y(x_0) = y_0$ for some values x_0 and y_0. We can use these initial conditions to find a particular solution of the ODE. An ODE, together with an initial condition, is called an **initial value problem (IVP)**.

Example 1.4. The IVP $y' = 2x$, $y(2) = 1$, has the general solution $y = x^2 + C$ from the previous problem. Since $y = 1$ when $x = 2$, we have $1 = 2^2 + C$ which means $C = -3$. Hence the solution to our IVP is $y = x^2 - 3$.

Problem 1.16 Consider the ordinary differential equation $y'' + 9y = 0$. By computing derivatives, show that $y(t) = A\cos(3t) + B\sin(3t)$ is a general solution to the ODE, where A and B are arbitrary constants. If we know that $y(0) = 1$ and $y'(0) = 2$, determine the values of A and B.

Problem 1.17 Consider the ordinary differential equation $y\dfrac{dy}{dx} = x^2$. Find a general solution to this ODE by integrating both sides with respect to x. State an interval on which your solution is valid.

They could introduce the entire method of separation by parts without me telling them what to do. I just need to ask them to do an integral. Afterward, I could ask them to solve an ODE. Put it in the same problem.

Problem 1.18 Consider the ODE given by $y' = 4ty$. Find a general solution to this ODE. [Hint: Rewrite y' as $\dfrac{dy}{dt}$. Then put all the terms that involve y on one side of the equation, and the terms that involve t on the other. Then it should be similar to the previous problem.]

Problem 1.19 Solve the IVP given by $y' = \dfrac{x^2 - 1}{y^4 + 1}$, where $y(0) = 1$.

1.4 General Functions and Derivatives

Recall that to compute partial derivatives, we hold all but one variable constant and then differentiate with respect to that variable. Partial derivatives can be organized into a matrix Df where columns represents the partial derivative of f with respect to each variable. This matrix, called the derivative or total derivative, takes us into our study of linear algebra. Some examples of functions and their derivatives appear in Table 1.1. When the output dimension is one, the matrix has only one row and the derivative is often called the gradient of f, written ∇f.

In multivariate calculus, we focused our time on learning to graph, differentiate, and analyze each of the types of functions in the table above. The next few problems ask you to review this.

Function	Derivative
$f(x) = x^2$	$Df(x) = [2x]$
$\vec{r}(t) = (3\cos(t), 2\sin(t))$	$D\vec{r}(t) = \begin{bmatrix} -3\sin t \\ 2\cos t \end{bmatrix}$
$\vec{r}(t) = (\cos(t), \sin(t), t)$	$D\vec{r}(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$
$f(x,y) = 9 - x^2 - y^2$	$Df(x,y) = \nabla f(x,y) = [-2x \quad -2y]$
$f(x,y,z) = x^2 + y + xz^2$	$Df(x,y,z) = \nabla f(x,y,z) = \begin{bmatrix} 2x + z^2 & 1 & 2xz \end{bmatrix}$
$\vec{F}(x,y) = (-y, x)$	$D\vec{F}(x,y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\vec{F}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$	$D\vec{F}(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\vec{r}(u,v) = (u, v, 9 - u^2 - v^2)$	$D\vec{r}(u,v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & -2v \end{bmatrix}$

Table 1.1: The table above shows the (matrix) derivative of various functions. Each column of the matrix corresponds a partial derivative of the function. When the output of a function is a vector, partial derivatives are vectors which are placed in columns of the matrix. The order of the columns matches the order in which you list the variables.

Problem 1.20 Let $\vec{r}(t) = \left\langle t^2 - 1, 2t + 3 \right\rangle$. Construct a graph of $\vec{r}(t)$, and compute the derivative $D\vec{r}(t)$.

Problem 1.21 Let $f(x,y) = 4 - x^- y^2$. Construct a 3D graph of $z = f(x,y)$. Also construct a graph of several level curves. Then compute the derivative $Df(x,y)$.

Recall that a level curve of $z = f(x,y)$ is curve in the xy plane where the output z is constant.

Problem 1.22 Let $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$. Construct a 3D graph of $\vec{r}(t)$, and compute the derivative $D\vec{r}(t)$.

Problem 1.23 Let $\vec{F}(x,y) = (y, -2x)$. Construct a 2D graph of this vector field, and compute the derivative $D\vec{F}(x,y)$.

1.4.1 The General Chain Rule

The chain rule in first semester calculus states that

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

You may remember this as "the derivative of the outside function times the derivative of the inside function." In multivariable calculus, most textbooks use a tree rule to develop the formula

$$\frac{df}{dt} = f_x x_t + f_y y_t$$

for a function $f(x, y)$, where x and y depend on t (so that $\vec{r}(t) = (x(t), y(t))$ is a curve in the xy plane). Written in matrix form, the chain rule is simply

$$\frac{df}{dt} = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = Df \cdot Dr,$$

which is the (matrix) product of the derivatives, just as it was in first semester calculus. You are welcome to tackle the following problems by using the tree rule or matrix product.

Problem 1.24 Suppose that $f(x, y) = x^2 + 3xy$, where $x = t^2 + 1$ and $y = \sin t$, so we could write $\vec{r}(t) = (t^2 + 1, \sin t)$.

1. Compute $Df(x, y)$, $\dfrac{dx}{dt}$, and $D\vec{r}(t)$. (You should have two matrices.)
2. Compute $\frac{df}{dt}$.

Problem 1.25 Suppose that $f(x, y) = x + 3y$ and that $\dfrac{dx}{dt} = \cos t$ and $\dfrac{dy}{dt} = e^t$. Compute $\frac{df}{dt}$.

Problem 1.26 Suppose that $z = f(x, y)$ and that $\dfrac{\partial f}{\partial x} = 3x^2 y$ and $\dfrac{\partial f}{\partial y} = x^3 y - e^y$. Also suppose that $x = \sqrt{t}$ and $y = \ln t$. Compute $\frac{df}{dt}$.

Problem 1.27 Suppose that $z = f(x, y)$ is a differential function of two variables. Suppose that $\vec{r}(t)$ is a parametrization of a level curve of f. We can write the level curve in vector form as $\vec{r}(t) = (x(t), y(t))$, or in parametric form $x = x(t)$ and $y = y(t)$.

1. If $f(\vec{r}(0)) = 7$, then what is $f(\vec{r}(2))$?
2. Why does $\frac{df}{dt} = \nabla f(x, y) \cdot \dfrac{d\vec{r}}{dt}$?
3. Why is the gradient of f normal to level curves?

Recall that the word normal means there is a 90 degree angle between the gradient and the level curve.

Before proceeding, let's practice with an examples to visually remind us that the gradient is normal to level curves. This key fact will help us solve most of the differential equations we encounter in the course.

Problem 1.28 Consider the function $f(x, y) = x^2 - y$. Start by computing the gradient. Then construct a graph which contains several level curves of f, as well as the gradient at several points on each level curve.

1.5 Potentials of Vector Fields and Differential Forms

When the output dimension of a function is one, so we would write $f:\mathbb{R}^n\to\mathbb{R}^1$, then we call the derivative the gradient and write $\vec{\nabla} f=(f_x,f_y,f_z)$. Notice that this is a vector field. Taking a derivative gives us a vector field. Is every vector field the derivative of some function? Hopefully you remember that the answer to this question is "No."

If a vector field $\vec{F}=(M,N)$ (or in 3D $\vec{F}=(M,N,P)$) is the gradient of some some function f (so that $\vec{\nabla} f=\vec{F}$), then we say that the vector field $\vec{F}$ is a gradient field (or conservative vector field). We say that f is a potential for the vector field $\vec{F}$ when $\nabla f=\vec{F}$. In this section, we'll review how to determine if a vector field has a potential, as well as how to find a potential.

Problem 1.29 Let $\vec{F}=(M,N)=(2x+y,x+4y)$. Find a potential for $\vec{F}$ by doing the following.

1. If we suppose $M=2x+y$ is the partial of f with respect to x, then $f_x=2x+y$. Find a function f whose partial with respect to x is M.
2. If we suppose $N=x+4y$ is the partial of f with respect to y, then $f_y=x+4y$. Find a function f whose partial with respect to y is N.
3. What is a potential for $\vec{F}$? Prove your answer is correct by computing the gradient of your answer.

By taking derivatives, there is a test that tells you if a function will have a potential. Some textbooks call it the test for a conservative field.

Problem 1.30: Test for a conservative vector field. Let's prove the test for a conservative vector field in both 2 and 3 dimensions.

The test for a conservative vector field states states more than what you showed in this problem. It states that if $\vec{F}$ is a continuously differentiable vector field on a simply connected domain, then (1) if $\vec{F}$ has potential, then certain pairs of partials must be equal, and (2) if those pairs of partial derivatives are equal, then the $\vec{F}$ has a potential. We will not prove part (2).

1. Suppose that $\vec{F}(x,y)=(M,N)$ is a continuously differentiable vector field on the entire plane. Suppose further that $\vec{F}$ has a potential f. The derivative of $\vec{F}$ is
$$D\vec{F}(x,y)=\begin{pmatrix} M_x & M_y\\ N_x & N_y\end{pmatrix}.$$
Some of the entries in this matrix must be equal? Which ones? Explain. [If you're not sure, try taking the derivative of the problem above.]
2. Suppose that $\vec{F}(x,y,z)=(M,N,P)$ is a continuously differentiable vector field on all of space. Suppose further that $\vec{F}$ has a potential f. State the derivative of $\vec{F}$, and then state which pairs of entries must be equal.

Problem 1.31 For each vector field below, either give a potential, or explain why no potential exists.

1. $\vec{F}=(4x+5y,5x+6y)$
2. $\vec{F}=(2x-y,x+3y)$
3. $\vec{F}=\left(4x+\dfrac{2y}{1+4x^2},\arctan(2x)\right)$
4. $\vec{F}=(3y+2yz,3x+2xz+6z,2xy+6y)$

We'll finish by introducing the vocabulary of differential forms. We'll use this vocabulary throughout the semester as we study differential equations. The vocabulary of vector fields parallels the vocabulary of differential forms.

Definition 1.5: Differential Forms. Assume that f, M, N, P are all functions of three variables x, y, z. Similar definitions hold in all dimensions.

- A differential form is an expression of the form $Mdx + Ndy + Pdz$ (just as a vector field is a function $\vec{F} = (M, N, P)$).
- The differential of a function f is the expression $df = f_x dx + f_y dy + f_z dz$ (just as the gradient is $\vec{\nabla} F = (f_x, f_y, f_z)$).
- If a differential form is the differential of a function f, then the differential form is said to be exact (just as we say a vector field is a gradient field). Again, the function f is called a potential for the differential form.

A differential form is exact precisely when the corresponding vector field is a gradient field.

Notice that $Mdx + Ndy + Pdz$ is exact if and only if $\vec{F} = (M, N, P)$ is a gradient field. The language of differential forms is practically the same as the language of conservative vector fields. Why do we have different sets of words for the same idea? That happens all the time when different groups of people work on seeming different problems, only to discover years later that they have been working on the same problem. If both sets of vocabulary stick, it's often because both have advantages. We have many different notations for the derivative (such as y', $\frac{dy}{dx}$, and Df), and each notation has advantages. The language of differential forms is best suited when studying differential equations.

Problem 1.32 For each differential form below, state if the differential form is exact. If it is exact, give a potential.

1. $(2x + 3y)dx + (4x + 5y)dy$
2. $(2x - y)dx + (3y - x)dy$
3. $\left(4x + \dfrac{3y}{1 + 9x^2}\right) dx + \arctan(3x)dy$
4. $(3y + 2yz)dx + (3x + 2xz + 6z)dy + (2xy + 5y)dz$

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Linear Algebra Arithmetic

This chapter covers the following ideas.

1. Be able to use and understand matrix and vector notation, addition, scalar multiplication, the dot product, matrix multiplication, and matrix transposing.

2. Use Gaussian elimination to solve systems of linear equations. Define and use the words homogeneous, nonhomogeneous, row echelon form, and reduced row echelon form.

3. Find the rank of a matrix. Determine if a collection of vectors is linearly independent. If linearly dependent, be able to write vectors as linear combinations of the preceding vectors.

4. For square matrices, compute determinants, inverses, eigenvalues, and eigenvectors.

5. Illustrate with examples how a nonzero determinant is equivalent to having independent columns, an inverse, and nonzero eigenvalues. Similarly a zero determinant is equivalent to having dependent columns, no inverse, and a zero eigenvalue.

The next unit will focus on applications of these ideas. The main goal of this unit is to familiarize yourself with the arithmetic involved in linear algebra.

2.1 Basic Notation

Most of linear algebra centers around understanding vectors, with matrices being functions which transform vectors from one vector space into vectors in another vector space. This chapter contains a brief introduction to the arithmetic involved with matrices and vectors. The next chapter will show you many of the uses of the ideas we are learning. You will be given motivation for all of the ideas learned here, as well as real world applications of these ideas, before the end of the next chapter. For now, I want you become familiar with the arithmetic of linear algebra so that we can discuss how all of the ideas in this chapter show up throughout the course.

Definition 2.1. A matrix of size m by n has m rows and n columns. We

Matrix size is row by column.

normally write matrices using capital letters, and use the notation

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{jk}],$$

where a_{jk} is the entry in the jth row, kth column.

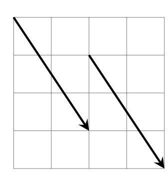
- We say two matrices A and B are equal if $a_{jk} = b_{jk}$ for all j and k.
- We add and subtract matrices of the same size entry wise. So we write $A + B = C$ where $c_{jk} = a_{jk} + b_{jk}$. If matrices do not have the same size, then we cannot add them.
- We can multiply a matrix A by a scalar C to obtain a new matrix cA. We do this multiplying every entry in the matrix A by the scalar c.
- If the number of rows and columns are equal, then we say the matrix is square.
- The main diagonal of a square $(n \times n)$ matrix consists of the entries $a_{11}, a_{22}, \ldots, a_{nn}$.
- The trace of a square matrix is the sum of the entries on the main diagonal $(\sum a_{jj})$.
- The transpose of a matrix $A = [a_{jk}]$ is a new matrix $B = A^T$ formed by interchanging the rows and columns of A, so that $b_{jk} = a_{kj}$. If $A^T = A$, then we say that A is symmetric.

Problem 2.1 Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$. Compute $2A - 3B$, and find the trace of both A and B.

Problem 2.2 Write down a 3 by 2 matrix, and compute the transpose of that matrix. Then give an example of a 3 by 2 symmetric matrix, or explain why it is not possible.

Vectors represent a magnitude in a given direction. We can use vectors to model forces, acceleration, velocity, probabilities, electronic data, and more. We can use matrices to represent vectors. A row vector is a $1 \times n$ matrix. A column vector is an $m \times 1$ matrix. Textbooks often write vectors using bold face font. By hand (and in this book) we add an arrow above them. The notation $\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$ can represent either row or column vectors. Many different ways to represent vectors are used throughout different books. In particular, we can represent the vector $\langle 2, 3 \rangle$ in any of the following forms

$$\langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j} = (2, 3) = \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



Both vectors represent $\langle 2, -3 \rangle$, regardless of where we start.

The notation $(2, 3)$ has other meanings as well (like a point in the plane, or an open interval), and so when you use the notation $(2, 3)$, it should be clear from the context that you are working with a vector. To draw a vector $\langle v_1, v_2 \rangle$, one option is to draw an arrow from the origin (the tail) to the point (v_1, v_2) (the head). However, the tail does not have to be placed at the origin.

The principles of addition and subtraction of matrices apply to vectors (which can be though of as row or column matrices). We will most often think of vectors as column vectors.

Definition 2.2. The magnitude (or length) of the vector $\vec{u} = (u_1, u_2)$ is $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$. In higher dimensions we extend this as

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2 + \cdots u_n^2} = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

A unit vector is a vector with length 1. In many books unit vectors are written with a hat above them, as $\hat{\mathbf{u}}$.

A unit vector $\hat{\mathbf{u}}$ has length $|\vec{u}| = 1$

We will need to be able to find vectors of any length that point in a given direction.

Problem 2.3 Find a vector of length 12 that points in the same direction as the vector $\vec{v} = (1, 2, 3, 4)$. Then give a general formula for finding a vector of length c that points in the direction of $\vec{u}$.

The simplest vectors in 2D are a one unit increment in either the x or y direction, and we write these vectors in any of the equivalent forms

$$\mathbf{i} = \vec{i} = \langle 1, 0\rangle = (1, 0) \quad \text{and} \quad \mathbf{j} = \vec{j} = \langle 0, 1\rangle = (0, 1).$$

We call these the standard basis vectors in 2D. In 3D we include the vector $\mathbf{k} = \vec{j} = \langle 0, 0, 1\rangle$ as well as add a zero to both $\vec{i}$ and $\vec{j}$ to obtain the standard basis vectors. The word basis suggests that we can base other vectors on these basis vectors, and we typically write other vectors in terms of these standard basis vectors. Using only scalar multiplication and vector addition, we can obtain the other vectors in 2D from the standard basis vectors.

The standard basis vectors in 3D
$\mathbf{i} = \vec{i} = \langle 1, 0, 0\rangle = (1, 0, 0)$
$\mathbf{j} = \vec{j} = \langle 0, 1, 0\rangle = (0, 1, 0)$
$\mathbf{k} = \vec{k} = \langle 0, 0, 1\rangle = (0, 0, 1)$

Problem 2.4 Write the vector $(2, 3)$ in the form $(2, 3) = c_1\vec{i} + c_2\vec{j}$.

If instead we use the non-standard basis vectors $\vec{u}_1 = (1, 2)$ and $\vec{u}_2 = (-1, 4)$, then write the vector $(2, 3)$ in the form $(2, 3) = c_1\vec{u}_1 + c_2\vec{u}_2$.

Definition 2.3. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an expression of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$, where c_i is a constant for each i.

A linear combination of vectors is simply a sum of scalar multiples of the vectors. We start with some vectors, stretch each one by some scalar, and then sum the result. Much of what we will do this semester (and in many courses to come) relates directly to understanding linear combinations.

Problem 2.5 The force acting on an object is $\vec{F} = (-3, 2)$ N. The object is in motion and has velocity vector $\vec{v} = (1, 1)$ and acceleration vector $\vec{a} = (-1, 2)$. Write the force as a linear combination of the velocity and acceleration vectors.

Problem 2.6 Write the vector $(2, 3, 1)$ as a linear combination of the standard basis vectors in $\mathbb{R}^3$. Then write $(2, 3, 1)$ as a linear combination of the vectors $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$.

One of the key applications of linear combinations we will make throughout the semester is matrix multiplication. Let's introduce the idea with an example.

Example 2.4. Consider the three vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\2\\-3\end{bmatrix}$, and $\begin{bmatrix}2\\1\\1\end{bmatrix}$. Let's multiply the first vector by 2, the second by -1, and the third by 4, and then sum the result. This gives us the linear combination

$$2\begin{bmatrix}1\\0\\1\end{bmatrix} - 1\begin{bmatrix}0\\2\\-3\end{bmatrix} + 4\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}10\\2\\9\end{bmatrix}$$

We will define matrix multiplication so that multiplying a matrix on the right by a vector corresponds precisely to creating a linear combination of the columns of A. We now write the linear combination above in matrix form

$$\begin{bmatrix}1&0&2\\0&2&1\\1&-3&1\end{bmatrix}\begin{bmatrix}2\\-1\\4\end{bmatrix} = \begin{bmatrix}10\\2\\9\end{bmatrix}.$$

Definition 2.5: A matrix times a vector. We define the matrix product $A\vec{x}$ (a matrix times a vector) to be the linear combination of columns of A where the components of $\vec{x}$ are the scalars in the linear combination. For this to make sense, notice that the vector $\vec{x}$ must have the same number of entries as there are columns in A. We can make this definition more precise as follows. Let $\vec{v}_i$ be the ith column of A so that $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}$, and let $\vec{x} = \begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix}$. Then the matrix product is the linear combination

The product $A\vec{x}$ gives us linear combinations of the columns of A.

$$A\vec{x} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix} = \vec{a}_1x_1 + \vec{a}_2x_2 + \cdots + \vec{a}_nx_n.$$

The definition above should look like the dot product. If you think of A as a vector of vectors, then $A\vec{x}$ is just the dot product of A and $\vec{x}$.

Problem 2.7 Write down a 2 by 4 nonzero matrix, and call it A (fill the matrix with some integers of your choice). Then write down a vector $\vec{x}$ such that the matrix product $A\vec{x}$ makes sense (again, fill the vector with integers of your choice). Then use the definition above to obtain the product $A\vec{x}$.

Definition 2.6: A matrix times a matrix. Let $\vec{b}_j$ represent the jth column of B (so $B = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n\end{bmatrix}$). The product AB of two matrices $A_{m\times n}$ and $B_{n\times p}$ is a new matrix $C_{m\times p} = [c_{ij}]$ where the jth column of C is the product $A\vec{b}_j$. To summarize, the matrix product AB is a new matrix whose jth column is a linear combinations of the columns of A using the entries of the jth column of B to perform the linear combinations.

Problem 2.8 Let $A = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$ and $B = \begin{bmatrix}0&1&4\\-1&2&-3\end{bmatrix}$. Use the definition given above to compute both AB and BA. Be prepared to show the class how you used linear combinations to get the matrix product. (If you are used to using the row dotted by column approach, then this problem asks you to do the matrix product differently.)

We introduced matrix multiplication in terms of linear combinations of column vectors. My hope is that by doing so you immediately start thinking of linear combinations whenever you encounter matrix multiplication (as this is what it was invented to do). There are many alternate ways to think of matrix multiplication. Here are two additional methods.

1. "Row times column approach." The product AB of two matrices $A_{m\times n}$ and $B_{n\times p}$ is a new matrix $C_{m\times p} = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ is the dot product of the of the ith row of A and the jth column of B. Wikipedia has an excellent visual illustration of this approach.

2. Rephrase everything in terms of rows (instead of columns). We form linear combinations of rows using rows. The matrix product $\vec{x}B$ (notice the order is flopped) is a linear combination of the rows of B using the components of x as the scalars. For the product AB, let $\vec{a}_i$ represent the ith row of A. Then the ith row of AB is the product $\vec{a}_iB$. We'll most often use the column definition instead of this, because we use the function notation $f(x)$ from calculus, and later we will use the notation $A(\vec{x})$ instead of $(\vec{x})A$ to describe how matrices act as functions.

Problem 2.9 Let $A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1 & 4\\-1 & 2 & -3\end{bmatrix}$. Use the two alternate definitions above to compute AB. Be prepared to show the class how you used both alternate definitions (You'll need to show your intermediate steps).

Problem 2.10 Do each of the following:

1. Solve the system of equations $x + 2y = 3$, $4x + 5y = 6$.

2. Write the vector $\begin{pmatrix}3\\6\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\4\end{pmatrix}$ and $\begin{pmatrix}2\\5\end{pmatrix}$.

3. Let $A = \begin{pmatrix}1 & 2\\4 & 5\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}3\\6\end{pmatrix}$. Find a vector $\vec{x}$ so that $A\vec{x} = \vec{b}$. This matrix A is called the coefficient matrix of the system in the first part.

How are these three questions related?

Prior to introducing Gaussian elimination, let's solve a system of equations using an elimination method. If $2x + 3y = 4$ and $5x + 7y = 0$, then we can eliminate x from the second equation by multiplying both sides of the first equation by 5, and both sides of the second equation by 2, and then subtracting. This would give us the equations $10x + 15y = 20$ and $10x + 14y = 1$. The first equation minus the second then gives $(10 - 10)x + (15 - 14)y = (20 - 0)$, or more simply $y = 20$. Similarly, you could multiply the first equation by 7, and the second by 3, to eliminate y.

Problem 2.11 Solve the system of equations

$$\begin{aligned}2x + 3y - 4z &= 4\\3x + 4y - 3z &= 8\\7x + 12y - 12z &= 19.\end{aligned}$$

Use elimination to find your solution. Eliminate x from the 2nd and 3rd equations (which will give you two equations that do not involve x). Then use one of these simplified equations to eliminate y from the other simplified equation. At this point you should have an equation that only involves z. Then use back substitution to give y and x.

Problem 2.12 Answer the following.

1. Suppose that $ax + by = c$ and $dx + ey = f$, where a, b, c, d, e, f are all constants. This is a system of equations with 2 equations and 2 unknowns. Each equation represents a line in the plane. How many solutions are there to this system? (You should have a few different cases.)

2. Suppose that $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$ and $a_{31}x + a_{32}y + a_{33}z = b_1$, where each a_{ij} is a constant. This is a system of equations with 3 equations and 3 unknowns. Each equation represents a plane in space. How many solutions are there to this system? (You should have a few different cases.)

3. Suppose that $a_{11}x + a_{12}y + a_{13}z = b_1$ and $a_{21}x + a_{22}y + a_{23}z = b_2$, where each a_{ij} is a constant. This is a system of equations with 2 equations and 3 unknowns. Each equation represents a plane in space. How many solutions are there to this system? (You should have a few different cases.)

Definition 2.7. We say that a system of linear equation is consistent, if it has at least one solution. We say it is inconsistent if there is no solution.

2.2 Gaussian Elimination

Gaussian elimination is an efficient algorithm we will use to solve systems of equations. This is the same algorithm implemented on most computers systems. The main idea is to eliminate each variable from all but one equation/row (if possible), using the following three operations (called elementary row operations):

1. Multiply an equation (or row of a matrix) by a nonzero constant,

2. Add a nonzero multiple of any equation (or row) to another equation,

3. Interchange two equations (or rows).

These three operations are the operations learned in college algebra when solving a system using a method of elimination. Gaussian elimination streamlines elimination methods to solve generic systems of equations of any size. The process involves a forward reduction and (optionally) a backward reduction. The forward reduction creates zeros in the lower left corner of the matrix. The backward reduction puts zeros in the upper right corner of the matrix. We eliminate the variables in the lower left corner of the matrix, starting with column 1, then column 2, and proceed column by column until all variables which can be eliminated (made zero) have been eliminated. Before formally stating the algorithm, let's look at a few examples.

Example 2.8. Let's start with a system of 2 equations and 2 unknowns. I will write the augmented matrix representing the system as we proceed. To solve

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ 2x_1 - 5x_2 &= 1 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 2 & -5 & 1 \end{array}\right]$$

we eliminate the $2x_1$ in the 2nd row by adding -2 times the first row to the second row.

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ x_2 &= -7 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 1 & -7 \end{array}\right]$$

The matrix at the right is said to be in **row echelon form**.

row echelon form

Definition 2.9: Row Echelon Form. We say a matrix is in row echelon form (ref) if

- each nonzero row begins with a 1 (called a leading 1),
- the leading 1 in a row occurs further right than a leading 1 in the row above, and
- any rows of all zeros appear at the bottom.

The position in the matrix where the leading 1 occurs is called a pivot. The column containing a pivot is called a pivot column.

pivot column

At this point in our example, we can use "back-substitution" to get $x_2 = -7$ and $x_1 = 4+3x_2 = 4-21 = -17$. Alternatively, we can continue the elimination process by eliminating the terms above each pivot, starting on the right and working backwards. This will result in a matrix where all the pivot columns contain all zeros except for the pivot. If we add 3 times the second row to the first row, we obtain.

$$\begin{aligned} x_1 &= -17 \\ x_2 &= -7 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & -17 \\ 0 & 1 & -7 \end{array}\right]$$

The matrix on the right is said to be in **reduced row echelon form** (or just rref). We can easily read solutions to systems of equations directly from a matrix which is in reduced row echelon form.

Definition 2.10: Reduced Row Echelon Form. We say that a matrix is in reduced row echelon form (rref) if

reduced row echelon form - rref

- the matrix is in row echelon form, and
- each pivot column contains all zeros except for the pivot (leading one).

Example 2.11. Let's now solve a nonhomogeneous (meaning the right side is not zero) system with 3 equations and 3 unknowns:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2 \\ x_1 - 2x_2 &= 3 \\ 4x_2 + 2x_3 &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right].$$

We'll encounter some homogeneous systems later on. To simplify the writing, we'll just use matrices this time. To keep track of each step, I will write the row operation next to the row I will replace. Remember that the 3 operations are (1)multiply a row by a nonzero constant, (2)add a multiple of one row to another, (3) interchange any two rows. If I write $R_2 + 3R1$ next to R_2, then this means I will add 3 times row 1 to row 2. If I write $2R_2 - R1$ next to R_2, then I have done two row operations, namely I multiplied R_2 by 2, and then added (-1) times $R1$ to the result (replacing $R2$ with the sum). The steps below read left to right, top to bottom. In order to avoid fractions, I wait to divide until the last step, only putting a 1 in each pivot at the very end.

$$\Rightarrow^{(1)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right] \begin{array}{c} \\ 2R_2 - R_1 \\ \\ \end{array} \quad \Rightarrow^{(2)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 4 & 2 & 1 \end{array}\right] \begin{array}{c} \\ \\ 5R_3 + 4R_2 \end{array}$$

$$\Rightarrow^{(3)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 14 & 21 \end{array}\right] \begin{array}{c} \\ \\ R_3/7 \end{array} \quad \Rightarrow^{(4)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -10 & 2 & 8 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} 2R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\Rightarrow^{(5)} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 7 \\ 0 & -10 & 0 & 5 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} \\ R_2/5 \\ \\ \end{array} \quad \Rightarrow^{(6)} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 7 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array}$$

$$\Rightarrow^{(7)} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 8 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} R_1/4 \\ R_2/-2 \\ R_3/2 \end{array} \quad \Rightarrow^{(8)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

Writing the final matrix in terms of a system, we have the solution $x_1 = 2, x_2 = -1/2, x_3 = 3/2$. Remember that this tells us (1) where three planes intersect, (2) how to write the 4th column $\vec{b}$ in our original augmented matrix as a linear combination of the columns of the coefficient matrix A, and (3) how to solve the matrix equation $A\vec{x} = \vec{b}$ for $\vec{x}$.

The following steps describe the Gaussian elimination algorithm that we used above. Please take a moment to compare what is written below with the example above. Most of the problems in this unit can be solved using Gaussian elimination, so we will practice it as we learn a few new ideas.

1. Forward Phase (row echelon form) - The following 4 steps should be repeated until you have mentally erased all the rows or all the columns. In step 1 or 4 you will erase a column and/or row from the matrix.

   (a) Consider the first column of your matrix. Start by interchanging rows (if needed) to place a nonzero entry in the first row. If all the elements in the first column are zero, then ignore that column in future computations (mentally erase the column) and begin again with the smaller matrix which is missing this column. If you erase the last column, then stop.

   Computer algorithms place the largest (in absolute value) nonzero entry in the first row. This reduces potential errors due to rounding that can occur in later steps.

   (b) Divide the first row (of your possibly smaller matrix) row by its leading entry so that you have a leading 1. This entry is a pivot, and the column is a pivot column. [When doing this by hand, it is often convenient to skip this step and do it at the very end so that you avoid fractional arithmetic. If you can find a common multiple of all the terms in this row, then divide by it to reduce the size of your computations.]

   (c) Use the pivot to eliminate each nonzero entry below the pivot, by adding a multiple of the top row (of your smaller matrix) to the nonzero lower row.

   (d) Ignore the row and column containing your new pivot and return to the first step (mentally cover up or erase the row and column containing your pivot). If you erase the last row, then stop.

   Ignoring rows and columns is equivalent to incrementing row and column counters in a computer program.

2. Backward Phase (reduced row echelon form - often called Gauss-Jordan elimination) - At this point each row should have a leading 1, and you should have all zeros to the left and below each leading 1. If you skipped step 2 above, then at the end of this phase you should divide each row by its leading coefficient to make each row have a leading 1.

   (a) Starting with the last pivot column. Use the pivot in that column to eliminate all the nonzero entries above it, by adding multiples of the row containing the pivot to the nonzero rows above.

   (b) Work from right to left, using each pivot to eliminate the nonzero entries above it. Nothing to the left of the current pivot column changes. By working right to left, you greatly reduce the number of computations needed to fully reduce the matrix.

Example 2.12. As a final example, let's reduce $\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 7 \\ 1 & 3 & 5 & 1 & 6 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right]$ to reduced row echelon form (rref). The first step involves swapping 2 rows. We swap row 1 and row 2 because this places a 1 as the leading entry in row 1.

(1) Get a nonzero entry in upper left

$$\Rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 7 \\ 1 & 3 & 5 & 1 & 6 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ \\ \\ \\ \end{array}$$

(2) Eliminate entries in 1st column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 + 2R_1 \end{array}$$

(3) Eliminate entries in 2nd column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & -6 & -6 & 1 & -20 \\ 0 & 7 & 7 & 2 & 17 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 6R_2 \\ R_4 - 7R_2 \end{array}$$

(4) Make a leading 1 in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & -11 & 22 \\ 0 & 0 & 0 & 16 & -32 \end{array}\right] \begin{array}{l} \\ \\ R_3/(-11) \\ R_4/16 \end{array}$$

(5) Eliminate entries in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

(6) Row Echelon Form

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

At this stage we have found a row echelon form of the matrix. Notice that we eliminated nonzero terms in the lower left of the matrix by starting with the first column and working our way over column by column. Columns 1, 2, and 4 are the pivot columns of this matrix. We now use the pivots to eliminate the other nonzero entries in each pivot column (working right to left).

(7) Eliminate entries in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R3 \\ R_2 + 2R_3 \\ \\ \\ \end{array}$$

(8) Eliminate entries in 2nd column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 8 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ \\ \\ \end{array}$$

(9) Reduced Row Echelon Form

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(10) Switch to system form

$$\Rightarrow \begin{aligned} x_1 + 2x_3 &= -1 \\ x_2 + x_3 &= 3 \\ x_4 &= -2 \\ 0 &= 0 \end{aligned}$$

Recall that a matrix is in reduced row echelon (rref) if:

1. Nonzero rows begin with a leading 1.
2. Leadings 1's on subsequent rows appear further right than previous rows.
3. Rows of zeros are at the bottom.
4. Zeros are above and below each pivot.

We have obtained the reduced row echelon form. When we write this matrix in the corresponding system form, notice that there is not a unique solution to the system. Because the third column did not contain a pivot column, we can write every variable in terms of x_3 (the redundant equation $x_3 = x_3$ allows us to write x_3 in terms of x_3). We are free to pick any value we want for x_3 and still obtain a solution. For this reason, we call x_3 a free variable, and write our infinitely many solutions in terms of x_3 as

Free variables correspond to non pivot columns. Solutions can be written in terms of free variables.

$$\begin{aligned} x_1 &= -1 - 2x_3 \\ x_2 &= 3 - x_3 \\ x_3 &= x_3 \\ x_4 &= -2 \end{aligned} \quad \text{or by letting } x_3 = t \quad \begin{aligned} x_1 &= -1 - 2t \\ x_2 &= 3 - t \\ x_3 &= t \\ x_4 &= -2 \end{aligned}.$$

By choosing a value (such as t) for x_3, we can write our solution in so called parametric form. We have now given a parametrization of the solution set, where t is an arbitrary real number.

parametric form

Problem 2.13 Each of the following augmented matrices requires one row operation to be in reduced row echelon form. Perform the required row operation, and then write the solution to the corresponding system of equations in terms of the free variables.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ -3 & -6 & 0 & 12 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 7 & 0 & 3 \\ 0 & 0 & 1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Problem 2.14 Use Gaussian elimination to solve

$$\begin{aligned} x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 + 3x_3 &= 4 \\ 4x_1 + x_2 + 4x_3 &= 5 \end{aligned}$$

by row reducing the matrix to reduced row echelon form. [Hint: Start by interchanging row 1 and row 2.]

Problem 2.15 Use Gaussian elimination to solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ -x_1 + 2x_2 + 3x_3 &= 8 \\ 2x_1 - 4x_2 + x_3 &= 5 \end{aligned}$$

by row reducing the matrix to reduced row echelon form. [Hint: You should end up with infinitely many solutions. State your solution by writing each variable in terms of the free variable(s).]

Problem 2.16 Use Gaussian elimination to solve

$$\begin{aligned} x_1 + 2x_3 + 3x_4 &= -7 \\ 2x_1 + x_2 + 4x_4 &= -7 \\ -x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 - 2x_3 - x_4 &= 4 \end{aligned}$$

by row reducing the matrix to reduced row echelon form.

2.3 Rank, Linear Independence, Inverses, and Determinants

Definition 2.13.

- The rank of a matrix is the number of pivot columns of the matrix. To find the rank of a matrix, you reduce the matrix using Gaussian elimination until you discover the pivot columns.
- The span of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is all possible linear combinations of the vectors. In terms of matrices, the span of a set of vectors is all possible vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ for some vector $\vec{x}$, where the vectors $\vec{v}_i$ are placed in the columns of A.
- We say that a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if the only solution to the homogeneous system $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. Otherwise we say the vectors are linearly dependent, and it is possible to write one of the vectors as a linear combination of the others. We say the vectors are dependent because one of them depends on (can be obtained as a linear combination of) the others.
- In terms of spans, we say vectors are linearly dependent when one of them is in the span of the other vectors.

As we complete each of the following problems in class, we'll talk about the span of the vectors, and the rank of the corresponding matrix. The key thing we need to focus on is learning to use the words "linearly independent" and "linearly dependent."

Problem 2.17 Are the vectors $\vec{v}_1 = (1, 3, 5)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (0, 3, 1)$ linearly independent? Solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Problem 2.18 Are the vectors $\vec{v}_1 = (1, 2, 0)$, $\vec{v}_2 = (2, 0, 3)$, and $\vec{v}_3 = (3, -2, 6)$ linearly independent? Solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Problem 2.19 Answer each of the following:

1. Suppose you have row reduced a 3 by 3 matrix, and discovered that the rank of the matrix is 2. Are the columns of the matrix independent or dependent? What if the rank was 3?
2. Now suppose you have row reduced a 7 by 7 matrix. If the columns are independent, what possible options do you have for the rank.
3. Now suppose you have row reduced a 7 by 5 matrix. If the columns are independent, what must the rank be.
4. Now suppose you have row reduced a 5 by 7 matrix. Explain why the columns cannot be independent.
5. If you have n vectors placed in the columns of a matrix, what must the rank of the matrix be in order to guarantee that the vectors are independent?

Problem 2.20 Is the vector $[2, 0, 1, -5]$ in the span of

$$\{[1, 0, -1, -2], [1, 2, 3, 0], [0, 1, -1, 2]\}?$$

If it is, then write it as a linear combination of the others. If it is not, then explain why it is not.

Problem 2.21 Find the reduced row echelon form of the matrix

$$B = \begin{bmatrix} 2 & -1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 3 & 3 \end{bmatrix}.$$

Use your result to answer the following questions.

1. Write both $(1, 0)$ and $(0, 1)$ as linear combinations of $(2, 1)$ and $(-1, 1)$.
2. Write $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then write $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
3. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. Find vectors $\vec{x}$ and $\vec{y}$ so that $A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
4. Find a matrix B so that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem: 21, revised Answer each of the following questions.

1. Find the reduced row echelon form of the matrix

$$B = \begin{bmatrix} 2 & -1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 3 & 3 \end{bmatrix}.$$

2. Write $(1, 0)$ as a linear combination of $(2, 1)$ and $(-1, 1)$. Remember, that when writing $c_1(2, 1) + c_2(-1, 1) = (1, 0)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
3. Write $(0, 1)$ as a linear combination of $(2, 1)$ and $(-1, 1)$. Remember, that when writing $c_1(2, 1) + c_2(-1, 1) = (0, 1)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
4. Continue to write each of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. [Hint:At some point, rather than row reducing $\begin{bmatrix} 2 & -1 & \vec{v} \\ 1 & 1 & \end{bmatrix}$, ask yourself how you could use part 1 to answer this.]

5. The following matrix row reduces to give

$$\begin{bmatrix}1&0&2&4&5&8\\0&2&5&2&-1&3\\0&-2&-1&0&2&1\end{bmatrix}\xrightarrow{\text{rref}}\begin{bmatrix}1&0&0&3&\frac{9}{2}&6\\0&1&0&-\frac{1}{4}&-\frac{9}{8}&-1\\0&0&1&\frac{1}{2}&\frac{1}{4}&1\end{bmatrix}.$$

Use this to write both $(4,2,0)$ and $(5,-1,2)$ as a linear combination of the first three columns.

Definition 2.14. The identity matrix I is a square matrix so that if A is a square matrix, then $IA = AI = A$. The identity matrix acts like the number 1 when performing matrix multiplication.

If A is a square matrix, then the inverse of A is a matrix A^{-1} where we have $AA^{-1} = A^{-1}A = I$, provided such a matrix exists.

Problem Let $A = \begin{bmatrix}1&3\\3&4\end{bmatrix}$. We now develop an algorithm for computing the inverse A^{-1}. If an inverse matrix exists, then we know it's the same size as A, so we could let $A^{-1} = \begin{bmatrix}\vec v_1 & \vec v_2\end{bmatrix}$ be the inverse matrix, where $\vec v_1$ and $\vec v_2$ are the columns of A^{-1}.

1. We know that $AA^{-1} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$. Explain why $A\vec v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $A\vec v_2 = \begin{pmatrix}0\\1\end{pmatrix}$.
2. Solve the matrix equations $A\vec v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $A\vec v_2 = \begin{pmatrix}0\\1\end{pmatrix}$. (This involves row reducing $\begin{bmatrix}1&3&1\\3&4&0\end{bmatrix}$ and $\begin{bmatrix}1&3&0\\3&4&1\end{bmatrix}$).
3. What is the reduced row echelon form of $\begin{bmatrix}1&3&1&0\\3&4&0&1\end{bmatrix}$. How is this related to your previous work.
4. State the inverse of A.

The previous problem showed you how to obtain a matrix B so that $AB = I$. You just had to row reduce that matrix $\begin{bmatrix}A & I\end{bmatrix}$ to the matrix $\begin{bmatrix}I & A^{-1}\end{bmatrix}$. The inverse shows up instantly after row reduction.

Problem 2.22 Use the algorithm describe immediately before this problem to compute the inverse of

$$A = \begin{bmatrix}3&0&3\\0&-1&1\\0&3&-4\end{bmatrix}.$$

Then use your work to write each of the standard basis vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ as a linear combination of the columns of A.

Problem 2.23 Let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$. Use Gaussian elimination to show that the inverse of A is

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}.$$

In computing the inverse of a 2 by 2 matrix, the number $ad - bc$ appears in the denominator. We call this number the determinant. If I asked you to compute the inverse of a 3 by 3 matrix, you would again see a number appear in the denominator. We call that number the determinant. This holds true in all dimensions.

Problem: Optional Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Use Gaussian elimination to find the inverse of A, and show that the common denominator is $a(ei - hf) - b(di - gf) + c(dh - ge)$.

Definition 2.15: Determinants of 2 by 2 and 3 by 3 matrices. The determinant of a 2×2 and 3×3 matrix are the numbers

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge) \end{aligned}$$

We use vertical bars next to a matrix to state we want the determinant. Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

In the examples above, we obtained the determinant of a 3 by 3 matrix by computing the determinant of several 2 by 2 matrices. We obtained each 2 by 2 matrix by removing a row and column from the original 3 by 3 matrix. We now add some language to extend the definition above to all dimensions.

Definition 2.16: Minors, Cofactors, and General determinants. Let A be an n by n matrix.

- The minor M_{ij} of a matrix A is the determinant of the the matrix formed by removing row i and column j from A.
- The cofactor C_{ij} is the product of the minor M_{ij} and $(-1)^{i+j}$, so we have $C_{ij} = (-1)^{i+j} M_{ij}$. So it's either the minor, or the opposite of the minor.
- To compute the determinant, first pick a row or column. We define the determinant to be $\sum_{k=1}^{n} a_{ik} C_{ik}$ (if we chose row i) or alternatively $\sum_{k=1}^{n} a_{kj} C_{kj}$ (if we chose column j).
- You can pick ANY row or ANY column you want, and then compute the determinant by multiplying each entry of that row or column by its cofactor, and then summing the results. (The fact that this works would require proof. That proof will be left to a course in linear algebra.)
- A sign matrix keeps track of the $(-1)^{j+k}$ term in the cofactor. All you have to do is determine if the first entry of your expansion has a plus or minus, and then alternate the sign as you expand.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

sign matrix

Problem 2.24 Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 4 & 2 & 5 \end{bmatrix}$ in 3 different ways. First, use a cofactor expansion using the first row (Definition 2.15). Then use a cofactor expansion using the 2nd row. Then finally use a cofactor expansion using column 3. Which of the was the quickest, and why?

Problem 2.25 Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 5 & -1 \\ 0 & 8 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & -5 & 3 & -1 \end{bmatrix}.$$

You can make these problems really fast if you use a cofactor expansion along a row or column that contains a lot of zeros.

Problem 2.26 Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix}.$$

Then find the inverse of A (or explain why it does not exist). Are the columns of A linearly independent or linearly dependent?

Problem 2.27 Compute the determinant of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Does A have an inverse? Are the columns of A linearly independent or linearly dependent? Answer both of the previous questions without doing any row reduction. Then row reduce $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$ to confirm your answer.

After completing the previous two problems, you should see that there is a connection between the determinant, inverse, and linear independence. Make a conjecture about what this connection is. We'll learn a little more about determinants and inverses, and then you'll have a chance to state your conjecture, as well as prove it.

Problem 2.28 Start by writing the system of equations

$$\begin{cases} -2x_1 + 5x_3 &= -2 \\ -x_1 + 3x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 3 \end{cases}$$

as a matrix product $A\vec{x} = \vec{b}$. (What are A, $\vec{x}$ and $\vec{b}$?) Then find the inverse of A, and use this inverse to find $\vec{x}$. [Hint: If we just have numbesr, then to solve $ax = b$, we multiply both sides by $\frac{1}{a}$ to obtain $\frac{1}{a}ax = \frac{1}{a}b$ or just $x = \frac{1}{a}b$.]

In the next problem, you'll prove that the determinant of a 2 by 2 matrix gives the area of a parallelogram whose edges are the columns of the matrix.

Problem 2.29 To find the area of the parallelogram with vertexes $O = (0, 0)$, $U = (a, c)$, $V = (b, d)$, and $P = (a + b, c + d)$, we would find the length of OU (the base b), and multiply it by the distance from V to OU. Complete the following:

1. Find the projection of $\vec{OV}$ onto $\vec{OU}$. (You may have to look up a formula from math 215.)

2. The vector $\vec{OV} - \text{proj}_{\vec{OV}}\vec{OU}$ is called the component of $\vec{OV}$ that is orthogonal to $\vec{OU}$. The length of this vector is precisely the distance from V to OU, which we'll call h. Find the length of this vector.

3. We now have the base $b = |OU|$ and height h of a parallelogram. Compute the product, and prove it equals $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = |ad - bc|$.

The result above extends to 3 dimensions. The determinant of a 3 by 3 matrix gives the volume of a parallelepiped whose edges are the columns of the matrix. We then use determinants to define nth dimensional volume.

Problem 2.30 Answer each of the following:

1. Let $\vec{u} = (2, 3)$. If you pick a vector $\vec{v}$ that is a linear combination of $\vec{u}$, what will the determinant of $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$ equal? First explain how you know the answer (before you have even chosen a vector $\vec{v}$). Then give us an example by picking a vector that is a linear combination of $\vec{v}$.

2. Let $\vec{u} = (1, 0, 2)$ and $\vec{v} = (0, -1, 1)$. If $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, what will the determinant equal? Explain. Then show us an example to confirm your conjecture.

3. We already computed the determinant of $A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$. Swap two columns of the matrix, and then compute the determinant. How does the determinant of your matrix with swapped columns relate to the determinant of the original matrix. If you swap two columns of a matrix, what happens to the determinant?

Problem 2.31 Construct a 2 by 2 matrix whose columns are linearly independent. What is the reduced row echelon form of your matrix? Compute the rank and the determinant, and finally find the inverse (if possible).

Now construct a 2 by 2 matrix whose columns are linear dependent. What is the reduced row echelon form of your matrix? Compute the rank and the determinant, and finally find the inverse (if possible).

Make a conjecture about the connection between (1) linear dependence, (2) rref, (3) rank, (4) determinant, and (5) inverses. Then use a computer to give two 3 by 3 examples similar to the examples above. You'll be asked to show us the computations on the computer in class.

Problem 2.32 Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$. Compute the determinant of A. Then create a matrix B so that the ijth entry of B is the cofactor C_{ij} (remove row i and column j, compute the determinant, and then times by an appropriate sign). This will require that you compute nine 2 by 2 determinants. Finally, compute the inverse of A (feel free to use a computer on this part). Make a conjecture about the connection between the determinant of A, this matrix B, and the inverse of A. We'll verify your conjecture is true on a 4 by 4 matrix in class.

2.4 Eigenvalues and Eigenvectors

The final computational skill we need to tackle is to compute eigenvalues and eigenvectors. Let's start by looking at an example to motivate the language we are about to introduce.

Example 2.17. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. When we multiply this matrix by the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we obtain $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}$. Multiplication by the matrix A was miraculously the same as multiplying by the number 3. Symbolically we have $A\vec{x} = 3\vec{x}$. Not every vector $\vec{x}$ satisfies this property, as letting $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the product $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which is not a multiple of $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Our main goal in this section is to answer the following two questions:

1. For which nonzero vectors $\vec{x}$ (eigenvectors) is it possible to write $A\vec{x} = \lambda\vec{x}$?
2. Which scalars λ (eigenvalues) satisfy $A\vec{x} = \lambda\vec{x}$?

Now for some definitions.

Definition 2.18: Eigenvector and Eigenvalue. Let A be a square $n \times n$ matrix.

- An eigenvector of A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. (Matrix multiplication reduces to scalar multiplication.) We avoid letting $\vec{x}$ be the zero vector because $A\vec{0} = \lambda\vec{0}$ no matter what λ is.
- If $\vec{x}$ is an eigenvector satisfying $A\vec{x} = \lambda\vec{x}$, then we call λ and eigenvalue of A.

Problem 2.33 Use the definition above to determine with of the following are eigenvectors of $\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, (1, 4), \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1, \\ -1 \end{pmatrix}, \begin{pmatrix} -2, \\ 2 \end{pmatrix}.$$

If the vector is an eigenvector, state the corresponding eigenvalue.

The next problem gives us an algorithm for computing eigenvalues and eigenvectors.

Problem 2.34: How to compute eigenvalues and eigenvectors Let A be a square matrix.

1. If λ is an eigenvalue, explain why we can find the eigenvectors by solving the equation $(A-\lambda I)\vec{x}=\vec{0}$. This means we can subtract λ from the diagonal entries of A, and then row reduce $\begin{bmatrix} A-\lambda I & \vec{0} \end{bmatrix}$ to obtain the eigenvectors. Note that you should always obtain infinitely many solutions.
2. Explain why we can obtain the eigenvalues of A by solving for when the determinant of $(A-\lambda I)$ is zero, i.e. solving the equation
$$\det(A-\lambda I)=0.$$

The algorithm above suggests the following definition.

Definition 2.19. If A is a square n by n matrix, then we call $\det(A-\lambda I)$ the characteristic polynomial of A. It is a polynomial in λ of degree n, and hence has n roots (counting multiplicity). These roots are the eigenvalues of A.

We now have an algorithm for finding the eigenvalues and eigenvectors of a matrix. We start by finding the characteristic polynomial of A. The zeros of this polynomial are the eigenvalues. To get the eigenvectors, we just have to row reduce the augmented matrix $\begin{bmatrix} A-\lambda I & \vec{0} \end{bmatrix}$. Finding eigenvalues and eigenvectors requires that we compute determinants, find zeros of polynomials, and then solve homogeneous systems of equations. You know you are doing the problem correctly if you get infinitely many solutions to the system $(A-\lambda I)\vec{x}=0$ for each lambda (i.e. there is at least one row of zeros along the bottom after row reduction). As another way to check your work, the following two facts can help.

- The sum of the eigenvalues equals the trace of the matrix (the sum of the diagonal elements).
- The product of the eigenvalues equals the determinant.

The trace and determinant are equal to the sum and product of the eigenvalues.

Problem 2.35 Consider the matrix $A=\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$ from problem 2.33.

1. Find the characteristic polynomial of A, and then find the zeros to determine the eigenvalues.
2. For each eigenvalue, find all corresponding eigenvectors.
3. Compute the trace and determinant of A.

Problem 2.36 Consider the matrix $A=\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

Problem 2.37 Consider the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

Problem 2.38 Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

2.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Linear Algebra Applications

This chapter covers the following ideas. (These are subject to change, as I write the notes.)

1. Find the currents in electrical systems involving batteries and resistors, using both Gaussian elimination and Cramer's rule.
2. Find interpolating polynomials. Use the transpose and inverse of a matrix to solve the least squares regression problem of fitting a line to a set of data.
3. Find the partial fraction decomposition of a rational function. Utilize this decomposition to integrate rational functions.
4. Describe a Markov process. Explain how an eigenvector of the eigenvalue $\lambda = 1$ is related to the limit of powers of the transition matrix.
5. Explain how to generalize the derivative to a matrix. Use this generalization to locate optimal values of the function using the second derivative test. Explain the role of eigenvalues and eigenvectors in the second derivative test.

The order of the following problems is not set. Feel free to work on them, but realize that the order may change as I continue writing over the weekend.

3.1 Vector Fields

In multivariate calculus, we studied vector fields of the form $\vec{F}(x, y) = (M, N)$, where M and N are functions of x and y. If you compute the derivative of a vector field, you obtain the square matrix

$$D\vec{F}(x, y) = \begin{bmatrix} \partial M/\partial x & \partial M/\partial y \\ \partial N/\partial x & \partial N/\partial y \end{bmatrix}.$$

The eigenvalues and eigenvectors of this matrix provide us with a wealth of information about the vector field. The next few problems have you discover many of these key ideas. We'll return to these problems throughout the semester, especially when we start studying systems of differential equations.

Problem 3.1 Consider the vector field $\vec{F}(x, y) = (2x + y, x + 2y)$.

1. At each of the 8 points given by $(\pm 1, \pm 1)$, $(0, \pm 1)$, $(\pm 1, 0)$, sketch the vector $\vec{F}(x, y)$ with it's base at the input point (so at point $(1, 0)$, sketch $(2, 1)$, a vector starting at $(1, 0)$ and ending at $(3, 1)$). This provides us with a rough sketch of the vector field.

2. Compute $A = D\vec{F}(x, y)$. It should be a 2 by 2 matrix.

3. Remember that we say a vector $\vec{x}$ is an eigenvector if $A\vec{x} = \lambda\vec{x}$. For any of the vectors from part 1., did you find that $A\vec{x} = \lambda\vec{x}$? Which ones (these are eigenvectors)? By how much was the vector $\vec{x}$ stretched (these are eigenvalues)?

4. Now compute the eigenvalues and eigenvectors of this matrix, using the algorithm from the previous chapter. You should obtain the same answer as part 3.

The problem above had two positive eigenvalues. In the next problem, your goal is to determine what a vector field looks like when you have a positive eigenvalues, and a negative eigenvalue.

Problem 3.2 Complete the following:

1. For the vector field $\vec{F} = (x, 2x - y)$, compute the eigenvalues and eigenvectors of $D\vec{F}(x, y)$.

2. For the vector field $\vec{F} = (x - 4y, -6x - y)$, compute the eigenvalues and eigenvectors of $D\vec{F}(x, y)$.

3. With each vector field, use a computer to construct a vector field plot. In the plot, please show us how to see the eigenvectors, together with which which eigenvector corresponds to a positive eigenvalues, and which corresponds to a negative eigenvalue. You can construct vector fields in Wolfram—Alpha by typing "vector field plot" in the input box, or just follow the link `http://www.wolframalpha.com/input/?i=vector+field+plot&lk=4&num=2`.

4. Add to your plots several trajectories, i.e. a path that a particle would follow if $\vec{F}$ represents the tangent vectors of the path. Think, "If I dropped a really light particle in this field, representing water current, where would the particle go?

Problem 3.3 The following three vector fields have imaginary eigenvalues. Compute the eigenvalues for each, construct a vector field plot, and on the plot add several trajectories (the path followed by a particle that is dropped into this field).

1. $\vec{F} = (-2y, x)$.

2. $\vec{F} = (-x + y, -x - y)$.

3. $\vec{F} = (x - y, x)$

Make a conjecture as to why one spirals in, one spirals out, and one just wraps around in ellipses. We'll address this conjecture in class.

The next problem requires that you are on a computer that can use Mathematica. These computers are available in the Ricks, Austin, Romney, and library. Alternately, you can download VMWare that will allow you to use Mathematica for free from your computer, provided you head to https://vdiview.byui.edu/. You can download step-by-step instructions from http://www.byui.edu/help-desk/categories/vdivmware. Please take a moment and make sure you can access Mathematica.

Problem 3.4 Start by downloading the Mathematica notebook VectorFields.nb. The goal of this problem is to make a connection between a vector field and it's corresponding eigenvalues/eigenvectors. Once the notebook is open, click somewhere in the text, hold down Shift, and then press Enter. This will evaluate the commands and produce a vector field plot. You can click on the bubbles with crosshairs in them to adjust the vectors (which are the columns of the matrix).

1. If the vector field pushes things outwards in all directions, what do you know about the eigenvalues?
2. How can you tell, by looking at a vector field plot, that one eigenvalue is positive, and the other is negative?
3. If the vector field spirals outwards, what do you know about the eigenvalues?
4. If the vector field pulls everything inwards, what do you know about the eigenvalues?
5. If the vector field spirals inwards, what do you know about the eigenvalues?
6. What happens when you have a repeated eigenvalue? (This one has lots of correct answers, and it a topic for much further discussion in chapter 10.)

3.1.1 Second Derivative Test

Have them compute the first and second derivative (call it the Hessian), and then have them compute the eigenvalues and eigenvectors of the Hessian. Have them draw this vector field as above (as an approximation to the gradient). Then ask them to state if you have a max or min. They should be able to tell the conditions under which you get a maximum by just asking about gradients. Have them do 3 problems like this, with the last having multiple critical values, maybe the last 2.

Let's start with a review from first semester calculus. If a function $y = f(x)$ has a relative extremum at $x = c$, then $f'(c) = 0$ or the derivative is undefined. The places where the derivative is either zero or undefined are called critical values of the function. The first derivative test allows you to check the value of the derivative on both sides of the critical value and then interpret whether that point is a maximum or minimum using increasing/decreasing arguments. The second derivative test requires you to compute the second derivative at $x = c$. If $f''(c) > 0$ (the function is concave upwards), then the function has a minimum at $x = c$. If $f''(c) < 0$ (the function is concave downwards), then the function has a maximum at $x = c$. If $f''(c) = 0$, then the second derivative test fails.

Example 3.1. The function $f(x) = x^3 - 3x$ has derivatives $f' = 3x^2 - 3$ and $f'' = 6x$. The first derivative is zero when $3(x^2 - 1) = 3(x - 1)(x + 1) = 0$, or $x = \pm 1$. The second derivative at $x = 1$ is 6 (concave upwards), so there is a minimum at $x = 1$. The second derivative at $x = -1$ is -6 (concave downwards), so there is a maximum at that point.

We're now ready to extend this idea to all functions of the form $f : \mathbb{R}^n \to \mathbb{R}$ (the output is 1 dimensional, so that it makes sense to talk about a largest or smallest number). We will only consider the case $n = 2$, as it simplifies the computations and provides all that is needed to extend to all dimensions. The first derivative test breaks down in every dimension past the first, because there are more than 2 ways to approach a point of the domain (you can't just look at the left side or right side). However, at a local extremum, the derivative is still zero, which often results in solving a system of equations. In higher dimensions, there are three classifications of critical points: maximum, minimum, or saddle point (a point where the tangent plane is horizontal, but in some directions you increase and in other directions you decrease).

The second derivative test does not break down. Consider the function $z = f(x, y)$. Its derivative $Df(x, y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$ is a function with two inputs and two outputs. The second derivative $D^2 f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ is a 2×2 square matrix called the Hessian of f. This matrix will always be symmetric, in that the transpose of the matrix equals itself (because $f_{xy} = f_{yx}$). At a critical point (where the first derivative is zero), the eigenvalues of $D^2 f$ give the directional second derivative in the direction of a corresponding eigenvector. The largest eigenvalue is the largest possible value of the second derivative in any direction and the smallest eigenvalue is the smallest possible value of the second derivative in any direction.

The **second derivative test** is the following. Start by finding all the critical points (places where the derivative is zero). Then find the eigenvalues of the second derivative. Each eigenvalue represents the 2nd directional derivative in the direction of a corresponding eigenvector. In every other direction, the directional 2nd derivative is between the smallest and largest eigenvalue.

the second derivative test with eigenvalues

1. If the eigenvalues are all positive at a critical point, then in every direction the function is concave upwards. The function has a minimum at that critical point.

2. If the eigenvalues are all negative at a critical point, then in every direction the function is concave downwards. The function has a maximum there.

3. If there is a positive eigenvalue and a negative eigenvalue, the function has a saddle point there.

4. If either the largest or smallest eigenvalue is zero, then the second derivative test fails.

Eigenvalues are the key numbers needed to generalize optimization to all dimensions. A proof of this fact is beyond the scope of this class.

Example 3.2. For the function $f(x, y) = x^2 + xy + y^2$, the gradient is $Df = \begin{bmatrix} 2x + y & x + 2y \end{bmatrix}$, which is zero only at $x = 0, y = 0$ (solve the system of equations $2x + y = 0, x + 2y = 0$). The Hessian is $D^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The eigenvalues are found by solving $0 = \det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 1 =$

$4 - 4\lambda + \lambda^2 - 1 = (\lambda - 3)(\lambda - 1)$, so $\lambda = 3, 1$ are the eigenvalues. Since both eigenvalues are positive, the function is concave upwards in all directions, so there is a minimum at $(0, 0)$.

The eigenvectors of the Hessian help us understand more about the graph of the function. An eigenvector corresponding to 3 is (1,1), and corresponding to 1 is (-1,1). These vectors are drawn in figure 3.1, together with two parabolas whose 2nd derivatives are precisely 3 and 1. The parabola which opens upwards the most quickly has a 2nd derivative of 3. The other parabola has a second derivative of 1. In every other direction, the 2nd derivative would be between 1 and 3.

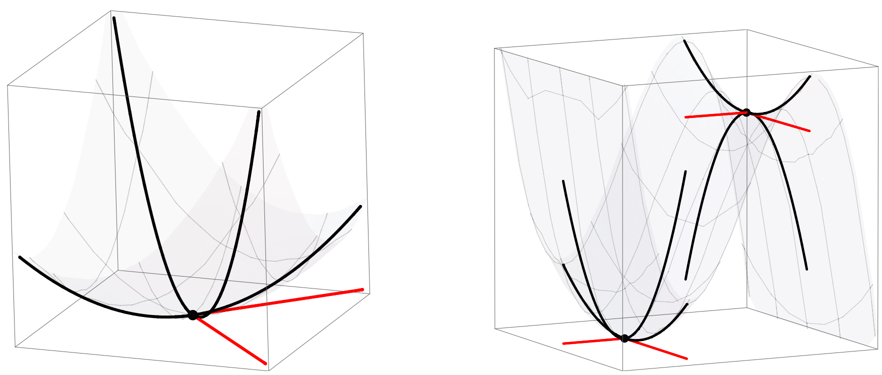


Figure 3.1: The eigenvectors of the second derivative tell you the directions in which the 2nd derivative is largest and smallest. At each critical point, two eigenvectors are drawn as well as a parabola whose second derivative (the eigenvalue) matches the second derivative of the surface in the corresponding eigenvector direction.

Example 3.3. For the function $f(x, y) = x^3 - 3x + y^2 - 4y$, the gradient is $Df = \begin{bmatrix} 3x^2 - 3 & 2y - 4 \end{bmatrix}$, which is zero at $x = 1, y = 2$ or $x = -1, y = 2$. Hence there are two critical points, so we have to find two sets of eigenvalues. The Hessian is $D^2 f = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}$. When $x = -1, y = 2$, the eigenvalues of $\begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda = -6, 2$. Since one is positive and one is negative, there is a saddle point at $(-1, 2)$. When $x = 1, y = 2$, the eigenvalues of $\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda = 6, 2$. Since both are positive, there is a minimum at $(-1, 2)$ (as in every direction the function is concave upwards).

Again the eigenvectors help us understand how the function behaves, as illustrated in figure 3.1. At (1,2) we have an eigenvector (1,0) corresponding to 6, and (0,1) corresponding to 2. In both eigenvector directions the function is concave upwards, but opens more stepply in the (1,0) direction as 6 is bigger than 2. At (-1,2) the we have an eigenvector (1,0) corresponding to -6, and (0,1) corresponding to 2. The functions opens stepply downwards in the (1,0) direction, and upwards in the (0,1) direction.

3.1.2 Systems of Differential Equations

Introduce the language of systems of differential equations. Then have them solve graphically a system by drawing the trajectories in the phase plane (it's the exact same as the vector field section, with no change). I could pull in a mutalism (bees and plants) or a competitive hunter, or an (xxxxxx done arms race example). All of which are great, but add more to the learning. This could be a really good section.

I could also introduce tank mixing (a conservation problem, to tie the two together, this is perfect right here at the end). The tank mixing problem connects the conservation part to the eigenvalues part. We wouldn't have a complete solution to the problem, but we could approximate the solution. After doing tank mixing, you should mention the huge applications, and how this is basically a flow problem (so relate to flux and divergence). Economic import/export. Immigration. Spread of Disease. Spreading of noxious weeds and more. This is a huge idea that we should start soon.

3.2 Conservation Laws

Many problems in nature arise from conservation laws. These laws generally focus on the principle that matter is neither is created nor destroyed, rather it is just moved, changed, or something. Any of the following are conservation laws:

- What comes in must come out.
- Voltage supplied equals voltage suppressed.
- Atoms before must equal atoms after.
- The change in a quantity is how much it increase minus how much it decreases.
- Current in equals current out.

All of the following subsections involve problems related to some conservation law. You'll see similar laws in your future classes, regardless of your discipline.

3.2.1 Network Flow

Imagine that

Problem 3.5 Have them set up a system of cars at specific intersections. Give them the inflows at some, and the outflows at others. Then ask them how many cars are on each road at any given time.

Problem 3.6 Use the same system as the previous, but now close one of the roads. Then ask them to solve the problem again. They are predicting what will happen to traffic patterns if you close down a road.

Connect this to network traffic on the internet, airplane routes, and more.

3.2.2 Markov Processes

Introduce the car rental company problem. The conservation law states.... Then have them set up a system

Problem 3.7 Have them setup a matrix equation. We wish to solve the problem... If this has a solution, then what is the corresponding eigenvalue of the coefficient matrix. Solve for the eigenvector.

Problem 3.8 Do the land problem (residential, commercial, economic).

Problem 3.9 Use Larry's problem. This connects to value of commodities.

Matrices can be used to model a process called a Markov Process. To fit this kind of model, a process must have specific states, and the matrix which models the process is a transition matrix which specifies how each state will change through a given transition. An example of a set of states is "open" or "closed" in an electrical circuit, or "working properly" and "working improperly" for operation of machinery at a manufacturing facility. A car rental company which rents vehicles in different locations can use a Markov Process to keep track of where their inventory of cars will be in the future. Stock market analysts use Markov processes and a generalization called stochastic processes to make predictions about future stock values.

Example 3.4. Let's illustrate a Markov Process related to classifying land in some region as "Residential," "Commercial," or "Industrial." Suppose in a given region over a 5 year time span that 80% of residential land will remain residential, 10% becomes commercial, and 10% becomes industrial. For commerical land, 70% remains commercial, 20% becomes residential, and 10% becomes industrial. For industrial land, 70% remains industrial, 30% becomes commercial, and 0% becomes residential. To find what happens at the end of a 5 year period, provided we know the current R, C, and I values, we could just compute

$$\begin{aligned} R_{\text{new}} &= .8R + .2C + 0I \\ C_{\text{new}} &= .1R + .7C + .3I \\ I_{\text{new}} &= .1R + .1C + .7I \end{aligned} \xrightarrow{\text{matrix form}} \begin{bmatrix} R_{\text{new}} \\ C_{\text{new}} \\ I_{\text{new}} \end{bmatrix} = \begin{bmatrix} .8 & .2 & 0 \\ .1 & .7 & .3 \\ .1 & .1 & .7 \end{bmatrix} \begin{bmatrix} R \\ C \\ I \end{bmatrix}$$

The matrix on the right above is called the transition matrix of the Markov process. It is a matrix where each column relates to one of the "states," and the numbers in that column are the proportions of the column state that will change to the row state through the transition (the ordering on row and column states is the same). We calculate the next "state" by multiplying our current state by the transition matrix. If current land use is about 50% residential, 30% commercial, and 20% industrial, then 5 years later the land use would be

$$\begin{array}{cc} & \begin{array}{ccc} R & C & I \end{array} \\ \begin{array}{c} \text{to } R \\ \text{to } C \\ \text{to } I \end{array} & \begin{bmatrix} .8 & .2 & 0 \\ .1 & .7 & .3 \\ .1 & .1 & .7 \end{bmatrix} \end{array}$$

Transition Matrix

$$\begin{bmatrix} .8 & .2 & 0 \\ .1 & .7 & .3 \\ .1 & .1 & .7 \end{bmatrix} \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix} = \begin{bmatrix} 46 \\ 32 \\ 22 \end{bmatrix}$$

If the same transitions in land use continue, we can multiply the previous projection (state) by the transition matrix to obtain a 10 and 15 year projection for land use:

$$\underbrace{\begin{bmatrix} .8 & .2 & 0 \\ .1 & .7 & .3 \\ .1 & .1 & .7 \end{bmatrix} \begin{bmatrix} 46 \\ 32 \\ 22 \end{bmatrix} = \begin{bmatrix} 43.2 \\ 33.6 \\ 23.2 \end{bmatrix}}_{\text{10 Year Projection}} \qquad \underbrace{\begin{bmatrix} .8 & .2 & 0 \\ .1 & .7 & .3 \\ .1 & .1 & .7 \end{bmatrix} \begin{bmatrix} 43.2 \\ 33.6 \\ 23.2 \end{bmatrix} = \begin{bmatrix} 41.28 \\ 34.8 \\ 23.92 \end{bmatrix}}_{\text{15 Year Projection}}$$

As we continue to multiply on the left by our transition matrix, each time we add 5 more years to our projection. This projection is valid as long as the same trends continue.

Consider the land use example from above. Let $\vec{x}_0$ be our initial state. If our transition matrix A remains the same forever, what will eventually happen to the proportion of land devoted to residential, commercial, or industrial use? We can write each new state as powers of the transition matrix A by writing $\vec{x}_1 = A\vec{x}_0$, $\vec{x}_2 = A\vec{x}_1 = AA\vec{x}_0 = A^2\vec{x}_0$, $\vec{x}_3 = A^3\vec{x}_0$, and $\vec{x}_n = A^n\vec{x}_0$. What happens to the product $A^n\vec{x}_0$ as $n \to \infty$? Can we reach a state $\vec{x} = (R, C, I)$ such that $A\vec{x} = \vec{x}$, the next state is the same as the current? If this occurs, then any future transitions will not change the state either. This state $\vec{x}$ is called a steady state, since it does not change when multiplied by the transition matrix (it remains steady).

Finding a steady state is an eigenvalue-eigenvector problem, as we are looking for a solution to the equation $A\vec{x} = 1\vec{x}$ (where the eigenvalue is 1). For any Markov process (where the columns of the matrix sum to 1), the number 1 will always be an eigenvalue. All we have to do is find the eigenvectors corresponding to the eigenvalue 1. The solution to the problem $\lim\limits_{n\to\infty} A^n\vec{x}_0$ is this steady state, and is an eigenvector. For the land use Markov process, an eigenvector (using technology) corresponding to 1 is $\begin{bmatrix}\frac{3}{2} & \frac{3}{2} & 1\end{bmatrix}^T$. Since any multiple of an eigenvector is again an eigenvector, we can multiply by a constant so that the proportions sum to 100. Multiplying by 2 we have $\begin{bmatrix}3 & 3 & 2\end{bmatrix}^T$, which means that the ratio of land will be 3 acres residential to 3 acres commercial to 2 acres industrial. To write this in terms of percentages, divide each component by 8 (the sum $3 + 3 + 2$) to obtain $3/8 : 3/8 : 2/8$ or multiplying by 100 we have $37.5\% : 37.5\% : 25\%$. These are the long term percentages of land use.

More examples are available in the handwritten solutions to problems, available online.

3.2.3 Kirchoff's Electrical Laws

Gustav Kirchoff discovered two laws of electricity that pertain to the conservation of charge and energy. To describe these laws, we must first discuss voltage, resistance, and current. Current is the flow of electricity, and often it can be compared to the flow of water. As a current passes across a conductor, it encounters resistance. Ohm's law states that the product of the resistance R and current I across a conductor equals the voltage V, i.e. $RI = V$. If the voltage remains constant, then a large resistance corresponds to a small current. A resistor is an object with high resistance which is placed in an electrical system to slow down the flow (current) of electricity. Resistors are measured in terms of ohms, and the larger the ohms, the smaller the current. Figure 3.2 illustrates two introductory electrical systems.

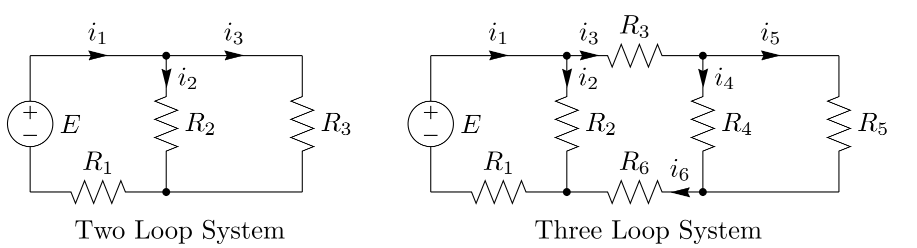


Figure 3.2: Electrical Circuit Diagrams.

In this diagram, wires meet at nodes (illustrated with a dot). Batteries and voltage sources (represented by or other symbols) supply a voltage of E volts. At each node the current may change, so the arrows and letters i represent the different currents in the electrical system. The electrical current on each wire may or may not follow the arrows drawn (a negative current means that the current flows opposite the arrow). Resistors are depicted with the symbol -/\/\/\- , and the letter R represents the ohms.

Kirchoff discovered two laws. They both help us find current in a system, provided we know the voltage of any batteries, and the resistance of any resistors.

1. Kirchoff's current law states that at every node, the current flowing in equals the current flowing out (at nodes, current in = current out).
2. Kirchoff's voltage law states that on any loop in the system, the directed sum of voltages supplied equals the directed sum of voltage drops (in loops, voltage in = voltage out).

Let's use Kirchoff's laws to generate a system of equations for the two loop system.

Problem 3.10 Set up the two loop problem in general. Then give them some coefficients and have them solve.

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ R_1 i_1 + R_2 i_2 &= E \\ -R_2 i_2 + R_3 i_3 &= 0 \end{aligned}$$

Problem 3.11 Set up the three loop problem in general, and then solve with some specific coefficients.

3.2.4 Stoichiometry

Chemical reaction stoichiometry is the study balancing chemical equations. A chemical reaction will often transform reactants into by-products. The by products are generally different compounds, together with either an increase or decrease in heat. One key rule in stoichiometry is that a chemical process neither creates nor destroys matter, rather it only changes the way the matter is organized. For simple reactions (with no radioactive decay), this conservation law forces the number of atoms entering a reaction to be the same as the number leaving. The next problem asks you to use this conservation law to create a balanced chemical reaction equation.

Problem 3.12 The chemical compound hydrocarbon dodecane ($C_{12}H_{26}$) is used as a jet fuel surrogate (see Wikipedia for more info). This compound reacts with oxygen (0_2), and the chemical reaction produces carbon dioxide (CO_2), water (H_20), and heat. Suppose we expose some dodecane to oxygen, and that a chemical reaction occurs in which the dodecane is completely converted to carbon dioxide and water. Conservation requires that the number of atoms (H, C, and 0) at the beginning of the chemical reaction must be the exact same as the number at the end. We could write the chemical reaction in terms of molecules as

$$x_1 C_{12}H_{26} + x_2 O_2 = x_3 CO_2 + x_4 H_2O \quad \text{or} \quad x_1 C_{12}H_{26} - x_2 O_2 = x_3 CO_2 - x_4 H_2O = 0,$$

where x_1 molecules of dodecane and x_2 molecules of oxygen were converted to x_3 units of carbon dioxide and x_4 units of oxygen. If we look at each atom (carbon, hydrogen, and oxygen) individually, we obtain three equations to relate the variables x_1, x_2, x_3, x_4. The carbon equation is simply

$$x_1(12) + x_2(0) = x_3(1) + x_4(0) \quad \text{or} \quad x_1(12) + x_2(0) - x_3(1) - x_4(0) = 0.$$

Your job follows:

1. Write the other two conservation equations (for hydrogen and oxygen).
2. Solve the corresponding system of equations by row reduction. As there are only 3 equations with 4 unknowns, you should obtain infinitely many solutions. Write each variable in terms of the free variable.
3. If about 10,000 molecules of water are present at the end of the reaction, about how many molecules of dodecane were burned?

3.3 Cramer's Rule

Gabriel Cramer developed a way to solve linear systems of equations by using determinants. For small systems, the solution is extremely fast. However, for large systems, the method looses it's power because of the complexity of computing determinants. Also, when the coefficients in the system are variables, Cramer's rule provides an extremely fast algorithm for computing determinants. I'll remind you occasionally throughout the problem set to apply Cramer's rule when the problem involves variable coefficients.

Theorem 3.5 (Cramer's Rule)**.** *Consider the linear system given by $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \vec{v}_n \end{bmatrix}$ is an n by n matrix whose determinant is not zero. Let $D = |A|$. For each i, replace vector $\vec{v}_i$ with $\vec{b}$, and then let D_i be the determinant of the corresponding matrix. The solution to the linear system is then*

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \cdots \quad x_n = \frac{D_n}{D}.$$

For the 2 by 2 system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

Cramer's rule states the solution is (provided $|A| \neq 0$)

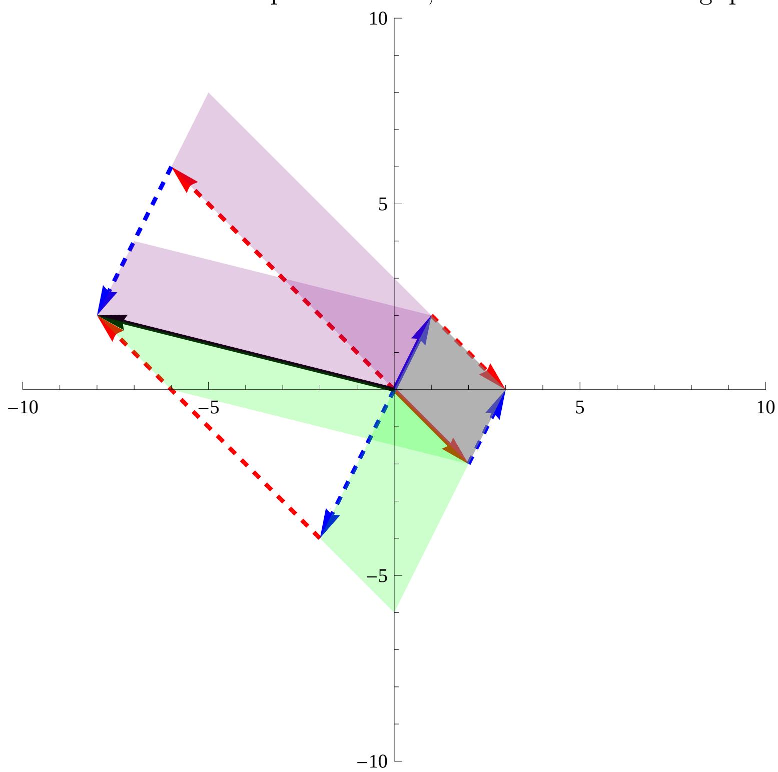
$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

$\boxed{\textbf{Problem 3.13}}$ Consider the system of equations $x + 2y = 3, 4x + 5y = 6$. Solve this system in 2 different ways.

1. Use Cramer's rule to solve the system. You just need to compute three 2 by 2 determinants.
2. Use row reduction to solve the system. Show the steps in class.

In the next problem, you'll provide a proof of Cramer's rule in 2D. Your proof will contain the key idea needed to prove the theorem in all dimensions. The key idea is to connect determinants to areas of parallelograms.

Problem 3.14: Proof of Cramer's Rule Let $\vec{v}_1 = (2, -2)$ and $\vec{v}_2 = (1, 2)$. Let $x_1 = -3$ and $x_2 = -2$, which means that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 = (-8, 2)$. In the picture below, the solid red vector is $\vec{v}_1$, the solid blue vector is $\vec{v}_2$, and the solid black vector is $\vec{b}$. Use the picture below, to answer the following questions.



[Hint: Each question can be answered by thinking about determinants as areas.]

1. Explain why $x_1 \begin{vmatrix} \vec{v_1} & \vec{v_2} \end{vmatrix} = \begin{vmatrix} x_1\vec{v_1} & \vec{v_2} \end{vmatrix}$. Then explain why $\begin{vmatrix} x_1\vec{v_1} & \vec{v_2} \end{vmatrix} = \begin{vmatrix} \vec{b_1} & \vec{v_2} \end{vmatrix}$. Finally, solve for x_1 to show

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

2. In a similar fashion, obtain a formula for x_2.

Problem 3.15 In problem .... we obtained the matrix equation .... Use Cramer's rule to obtain the solutions to this system of equations.

Cramer's rule is most useful when the coefficients in the linear system are variables, rather than numbers. Let's apply our knowledge to study the arms race (the building of armies - tanks, bombs, soldiers, etc. - between two countries). Consider two countries, country A and country B. As country B builds up their military, country A looks on and says "Hmm, we better build up our military." Similarly, as country A builds up their military, country B looks and says, "Hmm, we better build up our military." If country A has a grudge against country B, they will probably build up their military regardless of what country B does. Similarly, any past grievances and grudges that country B has against country A will increase the rate at which country B builds up their military. Building up a military costs money, so hopefully both countries have economic limitations that restrict the growth of their military. The real question behind the arms race is, "Will the two countries eventually decide they are spending enough on their military, or will their spending continue to grow without bound."

We now develop a system of differential equations that describes the above. The key principle is a general law of conservation:

> The change in a quantity equals the flow in of the quantity minus the flow out of the quantity, or more simply
>
> $$\text{Change} = \text{(Flow in) - (Flow out)}$$
>
> $$\text{Change} = \text{(Increase) - (Decrease)}$$

- Let x represent the dollar amount per year that country A spends on arms. Let y represent the dollar amount per year that country B spends on arms.
- When y is large, country A will respond by increasing their spending. We'll assume this change is proportional to y, so we see that x increases by an amount ay. Similarly, when x is large, country B responds by increasing their spending. Let's assume that y increases by an amount mx.
- The economy of each country tries to slow down the growth rate. The more money country A spends, the larger the effect of the economy. We'll assume that x decreases by an amount bx. Similarly, we'll assume y decrease by an amount ny.
- If the countries hold grudges against each other for past grievances, then they are inclined to increase their spending regardless of economic factors and the growth of the other country's army. Let c represent the amount that country A will increase their spending by, and let p represent the amount that country B will increase their spending by. These values might be zero (for example the US and Canada do not hold such grudges), but might not be zero at all (as was the cases during the cold war, between the US and USSR).

Problem 3.16 Read the arms race information above, and then answer the following questions.

1. There are three things causing x to change. The flow in (parts causing an increase) are ay and c, the response to the other country, and any grudges. The flow out (parts causing a decrease) is only bx, the economic restriction. We can write this as a differential equation

$$\frac{dx}{dt} = ay - bx + c.$$

Obtain a similar equation for $\frac{dy}{dt}$ (using the coefficients m, n, and p). Then write your system of ODEs in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b & -a \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ ? \end{bmatrix}.$$

2. An equilibrium solution to the system of differential equations above is a solution that remains stable. At equilibrium, there should not be any future change in x nor y, so we should have $dx/dt = 0$ and $dy/dt = 0$. Find the equilibrium solution for the arms race problem. [Cramer's rule should make this really fast.]

3. Find the eigenvalues of the square matrix from part 1. What conditions must be met so that both eigenvalues are negative? In class, we'll pick some positive values for a, b, c, m, n, p that satisfy the conditions you tell us, and then graph the vector field $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{p}$, along with some solution curves.

3.4 Curve Fitting

3.4.1 Interpolating Polynomials

Through any two points (with different x values) there is a unique line of the form $y = mx + b$. If you know two points, then you can use them to find the values m and b. Through any 3 points (with different x values) there is a unique parabola of the form $y = ax^2 + bx + c$, and you can use the 3 points to find the values a, b, c. As you increase the number of points, there is still a unique polynomial (called an interpolating polynomial) with degree one less than the number of points, and you can use the points to find the coefficients of the polynomial. In this section we will find interpolating polynomials, and show how the solution requires solving a linear system.

To organize our work, let's first standardize the notation. Rather than writing $y = mx + b$, let's write $y = a_0 + a_1x$ (where $a_0 = b$ and $a_1 = m$). For a parabola, let's write $y = a_0 + a_1x + a_2x^2 = \sum_{k=0}^{2} a_kx^k$. We can now write any polynomial in the form

$$y = a_0 + a_1x + \cdots + a_nx^n = \sum_{k=0}^{n} a_kx^k.$$

By standardizing the coefficients, we can use summation notation to express any degree polynomial by changing the n on the top of the summation sign.

Problem 3.17 Answer the following by row reducing an appropriate matrix. Please show us the steps in your row reduction. [Hint: Each point produces an equation.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1x$ that passes through the points $(1, 2)$ and $(3, 5)$. [We could have use m and b, but I chose to use a_0 and a_1 so you can see how this generalize quickly to all dimensions.]

2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1x^1 + a_2x^2$ that passes through the points $(0, 1)$, $(2, 3)$, and $(1, 4)$. [Hint: The second point produces the equation $3 = a_0 + a_1(2) + a_2(2)^2$.]

Problem 3.18 Give an equation of a cubic polynomial $y = a_0+a_1x^1+a_2x^2+a_3x^3$ that passes through the four points $(0, 1)$, $(1, 3)$, $(1, 4)$, and $(2, 4)$. Show us the steps in your row reduction. [Hint: Each point produces an equation. You should have a linear system with 4 equations and 4 unknowns.]

Problem 3.19 Solve the following. [Hint: Because the problem involves variable points, Cramer's rule will be much faster than row reduction.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1 x$ that passes through the points (x_1, y_1) and (x_2, y_2).
2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1 x^1 + a_2 x^2$ that passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Under what conditions will your solutions above not be valid?

If we collect 2 data points, then we can usually find an equation of a line that passes through them. If we collect 3 data points, we can usually find an equation of a parabola passing through them. Continuing in this fashion, if we collect $n+1$ data points, then we can usually find an equation of a polynomial of degree n that passes through them.

Problem 3.20 Suppose that we collect the 6 data points $(1,1)$, $(2,3)$, $(-1,2)$, $(0,-1)$, $(-2,0)$, $(3,1)$. We would like to find a polynomial that passes through all 6 points. State the degree n of this polynomial. Then find the coefficients $a_0, a_1, \ldots, a_n$ of this polynomial. Please use technology to do your row reduction. When you present in class, show us the matrix you entered into a computer, and then show us the reduced row echelon form together with the polynomial.

3.4.2 Least Squares Regression

Interpolating polynomials give a polynomial which passes through every point listed. While they pass through every point in a set of data, the more points the polynomial must pass through, the more the polynomial may have to make large oscillations in order to pass through each point. Sometimes all we want is a simple line or parabola that passes near the points and gives a good approximation of a trend in the data. When I needed to purchase a minivan for my expanding family, I gathered mileage and price data for about 40 cars from the internet. I plotted this data and discovered an almost linear downward trend (as mileage increased, the price dropped). Using this data I was able to create a line to predict the price of a car. I then used this data to talk the dealer into dropping the price of their car by over \$1000. Finding an equation of this line, called the least squares regression line, is the content of this section. In other words, if you have 3 or more points, how do you find a line that is closest to passing through these points? The least squares regression line is used to find trends in many branches of science, in addition to haggling for lower prices when buying a car. Statistics builds upon this idea to provide powerful tools for predicting the future.

Problem 3.21 Consider the three points $(2,4)$, $(0,1)$, and $(3,5)$. We wish to find a line $y = a_0 + a_1 x$ that fits this data.

1. What 3 equations do the points and line give. Write the linear system as a matrix equation by filling in A and $\vec{b}$ below:

$$A\vec{x} = \vec{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ ? \\ ? \end{bmatrix}.$$

   The first equation $4 = a_0 + a_1(2)$ is already on the first row.

2. Row reduce the corresponding augmented matrix to show that this system has no solution. The problem is that we have more equations than we do unknowns. The system is overdetermined.

3. If we multiply both sides of the equation $A\vec{x} = \vec{b}$ by a 2 by 3 matrix C, then the product CA will be a 2 by 2 matrix. We could then solve the system $CA\vec{x} = Cb$, as it would then have 2 equations and 2 unknowns.

   The only 2 by 3 matrix in the problem is the transpose of A. So compute A^TA and $A^T\vec{b}$. Then solve the system $(A^TA)\vec{x} = A^T\vec{b}$.

The previous problem suggests the following theorem. One proof of this theorem involves projecting $\vec{b}$ onto the plane spanned by the columns of A. This proof leads to the ideas behind inner product spaces, the Graham Schmidt orthogonalization process, and more, something you would study near the end of math 341 (Linear Algebra).

Theorem 3.6 (Least Squares Regression)**.** *When we collect n data points and notice the points follow a linear trend, the coefficients of the least square regression line $y = a_0 + a_1x$ are the solutions to the equation $A^TA\vec{x} = A^T\vec{b}$, where we have*

$$\vec{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \vec{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots x_n & \end{bmatrix}.$$

$\boxed{\textbf{Problem 3.22}}$ Suppose you collect the n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and you wish to find the least squares regression line $y = a_0 + a_1x$. Set up the matrices A, $\vec{x}$, $\vec{b}$, and A^T. Multiply together A^TA and $A^T\vec{b}$ (your result should involve sums of the form $\sum x_i$, $\sum y_i$, $\sum x_iy_i$, and $\sum x_i^2$). Then solve the equation $A^TA\vec{x} = A^T\vec{b}$ and state the coefficients a_0 and a_1. leas[Hint: Since the system involves variable coefficients, try using Cramer's rule. It will kick out the solution almost instantly.]

The key to solving the overdetermined system $A\vec{x} = \vec{b}$ is to multiply each side on the left by a matrix C, so that the produce CA is a square matrix. We then solve $CA\vec{x} = C\vec{b}$. The least square regression model comes by letting $C = A^T$. We obtain alternate data fitting models by using a matrix other than A^T (though this is a topic for another course). The next problem has you find the best fitting parabola, using the least square regression model.

$\boxed{\textbf{Problem 3.23}}$ Consider the 5 points $(-2, 3)$, $(-1, 1)$, $(0, -1)$, $(1, 2)$, $(2, 4)$, and . We would like to find an equation of a parabola $y = a_0 + a_1x + a_2x^2$ that approximates the trend in the data, using the least square regression model.

1. The 5 data points produce 5 equations in the three unknowns a_0, a_1, a_2. Write the linear system as a matrix equation by filling in A and $\vec{b}$ below:

$$A\vec{x} = \vec{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & -2 & 4 \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

2. Multiply both sides of the equation $A\vec{x} = \vec{b}$ by an appropriate 3 by 5 matrix C. Then solve the system $(CA)\vec{x} = C\vec{b}$. Feel free to use software to obtain your answer. In class, just show us CA, $C\vec{b}$, and the rref of $\begin{bmatrix} CA & C\vec{b} \end{bmatrix}$.

3. Plot the 5 data points and the parabola you found.

The next problem has the exact same solution as Problem 3.22, but does not require you to use a matrix transpose, nor matrix multiplication. Instead, it focuses on setting partial derivative equal to zero, which is the first step in locating minimums. You then just have to solve a system of linear equations.

Problem 3.24 Suppose you collect the n data points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), and you wish to find the least squares regression line $y = a_0 + a_1 x$. Each point (x_i, y_i) produces an error $y - y_i = (a_0 + a_1 x_i) - y_i$. The least squares regression line is the line that minimized the sum of the squares of these errors, which means we need to minimize

$$f(a_0, a_1) = \sum_{i=1}^{n} \left((a_0 + a_1 x_i) - y_i\right)^2.$$

1. Compute $\dfrac{\partial f}{\partial a_0}$ and $\dfrac{\partial f}{\partial a_1}$.

2. Since we seek the minimum of f, solve the system $\dfrac{\partial f}{\partial a_0} = 0$ and $\dfrac{\partial f}{\partial a_1} = 0$ for a_0 and a_1.

[Hint: Once you get each equation written in the form $(?)a_0 + (?)a_1 =?$, use Cramer's rule to kick out the answer almost instantly.]

3.5 Vector Spaces and Linear Transformations

Remind them of the idea about coordinate transformation from 215. Polar coordinates is an example of such a transformation. Now introduce them to linear transformations. Have them make connections about area, eigenvalues, and eigenvectors.

Problem 3.25 Give them a linear transformation. Have them compute the eigenvalues, eigenvectors, and determinant. Then have them draw the image of an object, and determine it's area. (How do connect eigenvalues — have to think of the best problem).

Problem 3.26 Give them a before/after image. Have them write down a linear transformation that would accomplish the transforming. Stay in 2D to 2D.

Problem 3.27 Jump to a 3D to 3D transformation. Give them two different ones. Have them determine volume in the new space. Have one of them have dependent columns. Ask them what the span is.

$\boxed{\textbf{Problem 3.28}}$ I need a problem that introduces the concept of rank and nullity of a linear transformation. What is the dimension of the image space. What is the dimension of the space that gets sent to zero. This will be the problem that introduces them to the kernel of a linear transformation.

Define carefully a linear transformation. Loosely define vector spaces.

$\boxed{\textbf{Problem 3.29}}$ Is the derivative a linear transformation? Is an integral a linear transformation? Is the Laplace Transform a linear transformation?

$\boxed{\textbf{Problem 3.30}}$ Have them solve a simple first order ODE, but make them use the language "What is the kernel of this linear transformation?" This should cement the language we'll use later one.
