Differential Equations with Linear Algebra
An inquiry based approach to learning

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Dedicated to Mr. Blair Nelson
My seventh grade algebra teacher
who trusted me enough to allow me to fail.

Contents

Introduction

Welcome to differential equations with linear algebra. One of our main objectives in this course is to learn to make powerful predictions about the future. We'll learn how our physical understanding of rates of change and forces will help us make predictions about the future. You've done this before when you studied how an object falls in a parabolic path under a constant force from gravity. In this class, we'll learn much more powerful ways to predict the future, and prepare you for your scientific career.

- If you can use a police scanner to determine someone's current speed, and then watch that speed change (measure acceleration), then you can predict where the object will be. Putting several of these sensors in car could allow you to build a self driven car (look up the Google Car for more info), or make a missile that can dodge anti-missile technology, and much more.
- At Napoleon's request, Fourier studied the flow of heat in cannons to make his army's cannons more durable. The cannons were warping due to extreme heat, and when the soldiers tried to shoot a ball through a warped cannon, the cannon would explode. Fourier unlocked the problem of predicting heat flow. We can use his work to study the flow of much more than heat. We can study immigration between countries, the spread of viruses, the import/export connection among several countries, the spread of a chemical pollutant (like an oil spill at sea), the weather, and so much more.
- Francis Hooke discovered a simple linear relationship between how far you stretch a spring, and the force exerted by a spring. Once you learn how one spring works, you have the power to build shock absorbers for a car. These principles apply directly to electrical systems. Once we can predict the current in a small electrical network, we can study complex electrical network (like a computer, or national power grid), and predict exactly what the current will be like anywhere in the network. Satellite communication relies on the ability to send electromagnetic waves, and because we can predict how the message will be received, we know what to send.

It all starts from some simple ideas. We'll delve into the beginning details of what makes these ideas work. I look forward to working with you this semester.

What is Inquiry Based Learning

This class may be unlike any math course you have taken in the past. We'll be learning through inquiry, rather than lecture. You'll have the chance to jump in and discover the big ideas. You are the artists in this course, and you'll discover what you decide to paint. My job as your teacher is to create the scaffolding that will enable you to discover centuries of learning. I will craft the problems you work on so that they start where you are at and get you to the knowledge of the old masters.

Here's what a typical day of class might look like:

- You have 8 problems to work on before coming to class. You crack 6 of them, but try on the other 2 and fail (which is OK - it will happen).

- When you come to class, during the first 10-15 minutes we'll work at the boards together in small groups to tackle some problems. These will often be related to the preparation you did for the current day, and to help prepare you for the next day's problems.

- While you're at the boards, I'll randomly select 8 of you to present your prep solutions to the class. Come to class with your solutions already written up (one solution per page). I'll take your solutions and make them available on the class projector.

- We'll turn the time over to you to share your work. You'll present, and then defend your work. Your peers will ask you questions. Sometimes you'll be spot on right, and sometimes you'll be wrong. As long as you can justify why you did what you did, we'll all learn and grow.

- We end class and have another 8 problem to tackle for the next day. You set up a group meeting time, where you learn to work together as a team.

As you progress in this course, you'll find that you enhance your ability to (1) reason critically, (2) present and defend your results,(3) work with a team, and (4) speak the language of mathematics. The entire structure of the course is designed to build these abilities in you.

As your instructor for this courses, I'm your coach and cheerleader. My goal is to build in you the knowledge of several centuries of work. If you'll jump in and start exploring, you'll find that you can learn so much with inquiry based learning, perhaps more than you've ever learned before.

Inquiry based learning has been around for hundreds of years, and has been shown over and over again to be more effective than lecture based learning. You can read more about it on the web at

- http://www.inquirybasedlearning.org/default.asp?page=Why_Use_IBL.

Adopting inquiry based learning requires that I turn our classroom over to you, the students. I've learned that you, my students, are capable of far more than I could ever have imagined.

Deep practice

To learn through inquiry, we have to be willing to explore ideas on our own. We have to be willing to allow ourselves to fail. Sometimes we'll tackle problems where there may not be a right answer. We have to formulate ideas, try them, and learn from the results (good or bad). We have to learn from our failures.

My first exposure to inquiry based learning occurred in the fourth grade. Our teacher gave us pretend money throughout the week for good behavior and then on Friday let us decide how to spend that money on goodies at our class store. We had to use our newly acquired arithmetic skills to figure out how to get the most bang for our buck. Some weeks I would purchase things, and then see another student's purchase and realize I had bought the wrong stuff. The next week I changed my purchasing plan. This was a weekly problem that I was allows to fail at, an then try again, with no punishment at all.

In the seventh grade, we learned an algorithm for solving absolute value inequalities. For two weeks, we had to repeat the algorithm over an over again on different problems. I wanted to find a faster way to do the problems, and discovered a way that relied on distances. Every time we learned a new concept, I tried to see if I could work the new concept into this faster approach. Some concepts took a little more trial and error to master with the faster approach, but eventually they all did. The time savings was amazing. I was super happy I had discovered something. When the test came for this material, I wanted to race my teacher to see if I could finish the exam before he finished passing out the exams. I beat him.

The next day of class, Mr. Nelson said he wanted to have me share with the class how I did the problems so quickly. He gave me the chalk and I got my first chance to stand in front of people and teach. After 5 minutes (or less), Mr. Nelson thanked me and told me to take a seat. No one in the room, not even my teacher, had a clue what I was doing. I failed.

This failure changed me. It changed entirely how I studied. I started working ahead of my class, because when they hit new material, I wanted to be able to answer questions for them. Ever time I learned an idea, I asked myself how I would share it with someone else. I was never again given a chance to teach an entire class, not even for a few minutes, but I was always ready. I didn't want to fail again.

I will always be grateful for a teacher, Mr. Nelson, who trusted me enough to allow me to try, and fail, at teaching his class. Failure is a key to success.

Why is Failure So Crucial?

I'm guessing that many of you have probably had all your math classes delivered in lecture form. Does this describe your typical math class.

> You show up to class, you take notes, you then go home and do every other odd at the end of the chapter (or something similar), and then come to class the next day to listen to a lecture again. This repeats every day, and you get used to the monotony. Your teacher comes to class and shows you the right way to do everything. You are asked to reproduce the method you were shown. If you can't do it right within a day, you get docked on your homework.

I like to refer to this as, "Monkey see, Monkey do."

Rarely did you have to struggle with the big ideas. They were handed to you already in perfect form. A few of you may have taken the time to thoroughly digest the proof in the book. More importantly, most of us don't know what doesn't work. This is the key to understanding why the current approach matters. Most valuable ideas have a plethora of important failures that helped shape the success.

These failures are perhaps the most important parts of the puzzle. Why should you do a problem the way the textbook says to? Is there a better way? If you knew the underlying issues, and tried to solve the problem yourself, you'd discover the reasons why the algorithms in your book as so awesome.

Most textbooks rarely focus on teaching you what doesn't work. They focus on success only. Perhaps our entire culture puts too much emphasis on success, and likes to forget the huge amount of knowledge that comes from our failures. This is where inquiry based learning comes in so handy. To truly master and

When we have to explore on our own, not only do we learn what works, but we learn what doesn't work. Sometimes knowing what not to do is the key.

appreciate an idea, we have to struggle with it. We have to attack the idea, struggle, fail, and then continue working. We have to learn to celebrate our failures.

Discovery takes time. It takes effort. It can at times be frustrating. But when you've struggled with an idea, failed some, and eventually crack it, there is so much joy that can come from this endeavor. This is the joy of discovery, and it fuels scientific research. My hope is that each of you will feel this joy multiple times this semester.

Our course allows you to discover. You'll be asked to try new things that you've never been shown. You'll be given a chance to try, fail, and eventually succeed. You'll come to class and present what you've done. Sometimes you'll be wrong, and we'll celebrate anyway. Your errors will have helped everyone in the class see why a certain approach does not work. Sometimes you'll be spot on right, and not even think you are before you share. The goal is to learn through inquiry, and share with each other what you've learned.

Your Peers are A Valuable Asset

I've learned over the past year that your peers might be one of your most valuable assets this semester. Every one of you comes to the course with a different background. Some of you are algebra masters. Some of you know how to do calculus really well, but don't know the right words. When you meet with peers, you'll find that you help each other over hurdles that would otherwise completely halt your progress. Your peers are perhaps your most valuable asset.

As you tackle problems this semester, you will get stuck at certain parts. I'll draw upon your knowledge from all your past classes, and almost every has holes somewhere. Some of you will want to weather this road alone. This might mean spending hours trying to relearn something you forgot. If you decide to weather the road alone, and you put in the time to master each idea that we explore, you'll have gained way more than I could ever hope if I had lectured and showed you the steps of how to do everything.

You don't have to do it all alone. If you study with peers, you can help each other over hurdles. I've seen time and time again where all you needed was a peer, not a tutor, to help you continue on. We all forget things occasionally, and our peers can be valuable assets. If you would like some suggestions about how to improve a group meeting, please see the following page online.

http://bmw.byuimath.com/dokuwiki/doku.php?id=group_study_suggestions

Chapter 1

Review

After completing this chapter, you should be able to:

1. Find the differential of a function, express it as a linear combination of partial derivatives, and then write this linear combination as the product of a matrix (the derivative) and a vector of differentials.

2. Explain how to construct the plot of a vector field and draw trajectories on that plot. You should also be able to locate graphically directions in which a vector field either pushes object directly away (or pulls objects directly towards) the origin along straight line paths.

3. Construct contour plots and gradient plots for functions of the form $z = f(x, y)$, and discuss the relationships between these two types of plots.

4. Use integration by substitution and/or integration by parts to find the potential of a vector field or differential form.

5. Solve an ordinary differential equation of the form $f(y)dy = g(x)dx$ by computing potentials of both sides and equating them.

6. Explain how to compute the Laplace transform of a function.

There are four different threads running through this chapter. They are

- Expressing Differentials as Linear Combinations,
- Visualizing Vector Fields,
- Finding a Potential with Integration, and
- Laplace Transforms through Limits and Integration.

All four topics build towards the same end goal, understanding how to recognize and solve differential equations. We will develop each idea in small increments each day of class. If you get stuck on a certain type of problem, try jumping to the next type.

We'll use several technology links throughout the chapter. Here's some links.

- Vector Field Plotter
- Level Curve Plotter
- Potential Calculator
- Parametric Curve Plotter
- First Order ODE Solver

Expressing Differentials as Linear Combinations

When Newton first invented calculus, he did not have the language of limits, instead he used differentials. He thought of dx as a really small change in x, and dy as a really small change in y. To compute the derivative of $y = x^2$, he would write

$$\begin{aligned} dy &= (x + dx)^2 - x^2 && \text{think } f(x + dx) - f(x) \\ &= x^2 + 2xdx + (dx)^2 - x^2 \\ &= 2xdx + (dx)^2 \end{aligned}$$

Because he assumed the quantity dx is extremely small, it seems reasonable to assume $(dx)^2$ is so small that it can be neglected. This yields

$$dy = 2xdx \quad \text{or} \quad \frac{dy}{dx} = 2x.$$

We now write these two expressions symbolically to work with any function $y = f(x)$, by writing $dy = y'dx$ or $\frac{dy}{dx} = y'$. The functions that the original masters worked with were generally polynomials, so the power rule $\frac{d}{dx}(x^n) = nx^{n-1}$ was the main rule they needed, which is quite simple to develop with differentials. It's not until almost 100 years after Newton's time that the limit, as we see it today, was invented.

One nice part about the approach of differentials is that the notion extends to higher dimensions almost instantly. Multivariable calculus looks the same as first semester calculus. Try the next problem.

Problem 1.1 Consider the equation $z = x^2 + xy + y^2$ which we would write today in function notation as $f(x, y) = x^2 + xy + y^2$.

1. Compute $f(x + dx, y + dy) - f(x, y)$, and use your result to obtain a formula for dz. If you encounter the product of two differentials, you should assume that product is so small that it can be ignored.
2. Compute the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Where do these partial derivatives show up in your formula for dz?
3. If the function were instead $f(x, y) = xy^2 + \sin(xy)$, use the idea from part 2 to rapidly compute dz.
4. Organize your work above to give a general formula that would tell you dz for any function $f(x, y)$. It should be in terms of the partials f_x, f_y and the differentials dx, dy.

If you didn't read the two paragraphs before this problem, please do so now. It provides some background that will help you complete this problem.
In general, you'll want to read the material in between problems, as that's where definitions, context, and hints will lie.

The same rules apply when we study a change of coordinates.

Problem 1.2 Recall the equations for polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$. We can use these to figure out x and y if we know r and θ. Can we compute dx and dy if we know dr and $d\theta$? In other words, you measure a force to have angle $\theta + d\theta$ and to have magnitude $r + dr$, where the $d\theta$ and dr are your possible errors. How much error will dr and $d\theta$ introduce when we then compute the x and y components? To answer this, complete the following:

1. We know $x = r\cos\theta$. Use this to obtain $dx = ?dr + ?d\theta$. Obtain a similar expression for dy. Write your two expressions in the vector form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} dr + \begin{pmatrix} ? \\ ? \end{pmatrix} d\theta.$$

This is precisely the formula we are after.

2. We can write the change of coordinates as the function
$$(x, y) = \vec{T}(r, \theta) = (r\cos\theta, r\sin\theta).$$
Compute the partials of $\vec{T}$ with respect to r and θ. How are these connected to your previous result? What is $d\vec{T}$ (you've already computed it)?

3. Let's now consider a different function, such as $(x, y, z) = (3u + 4v, 2u - 5v, uv)$. Using function notation we'd write this as $\vec{r}(u, v) = (3u+4v, 2u-5v, uv)$. Compute the partials of $\vec{r}$ with respect to u and v, and then use them to state the differential $d\vec{r}$.

Have you noticed in all the problems above that we are taking partial derivatives, multiplying them by scalars, and then summing the results. This occurs so often in so many different settings that mathematicians have given it a name. We could just keep saying, "Take the things you have, multiply each by a scalar, and then sum the result," or we could invent a word that says to do all this. We'll eventually start saying, "Form a linear combination." Let's make a formal definition.

Definition 1.1: Linear Combination. Given n vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ and n scalars $c_1, c_2, \cdots, c_n$ their linear combination is the sum
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Every time we introduce new words in this problem set, they'll show up as a definition. Look for the bold definitions. If you are not sure what a word means, use the search feature of your PDF viewer to hunt down the definition.

Given n functions $f_1, f_2, \cdots, f_n$ and n scalars $c_1, c_2, \cdots, c_n$ their linear combination is the sum
$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n.$$

In general, if we have n objects $o_1, o_2, \cdots, o_n$ where it makes sense to multiply each by a scalar c_i and then add them, then their linear combination is precisely
$$c_1 o_1 + c_2 o_2 + \cdots + c_n o_n.$$

Visualizing Vector Fields

Now that we've got a new definition, we need to practice using it. Let's practice using the definition of linear combination as we review the concept of a vector field. Remember that to draw a vector field, you should draw the vector $\vec{F}(x, y)$ with its base at the point (x, y). Please use the technology link on the side below to check your work with technology. Throughout the semester, I'll give you technology links where you'll have access to live Sage code that you can run directly in the web browser. I'll also provide Mathematica code and sometimes links to Mathematica notebooks.

Did you know that as a BYU-Idaho student you can download and install Mathematica on your personal computer for free? See I-Learn for details.

Problem 1.3 Consider the vector field $\vec{F}(x, y) = (x + 2y, 2x + y)$.

1. Construct by hand a plot of this vector field by plotting the field at the 8 points around the origin given by $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$, $(\pm 1, \mp 1)$. Remember, draw the vector $\vec{F}(x, y)$ with its base at (x, y), so at the point $(1, 0)$, you should draw the vector $\vec{F}(1, 0) = (1, 2)$. Then use software to construct a plot of this field and print your plot to share with the class.

Follow this link to a vector field plotter.

2. Are there any directions in which the vector field pushes things either straight out or straight in? Which directions? Explain how you know this.

3. Write the vector field as a linear combination of two vectors $\vec{v}_1$ and $\vec{v}_2$ with scalars x and y, by writing
$$\vec{F}(x,y) = \vec{v}_1 x + \vec{v}_2 y = \begin{pmatrix} \\ \end{pmatrix} x + \begin{pmatrix} \\ \end{pmatrix} y.$$

If you've forgotten the dot product, then please complete the following review problem. When you see a review problem in the problem set, the solution will always appear as a footnote on the page. These problems are there to help you quickly remember a concept that you may have forgotten.

Review Compute the dot product of the vectors $(1,2,3)$ and $(4,5,6)$. Then compute the dot product of $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$. See [1]

This next problem connects differentials to vector fields and the gradient.

Problem 1.4 Consider the function $z = f(x,y) = x^2 - y$.

1. Compute the differential dz. Then write the differential as the dot product $dz = (M,N) \cdot (dx,dy)$. Your job is to tell the class what the functions M and N are.

2. The function $\vec{F}(x,y) = (M,N)$ is a vector field. You've called it the gradient ∇f and/or the derivative Df of f. Draw this vector field by plotting several points by hand, and then use software to obtain a nice plot of the field.

Vector Field Plotter

3. Draw several level curves of the function by drawing $f(x,y) = -1$, $f(x,y) = 0$, and $f(x,y) = 4$.

Level curve plotter.

4. Now put your level curve plot and vector field plot on the same set of axes. What relationships exist between the level curves and the vector field?

You can check all your work with the level curve plotter.

Finding a Potential with Integration

We have been starting with a function that I gave you, and then from that function computing differentials and vector fields. We've seen that if $z = f(x,y)$, then the differential is $dz = f_x dx + f_y dy$ and the corresponding vector field (called the gradient and/or the derivative) is $\vec{\nabla} f = (f_x, f_y)$. In this course, one of main goals will be to look at a vector field and then from the vector field produce a function that would have given us this field. We can see and measure vector fields in nature (as one example, think about weather).

As you work on the two problems below, you'll need to review integration by substitution and integration by parts. These two methods of integration are so crucial to the development of further mathematical concepts, that it's worth our time to practice these ideas on example problems.

Problem 1.5 For each differential below, find a function $f(x,y)$ whose

Follow this link to a potential calculator in Sage to help you check your integration.

[1] The dot product is
$$(1,2,3)\cdot(4,5,6) = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 4 + 10 + 18 = 32.$$
You can think of this as a linear combination of the elements in $(1,2,3)$, where we use the scalars in $(4,5,6)$ to form the linear combination.
The dot product of $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$ is
$$\vec{a} = (a_1, a_2, \ldots, a_n) \cdot \vec{b} = (b_1, b_2, \ldots, b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

differential is given. You know you are correct if, after computing the differential, you get the the same result. We call this function $f(x,y)$ a potential.

1. $e^{2x}dx + y\cos(y^2)dy$
2. $(2x+3y)dx + (3x+\sin(2y))dy$
3. $\left(\frac{x}{1+x^2} + \arctan(y)\right) dx + \left(\frac{x}{1+y^2} + \frac{y}{\sqrt{1+y^2}}\right) dy$

Hint: This is really a question that asks you to review integration by substitution.

Problem 1.6 For each differential below, find a function $f(x,y)$ whose differential or gradient is given. Remember that we call f a potential for the differential, or a potential for the vector field.

1. $(x\sin(5x))\, dx + (1)\, dy$
2. $\left(x+y, x+y^2\sin(5y)\right)$

Hint: This is really a question that asks you to review integration by parts. Tabular integration by parts may help.

Check your work with the potential calculator

In this review unit, you'll find that many of the problems ask you to practice integration. You'll need those integration skills as the semester progresses. Learning to model the world around us and predict the future requires that we find a potential from a vector field. We can see and measure vector fields in the world around us. They appear as wind or magnetic forces that we can physically see and measure. Finding a potential for these fields is one of the keys to modeling the world around us.

Laplace Transforms through Limits and Integration

We now turn to a slightly different topic. By the end of the chapter, this topic will connect with all the ideas above. This section will introduce the Laplace transform. We first need to review some facts about limits and improper integrals.

Review: L'Hopital's rule Compute the following limits.

1. $\lim_{x\to\infty} \frac{4x+7}{e^{3x}}$ (Use L'Hopital's rule.)
2. $\lim_{x\to\infty} \frac{4x^2+3x+7}{3x^2+x+15}$ (Try dividing both the numerator and denominator by x^2.)

See [2] for an answer.

[2] L'Hopital's rule states that if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of the indeterminate form $0/0$ or ∞/∞, then you can compute the limit (under reasonable conditions) by using the formula

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

In the first problem we have $f(x) = 4x+7$ and $g(x) = e^{3x}$, both of which approach ∞ as $x \to \infty$. L'Hopital's rule gives

$$\lim_{x\to\infty} \frac{4x+7}{e^{3x}} = \lim_{x\to\infty} \frac{4}{3e^{3x}} = \frac{4}{\infty} = 0.$$

Problem 1.7 Which function grows more rapidly, the polynomial function x^n or the exponential function e^{ax}? Is there a value of n for which the function x^n grows faster, in the long run, than the exponential function. To answer this, please complete the following questions. In all your work below, you may assume that a is a positive constant.

- Use L'Hopital's rule to compute $\lim\limits_{x\to\infty}\dfrac{x}{e^{ax}}$, $\lim\limits_{x\to\infty}\dfrac{x^2}{e^{ax}}$, and $\lim\limits_{x\to\infty}\dfrac{x^3}{e^{ax}}$.
- What is the tenth derivative of x^{10} and e^{ax}? Use this, together with L'Hopital's rule, to compute $\lim\limits_{x\to\infty}\dfrac{x^{10}}{e^{ax}}$.
- What is the nth derivative of x^n and e^{ax}? Use this, together with L'Hopital's rule, to compute $\lim\limits_{x\to\infty}\dfrac{x^n}{e^{ax}}$.
- Is there a power of n for which x^n grows faster than e^{ax}? Use your answer to quickly compute $\lim\limits_{x\to\infty}\dfrac{8x^7+5x^2-3x+12}{e^{2x}}$.

Problem 1.8 On this problem, your job is to compute $\int_1^\infty \frac{1}{x^n}dx$. Please do the following:

1. Compute the integrals $\displaystyle\int_1^{10}\frac{1}{x^2}dx$ and $\displaystyle\int_1^{100}\frac{1}{x^2}dx$. Then compute $\displaystyle\lim_{b\to\infty}\int_1^b\frac{1}{x^2}dx$.
2. Compute the integrals $\displaystyle\int_1^{10}\frac{1}{x}dx$ and $\displaystyle\int_1^{100}\frac{1}{x}dx$. Then compute $\displaystyle\int_1^{\infty}\frac{1}{x}dx$, which is just shorthand for $\displaystyle\lim_{b\to\infty}\int_1^b\frac{1}{x}dx$.
3. Compute $\displaystyle\int_0^b e^{2x}dx$ and $\displaystyle\int_0^b e^{-2x}dx$. For each, state the limit as $b\to\infty$.

We are now prepared to define the Laplace transform, and use the definition to compute the Laplace transform for a few basic functions.

Definition 1.2: The Laplace Transform. Let $f(t)$ be a function that is defined for all $t\geq 0$. Using the function $f(t)$, we define the Laplace transform of f to be a new function F where for each s we obtain the value $F(s)$ by computing the integral

$$F(s)=\mathscr{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\lim_{b\to\infty}\int_0^b e^{-st}f(t)dt.$$

The domain of F is the set of all s such that the improper integral above has a finite limit. The function $f(t)$ is called the inverse Laplace transform of $F(s)$, and we write $f(t)=\mathscr{L}^{-1}(F(s))$.

Note that the Laplace transform of a function with independent variable t is another function with a different independent variable s. After integration, all t's will be removed from $F(s)$. You can of course use any other letters besides t and s.

We could also use L'Hopital's rule on the second example. Taking derivatives of the top and bottom still results in an ∞/∞ limit, so taking derivatives again yields 4/3. Alternately, recall that $x^n\to 0$ if $n<0$, which means we can write

$$\lim_{x\to\infty}\frac{4x^2+3x+7}{3x^2+x+15}=\lim_{x\to\infty}\frac{(4x^2+3x+7)/x^2}{(3x^2+x+15)x^2}=\lim_{x\to\infty}\frac{4+3/x+7/x^2}{3+1/x+15x^2}=\frac{4+0+0}{3+0+0}.$$

We will use the Laplace transform throughout the semester to help us solve many problems related to mechanical systems, electrical networks, and more. The mechanical and electrical engineers in this course will use Laplace transforms in many future courses. Our goal in the problems that follow is to practice integration-by-parts. As an extra bonus, we'll learn the Laplace transforms of some basic functions, and at the end of this chapter connect them to the other ideas.

Problem 1.9 Compute the definite integral $\int_0^\infty e^{-st}dt$ and state the values s for which the integral results in a finite limit. Now compute the the Laplace transform of $f(t) = 1$? (If the previous instruction seems redundant, then horray.) What is the Laplace transform of $f(t) = c$, a constant function? (Note, this is the same as asking, "What is the Laplace transform of a linear combination of 1?")

Problem 1.10 Compute the Laplace transform of $f(t) = e^{2t}$, and state the domain. Then compute the Laplace transform of $f(t) = e^{3t}$ and state the domain. Generalize your work to state the Laplace transform of $f(t) = e^{at}$ for any constant a, and state the domain. What is the Laplace transform of ce^{at} where a and c are constants? (Note, this is the same as asking, "What is the Laplace transform of a linear combination of e^{at}?")

Problem 1.11 Suppose $s > 0$ and n is a positive integer. Explain why

$$ds \lim_{t\to\infty} \frac{t^n}{e^{st}} = 0.$$

Then use this fact to prove that the Laplace transform of t^2 is

$$\mathscr{L}\{t^2\} = \frac{2}{s^3}.$$

[Hint: You'll need to do integration-by-parts twice. Try the tabular method.]

We'll come back to Laplace transforms later.

Expressing Differentials as Linear Combinations

It's time to review some facts about the connection between level curves, gradients, and differentials.

Problem 1.12 Consider the parametric curve given by $x = \cos t, y = 2\sin t$.

1. There are many ways to draw this curve. Please construct a graph of x versus t, a graph of y versus t, a graph of y versus x, and a 3D graph in xyt space. You should have 4 different graphs.

2. Show that this curve is a level curve of the function $f(x, y) = 4x^2 + y^2$. [Hint: plug the equations for x and y into this curve, and see if you get a constant.] What is $f(x, y)$ at points along this curve?

Recall that a level curve of a function $z = f(x, y)$ is a curve such that the output $f(x, y)$ is the same for every point (x, y) on the curve.

3. Compute the differential dz of $z = 4x^2 + y^2$. Then compute the differential $\begin{pmatrix} dx \\ dy \end{pmatrix}$ of the curve $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix}$.

4. If we require that we stay on the level curve $x = \cos t$, $y = 2\sin t$, then why must $dz = 0$? Show, by substitution, that $dz = 0$ when we replace x, y, dx, dy with what they equal in $dz = f_x dx + f_y dy$.

The trigonometric functions allow us to parameterize circles and ellipses. As the semester progresses, we'll need the functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are the hyperbolic trig functions, and we say the hyperbolic sine of x when we write $\sinh x$. These functions are very similar to sine and cosine functions, and have very similarly properties.

Problem 1.13 Consider the curve given by $x = \cosh t$ and $y = \sinh t$.

1. Compute $x(0) = \cosh(0)$ and $y(0) = \sinh(0)$.
2. Show that $\cosh^2 t - \sinh^2 t = 1$, which shows that the curve lies on the hyperbola $x^2 - y^2 = 1$.
3. Use the definition of $\cosh t$ and $\sinh t$ above to show that show that $\frac{d}{dt}\cosh t = \sinh t$ and $\frac{d}{dt}\sinh t = \cosh t$. In terms of differentials, you will have shown that

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = d\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} = \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} dt.$$

[Hint: Rewrite $\cosh t$ and $\sinh t$ in terms of exponentials and then differentiate.]

4. What's the angle between the gradient $(f_x, f_y) = (2x, -2y)$ and the tangent vectors (dx, dy) to the curve at points along the level curve? (This question has an answer regardless of the curve, and regardless of the function.)

Hint: Start by using the definitions above to rewrite each function in terms of exponentials. Then square each, expand, and subtract them. After a little algebra, you should get the result.

Visualizing Vector Fields

You've been using the derivative for at least a year to find the slope of a function. Because the derivative tells us slope, it tells us how a function moves. This means that we can use the derivative to produce a vector field.

Problem 1.14 Consider the derivative $y' = 2x$.

1. Give 2 different functions $y(x)$ so that $y' = 2x$. There are infinitely many right answers. Of those infinitely many, which one satisfies $y(0) = -4$?
2. Explain why writing $\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 \\ 2x \end{pmatrix} dx$ is equivalent to writing $y' = 2x$.

This gives us the vector field $\vec{F}(x, y) = (1, 2x)$. Construct a plot of this vector field, and add to your plot the graph of several of your curves from the first part.

3. Since $y' = 2x$ or alternately $dy = 2xdx$, we could also write

$$0 = 2xdx + (-1)dy = (2x, -1) \cdot (dx, dy).$$

This gives us another vector field $\vec{G}(x, y) = (2x, -1)$. Construct a plot of this vector field, and add to your plot the graph of several of your curves from the first part. Find a potential of this vector field.

In the previous problem, you solved your first differential equations. A differential equation is an equation which involves derivatives (of any order) of some function. For example, the equation $y'' + xy' + \sin(xy) = xy^2$ is a differential equation, where the function y depends on the variable x. Here's some formal definitions that we'll master as the semester progresses.

Definition 1.3: Differential Equation. A differential equation is an equation which involves derivatives (of any order) of some function.

- An **ordinary differential equation (ODE)** is a differential equation involving an unknown function y which depends on only one independent variable (often x or t).
- A partial differential equation involves an unknown function y that depends on more than one variable (such as $y(x,t)$).
- The order of an ODE is the order of the highest derivative in the ODE.
- A solution to an ODE on an interval (a,b) is a function $y(x)$ which satisfies the ODE on (a,b).
- Typically a solution to an ODE involves an arbitrary constant C. There is often an entire family of curves which satisfy a differential equation, and the constant C just tells us which curve to pick. A **general solution** of an ODE is an infinite collection of solutions which gives all solutions of the ODE. A **particular solution** is one of the infinitely many solutions of an ODE.
- An implicit solution to an ODE is an equation that relates the solution and the independent variable.
- Often an ODE comes with an **initial condition** $y(x_0) = y_0$ for some values x_0 and y_0. We can use these initial conditions to find a particular solution of the ODE. An ODE, together with an initial condition, is called an **initial value problem (IVP)**.

Here's a quick example of the proper use of the vocabulary above.

Example 1.4. The first order ODE $y'(x) = 2x$, or just $y' = 2x$, has unknown function y with independent variable x. A general solution on $(-\infty, \infty)$ is the collection of functions $y = x^2 + C$ for any constant C. An implicit solution to this ODE is $D = x^2 - y$ for any constant D (we didn't solve for y).

If $y' = 2x$ and $y(2) = 1$, then we have an initial value problem problem (IVP). Using $y = x^2 + C$, we know since $y = 1$ when $x = 2$ that $1 = 2^2 + C$ which means $C = -3$. Hence the solution to our IVP is $y = x^2 - 3$.

Problem 1.15 Consider the differential equation $y^2y' = x^3$, which we can rewrite in differential form as $y^2dy = x^3dx$.

1. Find a potential of both sides of $y^2dy = x^3dx$. Use your answer to give an implicit solution to $y^2y' = x^3$. How would you obtain all solutions? What solution passes through $(1,1)$?
2. We can rewrite $y^2dy = x^3dx$ as $0 = -x^3dx + y^2dy = Mdx + Ndy$. This gives us a vector field $\vec{F}(x,y) = (-x^3, y^2)$. Find a potential of this vector field and use that potential to give an implicit solution to the ODE $y^2y' = x^3$. How does this differ from the first part?

3. Use software to draw the vector field $\vec{F}(x, y)$. On the same graph, include several solutions to the ODE $y^2y' = x^3$. What patterns do you notice?

Vector Field Plotter Link

Finding a Potential with Integration

Finding the potential of a vector field is one of the key methods needed to solve differential equations. Remember, you can check any answer with software by using the potential calculator in Sage.

Problem 1.16 Consider the IVP $y' = \dfrac{t^2-1}{y^4+1}$ with $y(0) = 1$.

1. Rewrite the ODE in the differential form $f(y)dy = g(t)dt$. What are $f(y)$ and $g(t)$?

Get all the y's on one side, and all the t's on the other. We'll call this "separation of variables."

2. Find a potential for both sides, and state an implicit general solution to $y' = \dfrac{t^2-1}{y^4+1}$.
3. Use the initial condition to solve the IVP. You may leave your answer in implicit form.

Problem 1.17 Consider the IVP $\dfrac{dy}{dt} = ry$ with $y(0) = P$ where r and P are constants. Feel free to use $r = 5$ and $P = 7$ throughout the problem, if you would rather work with numbers.

1. If we write $dy = rydt$, why is there no potential for the right hand side?
2. Rewrite the ODE in the differential form $f(y)dy = g(t)dt$. What are $f(y)$ and $g(t)$?
3. Find a potential for both sides, and state a general solution to $y' = ry$.
4. Use the initial condition to solve the IVP. Make sure you solve for y. Where have you seen this solution before?

Laplace Transforms through Limits and Integration

Let's now return to Laplace transforms. We have already shown that

$$\begin{aligned}
\mathscr{L}\{1\} &= \frac{1}{s} \text{ for } s > 0 \\
\mathscr{L}\{e^{at}\} &= \frac{1}{s-a} \text{ for } s > a, \qquad \text{and} \\
\mathscr{L}\{t^2\} &= \frac{2}{s^3} \text{ for } s > 0.
\end{aligned}$$

Let's now add a few more facts to our list of Laplace transforms.

Problem 1.18 Show that the Laplace transform of t is $\mathscr{L}\{t^1\} = \dfrac{1}{s^2}$. Then compute the Laplace transforms of t^3, t^4, and so on until you see a pattern. Use this pattern to state the Laplace transform of t^{10} and t^n, provided n is a positive integer. [Hint: Try the tabular method of integration-by-parts. After evaluating at 0 and ∞, all terms but one should be zero.]

Theorem 1.5 (The Laplace transform of a linear combination)**.** *Since integration can be done term-by-term, and constants can be pulled out of the integral, we have the crucial fact that*

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$$

for functions f, g and constants a, b.

Problem The theorem above can be written in terms of linear combinations. We could have instead said that the Laplace transform of a linear combination of functions is the linear combination of the Laplace transform of each function.

You've seen this idea before in many settings, but perhaps never with these words. The operation is a familiar one that you have used many times in your past. If you perform an operation on a linear combination of objects, when is it the same as the linear combination of performing the operation on each object individually.

Can you think other instances when an operation applied to a linear combination of things is the same as the linear combination of the operation applied to each thing? What is the operation. What are the things. Please volunteer to share your answers in class.

Problem 1.19 Without integrating, rather using Theorem 1.5 above, compute the Laplace transform $\mathscr{L}\{3 + 5t^2 - 6e^{8t}\}$. State the values of s for which this is valid (i.e. the domain of the transformed function).

For the functions t^3, $2t$, and $\frac{1}{2}e^{5t}$ with constants c_1, c_2, and c_3, state the Laplace transform of the linear combination $c_1t^3 + c_2 2t + c_3\frac{1}{2}e^{5t}$.

Problem 1.20 Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}$ and $\sinh t = \dfrac{e^t - e^{-t}}{2}$. Use this to prove that

$$\mathscr{L}\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad \text{and} \qquad \mathscr{L}\{\sinh at\} = \frac{a}{s^2 - a^2}.$$

You can do this problem by using facts about the transform of e^{at}, and the fact that $\cosh at$ and $\sinh at$ are linear combinations of exponential functions.

Expressing Differentials as Linear Combinations

We saw in the previous section that the Laplace transform of a linear combination of functions can be done if we know the Laplace transform of each function (see Theorem 1.5 and problems 1.19 and 1.20). We've also seen that the differential of a function is a linear combination of the partials derivatives (see problem 1.2). We've also written a few vector fields as a linear combination of constant vectors (see problem 1.3).

When we want to solve an ODE, we can write the ODE in the differential form $M\,dx + N\,dy = 0$, which we write using the dot product as $(M, N) \cdot (dx, dy) = 0$. The scalars dx and dy are the numbers needed to create a linear combination of M and N that equals zero. If M and N are vectors, we can still write $(\vec{M}, \vec{N}) \cdot (dx, dy) = M\,dx + N\,dy$, which is a linear combination of the vectors. This gives us matrix multiplication.

Definition 1.6: The Product $A\vec{x}$ of a Matrix A and a vector $\vec{x}$. Suppose that $A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}$ is an ordered collection of n vectors of the same size (which we'll call a matrix). Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a vector of scalars. We define the product of a matrix A and a vector $\vec{x}$ to be the linear combination

$$A\vec{x} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \sum_{i=1}^{n} x_i\vec{v}_i.$$

With the definition of matrix multiplication above, we can now write differentials in terms of matrix multiplication. Let's practice this in the next problem.

Problem 1.21 Consider the vectors

$$\vec{v}_1 = \begin{pmatrix}1\\2\\3\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}2\\3\\-1\\1\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}0\\0\\2\\-2\end{pmatrix}, \vec{x} = \begin{pmatrix}-2\\1\\4\end{pmatrix}.$$

1. Compute the linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ using the scalars in $\vec{x}$.
2. Consider the matrix $A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}$ and compute the matrix product
$$A\vec{x} = \begin{bmatrix}1 & 2 & 0\\2 & 3 & 0\\3 & -1 & 2\\0 & 1 & -2\end{bmatrix}\begin{bmatrix}-2\\1\\4\end{bmatrix}.$$
3. Rewrite the vector quantity
$$\vec{v} = \begin{pmatrix}-2x+3y+4z\\x-y\\2y+3z\end{pmatrix}$$
as the linear combination $\vec{v} = x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3$. What are $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.
4. Then express $\vec{v}$ as the product $\vec{v} = A\begin{bmatrix}x\\y\\z\end{bmatrix}$. What is the matrix A?

You know you are correct if after multiplying your expression out you obtain $\vec{v}$.

Problem 1.22 Consider the function $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3s+2t\\s-4t\\st\end{pmatrix}$.

1. Compute the differentials dx, dy, and dz. Write your answer as a vector
$$\begin{pmatrix}dx\\dy\\dz\end{pmatrix} = \begin{pmatrix}?\\?\\?\end{pmatrix}.$$
2. Write the the previous vector as the linear combination
$$\begin{pmatrix}dx\\dy\\dz\end{pmatrix} = \begin{pmatrix}?\\?\\?\end{pmatrix}ds + \begin{pmatrix}?\\?\\?\end{pmatrix}dt.$$

3. Rewrite the previous part as a matrix A times $\begin{pmatrix} ds \\ dt \end{pmatrix}$. You should have a matrix A so that
$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = A \begin{pmatrix} dx \\ dy \end{pmatrix}.$$
In particular, how many rows and how many columns are in A?

Let's now examine how we can use matrices to rewrite some of the differential problems we encountered earlier in the chapter.

Example 1.7. For the function $z = x^2 + xy + y^2$, we computed the differential to be

$$\begin{aligned} dz &= (2x+y)dx + (x+2y)dy \\ &= \begin{bmatrix} 2x+y & x+2y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \end{aligned}$$

Do you see how the last step went from a linear combination to a matrix product?

For the polar coordinate transformation $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$, we computed the differential to be

$$\begin{aligned} d\vec{T} &= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} d\theta \\ &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}. \end{aligned}$$

Again, the last step was a conversion from a linear combination to a matrix product.

The key above is the conversion from a linear combination to the product of a matrix and a vector of differentials. This matrix has as its columns the partial derivatives of the function. We call this matrix the derivative or the total derivative.

The derivative is one of the most immediate applications of matrices. Just think of a matrix as bunch of partial derivatives placed side by side in columns. Each column of the matrix is a partial derivative, so if there are 3 different input variables, there will be three columns. If the output is vectors of size 2, then matrix will have 2 rows.

Some examples of functions and their derivatives Df appear in Table 1.1. When the output dimension is one, the matrix has only one row and we often just call Df the gradient of f and write $\vec{\nabla} f$ instead of Df. Both are acceptable.

In multivariate calculus, we focused our time on learning to graph, differentiate, and analyze each of the types of functions in the table above. We've been reviewing most of this throughout this chapter. Let's now practice one more problem where we get the total derivative from the differential.

$\boxed{\textbf{Problem 1.23}}$ In each problem below, find the differential of the function (writing it as a linear combination of the partial derivatives). Then write the differential as the product of a matrix (the total derivative) and a vector of differentials.

See Example 1.7 if you have not already.

1. $f(x,y,z) = xy^2 + z^3$
2. $\vec{r}(t) = (3\cos t, 2\sin t, t)$

Function	Derivative
$f(x) = x^2$	$Df(x) = [2x]$
$\vec r(t) = (3\cos(t), 2\sin(t))$	$D\vec r(t) = \begin{bmatrix} -3\sin t \\ 2\cos t \end{bmatrix}$
$\vec r(t) = (\cos(t), \sin(t), t)$	$D\vec r(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$
$f(x, y) = 9 - x^2 - y^2$	$Df(x, y) = \vec\nabla f(x, y) = [-2x \quad -2y]$
$f(x, y, z) = x^2 + y + xz^2$	$Df(x, y, z) = \vec\nabla f(x, y, z) = \begin{bmatrix} 2x + z^2 & 1 & 2xz \end{bmatrix}$
$\vec F(x, y) = (-y, x)$	$D\vec F(x, y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\vec F(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$	$D\vec F(r, \theta, z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\vec r(u, v) = (u, v, 9 - u^2 - v^2)$	$D\vec r(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & -2v \end{bmatrix}$

Table 1.1: The table above shows the (matrix) derivative of various functions. Each column of the matrix corresponds a partial derivative of the function. When the output of a function is a vector, partial derivatives are vectors which are placed in columns of the matrix. The order of the columns matches the order in which you list the variables.

3. $\vec r(u, v) = (u + 3v, 2u - v, uv)$
4. $\vec F(x, y, z) = (x + 3y, 2x - z, y + 4z)$

Have you noticed that sometimes I write the function with a vector above it, and sometimes I do not? Feel free to ask why in class. Curiosity is a great thing. Please ask questions. There's always a reason why.

Visualizing Vector Fields

Problem 1.24 Let A be the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$. We'll be analyzing the vector field given by the matrix product $\vec F(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix}$.

1. Compute the matrix product $A\begin{bmatrix} x \\ y \end{bmatrix}$ by considering linear combinations. Show how you used linear combinations of the columns of A. Then expand and simplify your work till you obtain $\vec F(x, y) = (M, N)$ where $M = x - y$ and $N = -2x$.
2. Use the vector field plotter in Sage (follow the link) to obtain and print a plot of this field. There is a line through the origin along which the field

pushes objects directly outwards away from the origin. On your printed plot, draw this line.

3. There is another line through the origin along which the field pulls objects directly inward toward the origin. On your printed plot, draw this line.

4. If you were to drop an object in this field at the point $(2,0)$, and allowed the object to move with the field, draw an approximation of the object's path on your print out. Then draw additional paths if you had instead dropped the object at the points $(-2,0)$, $(0,\pm 2)$, and a few more points of your choosing.

The curves you just drew are called trajectories and/or flow lines (even though they are not straight lines). We'll learn how to find the equations of these trajectories as part of our course. We can often visualize vector fields in nature by studying movement and forces. We'll eventually know how to predict exactly the path of an object that moves through a vector field. This gives us the power to predict the future.

Finding a Potential with Integration

Problem 1.25 For each matrix below, find a function that has this matrix as its derivative. Remember, the derivative of a function is a matrix whose columns are the partial derivatives.

1. $\begin{bmatrix} 2x+3yz & 3xz & 3xy+\sin(z)e^{\cos z+3} \end{bmatrix}$
2. $\begin{bmatrix} 2 & 3 \\ 2uv & u^2 \\ \sec^2 u & \dfrac{\cos v}{1+\sin v} \end{bmatrix}$

When we want to solve a differential equation such as $y' = 3y$, we've started by writing it in the differential form $dy = 3ydx$. The left hand side of this function has a potential, but the right hand side does not. If we divide both sides by y, then we have the expression $\frac{1}{y}dy = 3dx$. Now both sides have a potential, and we can quickly find a potential of both sides to get an implicit general solution of $\ln|y| = 3x + C$.

Alternately, we could have subtracted $3dx$ from both sides. This gives us $-3xdx + \frac{1}{y}dy = 0$. When we write the differential equation in this form, we can use matrices to understand the problem. We can write

$$0 = -3xdx + \frac{1}{y}dy = \begin{bmatrix} -3x & \frac{1}{y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

To solve the ODE, we just have to find the potential of the vector field $\begin{bmatrix} -3x & \frac{1}{y} \end{bmatrix}$. Because the dot product of the field and (dx, dy) is zero, we know the solution (x, y) must be a level curve of the potential. So we find the potential and make it equal a constant.

To solve first order ODEs, the key is to find a potential. Not every vector field has a potential. The next problem has you review when a vector field does.

Problem 1.26: Test for a potential Suppose we have a differential equation that we write in the form $M(x,y)dx + N(x.y)dy = 0$ (as done in the paragraph above). Our goal is to determine if the vector field $\vec{F}(x,y) = (\ M(x,y), N(x,y)\ )$ has a potential.

1. The derivative of a function $f(x,y)$ is the matrix $Df(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$. The second derivative of this function, and the derivative of $\vec{F}$ are
$$D^2f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \quad D\vec{F}(x,y) = \begin{bmatrix} M_x & M_y \\ N_x & N_y \end{bmatrix}.$$
What relationship must exist among the partial derivatives of M and N if $\vec{F}$ has a potential? (Two of them must be equal? Which two, and why?)
2. Suppose now that $\vec{F}(x,y,z) = (\ M(x,y,z), N(x,y,z), P(x,y,z)\ )$ is a vector field in space, and that $\vec{F}$ has a potential f. Compute the second derivative of f and the derivative of $\vec{F}$, and use your result to explain which pairs of partial derivatives of M, N, and P must be equal.
3. (Optional) If you remember learning about the curl of a vector field, then what is the curl of a vector field that has a potential?

Problem 1.27 Which vector fields, or differential forms, below have a potential? First use the test for a potential to determine this. If it has a potential, find it.

1. $\vec{F}(x,y,z) = (2x+3y+4z, 3x+5z, 5y+z^2)$
2. $(2t+3x+4y)dt + (3t+5y)dx + (4t+5x+y^2)dy$
3. $\vec{F}(x,y) = \left(\dfrac{1}{x(\ln x)^2}, \arctan y\right)$

The first two parts are just a quick check of understanding. The last one asks you to practice integration by substitution and integration by parts.

We've been solving differential equations by finding potentials. However, not every vector field has a potential. Sometimes a carefully chosen linear combination of the field may have a potential. For example, when we solved $y' = 5y$, we were able to write the ODE in the form $dy = 5ydx$ or $-5ydx + dy = 0$. While this differential form does not have a potential (check this), after we multiply both sides by $\frac{1}{y}$, we obtained the equation $-5dx + \frac{1}{y}dy = 0$. The linear combination $\frac{1}{y}(-5ydx + dy)$ has a potential, namely $-5x + \ln|y|$. An implicit solution to the ODE is $-5x + \ln|y| = C$.

Problem 1.28 Solve each ODE by finding a potential.

1. Consider the ODE $y' = 4xy$. Write this in the form $Mdx + Ndy = 0$. If you multiply both sides by $\frac{1}{y}$, you should be able to find a potential. Use the potential to state a general solution to the ODE $y' = 4xy$. Make sure you solve for y.

If you didn't read the paragraph before this problem, you might want to. It shows you an example quite similar to this one.

2. Consider the ODE $y' = f(x)g(y)$ (so any ODE where you can separate things as the product of a function involving x and a function involving y). After writing the ODE in the form $Mdx + Ndy = 0$, what should you multiply by so that you can find a potential. Use the test for a potential to show that a potential exists.

After you finish this, see page 21 in Schaums.

The process you developed in the previous problem is called "Separation of Variables." The goal is to write the ODE in the form $M(x)dx + N(y)dy$, as then you can find a potential to solve the ODE.

Problem 1.29 Solve each ODE below by first writing the ODE in the form $M(x)dx + N(y)dy = 0$. Give an implicit general solution. If there are initial conditions given, us them to find a particular solution to the ODE.

You'll find lots of practice of this idea in Chapter 4 of Schaum's.

1. Solve the ODE $y' = \dfrac{xe^x}{2y}$.
2. Solve the ODE $y' = 2 + 3y$, $y(0) = 5$.
3. Solve the ODE $(\tan x)y' = \cos^2 y$, $y(-\pi/6) = 0$.

Hint: After you separate variables, you'll either need integration by substitution, or by parts, to complete each piece. You can use the First Order ODE Solver in Sage to check your work.

Laplace Transforms through Limits and Integration

Let's now show the real reason why we care about Laplace transforms. The next theorem allows us to take the Laplace transform of a derivative, which turns a differential equation into an algebraic equations.

In the next chapter, you'll see where the formula for Laplace transforms comes from. It shows up when we use potentials to solve an ODE. The power behind the Laplace transform is that it can great simplify the work needed to solve a differential equation.

Theorem 1.8 (The Laplace Transform of a Derivative)**.** *Suppose that $y(t)$ is a differentiable function defined on $[0,\infty)$ such that $\lim_{t\to\infty} \dfrac{y(t)}{e^{st}} = 0$ for some s. We say that $y(t)$ does not grow faster than some exponential, as the function e^{st} grows faster that $y(t)$ (otherwise the limit would not be zero). If this is the case, then the Laplace transform of y' is*

$$\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0) = sY - y(0),$$

where Y is the Laplace transform of y.

Problem 1.30 Prove the previous theorem. In other words, show that $\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0) = sY - y(0)$. [Hint, use integration by parts once, and don't forget to use the bounds. The result should fall out immediately.]

Let's now use Theorem 1.8 to solve an ODEs. This first example shows the power behind this method.

Problem 1.31 Consider the IVP $y' = 7y$, $y(0) = 5$.

1. Apply the Laplace transform to both sides of this ODE. You should have an equation involving Y.
2. Solve for Y and show that $Y = \dfrac{5}{s-7}$.
3. Find the inverse Laplace transform of both sides. In other words, find a function whose Laplace transform is Y and a function whose Laplace transform is $\dfrac{5}{s-7}$? When you are done you should have the solution y to the ODE.
4. We know how to solve this ODE using separation of variables. Solve the ODE using separation of variables and show that you get the same answer.

Did you see the process above? Rather than integrate, we just (1) computed the Laplace transform of both sides, (2) solved an algebraic equation for Y, and then (3) obtained the inverse Laplace transform to get Y. Here's a parable to compare to using Laplace transforms.

> Imagine you are inside a house that has a single door leading to the downstairs. You are on the main floor, and need to open the door to the downstairs (you need to solve an ODE). However the door is locked and you don't have the key (you can't figure out how to solve the ODE). You (1) decide to walk out the front door (you apply the Laplace transform). Then you (2) walk around the house and find a back door entrance to the basement (you solve for Y). Then (3) you walk up to the locked door and unlock it from the other side (you find the inverse transform).

The Laplace transform replaces the problem of integrating with an algebraic problem where we have to solve for Y. Solving this equation with algebra is often easier. We'll be using the Laplace transform throughout the semester to help us see patterns and unlock difficult problems.

Wrap Up

In the context of a single, simpler example, let's illustrate all the pieces from this chapter.

Problem 1.32 Consider the ODE $xy' = 1 - y$.

1. We can rewrite the ODE in the differential form $(y-1)dx + (x)dy = 0$. Find a potential and state a general solution.
2. Use software to plot your vector field $(y-1, x)$ and several level curves of your potential. Make sure the vector field and the level curves are on the same plot.
3. We can separate variables by multiplying both sides by $\dfrac{1}{x(y-1)}$ to get $\dfrac{1}{x}dx + \dfrac{1}{y-1}dy = 0$. Find a potential and state a general solution. Then again use software to plot your vector field $(\frac{1}{x}, \frac{1}{y-1})$ and several level curves of your potential. To type $\ln|x|$ you'll need to write "log(abs(x))" in Sage.
4. Compare and contrast your vector fields in part 2 and 3. You should have the exact same level curves, which are hyperbolas that have been shifted away from the origin.

See the level curve plotter. If you just type in the potential, then it will graph the vector field.

To present this problem, you should have two plots, one for part 2, and one for part 3. You can copy the images from Sage into a Word document, and then put them on the same page. Then you can show how you got your potentials on this page.

Here's a summary of what we've done in this chapter.

- To solve an ODE, we rewrite the ODE as the linear combination $Mdx + Ndy = 0$ using differentials.

- Then we use integration to find a potential f of the vector field (M, N).
- The level curves of the potential are the solutions to our ODE. To solve the ODE, we find the potential f and make it equal a constant.
- We know the level curves of f are the solution because the tangent vectors (dx, dy) to our solution are orthogonal to the gradient of the potential. We know (dx, dy) and $\vec{\nabla} f$ are orthogonal because the dot product $(M, N) \cdot (dx, dy)$ equals zero, and because $\vec{\nabla} f = (M, N)$. (Make sure you can answer why?)
- If the field doesn't have a potential, we can sometimes multiply the vector field by a scalar (create a linear combination) so the rescaled field has a potential. If we can separate variables so that M depends only on x, and N depends only on y, then a potential exists.

Our approach above has one glaring error. What do we do if we can't find a potential, and we can't separate the variables? In the next chapter you'll learn how to overcome this obstacle in many instances, as well as learn how to set up differential equations that model the world around us.

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 1.33: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

As you create this lesson plan, consider the following:

- On the class period after making this plan, you'll have 30 minutes in class where you will get to teach a peer your examples. If you keep the examples simple, you'll be able to fully review the entire chapter.
- When you take the final exam, I give you access to your lesson plans. Put on your lesson plan enough reminders to yourself that you'll be able to use this lesson plan as a reference in the future. You'll want simple examples, together with notes to yourself about important parts.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 10 pages, you can have the entire course summarized and easy for you to recall.

Extra Practice

At the end of each chapter, I'll include some extra practice problems. These problems might come in the form of a reference to problems in Schaum's Outlines *Differential Equations* by Richard Bronson. Sometimes I'll point you to a freely available open source text. Generally, the problems will come with solutions where you can check your work. You can also use either Sage or Mathematica to check most solutions.

This chapter consists mostly of a review of concepts from calculus and multivariate calculus. As such, probably the best place to look for review problems is in your old calculus textbook. At some point, I'll either find, or create, a good collection of practice problems to help you. As of now, the best I can do is point you to problems in Thomas's Calculus 12 Edition, and Schaum's Outlines.

Concept	Source	Relevant Problems
Differentials	Thomas's	Section 14.6
Vector Fields	Thomas's	Section 16.2
Potentials	Thomas's	Section 16.3
Separable ODEs	Schaum's	Chapter 4: 1-8, 23-45
Exact ODEs (find a potential)	Schaum's	Chapter 5: 1-13, 24-40, 56-65
Laplace Transforms	Schaum's	Chapter 21: 4-7, 10-12, 27-35 (wait)

Chapter 2

First Order ODEs

After completing this chapter, you should be able to:

1. Identify and solve separable and exact ODEs by finding a potential.
2. Show how to obtain and use an integrating factor to solve an ODE.
3. Explain how to use a change of variables to solve an ODE.
4. Apply the modeling process and proportionality to analyze exponential growth and decay, Newton's law of cooling, mixing tank problems, Torricelli's law, the logistics model, and systems of first order differential equations.
5. Use Laplace transforms to solve first order ODEs, employing a partial fraction decomposition when needed.

When you've completed this chapter, you'll be able to make powerful predictions about the future. We'll do this by looking for linear relationships between the growth of a quantity and the quantity itself. We'll express this relationship as a differential equation, expand our ability to solve ODEs, and then use our results to obtain knowledge about the world around us. This chapter is a prototype for mathematics gets use in the sciences through modeling.

As you work on problems throughout this chapter, you can always check your work using technology. With Sage, the command "desolve" will provide you with answers to most problems. In Mathematica, the command is "DSolve." These technology links contain examples you can modify to solve most of the problems in this chapter. Please take the time to check your answers with technology.

- First Order ODEs
- Laplace Transforms
- Partial Fractions

For our convenience, the Laplace transforms we'll use most often are in Table 2.1. Feel free to use this table as you find Laplace transforms and their inverses. With practice, you will memorize this table.

$f(t)$	$F(s)$	provided
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^2	$\frac{2}{s^3}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$

$f(t)$	$F(s)$	provided
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s > \|\omega\|$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s > \|\omega\|$
y	$\mathscr{L}\{y\} = Y$	
y'	$s\mathscr{L}\{y\} - y(0)$ $= sY - y(0)$	

Table 2.1: Table of Laplace Transforms

Building a Mathematical Model

One of the key uses of differential equations is their ability to model the world around us. If we know how something is changing, then we can often use y' to represent that change. If we know a force is acting on an object, then $F = ma = my''$ allows us to build a second order differential equation whose solution is the position y of the object. As the semester progresses, we'll be making these connections in each chapter, and showing how to use differential equations to model our world. Many of the models we build will depend on observing that a change is proportional to something, or that a force is proportional to something. If you've forgotten what proportional means, here's a definition.

Definition 2.1: Proportional. We say that y is proportional to x if $y = kx$ for some constant k. We call the constant k the proportionality constant. When two quantities are proportional, then doubling one will double the other, tripling one will triple the other, and so on. A percentage change to one y results in the same percentage change to x.

My favorite way to determine if y is proportional to x is to ask, "If I double x, does y double? If I triple x, will y triple? If these are both yes, then I look to see if $y = kx$.

Here's a quick review of how to solve an ODE using separation of variables.

Review Give a general solution to the ODE $y' = 3y$. If $y(0) = 7$, state the particular solution to the IVP. See [1]

We're ready to build our first mathematical model. Suppose you go to the doctor's office to get a strep test done. They swab the back of your throat and then put a sample of tissue from your body in a petri dish. If you have strep, then the bacteria will grow inside the petri dish, and they'll be able to see the rapid growth of the strep bacteria in a fairly short amount of time.

Problem 2.1: Exponential Growth Suppose that you place some bacteria in a petri dish. Initially, there are P mg of the bacteria in the dish. The

[1] We rewrite the ODE in the differential form $dy = 3ydx$. We separate variables by dividing both sides by y to obtain $\frac{1}{y}dy = 3dt$. We compute the potential of both sides which gives $\ln|y| = 3t + C$ for any constant C. Exponentiating both sides gives $|y| = e^{3t+C} = e^{3t}C$, where we replaced e^C with the positive constant C. Removing the absolute values gives us $y = \pm Ce^{3t}$, or replacing $\pm C$ with C gives us the general solution $y = Ce^{3t}$. The initial condition $y(0) = 7$ means that $7 = Ce^{30} = C$. So the particular solution is $y = 7e^{3t}$.

bacteria begin to reproduce. Let $y(t)$ represent the mg of bacteria in the dish after t minutes. Then y' represents the growth rate of bacteria in the dish. The rate at which y grows depends on how much bacteria y there is. If you were to double the amount of bacteria y, then the growth rate y' should double as well (as long as there is space to grow, which initially there is). Similarly, if you tripled y, then the growth rate y' would triple as well. It seems reasonable to assume that y' is proportional to y.

1. Express the statement "y' is proportional to y" as a differential equation. What's the initial value $y(0)$?

2. Solve the differential equation above, obtaining a general solution to the ODE, and then a particular solution to the IVP.

Remember to check your answer with the First Order ODE Solver.

3. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). What will happen to y as t gets large?

4. Suppose initially that you measure 5 mg of the bacteria. Ten minutes later you measure 8 mg of the bacteria. Use this information to determine the constant of proportionality.

Let's change the setting from growth of a bacterial culture, to financial investments.

Problem 2.2 Suppose you invest $P = \$10,000$ dollars in an account, and that the account accumulates interest at a constant rate. Let $A(t)$ represent the accumulated worth of your investment after the investment has had t years to grow.

1. Express the connection between A and its growth as an initial value problem (state the ODE and initial value). Why can we assume that A' and A are proportional? What are the units of A', k and A?

2. Suppose that we decide to add an extra \$1000 per year to the account (with daily investments spread throughout the year). With this additional investment, explain with a sentence or two why we can express the connection between A' and A as the differential equation $A' = kA + 1000$.

3. Solve the IVP given by $A' = kA + 1000$, $A(0) = 10,000$.

Hint: Divide both sides by $kA + 1000$. Don't forget that you can check your work with the First Order ODE Solver.

4. Let's interpret the results. Suppose after 5 years that the value of the investment has reached about \$21,000. Approximately how long will it take for this investment to reach \$100,000? [Note: If you are having trouble solving for k, that's normal. It's actually a really hard problem. The key here is "Approximately." Trial and error is a valid way to solve a problem. Try some interest rates (5%, 6%, 7%, 8%, etc.)]

You've now seen two examples of how we can use differential equations to model our world. In your future courses, you'll be taking real world phenomenon and expressing the relationships you see as differential equations. Solving those differential equations gives us mathematical models we can use to interpret the world around us. There are three parts to this modeling process.

1. Express real world phenomenon in terms of a differential equation.

2. Solve the differential equation.

3. Interpret the solution in the context of the problem, which often involves using the results to predict behavior.

In our class we'll practice all three parts of this process. We'll focus more on the details in the "Solve" portion of the process than you will in future courses. You may find in some future courses that they focus on the "Express" and "Interpret" portions, and then refer you to some standard reference for the "Solve" part, or just ask you to use software. One goal of our course is to help you understand some of the key solution techniques. We'll add many problems that we can "Express" and "Interpret" without needing background specific to your majors.

Every time we've solved an ODE, we always did so by finding a potential of the differential form $Mdx + Ndy$. When a differential form has a potential, we'll start saying that it is exact.

Definition 2.2: Exact Differential Forms. Assume that f, M, N are all functions of two variables x, y.

- A differential form is an expression $Mdx + Ndy$.
- The differential of a function f is the expression $df = f_x dx + f_y dy$
- If the differential form $Mdx + Ndy$ is the differential of a function f, then we say the differential form is exact. The function f is called a potential for the differential form.

You've already spent plenty of time finding potentials to solve ODEs. Let's practice this again, using the new word "exact."

Problem 2.3 Complete both parts.

Remember to check your answer with the First Order ODE Solver.

1. Show that $(x + 2y)dx + (2x + 4y)dy$ is an exact differential form. Then give an implicit general solution to the ODE $y' = -\dfrac{x + 2y}{2x + 4y}$.
2. Show that the differential form associated with the ODE $3xy' + 3y = -2x$ is exact. Then state the solution if $y(2) = 1$.

Using Laplace Transforms to Solve ODEs

Recall that the Laplace transform of a function $y(t)$ defined for $t \geq 0$ is

$$Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)dt.$$

- We call the function $y(t)$ the inverse Laplace transform of $Y(s)$, and we write $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.
- As a notational convenience, we describe the original function $y(t)$ using a lower case y and we use the input variable t or x. We describe the transformed function $Y(s)$ using the same letter, but capitalized, and we use the input variable s.

We can use Table 2.1 to quickly compute both forward transforms and inverse transforms.

Problem 2.4 Use the table of Laplace transforms (Table 2.1) to do the following:

1. Compute the Laplace transform of both sides of $y(t) = 6 + 2t + 4t^2 - 5e^{7t} + 11\cosh(3t)$.
2. Compute the inverse Laplace transform of both sides of
$$Y(s) = \frac{5}{s} + \frac{4}{s^3} + \frac{3s}{s^2+16} - \frac{2}{s^2-9}.$$
Once you have a guess for the inverse Laplace transform, verify that your guess is correct by computing the Laplace transform.

Remember you can check your answers with the Laplace transform Sage sheet.

We now solve an ODE using Laplace transforms. Remember that the Laplace transform of a derivative y' is $sY - y(0)$.

See Table 2.1.

Problem 2.5 Consider the IVP given by $y' = 4y$ where $y(0) = 3$.

Remember you can check your answers with the Laplace transform Sage sheet.

1. Apply the Laplace transform to both sides of the ODE. You should have an equation involving $Y(s)$.
2. Solve this equation for Y to show that $Y = \dfrac{3}{s-4}$.
3. Find the inverse transform of both sides of this equation to obtain the solution $y(t)$ to the IVP.
4. Generalize your work to give a solution to $y' = ky$ where $y(0) = P$. Compare this with problem 2.1.

The first three steps above are the key steps to solving an ODE with Laplace transforms. Eventually we'll just say, "Solve the ODE with Laplace transforms," and you'll know that you need to use those 3 steps. As we develop different models, we'll revisit many of them and use Laplace transforms to obtain a solution.

First Order Systems of ODEs

Sometimes we need more than one differential equation to create a model. When an object moves in the plane, its position is given by $x(t)$ and $y(t)$. An equation for the velocity $\vec{v} = (x', y')$ is precisely a system of two differential equations.

Problem 2.6 A plane flies in a circle above a city (the center of the city is at $(0, 0)$). The plane's path is given by $(x, y) = (3\cos t, 3\sin t)$. The pilot places the plane on autopilot to continue this circular path. After the plane has been placed on autopilot, the wind starts blowing. The pilot does not adjust for the wind, which means the plane will start to veer off course. Your job on this problem is to figure out the path of the plane.

1. The velocity of the plane without the wind is $(x', y') = (-3\sin t, 3\cos t)$. The wind blows somewhat northeast and results in a new velocity vector for the plane of $(x', y') = (-3\sin t, 3\cos t) + (1/3, 1/5)$. Find equations for $x(t)$ and $y(t)$ that would give the position of the plane. Then graph the plane's position for t between 0 and 4π. You can use this parametric curve plotter to check your work. Follow the link.
2. Generalize your work to give the position (x, y) if
$$(x', y') = (-a\sin(bt), a\cos(bt)) + (c, d).$$
The radius of the circle is a, the angular velocity is b, and the wind contributes the extra (c, d).

Building a Mathematical Model

Let's look at another application before we introduce a new solution technique. Here's the scenario.

> You decide to cook a turkey for Thanksgiving. You turn the oven on to 350°F, and the package says that you need to get the turkey heated up to an internal temperature of 165°F. You followed the instructions and thawed the turkey so that currently it's about 40°F. How long will it take for the turkey to heat up?

If instead of heating a turkey, you wanted to heat a chicken patty, would the time vary? If you just wanted to heat a metal pan, how would the time vary? The next problem introduces a simplistic model, called Newton's law of cooling, to examine this question.

Newton's law of cooling works best when you assume that an increase in heat is evenly distributed throughout an object, such as heating an aluminum pan. When you heat a turkey, the heat is not evenly distributed. This uneven heat distribution complicates the model, and we'd need partial differential equations (PDEs) to obtain a better model for heat flow.

Problem 2.7: Newton's Law of Cooling Suppose that you place an object in an oven. The oven temperature is set to A (you can use Fahrenheit, Celsius, or Kelvin). The letter A is the temperature of the surrounding *a*tmosphere. The object's initial temperature is T_0. Let $T(t)$ represent the temperature of the object t minutes after we place the object in the oven. If $T(t)$ is really close to A, then the rate at which T increases should be pretty small, as the temperature of the object is almost the same as the temperature of the atmosphere. If T is really far from A, then the rate of temperature change should be a lot larger. It appears that T' depends on the difference $A - T$. Newton conjectured that the rate at which the temperature changes is proportional to the difference $A - T$.

1. Express the statement "the rate at which the temperature changes is proportional to the difference $A - T$" as a differential equation. What's the initial value?
2. Solve the IVP, obtaining a particular solution.
3. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large? Does this seem reasonable.

Remember to check your answer with the First Order ODE Solver.

Problem 2.8 You should have obtained the solution to Newton's law of Cooling as

$$T(t) = A + (T_0 - A)e^{-kt},$$

where k is the proportionality constant. Suppose that $T_0 = 45$°F and $A = 350$°F.

1. After 5 minutes, you check the temperature and observe $T(5) = 80$°F. What is k, and how long will it take for the object to reach 165°F.
2. After 5 minutes, you check the temperature and observe $T(5) = 120$°F. What is k, and how long will it take for the object to reach 165°F.
3. The number k depends on the material you are trying to heat. If k is large, what does that mean about the material? If you were to heat an aluminum pan versus a cast iron pan, what could you say about the constant k in each case?

Solving ODEs with an integrating factor

When we can't find a potential for an ODE, what do we do? Let's first examine a problem we've already solved and solve it in two different ways. From our example, we'll find an answer to this question.

Problem 2.9 Consider the ODE $y' = -3y + 7$ which we can write in differential form as $(3y - 7)dx + 1dy = 0$. To have a potential, we would need $M_y = N_x$. Since we have $M_y = 3$ and $N_x = 0$, this differential form is not exact.

See Problem 1.26 for a reminder of the test for a potential.

1. Multiply both sides of $(3y - 7)dx + 1dy = 0$ by $\dfrac{1}{3y-7}$. Show that the resulting differential form is exact (using the test for a potential, i.e. show $M_y = N_x$). Then find a potential and state a solution to the ODE.
2. Now multiply both sides of $(3y - 7)dx + 1dy = 0$ by e^{3x}. Show that the resulting differential form is exact (using the test for a potential). Then find a potential and state a solution to the ODE.

When we write an ODE in the form $Mdx + Ndy = 0$, the zero on right hand side gives us power. We can multiply both sides of the differential equation by some function F, called an integrating factor, so that the resulting differential $FMdx + FNdy$ is exact. A general solution to the ODE is then simply the level curves of the potential. Time for a formal definition.

Definition 2.3: Integrating Factor. An integrating factor for a differential form $M(x, y)dx + N(x, y)dy$ is a function $F(x, y)$ so that the product $FMdx + FNdy$ is exact.

In Problem 2.9, I gave you two different integrating factors. Where did they come from? The first one came from observing that the ODE was separable. The second one came from the formula in the next problem.

Problem 2.10 Let $M(x, y)dx + N(x, y)dy$ be a differential form. For simplicity, we just write $Mdx + Ndy$. Suppose that $F(x, y)$ is an integrating factor for this differential form.

1. For $(FM)dx + (FN)dy$ to be exact, explain why we must have
$$\frac{\partial F}{\partial y}M + F\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x}N + F\frac{\partial N}{\partial x}.$$

What does the product rule have to do with this part?

2. Assume that F only depends on x, so that $F(x, y) = F(x)$. Show that an integrating factor is
$$F(x) = e^{\int \frac{M_y - N_x}{N} dx} = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

Hint: Show that
$$\frac{1}{F}dF = \frac{M_y - N_x}{N}dx.$$
Then integrate (find a potential for) both sides.

3. (Optional) If we instead assume that F only depends on y, show that
$$F(y) = e^{\int \frac{N_x - M_y}{M} dy} = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

The problem above gives us a way to find integrating factors for many differential equations. We won't be able to find an integrating factor for every differential equation, but this method will give us integrating factors for almost every problem you'll see in undergraduate textbooks. Let's now use this technique on a problem we've already solved.

Problem 2.11 Consider the ODE $y' = ky + 1000$ from Problem 2.2.

1. Rewrite the ODE in the differential form $Mdx + Ndy = 0$.
2. Find the integrating factor $F(x) = e^{\int \frac{M_y - N_x}{N} dx} = \exp\left(\int \frac{M_y - N_x}{N} dx\right)$.
3. Multiply both sides of $Mdx + Ndy = 0$ by this integrating factor. Show that $FMdx + FNdy$ is exact and then solve the ODE by finding a potential.
4. Generalize your work to state a solution to the ODE $y' = ay + b$. (You shouldn't need to do any additional work.)

Remember you can check your answer with the First Order ODE Solver.

Problem 2.12 Solve each ODE by finding an appropriate integrating factor.

1. $y' + 4xy = 3x$
2. $2ydx + (8x + 4y)dy = 0$. Explain why $F(x)$ does not work. Use $F(y)$.

Remember to check your answer with the First Order ODE Solver. Note: You will need to simplify an expression like $e^{2 \ln x}$. Remember that $a \ln b = \ln b^a$, which means $e^{a \ln b} = b^a$. This shows up quite a bit in all our work.

Using a change of coordinates

Sometimes you won't be able to obtain an integrating factor with either formula we have for $F(x)$ or $F(y)$. Often, we can overcome this difficulty by using a change of coordinates. Just like we used polar coordinates, cylindrical coordinates, and spherical coordinates in multivariable calculus to simplify otherwise impossible problems, we'll now employ different coordinate systems to solve an ODE.

Problem 2.13 Consider the ODE $y' = (x + y)^2$.

1. Show that our formula for $F(x)$ results in a function that depends on both x and y. Show the same thing happens with $F(y)$. This means we can't use our integrating factor formulas.
2. Consider the change of coordinates $x = x$ and $u = x + y$. Show that we can rewrite the original ODE $y' = (x + y)^2$ in the differential form $du = (1 + u^2)dx$. [Hint: You need to compute the differential of u. Since $u = x + y$ we can compute $du = ?dx + ?dy$.]
3. Solve the ODE $du = (1 + u^2)dx$. Then replace u with $x + y$ and solve for y to get a general solution to this ODE. [Hint: The ODE is separable.]

Remember to check your answer with the First Order ODE Solver.

Let's try another problem where a simple substitution results in greatly simplifying the ODE.

Problem 2.14 Consider the ODE $xyy' = 4x^2 + 2y^2$. In this situation, if we let $u = y/x$ (so $y = xu$), show that we can rewrite the ODE as

$$\frac{u}{4 + u^2} du = \frac{1}{x} dx.$$

This is a separable ODE, which we can solve. Solve the ODE. Give an implicit general solution in terms of y and x.

[Hint: Since you have $y = xu$, you'll probably want to write $dy = ?dx + ?du$. This will allow you to replace dy in the original ODE.]

Notice that the coefficients xy, $4x^2$, and $2y^2$, all are second order monomial terms. When the coefficients of an ODE are monomials with the same degree, the substitution $u = y/x$ will always convert the ODE into a separable ODE. You have enough tools to prove this fact. If you do, I'll have you share it with the class.

Using Laplace Transforms to Solve ODEs

Let's now use Laplace transforms to tackle a problem similar to the one we used to introduce integrating factors.

Problem 2.15 Consider the IVP given by $y' = 3y + 7$ where $y(0) = 11$.

1. After computing the Laplace transform of both sides, show that
$$Y = \frac{11s+7}{(s)(s-3)}.$$

2. The right hand side above is not in our Table of Laplace transforms. However, if we could rewrite the right hand side as
$$\frac{11s+7}{(s)(s-3)} = \frac{A}{s} + \frac{B}{s-3}$$
for some constants A and B, then we could use an inverse transform.

   Find constants A and B so that the equation above is valid (as a suggestion, first multiply both sides by $(s)(s-3)$).

3. Solve the IVP by finding the inverse Laplace transform of
$$Y = \frac{A}{s} + \frac{B}{s-3}.$$

This process is called a partial fraction decomposition.Try this problem without looking for help from any outside source. If you are stuck, then try googling "partial fraction decomposition."

You can check your work with this partial fraction calculator.

Remember you can check your answers with the Laplace transform Sage sheet.

First Order Systems of ODEs

Let's now return to examining a system of first order differential equations. When we only have one differential equation, we have been writing it in the form $Mdx + Ndy = 0$, which we could also write in the matrix form
$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = 0.$$

How does this generalize to systems of first order differential equations?

Problem 2.16 Complete the following:

1. Consider the first order system of ODEs given by
$$2t\frac{dx}{dt} = 3 - 2x \quad \text{and} \quad (4t+5y)\frac{dx}{dt} + (5x+t)\frac{dy}{dt} = -4x - y.$$
Rewrite this system in differential form, and then obtain a 2 by 3 matrix A so that
$$A\begin{bmatrix} dt \\ dx \\ dy \end{bmatrix} = \begin{bmatrix} \underline{\hspace{2cm}} & \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2. Find a function $f(t, x, y)$ so that $df = A\begin{bmatrix} dt \\ dx \\ dy \end{bmatrix}$.

3. What do you think is a general solution to this system of ODEs? Why? It's OK if you are wrong. The goal here is to have you make a conjecture and be prepared to explain why you made your conjecture.

Building a Mathematical Model

Let's now analyze another type of model. In this case, we'll create the differential equation by studying flow in and flow out instead of looking for a proportionality. If we know how much y increases (flow in), and we know how much y decreases (flow out), then we know the rate of change of y which means we know

$$y' = (\text{flow in}) - (\text{flow out}).$$

In multivariable calculus, we studied the flux of a vector field across a curve or surface. This is precisely the study of flow in and flow out.

Problem 2.17: Tank Mixing Intro Suppose a 20 gallon tank contains an evenly mixed solution of water and salt. Initially, there are 4 lbs of salt mixed into the water. We start pumping in 3 gallons of water each minute, where the incoming water has 1/2 lb of salt per gallon. We'll assume that the salt remains evenly spread throughout the entire tank by constant stirring. At the same time, we allow 3 gallons per minute of the evenly stirred mixture to flow out through an outflow valve.

Let $y(t)$ represent the lbs of salt in the tank after t minutes. Our goal is to predict the amount of salt $y(t)$ in the tank after t minutes. We currently know $y(0) = 4$ lbs.

1. (Express) How many lbs of salt flow into the tank each minute? How many lbs of salt flow out of the tank each minute? State a differential equation that models the lbs of salt in the tank at any time t.
2. (Solve) Use software to give a general solution to the ODE and the particular solution to the IVP. See the First Order ODE Solver.
3. (Interpret) If we allowed t to run for a really long time, what would $y(t)$ approach? Does this seem reasonable?
4. What would you use for your ODE if the volume of the tank is V gal, the inflow/outflow rate is r gal/min, and the concentration of salt in the incoming water is c lbs/gal?

Hint: To get inflow and outflow of lbs of salt per min, you need to multiply some quantities together. Pay attention to units. For outflow, remember there are $y(t)$ lbs of salt in the 20 gallon tank, so we have $\frac{y \text{ lbs}}{20 \text{ gal}}$. What can you multiply this by to get lbs/min?

In our first model of this chapter, we analyzed the growth of a bacteria population in a petri dish. We could have applied this to any other population to predict things such as the number of deer in a forest, how many people will be on the Earth, the spread of cancer through the bloodstream, the number of cell phones users in Brazil, the speed of computer processors, etc. In our model, we assumed that the growth of the bacteria is proportional to the amount of bacteria currently present. This an assumption about the flow in. With this proportionality assumption, we obtained the ODE $y' = ky$ and solution $y = Ce^{kt}$. There is a glaring error with this model, namely that as t gets larger the population continues to grow without bound. The petri dish can not support this kind of growth. Our model needs to be improved. Let's now fix this, but let's change the setting to the spread of a virus.

Problem 2.18: Logistic Model Intro Suppose that a virus (like the bird flu) starts to spread in a city. Let $y(t)$ represent the number of people who have had the virus after t days. Initially, it seems reasonable to assume that y' is proportional to y, as if we double the number of people who have the virus, then the virus will spread twice as fast. However, the model $y' = ky$ needs to be altered because exponential growth cannot occur forever. There's only so many people. There are two ways to proceed.

1. As the virus affects more people, we know the growth rate should decrease. Let's assume there are M people in the town. If $y(t)$ ever equals M (so everyone is infected), then we'd have $y' = 0$. As y gets closer to M, the growth rate k should be small. Vice versa, if y is far from M, then the growth rate k should be large. Let's assume that $y' = ky$, but that k is proportional to the difference $M - y$ between the maximum population and the current population. Why does $y' = c(M - y)y$?

2. Let's analyze this problem in a different way. Viruses spread when sick people interact with non sick people. If y is the number of sick people, then $M - y$ is the number healthy people. The product $(M - y)y$ is the number of possible interactions between healthy and sick people. What assumption should we make to obtain $y' = c(M - y)y$.

3. Remember that if we know the slope y', then the vector field $\vec{F}(t, y) = (1, y')$ gives a field of tangent vectors to possible solution curves. Use software to construct a vector field plot of the the field

$$\vec{F}(t, y) = (1, \frac{1}{3}(4 - y)y)$$

where $0 \leq t \leq 10$ and $-2 \leq y \leq 6$. On your plot, draw several solution curves. This would model a scenario in which $M = 4$ million residents and $k = 1/3$ (about 1/3 of the time, an interaction between a sick and healthy person results in the healthy person getting sick).

See the Vector Field Plotter

Solving ODEs with an integrating factor

Problem 2.19 Suppose a 50 gallon tank contains a solution of fertilizer which initially contains 10 lbs of fertilizer. We start pumping in 4 gallons per minute of a solution where the concentration of fertilizer is 1/3 lb per gallon. Assume that the mixture remains evenly spread throughout the entire tank by constant stirring. At the same time, 4 gallons per minute of the evenly stirred mixture flow through the outflow valve. Let $y(t)$ represent the lbs of fertilizer in the tank after t minutes.

1. Explain why $y' = \frac{4}{3} - \frac{4}{50}y$ with $y(0) = 10$.

2. After rewriting the ODE in the differential form $Mdx + Ndy = 0$, find an integrating factor and use it to solve this IVP.

3. Plot your solution. Your plot should show the initial condition $y(0) = 10$, and you should be able to see what $y(t)$ approaches as t gets large.

Using Laplace Transforms to Solve ODEs

Problem 2.20 Suppose a 5 gallon tank contains a solution of fertilizer which initially contains 2 lbs of fertilizer. We start pumping in 3 gallons per minute of a solution where the concentration of fertilizer is 1/4 lb per gallon. Assume that the mixture remains evenly spread throughout the entire tank by constant stirring. At the same time, 3 gallons per minute of the evenly stirred mixture flow through the outflow valve. Let $y(t)$ represent the lbs of fertilizer in the tank after t minutes.

1. State an IVP (both the ODE and IV) that models this situation.
2. Use Laplace transforms to solve the ODE. After computing the Laplace transform of each side and solving for Y, you should obtain
$$Y = \frac{2s + (3/4)}{(s)(s + 3/5)}.$$
You'll need to perform the partial fraction decomposition
$$\frac{2s + (3/4)}{(s)(s + 3/5)} = \frac{A}{s} + \frac{B}{s + 3/5}.$$
Once you've found A and B, inverse Laplace transforms will get you the solution instantly.

You can check your work with this partial fraction calculator.

Using a change of coordinates

The logistics model $y' = a(M - y)y$ can be rewritten in the form $y' = aMy - ay^2$, or perhaps more simply as $y' = Ay + By^2$. This ODE is separable, however if we allow A and B to depend on x, then we have the ODE $y' = A(x)y + B(x)y^2$ which is not separable. Bernoulli discovered a way to solve any ODE of the form $y' = A(x)y + B(x)y^n$, by using the substitution $u = y^{1-n}$. The next problem has you solve the logistics model by using this substitution.

It's not easy to discover the right substitution that will convert an ODE into something we can solve. We call them Bernoulli ODEs because his discovery was quite clever.

Problem 2.21 Consider the ODE $y' = 3y + 5y^2$.

1. Use the substitution $u = y^{1-2} = y^{-1}$ to rewrite the ODE in the form $Mdx + Ndu = 0$. Show that $M = 3u + 5$ when $N = 1$.
2. Then obtain an integrating factor to solve this ODE. After finding a solution, replace u with $1/y$ and give an explicit solution by solving for y.
3. Generalize your work to state a general solution to $y' = Ay + By^2$. You have now solved every logistics model problem. In particular, what's the solution if $y' = aMy - ay^2$?

Since $u = y^{-1}$, we know $du = -y^{-2}dy$. Our ODE is $dy = (3y + 5y^2)dx$. If you combine these, you get $du = -y^{-2}(3y + 5y^2)dx$. Multiply the y^{-2} through and remember that $u = y^{-1}$.

Using Laplace Transforms to Solve ODEs

We've been looking at two main ways to solve ODEs. One approach involves rewriting the ODE in the form $Mdx + Ndy = 0$ and then finding a potential. Sometimes we have to use an integrating factors. Sometimes we have to change coordinates first. Sometimes we have to do both (as in the logistics model problem).

The second approach is to use Laplace transforms. This replaces the integration problem with an algebra problem, often involving a partial fraction decomposition. Let's practice this process again.

Problem 2.22 Solve each IVP with Laplace transforms.

1. $y' + 3y = 2t$ where $y(0) = 5$
2. $y' + 3y = e^{2t}$ where $y(0) = 5$

Check your work with this partial fraction calculator. The Laplace transform calculator will let you know if you have the correct answer.

Is there a connection between our two methods of solving ODEs? To answer this, let's solve a general problem with the integrating factor/potential approach.

Problem 2.23 Consider the ODE $y' = ay + f(t)$, where a is a constant and $f(t)$ represents any function of t.

1. Rewrite the ODE in differential form, and then use an appropriate integrating factor to solve the ODE. Because you do not know what $f(t)$ equals, your solution will involve an integral. However, you should be able to complete all integrals that do not involve $f(t)$, and then solve for y.

2. Compare and contrast the definition of the Laplace transform with your solution above.

3. Now consider the ODE $y' = a(t)y + f(t)$ where a is now a function of t. Show that

$$y(t) = e^{\int a(t)dt}C + e^{\int a(t)dt}\int \left(e^{-\int a(t)dt}f(t)\right) dt$$

where C is an arbitrary constant.

The find a potential approach to solving ODEs came first. The Laplace transform approach came much later. It wasn't until the 1900's that the Laplace transform approach gained a lot of momentum. Feel free to ask me in class about the history behind the Laplace transform.

In our table of Laplace transforms (Table 2.1) it states that

$$\mathscr{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \quad \text{and} \quad \mathscr{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}.$$

Problem 2.24 Pick one of the functions $\cos \omega t$ or $\sin \omega t$. Then use the definition of the Laplace transform to compute the Laplace transform and verify the above formula is correct. Only do one, as the other is similar.

[Hint: You'll want to use integration by parts twice.]

See the online text for a complete solution. It's in chapter 4 there.

Building a Mathematical Model

Systems of differential equations can model some pretty cool things. The next model uses our proportionality assumptions to create a model for describing the rise and fall of populations in a predator/prey relationship. If there are too many predators, or too much prey, can we model what will happen?

Problem 2.25: Predator-Prey In this problem, we'll build a mathematical model that describes the interaction between a predator and a prey, namely coyotes and deer. Let $x(t)$ and $y(t)$ represent the numbers of coyote and deer t years from now in a certain forest. To create a model, we have to make some assumptions. Suppose that in the absence of the deer, the coyote population cannot find enough other sources of food and will die off at a rate that is proportional to its current size (so $x' = -k_1x$). In the absence of the coyote population, the deer population will grow at a rate that is proportional to its current size (so $y' =?$). If there are a lot of deer, then the coyotes have plenty of food and their numbers will increase. Let's assume that this increase is proportional to the possible number of interactions (xy) between the coyote and deer population. Similarly, the deer population decreases at a rate that is proportional to this possible number of interactions.

We could similarly model whales versus plankton, or any other predator/prey relationship.

1. Using sentences (actually write them out) explain why we have the differential equations $x' = -k_1x + k_2xy$ and $y' = k_3y - k_4xy$. Explain why the negative signs appear in this model.

2. Let's visualize what this model looks like. To do so, we need to choose some values for the constants (which we could discover through measurements if we worked for wildlife management). Let's use the numbers $k_1 = .3$, $k_2 = .002$, $k_3 = .4$ and $k_4 = .005$. Plot the field $(x', y') = (-k_1x + k_2xy, k_3y - k_4xy)$, using the bounds $0 \leq x \leq 150$ and $0 \leq y \leq 300$.

3. If the current population numbers are 120 coyotes and 200 deer, what should happen to both populations in the next year? What if there are only 60 coyotes and 200 deer?

Using a change of coordinates

Let's practice another change of coordinates (substitution) problem. Remember that you need to get an equation that connects the differentials du and dy whenever you use a change of coordinates.

Problem 2.26 Let's solve the ODE $y' = (x - y)^2$ by using the change of coordinates $x = x$ and $u = x - y$. Remember to compute the differential du, and then separate variables to show that

$$\frac{1}{u^2 - 1} du = -dx.$$

Use a partial fraction decomposition to write $\frac{1}{u^2-1}$ as the sum of two simpler fractions (factor the denominator). After finishing the partial fraction decomposition, integrate and give an implicit general solution to the ODE.

You can check your work with this partial fraction calculator.

Using Laplace Transforms to Solve ODEs

Problem 2.27 Use Laplace transforms to solve the ODE $y' + 3y = \cos(2t)$ where $y(0) = 1$. Consider the following hints:

You can check your work with this partial fraction calculator.

- You'll need to use the partial fraction decomposition

$$\frac{s}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

  as part of your work. Remember that for a partial fraction decomposition, when the denominator is linear, you need a constant above, i.e. $A/(s + 3)$. When the denominator is quadratic, you need a linear expression above it, i.e. $(Bs + C)/(s^2 + 1)$.

- When you compute the inverse Laplace transform of $(Bs + C)/(s^2 + 4)$, remember that you can break this up as two fractions.

- If you end up with 13's in your denominators, you're on the right track.

The next problem applies Newton's law of cooling to examine what happens if the temperature of the surrounding environment changes. Recall that Newton's law of cooling suggests that the rate of change of temperature of an object is proportional to the difference between the current temperature and the surrounding atmosphere. If we let $y(t)$ be the temperature of the house at any time t, then we can write Newton's law of cooling as

$$y' = k(A - y), y(0) = y_0.$$

Problem 2.28 Suppose that during a summer day, the temperature outdoors fluctuates between 70°F and 110°F. We can approximate this with a sine wave. If we let $t = 0$ be noon, then we could obtain the temperature A outdoors after t hours using the formula $A(t) = 20\sin(\frac{2\pi}{24}t) + 90$. Suppose that the air conditioner breaks at noon (the house is at 70°F at noon), and then by 6 pm in the evening, the temperature rises to 90°F.

1. Use Newton's law of cooling to set up an IVP that would give the temperature of the house (see the paragraph before this problem).

2. Solving this ODE is quite involved, so let's simplify the computations. If we let $t = 2\pi$ correspond to 1 full day, then the temperature of the surrounding atmosphere is $A(t) = 20\sin(t) + 90$. If we let $k = 1$ and measure temperature by 10 degree increments, we could write our IVP as

$$y'(t) = 2\sin(t) + 9 - y, \quad y(0) = 7.$$

Solve this IVP with Laplace transforms.

[Hint: There are two partial fraction decompositions that we need to perform. One of them is $\dfrac{?}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$. The other is $\frac{?}{(s+1)(s^2+1)} = \frac{C}{s+1} + \frac{Ds+E}{s^2+1}$. Don't forget that the numerator is linear when the denominator is quadratic.]

In class, we'll solve this with technology for any k, as well as graph and interpret the solution.

You can check your work with this partial fraction calculator.

First Order Systems of ODEs

Let's now apply our knowledge about tank mixing problems to set up an IVP where there are two tanks. This gets interesting when we realize we can replace the tanks with countries and the salt with goods that we import/export (or deer immigrating between sections of a forest, or studying traffic flow between nearby cities, etc.)

Problem 2.29: Mixing Model System Imagine that we have two tanks. The first tank contains 6 lbs of salt in 10 gallons of water. The second tank contains no salt in 20 gallons of water. Each tank has an inlet valve, and an outlet value. We attach hoses to the tanks, and have a pump transfer 2 gallon/minute of solution from tank 1 to tank 2, and vice versa from tank 2 to tank 1. So as time elapses, there are always 10 gallons in tank 1 and 20 gallons in tank 2. Our goal is to find the amount of salt in each tank at any time t.

1. We know there are initially 6 lbs of salt in tank 1, and no salt in tank 2. If we allow the pumps to transfer salt for enough time, explain why the salt content in tank 1 will drop to 2 lb, and the salt content in tank 2 should increase to 4 lbs.

2. Let $y_1(t)$ and $y_2(t)$ be the lbs of salt in tanks 1 and 2. Explain why

$$y_1' = -\frac{2}{10}y_1 + \frac{2}{20}y_2.$$

Obtain a similar equation for y_2'.

3. Write your ODEs in the matrix form

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} -2/10 & 2/20 \\ ? & ? \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

4. Draw the vector field represented by your matrix (use bounds that include both $(6,0)$ and $(2,4)$). Then sketch the solution $(y_1(t), y_2(t))$ to your IVP by starting at the point $(6,0)$ and following the field until the vectors no longer tell you to move. Does your answer agree with your reasoning in the first part of this problem?

Remember you can use the vector field plotter to graph any vector field.

Problem 2.30 On October 14, 2012, Felix Baumgartner jumped from a helium balloon at about 39,000 m above sea level (the highest ever parachute jump made by man). Let $y(t)$ represent Felix's height t seconds after jumping. Let $v(t)$ represent his velocity. We know that $y'(t) = v(t)$ and that $v'(t) = a(t)$. The acceleration involves 2 parts. We know that the total force acting on an object, by Newton's second law of motion, is $F_T = ma$. We'll assume that the force from gravity $F_G = mg$ is constant (probably not the best assumption with such a large fall), and that the force due to air resistance F_R is proportional to Felix's velocity (so doubling his speed would provide twice as much resistance). Our goal is to find Felix's top speed, his terminal velocity.

Felix had to wear a pressurized space suit because the altitude was so high.

1. The total force $F_T = ma$ is the sum of the force from gravity $F_G = -mg$ and the force due to air resistance F_R, which we assumed was proportional to the velocity. Use this information to obtain the ODE $v' = -g - \frac{k}{m}v$.
2. Solve the IVP $v' = -g - \frac{k}{m}v$ where $v(0) = 0$. Solve for v and then state his maximum speed (what happens as $t \to \infty$). Your answer will be in terms of k, g, and m.
3. Integrate your solution for $v(t)$ to give Felix's height $y(t)$. Assume that $y(0) = h$.

My favorite approach on this one is an integrating factor.

Building a Mathematical Model

One of the main goals of this chapter is to help you see the huge range of applications where we can apply differential equations. The next application, Torricelli's law, allows us to understand how rapidly water will flow out of can that has a punctured hole in the bottom. This law connects the ideas that flow in and flow out must be the same, as well as providing another great application of proportionalities.

Problem 2.31: Torricelli's Law Suppose that we puncture a hole in the bottom of a cylindrical tank whose radius is r m. As the height of the water will slowly drop, let $h(t)$ represent the water level in the tank after t seconds. Assume that the hole we created has an area of a square meters.

1. The tank of water has a certain potential energy (measured from the bottom of the tank). As water leave the tank, this potential energy drops. For energy to be conserved, the kinetic energy of the water leaving must match the drop in potential energy of the water in the tank. The kinetic energy of a small mass m is $K = \frac{1}{2}mv^2$. The potential energy of a small mass located h units up is $P = mgh$. Use this information to explain why $v = \sqrt{2gh}$.
2. Let $V(t)$ be the volume of water in the tank at time t. If the water leaves at speed $v(t)$ through a hole whose area is a, explain why $\frac{dV}{dt} = -av$.

3. Because the can is cylindrical, we know that $V(t) = \pi r^2 h(t)$. Use the three equations $v = \sqrt{2gh}$, $\frac{dV}{dt} = -av$, and $V(t) = \pi r^2 h(t)$ to explain why h' is proportional to $\sqrt{h}$. What is the proportionality constant?

4. Solve the IVP $h' = -\dfrac{a\sqrt{2g}}{\pi r^2}\sqrt{h}$ where $h(0) = h_0$.

You can use your solution to determine how long it takes for tank to completely empty.

You can read more about Torricelli's law in this excellent online reference.

Problem 2.32 Let's analyze a deer population in a forested region. Data collection has shown that the forest can support about $M = 2000$ deer, and that the number of deer $y(t)$ after t years follows the logistics model $y' = k(M - y)y$ where $k = 1/5000$. Fish and game has decided to open the region up for hunting. They administer deer tags so that they can control how many deer die each year through hunting. Let's assume that the current number of deer is $y(0) = P$, and that fish and game issues tags to allow for about h deer to die each year from hunting.

1. Explain why an appropriate model for the deer population with hunting allowed is $y' = k(M - y)y - h$. What are the units of y, y', h, and k?

2. This ODE is rather complicated to solve. However, we can visualize the solution by looking at an appropriate vector field plot. Explain why the vector field $\vec{F}(t, y) = (1, y')$ gives the tangent vectors to the solution.

3. Remember that $k = 1/5000$ and $M = 2000$. Let $h = 100$, and then use the Sage vector field plotter to construct a plot of the vector field $F(t, y) = (1, k(2000-y)y-100)$. Discuss what you see and how it applies to the deer population (write a few sentences). In particular, what happens to the population of deer in the long run if the current population is $P = 1900$, versus $P = 1000$, versus $P = 400$.

4. Is there some level h at which hunting can cause the deer population to go extinct? Consider drawing the vector field with $h = 150$, and then with $h = 250$. What recommendation would you give to fish and game if you wanted to keep the deer population alive?

Using a change of coordinates

Let's return to practicing a few problems where we have to make a substitution. Remember that if we let $y = xu$, then we need the differential $dy = udx + xdu$ to get rid of dy in our ODE. If we make the substitution $u = y^{-3}$, then we need the differential $du = -3y^{-4}dy$ to get rid of dy in our ODE. The first step after making any substitution is to find appropriate differentials.

Remember that we say an ODE is a Bernoulli ODE if it can be written in the form $y' = a(x)y + b(x)y^n$. To solve this ODE, we use the substitution $u = y^{1-n}$. Bernoulli showed that with this substitution, you will always succeed in converting the ODE into an ODE that has an integrating factor.

Problem 2.33 Solve the ODE $y' = 3y + 7y^{12}$ by using a Bernoulli substitution (see the previous paragraph). Make sure you give an explicit solution (solve for y). Then generalize your work to give an explicit solution to $y' = ay + by^n$ where a, b, and n are constants.

[Hint: The first step after letting $u = y^{1-n}$ is to compute the differential du. Then get everything in terms of x and u. Find an integrating factor.]

Problem 2.34 Consider the ODE $(Ax+By)dx+(Cx+Dy)dy=0$ where A, B, C, and D are constants.

1. Why is this ODE not currently separable? Also, show that neither of our integrating factor formulas $F(x)$ or $F(y)$ are usable.

2. Use the substitution $u=y/x$, so $y=xu$, to rewrite the ODE as
$$\frac{C+Du}{A+(B+C)u+Du^2}du=-\frac{1}{x}dx.$$

3. If $A=4$, $B=3$, $C=2$, and $D=1$, then use a partial fraction decomposition to simplify the left side above, and finally solve the ODE. You may give an implicit solution.

Using Laplace Transforms to Solve ODEs

Let's practice one more problem with Laplace transforms before we end this chapter. Remember that when you perform a partial fraction decomposition, you need a constant above a linear denominator, and a linear expression above a quadratic denominator.

Problem 2.35 Use Laplace transform to solve the IVP $y'+2y=5\cos(3t)$ where $y(0)=1$.

If you wanted to use an integrating factor $F(x)$ on this problem, what integral would you have to perform? What does this have to do with Laplace transforms?

First Order Systems of ODEs

Let's now look at another position/velocity/acceleration model, but this time related to springs.

Problem 2.36 Suppose we attach an object with mass m to a spring. We place the spring horizontally, and put the mass on a frictionless track. We let go of the object, it starts to oscillate. We'll use the function $x(t)$ to keep track of the position of the spring at any time t, with $x=0$ corresponding to equilibrium (the mass is at rest). Robert Hooke (1635 – 1703) showed that the force F needed to displace an object attached to a spring is proportional to the displacement x.

1. Suppose the object has been displaced x units. Explain why the force of the spring on the object is $F_S=-kx$. Since newton's second law of motion says that the total force acting on an object is $F_T=ma$, explain why $v'=-\frac{k}{m}x$.

2. We now have the system of ODEs $x'=v$ and $v'=-\frac{k}{m}x$. Rewrite this system in the matrix form
$$\begin{pmatrix}x'\\v'\end{pmatrix}=\begin{bmatrix}\underline{\qquad} & \underline{\qquad}\\ \underline{\qquad} & \underline{\qquad}\end{bmatrix}\begin{pmatrix}x\\v\end{pmatrix}.$$

   If we let $k=3$ and $m=2$, draw the vector field associated with the matrix. What relationship do you see between x and v?

Use the vector field plotter to plot $\vec{F}(x,v)=(v,-\frac{3}{2}x)$.

3. Let $k = 3$ and $m = 2$. Also, let $x(0) = 5$ and $v(0) = 7$. Then compute the Laplace transform of both $x' = v$ and $v' = -\frac{k}{m}x$ (use X and V for the Laplace transforms of x and v). Solve for X in terms of s and then invert Laplace transform both sides. You can now predict the exact position $x(t)$ of the spring at any time t.

   [Hint: You should not need any partial fraction decompositions, though you'll have to take a square root of 3/2.]

Problem 2.37 Consider the system of first order differential equations given by $x' = 4y$ and $y' = x$.

1. Write the system as a matrix product (state A)
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \underline{\quad\quad} & \underline{\quad\quad} \\ \underline{\quad\quad} & \underline{\quad\quad} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

2. Create a vector field plot of $\vec{F}(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}$ (use software). Use your plot to guess a relationship between x and y. Draw several curves representing this relationship.

3. Compute the Laplace transform of both $x' = 4y$ and $y' = x$, where we'll use $x(0) = x_0$ and $y(0) = y_0$. Solve for X and Y to show that $X(s) = \dfrac{x_0 s + y_0}{s^2 - 4}$. What is $Y(s)$?

4. Use inverse Laplace transforms to state the solution $x(t)$ and $y(t)$ to this system. You can do this without needing a partial fraction decomposition if you use hyperbolic trig functions.

Wrap up

In this chapter, we've explored various different techniques to solve first order ODEs and systems. Here's a list.

- Separation of variables: The easiest, if you can separate.
- Exact: The ODE has a potential.
- Integrating Factors: Make the ODE exact.
- Substitution: Change variables so you can make the ODE exact.
- Laplace Transforms: Dodge integration. Replace it with algebra.

Problem 2.38 Which method would you use to solve each ODE below? If you opt for separation of variables, then show us how to separate. If the ODE is exact, show us how you know. If you decide to find an integrating factor, show us the integrating factor. If you will use a substitution, what substitution will you use? If you decide to use Laplace transforms, take the Laplace transform of both sides. In all cases, don't solve the ODE, rather just show us the first step in the solution process.

1. $x^2 y' = 4xy^2$, $y(2) = 1$.

2. $xy' = 3y + x$, $y(2) = 1$.
3. $3xy' = 3y + x$, $y(2) = 1$.
4. $y' + 8y = e^x$, $y(0) = 1$.
5. $y' + 8y = y^4$, $y(0) = 1$.

Question 2.4. Why can't we (yet) use a Laplace transform to solve $y' = -a(y - M)y$?

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 2.39: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

As you create this lesson plan, consider the following:

- On the class period after making this plan, you'll have 30 minutes in class where you will get to teach a peer your examples. If you keep the examples simple, you'll be able to fully review the entire chapter.
- When you take the final exam, I give you access to your lesson plans. Put on your lesson plan enough reminders to yourself that you'll be able to use this lesson plan as a reference in the future. You'll want simple examples, together with notes to yourself about important parts.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 10 pages, you can have the entire course summarized and easy for you to recall.

Extra Practice

Please use the problem list below to find extra practice problems to help you learn. All of these problems come from Schaum's Outlines *Differential Equations* by Richard Bronson.

Concept	Source	Suggested	Relevant
Separable Review	Schaum's Ch 4	42	1-8,23-45
Exact	Schaum's Ch 5	5,11,26,29,34	1-13,24-40,56-65
Integrating Factors	Schaum's Ch 5	21,22,41,47	21,22,41-42,47-49,51,55
Linear	Schaum's Ch 6	4,13,20,32,51	1-6,9-15,20-36,43-49,50-57
Homogeneous	Schaum's Ch 4,	11,12,48	11-17,46-54
Bernoulli	Schaum's Ch 6	16,53	16,17,37-42,53
Applications	Schaum's Ch 7	4[27],6[33],1[38] 10[48],17[67],7[88]	1-6 [26-44] 8-10 [45-50],16-18[65-70], 7[87-88]
Laplace Review	Schaum's Ch 21	19,32,33[use table]	4-7,10-12,27-35
Inverse Transforms	Schaum's Ch 22	1,2,3,6,13,15	1-3,6,15,17,20-28,42,42,45-47
Solving ODEs	Schaum's Ch 24	1,14,19(parfrac)	1,2,11,14,15,19-19,22,24,25,38-42

Remember that you can check almost all of your work with technology. Use the following technology links to help you check your understanding.

- First Order ODE Solver
- Laplace Transforms

Chapter 3

Linear Algebra Arithmetic

After completing this chapter, you should be able to:

1. Explain the difference between linearly independent and linearly dependent. When vectors are linearly dependent, write one of the vectors as a linear combination of the others.
2. Solve systems of equations by obtain the reduced row echelon form (rref) of a matrix (Gauss-Jordan elimination).
3. Explain how to compute the inverse of a matrix. Then use the inverse to solve various problems such as finding $\vec{x}$ in $A\vec{x} = \vec{b}$ or finding A in $AQ = QD$.
4. Show how to compute the determinant of a square matrix of any size. Be able to articulate the connection among determinants, linear dependence, and invertibility.
5. Explain how to see eigenvectors and the sign of eigenvalues in a vector field. Then use this knowledge to show how to obtain eigenvalues and eigenvectors from determinant and row reduction computations.

I've created video tutorials for many of the ideas in this chapter. You can view them by following this link to YouTube.

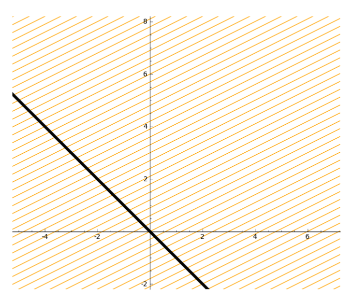
Linear Independence and Dependence

The heart of linear algebra has to deal with understanding relationships between vectors. Vectors give us directions of motion. They tell us how forces act on objects. We've seen already that vectors provide us with visual solutions to differential equations. One of our goals in this chapter is to become comfortable with working with linear combinations of vectors. Remember that given n vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ and n scalars $c_1, c_2, \cdots, c_n$ we say their linear combination is the sum

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

You may want to review the formal definition of a linear combination. See Definition 1.1 on page 3.

Problem 3.1 Sally is trying to find a treasure that's located in a corn field (she's geocaching). Her position is currently at $(0, 0)$, and she knows that the treasure is located at $(6, 8)$ (units are hundreds of yards). She can't walk in a straight line to the treasure, because that would damage the rows of corn. The corn is planted in rows that run parallel to the vector $(2, 1)$. She's currently on a road that moves parallel to the vector $(-1, 1)$. The farmer will only allow her to walk parallel to the rows of corn (if she crosses between rows, she might damage the crop). So she has to follow the road for some distance by following the vector $(-1, 1)$ along the road, and then enter the rows of corn and follow the vector $(2, 1)$.

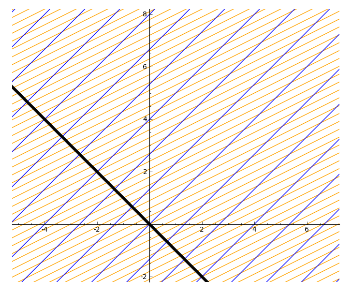


1. What does the vector equation
$$\begin{pmatrix}-1\\1\end{pmatrix}x+\begin{pmatrix}2\\1\end{pmatrix}y=\begin{pmatrix}6\\8\end{pmatrix}$$
have to do with Sally's problem.

2. How would you rephrase the above equation using the language of linear combinations. Which vector is a linear combination of which vectors?

3. Find values for x and y that make this equation valid.

The geocaching problem above requires that Sally find out how to obtain the vector $(6,8)$ as a linear combination of the vectors $(-1,1)$ and $(2,1)$. These two vectors (the road and corn rows) gave us two directions that are independent of each other. Each direction provides us with a new way to travel that we could not do before. There is only one way to get to the treasure at $(6,8)$ if these are Sally's only two ways to move. Let's examine what happens if we add a third direction, via some irrigation pipes.



Problem 3.2 Assume the same conditions as the previous problem. However, now let's assume that in the corn field there are irrigation pipes following the vector $(1,1)$. Sally now has the option to follow the road $(-1,1)$, the rows of corn $(2,1)$, or the irrigation pipes $(1,1)$. She still wants to get to the treasure at $(6,8)$, but now had 3 options for ways to travel.

1. To get to the treasure, Sally needs to write $(6,8)$ as a linear combination of the vectors $(-1,1)$, $(2,1)$, and $(1,1)$, i.e. she needs to solve
$$\begin{pmatrix}-1\\1\end{pmatrix}x+\begin{pmatrix}2\\1\end{pmatrix}y+\begin{pmatrix}1\\1\end{pmatrix}z=\begin{pmatrix}6\\8\end{pmatrix}.$$
One such option is $x=3$, $y=4$, and $z=1$. Find 3 different ways for Sally to get from the origin $(0,0)$ to the treasure at $(6,8)$.

2. Can you find a way to express every possible option that Sally has?

Solving Systems of Equations

Every time we want to solve a problem involving linear combinations, we can convert that problem into a system of equations. For example, if we want to write $\begin{pmatrix}-4\\-15\\9\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$, and $\begin{pmatrix}-2\\-10\\6\end{pmatrix}$, then we would write

$$\begin{pmatrix}1\\2\\-1\end{pmatrix}x_1+\begin{pmatrix}3\\4\\-2\end{pmatrix}x_2+\begin{pmatrix}-2\\-10\\6\end{pmatrix}x_3=\begin{pmatrix}-4\\-15\\9\end{pmatrix}\quad\text{or}\quad\begin{bmatrix}1&3&-2\\2&4&-10\\-1&-2&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-4\\-15\\9\end{bmatrix},$$

which as a system of equations becomes

$$\begin{aligned}x_1+3x_2-2x_3&=-4\\2x_1+4x_2-10x_3&=-15\\-1x_1-2x_2+6x_3&=9\end{aligned}$$

You've solved systems of this form in the past. In this chapter we'll learn an efficient algorithm for solving these systems.

Problem 3.3: Organized Substitution Our goal on this problem is to solve the system of equations

$$\begin{aligned} x + 3y - 2z &= -4 \\ 2x + 4y - 10z &= -15 \\ -1x - 2y + 6z &= 9. \end{aligned}$$

To solve this system, we'll use organized substitution.

1. The first equation is easy to solve for x. Solve for x and circle your result. Then replace x with this in both of the other equations. After substituting, you should be able to rewrite each equation in the form $0x + ?y + ?z = ?$.

2. One of these two simplified equations is easy to solve for y. Solve this equation for y (you shouldn't need fractions), and write your answer in the form $y = ?z + ?$. Circle this result. Use this to replace y in the other equation and simplify so you have $0x + 0y + ?z = ?$.

3. At this point you should be able to solve for z. Circle your result. Then use this result to find y in your circled equation for y. Then use both values for z and y to obtain x in your circled equation for x. If you ended up with $z = 3/2$, then you're on the right track.

Problem 3.4: Gaussian Elimination We'll now use elimination to solve the system of equations

$$\begin{aligned} x + 3y - 2z &= -4 \\ 2x + 4y - 10z &= -15 \\ -1x - 2y + 6z &= 9. \end{aligned}$$

1. The first equation has a 1 as the coefficient in front of x. Add a multiple of the first equation to every other equation so that you eliminate the x variable from the other equations. Write your system in the form

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x + ?y + ?z &= ? \\ 0x + ?y + ?z &= ?. \end{aligned}$$

Start by making sure that the coefficient in front of x on the top row is not zero. Swap rows if needed, and then multiply both sides of the top equation by a constant so that you have a 1 in this spot. Then add a multiple of this equation to every other equation to eliminate x from the other equations.

2. One of these two simplified equations is easy to solve for y. If you need to swap equations 2 and 3, or multiply both sides of an equation by some constant, do so now so that you can rewrite the system in the form

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x + 1y + ?z &= ? \\ 0x + ?y + ?z &= ?. \end{aligned}$$

If you ignore the top row and the variable x, then at this point we just repeat the above process with y. Make sure the coefficient in front of y is 1, and then add multiples of the second equation to each lower equation to eliminate y.

Then add a multiple of the second equation to the third equation so that you eliminate the y. Rewrite your system in the form

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x + y + ?z &= ? \\ 0x + 0y + ?z &= ?. \end{aligned}$$

3. Multiply the third equation by some nonzero constant so that the coefficient in front of z is a 1. Rewrite the system one final time as

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x + y + ?z &= ? \\ 0x + 0y + z &= ?. \end{aligned}$$

You now have z. Use your value for z to quickly obtain y from the second equation, and then x from the first equation.

We now ignore the top 2 rows and variables x and y, and then repeat the elimination process again. We make sure to get the coefficient in front of z to be a 1. Since there are no more rows beneath this third, we are done. If there had been more rows, we would just keep going.

Observation 3.1. When solving the above system with substitution, we

1. picked an equation for which it was easy to solve for a variable,
2. solved for that variable, and then
3. replaced that variable in the other equations and simplified each equation.

We then leave the picked equation alone, and repeat this process on the remaining simplified equations.

When solving the above system with elimination, we

1. picked an equation that we could use to eliminate a variable from the other equations, interchanging this equation with one higher up if needed,
2. multiplied the chosen equation by a nonzero constant to make the leading coefficient 1, and then
3. added a multiple of the chosen equation to the other equations to eliminate the variable from the other equations.

We then left the picked equation alone, and repeated this process on the remaining simplified equations below it.

These two processes (organized substitution and Gaussian elimination) are fundamentally the same process. You should have noticed that your intermediate steps were the same. We now develop a way to replicate both of these processes with matrices.

Problem 3.5: Gaussian Elimination with Matrices Let's solve the same system

$$\begin{aligned} x + 3y - 2z &= -4 \\ 2x + 4y - 10z &= -15 \\ -1x - 2y + 6z &= 9. \end{aligned}$$

but now we will only write the coefficients of our system in a matrix. The matrix helps us organize our work, and we have less to write.

1. Start by writing the coefficients of system in the matrix (fill in the blanks)

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 2 & ? & ? & -15 \\ ? & ? & 6 & ? \end{array}\right].$$

This matrix is called the augmented matrix of the system. The vertical bar you see is optional, and is often used to help people remember that there is an equal sign there.

2. Since the upper left entry is a 1, we are ready to reduce. Add a multiple of row 1 to both row 2 and row 3 (replacing the old row 2 and 3) so that you obtain a zero in both entries below the leading 1 in the top row. Write your work as

I often write the row operation next to the row I am about to replace.

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 2 & ? & ? & -15 \\ ? & ? & 6 & ? \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \Rightarrow \\ R_3 + ?R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & ? & -6 & ? \\ 0 & 1 & ? & ? \end{array}\right].$$

3. Swap R_2 and R_3. This should get you a 1 as the first nonzero entry in row 2. Then you can add a multiple of row 2 to row 3 to obtain a zero below this 1. Write your work in the form

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & ? & -6 & ? \\ 0 & 1 & ? & ? \end{array}\right] R_2 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & ? & ? \\ 0 & ? & -6 & ? \end{array}\right] \begin{array}{l} \\ \\ R_3 + ?R_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & ? & ? \\ 0 & 0 & ? & 3 \end{array}\right].$$

4. Multiply row 3 by some nonzero constant so that the first nonzero entry in row 3 is a 1. Write your work as

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & ? & ? \\ 0 & 0 & ? & ? \end{array}\right] \begin{array}{l} \\ \\ ?R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{array}\right]$$

5. Now rewrite your matrix as a system of equations. The last line row of your matrix, after rewriting it as a system, should be $z = 3/2$. Use this to find y and x.

In our work above, our process was precisely the same as when we used elimination without matrices. When working with equations, we can always (1) interchange the order of equations without changing the solution set to the system. We can also (2) multiply both sides of an equation by a nonzero number without affecting the solutions. Finally, (3) adding a multiple of one equation to another will not affect the solution set. When working with matrices, these three operations on equations become operations on rows of a matrix.

Definition 3.2: Row Operations. We define allowed row operations.

1. Interchange two rows.
2. Multiply a row of a matrix by a nonzero constant.
3. Add a nonzero multiple of a row to another row.

The row operations above are precisely what we used to row reduce our matrix. The elimination process begins with the first column. We obtain a 1 in the top of that column by (1) swapping rows and/or (2) multiplying a row by a nonzero scalar. We then (3) add multiples of the first row to the other rows to obtain zeros below this leading 1. We then ignore the row and column containing this leading 1, and repeat the reduction on the remaining part of the matrix. We can summarize the reduction algorithm with the diagram below.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right].$$

The matrix at the far right provides us with enough information to quickly obtain a solution. After reducing the matrix to this form, we say the matrix is in row echelon form.

Definition 3.3: Row Echelon Form, Leading 1, Pivot Column. We say a matrix is in row echelon form (ref) if it satisfies each of the following conditions:

- each nonzero row begins with a 1 (called a leading 1),
- the leading 1 in each row occurs further right than the leading 1 in the row above, and
- any rows of all zeros appear at the bottom.

The position in the matrix where the leading 1 occurs is called a pivot. The column containing a pivot is called a pivot column.

Problem 3.6: Gauss-Jordan Elimination Consider the three planes $2x+3y+4z=4$, $x+2y=6$, and $-x+y+2z=0$. Let's find the point of intersection by applying row operations to the augmented matrix

$$A=\begin{bmatrix}2&3&4&4\\1&2&0&6\\-1&1&2&0\end{bmatrix}.$$

We'll first obtain row echelon form, and then continue reducing the matrix until we obtain what is called reduced row echelon form.

1. Apply Gaussian elimination to obtain a row echelon form for A. You should start by interchanging the first and second rows, so that you have a 1 in the upper left. Remember the pattern

$$\begin{bmatrix}2&3&4&4\\1&2&0&6\\-1&1&2&0\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}1&*&*&*\\0&*&*&*\\0&*&*&*\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&*&*&*\\0&1&*&*\\0&0&*&*\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&*&*&*\\0&1&*&*\\0&0&1&*\end{array}\right]$$

We say a matrix is in row echelon form when (1) each nonzero row begins with a leading 1, (2) a leading 1 appears to the right of any leading one above it, and (3) any rows of all zeros appear at the bottom.

2. Let's now use row operations (instead of back substitution) to find the solution. Starting on the right, and working left, use the 1's in each pivot column to eliminate the nonzero numbers above each leading 1. Use the pattern

$$\left[\begin{array}{ccc|c}1&*&*&*\\0&1&*&*\\0&0&1&*\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&*&\mathbf{0}&*\\0&1&\mathbf{0}&*\\0&0&1&*\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&\mathbf{0}&0&*\\0&1&0&*\\0&0&1&*\end{array}\right]$$

to complete your reduction. State the point of intersection of the planes.

We say a matrix is in reduced row echelon form when the matrix is in row echelon form, and there are zeros above each pivot.

Check your answer with the technology link Visualizing Systems of Equations.

The process above is called Gauss-Jordan elimination. The forward phase of reduction results in a matrix in row echelon form. We then work backwards starting with the right most pivot column, and use the leading 1 to eliminate the zeros above it.

Definition 3.4: Reduced Row Echelon Form (rref). We say that a matrix is in reduced row echelon form (rref) if

- the matrix is in row echelon form, and
- each pivot column contains all zeros except for the leading 1 in the pivot.

The row reduction process we've described above may not always result in a unique solution.

Problem 3.7 On this problem, you'll be using software to obtain the rref of a matrix. The Sage command is "A.rref()" and the Mathematica command is "RowReduce[A]."

You can use the Sage RREF Calculator to check your result. Follow the link.

1. Consider the three planes $x+2y-z=3$, $2x-y+4z=0$, and $-x+2z=4$. Use software to obtain the reduced row echelon form (rref) for the matrix
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 4 & 0 \\ -1 & 0 & 2 & 4 \end{bmatrix}.$$
What does the reduced matrix tell you about how the planes intersect?

   When you present in class, you should always write the original matrix, draw an arrow to the rref of the matrix, and write rref above your arrow. This way the class can see what matrix you reduced, and what the rref is.

2. If I wanted to write the vector $(3,0,4)$ as a linear combination of the vectors $(1,2,-1)$, $(2,-1,0)$, and $(-1,4,2)$, then what should I let c_1, c_2, and c_3 equal so that
$$c_1(1,2,-1)+c_2(2,-1,0)+c_3(-1,4,2)=(3,0,4),$$
which we could rewrite in the easier to use column form
$$c_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}.$$
[Hint: What does this have to do with the first part?]

3. Now consider the three planes $x+2y-z=3$, $2x-y+4z=0$, and $-5y+6z=-6$. Set up an appropriate augmented matrix (make sure you show us the matrix), and use software to verify that the reduced row echelon form is
$$\begin{bmatrix} 1 & 0 & \frac{7}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{6}{5} & \frac{6}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Write the three equations represented by this rref (the third equation may seem silly).

4. Note that the third column does not have a pivot in it. If we added the equation $z=z$ to our work above, then we could solve for x, y, and z in terms of the variable z. Write your work in the form
$$\begin{aligned} x &= ? \\ y &= ? \\ z &= z \end{aligned} \qquad \text{and} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ 1 \end{pmatrix} z + \begin{pmatrix} ? \\ ? \\ 0 \end{pmatrix}.$$

Because we can choose the third variable to be anything we want, we call it a free variable.

Definition 3.5: Free Variable. The variables in a system of equations each correspond to column of the augmented matrix. Some of the columns are pivot columns, and some are not. The variables corresponding to the nonpivot columns we call free variables. We can choose these variables to be any value we want, and we can write the solution to a system of equations in terms of these free variables. The variables that correspond to pivot columns we call basic variables.

Problem 3.8 Each of the following augmented matrices requires one row operation to be in reduced row echelon form. Perform the single required row operation, and then write the solution to the corresponding system of equations in terms of the free variables.

Here's an applicable YouTube video.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]$ [Remember, you only get one row operation.]

2. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ -3 & -6 & 0 & 12 \end{array}\right]$ [The second column won't have a pivot, so include the equation $x_2 = x_2$.]

3. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 7 & 0 & 3 \\ 0 & 0 & 1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ [There are two free variables in this problem. One of the free variables is x_1.]

Throughout the remainder of this chapter, you'll be asked to obtain the rref of many matrices. Always start by using software to obtain the result. Even if the problem asks you to compute the rref by hand, please start by using software. This will save you hours of potentially wasted time. If you know what the final answer is, you will be able to recognize that you have made a mistake early in the reduction process.

You can use the Sage rref calculator to row reduce a matrix. You can use this on any device that can access a web browser.

Linear Independence and Dependence

Think back on the opening problems of this chapter. Sally starts at the origin $(0,0)$. Because Sally can follow the road $(-1,1)$, she has the ability to move away from $(0,0)$. Using the road, she can use linear combinations of $(-1,1)$ to reach any location on the line $y=-x$.

The rows of corn $(2,1)$ allow Sally to leave the road. She can use a linear combination of $(-1,1)$ and $(2,1)$ to arrive at any final destination in the plane that she wants. We say these two vectors $(-1,1)$ and $(2,1)$ are linearly independent because they each expand the places Sally can reach. Neither depends on the other. The vectors provide independent directions.

Introducing the third direction of travel $(1,1)$ along the irrigation pipes does not change where Sally can travel to, rather this third vector just increases her options for how to get there. Because of this, we say that $(1,1)$ linearly depends on $(-1,1)$ and $(2,1)$. The three vectors $(-1,1)$, $(2,1)$, and $(1,1)$ are dependent.

Problem 3.9 Read the three paragraphs before this problem. Then answer the following.

1. If Sally only uses the road and rows of corn, how many linear combinations of $(-1,1)$ and $(2,1)$ are there that will allow Sally to reach the origin? In other words, solve the linear combination equation
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
by reducing an appropriate matrix. Make sure you show your reduction steps by hand.

2. If Sally is also allowed to use the irrigation pipes, how many linear combinations of $(-1,1)$, $(2,1)$, and $(1,1)$ are there that will allow Sally to reach the origin? Obtain the reduced row echelon form of the matrix $\begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ to give your answer.

3. Write the vector $(1,1)$ as a linear combination of the vectors $(-1,1)$ and

Because this problem has an answer, we say that $(1,1)$ linearly depends on $(-1,1)$ and $(2,1)$.

(2, 1), i.e. solve the equation

$$\begin{pmatrix}-1\\1\end{pmatrix}x+\begin{pmatrix}2\\1\end{pmatrix}y=\begin{pmatrix}1\\1\end{pmatrix}.$$

4. Can you think of a different third vector so that using this vector would expand Sally's final destination points beyond where she can already get to with the road $(-1,1)$ and rows of corn $(2,1)$? Explain.

Definition 3.6: Linear Independence. We say that a set of vectors $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$ is linearly independent if the only solution to the homogeneous system

$$c_1\vec{v}_1+c_2\vec{v}_2+\ldots+c_n\vec{v}_n=\vec{0}$$

is the trivial solution $c_1=c_2=\cdots=c_n=0$. If the vectors are not independent, then we say that the vectors are linearly dependent.

When a collection of vectors is linearly dependent, it is always possible to write one of the vectors as a linear combination of the others. We say the vectors are linearly dependent because one of the vectors depends on (can be obtained as a linear combination of) the other vectors.

Problem 3.10 Are the vectors $\vec{v}_1=(1,3,5)$, $\vec{v}_2=(-1,0,1)$, and $\vec{v}_3=(0,3,1)$ linearly independent? Solve the system $c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=\vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Are the vectors $\vec{v}_1=(1,2,0)$, $\vec{v}_2=(2,0,3)$, and $\vec{v}_3=(3,-2,6)$ linearly independent? Solve the system $c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=\vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

[Hint: Rewrite each of these problems as a system of 3 equations. From that system of equations, write down the corresponding augmented matrix (it will have a column of all zeros at the right). Then use software to answer each problem. You do not need to show your reduction steps, rather show the matrix you reduced, and the rref.]

Problem 3.11 Imagine you are in a rocket traveling through space. The rocket has 4 boosters on it. The boosters provide thrust in a specific direction (vector), with the ability to adjust how strong the push should be in each direction (possibly even moving backwards in that direction - a two sided booster). The 4 boosters allow movement in the directions $(1,1,2)$, $(0,1,3)$, $(2,1,1)$, and $(-2,1,0)$.

Imagine that each booster provides a thrust through the center of mass, so no rotation occurs.

1. Start by row reducing the matrix $\begin{bmatrix}1&0&2&-2\\1&1&1&1\\2&3&1&0\end{bmatrix}$ to determine which columns are pivot columns. Use technology to get an answer. Then show your row reduction steps by hand.

   Use the Sage RREF Calculator.

   The rest of this problem deals with interpreting your rref. Please give answers with sentences.

2. If the 4th booster breaks, could some linear combination of the first three rocket thrusts allow you to move in the direction of the 4th rocket? In other words, is it possible to write $(-2,1,0)$ as a linear combination of $(1,1,2)$, $(0,1,3)$, and $(2,1,1)$? Explain.

3. If the 3rd booster breaks, show that some linear combination of the other three rocket thrusts allows you to move in the direction of the 3rd rocket. What matrix should you row reduce to answer this. Show the class the matrix you started with, and its rref. You do not need to show by hand any reduction steps. Then write $(2,1,1)$ as a linear combination of the other three vectors.

4. You have been asked to give advice on a new rocket design. The designers figure that as long as they pick 3 directions in which to provide thrust, they should be able to fly in any direction they want. They attach boosters which allow movement in the directions $(1,3,2)$, $(-3,1,4)$, $(0,1,1)$. Set up an appropriate matrix and use software to row reduce the matrix. What advice would you give the designers?

5. What does any of the above have to do with linear independence and linear dependence?

Problem 3.12 Start by finding the reduced row echelon form of the matrix

Use the Sage RREF Calculator.

$$B = \begin{bmatrix} 2 & 6 & -1 & 2 & 0 & 1 & 3 \\ 1 & 3 & 1 & 5 & 1 & 0 & 3 \end{bmatrix}.$$

Show the steps you used to row reduce this matrix. The point to this problem is to help you see how this single row reduction can answer all of the questions below.

1. Write $(2,5)$ as a linear combination of $(2,1)$ and $(-1,1)$. Remember, that when writing $c_1(2,1) + c_2(-1,1) = (1,0)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 1 & 5 \end{bmatrix}$ with technology. You don't need to show any steps of the computation.

2. Write $(0,1)$ as a linear combination of $(2,1)$ and $(-1,1)$. Remember, that when writing $c_1(2,1) + c_2(-1,1) = (0,1)$, you must solve for the unknown constants. If you decide to row reduce the matrix $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, then use technology and don't show us any of the intermediate steps.

3. Continue to write each of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. [Hint: At some point, rather than row reducing $\begin{bmatrix} 2 & -1 & a \\ 1 & 1 & b \end{bmatrix}$, ask how you could use the larger matrix to answer this.]

4. The following matrix row reduces to give

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 5 & 8 \\ 0 & 2 & -6 & 2 & -1 & 3 \\ 0 & -2 & 6 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 2 \end{bmatrix}.$$

Write $(5,-1,2)$ as a linear combination of the pivot columns.

Question 3.7. What connection is there between the rref of a matrix and the columns of the matrix?

Definition 3.8: Coordinates of a vector relative to independent vectors. Suppose that $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$ are a set of linearly independent vectors. Suppose that

$$\vec{v} = c_1\vec{p}_1 + c_2\vec{p}_2 + \cdots + c_k\vec{p}_k.$$

Then we call the coefficients c_1, c_2, ..., c_k, the coordinates of $\vec{v}$ relative to the vectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$.

Theorem 3.9 (Coordinates of a vector relative to the pivot columns)**.** *Suppose that we row reduce an augmented matrix A to obtain the reduced row echelon form R. Suppose that the pivot columns of A are $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$. Then these vectors are linearly independent. Furthermore, if $\vec{v}$ is any column in the matrix A, then we can write*

$$\vec{v} = c_1\vec{p}_1 + c_2\vec{p}_2 + \cdots + c_k\vec{p}_k,$$

where the numbers c_1, c_2, ..., c_k, we call the coordinates of $\vec{v}$ relative to the pivot columns of A. These coordinates are precisely the entries in the column of the reduced row echelon form R that corresponds to the original column $\vec{v}$.

In summary, to obtain the coordinates of jth column of A relative to the pivot columns of A, we just obtain the rref R and then the coordinates are precisely the first k entries in the jth column of R.

Seeing Eigenvectors in Vector Fields

Problem 3.13 The following parts ask you to look for points in a vector field where the vector field pushes either straight outwards from the origin, or pulls straight towards the origin.

1. Consider the vector field $\vec{F}(x,y) = \begin{bmatrix}3 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3x+2y\\2y\end{bmatrix}$ which we could also write in the form $\vec{F}(x,y) = (3x+2y, 2y)$. Compute $\vec{F}(x,y)$ for each of (x,y) equal to $(2,2)$, $(2,1)$, $(2,0)$, $(2,-1)$, and $(2,-2)$. Then circle the two vectors (x,y) where the output $F(x,y)$ is a linear combination of the input (x,y). For example, if $(x,y) = (-4,2)$, then we compute $\vec{F}(-4,2) = (-8,4)$ and we see that $(-8,4) = 2(-4,2)$. We have $\vec{F}(x,y) = 2(x,y)$.

2. Suppose you knew that there was a direction in which the vector field $\vec{F}(x,y) = \begin{bmatrix}2 & 5\\4 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x+5y\\4x+y\end{bmatrix}$ causes a radial push outwards of 6 units. This would mean there exists $(x,y) \neq (0,0)$ such that

$$\begin{bmatrix}2 & 5\\4 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 6\begin{bmatrix}x\\y\end{bmatrix}.$$

   Find a nonzero vector (x,y) that satisfies this equation.

   [Hint: Multiply out the left hand side, and then subtract $6\begin{bmatrix}x\\y\end{bmatrix}$ from both sides. Combine terms to get a new matrix to row reduce, and then row reduce the matrix. You should find there are infinitely many correct answers. You just need to give one answer.]

Problem 3.14 Consider the matrix $A = \begin{bmatrix}1 & -1\\3 & 5\end{bmatrix}$.

1. Explain why solving the problem $A\vec{x} = c\vec{x}$ can be done by row reducing the matrix $\left[\begin{array}{cc|c} 1-c & -1 & 0 \\ 3 & 5-c & 0 \end{array}\right]$.
2. Let $c = 3$. Solve $A\vec{x} = 3\vec{x}$ by row reducing an appropriate matrix. How many solutions are there?
3. Let $c = 2$. Solve $A\vec{x} = 2\vec{x}$ by row reducing an appropriate matrix. How many solutions are there?
4. When you row reduce a matrix, what must occur for there to be infinitely many solutions? Can you find another value of c where there are infinitely many solutions to this problem?

Problem 3.15 Consider the matrix $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$. This matrix gives us the vector field $\vec{F}(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}$. We would like to find the directions in which the vector field either pulls a point (x, y) directly towards the origin, or pushes the point (x, y) directly away from the origin.

1. Explain why we seek a solution to
$$A \begin{bmatrix} x \\ y \end{bmatrix} = c \begin{bmatrix} x \\ y \end{bmatrix}$$
where c is some constant. Is $(x, y) = (0, 0)$ a solution to this equation. We call this the trivial solution.
2. Subtract $c \begin{bmatrix} x \\ y \end{bmatrix}$ from both sides above. Show that to find a nonzero solution (x, y), we need to row reduce the matrix $\left[\begin{array}{cc|c} 3-c & 4 & 0 \\ 2 & 1-c & 0 \end{array}\right]$. Then use row operations to eliminate the 2 in the lower left of the matrix. [Hint: Take row 2 and multiply it by $(3 - c)$. Then add -2 times row 1 to row 2.]
3. We already know that $(x, y) = (0, 0)$ is a solution. We want a nonzero solution (x, y). Explain why the bottom row must reduce to be all zeros?
4. By forcing the bottom row to consist of all zeros, you should have a quadratic equation involving c. Solve this equation for c. These are the scalars for which you can find a vector that either pushes directly out or pulls directly in.

The numbers c that you computed above are called eigenvalues. Note that to find the eigenvalues, we wanted to row reduce a matrix and obtain infinitely many solutions. We'll return to this idea throughout the chapter.

Matrix Multiplication and Inverses

When we solve equations of the form $ax = b$ with numbers, we simply multiply both sides by $\frac{1}{a}$ to obtain $x = \frac{1}{a}b$. This is because for any nonzero number a, we have an inverse a^{-1} such that $a^{-1}a = 1 = aa^{-1}$.

Definition 3.10: I_n and A^{-1}. The identity matrix I is a square matrix so that if A is a square matrix, then $IA = AI = A$. The identity matrix acts like the number 1 when performing matrix multiplication. We write

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{etc.}$$

If A is a square matrix, then the inverse of A is a matrix A^{-1} where we have $AA^{-1} = A^{-1}A = I$, provided such a matrix exists.

Problem 3.16 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. We now develop an algorithm for computing the inverse A^{-1}. If an inverse matrix exists, then we know it's the same size as A, so we could let $A^{-1} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ be the inverse matrix, where $\vec{v}_1$ and $\vec{v}_2$ are the columns of A^{-1}.

1. We know that $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Explain why $A\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
2. Solve the matrix equations $A\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by row reducing $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right]$ and $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right]$.
3. What is the reduced row echelon form of $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$? How is this related to your previous work?
4. State the inverse of A.

In the previous problem we showed how to obtain a matrix B so that $AB = I$. We now have an algorithm for finding the inverse matrix A^{-1}. We augment A by the identity matrix, and then row reduce $[A|I]$ to the matrix $[I|A^{-1}]$. The inverse shows up instantly after row reduction.

Problem 3.17 Use the algorithm described immediately before this problem to compute the inverse of

$$A = \begin{bmatrix} 3 & 1 & -11 \\ 0 & -1 & 1 \\ 1 & 0 & -4 \end{bmatrix}.$$

Use technology to show you the rref of $[A|I]$, or just use A.inverse() in Sage, or Inverse[A] in Mathematica. Then show your row reduction steps by hand.

Once you have obtained the inverse, can you use your work to write $(1, 0, 0)$ as a linear combination of the columns of A. In other words, what are the coordinates of $(1, 0, 0)$ relative to the columns of A?

Linear Independence and Dependence

Problem 3.18 For each collection of vectors, use software to determine if the collection of vectors is linearly independent or linearly dependent. If the vectors are linearly dependent, write one of the vectors as a linear combination of the others. Do not row reduce the matrices by hand, rather on each problem first show the matrix you would row reduce, and then give the reduced row echelon form by using technology.

1. $(1,0,0)$, $(0,1,1)$, $(2,3,2)$, and $(0,1,-1)$ [Remember, the vectors are linearly independent if the only solution to

$$\begin{pmatrix}1\\0\\0\end{pmatrix} c_1 + \begin{pmatrix}0\\1\\1\end{pmatrix} c_2 + \begin{pmatrix}2\\3\\2\end{pmatrix} c_3 + \begin{pmatrix}0\\1\\-1\end{pmatrix} c_4 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

is the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$.]

2. $(1,0,2,0)$, $(0,1,3,1)$, and $(0,1,2,-1)$

3. $(1,1,2,-1)$, $(-3,1,4,1)$, and $(-1,1,3,0)$

4. Suppose you have 5 vectors that are each 7 tall. Row reducing the 7 by 5 matrix obtained by placing these vectors in columns results in a matrix that has 3 rows of zeros at the bottom. Why are the vectors linearly dependent?

5. If you have a matrix with n rows and m columns, what must happen for the column vectors to be linearly independent? How many rows of zeros would be at the bottom if the vectors are linearly independent.

[Hint: For all parts, think about the number of pivot columns. You are looking to see if there are any nonpivot columns.]

Problem 3.19: Rocket Booster Design Three teams have been asked to design a space suit that allows for travel in space. As part of the project requirements, the teams are required to use 4 two-way boosters for propulsion. The 4th booster is there to allow for redundancy in case any of the the other boosters break.

If you are worried about rotation that might occur from firing these boosters, then please imagine that each booster applies a force through the center of mass of the object, so that no rotation occurs.

- Team 1 decides to add boosters to their suit that allows for travel in the directions $[1,-1,1], [1,2,-1], [3,-1,2], [1,1,0]$.
- Team 2 decides to add boosters to their suit that allows for travel in the directions $[1,-3,2], [0,1,1], [-1,3,2], [1,-1,3]$.
- Team 3 decides to add boosters to their suit that allows for travel in the directions $[1,1,-2], [3,-1,4], [2,0,1], [1,-3,8]$.

For each team, use software to row reduce the appropriate 3 by 4 matrix (remember to put vectors in columns, not rows) that would tell the dependence relationships among the vectors. If you were in charge of picking a winning design, which team would you pick, and why?

Matrix Multiplication and Inverses

Problem 3.20 Start by writing the system of equations

$$\begin{aligned} -2x_1 + 5x_3 &= -2 \\ -x_1 + 3x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 3 \end{aligned}$$

as a matrix product $A\vec{x} = \vec{b}$. (What are A, $\vec{x}$ and $\vec{b}$?)

1. Use software to find the inverse of the matrix A (state the matrix you row reduced, and the rref of the matrix).

2. Use software to row reduce the augmented matrix $\left[A \mid \vec{b}\right]$. State the rref.

3. To solve the problem $ax = b$ where a, x, and b are numbers, we multiply both sides by $\frac{1}{a}$ to obtain $\frac{1}{a}ax = \frac{1}{a}b$, or because $\frac{1}{a}a = 1$, we simplify to get $x = \frac{1}{a}b$. How can you use this idea to solve the matrix problem $A\vec{x} = \vec{b}$? Show how to obtain the solution to this system by using the matrix inverse.

4. Does it matter if you compute $\vec{b}A^{-1}$ or $A^{-1}\vec{b}$?

You should use technology to rapidly compute the inverse and also row reduce the augmented system. Show by hand any matrix computations you do on part 3.

Problem 3.21 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Obtain the rref of $[A|I]$ to show that the inverse of A is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Are there any conditions under which a matrix would not have an inverse? What are they, and why? Is there a number you could check to *determine* if a matrix has an inverse? If you are struggling with reducing this symbolic matrix, please see the hint on the right.

[Hint: Because the matrix has variables in it, you may want to try a different scheme for row reducing. Multiply the top row by c and the bottom row by a. Then subtract the top row from the bottom. This gets you a zero below the pivot in the first column. Then multiply the top row by $ad - bc$ and the bottom row by something else.]

Applications of Determinants

In computing the inverse of a 2 by 2 matrix, the number $ad - bc$ appears in the denominator. We call this number the determinant. If I asked you to compute the inverse of a 3 by 3 matrix, you would again see a number appear in the denominator. We call that number the determinant. This holds true in all dimensions.

Take a guess as to why we call this number the determinant. What does it help determine?

Problem: Optional Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Use Gauss-Jordan elimination to find the inverse of A, and show that the common denominator is $a(ei - hf) - b(di - gf) + c(dh - ge)$.

Definition 3.11: Determinants of 2 by 2 and 3 by 3 matrices. The determinant of a 2×2 and 3×3 matrix are the numbers

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge) \end{aligned}$$

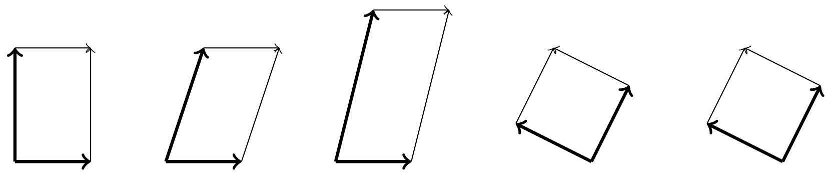
In Sage, we've been using A.rref() to get the reduced row echelon form of A. You can type A.determinant() to get the determinant. Similarly, A.inverse() will get you the inverse.

We use vertical bars next to a matrix to state we want the determinant. Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we can compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

This approach generalizes to give the determinant of any square matrix. More on this soon.

Problem 3.22 The columns of each matrix below provide the edges of the parallelogram beneath the matrix.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$



1. Compute the determinant of each matrix above. What happens to the determinant when you switch the order of the columns?
2. Use geometric reasoning to compute the area of each parallelogram ($A = bh$). For the last two, note that the vectors $(-2, 1)$ and $(1, 2)$ are orthogonal, so the parallelogram is a square. Find the length of each side.
3. For each parallelogram above, decide if you have to rotate clockwise or counterclockwise to get from the vector in the first column to the vectors in the second column. What does this have to do with the sign of the determinant?
4. Consider the matrix $F = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$. Draw the corresponding parallelogram and make a guess as to whether or not the determinant is positive or negative (without computing it). Then compute the determinant and use it to guess the area of the triangle with vertices $(0, 0)$, $(3, 2)$, and $(4, -1)$.

The problem above uses inductive reasoning (lots of examples) to suggest that the determinant of a matrix (up to a sign) is the area of a parallelogram. This next problem asks you to use deductive reasoning to prove that the determinant of a 2 by 2 matrix gives the area of a parallelogram whose edges are the columns of the matrix.

Problem 3.23 To find the area of the parallelogram with vertexes $O = (0, 0)$, $P = (a, c)$, $Q = (b, d)$, and $R = (a + b, c + d)$, we need to find the length of OP (the base b), and multiply it by the distance from Q to OP (the height h). Let $\vec{b} = \vec{OP}$ and let $\vec{h}$ be the shortest vector from the line OP to the point Q. Complete the following:

1. Find the projection of $\vec{OQ}$ onto $\vec{OP}$. (You may need to look up a vector projection formula.) Part of this formula requires that you compute the length of $\vec{b}$.
2. Recall that we can obtain the vector $\vec{h}$ by computing $\vec{h} = \vec{OQ} - \text{proj}_{\vec{OP}}\vec{OQ}$. We call this the vector component of $\vec{OQ}$ that is orthogonal to $\vec{OP}$. Compute $\vec{h}$.
3. The length of $\vec{h}$ is the distance h from Q to OP. Find the length of $\vec{h}$.
4. We now have b and h. Compute the product and simplify to show that the area of the parallelogram is $|ad - bc|$.

The algebra on this problem can get quite messy.

The result above extends to 3 dimensions. The determinant of a 3 by 3 matrix gives the volume of the parallelepiped whose edges are the columns of the 3 by 3 matrix. Because this result holds true in 1, 2, and 3 dimensions, we can use the determinant to define an nth dimensional volume. This is precisely what happens in practice.

Seeing Eigenvectors in Vector Fields

There is a connection between linear independence and the determinant.

Problem 3.24 Consider the matrices $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} -2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}$. (Note that $C = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

1. Compute the determinants of A, B, and C.
2. Are the columns of A linearly independent or linearly dependent? Explain.
3. Are the columns of B linearly independent or linearly dependent? Explain.
4. Make a conjecture about determinants and linear independence.
5. Find two different values λ so that C has linearly dependent columns. (Your answer should involve irrational numbers.)

The determinant of C is called the characteristic polynomial of $\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$.

A main goal in this chapter has been to answer the following two questions:

1. For which nonzero vectors $\vec{x}$ (eigenvectors) is it possible to write $A\vec{x} = \lambda\vec{x}$?
2. Which scalars λ (eigenvalues) satisfy $A\vec{x} = \lambda\vec{x}$?

These questions are precisely connected to when a vector causes a radial push away or pull towards the origin. Let's give some formal definitions.

Definition 3.12: Eigenvector, Eigenvalue, Characteristic Polynomial. Let A be a square $n \times n$ matrix.

- An eigenvector of A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. (Matrix multiplication reduces to scalar multiplication.) We avoid letting $\vec{x}$ be the zero vector because it is trivially true that $A\vec{0} = \lambda\vec{0}$ no matter what λ is.
- If $\vec{x}$ is an eigenvector with $A\vec{x} = \lambda\vec{x}$, then we call λ an eigenvalue of A.
- We call $\det(A - \lambda I)$ the characteristic polynomial of A. It is a polynomial in λ of degree n, hence has n roots (counting multiplicity). These roots are the eigenvalues of A.

Problem 3.25 Consider the matrix $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$.

1. Show that the eigenvalues are $\lambda = 7$ and $\lambda = -4$. You'll want to compute the determinant of $A - \lambda I = \begin{bmatrix} 5-\lambda & 6 \\ 3 & -2-\lambda \end{bmatrix}$
2. If we let $\lambda = 7$, find a nonzero vector $\vec{x} = (x, y)$ such that $A\vec{x} = 7\vec{x}$. You'll need to row reduce $\left[\begin{array}{cc|c} 5-7 & 6 & 0 \\ 3 & -2-7 & 0 \end{array}\right]$.
3. If we let $\lambda = -4$, find a nonzero vector $\vec{x} = (x, y)$ such that $A\vec{x} = -4\vec{x}$.
4. If $\lambda = 6$, then what is the only solution $\vec{x} = (x, y)$ to $A\vec{x} = 6\vec{x}$?

Problem 3.26 Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$.

1. Find the characteristic polynomial of A, and use it to determine the eigenvalues of A.

2. For each eigenvalue, find all corresponding eigenvectors.

Problem 3.27 Consider the matrix $A = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. Find the eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors.

Matrix Multiplication and Inverses

Problem 3.28: Encryption Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 5 & 2 & -3 \\ 0 & 2 & 1 \end{bmatrix}$. Joe decides to send a message to Ben by encrypting the message with the matrix A. He first takes his message and converts it to numbers by replacing A with 1, B with 2, C with 3, and so on till replacing Z with 26. He uses a 0 for spaces. After replacing the letters with numbers, he breaks the message up into chunks of 3 letters. He then multiplies each chunk of 3 by the matrix A, resulting in a coded message. For example, to send the message "good job ben" he firsts converts the letters to the numbers and places them in a large matrix M (top to bottom, left to right)

$$\left[\begin{bmatrix} g \\ o \\ o \end{bmatrix}, \begin{bmatrix} d \\ \\ j \end{bmatrix}, \begin{bmatrix} o \\ b \\ \\ \end{bmatrix}, \begin{bmatrix} b \\ e \\ n \end{bmatrix}\right] \to \left[\begin{bmatrix} 7 \\ 15 \\ 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 15 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}\right] = M = \begin{bmatrix} 7 & 4 & 15 & 2 \\ 15 & 0 & 2 & 5 \\ 15 & 10 & 0 & 14 \end{bmatrix}.$$

To encode the matrix, he computes

$$AM = \begin{bmatrix} 14 & -2 & 32 & -5 \\ 20 & -10 & 79 & -22 \\ 45 & 10 & 4 & 24 \end{bmatrix}.$$

and then sends the numbers $[[14, 20, 45], [-2, -10, 10], [32, 79, 4], [-5, -22, 24]]$ to Ben. Ben uses the inverse of A to decode the message.

1. Find the inverse of A.

2. Use A^{-1} to decode $[[14, 20, 45], [-2, -10, 10], [32, 79, 4], [-5, -22, 24]]$ and show the message is "good job ben".

3. Decode the message $[[39, 89, 22], [20, 48, 4], [39, 88, 33]]$.

Problem 3.29 The eigenvalues of the matrix $A = \begin{bmatrix} 2 & 6 \\ 18 & 5 \end{bmatrix}$ are $\lambda_1 = 14$ and $\lambda_2 = -7$. An eigenvector corresponding to $\lambda_1 = 14$ is $\vec{x}_1 = (1, 2)$. An eigenvector corresponding to $\lambda_2 = -7$ is $\vec{x}_2 = (-2, 3)$.

1. What is the product $A\vec{x}_1$? What is the product $A\vec{x}_2$? Can you explain how to get these products without actually doing matrix multiplication.

2. What is the product AQ where $Q = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$. You can do this product by using your answer to the first part.

We place the eigenvectors of A into the columns of Q.

3. Find a matrix D so that $AQ = QD$. Any idea why we use D for this matrix? See the hint on the side.

4. Suppose A is a 3 by 3 matrix with eigenvectors $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$, corresponding to the eigenvalues $\lambda = 2, 4, -5$, respectively. If we let $Q = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix}$, then make a guess as to what D should equal so that $AQ = QD$. Explain your guess. Guess what D equals if we instead place the eigenvectors into Q in the order $Q = \begin{bmatrix} \vec{x}_2 & \vec{x}_3 & \vec{x}_1 \end{bmatrix}$?

[There are several ways to do this problem. You could multiply both sides on the left by the inverse of Q to solve for D. Another way is to reason about the connection between eigenvalues, eigenvectors, the matrix A, and linear combinations.]

In the problem above, we wrote $AQ = QD$ where D is a diagonal matrix whose diagonal entries are the eigenvalues of A. The columns of Q are eigenvectors of A, which we place in the same order as the eigenvalues on the diagonal of D. We can use this idea to obtain a matrix with any desired eigenvalue/eigenvector pairs. In particular, this means we can observe something in nature and look for outward/inward pushes/pulls. From these observations we know Q and D, which means we can solve for A.

Problem 3.30 Suppose you know that the matrix A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$ with corresponding eigenvectors $\vec{x}_1 = (2, -5)$ and $\vec{x}_2 = (1, 3)$.

1. We can write $AQ = QD$ where $Q = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. Use this information to solve for A. You can get Q^{-1} quickly from Problem 3.21. Your matrix A should have fractional values in it, with a denominator equal to the determinant of Q.

Since $AQ = QD$, what does multiplying both sides by Q^{-1} on the right yield?

2. We could have instead written $AQ = QD$ where $Q = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ (reversing the order we put things into Q and D). Use this information to solve for A.

3. Suppose we know that a vector field $\vec{F}$ applies the forces $F(4, -2) = (8, -4)$ and $F(-1, -3) = (3, 9)$. Explain why we know two eigenvectors are $(4, -2)$ and $(3, 9)$ with corresponding eigenvalues $\lambda = 2$ and $\lambda = -3$. Use this information to state Q and D, and then use $AQ = QD$ to find the matrix A such that $F(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix}$.

4. You should have gotten the exact same matrix A in each problem above, though the Q and D used on each part was different. Guess another choice for Q and D, different than the three above, so that $AQ = QD$. Why did you make this guess?

Applications of Determinants

Recall that the determinant of a 2×2 and 3×3 matrix are the numbers

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge) \end{aligned}$$

Notice that we compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3. We now extend this to give a way to compute determinants of any matrix.

Definition 3.13: Minors, Cofactors, and General Determinants. Let A be an n by n matrix.

- The minor M_{ij} of a matrix A is the determinant of the the matrix formed by removing row i and column j from A.
- The cofactor C_{ij} is the product of the minor M_{ij} and $(-1)^{i+j}$. This gives $C_{ij} = (-1)^{i+j} M_{ij}$.
- To compute the determinant, first pick a row or column. Then take each entry a_{ij} in that row or column and multiply the entry by its cofactor C_{ij}. The determinant is the sum of these products.

  The determinant is a linear combination of the cofactors of any row or column of the matrix, where we use the entries of that row or column as the scalars.

  Using summation notation, we can write $|A| = \sum_{k=1}^{n} a_{ik} C_{ik}$ (if we chose row i) or alternatively $|A| = \sum_{k=1}^{n} a_{kj} C_{kj}$ (if we chose column j).

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This sign matrix keeps track of the $(-1)^{i+j}$ portion in the cofactor.

Problem 3.31 Compute the determinant of $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 4 & 2 & 5 \end{bmatrix}$ in 3 ways.

1. Compute the cofactors of the first row of the matrix, and use them to obtain the determinant. Please show each step of your work, don't just skip straight to an answer. You'll need to explain what you did in class, and you can't do this if you just skip all the steps in between.
2. Use the cofactors of the second row to obtain the determinant.
3. Find the determinant by using a linear combination of the cofactors of the third column.

Watch this relevant YouTube video.

Note: The determinant, using a linear combination of the cofactors along the second column, is

$$\begin{aligned} 3C_{1,2} + 0C_{2,2} + 2C_{3,2} &= (3)(-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} + (0)(-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} + (2)(-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \\ &= -(3) \begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} + (0) \begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} - (2) \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \\ &= \cdots \text{(you can finish).} \end{aligned}$$

Problem 3.32 Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 5 & -1 \\ 0 & 8 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & -5 & 3 & -1 \end{bmatrix}.$$

You can make these problems really fast if you use a linear combination of cofactors where most of the scalars are zero.

Problem 3.33 Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}.$$

These matrices are related to the rocket booster problem (Problem 3.11).

You can use Sage to perform all your work. First store the matrices as A and B. Then use A.inverse(), B.inverse(), A.determinant(), and B.determinant(), etc. to check.

1. Compute the determinants of A and B. Show this part by hand, though you should use software to check your answer.
2. Row reduce A and B. Use software and just show us the rref.
3. Are the columns of A linearly independent or linearly dependent? What about the columns of B?
4. Row reduce both $[A|I]$ and $[B|I]$, and then state the inverse of each (or explain why it does not exist). Use software and just show us the rref.
5. How many solutions are there to $A\vec{x} = \vec{0}$? How many solutions are there to $B\vec{x} = \vec{0}$?
6. How many solutions does $x\begin{pmatrix}1\\1\\2\end{pmatrix} + y\begin{pmatrix}0\\1\\3\end{pmatrix} + z\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ have?

   How many solutions does $x\begin{pmatrix}1\\1\\2\end{pmatrix} + y\begin{pmatrix}0\\1\\3\end{pmatrix} + z\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ have?
7. Make some conjectures about the relationships you see above.

Definition 3.14: Singular. We say that a matrix is singular when the determinant of A equals zero, which is precisely when the matrix does not have an inverse.

Seeing Eigenvectors in Vector Fields

Remember, to find the eigenvalues and eigenvectors of a matrix, we need to find nonzero vector solutions to $(A-\lambda I)\vec{x} = \vec{0}$. This means the determinant of $A-\lambda I$ must be zero, which is the quick formula we use for computing eigenvalues.

Problem 3.34 Consider the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Show that the characteristic polynomial of A is $(4-\lambda)(\lambda-3)(\lambda-1)$, and then state the eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. Show your row reduction steps to get the eigenvectors. (You'll need to row reduce 3 matrices, but with each one the row reduction should be quite fast as the bottom row should reduce to all zeros.)

Problem 3.35 Consider the matrices $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Feel free to use software on this problem to perform any needed row reductions.

1. Show that the characteristic polynomial for both B and C is $(3-\lambda)(\lambda-3)(\lambda-1)$. What are the eigenvalues?

2. For each eigenvalue of B, state all the corresponding eigenvectors. Show us the matrix you need to row reduce, show the rref (from software), and then state the eigenvectors by writing (x, y, z) in terms of the free variables.

3. For each eigenvalue of C, repeat the previous step.

4. If you have a repeated eigenvalue, how many linearly independent eigenvectors should you expect to find?

When you are done, you should have written down 4 matrices (2 for each part), each matrix's rref, and then stated the eigenvectors by writing (x, y, z) in terms of the free variables on each part.

Solving Systems of Equations

Recall that there are three types of row operations, namely (1) swap rows, (2) multiply a row by a nonzero constant, and (3) add a multiple of a row to another row.

When you row reduce the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ using Gauss-Jordan elimination (or any 2 by n matrix), we can count the largest number of row operations we'll ever need to perform. Assume at each stage we decide to swap two rows to get a nice nonzero number in the pivot, and then we multiply that row by a nonzero number to obtain a leading 1. With a 2 by n matrix, we would swap rows 1 and 2, and then multiply row 1 by a nonzero number. Then we would add a multiple of row 1 to row 2 to eliminate the entry below this leading 1. We would then multiply row 2 by a nonzero number to obtain a leading 1. Then we'd need one more row operation to eliminate the number above the pivot in column 2. This is a total of 5 operations (we swapped 1 time, multiplied a row 2 times, and added a multiple of a row to another 2 times).

For larger matrices, how many row operations are needed to perform Gauss-Jordan elimination? This question is extremely important to computer programming, as the answer related to the time needed for a computer to row reduce a million by million matrix, something that happens all the time since the advent of computers.

Problem: Number of operations - Challenge Here's your challenge: How many row operations are needed to fully reduce an m by n matrix, where $n > m$.

- For a 3 by 4 matrix, we would swap rows a maximum of 2 times (once for each row but the last). We would need to multiply a row by some number 3 times (once for each leading 1). We would add a multiple of a row to another 3 times in the forward elimination and 3 times in the backward elimination (this puts the zeros above and below each pivot). We now have $2 + 3 + 6 = 11$ row operations.

- Now consider a 4 by 5 matrix. How many row swaps at most will you need? How many times will you need to multiply to get a leading 1? How many times will you need to add a multiple of a row to another to get a zero above or below a pivot? List all these out, then write the number of row operations needed.

- Now consider a 5 by 6 matrix. Repeat the above.

- Now consider a 6 by 7 matrix. Repeat the above. Did you get 41 row operations?
- Give a formula $f(n)$ so that for each n, the number $f(n)$ is the maximum number of row operations. We currently know $f(1) = 1$, $f(2) = 5$, $f(3) = 11$, $f(4) = ?$, $f(5) = ?$, $f(6) = 41$.

If you've never see the online encyclopedia of integer sequences, please head to http://oeis.org/. Try the sequence 1, 5, 11, ...

Problem 3.36 Let $A = \begin{bmatrix} 6-c & 2 \\ 2 & 3-c \end{bmatrix}$ where c is a real number.

1. For which values of c are the columns of A linearly independent?
2. For which values of c is the matrix A invertible?
3. For which values of c are there infinitely many solutions to $A\vec{x} = \vec{0}$?
4. For which values of c is the determinant of A equal to zero?
5. For which values of c does the rref of A equal the identity matrix?
6. For which values of c is the matrix A singular?
7. For which values of c is $\lambda = 0$ an eigenvalue of A?

Seeing Eigenvectors in Vector Fields

There are many connections between vector fields and eigenvalues/eigenvectors. The next three problems have you explore this topic, and make some conjectures.

Problem 3.37 The following three vector fields are represented by matrices with imaginary eigenvalues. Compute the eigenvalues for each, construct a vector field plot, and on the plot add several trajectories (the path followed by a particle that is dropped into this field).

1. $\vec{F}(x, y) = \begin{bmatrix} -1 & 3 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + 3y \\ -2x - 4y \end{bmatrix} = (-x + 3y, -2x - 4y)$.
2. $\vec{F}(x, y) = (x - y, x)$
3. $\vec{F}(x, y) = (-2y, x)$.

Make a conjecture as to why one spirals in, one spirals out, and one just wraps around in ellipses.

Problem 3.38 Start by downloading the Mathematica notebook VectorFields.nb (click on the link). The goal of this problem is to make a connection between a vector field and its corresponding eigenvalues/eigenvectors. Once the notebook is open, click somewhere in the text. Hold down Shift and press Enter to evaluate the commands and produce a vector field plot. The eigenvector directions are drawn in green. You can click on the bubbles with crosshairs to adjust the vector field (the adjustable vectors are are the columns of the matrix). Play around with the animation until you feel like you can answer each of the following questions. Write your answers to the first 4 by writing complete sentences, and provide a rough hand sketch of a vector field for each case to match your sentences.

1. If the vector field pushes things outwards in all directions, what do you know about the eigenvalues?
2. If the vector field pulls things inwards in all directions, what do you know about the eigenvalues?
3. How can you tell, by looking at a vector field plot, that one eigenvalue is positive and the other is negative?
4. If the vector field involves swirling motion, what do you know about the eigenvalues? What makes the difference between spiraling inwards, outwards, or just spinning in circles?
5. (Challenge) What happens when you have a repeated eigenvalue? This one has lots of correct answers, and it is a topic for much further discussion. See if you can get an example of a repeated eigenvalue with a behavior that's different from the above.

If you have the first 4, you can present in class. We'll have you come up to the computer and show us what you did.

We've already seen how to visualize the solution to a first order systems of ODEs. All we have to do is draw the corresponding vector field. The solutions to the ODE are the trajectories that follow the vectors in the field. In the previous chapter we visualized solutions by drawing a vector field. The next problem has you construct visual graphical solutions by only considering the eigenvalues and corresponding eigenvectors. The vector field plot is not needed.

$\boxed{\textbf{Problem 3.39}}$ Consider the system of ODEs $x' = y$, $y' = 8x - 2y$. This is a system of ODEs whose solution would give the position (x, y) of a particle whose tangent vectors are $(x', y') = (y, 8x - 2y)$. In other words, solving this ODE will tell us the trajectories we can see from a plot of the vector field $\vec{F}(x, y) = (y, 8x - 2y)$. On this problem, do not draw the vector field, as the goal is to answer all the questions below by just knowing the eigenvalues and eigenvectors.

Once you have finished, you should look at a vector field plot of $\vec{F}(x, y) = (y, 8x - 2y)$. Your trajectory plots should follow the vectors in the vector field plot, but you didn't ever have to make the vector field plot.

1. Write this system of ODEs in the form
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
Find the eigenvalues of A, and then give an eigenvector for each eigenvalue.
2. The eigenvectors determine two lines through the origin. Draw these lines on the same plot. This will divide the plane into 4 parts.
3. If an object starts at $(2, 4)$, draw the path the object will follow. Draw your path in the same plot that contains the lines determined from your eigenvectors. Then repeat this part for each initial condition $(0, 4)$, $(-2, 4)$, $(-1, 4)$, and $(2, -2)$.

You should have single plot that contains 2 lines, and then 5 trajectories. Two of your trajectories will match up with your lines, but it's important that you note which trajectory moves radially outwards, and which moves radially inwards.

Wrap up

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 3.40: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

Extra Practice

Please use the problem list below to find extra practice problems to help you learn. You'll find the problems listed below at the end of Chapter 1 (pages 23-28, including solutions) in *Linear Algebra* by Ben Woodruff. This text is freely available online. The text also references Schaum's Outlines Beginning Linear Algebra by Seymour Lipschutz for even more practice.

- https://content.byui.edu/file/c2f91762-7a1e-4d0b-a1ae-8d5f5f548e17/1/341-Book.pdf

Concept	Suggested	Relevant
Basic Notation	1bcf,2abehln	1,2
Gaussian Elimination	3all,4acf	3,4
Rank and Independence	5ac,6bd	5,6
Determinants	7adgh	7
Inverses	8ag,9ac	8,9
Eigenvalues	10abdghi	10
Summarize	11(multiple times)	11

Remember that you can check almost all of your work with technology. Use the following technology links to help you check your understanding.

- Sage RREF calculator

Chapter 4

Linear Algebra Applications

After completing this chapter, you should be able to:

1. Explain the connection between vector fields and their corresponding eigenvalues and eigenvectors. Use this knowledge to apply the second derivative test and explore systems of ODEs at equilibrium points.
2. Show how to solve various problems relating to conservation laws (such as stoichiometry, Kirchoff's electrical laws, Markov Processes, etc.) by finding the kernel of a matrix.
3. Use Cramer's rule to solve systems, and explain when you would choose Cramer's rule over row reduction.
4. Find interpolating polynomials, and use the transpose to solve the least squares regression problem.
5. Appropriate apply the words span, basis, vector space, dimension, eigenspace.

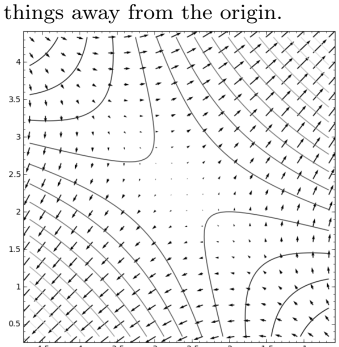
Nonconservative Eigenvector Problems

Vector fields and eigenvalues provide us with precisely the key information needed to locate maximums, minimums, and saddles for functions of the form $z = f(x, y)$.

Problem 4.1 Consider the function $f(x, y) = x^2 + 4xy - 4x + y^2 + 6y$. The derivative (gradient) is the vector field $Df(x, y) = (2x + 4y - 4, 4x + 2y + 6)$.

1. At what point(s) does $Df(x, y) = \vec{0}$? These are the potential locations of maximums, minimums, or saddles.
2. Compute the second derivative of f, which should give you a 2 by 2 symmetric matrix. This matrix is called the Hessian.
3. By looking at the plot to the right, are the eigenvalues of $D^2f(x, y)$ both positive, both negative, or do they differ in sign? How can you tell? Then confirm you are correct by computing the eigenvalues and eigenvectors of $D^2f(x, y)$.
4. Recall that the gradient points in the direction of greatest increase. Using this information alone, does the function have a maximum, minimum, or saddle point.

Here's a plot of several level curves of $f(x, y) = x^2 + 4xy - 4x + y^2 + 6y$ and its gradient. In one direction the gradient is pulling things towards the origin. In another direction, the gradient is pushing things away from the origin.



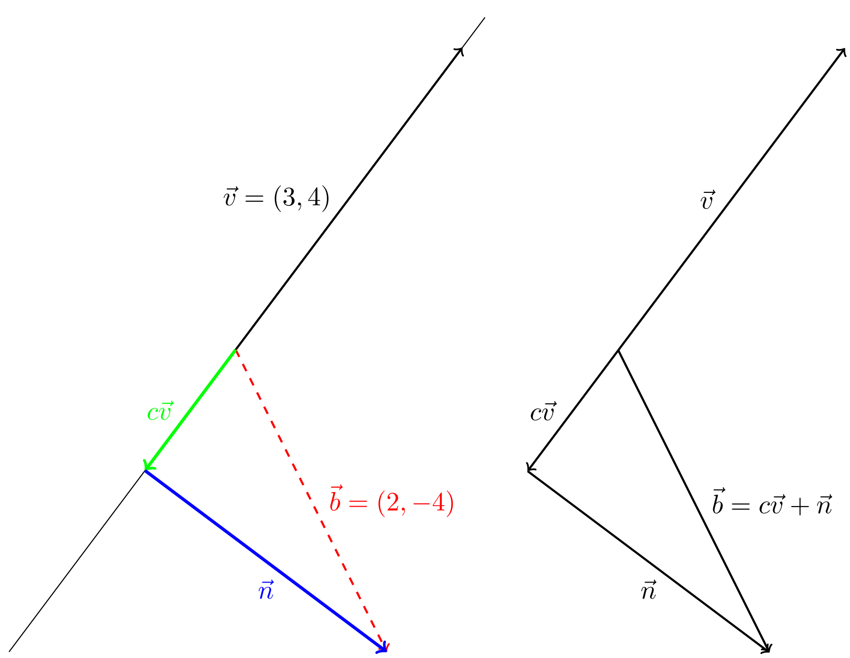
We can summarize the results from the problem above into a theorem from multivariate calculus.

Theorem 4.1. *Let $f(x, y)$ be a function that is twice continuously differentiable. Suppose that $Df(x, y) = (0, 0)$ when $(x, y) = (a, b)$, so that (a, b) is a critical point. To determine if the point (a, b) corresponds to a maximum, minimum, or saddle point, we compute the eigenvalues of $D^2f(a, b)$ (the second derivative is called the Hessian).*

- *If all eigenvalues are positive, then f has a minimum at (a, b).*
- *If all eigenvalues are negative, then f has a maximum at (a, b).*
- *If the eigenvalues differ in sign, then f has a saddle at (a, b).*
- *If zero is an eigenvalue, then the second derivative test fails.*

Projections and Linear Regression

Problem 4.2 Sally found the treasure in the corn field. She's now looking for treasure in a swamp. There's a road through the swamp that runs parallel to the vector $\vec{v} = (3, 4)$. Her current location is $(0, 0)$ and the treasure (geocache) is located at the position $\vec{b} = (2, -4)$ (units are hundreds of yards). When Sally decides to leave the road, she'll have to wade through some swamp water. She would prefer to spend as little time in the swamp as possible. Her goal is to walk along the road until she reaches the point closest to the treasure, and then wade straight to the treasure. This means she needs to find a scalar c so that $c\vec{v}$ gets her as close to the treasure as possible. See the picture below.



The vector $\vec{n}$ represents the path she must take through the swamp. Her goal is to find a scalar c so that $\vec{b} = c\vec{v} + \vec{n}$ and $\vec{n}$ is as short as possible.

1. What is the angle between $\vec{v}$ and $\vec{n}$? Why does $\vec{v} \cdot \vec{n} = 0$?

2. Sally knows that the treasure is at $\vec{b} = c\vec{v} + \vec{n}$. Since $\vec{v} \cdot \vec{n} = 0$, she decides to dot both sides of this equation, on the left, by the vector $\vec{v}$ to get $\vec{v} \cdot \vec{b} = \vec{v} \cdot (c\vec{v} + \vec{n})$. Show that in general

$$c = \frac{\vec{v} \cdot \vec{b}}{\vec{v} \cdot \vec{v}}$$

Then show that with Sally's specific road vector $\vec{v}$ and treasure vector $\vec{b}$, the constant c is $c = -2/5$.

The vector $c\vec{v}$ above is often called the orthogonal projection of $\vec{b}$ onto $\vec{v}$. The word orthogonal means that $\vec{v} \cdot \vec{n} = 0$, i.e. that there is a 90 degree angle between $\vec{v}$ and $\vec{n}$.

Problem 4.3 Now assume that Sally is an astronaut in space. She's moving through an asteroid field and knows there is safe passage if she follows the vector $\vec{v} = (1, -2, -3)$. She needs to get to the point $\vec{b} = (3, -6, -11)$. She already knows that if she follows $\vec{v}$ three times, she'll end up pretty close by arriving at $(3, -6, -12)$. However, she wants to follow $\vec{v}$ until she is as close to $\vec{b}$ as possible, as leaving the known safe path could be dangerous.

1. Determine the scalar c so that $\vec{v}c$ is as close to $\vec{b}$ as possible. Your answer should be close to 3. Use the formula from the previous problem.

2. Let's swap to a different question. Suppose we would like to find an equation of a line $y = mx$ through the origin that passes through the three points $(1, 3)$, $(-2, -6)$, and $(-3, -11)$. To pass through all three points we need to solve the system of equations $3 = m(1)$, $-6 = m(-2)$, and $-11 = m(-3)$. Rewrite this system of equations as the vector equation (state $\vec{v}$ and $\vec{b}$)

$$\vec{v}m = \vec{b} \qquad \Rightarrow \qquad \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} m = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}.$$

Explain why there is no solution to this problem.

3. What should we choose the slope m to equal so that $\vec{v}m$ will be as close to $\vec{b}$ as possible?

Conservation Laws through Eigenvectors and Kernels

Many problems in nature arise from conservation laws. These laws generally focus on the principle that matter is neither created nor destroyed, rather it is just moved, changed, or something. Any of the following could be viewed as a conservation law:

- What comes in must come out.
- Voltage supplied equals voltage suppressed.
- Atoms before equal atoms after.
- The change in a quantity is how much it increases minus how much it decreases.

- Current in equals current out.
- The sum of the forces in every direction must match the total force.
- The force and moments must sum to zero when an object is at rest.
- This list could go on for a while.

Throughout this chapter, we'll see several different conservation laws. You'll focus on understanding these conservation laws in your major classes. We'll see that almost every one of these problem can be written in the matrix form $A\vec{x} = \vec{0}$. We'll see that $\lambda = 0$ is an eigenvalue, which means that when we follow the eigenvector direction, the underlying vector field neither pushes outward nor inward. In this eigenvector direction, the system is conserving something.

Definition 4.2: Homogeneous System, Kernel. Because we'll encounter problems of the form $A\vec{x} = \vec{0}$ quite often, we make some definitions.

- We say that the linearly system $A\vec{x} = \vec{b}$ is homogeneous if $\vec{b} = \vec{0}$.
- The set of solutions to $A\vec{x} = \vec{0}$ is called the kernel (or null space) of A.

Chemical reaction stoichiometry is the study of balancing chemical equations. A chemical reaction will often transform reactants into by-products. The by products are generally different compounds, together with either an increase or decrease in heat. One key rule in stoichiometry is that a chemical process neither creates nor destroys matter, rather it only changes the way the matter is organized. For simple reactions (with no radioactive decay), this conservation law forces the number of atoms entering a reaction to be the same as the number leaving. The next problem asks you to use this conservation law to create a balanced chemical reaction equation.

Problem 4.4: Stoichiometry The chemical compound hydrocarbon dodecane ($C_{12}H_{26}$) is used as a jet fuel surrogate (see Wikipedia for more info). This compound reacts with oxygen (O_2), and the chemical reaction produces carbon dioxide (CO_2), water (H_2O), and heat. Suppose we expose some dodecane to oxygen, and that a chemical reaction occurs in which the dodecane is completely converted to carbon dioxide and water. Conservation requires that the number of atoms (H, C, and O) at the beginning of the chemical reaction must be the exact same as the number at the end. We could write the chemical reaction in terms of molecules as

$$x_1C_{12}H_{26}+x_2O_2 = x_3CO_2+x_4H_2O \quad \text{or} \quad x_1C_{12}H_{26}+x_2O_2-x_3CO_2-x_4H_2O = 0,$$

where x_1 molecules of dodecane and x_2 molecules of oxygen were converted to x_3 units of carbon dioxide and x_4 units of water. If we look at each atom (carbon, hydrogen, and oxygen) individually, we obtain three equations to relate the variables x_1, x_2, x_3, x_4. The carbon equation is simply

$$x_1(12) + x_2(0) = x_3(1) + x_4(0) \quad \text{or} \quad x_1(12) + x_2(0) - x_3(1) - x_4(0) = 0.$$

Your job follows:

1. Write the other two conservation equations (for hydrogen and oxygen), and then organize your work into the matrix product $A\vec{x} = \vec{0}$. This means you are working with a homogeneous system.
2. Solve the corresponding system of equations by row reduction. As there are only 3 equations with 4 unknowns, you should obtain infinitely many solutions. Write each variable in terms of the free variable. You have found the kernel of the matrix A.

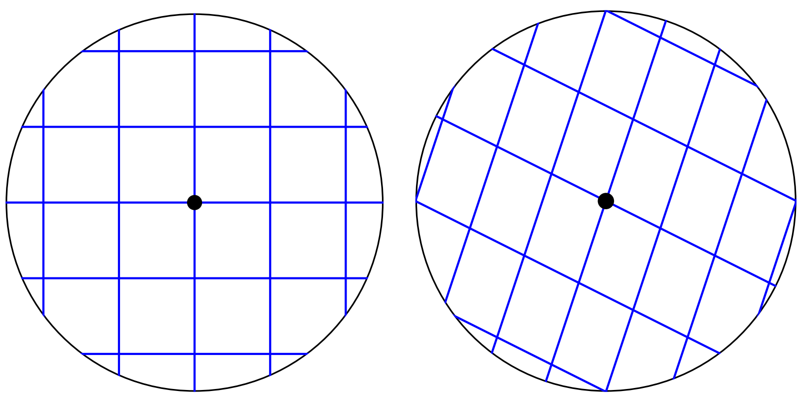
3. If about 10,000 molecules of water are present at the end of the reaction, about how many molecules of dodecane were burned?

4. The matrix A was not square. We can make the system square by adding the trivial equation $0 = 0$, a row of zeros, to the bottom of the matrix. Let B be this matrix. Why do we know $\lambda = 0$ is an eigenvalue of this matrix? Find an eigenvector corresponding to $\lambda = 0$.

If your answer on part 4 looks like your answer on the previous part, good. The point is that finding an eigenvector corresponding to $\lambda = 0$ is the exact same as finding the kernel.

Visualizing Linear Transformations between Vector Spaces

Problem 4.5 After getting the treasure in the swamp, Sally moves on to find a treasure located in a small town. She has a map of the town that shows the city blocks. However, when she looks at a satellite image of the city it's slightly different than her map. Here are the two maps (the city map is on the left, the satellite on the right).



City Map Satellite View

The city grid is not lined up with compass directions. When the city map tells her to go up one block, this really means her (x, y) position should follow the vector $\vec{v}_1 = (1, 3)$. To go right 1 block, she follows the vector $\vec{v}_2 = (2, -1)$. She has to learn to work with two different coordinate systems, namely the city coordinates (given in blocks) and the (x, y) satellite coordinates (given in hundreds of yards). Assume that Sally is currently at the origin $(0, 0)$.

1. If Sally goes up 2 blocks, and right 3 blocks, what are here (x, y) coordinates (in hundreds of yards)?

2. If Sally moves up c_1 blocks and right c_2 blocks, then her (x, y) coordinates are
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ -1 \end{bmatrix} c_2.$$
Rewrite this linear combination as a matrix product $A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ (What is the matrix A?). You can check if you are correct by computing $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

3. If the treasure is located at the (x, y) coordinates $(0, 7)$, what directions would you give her in terms of blocks? If the treasure is located at $(1, -7.5)$, what directions would you give?

One way we can use a matrix is to think of the matrix as a map. When Sally was walking through the city in Problem 4.5, she had a map of the city in her hands. This map gave her the coordinates of locations in the city, but did so in a much simplified way. Going right on the map 1 block resulted in following the vector $(2, -1)$. Going up 1 block resulted in following the vector $(1, 3)$. It's much easier to give directions in terms of blocks.

If Sally walks 2 blocks right, and 1 block up, then she arrives at $\begin{bmatrix}2\\-1\end{bmatrix}(2) + \begin{bmatrix}1\\3\end{bmatrix}(1) = \begin{bmatrix}7\\1\end{bmatrix}$. In the city map, we base our all movement on the vectors $(1, 0)$ and $(0, 1)$. When looking at actual (x, y) position, we base all our movements on the vectors $(2, -1)$ and $(1, 3)$. We call each of these collections of independent vectors a basis. We call $(c_1, c_2) = (2, 1)$ the coordinates of the point $(x, y) = (7, 1)$ relative to the basis $\{(2, -1), (1, 3)\}$. We can describe any point (x, y) using the simplified coordinates (c_1, c_2) relative to this basis.

Definition 4.3: Basis and Coordinates Relative to a Basis. If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, then we'll say these vectors form a basis. We use the word "basis" because we can write (base) other vectors uniquely as a linear combination of these basis vectors. You have been using the standard basis vectors $(1, 0)$ and $(0, 1)$ your entire life to talk about vectors in the plane. To plot the point $(2, 3)$, we think "right 2, up 3" which is the same as the vector equation $(2, 3) = 2(1, 0) + 3(0, 1)$.

Suppose $\mathscr{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis, and $\vec{x}$ is the linear combination

$$\vec{x} = \vec{v}_1 c_1 + \vec{v}_2 c_2 + \cdots + \vec{v}_n c_n.$$

Then we call $c_1, c_2, \ldots, c_n$ the coordinates of $\vec{x}$ relative to the basis $\mathscr{B}$.

In terms of matrices, when the columns of A are linearly independent and $A\vec{c} = \vec{x}$, we say that $\vec{c}$ is the coordinates of $\vec{x}$ relative to the columns of A.

A matrix A takes each coordinate (c_1, c_2) and transforms it to the point $\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}c_1\\c_2\end{bmatrix}$. You'll see in the next problem that lines get transformed to lines. For this reason, and others we'll soon see, we call this coordinate transformation map a linear transformation.

Problem 4.6 Consider the matrices $A = \begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}$ and $B = \begin{bmatrix}1 & 3\\2 & -2\end{bmatrix}$.

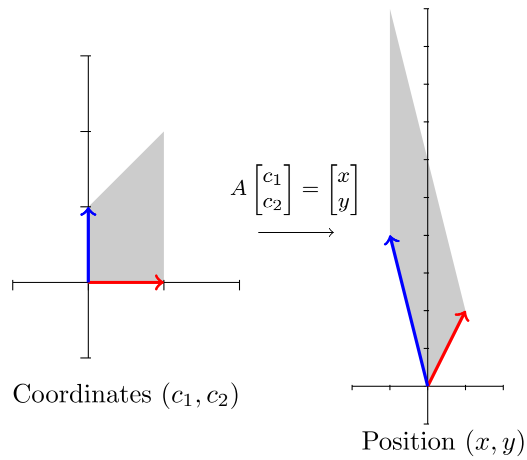
1. Consider $\begin{bmatrix}1 & -1 & -1 & 3 & 1 & 1\\2 & 4 & 10 & 0 & 2 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1 & 2 & 1 & 2/3\\0 & 1 & 2 & -1 & 0 & -1/3\end{bmatrix}$.

   If we want to write $(x, y) = (-1, 10)$ as a linear combination of the columns of A, what scalars (coordinates) c_1 and c_2 give $\begin{pmatrix}1\\2\end{pmatrix}c_1 + \begin{pmatrix}-1\\4\end{pmatrix}c_1 = \begin{pmatrix}-1\\10\end{pmatrix}$?

   What are the coordinates of $(x, y) = (3, 0)$ relative to the columns of A?

   What are the the coordinates of $(1, 0)$ relative to the basis $\{(1, 2), (-1, 4)\}$?

2. Alice decides to walk around using coordinates relative to the columns of A. She starts at $(0, 0)$, and then walks to the (c_1, c_2) coordinates $(1, 0)$, then $(1, 2)$, then $(0, 1)$, and then back to $(0, 0)$. Her path in the (x, y) plane is shown below on the right.



Bob decides to follow the same coordinate path, but he's using coordinates relative to the columns of B. Draw Bob's path in the (x, y) plane. Remember that since we know coordinates (c_1, c_2), we can get the (x, y) position from $B\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$.

3. Candice is staring at a treasure map. She doesn't have a matrix C to help her translate from the treasure map to actual (x, y) points. However, she does have a few bits of information. There are two trees on her map with coordinates (c_1, c_2) at $(2, 1)$ and $(-4, -3)$. The actual (x, y) location of these trees is $(-4, 5)$ and $(7, -10)$ This means if Candice knew C, then she could compute $C\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-4\\5\end{bmatrix}$ and $C\begin{bmatrix}-4\\-3\end{bmatrix}=\begin{bmatrix}7\\-10\end{bmatrix}$. Combining these two products together into one matrix means
$$C\begin{bmatrix}2 & -4\\1 & -3\end{bmatrix}=\begin{bmatrix}-4 & 7\\5 & -10\end{bmatrix}.$$
Use this to find C. [Hint: Try using an inverse matrix.]

4. The treasure on Candice's map has the map coordinates $(-1, 5)$. Give Candice the (x, y) location of the treasure.

Problem 4.7 Consider the matrix $A=\begin{bmatrix}2 & 1 & -1\\1 & 2 & 0\\0 & 4 & 3\end{bmatrix}$.

1. Compute the determinant of A.

2. Now create a matrix B so that the ijth entry of B is the cofactor C_{ij} (remove row i and column j, compute the determinant, and then times by an appropriate sign). This will require that you compute nine 2 by 2 determinants.

If you forgot what a cofactor is, you'll want to review the definition. See Definition 3.13 on page 61.

3. Compute the inverse of A with software.

4. Make a conjecture about the connection between the determinant of A, this matrix of cofactors B, and the inverse of A.

In your work above, you should have noticed that you had to interchange the rows and columns of B to make your conjecture. This process of interchanging rows and columns, called transposing a matrix, will show up in so many of our applications that we make a definition.

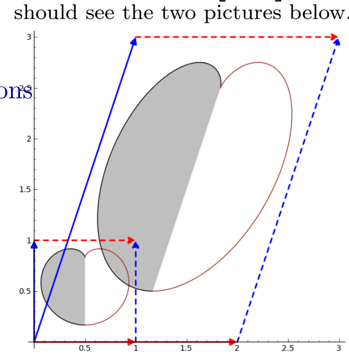
Definition 4.4: The Transpose A^T. Symmetric Matrix. If A is an m by n matrix, then the transpose of A is the n by m matrix formed by interchanging the rows and columns. Row 1 is now column 1. Row 2 is now column 2. Just think of each row as a vector, and then place those vectors in the columns of a new matrix. We use the symbol A^T to stand for the transpose of a matrix.

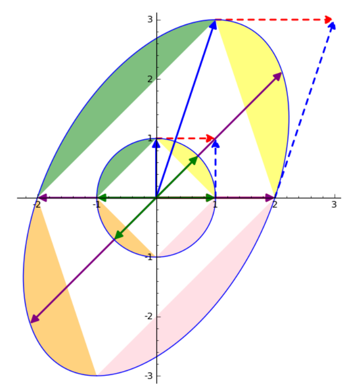
We'll often encounter square matrices where the transpose of the matrix is the matrix itself. If $A = A^T$ then we say the matrix is symmetric. When a vector field has a potential, its derivative satisfies this property.

Problem 4.8 On this problem, you will explore graphs of several linear transformations. Your job is to look for patterns and explanations. Please head to the following webpage:

- http://bmw.byuimath.com/dokuwiki/doku.php?id=2d_linear_transformations

For the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, you should see the two pictures below.





You may find this problem easiest if you create your own account at sagemath.org, and then copy the code from the URL above to your own notebook. Then you can put 10+ linear transformation graphs in the same document so you can compare them all side by side.

Here's your job. In the software link above, change the matrix A to be several different matrices. Look for an answer to each question below.

- What does the determinant tell you about the map? Can you see the determinant in the pictures?
- If the determinant is zero, what does that mean about your map?
- What do the eigenvalues tell you about your map? This is perhaps easiest to tell with the circle map.
- Is there a connection between the eigenvalues and the determinant?
- What do the eigenvectors tell you?
- Does anything special happen if the matrix is symmetric?

To be prepared for class, write answers to at least 4 of these questions using complete sentences. There are many great answers to the questions above.

Here are some matrices that might be useful to look at. Please type each of these in. As you discover patterns, test them against these matrices.

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix},$$

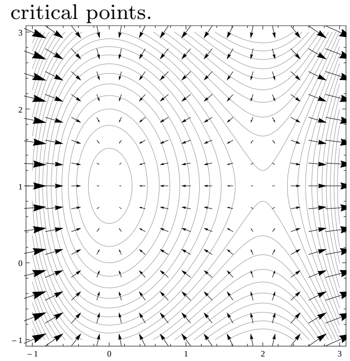
$$\begin{bmatrix} \cos(pi/3) & -\sin(pi/3) \\ \sin(pi/3) & \cos(pi/3) \end{bmatrix}, \begin{bmatrix} 3\cos(pi/6) & -5\sin(pi/6) \\ 3\sin(pi/6) & 5\cos(pi/6) \end{bmatrix}.$$

Nonconservative Eigenvector Problems

Problem 4.9 Consider the function $f(x, y) = x^3 - 3x^2 - y^2 + 2y$

1. At what point(s) does $Df(x, y) = \vec{0}$? You should obtain two points. These are the potential locations of maximums, minimums, or saddles.
2. Compute the second derivative of f, which is a 2 by 2 symmetric matrix.
3. Pick one of the critical points. Use the vector field plot to the right to decide if the eigenvalues of $D^2 f(x, y)$ are both positive, both negative, or differ in sign at that critical point. Then state if the function has a maximum, minimum, or saddle at that point. Then repeat with the other critical point.
4. Now compute the eigenvalues of the Hessian at each critical value. You'll need to find the eigenvalues of two different matrices. This should confirm your answer to part 3. (The matrix is diagonal, so computing eigenvalues should be quick.) Don't forget that you are finding eigenvalues of $D^2 f(a, b)$, not $D^2 f(x, y)$.

Here's a plot of several level curves of $f(x, y) = x^3 - 3x^2 - y^2 + 2y$ and the gradient. There are two critical points.



The following example adds a little more information to this discussion. I've included it to give you one additional piece of information, namely how the eigenvalues and eigenvectors connect to the concavity of the function.

Example 4.5. For the function $f(x, y) = x^2 + xy + y^2$, the gradient is $Df = \begin{bmatrix} 2x + y & x + 2y \end{bmatrix}$, which is zero only at $x = 0, y = 0$ (solve the system of equations $2x + y = 0, x + 2y = 0$). The Hessian is $D^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The eigenvalues are found by solving $0 = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = (\lambda - 3)(\lambda - 1)$, so $\lambda = 3, 1$ are the eigenvalues. Since both eigenvalues are positive, the gradient pushes things away from the origin in all direction, which means in every direction you move from the critical point, you'll increase in height. There is a minimum at $(0, 0)$.

The eigenvectors of the Hessian help us understand more about the graph of the function. An eigenvector corresponding to 3 is $(1, 1)$, and corresponding to 1 is $(-1, 1)$. These vectors are drawn in figure 4.1, together with two parabolas whose 2nd derivatives are precisely 3 and 1. The parabola which opens upwards the most quickly has a 2nd derivative of 3. The other parabola has a second derivative of 1. In every other direction, the 2nd derivative would be between 1 and 3.

Projections and Linear Regression

Problem 4.10 Jimmy is using a rocket suit to travel out in space. His rocket suit had 4 good boosters that allowed travel in any direction, with a backup booster in case one got damaged. However, some tiny meteorites happened to pass by and take out two of his boosters, as well as his radio to call for help. He's now only able to move in the directions $\vec{v}_1 = (1, 1, 1)$ and $\vec{v}_2 = (-1, 1, 2)$. His space ship is sitting at $\vec{b} = (-1, 4, 7)$.

1. Show that Jimmy cannot arrive at his ship using his two working boosters. In other words, show that we cannot write $\vec{b}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Set up an appropriate matrix equation, row reduce the equation, and use your row reduction to give an answer.

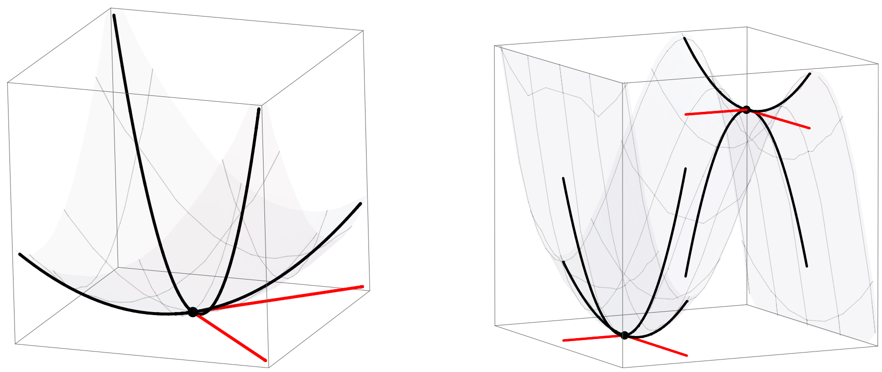
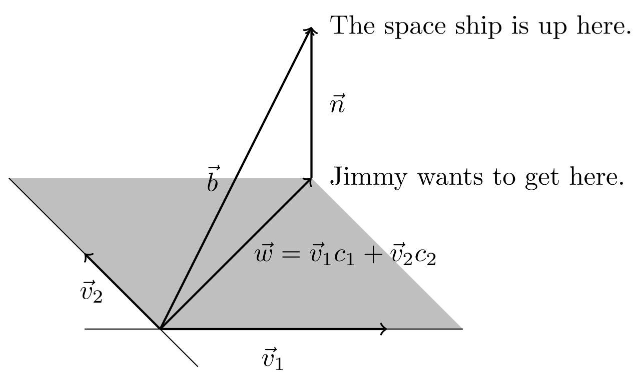


Figure 4.1: The eigenvectors of the second derivative tell you the directions in which the 2nd derivative is largest and smallest. At each critical point, two eigenvectors are drawn as well as a parabola whose second derivative (the eigenvalue) matches the second derivative of the surface in the corresponding eigenvector direction.

2. Jimmy has a one shot back up gun. This gun will propel him towards the ship if he points the gun directly away from the ship and fires. It's easy to miss aim, so he would like to get as close to the ship as possible before he fires the gun. He needs to find c_1 and c_2 so that $\vec{w} = \vec{v}_1 c_1 + \vec{v}_2 c_2$ gets him as close to the ship as possible. The picture below illustrates the general idea. The vectors $\vec{v}_1$ and $\vec{v}_2$ give Jimmy a plane of possible movements.



When Jimmy has arrived at the closest spot to the ship, he'll have the smallest $\vec{n}$ so that

$$\vec{v}_1 c_1 + \vec{v}_2 c_2 + \vec{n} = \vec{b} \qquad \text{or} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} c_2 + \vec{n} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}.$$

Why must $\vec{v}_1^T \vec{n} = 0$ and $\vec{v}_2^T \vec{n} = 0$?

3. Since there are two unknown constants c_1 and c_2, we need two equations. Multiply both sides of the above equation on the left by $\vec{v}_1^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Why does $\vec{n}$ vanish from the equation? This gets us one equation.

4. To get a second equation, we multiply both sides by $\vec{v}_2^T = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}$. We now have two equations with two unknowns c_1 and c_2. Solve and show that $c_1 = 11/7$ and $c_2 = 37/14$.

The problem above asked you to find the point in a plane that was closest to a point not on the plane. This is called the orthogonal projection of $\vec{b}$ onto the plane formed by the vectors $\vec{v}_1$ and $\vec{v}_2$. If we let $A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$, then the projection of $\vec{b}$ onto this plane is $w = \vec{v}_1 c_2 + \vec{v}_2 c_2 = A\begin{bmatrix}c_1\\c_2\end{bmatrix}$. Our goal is to find $\begin{bmatrix}c_1\\c_2\end{bmatrix}$ such that $\vec{n} = \vec{b} - A\begin{bmatrix}c_1\\c_2\end{bmatrix}$ is as short as possible. The next problem shows that you can accomplish this by solving $A^TA\vec{x} = A^T\vec{b}$ for $\vec{x}$. We just take the problem $A\vec{x} = \vec{b}$ which has no solution, multiply both sides by A^T, and then solve.

Problem 4.11 Suppose we would like to find an equation of a line $y = a_0 + a_1x$ that passes through the three points $(-1,-1)$, $(1,4)$ and $(2,7)$. If such a line does not exist, we'd like to find a line that passes close to these three points.

1. The three points give us three equations that involve the unknown constants a_0 and a_1. Show that we can write these equations in the matrix form $A\vec{x} = \vec{b}$ and vector form $\vec{v}_1 a_0 + \vec{v}_2 a_1 = \vec{b}$

$$\begin{bmatrix}1 & -1\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix} = \begin{bmatrix}-1\\4\\7\end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix}1\\1\\1\end{bmatrix}a_0 + \begin{bmatrix}-1\\1\\2\end{bmatrix}a_1 = \begin{bmatrix}-1\\4\\7\end{bmatrix}.$$

   Then show that there is no solution to this problem.

2. We now know that no linear combination of $\vec{v}_1 = (1,1,1)$ and $\vec{v}_2 = (-1,1,2)$ will give us the vector $\vec{b} = (-1,4,7)$. We would like to find scalars a_0 and a_1 so that $\vec{v}_1 a_1 + \vec{v}_2 a_1$ is as close to $\vec{b}$ as possible. This is the exact same question as the previous problem (where Jimmy could not get to his space ship). Multiply both sides of the inconsistent equation $A\vec{x} = \vec{b}$ on the left by $\begin{bmatrix}1\\1\\1\end{bmatrix}^T = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$, and then multiply both sides by $\begin{bmatrix}-1\\1\\2\end{bmatrix}^T = \begin{bmatrix}-1 & 1 & 2\end{bmatrix}$. This should get you two different equations that involve c_1 and c_2.

3. Compute $A^TA = \begin{bmatrix}1 & 1 & 1\\-1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\\1 & 2\end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix}1 & 1 & 1\\-1 & 1 & 2\end{bmatrix}\begin{bmatrix}-1\\4\\7\end{bmatrix}$.
   You should see that both of your equations above are in the matrix equation

$$A^TA\vec{x} = A^T\vec{b} \quad \Rightarrow \quad \begin{bmatrix}1 & 1 & 1\\-1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\-1 & 1 & 2\end{bmatrix}\begin{bmatrix}-1\\4\\7\end{bmatrix}.$$

   Make sure you simplify the matrix products A^TA and $A^T\vec{b}$, as this should become a system of 2 equations and 2 unknowns.

4. Now solve $A^TA\vec{x} = A^T\vec{b}$ for $\vec{x} = \begin{bmatrix}a_0\\a_1\end{bmatrix}$. Use your answer to state the line $y = a_0 + a_1x$ that passes nearest these three points.

The transpose of a matrix plays a crucial role in finding projections. When the problem $A\vec{x} = \vec{b}$ has no solution, it is impossible to write $\vec{b}$ as a linear combination of the columns of A. If we multiply both sides on the left by A^T, then we have an equation that we can solve to obtain the coefficients $\vec{x}$ so that $A\vec{x}$ is as close to $\vec{b}$ as possible. This is the key idea to regression.

Conservation Laws through Eigenvectors and Kernels

When we perform a partial fraction decomposition, Our goal is to rewrite a complicated fraction as the sum of simpler fractions. We are not changing the quantity that the fraction represents, rather we are just changing how we express the fractions. This is a conservations law, as the fractional quantity is conserved. Can we answer this problem by looking for the kernel of some matrix, or an eigenvector corresponding to $\lambda = 0$?

Problem 4.12 Consider the partial fraction decomposition

$$\frac{8s+7}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

which we can rewrite in the form

$$8s + 7 = A(s+3) + B(s-2).$$

Let's compare several different ways of solving this problem.

1. Complete this partial fraction decomposition. Use any method you like.

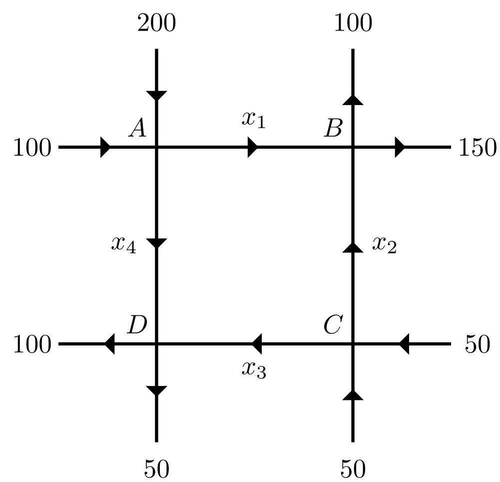
2. Now let's solve
$$\frac{cs+d}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$
Rather than thinking of c and d as known constants, let's make them variables in our linear system of equations. Our goal is to solve
$$(A + B - c)s + (3A - 2B - d) = 0$$
which we can rewrite in the matrix form
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Why did I save the last two columns for c and d?

This is a matrix equation of the form $A\vec{x} = \vec{0}$, so the solutions are the kernel of A. Solve this matrix equation (find the kernel of A) and write your answer in terms of the free variables. Please use software to row reduce, and just share the key parts of your work (as shown below).
$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{array}\right] \Rightarrow \begin{pmatrix} A \\ B \\ c \\ d \end{pmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} c + \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} d.$$

3. Change the 2 by 4 matrix above to solve the partial fraction decomposition
$$\frac{cs+d}{(s-p)(s-q)} = \frac{A}{s-p} + \frac{B}{s-q}.$$
Then use software to solve, giving A and B in terms of c, d, p, q.

Problem 4.13: Traffic Flow Consider the following traffic flow grid.



The numbers on the edges represent the number of vehicles that either enter or leave the system each hour. The variables x_1, x_2, x_3, and x_4 represent the number of cars on each road. Assume that all streets are one-way streets where the arrows give the direction of traffic flow.

1. How do you know there are 400 total cars entering this network of roads each hour? Are all these cars leaving? This is a conservation problem?

2. The number of cars entering an intersection must match the number of cars leaving an intersection. We can use this to build a system of equations for the traffic flows x_1, x_2, x_3, x_4. Every hour at node A there are 300 cars entering the intersection and $x_1 + x_4$ cars leaving the intersection. This gives us an equation $x_1 + x_4 = 300$. Continue in this fashion to obtain an equation at each intersection point. You should have a system of 4 equations with 4 unknowns.

3. Write your system of equations in the matrix form $A\vec{x} = \vec{b}$. What is A, what is $\vec{x}$, and what is $\vec{b}$? Is this system homogeneous or non homogeneous?

4. Solve your system of equations. When you are presenting this kind of information in class, you should use the pattern $B \xrightarrow{\text{rref}} R \Rightarrow \vec{x} = ...$, so show us the augmented matrix B, show us its rref R, and then state the solution $\vec{x}$ as a linear combination of vectors, namely

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} x_4 + \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}.$$

5. (Challenge - we'll answer in class if you're unable) What's the minimum number and maximum number of cars that can be on the road AD each hour? Explain.

Whenever I see a problem that involves a conservation law, I think two things. For one, there is probably a homogeneous system $A\vec{x} = \vec{0}$ somewhere in the background whose kernel is the solution. Two, if I make sure A is a square matrix (possibly adding rows of zeros), then I can rephrase "find the kernel" as "find the eigenvectors corresponding to zero." To accomplish finding this matrix A, we'll often have to think of given constants as variables. Let's do this with the previous problem.

Problem 4.14 Use the same setting as the previous traffic flow problem, however, let's change the given values to be variables. Starting in the upper left corner and moving clockwise, replace the numbers 100, 200, 100, 150, etc., with the variables a, b, c, d, e, f, g, h. We now have 12 unknowns, namely x_1, ..., x_4, a, b, ..., h.

1. At node A, our equation is now $x_1 + x_4 - a - b = 0$. Write the other 3 equations and express the homogeneous system in the form $A\vec{x} = \vec{0}$ where A is a 4 by 12 matrix. State the matrix A.

2. Find the kernel of A, and write your solution as a linear combination of vectors where the scalars are the free variables (use the $B \xrightarrow{\text{rref}} R \Rightarrow \vec{x} = ...$ pattern). Your solution should look like

$$\begin{bmatrix} x_1\\x_2\\x_3\\x_4\\a\\b\\c\\d\\e\\f\\g\\h \end{bmatrix} = \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} x_4 + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} b + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} c + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} d + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} e + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} f + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} g + \begin{bmatrix} *\\*\\*\\*\\*\\*\\*\\*\\*\\*\\*\\* \end{bmatrix} h.$$

3. The 4th row of your rref is not zero (which is why a is not a free variable). Write the equation given by this 4th row. Can you interpret this 4th equation as a conservation law?

Visualizing Linear Transformations between Vector Spaces

Have you noticed in every matrix problem we can always write the solution as a linear combination of vectors? When the system is homogeneous, the solution to $A\vec{x} = \vec{0}$ (the kernel) is always all linear combinations of a few vectors. We take a vector times a free variable, plus a vector times a free variables, etc. The solution is the set of all linear combinations of a few vectors. It would be nice to say "all linear combinations of" in an efficient way. We'll use the word span to talk about forming all linear combinations, the word vector space to talk about the vectors in the span, and then dimension to talk about the geometric size of the span (is it a line, a plane, a 3D space, etc.).

Definition 4.6: Span, Basis, Vector Space, and Dimension. Consider the set of vectors $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.

- The span of S is the set of all linear combinations of the vectors in S.
- When the vectors in S are linearly independent, we say that the vectors form a basis for their span.

- A vector space is the span of a collection of objects (we'll focus on vectors and functions).
- A basis for a vector space is a collection of linearly independent objects whose span is the vector space.
- The dimension of a vector space is the number of vectors in a basis for the vector space.

Problem 4.15 We've seen each of the following problems before. On this problem, you'll practice using the words span, vector space, basis, and dimension.

1. In problem 3.1 on page 42, Sally could move along the road $(-1, 1)$ and the rows of corn $(2, 1)$. Is the span of these two vectors the entire plane? [Hint: You can row reduce $\left[\begin{array}{cc|c} -1 & 2 & x \\ 1 & 1 & y \end{array}\right]$, or you can come up with another explanation as to why any vector (x, y) must be a linear combination of the given two.]
2. Suppose our astronaut Jimmy has 4 boosters (see Problem 3.11) that allow bidirectional movement in the directions $(1, 1, 2)$, $(0, 1, 3)$, $(2, 1, 1)$, and $(-2, 1, 0)$. Show that the span of these vectors is all of three dimensional space. Then select from these boosters a basis for $\mathbb{R}^3$. [You'll want to row reduce a matrix to answer this. Show the matrix and its rref.]
3. If the 4th booster breaks, what kind of object is the span of the remaining three directions? Is it all of space, a plane, a line, a circle, a parallelogram, etc.? Then state the dimension of and give a basis for the vector space obtained as the span of these three vectors.

The set of vectors (x, y) in the plane forms a vector space of dimension 2. We know this because the vectors $(1, 0)$ and $(0, 1)$ are linearly independent and we can obtain any point (x, y) in the plane as the linear combination $(1, 0)x + (0, 1)y$. This shows that the two vectors $(1, 0)$ and $(0, 1)$ form a basis for the set of vectors in the plane. We call this vector space $\mathbb{R}^2$.

Problem 4.16 Read the preceding paragraph (if you have not already). For each vector space below, produce a collection of independent vectors (or functions) whose span is the space. You might need to rref a matrix and obtain a solution first. State the dimension of the vector space.

1. The set of vectors (x, y, z) in space ($\mathbb{R}^3$).
2. The kernel of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & -2 & 0 & -1 \end{bmatrix}$ from Problem 4.12.

   Your work here generalizes to show that the kernel of any matrix is always a vector space.
3. The solutions to $B\vec{x} = 3\vec{x}$ for $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ from Problem 3.35.
4. The solutions to $C\vec{x} = 3\vec{x}$ for $C = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ from Problem 3.35.
5. (Challenge) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ of degree 3 or less.

You can present in class if you got the first four.

From the examples above, we see that the solutions to $A\vec{x} = \lambda\vec{x}$ form a vector space. This is easy to see when we realize that $A\vec{x} = \lambda\vec{x}$ is the same equation as $(A - \lambda I)\vec{x} = \vec{0}$, which means that to find eigenvectors, we must find the kernel of $A - \lambda I$. Let's give a name to this vector space of eigenvectors.

Definition 4.7: Eigenspace. Let λ be an eigenvalue of A. The eigenspace of A corresponding to λ is the set of solutions to $A\vec{x} = \lambda\vec{x}$. We write $E_A(\lambda)$ for the eigenspace of A corresponding to λ. The geometric multiplicity of λ is the dimension of $E_A(\lambda)$.

Once we have a collection of vectors in the kernel of a matrix, or a collection of eigenvectors corresponding to the same eigenvalue, the span of these vectors gives us an entire vector space full of vectors that will still be in the kernel. The next problem asks you to show why.

Problem 4.17 Recall that the kernel of A is the set of solutions $\vec{x}$ to $A\vec{x} = \vec{0}$. Suppose that $\vec{y}$ and $\vec{z}$ are both in the kernel of a matrix A. Show that any linear combination of $\vec{y}$ and $\vec{z}$ is also in the kernel of A. In other words, show that $a\vec{y} + b\vec{z}$ is also in the kernel of A.

[Hint: Why does $A\vec{y} = \vec{0}$? What is $A\vec{z}$? Then compute $A(a\vec{y}+b\vec{z})$ (distribute and simplify). Make sure you show each step of your work.]

Because of the previous problem, we say that the kernel is closed under linear combinations. We can't get out of the kernel by performing linear combinations of things that are in the kernel. This fact is true in any vector space.

> Any linear combination of vectors in a vector space will still be in the vector space. Vectors spaces are closed under linear combinations.

Conservation Laws through Eigenvectors and Kernels

Consider the following scenario. We attach a beam to a wall with a pin, and use pins to attach a support rod to the beam and wall. We then apply a force to the end of the beam. We'd like to understand the forces that the pins apply to the beam and rod. See figure 4.2. This is a typical problem that engineers encounter in first course in statics.

To solve this type of problem, engineers apply two conservation laws.

1. The first law is that the net force acting on the beam must equal the sum of individual forces acting on the beam. Since the beam is not moving (zero total net force), the sum of all forces acting on the beam is zero.

2. The second law is that the net torque (tendency to rotate) at every point must be zero. Engineers use the word "moment" to compute torques. The process involves picking any point in the system. The moment about this point contributed by force $\vec{F}_i$ whose displacement from the point is $\vec{d}_i$ is the cross product $\vec{d}_i \times \vec{F}_i$. If we sum these torques, it must equal zero or else the rigid body will rotate.

Engineers spend a year practicing these ideas, and become quite fast at solving these kind of computations. Let's walk through a computation, and then see how kernels, bases, and eigenspaces simplify the work and allow us to rapidly compute rather complex problems with ease.

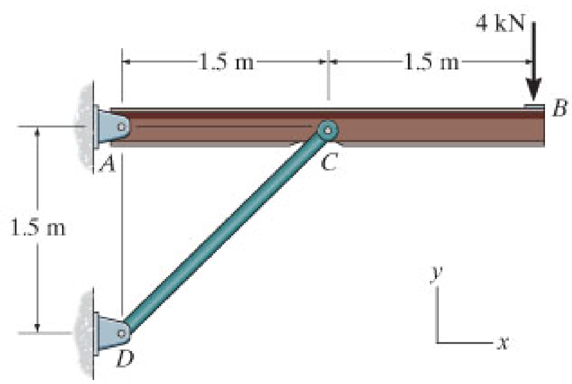


Figure 4.2: This is a typical example of a statics problem encountered by engineers. The goal is to understand the reactions of the pins at A and D. Courtesy of chegg.com.

Problem 4.18: Statics Consider the beam diagram in Figure 4.2. We'll first solve this exact problem for the reactions at A and D. Then we'll make the force B_y unknown, and vary the distances. On this problem your job is to write a system of equations, and then row reduce 4 matrices. Use software.

1. The pins at A and D apply the forces $\vec{F}_A = (A_x, A_y)$ and $\vec{F}_D = (D_x, D_y)$ to the system consisting of the beam and rod. The third force at B is $\vec{F}_B = (0, B_y) = (0, -4)$ kN. Summing the forces in the x direction gives the equation $A_x + D_x + 0 = 0$. What equation do you get from summing the y components?

2. We know $D_x = D_y$ because the rod is attached to the wall at a 45 degree angle. If instead the segment AD has distance d and the segment AC has distance c, explain why $D_x d - D_y c = 0$. [Think similar triangles.]

3. We can sum the moments about any point we choose. The simplest point in this problem might be C. Summing the moments about this point gives us $(-3/2)A_y + (3/2)B_y = 0$. We now have the system of equations

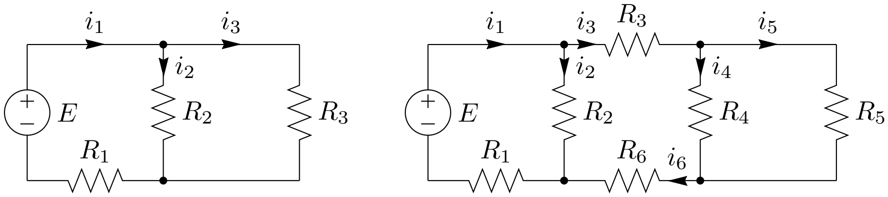
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -3/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ D_x \\ D_y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ (-3/2)(-4) \end{bmatrix}.$$

   Solve this system to get the reactions at A and D.

4. Instead of using $B_y = -4$, let's now assume that B_y is an unknown force (use it as the last variable so it will become the free variable). Show how to rewrite the equation above as the homogeneous matrix equation

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -3/2 & 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ D_x \\ D_y \\ B_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

   Solve this system. Give a basis for kernel. If $B_y = -8$, what is A_x?



Two Loop System Three Loop System

Figure 4.3: Electrical Circuit Diagrams.

5. Continue to assume that B_y is an unknown force. Let's change the distance AD to be 8, the distance AC to be 4, and the distance CB to be 5. We can then write the system of equations as

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 8 & -4 & 0 \\ 0 & -4 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ D_x \\ D_y \\ B_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the system by giving a basis for the kernel. If $B_y = -8$, what is A_x?

Kirchoff's Electrical Laws

Gustav Kirchoff discovered two laws of electricity that pertain to the conservation of charge and energy. To describe these laws, we must first discuss voltage, resistance, and current.

- Current is the flow of electricity. We'll often compare it to water flow or traffic flow.
- As a current passes across a conductor, it encounters resistance. Ohm's law states that the product of the resistance R and current I across a conductor equals the voltage V, i.e. $RI = V$. If the voltage remains constant, then a large resistance corresponds to a small current.
- A resistor is an object with high resistance which is placed in an electrical system to slow down the flow (current) of electricity. Resistors are measured in terms of ohms. The larger the ohms, the smaller the current.

Figure 4.3 illustrates two introductory electrical systems. In this diagram, wires meet at nodes (illustrated with a dot). Batteries and voltage sources (represented by or other symbols) supply a voltage of E volts. At each node the current may change, so the arrows and letters i represent the different currents in the electrical system. The electrical current on each wire may or may not follow the arrows drawn (a negative current means that the current flows opposite the arrow). Resistors are depicted with the symbol , and the letter R represents the ohms.

Kirchoff discovered two laws. They both help us find current in a system, provided we know the voltage of any batteries, and the resistance of any resistors.

1. Kirchoff's current law states that at every node, the current flowing in equals the current flowing out (at nodes, current in = current out).

2. Kirchoff's voltage law states that on any loop in the system, the directed sum of voltages supplied equals the directed sum of voltage drops (in loops, voltage in = voltage out). To use this law, pick a spot in the system. Then move around the system following a path that eventually gets you back to where you began (a closed curve). If you encounter a battery (a voltage source), then it counts as voltage in. If you encounter a resistor as you move with the current, then the voltage drop is Ri. If you encounter a resistor while moving opposite the current, then times by a negative to get a voltage drop of $-R_i$.

Let's use Kirchoff's laws to generate a system of equations for the two loop system. Remember that every time a current encounters a resistor, the voltage drop is $V = RI$, the product of the resistance and the current.

Problem 4.19 Consider the two loop system in figure 4.3. Assume that the voltage supplied from the battery E, as well as the ohms R_1, R_2, and R_3, on the resistors are known. The currents i_1, i_2, and i_3 are unknown.

1. Use Kirchoff's laws to explain how to obtain each of the equations below:

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ -i_1 + i_2 + i_3 &= 0 \\ R_1 i_1 + R_2 i_2 - E &= 0 \\ -R_2 i_2 + R_3 i_3 &= 0. \\ R_1 i_1 + R_3 i_3 - E &= 0. \end{aligned}$$

   [Hint: If you encounter a resistor while moving backwards along a loop, then the voltage drop becomes a voltage gain (times by a negative).]

2. Some of the equations above are linear combinations of the other equations. How could you obtain the 2nd and 5th as a linear combination of the others?

3. Suppose $R_1 = 2$, $R_2 = 3$, and $R_3 = 6$ ohms. Solve the system of equations above by row reducing an appropriate matrix (think of E as an unknown and find the kernel of a matrix). State a basis for the solutions.

4. If we know the power source is $E = 12$ V, what is i_1? If we measure the current in the first wire to be $i_1 = 10$ amps, then what is E?

Projections and Linear Regression

When we want to find the coefficients of an equation such as $y = mx + b$ or $y = ax^2 + bx + c$ that passes through several points, remember that the key idea is to write the linear system of equations $A\vec{x} = \vec{b}$ that we wish to solve. If the equation has no solution, then we multiply both sides by A^T and then solve the corresponding system. This gets us the linear combination of the columns of A that is closest to $\vec{b}$. We call this linear regression.

Problem 4.20 Consider the 5 points

$$(-1, 1), (0, -1), (1, -2), (2, -1), (2, -2)$$

1. Use linear regression to give an equation of the line $y = a_0 + a_1 x$ that best fits these 5 points. (Remember to set up the system $A\vec{x} = \vec{b}$, and then multiply on the left by the transpose A^T.)

2. Use linear regression to give an equation of the parabola $y = a_0 + a_1x + a_2x^2$ that best fits these 5 points. For this one, your system $A\vec{x} = \vec{b}$ looks like
$$\begin{bmatrix}1 & -1 & ?\\ 1 & 0 & ?\\ 1 & 1 & ?\\ 1 & 2 & ?\\ 1 & 2 & ?\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} = \begin{bmatrix}1\\ -1\\ -2\\ -1\\ -2\end{bmatrix}.$$
Just multiply both sides by A^T and then solve the system of equations. The coefficients are rather ugly (one is $-115/78$).

3. Use linear regression to give an equation of the cubic $y = a_0 + a_1x + a_2x^2 + a_3x^3$ that best fits these 5 points. Your answer should have some 1/12ths in it. Graph your solution.

4. Why is there no quartic that passes exactly through these points?

When you finish this problem, you should have three setups of the form $A\vec{x} = \vec{b}$. You should also show what equation you get after multiplying by A^T on both sides. Then show the rref of the resulting system, and write the equation of the line, parabola, and cubic that you obtain.

When the number of points matches the number of unknown coefficients, we can find an equation of the model without using linear regression. To organize our work, let's first standardize the notation. Rather than writing $y = mx + b$, let's write $y = a_0 + a_1x$ (where $a_0 = b$ and $a_1 = m$). For a parabola, let's write $y = a_0 + a_1x + a_2x^2 = \sum_{k=0}^{2} a_kx^k$. We can now write any polynomial in the form
$$y = a_0 + a_1x + \cdots + a_nx^n = \sum_{k=0}^{n} a_kx^k.$$
By standardizing the coefficients, we can use summation notation to express any degree polynomial by changing the n on the top of the summation sign.

Problem 4.21 Answer the following by row reducing an appropriate matrix (just use software). [Hint: Each point produces an equation.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1x$ that passes through the points $(1, 2)$ and $(3, 5)$. [We could have used m and b, but I chose to use a_0 and a_1 so we can see how this generalizes to all dimensions.]

2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1x^1 + a_2x^2$ that passes through the points $(0, 1)$, $(2, 3)$, and $(1, 4)$. [Hint: The second point produces the equation $3 = a_0 + a_1(2) + a_2(2)^2$.]

3. Give an equation of a cubic polynomial $y = a_0 + a_1x^1 + a_2x^2 + a_3x^3$ that passes through the four points $(0, 1)$, $(1, 3)$, $(1, 4)$, and $(2, 4)$. [You should have a linear system with 4 equations and 4 unknowns.]

4. Suppose that we collect the 6 data points $(1, 1)$, $(2, 3)$, $(-1, 2)$, $(0, -1)$, $(-2, 0)$, $(3, 1)$, and we would like to find a polynomial that passes through all 6 points. State the degree n of this polynomial. Then find the coefficients $a_0, a_1, \ldots, a_n$ of this polynomial. Use technology to do your row reduction. When you present in class, show us the matrix you entered into a computer, and then show us the reduced row echelon form together with the polynomial.

Visualizing Linear Transformations between Vector Spaces

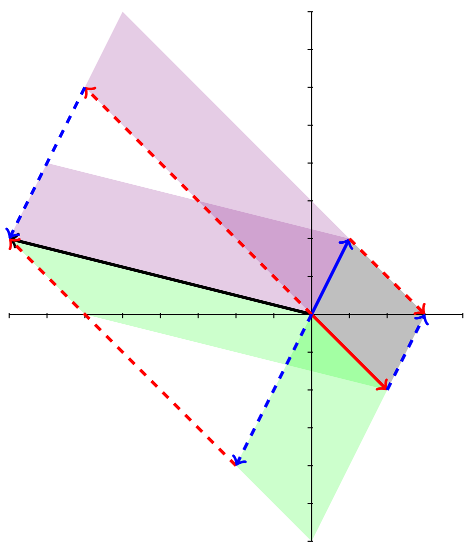
Cramer's Rule

Gabriel Cramer developed a way to solve linear systems of equations by using determinants. For small systems, the solution is extremely fast. For large systems the method looses its power because of the complexity of computing determinants. However, when the coefficients in the system are variables, Cramer's rule provides an extremely fast algorithm for obtaining solutions. I'll remind you occasionally throughout the problem set to apply Cramer's rule when the problem involves variable coefficients.

Problem 4.22: Cramer's Rule Our goal on this problem is to find a quick way to solve the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Let's look at an example and from it develop a general rule. Let $\vec{v}_1 = (a_{11}, a_{21}) = (2, -2)$ and $\vec{v}_2 = (a_{12}, a_{22}) = (1, 2)$, so $A = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$. If we know that $x_1 = -3$ and $x_2 = -2$, the we have $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 = (-8, 2)$. In the picture below, the solid red vector is $\vec{v}_1$, the solid blue vector is $\vec{v}_2$, and the solid black vector is $\vec{b}$. Use the picture below, to answer the questions that follow.



[Hint: Each question can be answered by thinking about determinants as areas.]

1. Explain why $x_1 \begin{vmatrix} \vec{v_1} & \vec{v_2} \end{vmatrix} = \begin{vmatrix} x_1\vec{v_1} & \vec{v_2} \end{vmatrix}$.

2. Now explain why $\begin{vmatrix} x_1\vec{v_1} & \vec{v_2} \end{vmatrix} = \begin{vmatrix} \vec{b} & \vec{v_2} \end{vmatrix}$. [Hint: Why do the two purple parallelograms have the same area?]

3. Finally, solve for x_1 to show that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Remember that when we put vertical bars on a matrix, that means we compute the determinant.

4. In a similar fashion, show that

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

5. Consider the system of equations $x+2y=3, 4x+5y=6$. Use the formulas you just developed to solve this system. You'll need to compute three determinants.

The previous problem is a proof by picture of Cramer's rule in 2D. The proof of the theorem is similar in all dimensions. The key idea is to connect determinants to area. Here's a formal statement of Cramer's Rule.

Theorem 4.8 (Cramer's Rule)**.** *Consider the linear system given by $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \vec{v}_n \end{bmatrix}$ is an n by n matrix whose determinant is not zero. Let $D = |A|$. For each i, replace vector $\vec{v}_i$ with $\vec{b}$, and then let D_i be the determinant of the corresponding matrix. The solution to the linear system is*

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \cdots \quad x_n = \frac{D_n}{D}.$$

For the 2 by 2 system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

Cramer's rule states the solution is (provided $|A| \neq 0$)

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Projections and Linear Regression

Problem 4.23 Solve the following. [Hint: Because the problem involves variable points, Cramer's rule will be much faster than row reduction.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1 x$ that passes through the points (x_1, y_1) and (x_2, y_2).

2. Use Cramer's rule to state the coefficients a_1 of a parabola $y = a_0 + a_1 x^1 + a_2 x^2$ that passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). You could similarly find a_0 and a_2, but don't worry about it.

Can you think of any conditions where your solutions above will not be valid?

We've seen how to linear regression to find an equation of lines, parabolas, cubics, and more that best fit several data points. The key is to set up a system which has no solution, multiply both sides on the left by the transpose, and then solve. Let's use this idea to obtain a general solution for finding an equation of the linear regression line that best approximates some arbitrary points.

Problem 4.24 Consider the five points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

We would like to find an equation of the least squares regression line $y = a_0 + a_1 x$ that best fits these points. Set up the matrices A, $\vec{x}$, $\vec{b}$, and A^T. Multiply together $A^T A$ and $A^T \vec{b}$ (your result should involve sums of the form $\sum x_i$, $\sum y_i$, $\sum x_i y_i$, and $\sum x_i^2$). Then solve the equation $A^T A\vec{x} = A^T\vec{b}$ and state the coefficients a_0 and a_1.

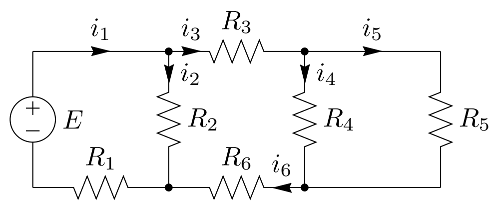
[Hint: Since the system involves variable coefficients, try using Cramer's rule. It should kick out the solution almost instantly with 3 two by two determinants. One of these determinants should be $(5)\left(\sum x_i y_i\right) - \left(\sum x_i\right)\left(\sum y_i\right)$.]

The formula you developed above is the formula found in software programs. It's also the formula you'll find in statistic textbooks, high school textbooks, online help sites, etc. They just change the 5 to an n.

Conservation Laws through Eigenvectors and Kernels

Let's return to another problem involving Kirchoff's electrical laws.

Problem 4.25 Consider the three loop system below.



Assume that the voltage supplied from the battery E and that the ohms R_j on the resistors are known. The currents are unknown. Even though E is known, treat it as an unknown so that it can act as the free variable in our final solution.

1. There are 4 nodes in this system. Write the 4 equations we obtain from Kirchoff's current law (flow in equal flow out at a node).

2. There are three inner loops in the system above. Write the equations formed by going around each inner loop using Kirchoff's voltage law (current in equals current out along any loop). As a reminder, here's how to get the equation from the middle loop. Start at the node in the upper left corner and move clockwise. We encounter R_3 while moving with i_3. We then move down i_4 and encounter R_4. Along i_6 at the bottom we move left and encounter R_6. We then move up (against) i_2 and encounter R_2. Our equation is

$$-R_2 i_2 + R_3 i_3 + R_4 i_4 + R_6 i_6 = 0,$$

where the negative on $R_2 i_2$ comes because we encountered R_2 while moving against the flow of i_2.

3. You should have 7 equations with 7 unknowns (treating E as the last unknown). Write your system of equations in the form $A\vec{x} = \vec{0}$. Your matrix will have R_i's in it, lots of zeros, and some 1's and -1's.

4. If $R_1 = 1$, $R_2 = 1$, $R_3 = 1$, $R_4 = 1$, $R_5 = 1$, $R_6 = 1$, find the unknown currents by finding an eigenvector of A corresponding to $\lambda = 0$ (i.e., give a basis for the eigenspace $E_A(0)$, which just means find the kernel of A or just solve the system).

5. If $E = 12$ V, what is i_1? If $i_1 = 12$ amps, what is E?

Visualizing Linear Transformations between Vector Spaces

We've been using linear combinations to organize almost all our work. The solutions to $A\vec{x} = \vec{b}$ are always a linear combination of some vectors. The matrix product $A\vec{x}$ is a linear combination of the columns of A. We can use the rref of a matrix to write each column as a linear combination of the pivot columns. Once we have a basis for the kernel, every other solution is a linear combination of these basis vectors. The list goes on.

You've been using operations that preserve linear combinations for quite some time. This next problem has you show this.

Problem 4.26: Optional Complete the following:

1. If we think of A as a coordinate map $T(\vec{x}) = A\vec{x}$, then does
$$A(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1A(\vec{x}_1) + c_2A(\vec{x}_2)?$$
Explain. (Your answer can be really short). This shows that a matrix coordinate transformation preserves linear combinations of vectors.

2. Explain why $\frac{d}{dx}(c_1f_1(x) + c_2f_2(x)) = c_1\frac{d}{dx}(f_1) + c_2\frac{d}{dx}(f_2)$. What two differentiation rules are needed to explain why this is true? Once you are finished, you'll have shown that the derivative operator preserves linear combinations of functions.

3. Explain why $\displaystyle\int_a^b c_1f_1 + c_2f_2dx = c_1\int_a^b f_1dx + c_2\int_a^b f_2dx$. Again, this shows that the integral operator preserves linear combinations of functions.

4. Does the Laplace transform preserve linear combinations of functions?

Each of the examples above provided an example of a function, operation, or transformation that preserved linear combinations. When this occurs, we can perform the linear combination either before or after we perform the operation. Let's make a definition to isolate this pattern.

Definition 4.9: Function, Transformation, Operator. A function f has a domain D and range R. The domain D is the set of inputs to the function. The range is the set of outputs.

- When the domain D is a collections of vectors, we'll often say that f is a transformation of vectors and write $T(\vec{x})$ instead of $f(x)$. An example is $T(\vec{x}) = A\vec{x}$ where A is a matrix.

- When the domain D is a collection of functions, we'll often say that f is an operator on functions and write $L(g)$ instead of $f(g)$. An example is $L(g) = \frac{d}{dx}g$ of $L(g) = \int_a^b gdx$.

The words function, transformation, and operator are all synonyms. We just typically use transformation to talk about functions when the domain is vectors, and operator to talk about functions when the domain is functions.

Definition 4.10: Linear function, Linear Transformation, Linear Operator. When the domain D and range R of a function (transformation, operator) are vector spaces (so we can perform linear combinations), then we say that the function f, transformation T, or operator L is linear if it preserves linear combinations. This means that

$$\begin{aligned} f(c_1x_1 + c_2x_2) &= c_1f(x_1) + c_2g(x_2) \quad \text{or} \\ T(c_1\vec{x}_1 + c_2\vec{x}_2) &= c_1T(\vec{x}_1) + c_2T(\vec{x}_2) \quad \text{or} \\ L(c_1f_1 + c_2f_2) &= c_1L(f_1) + c_2L(f_2). \end{aligned}$$

We can apply linear combinations either before or after we apply the function.

In problem 4.26, we showed that $T(\vec{x}) = A\vec{x}$ is a linear transformation and that the derivative, integral, and Laplace transform are linear operators. We can differentiate a sum by differentiating each piece separately (term-by-term differentiation) and we can pull constants out. Similarly, we can integrate term-by-term, and pull constants come out. These are precisely the key properties behind a linear function.

> If you ever find yourself saying, "Just do each part individually," chances are pretty high that you are using linearity.

If A is a matrix, then the product $A\vec{x}$ is a linear transformation. We'll often write this as $T(\vec{x}) = A\vec{x}$. Do you remember Candice's treasure map in Problem 4.6. Once she knew how to locate 2 linearly independent object on her map (the two trees), she could translate the entire map. Once we understand how the map transforms a basis for the domain, we understand the entire linear transformation.

Problem 4.27: Optional Suppose that we have a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. Since we are mapping vectors from 3D to 2D, we could think of this as a way of portraying a three dimensional world on a flat 2D screen (so computer animation).

We've been told that $T(1, 0, 0) = (1, 3)$, $T(0, 1, 0) = (-2, 4)$, and that $T(1, 1, 1) = (3, 1)$.

1. Show that $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$ are a basis for $\mathbb{R}^3$.
2. Write $(0, 0, 1)$ as a linear combination of $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$, and then use the fact that T is linear to compute $T(0, 0, 1)$.
3. Since $(x, y, z) = (1, 0, 0)x + (0, 1, 0)y + (0, 0, 1)z$, and we know T at each of these three vectors, compute $T(x, y, z)$.
4. Find a matrix A so that $T(x, y, z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
5. Find the kernel of the linear transformation T. Ask me in class to talk about what this means in terms of 3D animation.

Not every function, transformation, or operator is linear. The next problem has you distinguish between a few examples.

Problem 4.28: Optional Complete the following. When a variable is not listed as part of the domain, we assume it is constant.

1. Show that $f(x) = ax^2$ is not linear (a is a constant). [Does $f(c_1x_1+c_2x_2) = c_1f(x_1) + c_2f(x_2)$?]

2. Show that $f(a) = ax^2$ is linear (x is a constant). [Does $f(c_1a_1 + c_2a_2) = c_1f(a_1) + c_2f(a_2)$?]

3. Consider $f(x) = mx + b$. Show that f is not a linear function of x.

4. Consider $f(m, b) = mx + b$. Show that f is a linear function of m and b. [Does $f(c_1(m_1, b_1) + c_2(m_2, b_2)) = c_1f(m_1, b_1) + c_2f(m_2, b_2)$?]

5. Which do you think is linear, $f(x) = ax^2+bx+c$ or $f(a, b, c) = ax^2+bx+c$?

Wait! So $f(x) = mx + b$ is not linear? Ask me about this in class.

This last examples explains why we use the phrase "linear" regression to find the coefficients of any degree polynomial that passes through some given data points.

We'll come back to linear transformations all semester long. We'll soon see that solving differential equations requires that we find the kernel of a linear operator.

Conservation Laws through Eigenvectors and Kernels

Problem 4.29: Google PageRank (Thanks to David Stowell for this problem) The Google Search Engine uses an algorithm called PageRank. The basic idea is that the world wide web contains a number of documents with links connecting them all. Each document is ranked according to its importance. A document's importance score depends on how many other pages have links pointing to it. To fix ideas,suppose that we have four pages in our web: P_1, P_2, P_3, and P_4. Now suppose that this web has the following links:

Many people think that we use the word PageRank because we are ranking web pages. The name comes from its creator, Larry Page. You can read more via a Google search (ironic), or with the article http://epubs.siam.org/doi/abs/10.1137/050623280.

- P_1 has outgoing links to all other pages.
- P_2 has outgoing links to P_3 and P_4.
- P_3 has outgoing links to P_1.
- P_4 has outgoing links to P_1 and P_3.

To determine the importance of a particular page, we simply need to count the number of times all the other pages have voted for that page. In addition, each page has only one vote, or point, to give. It can give that one point to one page, by voting for only one page, or it can also choose to divide its vote among all the pages it votes for. In our example above, P_1 has two incoming links, called backlinks. Its importance score we'll denote by x_1 and we compute it with $x_1 = (1)x_3 + \frac{1}{2}x_4$. Notice that the only links coming into P_1 are from P_3 and P_4. Moreover, P_3 only votes once, while P_4 splits its vote in two ways – half of its vote goes to P_1, the other half to P_3.

1. Obtain an equation for x_2, x_3, and x_4, similar to the one above. Then write your system of equations in the form
$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ 1/3 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

2. Look at the structure of the matrix. In particular, what do you notice about the columns of the matrix?

3. Notice that the above equation can be written as $A\vec{x} = \lambda\vec{x}$. What is the eigenvalue λ? (If we write $y = x$, then the slope is there, even though it appears to be missing.)

4. Compute the eigenvector associated with this eigenvalue. From your computation, which page is the most important?

The world wide web consists of billion to trillions of pages. Modern computers can find eigenvectors of this size of a matrix extremely quickly.

Problem 4.30: Markov Process Suppose we own a car rental company which rents cars in Idaho Falls and Rexburg. The last few weeks have shown a weekly trend that 60% of the cars which are rented in Rexburg will remain in Rexburg (the other 40% end up in Idaho Falls). About 80% of the cars which are rented in Idaho Falls will remain in Idaho Falls (the other 20% end up in Rexburg).

1. If there are currently 60 cars in Rexburg and 140 cars in IF, how many will be in each city next week? If this trend continues, how many will be in each city in 2 weeks?

2. Let R_n and I_n be the number of cars in Rexburg and Idaho Falls, respectively, at the beginning of the nth week, where $R_0 = 60$ and $I_0 = 140$. We know that we can compute R_{n+1} by summing of 60% of R_n and 20% of I_n. This gives us the equation $R_{n+1} = 0.6R_n + 0.2I_n$. Write a similar equation for I_{n+1} and then organize your work into the matrix form
$$A\begin{pmatrix}R_n\\ I_n\end{pmatrix} = \begin{pmatrix}R_{n+1}\\ I_{n+1}\end{pmatrix}.$$
You can check your work by computing $A\begin{pmatrix}R_0\\ I_0\end{pmatrix} = \begin{pmatrix}R_1\\ I_1\end{pmatrix}$, which you computed above.

3. We would like to know if the number of cars will stabilize in each city. This would mean that if the current week's car totals are R and I, then we could find the next week's totals by solving the system
$$A\begin{pmatrix}R\\ I\end{pmatrix} = \begin{pmatrix}R\\ I\end{pmatrix}.$$
The totals don't change, so we call this a steady state solution. Find the steady state solution by solving $A\begin{pmatrix}R\\ I\end{pmatrix} = \begin{pmatrix}R\\ I\end{pmatrix}$.

4. In the long run, what proportion of the cars will end up in Rexburg?

5. Because the system $A\begin{pmatrix}R\\ I\end{pmatrix} = \begin{pmatrix}R\\ I\end{pmatrix}$ had a nonzero solution, we know something about the eigenvalues of the matrix A. Can you spot an eigenvalue of A without doing any computations?

Recall that an eigenvalue satisfies the equation $A\vec{x} = \lambda\vec{x}$.

(We'll answer 4 and 5 in class if you are unable. The key parts are 1-3.)

In the problem above, each week we could assign a car a state (Rexburg or IF). The matrix A above helped us get from one state to another. Other examples of states are "open" or "closed" in an electrical circuit, or "working properly" and "working improperly" for operation of machinery at a manufacturing facility. Stock market analysts use Markov processes and a generalization called stochastic processes to make predictions about future stock values. A car rental company which rents vehicles in different locations can use a Markov Process to keep track of where their inventory of cars will be in the future. Imagine if you worked for Alamo and had thousands of car rental spots. Knowing where your cars will end up will let you know where to hire drivers, so you can move the cars to where they are needed.

We call the matrix A in a Markov process a transition matrix. It's the matrix which tells you how to move from the current state $\vec{x}_n$ to the next state $\vec{x}_{n+1}$. This means we have

$$\begin{aligned}
\vec{x}_1 &= A\vec{x}_0 \\
\vec{x}_2 &= A\vec{x}_1 = A(A\vec{x}_0) = A^2\vec{x}_0 \\
\vec{x}_3 &= A\vec{x}_2 = A(A\vec{x}_1) = \cdots = A^3\vec{x}_0 \\
\vec{x}_4 &= A\vec{x}_3 = A(A\vec{x}_2) = \cdots = A^4\vec{x}_0 \\
&\vdots
\end{aligned}$$

You can find the nth state by computing $\vec{x}_n = A^n\vec{x}_0$. We just raise the matrix to a power, and times by the initial state. The next problem has you examine what happens when you raise a matrix to a power.

Problem 4.31 Raising a matrix to a power A^n can be rather time consuming. There's a really simple way to do it if you know the eigenvalues and eigenvectors. First write $AQ = QD$ and then solve for A. We can then write $A^2 = AA = (QDQ^{-1})(QDQ^{-1})$.

1. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Compute D^2, D^3, and D^n. Make a guess for $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^n$.
2. Explain why $A^2 = QD^2Q^{-1}$. (See the margin for a hint.) Then explain why $A^3 = QD^3Q^{-1}$ and $A^n = QD^nQ^{-1}$.
3. Suppose that the eigenvalues of A are $\lambda = 1$ and $\lambda = 1/2$, with corresponding eigenvectors $(1, 2)$ and $(3, 4)$. Explain why $\lim_{n\to\infty} D^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and then compute $\lim_{n\to\infty} A^n$.

We know $A^2 = QDQ^{-1}QDQ^{-1}$. Does anything cancel?

We'll soon start seeing partial fraction decomposition problems where the denominator consists of repeated roots. For example, we've already seen problems of the form

$$\frac{1}{s^3(s-1)} = \frac{As^2 + Bs + C}{s^3} + \frac{D}{s-1} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1}.$$

What form should we use for the partial fraction decomposition of $\dfrac{1}{s(s-1)^3}$? We could use

$$\frac{1}{s(s-1)^3} = \frac{A}{s} + \frac{Bs^2 + Cs + D}{(s-1)^3},$$

but then we can't simplify the complex fraction on the right. What if instead we shifted our polynomial so that it was centered at $s - 1$. All we would need to do is replace each s in the numerator with $s - 1$. This gives us

$$\frac{1}{s(s-1)^3} = \frac{A}{s} + \frac{B(s-1)^2 + C(s-1) + D}{(s-1)^3} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}.$$

This new option produces 4 quite simple fractions.

Problem 4.32: Partial Fractions with Repeated Roots We can write

$$\begin{aligned}\frac{1}{(x+1)^3(x-3)} &= \frac{A(x+1)^2+B(x+1)+C}{(x+1)^3}+\frac{D}{x-3}\\ &= \frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{x-3}.\end{aligned}$$

1. Multiply both sides by the denominator of the original. Use software if needed to expand the right hand side. Then set up a system of equations by equating coefficients. Finally, solve this system for the unknown constants A, B, C, and D. Show us the matrix you row reduced, and the rref.

2. If instead we wanted to solve the partial fraction decomposition problem

$$\frac{mx^3+nx^2+px+q}{(x+1)^3(x-3)} = \frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{x-3}$$

   where we treated m, n, p, q as free variables, what 4 by 8 matrix A should we row reduce to solve the matrix equation $A\vec{x} = \vec{0}$. Row reduce this matrix, and then state A, B, C, and D in terms of m, n, p, q.

Visualizing Linear Transformations between Vector Spaces

Problem 4.33: Optional Let $A = \begin{bmatrix}1 & 3 & 4\\ -2 & 0 & -2\\ 0 & 1 & 1\end{bmatrix}$ and let T be the transformation $T(\vec{x}) = A\vec{x}$.

1. What are $f(1,0,0)$, $f(0,1,0)$, $f(0,0,1)$, and $f(2,3,0)$?

2. Is T linear? [Does $T(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1T(\vec{x}_1) + c_2T(\vec{x}_2)$?]

3. Find (x, y, z) such that $f(x, y, z) = (5, -2, 1)$, or explain why it is not possible.

4. The set of possible outputs of T is an object in 3D. It is the span of the columns of A. Describe that object (is it a line, a plane, all of space, something else). [Hint: row reduce the matrix. How many pivots are there.]

5. Find the kernel of T, i.e. solve $T(\vec{x}) = \vec{0}$. [Your rref above should give this to you.]

Matrices provide us with the key examples to understanding linear transformations. However, a matrix by nature requires that we look at functions between finite dimensional spaces. The key linear transformations we will study throughout the semester will involve infinite dimensional spaces (like the space of all differentiable functions). Most of the ideas we have learned will still be useful to us as we explore functions between infinite dimensional vector spaces. Near the end of the semester, we'll even start discussing eigenvalues and eigenfunctions of linear transformations between infinitely dimensional vector spaces. This is where most modern innovations come from. You'll explore these concepts in greater detail in future classes.

Conservation Laws through Eigenvectors and Kernels

Problem 4.34 In a certain town, there are 3 types of land zones: residential, commercial, and industrial. The city has been undergoing growth recently, and the city has noticed the following 5 year trends.

- Every 5 years, they've notice that 10% of the residential land gets rezoned as commercial land, while 5% of the residential land gets rezoned as industrial. The other 85% of residential land remains residential.
- For commercial land, 70% remains commercial, while 10% becomes residential and 20% becomes industrial.
- For industrial land, 60% remains industrial, while 25% becomes commercial and 15% becomes residential.
- Currently the percent of land in each zone is 40% residential, 30% commercial, and 30% industrial.

Let's assume that these trends continue over an extended period of time.

1. The current state is $\vec{x}_0 = (40, 30, 30)$. After 5 years, what percentage of land will be zoned residential? Commercial? Industrial? Answering this question should give you the transition matrix A so that $\vec{x}_1 = A\vec{x}_0$.
2. Use software to find $\vec{x}_2$, $\vec{x}_3$, and $\vec{x}_4$ (the land use percentages after 10, 15, and 20 years).
3. Find the steady state solution to this Markov Process by solving $A\vec{x} = 1\vec{x}$ (i.e., the eigenvector corresponding to the eigenvalue $\lambda = 1$.)

Wrap up

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 4.35: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

Extra Practice

Please use the problem list below to find extra practice problems to help you learn. You'll find the problems listed below at the end of Chapter 2 (pages 55-61, including solutions) in *Linear Algebra* by Ben Woodruff. This text is freely available online. The text also references Schaum's Outlines Beginning Linear Algebra by Seymour Lipschutz for even more practice.

- https://content.byui.edu/file/c2f91762-7a1e-4d0b-a1ae-8d5f5f548e17/1/341-Book.pdf

Concept	Suggested	Relevant
Kirchoff's Laws		1
Cramer's Rule		2
Interpolating Polynomials		3
Least Squares Regression		4
Partial Fraction Decomposition		5
Markov Process		6
2nd Derivative Test		7

Remember that you can check almost all of your work with technology. Use the following technology links to help you check your understanding.

- Sage RREF calculator

Chapter 5

Homogeneous ODEs

After completing this chapter, you should be able to:

1. Explain Hooke's Law in regards to mass-spring systems. Construct and solve differential equations that model spring motion, with or without the presence of a damper.
2. Speak using the vocabulary and language of higher order ODEs, such as homogeneous, linear, coefficients, superposition principle, basis of solutions, Wronskian, etc.
3. Use Laplace transforms and s-shifting to solve homogeneous IVPs.
4. Use eigenvalues and eigenvectors to solve homogeneous ODE's. Show how to solve IVPs with $\vec{y} = QDQ^{-1}\vec{y}(0)$.
5. Explain how to immediately jump from the characteristic polynomial to the solution of a homogeneous ODE.

Building Mathematical Models

In this chapter, we're going to learn how to solve a huge collection of higher order differential equations. Before diving into the details, let's make sure we know WHY we would even want to do so. If I knew you all had the same background, we could dive into lots of examples directly related to your field (you'll do that in future classes in your major). Since we have a diverse background in our class, we'll stick mostly to models that connect velocity, position, and acceleration. Before the next chapter ends, we'll add to this some information about electrical circuits.

For our first model, let's look at how we can obtain the position of an object in projectile motion from knowledge about the acceleration and velocity. You've solve this problem before, but the solution required neglecting air resistance.

Example 5.1. In multivariate calculus, we encountered the differential equation $y'' = -g$. In this differential equation, the only force $F_T = my''$ acting on an object in projectile motion is the force of gravity $F_G = -mg$. Equating these two gives us the ODE $my'' = -mg$, or just $y'' = -g$. If we have initial position $y(0) = y_0$ and initial speed $y'(0) = v_0$, then the solution is $y = -\frac{1}{2}gt^2 + v_0t + y_0$. We found that solution by integrating twice.

We don't have to neglect air resistance anymore. We could talk about sky diving (risky), dropping bombs (deadly), throwing math books off a roof (illegal), putting a satellite into geosynchronous orbit (useful), or dropping a pebble from the top of a waterfall (head to Yellowstone and try it - sounds like we need a field trip). The next problem asks you to revisit the example above, but now add in air resistance.

Problem 5.1 Joe hikes up to the top of Lower Falls in Yellowstone. His hope is to gauge the height h of the waterfall. He plans to drop a pebble from the top, and time how long it takes for the pebble to hit the ground. He'll need a model that predicts the height of the pebble at any time t.

For this to work, Joe has to make some assumptions. His assumptions might be way off, but that's how science works. We start with assumptions and then turn those assumptions into differential equations. Here's what Joe assumes:

- He assumes Newton's second law of motion, namely that $F = ma$ (the total force is the mass times the acceleration).
- He assumes that the total force is comprised of two parts.
- The first force F_G comes from a constant acceleration due to gravity. He assumes that gravity is constant $a = -g$. The negative sign comes because the acceleration causes a decrease in height.
- The second part comes from air resistance. He assumes that the faster the pebble goes, the greater this force will be. If the pebble's speed were to double, then this force should double. So he assumes that the force due to air resistance F_R is proportional to the pebble's velocity.

Let $y(t)$ represent the height, above ground, of the pebble after t seconds. Use Joe's assumptions to answer the following:

1. Rewrite Newton's second law of motion in terms of y, y', and/or y''.
2. What is the constant force F_G due to gravity?
3. Rewrite Joe's assumption about air resistance in terms of y, y' and/or y''.
4. The total force F is the sum of the two forces, i.e. we can write $F = F_G + F_R$. Use this fact, together with your answers from the previous parts, to obtain a second order ODE. You don't have to solve the ODE, rather you just need to obtain it.
5. If we let $y' = v$ so that $v' = y''$, then show how to rewrite your ODE above in the matrix form

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -k/m \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix}.$$

If you need any hints, try searching the web for "modeling motion if we assume that air resistance is proportional to speed."

Congrats. You've just set up your first second order ODE. Let's now look at another position/velocity/acceleration model, but this time related to springs. We'll start by considering the following scenario. We attach an object with mass m to a spring. We place the spring horizontally, and put the mass on a frictionless track. We let go of the object, and allow it to come to rest. We'll use the function $x(t)$ to keep track of the position of the spring at any time t, with $x = 0$ corresponding to equilibrium (the mass is at rest). Robert Hooke (1635 – 1703) developed the following law, called Hooke's law:

In the next chapter, we'll hang the spring from a ceiling. In this case, we'll have an additional force $F_g = -mg$ acting on the spring.

The force needed to extend (or compress) a spring a distance x is proportional to the distance x. Note that the force acts opposite the displacement.

Problem 5.2 Read the preceding paragraph. Then answer the following:

- Draw a picture of a horizontal track. On the left end of the track, put a wall. Put an object, like a square block, in the center of your track and draw a spring that connects the wall to the block.
- Explain why $mx''(t) = -kx(t)$. We generally just write $mx'' = -kx$ (the t is assumed).
- If it takes 8 N = 8kg m/s^2 to move an object whose mass is 4 kg about .3 m, what is the spring constant k? How far would a 12 N force cause the same object to move?

Hooke's law is not a perfect model for all springs, but it does a good job for most, provided the displacement is not too large. If the displacements are too large, then the spring may deform, which changes the properties of the spring in all future computations. If you take your car out onto extremely bumpy roads, and purposefully hit some nasty bumps, you could permanently damage the shocks. In this case, you would want to replace your springs.

Every linear spring has a spring constant k. This constant has many names, such as the spring modulus, Young's modulus, Young's constant, and more. The next problem shows you how to obtain the spring constant k.

Problem 5.3 You attach a spring to the ceiling. You attach a mass of 10 kg on the end, and the spring elongates 3 cm.

1. You now attach a mass of 20 kg. How long will the spring elongate?
2. What is the spring constant k? Give the units.
3. We attach a different spring, and hang the same 10 kg on the end, but this time the spring elongates 2 cm. Is the spring constant larger or smaller?
4. If a spring has really large modulus, will it be easy or hard to elongate it?

We need one more model before we start solving some ODEs. We'll use the exact same spring model as before. Place a horizontal spring whose modulus is k on a frictionless track. Attach an object whose mass is m to the end of the spring. We now place the entire mass-spring system underwater. When it was exposed to air, we neglected air resistance. Now we'll have to take resistance into account.

We don't have to place the spring underwater to get the same affect. We could use a dashpot to resist the motion. One type of dashpot is a cylindrical tube placed around a cylindrical object, so that as the object moves, its sides come in contact with the dashpot, resulting in friction that resists motion. See Wikipedia for more info.

Problem 5.4 When we have no resistance, the mass-spring system ODE is $F_T = F_S$, or $mx'' = -kx$. Let's place the mass-spring system underwater, and assume that the liquid applies a resistive force that is proportional to the velocity of the object. If the object is resting, the liquid doesn't apply a force. If you double the speed, then the resistive force doubles. If you triple the speed, the resistive force triples.

1. Modify the second order ODE $mx'' = -kx$ to account for the resistive force of water.
2. Then show that we can write this 2nd order ODE as a system of ODEs in the matrix form
$$\begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}.$$

Solving Systems of ODEs with Eigenvalues and Eigenvectors

We will need to perform lots of partial fraction decompositions in the next few problems. It would be nice if we had a really easy way to perform them symbolically. Row reduction and Cramer's rule will be our friends. Start by completing the following problem, which is a slight adaptation of a problem from the previous chapter.

Problem 5.5 Consider the partial fraction decomposition

$$\frac{8s+7}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

which we can rewrite in the form

$$8s+7 = A(s+3) + B(s-2).$$

1. Complete this decomposition (find A and B). Use any method you like.
2. Now let's solve
$$\frac{cs+d}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$
Rather than thinking of c and d as known constants, let's make them variables in our linear system of equations. Our goal is to solve
$$cs+d = (A+B)s + (3A-2B) \quad \text{or} \quad (A+B-c)s + (3A-2B-d) = 0$$
which we can rewrite in the matrix form
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
This is a matrix equation of the form $M\vec{x} = \vec{0}$. Find a basis for the kernel of the matrix M.
3. Change the 2 by 4 matrix above to solve the partial fraction decomposition
$$\frac{cs+d}{(s-p)(s-q)} = \frac{A}{s-p} + \frac{B}{s-q}.$$
Find the kernel of this 2 by 4 matrix (use software). Then state A and B in terms of c, d, p, and q.
4. Use Cramer's rule to quickly solve
$$\begin{bmatrix} 1 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$
and verify that your answer in part 3 is correct. Under what circumstances would your answer not be valid?

Why did I save the last two columns for c and d?

You'll find that either row reduction with software that can handle symbolic entries (both Sage and Mathematica can do this), or using of Cramer's Rule, will greatly simplify lots of the work we need to do below. You've proven the following theorem.

Theorem 5.2. *If $p \neq q$, the solution to the partial fraction decomposition*

$$\frac{cs+d}{(s-p)(s-q)} = \frac{A}{s-p} + \frac{B}{s-q} \quad \textit{is} \quad A = \frac{-cp-d}{q-p}, \quad B = \frac{cq+d}{q-p}.$$

Problem 5.6 Consider the system of ODEs

$$\begin{aligned} x' &= 2x+3y \\ y' &= 5x+4y \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad \frac{d\vec{x}}{dt} = A\vec{x}.$$

1. Find the eigenvalues of A. Then find a basis for each eigenspace (i.e. find an eigenvector corresponding to each eigenvector). Pick vectors for your bases that do not have fractions in them (rescale if needed).

2. Recall that the Laplace transforms of each equation are

$$\begin{aligned} sX - x_0 &= 2X+3Y \\ sY - y_0 &= 5X+4Y \end{aligned} \quad \Rightarrow \quad \begin{aligned} -x_0 &= (2-s)X + 3Y \\ -y_0 &= 5X + (4-s)Y \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} -x_0 \\ -y_0 \end{bmatrix} = \begin{bmatrix} (2-s) & 3 \\ 5 & (4-s) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix},$$

   where X is the Laplace transform of $x(t)$ and Y is the Laplace transform of $y(t)$. Solve for X and Y (Cramer's rule should make this super fast). Do you see any connections between your solution here and your work with eigenvalues?

3. After factoring the denominators and setting up some partial fraction decompositions, show that

$$\begin{aligned} X &= \frac{sx_0 - 4x_0 + 3y_0}{(s-7)(s+1)} = \frac{A}{s-7} + \frac{B}{s+1} \\ Y &= \frac{sy_0 + 5x_0 - 2y_0}{(s-7)(s+1)} = \frac{C}{s-7} + \frac{D}{s+1} \end{aligned}$$

   Then complete both partial fraction decompositions and state the values of A, B, C, and D. See the margin for a hint.

To complete these partial fraction decompositions, use Theorem 5.2. Just label what c, d, p, q are, and the write down the solution. You should find that

$$A = \frac{-x_0(7) - (-4x_0 + 3y_0)}{-8} = \frac{3}{8}(x_0 + y_0),$$

$$B = \frac{x_0(-1) + (-4x_0 + 3y_0)}{-8} = \frac{1}{8}(5x_0 - 3y_0).$$

I'll let you get C and D.

4. Recall that the inverse Laplace transforms of X and Y are

$$\begin{aligned} x(t) &= Ae^{7t} + Be^{-1t} \\ y(t) &= Ce^{7t} + De^{-1t} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} e^{7t} + \begin{bmatrix} B \\ D \end{bmatrix} e^{-1t}$$

   How are the vectors (A, C) and (B, D) related to the eigenvectors in your bases from part 1?

The previous problem shows us that if we know the eigenvalues $\lambda_1 \neq \lambda_2$ and a corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$ of a 2 by 2 matrix A, then the solutions to $\frac{d\vec{x}}{dt} = A\vec{x}$ are of the form $\vec{x} = \vec{x}_1 c_1 e^{\lambda_1 t} + \vec{x}_1 c_2 e^{\lambda_2 t}$. Let's check this pattern one more time.

Problem 5.7 Consider a mass spring system where $m = 1$ kg, $c = 5$ kg/s, and $k = 6$ kg/s^2. The corresponding ODE from Problem 5.4 is $x'' + 5x' + 6x = 0$. If we let $v = x'$, then the second order ODE is the same as $v' + 5v + 6x = 0$. We can write all this as the first order system

$$\begin{aligned} x' &= v = 0x + 1v \\ v' &= -6x - 5v \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \quad \Rightarrow \quad \frac{d\vec{x}}{dt} = A\vec{x}.$$

1. Find the eigenvalues of A. Then give a basis for each eigenspace, avoiding fractions.

2. Compute the Laplace transforms of both equations. Show that
$$\begin{aligned} -x_0 &= -sX + 1V \\ -v_0 &= -6X + (-5-s)V \end{aligned}$$
and then solve for X and V. (Use Cramer's rule and Theorem 5.2.)

Remember that we use capital letters often to stand for the transformed function of the corresponding lowercase letter. So here X is the transform of x, and V is the transform of V.

3. Perform appropriate partial fraction decompositions on X and B. Use A and B as the numerators for X, and C and D for the numerators of V.
4. Compute the inverse Laplace transform of both X and V. You should at this point have $x = Ae^{-3t} + Be^{-2t}$ and $V = Ce^{-3t} + De^{-2t}$, where you solved for A, B, C, and D when you computed the partial fraction decompositions. See the margin for a hint.

One easy way to check if you are right on this problem is that v' should be x, which is why $C = -3A$ and $D = -2B$. This should also match your eigenvectors $(1, -3)$ and $(1, -2)$.

Problem 5.8 Consider the system of mixing tanks in Problem 2.29. There were two tanks. The first tank contained 6 lbs of salt in 10 gallons of water, and the second tank contained no salt in 20 gallons of water. We attached hoses to the tanks and transfered 2 gallon/minute of solution from tank 1 to tank 2, and vice versa. Our goal is to find the amount of salt in each tank at any time t.

1. If we let $y_1(t)$ and $y_2(t)$ represent the amount of salt in each tank after t minutes, then we've already shown that we can write this as the system of differential equations
$$\begin{aligned} y_1' &= -\tfrac{2}{10}y_1 + \tfrac{2}{20}y_2 \\ y_2' &= \tfrac{2}{10}y_1 - \tfrac{2}{20}y_2 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} -1/5 & 1/10 \\ 1/5 & -1/10 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \frac{d\vec{y}}{dt} = A\vec{y}.$$
Find the eigenvalues of A. Then give a basis for each eigenspace (find an eigenvector corresponding to each eigenvalue).
2. Based on our previous two problems, a general solution to this ODE should be $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix} c_1e^{*t} + \begin{pmatrix} * \\ * \end{pmatrix} c_2e^{*t}$, where the asterisks come from the eigenvalues and eigenvectors. Give the general solution by replacing the asterisks with appropriate values.
3. We were told that when $t = 0$, we have $y_1(0) = 6$ lbs and $y_2(0) = 0$ lbs. Use this information to find the scalars c_1 and c_2.

Did you notice on this previous problem that we completely skirted around Laplace transforms and partial fraction decompositions? Instead, we just used eigenvalues and eigenvectors to get a general solution. We found the scalars c_1 and c_2 only after writing down the general solution. We'll come back to this more soon.

The theory of homogeneous differential equations

In the previous section, we found that eigenvalues and eigenvectors played a huge role in solving first order systems of the form $\dfrac{d\vec{x}}{dt} = A\vec{x}$. The ODEs from the first problems we can write as
$$my'' + ky' = -mg, \quad mx'' + kx = 0, \quad \text{and} \quad mx'' + cx' + kx = 0.$$
If we divide each ODE by m, then we can write each ODE in the general form
$$y'' + p(t)y' + q(t)y = r(t).$$
This introduces our next definitions.

Definition 5.3: Linear, Constant Coefficient, and Homogeneous.

- If we can write an ODE in the form $y'' + p(t)y' + q(t)y = r(t)$, then we say the ODE is a second order linear ODE.
- The functions $p(t)$ and $q(t)$ we call the coefficients of the linear ODE.
- If the coefficients are constant, the we say the ODE is a constant coefficient linear ODE.
- When the right hand side equals zero, so $r(t) = 0$, then we say the linear ODE is homogeneous. Otherwise we say the ODE is non homogeneous.
- We use the words nth order linear ODE to talk about any ODE that we can write in the form
$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = r(t),$$
where $y^{(n)}$ is the nth derivative of y. We define coefficients, constant coefficient, homogeneous, and non homogeneous, in the same way.

We just introduced a few new words. In the problems that follow, let's practice using these words. The next problem has you explain why we use the word "linear."

Problem 5.9 Consider the second order ODE $y'' + 7y' + 6y = 0$.

- Why is this ODE linear? Modify it so it is no longer linear, and show us in class what would make it non linear.
- Is this ODE homogeneous? Explain.
- Is this ODE constant coefficient? Explain.

To solve second order linear homogeneous constant coefficient ODEs, we'll discover a pattern using the Laplace transform. In the previous chapter, we showed that
$$\mathscr{L}(y') = s\mathscr{L}(y) - y(0) = sY - y(0).$$
We need a rule for second derivatives. Repeated application of the single derivative rule will give you all the rules we need.

Problem 5.10 Show that under suitable conditions, we can compute the Laplace transform of the second derivative of y by using the formula
$$\mathscr{L}(y'') = s^2\mathscr{L}(y) - sy(0) - y'(0) = s^2Y - sy(0) - y'(0).$$
Then show that
$$\mathscr{L}(y''') = s^3\mathscr{L}(y) - s^2y(0) - sy'(0) - y''(0).$$
Conjecture a formula for the Laplace transform of the 7th derivative of y. [Hint: As stated in the paragraph before this problem, apply the rule $\mathscr{L}(y') = s\mathscr{L}(y) - y(0)$ multiple times.]

Solving Higher Order ODEs

We are now ready to solve a second order ODE with Laplace transforms.

Problem 5.11 Consider the IVP $x'' + 3x' + 2x = 0$, $x(0) = x_0$, $x'(0) = v_0$. This models a mass-spring system where we've attached a mass of $m = 1$ kg to a spring whose modulus is $k = 2$ kg/s^2. A dashpot has been added to provide friction with a coefficient of fraction $c = 2$ kg/s. We displace the spring right y_0 cm, and give it an initial velocity of v_0 cm/s.

1. Is the ODE linear? Is it homogeneous? Are the coefficients constant?
2. Compute the Laplace transform of both sides and show that
$$X = \frac{sx_0 + v_0 + 3x_0}{s^2 + 3s + 2}.$$
3. Use a partial fraction decomposition to write
$$X = \frac{A}{s+1} + \frac{B}{s+2},$$
where you give the constants A and B.

Feel free to use Cramer's rule, or Theorem 5.2.

4. Find the solution $x(t)$ to this IVP by computing the inverse Laplace transform of X.
5. How are solutions to $s^2 + 3s + 2 = 0$ connected to your solution?

Let try another problem, but this time let's not give any initial conditions. We should notice a connection between the zeros of a polynomial and the final solution.

Problem 5.12 Consider the ODE $y'' + 7y' + 10y = 0$. No initial conditions are given, so we need a general solution.

1. Is the ODE linear? Is it homogeneous? Are the coefficients constant?
2. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$. Since we don't have initial conditions, you'll have to use some place holder for $y(0)$ and $y'(0)$, like c and d. However, these variables c and d represent arbitrary constants.
3. If we used a partial fraction decomposition, explain why we could write
$$Y = \frac{A}{s+2} + \frac{B}{s+5}.$$
Without doing the partial fraction decomposition, explain why the general solution to $y(t)$ is a linear combination of e^{-2t} and e^{-5t}, i.e. explain why $y(t) = Ae^{-2t} + Be^{-5t}$?

The theory of homogeneous differential equations

In the two examples above, we took an ODE $ay'' + by' + cy = 0$, applied a Laplace transform, and obtained the polynomial $as^2 + bs + c$. The zeros of this polynomial seem to be intimately connected to the solution. Let's give this polynomial a name.

Definition 5.4: Characteristic Polynomial (Equation). Consider the homogeneous constant coefficient ODE $ay'' + by' + cy = 0$.

- The characteristic polynomial of our ODE is $as^2 + bs + c$. We could alternately use $a\lambda^2 + b\lambda + c$.
- The characteristic equation of our ODE is $as^2 + bs + c = 0$. We could alternately use $a\lambda^2 + b\lambda + c = 0$.
- For higher order ODEs, we define the characteristic polynomial and characteristic equation in the same way.

Some students of mine in previous years have created a verb to get from an ODE to its characteristic polynomial. They coined the phrase, "lambdacize the ODE." I think it describes the process perfectly.

With these new words, we now have the correct vocabulary to discuss solving ODEs. We noticed a pattern in the first few problems. From that pattern, we developed some new words. Now we can use these words to simplify our solution techniques.

Problem 5.13 For each ODE below, state the characteristic equation of the ODE, find the zeros of the characteristic polynomial, and then state a general solution to the ODE.

1. $y'' + 9y' + 20y = 0$, where y is a function of x. Note: your answer should look like $y(x) = c_1e^{?x} + c_2e^{?x}$.
2. $2x'' + 10x' + 12x = 0$, where x is a function of t.
3. $p'' - p' - 12p = 0$, where p is a function of q.

The definition of the characteristic equation allows us to alternately use the variable λ instead of s. We've briefly seen why this is so when we completed Problem 5.7. This next problem firmly connects what we are doing to eigenvalues and eigenvectors.

Problem 5.14 Consider the ODE $y'' + 9y' + 20y = 0$ from the previous problem. If we let $y_1 = y$ and $y_2 = y'$, then we can write the ODE in the form $y_2' + 9y_2 + 20y_1 = 0$. We can write this as the system of ODEs

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -20y_1 - 9y_2. \end{aligned}$$

1. Write the system above in matrix form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.
2. Find the eigenvalues and eigenvectors of A.
3. We showed in the previous problem that the solution to $y'' + 9y' + 20y = 0$ is $y = c_1e^{-4t} + c_2e^{-5t}$. We know that $y_2' = y' = -4c_1e^{-4t} - 5c_2c_2e^{-5t}$. We can organize this work the vector form
$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{\lambda_1 x} + c_2\begin{pmatrix} 1 \\ -5 \end{pmatrix}e^{\lambda_2 x}.$$
What does this this have to do with the eigenvalues and eigenvectors of the matrix?

Hint: Don't forget that you can write $y_1' = y_2$ as $y_1' = 0y_1 + 1y_2$. It's really easy to miss the 0 and 1 that are sitting in the problem.

We can now solve any 2nd order ODE provided the eigenvalues are real and distinct. What do we do when they are not distinct, or are complex? To answer these questions we need to learn a new Laplace transform rule that allows us to work with problems involving repeated roots. In other words, how do we find the inverse transform of any of the following:

$$\frac{1}{(s+3)^2}, \frac{1}{(s+3)^3}, \frac{1}{s^2+6s+25} = \frac{1}{(s+3)^2+4^2}, \frac{1}{s^2+6s-7} = \frac{1}{(s+3)^2-4^2}, \text{etc.}$$

The key is that we already know how to find the inverse transform of each of the following:

$$\frac{1}{(s)^2}, \frac{1}{(s)^3}, \frac{1}{(s)^2+4^2}, \frac{1}{(s)^2-4^2}, \text{etc.}$$

The more complicated versions are just shifts of the problems we already understand. If we already know how to compute the inverse Laplace transform of $Y(s)$, then what is the inverse Laplace transform of $Y(s-a)$ where we replace each s with $s-a$?

Problem 5.15: The s-shifting Theorem In this problem we'll develop a rule for the inverse transform of $Y(s-a)$.

1. We know that $Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty e^{-st}[y(t)]dt$. Replace s with $s-a$ and obtain a formula

   You'll want to rewrite $e^{-(s-a)t}$ as the product of two exponentials.

   $$Y(s-a) = \int_0^\infty e^{-st}[?]dt.$$

   This gives you a formula $\mathscr{L}\{?\} = Y(s-a)$. [Hint: The s-shifting theorem is now in Table 5.1. Tackle this part of the problem without referring to the table, and then check your answer before completing the last two parts.]

2. What is the inverse Laplace transform of $1/s^2$?

   What is the inverse Laplace transform of $1/(s-4)^2$ and $1/(s+5)^2$?

   Use software to check your answer. See the Laplace transform calculator or use Mathematica.

3. What is the forward Laplace transform of $\cos(bt)$ and $e^{at}\cos(bt)$?

   What is the forward Laplace transform of $e^{7t}t^3$ and $e^{-7t}t^3$?

   Again, you can use software to check your answer.

Before we get too far, let's practice the s-shifting theorem for Laplace transforms. To apply the s-shifting theorem, we'll need to become good at completing the square. If we know the transform is $\dfrac{2}{s^2+4}$, then the inverse transform is $\sin(2t)$. If we know the transform is $\dfrac{2}{(s+3)^2+4}$, then the inverse transform is $e^{-3t}\sin(2t)$. However, we would normally have a characteristic polynomial in the form $s^2+6s+13$, which we must complete the square on to obtain $(s+3)^2+4$. Once we complete the square, we can apply the s-shifting theorem immediately.

Problem 5.16 Complete the following:

$y(t)$	$Y(s)$	provided
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
y'	$sY - y(0)$	
y''	$s^2Y - sy(0) - y'(0)$	

$y(t)$	$Y(s)$	provided
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s > \lvert\omega\rvert$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s > \lvert\omega\rvert$
$y(t)$	$Y(s)$	
$e^{at}y(t)$	$Y(s-a)$	

Table 5.1: Table of Laplace Transforms

1. Find the Laplace transform of the following:
   (a) t^3 and t^3e^{4t}
   (b) $\cos(2t)$ and $e^{-3t}\cos(2t)$
   (c) $3\sin(7t)$ and $3e^{-5t}\sin(7t)$
2. Find the inverse Laplace transform of the following:
   (a) $\dfrac{3}{s^4}$ and $\dfrac{3}{(s-5)^4}$
   (b) $\dfrac{s+3}{(s+3)^2+4}$ and $\dfrac{1}{(s+3)^2+4}$
   (c) $\dfrac{s}{(s+3)^2+4}$. [Note: You might want to add and subtract 3 from the numerator, so that you have $s = (s+3) - 3$.

You can quickly check all your answers with software. See the Laplace transform calculator or use Mathematica.

Let's now tackle some problem where the characteristic equation does not have real zeros, or has repeated zeros.

Problem 5.17 Consider the ODE $x'' + 16x = 0$.

1. Compute the Laplace transform of both sides of the ODE and solve for $\mathscr{L}(x) = X$. You'll have $x(0)$ and $x'(0)$ in the numerator of your solution.
2. Compute the inverse Laplace transform of X. Your answer will involve $x(0)$ and $x'(0)$, and it should involve sines and cosines.
3. What is the characteristic polynomial of the ODE? Show that its roots are purely imaginary.

The previous problem showed us how to tackle a problem where the roots of the characteristic polynomial are purely imaginary. What do we do if the roots repeat, or if they are complex of the form $a \pm bi$? The next two problems address this.

Problem 5.18: Repeated Roots Consider the ODE $y'' + 6y' + 9y = 0$.

1. What are the zeros of the characteristic equation? From these zeros, guess a general solution. (It's OK if you're wrong.)
2. Compute the Laplace transform of both sides of ODE. Then solve for Y.
3. Using a partial fraction decomposition, we can write
$$Y = \frac{A(s+3)+B}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2}.$$
Explain why this means we know
$$y(t) = Ae^{-3t} + Bte^{-3t}.$$
4. If $y(0) = 7$ and $y'(0) = 0$, then what are A and B?

Problem 5.19: Complex Roots Consider the ODE $y'' + 4y' + 13y = 0$.

1. What are the zeros of the characteristic equation?
2. Compute the Laplace transform of both sides of ODE. Then solve for Y and complete the square of the denominator.
3. Using a partial fraction decomposition, we can write
$$Y = \frac{A(s+2)+B}{(s+2)^2+3^2} = \frac{A(s+2)}{(s+2)^2+3^2} + \frac{B}{(s+2)^2+3^2}.$$
Use the s-shifting theorem to solve for $y(t)$. Use software to check if you are correct (Wolfram Alpha can give you the solution to this one).

We can now solve EVERY second order homogeneous constant coefficient ODE. All we have to do is find the characteristic equation. The zeros unlock a general solution of the ODE.

Problem 5.20 Suppose that we have a second order ODE, and we have already computed the roots of the characteristic polynomial to be λ_1 and λ_2.

1. If $\lambda_1 = -3$ and $\lambda_2 = -5$, then $y(t) =$_______________.
If the roots are real and $\lambda_1 \neq \lambda_2$, then $y(t) =$_______________.
2. If $\lambda_1 = -3$ and $\lambda_2 = -3$, then $y(t) =$_______________
If the roots are real and $\lambda_1 = \lambda_2$, then $y(t) =$_______________.
3. If $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$, then $y(t) =$_______________.
If the roots are complex where $\lambda = a \pm bi$, then $y(t) =$_______________.
4. If $\lambda_1 = 5i$ and $\lambda_2 = -5i$, then $y(t) =$_______________.
If the roots are purely imaginary so that $\lambda = \pm bi$, then $y(t) =$_______________.

Problem 5.21 Complete each of the following:

1. Consider the ODE $y'' + 4y' + 3y = 0$. Find the characteristic polynomial, complete the square (if needed), and state a general solution.

2. Consider the ODE $y'' + 6y' + 9y = 0$. Find the characteristic polynomial, complete the square (if needed), and state a general solution.

What do you do with a repeated root?

3. Consider the ODE $y'' + 2y' + 5y = 0$. Find the characteristic polynomial, complete the square (if needed), and state a general solution.

Your answer should involve sines and cosines.

Problem: Optional Consider the ODE $ay'' + by' + cy = 0$.

1. Obtain the characteristic equation. Complete the square. State the zeros of the characteristic equation. [When you finish this problem, you will have proved the quadratic formula.]
2. If we let $y_1 = y$ and $y_2 = y'$, we obtain the system of ODE $y_1' = y_2$ and $ay_2' + by_2 + cy_1 = 0$. Write this system in the matrix form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (state the matrix A), and then find the eigenvalues of A.

Now that we have a general solution, let's show how to quickly obtain the solution to an IVP. The key principle, is to first obtain a general solution. Differentiate your general solution, and then use your initial conditions to find the unknown constants.

Problem 5.22 Consider the IVP $y'' + 6y' + 5y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Plug the initial conditions into both y and y' to solve for the unknown constants in your general solution.

Problem 5.23 Consider the IVP $y'' + 6y' + 9y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Use the initial conditions to solve for the unknown constants in your general solution.

Problem 5.24 Consider the IVP $y'' + 2y' + 5y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Use the initial conditions to solve for the unknown constants in your general solution.

Recall from the introductory examples that we can model the position of a spring using the ODE

$$mx'' + cx' + kx = 0$$

The constants m, c, and k are physical constants related to the mass-spring system.

- The mass of the object attached to the spring is m.
- The spring constant, or modulus, is k.
- The coefficient of friction of any attached dashpot is c. If no dashpot is attached, then we just let $c = 0$.

Problem 5.25 Attach a mass of m kg to a spring with modulus k kg/s^2.

1. Set up a differential equation whose solution gives the position $x(t)$ of the spring at any time t. Solve this ODE and give a basis for the set of solutions.

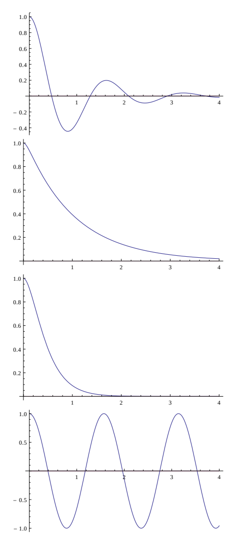
2. If we displace the mass x_0 cm from equilibrium, and give the object an initial velocity of v_0 cm/s to the right, then determine the position of the spring at any time t.

3. You should see that the solution is a linear combination of trig functions, hence will oscillate indefinitely. What is the period of oscillation?

4. If you doubled the spring constant k, would the period increase or decrease?

Problem: Optional Suppose we attach a mass of m kg to a spring with modulus k kg/s^2. We displace the object y_0 cm from the equilibrium position of the spring, and give the object an initial velocity of v_0 cm/s away from equilibrium. In the absence of friction, the mass-spring system will oscillate in a regular pattern. Give a formula for the amplitude of the oscillation. [Hint: If you write your solution in the form $y(t) = C\sin(\omega t + \phi)$, then you can quickly read off the amplitude. How do you write $y(t) = A\cos(bt) + B\sin(bt)$ in the form $C\sin(\omega t + \phi)$?]

Each of the problems above dealt with undamped motion, there was no friction to slow down the motion. What if the system includes a dashpot, something placed around the mass-spring system that adds friction to the system. Wikipedia has some excellent pictures of what a dashpot could look like. I like to think of an old screen door, and the cylindrical tube at the bottom of the door that helps close the door and prevent it from smashing little fingers. Ask me in class to show you a dashpot on our classroom door.



Problem 5.26 Recall from the introductory examples that we can model the position of a spring using the ODE $mx'' + cx' + kx = 0$. We now attach a mass of 1 kg to a spring with $k = 16$ kg/s^2. We extend the spring 1 cm and then release it. The spring is inside a dashpot, to add friction to the system, and the dashpot has a coefficient of friction equal to c kg/s.

1. If $c = 0$ then a basis for the general solution is $\cos 4t$ and $\sin 4t$. For each of $c = 2, 8, 17$, state a basis for the general solution (you'll see the irrational number $\sqrt{15}$ when $c = 2$).

2. The graph of the IVPs for $c = 0, 2, 8, 17$ are shown on the right. Match each graph to the correct coefficient of friction. Explain.

3. As the coefficient of friction increases, describe with a sentence or two what happens to the graph of the solution.

We already know that the general solution to the ODE $y'' + y = 0$ is $y = c_1\cos t + c_2\sin t$. In this problem, we'll discover a crucial identity that connects the trigonometric functions to complex exponentials.

Problem 5.27: Euler's Identity Consider the ODE $y'' + y = 0$. This models the undamped motion of a spring where $m = 1$ and $k = 1$.

1. Explain why both $y(t) = c_1\cos t + c_2\sin t$ and also $y(t) = c_3e^{it} + c_4e^{-it}$ are a general solution to this ODE.

2. If $y(0) = 1$ and $y'(0) = 0$, show that $y(t) = \cos t = \dfrac{e^{it} + e^{-it}}{2}$.

3. If $y(0) = 0$ and $y'(0) = 1$, show that $y(t) = \sin t = \dfrac{e^{it} - e^{-it}}{2i}$.

4. The equations $\cos t = \dfrac{e^{it} + e^{-it}}{2}$ and $\sin t = \dfrac{e^{it} - e^{-it}}{2i}$ show us how to write $\cos t$ and $\sin t$ as a linear combination of e^{it} and e^{-it}. Use these equations to explain why

$$e^{it} = \cos t + i \sin t.$$

This formula is called Euler's identity.

It's time to get back to $AQ = QD$, and show a powerful solution technique.

$\boxed{\textbf{Problem 5.28}}$ Two tanks are connected with hoses and pumps so that 3 gallons/second flows back and forth between the tanks. The first tank is a 60 gallon tank, with 2 lbs of salt inside. The second tank is a 90 gallon tank with 23 lbs of salt in it. Please find the amount of salt in each tank at any time t.

1. Write a linear system of ODEs in the form

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

whose solution will give the amount of salt in each tank at any time t.

2. Compute the eigenvalues and eigenvectors of A, and then write a general solution to this system of ODEs. Your solution should involve arbitrary constants c_1 and c_2.

3. Use the initial conditions to solve for c_1 and c_2.

4. Construct a graph that contains the the vector field representing the coefficient matrix and the parametric plot $\vec{y}(t) = (y_1(t), y_2(t))$ of your solution.

You can check your work with technology. Follow this link.

$\boxed{\textbf{Problem 5.29}}$ Consider the linear system of ODEs given by $y_1' = 2y_1 + y_2$ and $y_2' = 3y_1 + 4y_2$. Let $\vec{y} = (y_1, y_2)$. We can write this ODE in the form $\vec{y}' = A\vec{y}$, where $\vec{y} = (y_1, y_2)$.

1. Find the eigenvalues of A. Then give a basis for each eigenspace.

2. We know that we can write a general solution to this system of ODEs as

$$\vec{y} = c_1\vec{x}_1 e^{\lambda_1 t} + c_2\vec{x}_2 e^{\lambda_2 t}.$$

Find a 2 by 2 matrix Q so that we can write this solution in the form

$$\vec{y} = Q \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = QD\vec{c},$$

where we have $D = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

3. We now have $\vec{y} = QD\vec{c}$. When we let $t = 0$, explain why D equals the identity matrix. This means that $\vec{y}(0) = Q\vec{c}$. Solving for $\vec{c}$ gives us $\vec{c} = Q^{-1}\vec{y}(0)$.Since we know $\vec{y} = QD\vec{c}$ and $\vec{c} = Q^{-1}\vec{y}(0)$, this means

$$\vec{y} = QDQ^{-1}\vec{y}(0).$$

Compute Q^{-1} and then compute the matrix product QDQ^{-1}.

You can check your work with technology. Follow this link.

4. Let $y_1(0) = a$ and $y_2(0) = b$ which means $\vec{y}(0) = (a, b)$. Multiply out the matrix product $QDQ^{-1}\vec{y}(0)$ and state the solution to this IVP.

In the problem above, we solved the linear system of ODEs in the form $\frac{d\vec{y}}{dt} = A\vec{y}$ with initial conditions $\vec{y}(0)$ by just writing

$$\vec{y} = QDQ^{-1}\vec{y}(0).$$

The columns of Q are the eigenvectors. The nonzero entries of the diagonal matrix D contain $e^{\lambda t}$ where λ is an eigenvalue. Does this pattern work in other places?

Problem 5.30 Solve the system of ODEs $\frac{d\vec{y}}{dt} = A\vec{y}$ where $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$. Do so by stating Q, D, and Q^{-1}, and then perform the matrix product QDQ^{-1}. Finally, if we assume $\vec{y}(0) = (a, b)$, then give the solution to this system of IVPs by stating what $y_1(t)$ equals, and what $y_2(t)$ equals (hint, multiply out $QDQ^{-1}\vec{y}(0)$). Please use technology to perform as much of the computations as you want. Just be prepared to tell us how you got each part.

You can check your work with technology. Follow this link.

Does the pattern above continue to work if we increase the size of the matrix?

Problem: Optional Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Do so by stating Q, D, and Q^{-1}, and then perform the matrix product QDQ^{-1}. Finally, if we assume $\vec{y}(0) = (a, b, c)$, then give the solution to this system of IVPs by stating what $y_1(t)$ equals, what $y_2(t)$ equals, and what $y_3(t)$ equals. Please use technology to perform as much of the computations as you want. Just be prepared to tell us how you got each part.

You can check your work with technology. Follow this link.

Does the pattern even work if the eigenvalues are complex?

Problem: Optional Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

1. Find the eigenvalues by hand. Then for each eigenvalue, compute an eigenvector by hand. State Q and D from this information. You are welcome to use e^{it} in your work as needed.

2. Compute Q^{-1} by hand. Then compute the product QDQ^{-1} by hand.

3. Use Euler's formula $e^{it} = \cos t + i \sin t$ to simplify the product QDQ^{-1}. You should be able to simplify the product to remove all complex terms. Note that $e^{-it} = e^{i(-t)} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$.

Why does $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$?

4. If $y_1(0) = 5$ and $y_2(0) = 7$, then what are $y_1(t)$ and $y_2(t)$. Then use software to check your answer. You should see that the solution in the y_1y_2 plane, along with the appropriate vector field, gives circular motion.

You can check your work with technology. Follow this link.

Problem 5.31 Suppose that $\frac{d\vec{y}}{dt} = A\vec{y}$ is a linear system of ODEs. Also suppose that $\vec{y} = \vec{x}e^{ct}$ is a nonzero solution to this system. Explain, using the definition of eigenvalues and eigenvectors, why we must have that c is an eigenvalue, and $\vec{x}$ is an eigenvector corresponding to c. [Hint: Look up the definition of eigenvalues and eigenvectors. If you compute $\frac{d\vec{y}}{dt}$ and then place both $\vec{y}$ and $\frac{d\vec{y}}{dt}$ into the system $\frac{d\vec{y}}{dt} = A\vec{y}$, you should see the definition appear.]

In the previous sections, we focused mainly on second order ODEs. We started by using Laplace transforms to find the exact solutions. The accompanying partial fraction decomposition was sometimes rather ugly, so we opted for guessing the form of the solution, and then taking derivatives to determine the unknown constants.

Problem 5.32 Consider the ODE $y''' + 3y'' + 3y' + y = 0$ (where y is a function of x, not t). Compute the Laplace transform of both sides, and solve for Y. The denominator factors as $(s+1)^3$. Explain why a general solution is $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$.

For the ODE $y'''' + 4y''' + 6y'' + 4y' + y = 0$, whose characteristic equation is $(s+1)^4 = 0$, make a guess as to the solution. Then use a computer and dsolve to check that your answer is correct. (Wolfram Alpha can solve this.)

If you encounter a repeated root, what does that contribute to the solution? Explain this in a way that you and others can remember it. Write a few sentences, and give an example.

Problem 5.33 In each problem below, you are given the characteristic equation of an ODE. State the general solution of the ODE.

1. $(s+3)(s+2)(s+1) = 0$
2. $(s+3)(s+3)(s+1) = 0$
3. $(s+3)^3(s^2+9) = 0$
4. $(s+3)^2(s^2+9)^2 = 0$
5. $(s+3)^2(s^2-9)^2 = 0$ (Note the sign change.)

When we find a basis of solutions for an ODE, we need to know that the functions are linearly independent. If they are not linearly independent, then they are not a basis. Is there a simple way to determine if solutions are linearly independent? For example, how do we know that for a 3rd order ODE whose characteristic equation is $(s+3)(s+2)(s+1) = 0$ that the functions e^{-t}, e^{-2t}, and e^{-3t} are linearly independent? Can we use matrices and determinants to tackle this? The answer is yes, and its proof is beyond the scope of our class. The problem is that we need a 3 by 3 matrix matrix from the three functions.

Definition 5.5: Wronskian. The Wronskian of a collection of n functions is the determinant of an n by n matrix that we form by placing the functions in the first row and then each other row is the derivative of the row above. This gives us the Wronskian as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad \text{etc.}$$

Problem 5.34 For each collection of functions below, compute the Wronskian. Then decide if the functions are linearly independent or linearly dependent.

1. e^{3t} and $4e^{3t}$
2. $\cos t$ and $\sin t$

3. e^t, e^{2t}, and e^{3t}
4. $\cosh t$, $\sinh t$, and e^t (Simplify this one fully.)
5. What does the Wronskian equal if the functions are linearly dependent?

You should have found that two were dependent, and two were independent.

Wrap up

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 5.35: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

Extra Practice

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the preparation problem set, please use this list as a guideline to find related practice problems.

Concept	Sec	Suggestions	Relevant Problems
Vocabulary of ODEs	8*	33-35	1-3,33-35
2nd Order Homogeneous	9*	1,7,12,21,27,40	1-15, 17-45
nth Order Homogeneous	10*	3,7,8,9,12,18,37,41,44,49	All
IVPs (Homogeneous)	13	9	4,9,13
Applications	14	2,3,5,29,31,34,41-43	1-8,26-43
Laplace Transforms	21*	26, 54	14(c),15(b),25,26,54-58,
Inverse Transforms	22*	7, 34-36,38,read 12 and 18,44	6-10,15-19,29-30,32-53
Solving ODES	24	26,44	5,26,31,36,43,44
Wronskian and Theory	8*	9,10,18,20,43,48,53,58	5-10, 13-20, 31,36-64

*The problems in these sections are quick problems. It is important to do lots of them to learn the pattern used to solve ODEs. You may be able to finish 7 or more problems in 15 minutes or less. Please a bunch so that when you encounter these kinds of problems in the future you can immediately give an answer and move forward.

Remember that you can check almost all of your work with technology. Use the following technology links to help you check your understanding.

- Links are coming.
- Links are coming.

Chapter 6

Non Homogeneous ODEs

After completing this chapter, you should be able to:

1. Explain Hooke's Law in regards to mass-spring systems, where there is an external force. Construct and solve differential equations which represent this physical model, with or without the presence of a damper and be able to interpret how solutions change based on changes in the model.

2. Explain how to obtain solutions to non homogeneous ODEs by finding a particular solution y_p and the kernel y_h to the corresponding homogeneous ODE.

3. Use the method of undetermined coefficients to solve non homogeneous linear ODEs.

4. Explain Kirchhoff's voltage law, Ohm's law, and how to model electrical circuits using 2nd order non homogeneous linear ODEs. Illustrate how results about circuits can be translated into results about mass-spring systems.

6.1 Non Homogeneous Linear Systems

In the previous chapter, we focused on solving homogeneous ODEs of the form $L(y) = 0$. We need to learn how to solve non homogeneous ODEs, namely $L(y) = r(t)$ where $r(t) \neq 0$. Before solving these kinds of ODEs, let's revisit solving matrix equations of the form $A\vec{x} = \vec{0}$ (homogeneous) and $A\vec{x} = \vec{b}$ (non homogeneous). As we compare the solutions to these matrix equations (where solving is much easier), we'll glean some patterns that help us with solving non homogeneous ODEs.

I highly suggest you complete this review problem, and check your answer, to make sure you are comfortable with how to express infinitely many solutions in vector form.

$\boxed{\textbf{Review}}$ Suppose we want to solve the system $A\vec{x} = \vec{b}$. We row reduce the augmented matrix and obtain

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -3 & 4 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What's the solution? Give a basis for the kernel of A? See [1] for an answer.

If the vector $\vec{b}$ above had been $\vec{b} = \vec{0}$, what would the right most column have been after row reduction? If you said, "A column of all zeros," then congrats. In this case, how would it have affected the solution? It would just remove the constant vector from the solution.

Problem 6.1 Consider the homogeneous linear system $\begin{cases} x + 2y - 3z = 0 \\ 2x + 4y - 6z = 0 \\ -x - 2y + 3z = 0 \end{cases}$.

1. Solve this homogeneous system. Show us your rref, and then your solution (in vector form).
2. Solve the non homogeneous system $\begin{cases} x + 2y - 3z = 4 \\ 2x + 4y - 6z = 8 \\ -x - 2y + 3z = -4 \end{cases}$.
3. Compare and contrast the two solutions.

Problem 6.2 Consider the matrix equation $A\vec{x} = \vec{b}$ given by

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ -1 & 3 & 5 & 5 \\ 2 & 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1. Give a basis for the kernel of A. In other words, solve this homogeneous matrix equation.
2. Solve the non homogeneous equation $\begin{bmatrix} 1 & 0 & 1 & -2 \\ -1 & 3 & 5 & 5 \\ 2 & 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \\ 1 \end{bmatrix}$.
3. How is your solution to the homogeneous problem related to your solution of the non homogeneous problem? Compare and contrast the two.

Problem 6.3 Consider the matrix equation $A\vec{x} = \vec{b}$. Suppose that $\vec{x}_1$ and $\vec{x}_2$ are both solutions to this non homogeneous equation.

1. Why is $\vec{x}_1 - \vec{x}_2$ a solution to the homogeneous equation $A\vec{x} = 0$? [Hint: If $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$, then what is $A(\vec{x}_1 - \vec{x}_2)$?

[1] The free variables are x_3 and x_4, as these columns don't have a pivot). The three nonzero rows of our rref give the equations $x_1 - 3x_3 + 4x_4 = 5$, $x_2 + 2x_3 + x_4 = -2$, and $x_5 = 3$. We solve for each variable in terms of the free variables to obtain

$$\begin{aligned} x_1 &= 3x_3 - 4x_4 + 5 \\ x_2 &= -2x_3 - 4x_4 - 2 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= x_3 \end{aligned} \qquad \text{or in vector form} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 5 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

The kernel of A comes by making the last column all zeros. A basis for the kernel is $(3, -2, 1, 0, 0)$ and $(-4, -1, 0, 1, 0)$ (the kernel does not include the last vector).

2. If $\vec{x}_1$ and $\vec{x}_2$ are solutions to $A\vec{x} = \vec{b}$, explain why $\vec{x}_2 - \vec{x}_1$ must be in the kernel of A.

3. If $\vec{x}_p$ is a single particular solution to $A\vec{x} = \vec{b}$, and we know that a basis for the kernel is $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$, then explain why all solutions to $A\vec{x} = \vec{b}$ can be written in the form
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n + \vec{x}_p = \vec{x}_h + \vec{x}_p,$$
where $\vec{x}_h = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ is a general solution to the homogeneous equation $A\vec{x} = \vec{0}$.

Look back at the last few problems. Do you notice how we solved a few problems using matrices and noticed a pattern, namely that the solution to $A\vec{x} = \vec{b}$ is simply $\vec{x}_h + \vec{x}_p$, where $\vec{x}_h$ is the general solution to the homogeneous equation $A\vec{x} = \vec{0}$ (i.e. the kernel), and $\vec{x}_p$ is a single solution to the original equation. The last problem showed that this pattern continued for ANY linear function (operator, transformation). So if we can show something is linear, then the solution follows the same technique.

6.2 Solving Non Homogeneous ODEs

From the last section, we now know that the solutions to a non homogeneous ODE $L(y) = r(t)$ must look like $y = y_h + y_p$ where y_h is the general solution to the homogeneous ODE, and y_p is a single particular solution.

Definition 6.1: Particular y_p and Homogeneous y_h Solution. Given a non homogeneous ODE, a particular solution to the ODE is any one single solution y_p to the ODE. The homogeneous solution which we call y_h is a general solution to the corresponding homogeneous ODE.

We've already seen a couple of ODEs that are non homogeneous in the previous section. We have the tools to solve them with Laplace transforms. Let's look at a few examples, discover some patterns, and then speed things up.

Review We'll occasionally need to solve inverse transforms with rather ugly coefficients. Let's review this. Find the inverse Laplace transform of $\dfrac{fs^2+gs+h}{s^2(s+3)}$. See [2] for an answer.

Problem 6.4 Consider the ODE $my'' = -ky' - mg$ from Problem 5.1. This ODE models the position of a pebble (or any other object) as it falls through the air. With this problem, we assumed that gravity pulls the object down (the $-mg$ term), and that air resistance is proportional to velocity (the $-ky'$ terms).

Please use dsolve with WolframAlpha to check EVERY problem you do in this chapter. Class will go much better if you do.

1. For the homogeneous ODE $my'' + ky' = 0$, what are the zeros of the characteristic polynomial? Show that a general solution to this homogeneous ODE is $y_h(t) = c_1 e^{-kt/m} + c_2$. We use the h in the subscript to just remind us that the solution came from solving the homogeneous ODE.

2. For simplicity, let's have $m = 1$, $k = 5$, and $g = 32$. Use Laplace transforms

See WolframAlpha

[2] We use a partial fraction decomposition to write
$$\frac{fs^2+gs+h}{s^2(s+3)} = \frac{As+B}{s^2} + \frac{C}{s+3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}.$$
The inverse transform is $A + Bt + Ce^{-3t}$. We can obtain A, B, and C in terms of f, g, and h. Since $fs^2 + gs + h = (As+B)(s+3) + Cs^2$, we would solve the system $f = A + C$, $g = 3A + B$, $h = 3B$, which gives $B = h/3$, $A = g/3 - h/9$, and $C = h/9 - g/3$.

to solve the ODE $y'' + 5y' = -32$ if $y(0) = 0$ and $y'(0) = 0$. Make sure you check the link on the side to see if you have the correct answer.

3. Your solution involves 3 terms. Which terms are part of the homogeneous solution? Which term is not part of the homogeneous solution? We call this term a particular solution and write y_p. Circle y_p in your solution above.

4. Use software to obtain a general solution to the ODE $y'' + 5y' = -32$ with no initial values given (see the link on the side). How can you use your work above to obtain this solution? See WolframAlpha

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For our mass spring systems in the last chapter, we placed the system horizontally so that we could ignore the force due to gravity. Let's now hang the mass-spring system vertically.

Problem 6.5 Consider a vertical mass-spring system inside a dashpot. The object's mass is m kg. The dashpot applies a frictional force proportional to the velocity and has a coefficient of friction equal to c kg/s. The spring constant is k kg/s^2.

1. Use Newton's second law of motion to explain why $my'' = -cy' - ky - mg$, or $my'' + cy' + ky = -mg$.

2. For simplicity, suppose $m = 1$ kg, $c = 5$ kg/s, and $k = 4$ kg/s^2. Give the solution y_h to the homogeneous ODE $y'' + 5y' + 4y = 0$.

3. Using the same conditions, use Laplace transforms to solve the non homogeneous ODE $y'' + 5y' + 4y = -32$. [You might want to use $y(0) = h$ and $y'(0) = v$, but remember they are arbitrary.] See WolframAlpha

4. As $t \to \infty$, what happens to $y(t)$?

Review Find the inverse Laplace transform of $\dfrac{1}{s(s^2 + 9)}$. See [3] for an answer.

Problem 6.6 Consider a vertical mass-spring system without friction, so the corresponding ODE is $my'' = -0y' - ky - mg$, or $my'' + ky = -mg$.

1. Let $m = 1$ kg and $k = 4$ kg/s^2. Solve the homogeneous ODE $y'' + 4y = 0$.

2. Use Laplace transforms to solve the non homogeneous ODE $y'' + 4y = -32$. [You might want to use $y(0) = y_0$ and $y'(0) = v_0$, but remember that they are arbitrary.] See WolframAlpha

3. Compare your solutions to the homogeneous and non homogeneous ODEs.

[3] The quadratic $s^2 + 9$ does not factor over the reals, so we write

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9},$$

whose inverse transform is $A + B\cos(3t) + \frac{C}{3}\sin(3t)$. Multiplying both sides by $s(s^2 + 9)$ gives $1 = A(s^2 + 9) + (Bs + C)(s)$. Equating coefficients gives the system $0 = A + B$, $0 = C$, $1 = 9A$. Solving this system yields $A = 1/9$, $B = -1/9$, and $C = 0$.

In all three problems above, we applied an external force to the physical system. This constant force $-mg$ showed up on the right had side of the ODE. Whenever we apply an external force to a problem, it shows up on the right hand side of an ODE. If all the forces are internal, then we are solving a homogeneous ODE. Any external forces change it to a non homogeneous ODE.

The forces above are all constant. What do we do with a non constant force? The same thing! It just might get messier. Because we know how to compute Laplace transforms of polynomials, exponentials, cosines and sines, and products of these, we'll focus our attention on external forces that involve these kinds of functions.

Learning to Guess Appropriately

In the last chapter, we discovered that we can solve homogeneous ODEs by simply finding the zeros of a polynomial. We gleaned all this information by studying Laplace transforms. In this section, let's tackle a few problems and look for patterns that should greatly simply our ability to solve non homogeneous ODEs. In all these problems, we'll be solving second order ODEs of the form $L(y) = r(t)$.

Problem 6.7 Consider the ODE $y'' + 5y' + 4y = r(t)$, which we could write as $L(y) = r(t)$, where $L(y)$ is a linear operator.

1. Find a general solution to the homogeneous ODE $y'' + 5y' + 4y = 0$. Remember, we use the notation y_h as a name for the general solution to the homogeneous ODE.

2. Let $r(t) = 3t$. The Laplace transform yields (check if you want)
$$\begin{aligned} Y &= \frac{sy(0) + y'(0) + 5y(0)}{s^2 + 5s + 4} + \frac{3}{s^2(s^2 + 5s + 4)} \\ &= \left[\frac{a}{s+4} + \frac{b}{s+1}\right] + \left[\frac{c}{s+4} + \frac{d}{s+1} + \frac{es+f}{s^2}\right]. \end{aligned}$$
Don't spend any time doing the partial fraction decompositions. Instead, explain why a solution to $y'' + 5y' + 4y = 3t$ must be
$$y = c_1e^{-4t} + c_2e^{-t} + At + B,$$
where c_1 and c_2 are arbitrary (depending on the initial conditions), but A and B are the same regardless of the initial conditions (they could be determined by doing a partial fraction decomposition).

3. Since c_1 and c_2 are arbitrary constants, let them be zero. This means a particular solution to our ODE is $y_p = At + B$. Substitute $y_p = At + B$, $y_p' = A$ and $y_p'' = 0$ into the ODE $y'' + 5y' + 4y = 3t$ to get $0 + 5(A) + 4(At + B) = 3t + 0$. Use this to find A and B.

4. We now have y_h and y_p. State a general solution to the ODE. See WolframAlpha

5. If $r(t) = 7t^3 - 4t$, make a guess as to what form y_p would take. Use dsolve to check if you are correct. See WolframAlpha

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Did you notice that if the external force $r(t)$ is a polynomial, then a particular solution y_p is a polynomial of the same degree?

Problem 6.8 Again consider the ODE $y'' + 5y' + 4y = r(t)$, or $L(y) = r(t)$. We know the solution to $L(y) = 0$ (the kernel of L) is $y_h = c_1e^{-4t} + c_2e^{-t}$. The kernel is two dimensional with basis e^{-4t} and e^{-t}.

1. Let $r(t) = 2\cos 3t$. The Laplace transform yields

$$Y = \frac{sy(0) + y'(0) + 5y(0)}{s^2 + 5s + 4} + \frac{s}{(s^2 + 9)(s^2 + 5s + 4)}$$
$$\left[\frac{a}{s+4} + \frac{b}{s+1}\right] + \left[\frac{c}{s+4} + \frac{d}{s+1} + \frac{es+f}{s^2+9}\right].$$

   Don't complete the partial fraction decomposition, rather explain why a solution to $y'' + 5y' + 4y = 2\cos 3t$ must be

$$y = c_1e^{-4t} + c_2e^{-t} + A\cos(3t) + B\sin(3t),$$

   where c_1 and c_2 are arbitrary (depending on the initial conditions), but A and B are the same regardless of the initial conditions (they could be determined by doing a partial fraction decomposition).

2. Since c_1 and c_2 are arbitrary constants, let them be zero. This means a particular solution to our ODE is $y_p = A\cos 3t + B\sin 3t$. Compute two derivatives and then substitute y_p, y_p' and y_p'' into the ODE to get

$$(A\cos 3t+B\sin 3t)''+5(A\cos 3t+B\sin 3t)'+4(A\cos 3t+B\sin 3t) = 2\cos 3t.$$

   Use this equation to show $A = -\frac{1}{25}$ and $B = \frac{3}{25}$. [Hint: The right hand side is $2\cos 3t + 0\sin 3t$.]

3. We now have y_h and y_p. State a general solution to the ODE. Use the link to the right to check if you are correct. See WolframAlpha

4. If $r(t) = \sin(7t)$, make a guess as to what form y_p would take. Use software to check that you are correct.

Did you notice that if the external force $r(t)$ involves a sine or a cosine, then a particular solution will be a linear combination of both sine and cosine with the same frequency?

Let's look at one more, but this time let's have $r(t)$ involve exponentials. Something different happens when $r(t)$ is actually part of the kernel of L.

Problem 6.9 Again consider the ODE $y'' + 5y' + 4y = r(t)$, or $L(y) = r(t)$. We know the kernel of L is $y_h = c_1e^{-4t} + c_2e^{-t}$.

1. Let $r(t) = 7e^{-3t}$. Compute the Laplace transform of both sides, and solve for Y. Explain why a solution to $y'' + 5y' + 4y = 7e^{-3t}$ must be $y = c_1e^{-4t} + c_2e^{-t} + Ae^{-3t}$, where c_1 and c_2 are arbitrary, but A could be determined by doing a partial fraction decomposition (just set the decompositions up, don't spend time finding the constants).

2. Since c_1 and c_2 are arbitrary constants, let them be zero. This means a particular solution to our ODE is $y_p = Ae^{-3t}$. Substitute y_p, y_p' and y_p'' into the ODE to get

$$(Ae^{-3t})'' + 5(Ae^{-3t})' + 4(Ae^{-3t}) = 7e^{-3t}.$$

   Use this equation to find A.

3. We now have y_h and y_p. State a general solution to the ODE. See WolframAlpha
4. If $r(t) = e^{-2t}$, make a guess as to what form y_p would take. Use WolframAlpha to check if you are correct.
5. If $r(t) = e^{-4t}$, make a guess as to what form y_p would take. Use WolframAlpha to check if you are correct. You should notice that this answer takes on a slightly different form than the others. What makes it different? Why do you think this difference occurred?

Did you notice that when the external force is an exponential, then a particular solution is an exponential. Also, did you notice that if the external force is part of kernel (the solution to the homogeneous solution), then our particular solution was multiplied by t?

The previous three problems developed some key ideas we need to expand. Whenever we need to solve an ODE of the form $L(y) = r(t)$, we have a few things to consider.

- First, we need the solution y_h to the homogeneous ODE $L(y) = 0$. This is the kernel of the linear function L.
- Then we need to find a particular solution y_p to $L(y) = r(t)$. The previous 3 problems suggest that we can guess a form for y_p (based off r), and then take derivatives to determine the value of any undetermined coefficients in our guess.
- The last problem suggested that if $r(t)$ is actually part of the kernel, then we have to modify our guess slightly (multiply by t) to get y_p.

We need to make sure this pattern works on more problems. Here's where software comes in handy. We can use dsolve with WolframAlpha to kick out solutions to ODEs really fast. What we need is to practice guessing a particular solution with lots of ODEs, and make sure we build up the right patterns. Then, we can start tackling every non homogeneous ODE in a consistent, fast, clean, way. The next problem asks you to do this. Here's a pattern of what I expect.

- For the ODE $y'' + 5y' + 6y = t + e^{-7t}$, the characteristic equation is $\lambda^2 + 5y + 6 = (\lambda + 2)(\lambda + 3) = 0$. This gives $y_h = c_1e^{-2t} + c_2e^{-3t}$. Since $r(t) = t + e^{-7t}$, I'm going to guess that $y_p = (At + B) + (Ce^{-7t})$. I guessed $At + B$ because of the first degree polynomial t in $r(t)$. I guessed Ce^{-7t} because of the exponential e^{-7t} in $r(t)$. Checking my guess with WolframAlpha shows I'm correct, where $A = 1/6$, $B = -5/36$ and $C = 1/20$. See WolframAlpha

Problem 6.10 Consider the ODE $y'' + 6y' + 9y = \sin(4t) + e^{-t}$.

1. Find the kernel of $L(y) = y'' + 6y' + 9y$ (i.e. state y_h).
2. Make a guess for y_p (with undetermined coefficients), and explain why you made this guess.
3. Use WolframAlpha to find the coefficients in your guess y_p. Update your guess above (with any reasons for the changes needed). Then state a general solution. See WolframAlpha
4. Repeat parts 2 and 3 if instead you needed to solve $y'' + 6y' + 9y = e^{-3t}$. What makes this different?

Problem 6.11 For each problem below, (1) state y_h, (2) make a guess for y_p (your guess will involve undetermined coefficients), and then (3) check your answer using software (giving the value of any coefficients in your guess). If your guess was wrong, please tell us your original guess, and why it was wrong.

1. $y'' + 3y' + 2y = t + e^{-2t} + e^{-5t}$ See WolframAlpha
2. $y'' + 4y = 4t^2 - 3\cos(3t) + 5\sin(2t)$ (Some constants could be zero.) See WolframAlpha
3. $y'' + 7y' + 10y = 5e^{-2t}\cos(3t) - 6e^{-5t}$ See WolframAlpha
4. $y'' + 6y' + 25y = 7e^{-3t} - 2\cos(4t) + 6e^{-3t}\cos(4t)$ (Wolfram's solution is factored. Expand it.) See WolframAlpha

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Let's summarize the patterns we've seen.

Problem 6.12 Suppose that $L(y) = r(t)$ is a linear ODE.

1. If $r(t)$ is in the table below, what would you guess for y_p?

Form of $r(t)$	Guess for y_p
ke^{at}	
kt^n	
$k\cos(\omega t)$	
$k\sin(\omega t)$	
$ke^{at}\cos(\omega t)$	
$ke^{at}\sin(\omega t)$	

2. If $r(t)$ involves a sum of terms in the table above, what do you guess for y_p? Write sentence or two to explain what you should do. In particular if $r(t) = 7\cos(2t) - 3\sin(2t) + 8e^{3t}\cos(2t)$, then tell us your guess for y_p.
3. If part of your guess is in the kernel of L, how should you modify your guess? Again, write a sentence or two to explain what you should do. In particular, if y_h involves c_1e^{-3t} and $r(t) = 7e^{-3t} + t^3$, then tell us your guess for y_p.

The ideas above work with higher order ODEs as well. Let's try this on a 5th order ODE.

Problem 6.13 Suppose we have a constant coefficient linear differential equation of the form $L(y) = r(t)$. It's a 5th order ODE, and the characteristic equation has zeros $2, -3, -3, -4+5i, -4-5i$. State the solution y_h to the homogeneous ODE, and then explain how to make a guess y_p if $r(t) = 3e^{-2t} - 7e^{2t} + 5e^{-3t} + \cos(5t) - 12e^{-4t}\sin(5t)$.

At this point, we've got the basic idea to solve $L(y) = r(t)$, provided L is a constant coefficient linear operator. We find y_h, and then guess a particular solution y_p, with undetermined coefficients. We then take a couple derivatives to figure out the unknown constants. We call this "The Method of Undetermined Coefficients." Here's a few observations:

- We could just solve all the ODEs using Laplace transforms. The problem is that if we need a general solution, the solution might get ugly really fast. Laplace transforms work best when we have initial conditions.
- If we have initial conditions, maybe we should just do a Laplace transform flat out. No guessing is needed. We'll have to decide which is faster.
- These ideas work on higher order ODEs, in the exact same way they work on second order ODEs.

6.3 Applications

At this point, we need to practice the method of undetermined coefficients. Rather than just solve a bunch of ODEs with no application, let's connect each one to a physical problem.

Problem 6.14 Consider a vertical mass spring system with $m = 1$ kg, $c = 5$ kg/s, and $k = 6$ kg/s^2. Let's assume $g = -10$ m/s^2 (it makes the arithmetic simpler). The 1 kg object is a magnetic brick. We turn on an electromagnet underneath the brick, and then slowly ramp up the force of the magnet be $2t$ N. When the magnet was turned on, the brick was motionless. Where is the brick after t seconds?

1. Solve by hand the IVP $y'' + 5y' + 6y = -10 - 2t$, where $y(0) = -5/3$ and $y'(0) = 0$. Do so by finding y_h, guessing y_p, and then use derivatives to find the undetermined coefficients in your guess. Then use the initial conditions to find c_1 and c_2. Check your answer with software (see the link on the right). See WolframAlpha
2. Use software (see the link to the right) to solve the ODE $y''+5y'+6y = -10$ (gravity with no magnet). Explain why after sufficient time, the brick will stabilize 5/3 m below equilibrium. See WolframAlpha
3. Use software to solve the IVP $y'' + 5y' + 6y = -2t$, where $y(0) = 0$ and $y'(0) = 0$. What does this have to do with the previous part? See WolframAlpha

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One nice consequence of the previous problem is that when examining a vertical mass spring system, we can shift the equilibrium point to where the spring stabilizes with gravity, and then we can still use $my'' + cy' + ky = 0$ to analyze vertical mass spring systems.

Problem 6.15 Complete the following:

1. Solve the IVP $y'' + 5y' + 6y = 20e^{-t}$, where $y(0) = 0$, $y'(0) = 0$. See WolframAlpha
2. Solve the IVP $y'' + 5y' + 6y = 20e^{-2t}$, where $y(0) = 0$, $y'(0) = 0$. See WolframAlpha
3. Check both your answers with Wolfram Alpha. You may have to expand WolframAlpha's solution to get it to match yours.

Problem 6.16 We build a rocket and attach an engine. In free fall, we already know the ODE which models the motion is $my'' + ky' = -mg$. The engine adds an external force $r(t)$ to this system. Because the engine burns fuel as it propels upwards, the mass $m(t)$ now depends on time. This gives us the ODE $m(t)y'' + ky' = -m(t)g + r(t)$. If we fire the rocket in space, then we

Most rocket engines have a three part thrust. The engine first ramps up (linearly) to some constant thrust, stays at that constant thrust for a time, and then ramps down linearly. We'll revisit this again in the next chapter, when we have some powerful tools for working with piecewise defined functions.

could neglect the $-m(t)g$ part (but then k would probably also be zero). We need a good model for engine thrust.

Let's fire a toy rocket from the earth's surface. Suppose $m = .2$ kg and $k = .6$ kg/s. For simplicity, use $g = 10\text{m/s}^2$. Let's assume the rocket thrust starts out fast, and drops to zero exponentially. We'll also assume that the fuel is extremely light, so that we can assume $m(t)$ is just the constant .2 kg. This gives us an external force $f(t) = ae^{bt}$ for some known constants a and b.

1. In this chapter, we are studying linear constant coefficient non homogeneous ODEs. If we allowed m to change with t, why does the material in this chapter no longer apply?

2. Why are the initial conditions $y(0) = 0$ and $y'(0) = 0$.

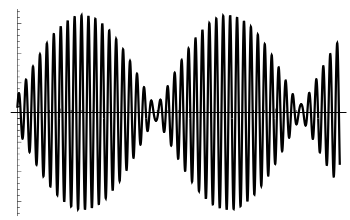
3. If $r(t) = 7e^{-5t}$, determine the rocket's height $y(t)$ after t seconds. Here I gave you some specific numbers to work with. This is often the key to working on a problem with symbols.

4. If $r(t) = ae^{bt}$, determine the rocket's height $y(t)$ after t seconds.

For the next problem, let's imagine attaching the left end of a spring to a wheel, and then rotate the wheel. We'll keep the mass-spring system horizontal, so we can neglect gravity. Please look at the links to

- LearnersTV or
- Wolfram Demonstrations Project

to see a picture of such a system. This rotating wheel applies a periodic external force to the mass-spring system. This force is often called a driving force.

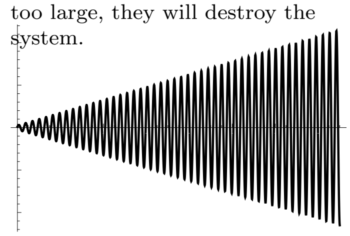


Notice the beats. In this example, $\omega = 2$, $m = 1.1$, $c \approx 0$, and $k = 4$. Since $\omega_0 = \sqrt{4/1.1} \approx 2 = \omega$, the solution results in large periodic oscillations. If the oscillations are too large, they will destroy the system.

$\boxed{\textbf{Problem 6.17}}$ Let's attach a mass-spring system to a wheel. Suppose $m = 1$ and $k = 4$ (with no dashpot). The driving force, $r(t)$, is periodic.

1. Assume the driving force is $r(t) = 7\sin(5t)$. Solve the IVP $y'' + 4y = 7\sin(5t)$, where $y(0) = 0$ and $y'(0) = 0$.

2. Assume m and k are known constants, and the driving force is $r(t) = F\sin(\omega t)$, where $\omega \neq \sqrt{k/m}$. Solve the IVP $my'' + ky = F\sin(\omega t)$, where $y(0) = 0$ and $y'(0) = 0$.



When $\omega_0 = \omega$ and friction is negligible, a system will oscillate with an amplitude that grows without bound. Beware of this situation, as any mechanical system which undergoes this kind of oscillation will self destruct.

$\boxed{\textbf{Problem 6.18}}$ Again, let's attach a mass-spring system to a wheel. Suppose $m = 1$ and $k = 4$ (with no dashpot). The driving force, $r(t)$, is periodic.

1. Assume the driving force is $r(t) = 7\sin(2t)$. Solve the IVP $y'' + 4y = 7\sin(2t)$, where $y(0) = 0$ and $y'(0) = 0$.

2. Assume m and k are now some constant, and the driving force is $r(t) = A\sin(\omega t)$, where $\omega = \sqrt{k/m}$. Solve the IVP $my'' + ky = F\sin(\omega t)$, where $y(0) = 0$ and $y'(0) = 0$.

Make sure you ask me in class to graph your two solutions above in Mathematica. The plots get interesting when $\omega \approx \sqrt{k/m}$, and the solution produces steady beats. You should see a rhythmic rise and fall in the amplitude of the solution. When $\omega = \sqrt{k/m}$, the solution grows without bound. The following YouTube videos show the collapse of the Tacoma Narrows bridge, and airplane flutter. The points to these videos is to show you the dangerous things that can (and do) happen to a structure when the designer forgets to take into account how external driving forces might interact with the internal frequencies of the mechanical system.

- The Tacoma Narrows bridge collapses.
- The tail of a small airplane begins to flutter.
- Watch a collection of flutter examples.
- An RC airplane looses its wing midflight.

In both problems above, we assumed there was no friction ($c = 0$). Can we produce beats or resonance when there is friction? Let's analyze this problem and show that YES, bad things can still happen when friction is involved. To discover when disaster might occur, we have to work with symbolic answers and then ask, "What would it take to produce large oscillations?" Let's analyze this first with some specific numbers (to notice patterns) and then we'll analyze what happens symbolically.

Problem 6.19 Consider the ODE $y'' + 2y' + 5y = 5\sin(3t)$.

1. Find a general solution to this ODE. See WolframAlpha
2. As t increases, what happens to the homogeneous solution?
3. If $y(0) = 0$ and $y'(0) = 0$, solve the IVP.

When friction enters a mass spring system, the homogeneous solution will always die out over time. The particular solution y_p is called the "steady-state", or "steady periodic" solution. As time moves on, friction will damp out all oscillation except for the steady-state solution, y_p.

Problem 6.20 Consider the ODE $my'' + cy' + ky = F\sin(\omega t)$.

1. What are the roots of the characteristic polynomial?
2. We guess the steady-state solution (particular solution) is $y_p = A\cos(\omega t) + B\sin(\omega t)$. Why do we never have to multiply the guess by t?
3. Find the steady-state solution. As a hint, you'll probably find Cramer's Rule useful when solving for A and B (because you'll get a linear system with variables as coefficients). See WolframAlpha

In class, we'll examine what it would take to get a really large amplitude for the steady-state solution (thus destroying the mechanical system).

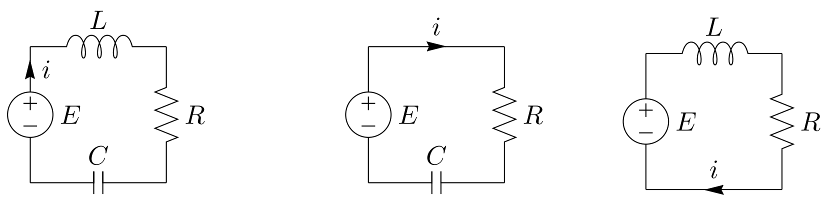
6.3.1 Electric circuits

Remember that Kirchoff's voltage law states that the voltage (electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop. We've summarized this by saying "voltage in equals voltage out." Because we've been using complex numbers in our work, we'll use $I(t)$ to represent the current in a loop instead of $i(t)$. We've already used Kirchoff's voltage law in connection with resistors. Let's now add an inductor and a capacitor to a single loop. Each element (resistor, inductor, capacitor) produces the voltage drop given in the table below.

Component	Voltage drop	Other
Resistor	RI	Ohm's law, where R is in ohms
Inductor	$LI' = L(dI/dt)$	L is in henrys
Capacitor	$\frac{Q}{C} = \frac{1}{C}\frac{dI}{dt}$	Q is in coulombs, C in farads.

The charge Q on a capacitor is related to the current by $I(t) = \frac{dQ}{dt}$, or $Q = \int I(t)dt$.

We'll be studying RC, RLC, and RL circuits in the next few chapters. Table 6.1 shows the differential equations corresponding to each type of circuit.



An RLC-circuit	An RC-circuit	An RL-circuit
$LI' + RI + \frac{1}{C}\int I(t)dt = E(t)$	$RI + \frac{1}{C}\int I(t)dt = E(t)$	$LI' + RI = E(t)$
$LQ'' + RQ' + \frac{1}{C}Q = E(t)$	$RQ' + \frac{1}{C}Q = E(t)$	
$LI'' + RI' + \frac{1}{C}I = E'(t)$	$RI' + \frac{1}{C}I = E'(t)$	

Table 6.1: Typical diagrams of RCL, RC, and RL circuits, and their corresponding ODEs. The first row is an integro-differential equation for the current $I(t)$. The second row is the ODE for the charge Q on the capacitor. The third row is the derivative of the first row.

In a circuit with one resistor, one inductor, and one capacitor (an RLC circuit), if the electromotive force is $E(t)$, then Kirchoff's Voltage law gives the integro-differential equation

$$LI' + RI + \frac{1}{C}Q(t) = E(t) \quad \text{or} \quad LI' + RI + \frac{1}{C}\int I(t)dt = E(t).$$

Differentiating both sides removes the integral and gives

$$LI'' + RI' + \frac{1}{C}I(t) = E'(t),$$

which is a second order linear differential equation with constant coefficients. However, the initial conditions are in terms of initial charge $Q(0)$ and initial current $I(0)$. To solve the differential equation for I, we need $I'(0)$, which we can get from the equation $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$. Problem 14.13 in Schaum's provides an excellent example that summarizes the solution technique.

$\boxed{\textbf{Problem 6.21}}$ Consider an RLC circuit with $L = 1/2$, $R = 2$, and $C = 2/3$. Let's plug the circuit into an alternating current power source (like a wall outlet), which means we might have something like $E(t) = 2\cos(3t)$. Initially, assume that the current is zero and the charge on the capacitor is zero. We'd like to find the current at any time t in the circuit.

1. Explain why the current satisfies $I'' + 4I' + 3I = -12\sin(3t)$. Find a general solution to this ODE. See WolframAlpha

2. We know that $I(0) = 0$ and $Q(0) = 0$. Use the equation $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$ to explain why $I'(0) = 4$.

3. Find the current in the wire at any time t by solving the corresponding IVP. Use the initial conditions you found in the previous part.

See WolframAlpha

4. What is the steady-state current? (Which part of your solution above does not vanish after sufficient time has passed? This would be the current flowing through the circuit after the initial conditions have died out.)

Problem 6.22 Consider an RLC circuit with $L = 1$, $R = 8$, and $C = \frac{1}{25}$. Let's plug the circuit into a 12 V battery, so we have $E(t) = 12$. Initially, assume that the current is zero and the charge on the capacitor is zero. We'd like to find the current at any time t in the circuit.

1. State the IVP whose solution would give the current at any time t. (What's the ODE, and what are the initial conditions $I(0)$ and $I'(0)$). [Hint: Use the equation $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$ to find $I'(0)$.]

2. Find the current in the wire at any time t. Check your answer with WolframAlpha (you'll want to use y instead of I).

3. What's the steady-state current?

Observation 6.2. Mechanical models are expensive to build. Electrical models are fairly simple to build and measure. If you need to create a mechanical system, it may prove beneficial financially to start with an electrical model. Engineers spend another semester on this idea in system dynamics. Hydraulic systems are also very closely related. In bridging between mechanical and electrical systems, we compare the following variables.

Mechanical System	m	c	k	$r(t) = F_0 \cos \omega t$	$y(t)$
Electrical System	L	R	$1/C$	$E'(t) = E_0 \omega \cos \omega t$	$I(t)$

Solving a problem in one system (either mechanical or electrical) can provide useful results in the other.

Wrap up

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 6.23: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

Extra Practice

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the preparation problem set, please use this list as a guideline to find related practice problems.

Concept	Sec	Suggestions	Relevant Problems
Theory	8	21,65	21-23,65-67
Undetermined Coef	11	1,2,3,8,10,24,26,34,36,41,46,47,48	All
IVP	13	1,7,14	1,3,7,8,10,11,14
Applications	7	19,76	19-22,71-81
Applications	14	10,11,13,14,17,46,50,51,52,54,57	9-18,44-65

Remember that you can check almost all of your work with technology. Wolfram Alpha provides a great resource for rapidly checking your work in this chapter.

Chapter 7

Laplace Transforms

After completing this chapter, you should be able to:

1. Explain how to compute Laplace transforms and inverse Laplace transforms. Use Laplace transforms to solve IVPs.

2. Use and prove both both the s-shifting and t-shifting theorems.

3. Express discontinuous functions with the Heaviside function, and use the Heaviside to set up and solve ODEs.

4. Express impulses in terms of the Dirac delta distribution, and use it to set up and solve solve ODEs.

5. Compute convolutions, and show how to use the convolution theorem as an inverse product rule for Laplace transforms.

You can find additional practice problems in Schaum's Outlines *Differential Equations* by Richard Bronson. You'll find relevant problems in chapters 21 -24, as well as some extra practice problems at the end of this chapter. Do enough of each type that you feel comfortable with the ideas.

You may want to download this Mathematica Technology Introduction to check your work throughout the entire chapter.

You must practice lots of problems to gain a feel for patterns. Many of the problems in 21-23 are fast. Please take a few minutes every day to just flat out practice with the basics (kind of like when you were learning the times tables - they get really fast if you just practice them). When you feel like you have the basics down, see if you can complete chapters 21 and 22 in less than an hour. If one stumps you, skip it and come back later.

Once you feel confident, chapters 23 (on convolutions and the heaviside function) and 24 (solving IVPS) will help you use the Laplace transforms to solve ODEs. At the end of this chapter are some additional problems to help you cement your understanding. Table 7.1 summarizes the transforms we use most often.

Laplace Transform Practice

In this chapter we'll examine how to handle non differentiable changes in an external driving force. This corresponds to hitting a mass-spring system with a hammer, marching across a bridge, flipping a switch in an electrical network, etc. We'll find that Laplace transforms provide us with extremely nice tools to solve these types problems. Before we jump in, let's review how to solve a couple ODEs with Laplace transforms, and perhaps make some connections that we haven't yet made.

$f(t)$	$F(s)$	provided
1	$\frac{1}{s}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
y'	$sY-y(0)$	
y''	$s^2Y-sy(0)-y'(0)$	
$e^{at}f(t)$	$F(s-a)$	
$f(t)*g(t)$	$F(s)G(s)$	

$f(t)$	$F(s)$	provided
$\cos(wt)$	$\frac{s}{s^2+\omega^2}$	$s>0$
$\sin(wt)$	$\frac{\omega}{s^2+\omega^2}$	$s>0$
$\cosh(wt)$	$\frac{s}{s^2-\omega^2}$	$s>\|\omega\|$
$\sinh(wt)$	$\frac{\omega}{s^2-\omega^2}$	$s>\|\omega\|$
$u(t-a)$	$\frac{1}{s}e^{-as}$	
$\delta(t-a)$	e^{-as}	
$f(t-a)u(t-a)$ $f(t)u(t-a)$	$\mathscr{L}(f(t))e^{-as}$ $\mathscr{L}(f(t+a))e^{-as}$	

Table 7.1: Table of Laplace Transforms. Note that the s shifting theorem $\mathscr{L}(e^{at}f(t))=F(s-a)$ has a positive a in the exponent, while the t shifting theorem $\mathscr{L}(f(t-a)u(t-a))=\mathscr{L}(f(t))e^{-as}$ has a negative a in the exponent.

Review Compute the inverse Laplace transform of $\dfrac{3s+8}{(s+1)^2+16}$. See [1].

Problem 7.1 Compute the inverse Laplace transform of

$$Y=\frac{5}{(s+3)^3}+\frac{2s+3}{(s+4)^2+9}+\frac{3s+1}{(s+2)^2-49}.$$

Use the rules for $\cosh(t)$ and $\sinh(t)$ to tackle the last terms, rather than doing a partial fraction decomposition. The goal of this problem is to make sure you have the s-shifting theorem mastered.

We've solved inverse transforms such as this one multiple times. If you need to refresh, please head to chapters 21 and 22 in Schaum's, and just practice the problems where answers are provided.

Problem 7.2 Consider the IVP $y''+6y'+8y=0$, $y(0)=2$, $y'(0)=3$.

1. Use Laplace transforms to solve the IVP. Use a partial fraction decomposition to write $Y=\frac{A}{s+2}+\frac{B}{S+4}$ and then show that $y(t)=\dfrac{11}{2}e^{-2t}-\dfrac{7}{2}e^{-4t}$.
2. Instead of using a partial fraction decomposition, instead notice that we could complete the square in the denomoinator to write $s^2+6s+8=(s+3)^2-1$. Use this approach to solve for $y(t)$ and write $y(t)$ as a linear combination of $e^{-3t}\cosh(t)$ and $e^{-3t}\sinh(t)$.

---

[1] We can rewrite Y as

$$Y=\frac{3(s+1)-3+8}{(s+1)^2-16}=\frac{3(s+1)}{(s+1)^2-16}+\frac{5}{(s+1)^2-16}\frac{4}{4}.$$

The inverse Laplace transform is then

$$y(t)=3e^{-t}\cosh(4t)+\frac{5}{4}e^{-t}\sinh(4t).$$

3. Remember that $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$ and $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$. Use this to show how your second solution involving hyperbolic trig functions is really the same as $y(t) = \dfrac{11}{2}e^{-2t} - \dfrac{7}{2}e^{-4t}$.

We can use Laplace transforms to obtain quick solutions to problems with discontinuous external forces. Let's start by examining an RL circuit with a battery, because it keeps the computations simple.

Problem 7.3 Consider an RL circuit with $R = 1$ ohms and $L = 1$ Henry. At time zero, there is no battery in the system. After 2 seconds, we connect a battery $E = 12V$ to the circuit. Two seconds after connecting the battery, we disconnect it. Our goal is to determine the current in the wire exactly 2 second after we disconnect the battery.

1. During the first two seconds, we need to solve the IVP $I' + I = 0$ where $I(0) = 0$. Solve this IVP and use your solution to show that the current after 2 seconds is $I(2) = 0$.

2. Between $t = 2$ and $t = 4$, we know $E = 12$. Solve the IVP $I' + I = 12$, $I(2) = 0$. Show that $I(4) = 12 - 12e^{-2}$. This is the current as the battery gets removed. See WolframAlpha

3. When we remove the battery, the ODE is $I' + I = 0$. We know the initial condition $I(4) = 12 - 12e^{-2}$ from the previous part. Solve the IVP, and state $I(6)$.

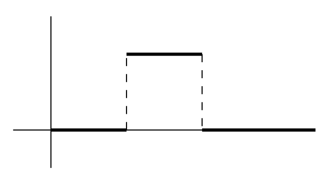
4. We can now predict the current at any time t. Use piecewise function notation to state the current in the form

$$I(t) = \begin{cases} 0 & 0 \le t < 2 \\ 12 - 12e^2e^{-t} & 2 \le t < 4 \\ ? & 4 \le t \end{cases}.$$

The Heaviside models discontinuities

The electromotive force $E(t)$ in Problem 7.3 was a piecewise defined force. It was zero, then 12, then 0. Using piecewise notation, we would write this as

$$E(t) = \begin{cases} 0 & 0 \le t < 2 \\ 12 & 2 \le t < 4 \\ 0 & 4 \le t \end{cases},$$



and we could graph the function $E(t)$ using the figure to the right. We need a nice clean way to work with piecewise defined external forces. We also need to become comfortable graphing and working with these kinds of forces.

Problem 7.4 Consider the functions

$$f(t) = \begin{cases} 2t & 0 \le t < 2 \\ -t^2 + 4t & 2 \le t < 4 \\ 18 - 3t & 4 \le t < 6 \\ t^2 - 12t + 35 & 6 \le t < 8 \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 2t & 0 \le t < 2 \\ 4 - (t-2)^2 & 2 \le t < 4 \\ 6 - 3(t-4) & 4 \le t < 6 \\ (t-6)^2 - 1 & 6 \le t < 8 \end{cases}.$$

1. Graph $f(t)$ in the ty plane.
2. Graph $g(t)$ in the ty plane.
3. Graph $y = 2t$, $y = 4 - t^2$, $y = 6 - 3t$, and $y = t^2 - 1$ in the ty plane. What does this have to do with the above?

We now define the key function that allows us to work with piecewise defined functions. Some people call this the Heaviside function, some call it the unit step function. This is a simple function that jumps up a single unit at a specified value of t.

Definition 7.1: Heaviside or Unit Step Function. We define the Heaviside, or unit step function, to be the function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

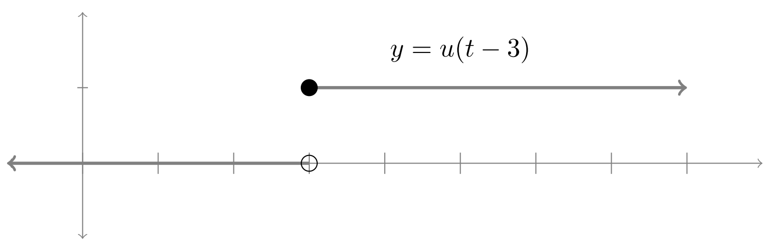
In our work, it won't matter what we define $u(0)$ to equal. Here, we define $u(0) = 1$, but we could have just as easily define $u(0) = 0$ or $u(0) = 1/2$. This last option, the $1/2$, comes in handy when working with Fourier series.

We'll most often shift this function right a units, so we replace t with $t - a$, which means we could write

$$u(t-a) = \begin{cases} 0 & t-a < 0 \\ 1 & t-a \geq 0 \end{cases} \quad \text{or} \quad u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}.$$

Why does this function matter. It's like an on/off function. If you multiply $f(t)$ by $u(t-a)$, then the function $f(t)u(t-a)$ is zero to the left of a, and is equal to $f(t)$ after a. If $a = 3$, then look below for the graph of $y = u(t-3)$.

Mathematica uses the name "HeavisideTheta" for the Heaviside function. You'll see the symbol θ show up as the name of a function.



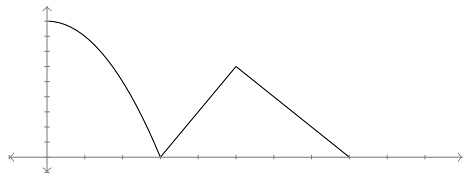
Problem 7.5 Construct a graph of each of the following:

1. $f(t) = u(t-4) - u(t-7)$
2. $g(t) = (10-t)(u(t-4) - u(t-7))$
3. $h(t) = (10-(t-4))(u(t-4) - u(t-7))$ (How does this differ from the previous?)
4. $k(t) = t^2(u(t-3) - u(t-5))$
5. $l(t) = (t-3)^2(u(t-3) - u(t-5))$ (How does this differ from the previous?)

Make sure you check your solution by following the link to Sage.

WolframAlpha and I are having issues when it comes to plotting Heavisides. I can plot the first one just fine, but as soon as I times it by $(10-t)$, it tries to plot a surface. As such, please use this link to Sage to check your work. Make sure you can explain how the graphs are made (not just give them).

Problem 7.6 The graphs of $f(t) = 9 - t^2$ for $0 \leq t \leq 3$, and $g(t) = 3t$ for $0 \leq t \leq 2$, and $h(t) = 6 - 2t$ for $0 \leq t \leq 2$ are connected together (when one ends, the others starts) to give the following graph.



1. Write this function using piecewise function notation.
2. Write this function using Heaviside notation. You'll want to use the idea that $u(t-a) - u(t-b)$ turns a function on at a and off at b.
3. When you think you have the function, use this Sage link to check if you are correct (you'll have to type in your function).

   Feel free to use Mathematica instead, if you have downloaded and installed it. Remember that BYU-I students can now install Mathematica on their personal machines for free. Please head to I-Learn for instructions. You can then download this Mathematica Technology Introduction, and you'll see how to code HeavisideTheta functions in Mathematica.

Did you notice in your work above that it was a lot easier to graph a piecewise defined function when everything was shifted to the starting point. It's much easier to graph $f(t-a)u(t-a)$ than it is to graph $f(t)u(t-a)$. We'll find that this remains true as well, when we start applying Laplace transforms.

Convolutions provide an inverse product rule

We've already seen that the Laplace transform of the product $f \cdot g$ of two functions is not the product of the Laplace transform of each ($L(fg) \neq L(f)L(g)$). Is there some kind of product rule for transforms? This question led mathematicians to invent what we now call a convolution. They discovered that the Laplace inverse of $H(s) = L(f)L(g)$ is equal to the quantity $h(t) = (f*g)(t) = \int_0^t f(p)g(t-p)dp$. We call this the convolution of f and g. The variable p is a dummy variable of integration, and we could call it anything else. Some books use τ, but I find it really hard to distinguish between t and τ when I'm writing on paper, so I use p instead.

Problem 7 has you prove this theorem, which requires that you review double integration and swapping the order of integration.

Definition 7.2: Convolutions. If $f(t)$ and $g(t)$ are function, then we define the convolution of f and g to be

$$(f*g)(t) = \int_0^t f(p)g(t-p)dp.$$

Theorem 7.3 (The Convolution Theorem)**.** *If $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$, then the inverse Laplace transform of $F(s)G(s)$ is the convolution*

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f*g)(t).$$

This is as close as we get to an inverse product rule for Laplace transforms.

We need to practice the convolution theorem. It's just an integral.

Problem 7.7 Do the following:

1. Show that $1*1 = t$. You'll need to compute $\displaystyle\int_0^t f(p)g(t-p)dp = \int_0^t 1 \cdot 1 dp$.

2. Compute $1 * t$ and $t * 1$. You'll need to compute $\displaystyle\int_0^t f(p)g(t-p)dp = \int_0^t 1\cdot(t-p)dp$ and $\displaystyle\int_0^t f(p)g(t-p)dp = \int_0^t p\cdot 1dp$.

3. Compute $t * t$. Compare this to the inverse transform of $\frac{1}{s^2}\frac{1}{s^2}$.

4. Compute $\sin(t) * t = \int_0^t \sin(p)(t-p)dp$. You'll want to use integration by parts.

If you'd like to know why the convolution theorem works, please complete the next problem. It requires that you can swap the order of integration on a double integral. Otherwise, feel free to move on to the next problem. If you want a fun puzzle, do this one.

Problem Prove the convolution theorem (Theorem 7.3). Here are some hints.

- Let $F(s) = \int_0^\infty f(t)e^{-st}dt$ and then use $G(s) = \int_0^\infty g(w)e^{-sw}dw$. (Why can I use w instead of t?)
- Explain why
$$F(s)G(s) = \int_0^\infty \int_0^\infty f(t)g(w)e^{-s(t+w)}\,dw\,dt.$$
- Do a p substitution $p = t + w$ on the inside integral. You should have something like
$$F(s)G(s) = \int_0^\infty \int_t^\infty ?e^{-s(p)}\,dp\,dt.$$
- Swap the order of integration so that t is inside and p is outside. This will require that you draw the region of integration in the tp plane.
- Show that
$$F(s)G(s) = \int_0^\infty \left[\int_0^p f(t)g(t-p)\,dt\right] e^{-sp}\,dp.$$
Why does this complete the theorem?

Problem 7.8 Compute $t * e^{-3t}$ and $e^{-3t} * t$. Which is easier? What is the Laplace inverse of $\dfrac{1}{s^2}\dfrac{1}{s+3}$?

The Heaviside models discontinuities

It's time to look at the Laplace transform of the Heaviside function. This problem is the key to why Laplace transforms work so nicely with piecewise defined functions. We'll compute the Laplace transform of both $f(t-a)u(t-a)$ and $f(t)u(t-a)$. Then we'll practice on a few problems.

Problem 7.9: t-shifting Theorem Suppose that $y = f(t)$ is a function for which you can find the Laplace transform.

1. Show, using the definition of the Laplace transform, that we have

$$\mathscr{L}\{f(t-a)u(t-a)\} = \mathscr{L}\{f(t)\}e^{-as}.$$

   In particular, this means that $\mathscr{L}\{1u(t-a)\} = \frac{1}{s}e^{-as}$.

2. Then show that

$$\mathscr{L}\{f(t)u(t-a)\} = \mathscr{L}\{f(t+a)\}e^{-as}.$$

[Hint: For the first part, just write down the definition of the Laplace transform. You'll have to do a substitution $w = t - a$. Remember that $u(t-a) = 0$ if you are below a, which should allow you to remove $u(t-a)$ from any integral, after updating the bounds. In particular, you know that

$$\int_0^\infty e^{-st}f(t-a)u(t-a)dt = \int_0^a e^{-st}f(t-a)(0)dt + \int_a^\infty e^{-st}f(t-a)(1)dt = 0 + \int_a^\infty e^{-st}f(t-a)dt.$$

]

Problem 7.10 Compute the Laplace transforms of each of the following functions.

See Schaum's chapter 24 for lots more practice. Please do a bunch of these until you feel like you have the idea down. Each problem takes just a tiny bit of time. Unless you practice this a bunch, you'll be lost and spend gobs of time on the upcoming problems.

1. $f(t) = 3u(t-4)$
2. $f(t) = 3(t-4)\,u(t-4)$
3. $f(t) = 3t\,u(t-4)$ [Hint: $3t = 3(t-4) + 12$]
4. $f(t) = (t-3)^2u(t-3)$
5. $f(t) = t^2(u(t-2) - u(t-5))$

Problem 7.11 Compute the inverse Laplace transform of each of the following functions.

1. $\dfrac{4}{s^3}e^{-2s}$ 2. $\dfrac{4}{(s+5)^3}e^{-2s}$ 3. $\dfrac{2s+1}{s^2+9}e^{-\pi s/6}$ 4. $\dfrac{3s+4}{(s+2)^2+16}e^{-5s}$

Laplace Transform Practice

We are now ready to solve Laplace transform problems with the Heaviside function. The simplest example is an RL circuit.

Problem 7.12 Consider an RL circuit with $R = 4$ and $L = 1$. At $t = 0$ there is no current in the wire. Two seconds later ($t = 2$), we connect a 9V battery to the circuit. Three seconds later ($t = 5$) we remove the battery. This gives us the electromotive force as $E(t) = 9(u(t-2) - u(t-5))$. We need to solve the IVP

$$1I' + 4I = 9(u(t-2) - u(t-5)), \quad I(0) = 0.$$

Use Laplace transforms to predict the current $I(t)$ at any time t. Check your answer with technology. [Hint: You'll want to ignore the e^{-as} terms when you perform any needed partial fraction decompositions.]

Problem 7.13 Consider a vertical mass spring system, where we attach a magnetic brick to the bottom of the spring. Let $y = 0$ be the equilibrium height of the brick after accounting for gravity (so we can ignore the force of gravity). Suppose that $m = 1$, $c = 0$, and $k = 4$. The spring is pushed upwards 1 unit, and then let go from rest. After 3 seconds, an electromagnet pulls down on the spring with a force of 7 (the units all agree). We can model this using $r(t) = -7u(t-3)$. Solve the ODE $y'' + 4y = -7u(t-3)$. Check your solution with Mathematica or Sage.

Convolutions provide an inverse product rule

Problem 7.14 Let $f(t)$ be a differentiable function.

1. Show that $f'(t) * 1 = f(t) - f(0)$.
2. Use the previous part to quickly compute the convolutions $1 * 1$, $t * 1$, $t^2 * 1$, $\cos(t) * 1$, $\sin(\omega t) * 1$, and $e^{at} * 1$.
3. The convolution theorem $\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)$ allows us to quickly rephrase each of the preceding computations. For example, we can write the rule $1 * t = t^2/2$ using the convolution theorem as $\mathscr{L}^{-1}\left\{\frac{1}{s}\frac{1}{s^2}\right\} = t^2/2$. Rewrite the rule $\sin(\omega t) * 1 = \frac{1}{\omega} - \frac{1}{\omega}\cos(\omega t)$ in this way.

The Dirac Delta models impulses

Problem 7.15: Discovering the Dirac Delta $\delta(t-a) = u'(t-a)$ In problem 7.14, we showed that $f'(t) * 1 = f(t) - f(0)$ provided f is a differentiable function. Let's ignore this assumption, and just assume the result $f'(t) * 1 = f(t) - f(0)$ holds always. Ignoring assumptions and using results blindly can be dangerous and lead to bogus results. However, it's also a way to make conjectures and find new results that actually do work. When mathematicians find useful results in this fashion (by blinding using tools that have no theoretical backing), they'll search for alternate proofs that the useful results work. This can sometimes take decades to fully complete, during which time other branches of science will often just start using the useful results without having a firm theoretical foundation. Ask me in class about examples of this if you would like more information.

1. Let $f(t) = u(t-a)$, the Heaviside function, where $a > 0$. Explain why $u'(t-a) * 1 = u(t-a)$. We'll use the notation $\delta(t) = u'(t-a)$.
2. Using the convolution theorem $\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)$, compute the transform of both sides of $u'(t-a)*1 = u(t-a)$. You know the transform of both 1 and $u(t-a)$. Use this to show that $\mathscr{L}\{u'(t-a)\} = e^{-as}$.
3. Construct a graph of $u(t-3)$. From your graph, explain why the derivative of $u(t-3)$ at $t = 2$ is zero. What's the derivative at $t = 5$? What's the derivative at any point other than $t = 3$?

4. If you were forced to state a value for the derivative of $u(t-3)$ at $t=3$ and saying, "it's not differentiable at $t=3$," is not an allowed answer, what answer would you give? It's OK to be wrong here. Just make sure you give an answer, and then write a sentence or two to explain why you gave this answer.

Because of our work above, let's make a definition

Definition 7.4: Dirac Delta Distribution (or function). We define the Dirac delta distribution to be the "function"

$$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}.$$

- The Dirac delta $\delta(t-a)$ is not really a function, because the output is ∞ (not a real number) at a single point.
- The derivative of a Heaviside $u(t-a)$ is the Dirac delta $\delta(t-a)$, which we'll write as
$$\frac{d}{dt}u(t-a) = \delta(t-a).$$
- We'll require that the Dirac Delta distribution satisfy the integral sifting property
$$\int_0^\infty g(t)\delta(t-a)dt = g(a).$$
This just means when you multiply a function by the Dirac delta and integrate, you eliminate everything except the function at that single point. The Dirac delta provides a pulse at a single point.

Engineers often refer to both u and δ as singularity functions. They use the notation $\langle x-a\rangle^0$ for the Heaviside, and $\langle x-a\rangle^{-1}$ for the Dirac delta. In future engineering courses, you'll eventually use $u''(t-a) = \langle x-a\rangle^{-2}$ and $u''' = \langle x-a\rangle^{-3}$. The last part shows up when analyzing the displacement of a beam when two beams are connected with a pin.

What happens if we hit a mass-spring system with a hammer? A hammer blow applies a really large force over a very small amount of time. To analyze this scenario, we could instead place a magnet underneath a magnet brick that is attached to a spring, and the pull the brick down. We'd have to leave the magnet on for a while to get the brick to come down. With the magnet, we can apply a small force over a long period of time. If we hit the brick with a hammer, the blow occurs almost instantly, and yet could result in the exact same downward pull as the magnet.

This problem has you develop the connection between magnets and hammer blows. You should see that a hammer blow is like having an infinitely strong magnet on for no time, which is precisely how we defined the dirac delta.

Problem 7.16 Consider a mass-spring system with $m = 1$ kg, $c = 3$ kg/s, and $k = 2$ kg/s^2. Initially the system is at rest (so $y(0) = 0$ and $y'(0) = 0$). At $t = 1$, we turn on a magnet underneath the brick. Let's vary the strength of the magnet, and the length of time we leave the magnet on, and then solve the IVP $y'' + 3y' + 2y = r(t)$, $y(0) = 0$, $y'(0) = 0$.

1. Solve the IVP if the magnet pulls down with a force of 10 N for 1 second so that $r(t) = -10[u(t-1) - u(t-2)]$. This is the only one you should solve by hand.

2. If we doubled the force of the magnet and left the magnet on for half the time, we'd need use $r(t) = -20[u(t-1) - u(t-(1+1/2))]$. If we quadrupled the force, and quartered the time the magnet was on, we'd need $r(t) = -40[u(t-1) - u(t-(1+1/4))]$. If we times the force by 20, but leave the magnet on for one twentieth the time, what should $r(t)$ equal?

3. If we leave the magnet on for h seconds, what should $r(t)$ equal?

4. Our goal is to understand what happens to a mass spring system if we continue to decrease the time the magnet is on, but correspondingly increase the force. To do this, we need to solve the ODE

$$y'' + 3y' + 2y = -\frac{10}{h}[u(t-h) - u(t-(1+h))].$$

Rather than solve this ODE by hand, let's use technology to visualize the solution. Open the Sage worksheet "Hammer Blow and Magnet Comparision." This worksheet shows the position of the spring during the first 10 seconds if we let $h = 1, 1/2, 1/4, 1/20$. Try changing it so $h = 1/100$ and $h = 1/10000$. Describe what happens to the graph of the solution as $h \to 0$.

Laplace Transform Practice

Problem 7.17 Consider the IVP $y'' + 7y' + 10y = 0$, $y(0) = 4$, $y'(0) = -3$.

1. Use Laplace transforms and a partial fraction decomposition to solve the IVP. Write your answer as a linear combination of e^{-2t} and e^{-5t}.

2. Complete the square on $s^2 + 7s + 10$ and use the Laplace transform with cosh and sinh rules to solve the IVP.

3. Which is easier?

Problems 7.2 and 7.17 reviewed the main ideas used in solving ODEs with Laplace transforms. In addition, these problems illustrate some of the advantages of using the hyperbolic trigonometric functions $\cosh \omega t$ and $\sinh \omega t$. They can greatly simplify computations.

The Heaviside models discontinuities

Problem 7.18 Consider an RL circuit with $R = 5$ and $L = 1$. We connect a variable voltage source to the circuit and start to ramp up the power. Suppose our electromotive force is $E(t) = t$ volts. After 12 seconds, we loose power and $E(t)$ drops to zero. Solve for the current in the wire at any time t.

Remember that the key ODE is $LI' + RI + \frac{1}{C}Q = E$. We only have to compute E' if there is a capacitor in the problem.

[Hint: The electromotive force is $E(t) = t - tu(t-12)$, which means we solve

$$I' + 5I = t - tu(t-12), \quad I(0) = 0.$$

Use Laplace transforms and the t-shifting theorem to complete this.]

The Dirac Delta models impulses

We now have all the tools we need to solve some pretty cool electricity and mass-spring problems. We couldn't tackle any capacitors in our discontinuous problems before now, because we would have needed to compute the derivative $E'(t)$. Now that we have derivatives of Heavisides (namely Dirac deltas), we can solve these problems.

Problem 7.19 Consider an RC circuit with $R = 2$ and $C = 1/8$. Suppose that the capacitor has no charge on it, and there is no current flowing at $t = 0$. At $t = 2$, we connect a 12 V battery to the circuit. Then at $t = 7$ we remove the battery. Set up an IVP that would give the current at any time t, and then solve the IVP. Use software to construct a graph of your solution. [Hint: There are no partial fraction decompositions in this one, so it should be REALLY fast.]

Remember that the key ODE is $LI' + RI + \frac{1}{C}Q = E$. We have to compute E' as there is a capacitor in the problem.

Problem 7.20 Consider a mass spring system with no friction. Let $m = 1$ and $k = 9$. We'll examine what happens if a hammer hits the system. Suppose the spring is initially at rest at equilibrium. After 3 seconds, a hammer hits the spring downwards with a force of 10 N (so $r(t) = -10\delta(t-3)$). The mass-spring system starts to oscillate. Set up and solve an IVP what would give the position of the spring at any time t, and state the amplitude of the oscillation. [Again, there are no partial fraction decompositions on this problem, so it should go quite fast.]

Convolutions provide an inverse product rule

Problem 7.21 Complete each of the following:

1. Compute the convolution $\sin(2t) * 1$. Then solve the IVP $y'' + 4y = 1$, $y(0) = 0$, $y'(0) = 0$.
2. Compute the convolution $\sin(3t) * t$. Then solve the IVP $y'' + 9y = t$, $y(0) = 0$, $y'(0) = 0$.

Problem 7.22 We've been using the modification rule when we have double complex roots. Use convolutions to find the Laplace inverse of

$$\frac{1}{(s^2+9)^2} = \frac{1}{s^2+9}\frac{1}{s^2+9}.$$

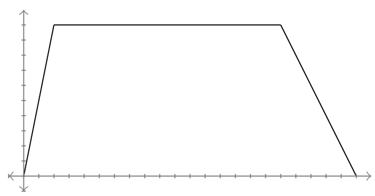
Laplace Transform Practice

Problem 7.23 Consider an RLC circuit, with $R = 6$, $L = 1$, and $C = 1/10$. Suppose that at $t = 0$ there is no charge on the capacitor, and no current in the wire. We attach a variable voltage source to the RLC circuit, $E(t) = 2t$, and ramp up the power for the first 6 seconds. After 6 seconds, the voltage drops to zero. This gives us $E(t) = 2t(1 - u(t-6))$. Set up and solve an IVP that would give the current in the wire after t seconds.

Remember that the key ODE is $LI' + RI + \frac{1}{C}Q = E$. We have to compute E' as there is a capacitor in the problem.

Problem: Optional We decide to launch a rocket. We attach an engine to the rocket, and light it. Neglect the mass of the fuel. The mass of the rocket and engine we'll assume is 4 kg. Let's launch the rocket in space, so we can ignore the force due to gravity. Since we are in space, we don't have air resistance, so instead let's assume that we've put some Jello out in space, and the rocket plans to fly through the Jello (something to slow it down). Assume that the force due to the Jello's resistance is proportional to the velocity of the rocket, with proportionality constant 8 kg/s. When we light the engine, for the first 2 seconds the force ramps up, following a linear path until it gives a force of 10 N after 2 seconds. Then for 15 more seconds the rocket maintains a force of 10 N. The force then starts to ramp down linearly, taking an additional 5 seconds until the force drops to zero. A picture of this force is below.

Ask me in class to show you how to modify this to add in gravity, or come by and show me what you would do. It's a pretty fun switch.)



1. Set up an initial value problem whose solutions is the position of the rocket after any time t. The right hand side should be written in terms of Heaviside functions, where the discontinuities occur after 2, 17, and 22 seconds.
2. Use Laplace transforms and technology to solve your IVP. Let the computer do all the solving. The goal is to just GET a solution, and then we'll interpret it.
3. The rocket will eventually get stuck in the Jello. How far will it travel before the rocket stops moving? See the side for a hint.

Why must every term in the problem that involves an exponential eventually vanish? What's left over after you ignore all these terms?

Problem 7.24 Consider the IVP $y'' + 5y' + 6y = 0$, $y(0) = 0$, and $y'(0) = 1$. Solve this IVP in three ways.

1. Laplace both sides, and then use a convolution (no partial fraction decomposition) to obtain a solution.
2. Laplace both sides, but use a partial fraction decomposition.
3. Obtain the homogeneous solution using the characteristic equation, and then use the initial conditions to obtain the constants.

We'll compare your three solutions in class, and discuss why someone would care about the convolution approach (if you don't see why it's so cool as you use it).

Problem If you have a beam, and you know there is a distributed load on the beam, with a point force applied at another spot on the beam, can you compute the shear stress and the moment at any point in the beam? This is exactly the same question as finding the Velocity and position of an object, provided you know the acceleration on the rocket (some external driving force turned on/off based on time) and the rocket is hit by hammers, or meteors (dirac delta) at various points along its path.

I'll be talking about this problem on the next optional day (the day I cancel class while you are getting ready for the exam).

If you have had strength of materials, then this problem connects how you use use Heaviside and Dirac delta to tackle all the problems in strengths. You can also use this to solve problems in Statics. You can use it to solve any problem where you have a distributed load (use Heavisides to turn it on and off), and a point force (Dirac delta).

Wrap up

This concludes the chapter. Look at the objectives at the beginning of the chapter. Can you now do all the things you were promised?

Problem 7.25: Lesson Plan Creation Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your one-page lesson plan. You may use both sides. The objectives at the beginning of the chapter give you a list of the key concepts. Once you finish your lesson plan, scan it into a PDF document (use any scanner on campus), and then upload the document to I-Learn.

This counts as 4 prep problems. My hope is that you spend at least an hour creating your one-page lesson plan.

Extra Practice

You can find additional practice problems in Schaum's Outlines *Differential Equations* by Richard Bronson. You'll find relevant problems in chapters 21 -24, as well as some extra practice problems related to mass spring systems and electrical networks below. Do enough of each type that you feel comfortable with the ideas.

You must practice lots of problems to gain a feel for patterns. Many of the problems in 21-23 are fast. Please take a few minutes every day to just flat out practice with the basics (kind of like when you were learning the times tables - they get really fast if you just practice them). When you feel like you have the basics down, see if you can complete chapters 21 and 22 in less than an hour. If one stumps you, skip it and come back later.

Once you feel confident, chapters 23 (on convolutions and the heaviside function) and 24 (solving IVPS) will help you use the Laplace transforms to solve ODEs. Table 7.1 at the beginning of this chapter summarizes the transforms we use most often.

Make sure you try some of each type of problem from chapters 21-24 (except for the last set of problems in 23). The new ideas involve convolutions and the Heaviside (unit step) function in 23. Once you have tried some of each of these, use the problems below to give you more practice.

I Find the Laplace transform of each of the following, and use the Mathematica Tech Intro to check your answer.

1. $f(x) = 8e^{-3x}\cos 2x - 4e^{4x}\sin 5x + 3e^{7x}x^5$
2. $f(x) = xu(x-4) + \delta(x-6)$
3. $f(x) = e^{3x}u(x-2) + 7\delta(x-4)$

II Find the inverse Laplace transform of each of the following, and use Mathematica to check your answer. Many of these will require you to use a partial fraction decomposition.

4. $\dfrac{s}{(s+3)^2+25} + \dfrac{2}{(s-2)^4}e^{-3s}$
5. $\dfrac{s}{s^2+4s+13}e^{-4s}$
6. $\dfrac{1}{s(s^2+1)}e^{-5s}$
7. $\dfrac{1}{s^2(s^2+1)}e^{-3s}$
8. $\dfrac{2s+1}{(s-1)^2(s+1)}e^{-7s}$
9. $\dfrac{1}{(s-1)(s+2)(s-3)}e^{-4s}$

III Use Laplace transforms to find the position $y(t)$ of an object or current current $I(t)$ in each of the following scenarios. I will give you the constants m, c, k and the driving force $r(t)$, or I will give you the inductance L, resistance R, capacitance C, and voltage source $E(t)$, as well as any relevant initial conditions. Your job is to use Laplace transforms to find the solution. Use Mathematica to check your solution, and draw the graph of $y(t)$ or $I(t)$ and the steady-state (steady periodic) solution to see how the Heaviside and Dirac delta functions affect the graph. The point here is to see these two new functions affect solutions. I suggest that you do all of these problems with the computer, so you can quickly see the effects of a Heaviside function or Dirac delta distribution.

10. $m = 1, c = 0, k = 4, r(t) = u(t-1), y(0) = 1, y'(0) = 0$
11. $m = 1, c = 0, k = 4, r(t) = \delta(t-3), y(0) = 1, y'(0) = 0$

12. $m = 1, c = 0, k = 4, r(t) = 7u(t-3), y(0) = 1, y'(0) = 0$
13. $m = 1, c = 0, k = 4, r(t) = 7u(t-3) + 11\delta(t-5), y(0) = 1, y'(0) = 0$
14. $m = 1, c = 0, k = 4, r(t) = 7tu(t-3), y(0) = 1, y'(0) = 0$
15. $m = 1, c = 0, k = 4, r(t) = 7, y(0) = 1, y'(0) = 0$
16. $m = 1, c = 0, k = 4, r(t) = 7, y(\pi) = 1, y'(\pi) = 0$
17. $m = 1, c = 3, k = 2, r(t) = u(t-2), y(0) = 0, y'(0) = 0$
18. $m = 1, c = 3, k = 2, r(t) = \delta(t-2), y(0) = 0, y'(0) = 0$
19. $m = 1, c = 3, k = 2, r(t) = 4u(t-1), y(0) = 0, y'(0) = 0$
20. $m = 1, c = 3, k = 2, r(t) = 4u(t-1) + 10\delta(t-2), y(0) = 0, y'(0) = 0$
21. $m = 1, c = 3, k = 2, r(t) = 4tu(t-1), y(0) = 0, y'(0) = 0$
22. $L = 0, R = 2, C = 1/5, E(t) = 12u(t-2), Q(0) = 0$ (use first order ODE)
23. $L = 1, R = 2, C = 0, E(t) = 12u(t-2), I(0) = 0$ (use first order ODE)
24. $L = 1, R = 2, C = 1/5, E(t) = 12, Q(0) = 0, I(0) = 0$ (first find $I'(0)$.)
25. $L = 1, R = 2, C = 1/5, E(t) = 12u(t-2), Q(0) = 0, I(0) = 0$
26. $L = 1, R = 2, C = 1/5, E(t) = e^{3t}u(t-2), Q(0) = 0, I(0) = 0$
27. $L = 1, R = 2, C = 1/5, E(t) = 4\cos(3t), Q(0) = 0, I(0) = 0$
28. $L = 1, R = 4, C = 1/4, E(t) = u(t-3), Q(0) = 0, I(0) = 0$
29. $L = 1, R = 4, C = 1/4, E(t) = e^{-2t}, Q(0) = 0, I(0) = 0$

Chapter 8

Power Series

This chapter covers the following ideas. When you create your lesson plan, it should contain examples which illustrate these key ideas. Before you take the quiz on this unit, meet with another student out of class and teach each other from the examples on your lesson plan.

1. Compute MacLaurin series for various common functions, either directly by taking derivatives, or by solving ODEs.
2. Use the power series method to solve ODEs where $x = 0$ is an ordinary point.
3. Explain the Frobenius method and use it to solve ODEs where $x = 0$ is a regular singular point.
4. Define the Gamma function and show how it generalizes the factorial. Be able to compute the Gamma function at any multiple of $\frac{1}{2}$.

You'll find extra practice problems at the end of this chapter. You can use these to gain practice with the ideas. Handwritten solutions are available online. Click for solutions.

8.1 MacLaurin Series

8.1.1 MacLaurin Series and ODEs

As we proceed in this unit, we'll be looking for patterns. When you are looking for patterns, one key rule is to avoid simplifying. Instead of writing $2 \cdot 3 = 6$, just leave it as $2 \cdot 3$. If notice a pattern, like $2 \cdot 3 \cdot 4 \cdot 5$, then write $5!$ instead of 120. If you will resist the urge to simplify, you'll find a lot of patterns immediately pop out.

Problem 8.1 Consider the function $f(x) = e^x$. In this problem we would like to approximate $f(x)$ using various polynomials. We'd like to make sure that the function and its derivatives match the polynomial and its derivatives.

1. Let's approximate $f(x)$ using a 4th degree polynomial. Write the polynomial as
$$P_4(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4,$$
where the coefficients $a_0, a_1, \ldots, a_4$ are unknown (we'll discover them in a bit). Compute the first 4 derivatives of $P_4(x)$ and the first 4 derivatives of $f(x)$. As there are 5 unknowns, we need 5 equations. Let's require that f

and P_4, together with their derivatives, match at $x = 0$. This gives us the 5 equations

$$\begin{aligned} f(0) &= P_4(0), \\ f'(0) &= P_4'(0), \\ f''(0) &= P_4''(0), \\ f'''(0) &= P_4'''(0), \text{ and} \\ f''''(0) &= P_4''''(0). \end{aligned}$$

Use these equations to solve for the unknown constants.

2. If you wanted a 7th degree polynomial, what should the coefficients a_5, a_6, and a_7 equal?

In this chapter, our goal is to solve ODEs where the coefficients are no longer constant. We'll learn how to solve a mass-spring problem where the mass is changing, the spring constant errodes over time, or the friction coefficient increases as we tighten a dashpot. We'll also gain the key ideas need to deal with rocket problems where the mass decreases because fuel burns up. To solve these problems, we're going to start approximating functions with polynomials. We'll be using really large polynomials. We'll then solving the problems using these polynomials. The only catch is that we'll start using polynomials that are arbitrarily large. These polynomials are called Taylor polynomials. When we consider an infinitely long polynomial, we call it a Taylor series, or MacLaurin series. We'll get a formal definition in a bit.

Problem 8.2 We already know the solution to the ODE $y' - y = 0$ (see part 1). Let's find this solution using a series approach. Suppose we write

$$y = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots,$$

where the polynomial continues on for as long as we want (why not forever). We'll use this polynomial to find a solution.

1. Solve the ODE $y' - y = 0$ by any method you would like. The characteristic equation might make this really fast.

2. Now consider the series (infinitely long polynomial) above. Compute y' by computing the derivative (so $y' = 0 + a_1 + 2a_2x + \cdots$). Write out the first 7 terms or so.

3. Now subtract y from y'. You can combine the two infinite sums by adding coefficients that are multiplied by collecting the coefficients on the same powers of x. You'll get an infinitely long sum of the form

$$(a_1 - a_0) + (2a_2 - a_1)x + (?)x^2 + (?)x^3 + \cdots.$$

Carry this out 7 terms. What pattern do you see?

4. Because $y' - y = 0$, and $0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$, you should now have an infinitely large system of equations by equating coefficients. The first two equations are $a_1 - a_0 = 0$ and $2a_2 - a_1 = 0$. If you let $a_0 = c$, then solve for a_1, a_2, a_3, and so on in terms of c. What is a_n in terms of c?

The last two problems dealt with the function e^x. Let's now turn our attention to $\cos x$ and $\sin x$.

Problem 8.3 Let $f(x) = \cos x$.

1. Find a 6th degree polynomial to approximate cosine. So let

$$P(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6.$$

   Now require that f and P have the same values at $x = 0$, and that the first 6 derivatives of both f and P have the same values at $x = 0$. You might want to organize your work in table (keep track of f, its first 6 derivatives, and their values at $x = 0$, as well as P, its first 6 derivatives, and their values at $x = 0$). What pattern do you see?

2. Guess what the 20th degree polynomial would be.

3. If $x = 2$, use a calculator to compute $\cos(2)$ as well as $P(2)$ for your 6th degree polynomial. We'll compute $P(2)$ for your 20th degree polynomial in class.

Problem 8.4 We know the solution to the IVP $y'' + y = 0$, $y(0) = c$, $y'(0) = d$ is $y(t) = c\cos(t) + d\sin(t)$. Suppose that

$$y = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$

1. Compute both y' and y'' by taking the derivative, term-by-term, of the infinitely long series. REMEMBER, DO NOT SIMPLIFY. So you should have something like

$$y'' = 2 \cdot 1a_2 + 3 \cdot 2a_3x + \cdots.$$

   Continue this out 7 terms.

2. We want to solve $y'' + y = 0$, so add together y'' and y. Group together terms that are multiplied by the same power of x, so your answer will look something like

$$y'' + y = (2 \cdot 1a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (?)x^2 + (?)x^3 \cdots.$$

   Carry this out 7 terms.

3. If $y(0) = c$, then what is a_0? If $y'(0) = d$, then what is a_1?

4. Write a_2, a_3, a_4, and so on, in terms of c and d. Can you guess the Taylor series for $\sin x$?

Problem 8.5: MacLaurin Series Suppose we write $f(x)$ as the series

$$f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$

1. Compute the first 4 derivatives of f and evaluate them at 0. What pattern do you see? State the nth derivative of f evaluated at $x = 0$, which we write as $f^{(n)}(0)$.

2. Solve for the coefficient a_n in terms of the nth derivative of f.

3. Let $f(x) = \sin x$. Compute the first 8 derivatives of $\sin x$ and evaluate each at $x = 0$. Then use the pattern you see to state what a_n equals for each n if we write
$$\sin x = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots .$$
Carry out your sum until you hit x^9. If you continue forever, we call this infinite polynomial the MacLaurin series of $\sin(x)$.

Based on your results to the previous problem, we make the following definitions.

Definition 8.1: MacLaurin Series. Let $f(x)$ be a function. We define the MacLaurin series of $f(x)$ to be the infinite series

$$a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

where $a_n = \dfrac{f^{(n)}(0)}{n!}$. We use the notation $f^{(n)}(x)$ to denote the nth derivative. Note that $0! = 1$, and that $f^{(0)}(x)$ is the 0th derivative (so original function). With this notation, we could write the MacLaurin series as

$$f(0)+f'(0)x^1+\frac{f''(0)}{2!}x^2+\frac{f''''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Definition 8.2: Power Series. A power series is an expression of the form

$$a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots = \sum_{n=0}^{\infty} a_nx^n,$$

where a_n is any real number. It's a power series because we create an infinite series using powers of x.

A MacLaurin series is a power series. We'll often start with a power series, and then look for the function $f(x)$ whose MacLaurin series is the power series we started with.

The MacLaurin series of a function depends on the value of the function and its derivatives at $x = 0$. Sometimes, you would rather compute the function and its derivatives at another spot. We won't have much use for doing this in our course, but for completeness, you should see the full definition of a Taylor series centered at $x = c$.

Definition 8.3: Taylor Series centered at $x = c$. Let $f(x)$ be a function. We define the Taylor series of $f(x)$ centered at $x = c$ to be the infinite series

$$a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + a_(x-c)x^4 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

where $a_n = \dfrac{f^{(n)}(c)}{n!}$. The MacLaurin series is the Taylor series centered at $x = 0$.

Let's compute a few more MacLaurin series.

Problem 8.6: MacLaurin series for $\cosh x$ and $\sinh x$ Obtain the first 10 terms of the MacLaurin series for both $\cosh x$ and $\sinh x$. Do so by using the formula $a_n = \dfrac{f^{(n)}(0)}{n!}$. Write out the two series. What patterns do you see. Write down a formula for the coefficient a_n for both $\cosh x$ and $\sinh x$.

The next problem shows you how to obtain the MacLaurin series for $\cosh x$ and $\sinh x$ in a different way.

Problem 8.7 Consider the IVP given by $y'' - y = 0$, with $y(0) = A$ and $y'(0) = B$.

1. Show, using Laplace transforms, that the solution to this IVP is $y(x) = A\cosh x + B\sinh x$.
2. We'll now obtain the solution using power series. Suppose

$$y = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$

Compute y' and y''. Substitute y and y'' into the ODE $y'' - y = 0$, and group together terms that are multiplied by the same power of x. You should have something of the form

$$(2a_2 - a_0) + (3 \cdot 2a_3 - a_1)x + (?)x^2 + \cdots = 0 + 0x + 0x^2.$$

3. Use the initial conditions to explain why $a_0 = A$ and $a_1 = B$. Then solve for a_2, a_3, and so on, in terms of A and B. Keep going until you see a pattern for a_n.
4. You now have the solution y. Some of the coefficients depend on A. Some depend on B. Group together the terms that involve A and the terms that involve B, and write your solution in the form

$$y = A(1 + \frac{1}{2!}x^2 + \cdots) + B(x + \frac{1}{3!}x^3 + \cdots).$$

Please carry out each series at least 5 terms.

We have so far developed the following MacLaurin series:

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \\
\cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots \\
\cosh(x) &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \\
\sinh(x) &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots.
\end{aligned}$$

Problem 8.8: Euler's Formula Use the MacLaurin series above to show that we can express e^{ix} in the form

$$e^{ix} = \cos x + i\sin x$$

and that

$$\cosh(ix) = \cos x.$$

Then use the first fact to compute $e^{i\pi}$. What does $\sinh(ix)$ equal, if written in terms of $\sin x$?

The equation

$$e^{ix} = \cos x + i\sin x$$

is called Euler's formula.

We'll return to Euler's formula in a minute. Before we do so, let's examine a different function.

Problem 8.9 Consider the IVP $(x+1)y' = 1$, $y(0) = 0$. Is it linear? Is it homogeneous? Does it have constant coefficients? Solve the ODE first by using separation of variables. Then solve the ODE using a power series (assume $y = a_0 + a_1x + a_2x^2 + \cdots$, compute y', plug these into the ODE, and then solve for the unknown constants a_0, a_1, a_2, etc.). From your answer, what are the first 5 terms in the the MacLaurin series of $\ln(x+1)$?

Problem 8.10 Find a 9th degree polynomial to approximate $\ln(x+1)$. Do this by computing the first 9 derivatives of $\ln(x+1)$ or by computing the first 4 and noticing a pattern when you plug in 0. The formula $a_n = \frac{f^{(n)}(0)}{n!}$ will give you to coefficients. Then use your polynomial to estimate $\ln 1.2$, $\ln(1-.8) = \ln(.2)$ and $\ln 2.5$. Use a computer to draw $\ln(x+1)$ and your 9th degree polynomial. For which values of x do you think the polynomial will do a poor job approximating $\ln(x+1)$. Why do you think this? Will increasing the degree of your approximation ever help you approximate $\ln 2.5$?

We've now turned multiple functions into power series. Polynomials are extremely easy to differentiate and integrate. What happens if we differentiate or integrate a power series. Do we get the power series of the derivative of the function?

Problem 8.11 For each function below, compute the derivative of the function, and the derivative of the power series. Write your solution in summation notation. Then answer the question at the end.

1. $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$
2. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$
3. $\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}x^n.$
4. What function has the MacLaurin series $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$. [Hint: If you modify the derivative on part 3 slightly, you should get this.]

Problem 8.12 In the last problem you showed that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$. Let's obtain the MacLaurin series for $\arctan x$.

1. Compute the first 3 derivatives of $f(x) = \arctan x$. Use this to obtain the third degree Taylor polynomial of $\arctan x$, centered at $x = 0$.
2. We know that $\int \frac{1}{1+x^2}dx = \arctan x$. Replace each x with x^2 in $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$, and then integrate to obtain a power series for $\arctan x$. Write your answer with summation notation. Does it match your first answer?

Problem 8.13 Let's solve the IVP $(x^2+1)y' = 1$, $y(0) = 0$ in two ways.

1. Use the power series method. Let $y = \sum_{n=0}^{\infty} a_n x^n$, compute y', plug these into the ODE, collect coefficients of the same powers of x, and then solve for the unknowns a_n.
2. Use separation of variables.

Compare your solution here with the previous problem.

Problem 8.14 Solve the ODE $y' + 2xy = 0$ by using power series. Your initial condition will just be $y(0) = a_0$. After you have a solution, look at the table of known power series and try to match the solution you got to one of our known power series (you might have to replace x with something). Then use separation of variables to solve the ODE, and check if you are correct.

In the previous problem, the power series solution results in a series that we can match with a series we already recognize. We might have to replace x with x^2, but the power series is still quite manageable. Things won't always be this nice.

Problem 8.15 Solve the ODE $y'' + 2xy' + y = 0$ by using power series. Your initial conditions are $y(0) = a_0$ and $y'(0) = a_1$. When you're done, write your solution as

$$y(x) = a_0(y_1(x)) + a_1(y_2(x))$$

where y_1 and y_2 are power series. Just give the first 4 terms of y_1 and y_2, together with a rule that would allow us to compute more terms if needed (so how could I find a_{10} if I knew a_8 and a_9, or better yet, how could I find a_{n+2} if I knew a_n and a_{n+1}).

You won't find either y_1 or y_2 on the list of power series we recognize.

Problem 8.16 Consider the ODE $y'' + 3xy' + 2y = 0$. Solve this ODE using power series methods. Write your answer by give the first 6 nonzero terms of the series, and make sure you state a recurrence relation that will give more coefficients of the series. Write your answer in the form $y(x) = a_0(y_1(x)) + a_1(y_2(x))$, so give the first three nonzero terms of y_1 and y_2.

Problem 8.17 Consider the ODE $(1+x^3)y'' + 3x^2y' + 2xy = 0$. Solve this ODE using power series methods. Write your answer by give the first 6 nonzero terms of the series. State a recurrence relation that will give more coefficients of the series. What is a_{62}, the coefficient in front of x^{62}?

You've now seen the power series technique used to solve many problems. Sometimes the power series does a good job of approximating the function. Sometimes, it does a really bad job. How can we determine the difference between when a power series does a good job, and when it does a bad job? That's the content of this section.

Problem 8.18: Geometric Series Consider the infinite series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n.$$

This series is called a geometric series. You can obtain the next terms in the series by multiplying by the common ration r.

If we add up the first k terms, we obtain the kth partial sum

$$S_k = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{k-1} = \sum_{n=0}^{k-1} ar^n.$$

Our goal on this problem is to determine for which a and r we can compute the limit as $k \to \infty$ of the partial sums, and obtain a value.

1. Show that $S_k = \frac{a(1-r^k)}{1-r}$. [Hint: Consider the difference $s_k - rs_k$. Just write out each. Lots should cancel.]
2. Compute $\lim\limits_{k\to\infty} S_k$. For which a and r does this limit exist, and for which does it not exist. Explain.
3. We have seen the power series given by

   $$1 + x + x^2 + x^3 + x^4 + \cdots .$$

   If we want to know what this infinite sum approaches, we could compute the partial sums and then find the limit of the partial sums. The partial sums are $S_1 = 1$, $S_2 = 1 + x$, $S_3 = 1 + x + x^2$, etc. What is the limit as $k \to \infty$ of these partials sum? For which x does this limit not exist? [Hint: What are a and r from part 2. You already did this in part 2.]

We now have a way to determine the sum of an infinite series. We just look at the partial sums, and then compute their limit (provided it exists). This means we can go back to all the power series we created and ask, "Which of these power series actually have sum that matches the function we started with." Let's make some definitions.

Definition 8.4: Converges, Diverges, Analytic. Consider the infinite series

$$b_0 + b_1 + b_2 + b_3 + b_4 + \cdots = \sum_{n=0}^{\infty} b_n.$$

- The kth partial sum of this series is the sum of the first k terms. So we have $S_1 = b_0$, $S_2 = b_0 + b_1$, $S_3 = b_0 + b_1 + b_2$, etcetera.
- We say the series converges if $\lim\limits_{k\to\infty} S_k$ exists. In this case, we say the series converges to this limit.
- We say the series diverges if $\lim\limits_{k\to\infty} S_k$ does not exist.
- We say that a function is analytic at $x = c$ if it has a power series representation centered at $x = c$ that converges for values of x other than c itself.
- If a function is not analytic at $x = c$, then we say the function is singular at $x = c$.

8.2 Special Functions

We've seen the power series method work for many problems now where the coefficients are not constants. Will this method work for every problem with variable coefficients? No. Why would it fail? Let's considering a few more power series problems, and discover why it would fail. For some, the power series method will work. For others, it will not. By the time we're done with this section, we'll know what to look for. It all has to do with certain coefficients being analytic at $x = 0$.

Problem 8.19: Legendre Polynomials Consider the ODE

$$(1-x^2)y'' - 2xy' + 20y = 0.$$

1. Use the power series method to solve this ODE. Give all the coefficients up to a_6. What is a_8? What is a_{20}?
2. Write your solution in the form $y(x) = a_0 y_1(x) + a_1 y_2(x)$. If $y(0) = 1$ and $y'(0) = 0$, what is the solution?
3. Modify your work above slightly to solve the IVP $(1-x^2)y''-2xy'+12y = 0$, $y(0) = 0$, $y'(0) = 1$. Show that the solution is a polynomial.

Legendre's ODE is

$(1-x^2)y''-2xy'+(n)(n+1)y = 0.$

This ODE shows up when solving Laplace's equation in spherical coordinates (studying heat, waves, gravity, and/or electric/static potentials). When n is an integer, one of the solutions will terminate in a polynomial of degree n. These polynomials are called Legendre polynomials.

As in the problem above, sometimes the power series method gives you a polynomial, because the series stops. In the next problem, the power series method will fail, but you should find that with a slight modification (multiply the power series by x^λ), you quickly get two solutions that each have only one term. The entire solution is a linear combination of these two solutions.

Review Suppose that a 2nd order homogeneous ODE has a solution $y_1(x) = e^{-3x}$. Suppose that another solution is $y_2(x) = e^{-2x}$. State a general solution to this ODE. See [1].

Problem 8.20: Euler-Cauchy Equation Consider the ODE

$$2x^2y'' + 5xy' + y = 0.$$

1. Let's first try the power series, so suppose $y = \sum_{n=0}^{\infty} a_n x^n$. Compute both derivatives and plug them into the ODE. Use this to explain why the only solution that the power series method will get you is $y = 0$.
2. Earlier in the semester we noticed that sometimes to get a solution, we had to multiply by a power of x. Let's see if this works with power series as well. Suppose instead that

$$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\lambda} = a_0 x^\lambda + a_1 x^{\lambda+1} + \cdots.$$

Compute both derivatives and plug them into the ODE.

Any ODE of the form $ax^2y'' + bxy' + cy = 0$, where a, b, c are constants, is called an Euler-Cauchy ODE.

Frobenius suggested that we multiply a power series by x^λ to get a solution. He also gave conditions on the ODE that state when this method is needed, and when it will succeed.

[1] If the ODE is homogeneous, then the solution is a linear combination of two linearly independent solutions, namely

$$y(x) = c_1 e^{-3x} + c_2 e^{-2x}.$$

The solutions y_1 and y_2 are linearly independent, because the only solution to $c_1e^{-3x} + c_2e^{-2x} = 0$ is $c_1 = c_2 = 0$. This is because it is impossible to write one of the functions as a multiple of the other. We obtain solutions by summing together linearly independent solutions.

3. Look at the coefficients in front of x^{lambda}. You should get that either $a_0 = 0$, or that a polynomial equals zero. If we set this polynomial equal to zero, we call the corresponding equation the indicial equation. Find the values of λ that solve the indicial equation. You should get two values for λ. Let's call the largest value λ_1, and the smallest value λ_2.

4. If you replace each λ with λ_1, show that $a_n = 0$ for every $n \geq 1$. Then repeat with $\lambda = \lambda_2$ and show that $a_n = 0$ for $n \geq 1$.

5. You should now have two solutions to this ODE. Use the superposition principle to state a solution to the ODE. Make sure you check your work with the link to WolframAlpha, or use Mathematica.

See Problem **??** if you forgot the superposition principle. You can check your work with Mathematica, or here's a link to WolframAlpha.

Why did the power series method fail in the previous problem? The answer lies in a quick computation. If we take the ODE $2x^2y'' + 5xy' + y = 0$ and divide by the leading coefficient of y'', we obtain

$$y'' + \frac{5}{2x}y' + \frac{1}{2x^2}y = 0.$$

The coefficients of the ODE, namely $\dfrac{5}{2x}$ and $\dfrac{1}{2x^2}$ are now not defined at $x = 0$, hence not analytic at $x = 0$. To guaranteed that the power series method will succeed and give the entire general solution, these coefficients must be analytic at $x = 0$. Let's try one more, and then introduce some vocabulary.

Problem 8.21: Bessel Equation Consider the ODE

$$x^2y'' + xy' + (x^2 - 9)y = 0.$$

1. Rewrite the ODE so that the coefficient in front of y'' is a one. Then state the other coefficients, and show that they are not analytic at $x = 0$. [Hint: See the previous paragraph.]

2. Since the power series method may not give both solutions, let's multiply the series by x^λ (Frobenius's idea) and suppose that

$$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\lambda}.$$

   Compute both derivatives and plug them into the ODE. Multiply the coefficients x^2, x, and $x^2 - 9$ into the sums, splitting the $x^2 - 9$ product into two sums.

3. After collecting common coefficients, the equation containing the coefficients in front of x^λ gives you the indicial equation. Show that $\lambda = \pm 3$. We'll let $\lambda_1 = 3$ and $\lambda_2 = -3$. Frobenius always chose λ_1 to be the larger of these roots.

4. Let $\lambda = 3$, and then solve for the other coefficients a_1, a_2, a_3, a_4, etc. State the solution, making sure to list the first 4 nonzero terms.

In the previous problem, we were only able to obtain one solution y_1 to the ODE. Frobenius showed how to obtain another linearly independent solution, and gave an algorithm for obtaining that solution. If the roots of the indicial equation have a difference that is not an integer, then our current method will give the second solution. However with Bessel's equation above, we got the roots to be ± 3, which differ by the integer 6. This is why we did not find a second solution. You are welcome to study this topic more on your own, if and when you need it.

Let's now focus on when we should use the power series method, and when we should use the Frobenius method. Let's introduce some vocabulary, and state some facts without proof.

Definition 8.5: Ordinary, Singular, Regular Singular. Consider the ODE

$$y'' + b(x)y' + c(x)y = 0.$$

Notice that the coefficient in front of y'' is one. Some people call this form of an ODE the standard form.

- As a reminder, we say that a function is analytic at $x = c$ if it has a power series solution centered at $x = c$ with a positive radius of convergence. Polynomials, exponentials, trig function, and rational functions whose denominator is not zero at $x = c$ are all analytic.
- If $b(x)$ and $c(x)$ are both analytic at $x = 0$, then the solution y to the ODE is analytic. We say that $x = 0$ is an ordinary point of the ODE. The power series method will yield a complete solution.
- If either $b(x)$ or $c(x)$ are not analytic at $x = 0$, then we say that $x = 0$ is a singular point of the ODE. The power series method is not guaranteed to work. You can try it, and you might get lucky.
- If $x = 0$ is a singular point, and both $xb(x)$ and $x^2c(x)$ are analytic, then we say $x = 0$ is a regular singular point of the ODE. We can use the Frobenius method to solve ODEs at regular singular points. The big idea is to guess a solution of the form $y = x^\lambda \sum_{n=0}^{\infty} a_n x^n$ and then solve for λ and the remaining coefficients as in the power series method.
- The indicial equation is the first equation resulting from matching coefficients in the Frobenius method. It's roots λ_1 and λ_2 are sometimes called the exponents of the ODE.

We could also define ordinary, singular, and regular singular points at $x = c$ by considering power series representations centered at $x = c$ instead of $x = 0$.

The next problem asks you to use the vocabulary above to determine which method you should use to solve the ODE.

Problem 8.22 For each ODE, write the ODE in standard form. Then determine if the point $x = 0$ is an ordinary point or a singular point. If it is an ordinary point of the ODE, determine if it is a regular singular point. To solve the ODE, should you use the power series method, the Frobenius method, or neither?

1. $x^2y'' + xy' + (x^2 - v^2)y = 0$ where v is a constant.
2. $x^2y'' + ax^3y' + x^2e^{bx}y = 0$ where a and b are constants.
3. $x^2y'' + cy' + x^ny = 0$ where c is a constant, and n is a positive integer.
4. $\dfrac{d}{dx}\left((1 - x^2)y'\right) = -\lambda y$ where λ is a constant. [Hint: Use the product rule to expand the derivative. Then write the ODE in standard form.]

Ask me in class about how this relates to eigenvalues and Legendre's equation.

Let's end this section with one final problem. In this problem, the difference between the roots of the indicial equation will not differ by an integer.

Problem 8.23 Consider the ODE

$$8x^2y'' + 10xy' + (x-1)y = 0.$$

1. Show that $x = 0$ is a regular singular point of this ODE.
2. State the indicial equation, and obtain the zeros. You should have $\lambda_1 = 1/4$. What is λ_2?
3. When $\lambda = \lambda_1$, obtain the first 3 nonzero terms of the solution, which we'll call y_1.
4. When $\lambda = \lambda_2$, obtain the first 3 nonzero terms of the solution, which we'll call y_2.
5. State the general solution to this ODE.

The Mathematica technology introduction will help you check your work. Just look in the Special Functions section.

There are a lot of special functions that we have not even touched on. You could spend years studying all the special functions that have already been discovered and classified. This section gave you an introduction to the techniques needed to solve these ODEs.

8.2.1 The Gamma Function

We'll end this chapter with one last special function, the Gamma function $\Gamma(x)$. This function generalizes the factorial. We've already learned that the Laplace transform of t^n is $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$. This formula only works if we require n to be an integer. So what about the Laplace transform of something like $\sqrt{t}$? Once we've defined the Gamma function, we'll have the formula

$$\mathscr{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}.$$

The symbol Γ is the uppercase greek letter Gamma. That's why we capitalize the "G" in the Gamma function.

Definition 8.6: The Gamma Function $\Gamma(t)$. We define the Gamma function to be

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$

As x is a dummy variable, we could have also written $\Gamma(t) = \int_0^\infty p^{t-1}e^{-p}dp$ or $\Gamma(x) = \int_0^\infty p^{x-1}e^{-p}dp$.

Problem 8.24 Do the following:

1. Show that $\Gamma(1) = 1$. You are welcome to skip to the last part right now.
2. Show that $\Gamma(2) = 1$ and that $\Gamma(3) = 2$.
3. Compute $\Gamma(4)$ and then make a conjecture for $\Gamma(5)$, $\Gamma(6)$, and $\Gamma(7)$. Use software to check if you are correct.
4. Now show that for any n, we know that $\Gamma(n+1) = n\Gamma(n)$. Now use this rule to repeat parts 2 and 3 above.

The Gamma function is a generalization of the factorial function. In order to evaluate the gamma function at non integers, we would need to compute the integral that defines the Gamma function. This is in general a very nontrivial task. The next problem shows you how to do this. If you've forgotten how to

Problem 8.25: $\Gamma(1/2) = \sqrt{\pi}$ In this problem we'll prove that $\Gamma(1/2) = \sqrt{pi}$. First, notice that by definition we have

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{x} dx.$$

1. Let $u = x^{1/2}$. Use this u-substitution to explain why

   $$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx = 2\int_0^\infty e^{-u^2} du = 2I.$$

   If we could compute the integral $I = \int_0^\infty e^{-u^2} du$, we'd be done. There is no way however to compute this integral exactly, unless we employ higher dimensional tools.

2. Explain why we can write

   $$I^2 = \left(\int_0^\infty e^{-u^2} du\right)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

3. Convert this integral to a double integral in polar coordinates (what is the Jacobian) and then evaluate the integral. This gives you I^2. Solve for $\Gamma(1/2)$.

Problem 8.26 We know that $\Gamma(1/2) = \sqrt{\pi}$, and we know that $\Gamma(n+1) = n\Gamma(n)$. Use this to compute $\Gamma(3/2)$, $\Gamma(5/2)$, and $\Gamma(11/2)$. Then state the Laplace transform of $t^{9/2}$. [Hint: You may have to repeatedly apply the rule $\Gamma(n+1) = n\Gamma(n)$, as we have $3/2 = 1/2 + 1$, and $5/2 = 3/2 + 1$, and so on.]

8.3 Extra Practice Problems

Extra homework for this unit is right here. Make sure you try a few of each type of problem, ASAP. I suggest that the first night you try one of each type of problem. It's OK if you get stuck and don't know what to do, as long as you decide to learn how to do it and then return to the ones where you got stuck. Eventually do enough of each type to master the ideas. The only section in Schaum's with relevant problems is chapter 27. Handwritten solutions are available online. Click for solutions.

Most engineering textbooks assume you have seen Taylor series and power series before (in math 113), but many of you have not. If you have your old Math 215 book, you can find many relevant problems and explanations in the section on the Ratio Test and Taylor Series.

Here are a few key functions and their Taylor series centered at $x = 0$ (their MacLaurin series).

$f(x)$	MacLaurin Series	Radius	$f(x)$	MacLaurin Series	Radius
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!}x^n$	$R = \infty$	$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$R = 1$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$	$R = \infty$	$\cosh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$	$R = \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$R = \infty$	$\sinh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$	$R = \infty$

(I) For each of the following, find a Taylor polynomial of degree n centered at $x = c$ of the function $f(x)$.

1. $e^{4x}, n = 3, c = 0$
2. $\cos(x), n = 4, c = \pi$
3. $\cos(2x), n = 4, c = 0$
4. $\sin(\frac{1}{2}x), n = 5, c = 0$
5. $\frac{1}{x}, n = 3, c = 1$
6. $\ln x, n = 3, c = 1$
7. $\ln(1-x), n = 4, c = 0$
8. $\ln(1+x), n = 4, c = 0$

(II) Find the radius of convergence of each power series.

9. $\sum_{n=0}^{\infty} \frac{1}{3^n}x^n$
10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}}x^n$
11. $\sum_{n=0}^{\infty} \frac{n}{2^n}x^{3n}$
12. $\sum_{n=0}^{\infty} \frac{3n+1}{n^2+4}x^n$
13. $\sum_{n=0}^{\infty} \frac{(-4)^n n}{n^2+1}x^{2n}$
14. $\sum_{n=0}^{\infty} \frac{n}{2^n}x^{2n}$
15. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$
16. $\sum_{n=0}^{\infty} \frac{n!}{10^n}x^{2n}$

(III) For each function, find the MacLaurin series and state the radius of convergence.

17. $f(x) = e^x$
18. $f(x) = \cos x$
19. $f(x) = \sin x$
20. $f(x) = \frac{1}{1-x}$
21. $f(x) = \frac{1}{1+x}$
22. $f(x) = \cosh x$
23. $f(x) = \sinh x$

(IV) Prove the following formulas are true by considering power series. These formulas will allow us to eliminate complex numbers in future sections.

24. $e^{ix} = \cos x + i \sin x$ (called Euler's formula)
25. $\cosh(ix) = \cos x$
26. $\cos(ix) = \cosh x$
27. $\sinh(ix) = i \sin x$
28. $\sin(ix) = i \sinh x$

(V) Use MacLaurin series of known functions to find the MacLaurin series of these functions (by integrating, differentiating, composing, or multiplying together two power series). Then state the radius of convergence.

29. $f(x) = x^2e^{3x}$
30. $f(x) = \frac{x^2}{e^{3x}}$ [hint, use negative exponents]
31. $f(x) = \cos 4x$
32. $f(x) = x \sin(2x)$
33. $f(x) = \frac{x}{1+x}$

34. $f(x) = \frac{1}{1+x^2}$
35. $f(x) = \arctan x$ [hint, integrate the previous]
36. $f(x) = \arctan(3x)$

(VI) Shift the indices on each sum so that it begins at $n = 0$.

37. $\sum_{n=3}^{6} n + 2$
38. $\sum_{n=2}^{8} n^2$
39. $\sum_{n=4}^{\infty} 2^n$
40. $\sum_{n=2}^{\infty} x^n$
41. $\sum_{n=1}^{\infty} na_nx^n$
42. $\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$

(VII) Solve the following ODEs by the power series method. With some, initial conditions are given (meaning you know $y(0) = a_0$ and $y'(0) = a_1$). Identify the function whose MacLaurin series equals the power series you obtain.

43. $y' = 3y$
44. $y' = 2xy$
45. $y'' + 4y = 0$
46. $y'' - 9y = 0, y(0) = 2, y'(0) = 3$
47. $y'' + 4y' + 3y = 0, y(0) = 1, y'(0) = -1$

(VII) Determine whether the given values of x are ordinary points or singular points of the given ODE.

48. Chapter 27, problems 26-34 (these are really quick).

(VIII) Solve the following ODEs by the power series method. State the recurrence relation used to generate the terms of your solution, and write out the first 5 nonzero terms of your solution.

49. Chapter 27, problems 35-47 (or from the worked problems).

8.4 Extra Practice Solutions

Handwritten solutions are available online. Click for solutions.

8.5 Special Functions

Here are some extra practice problems related to the Frobenius method and other special functions. Section numbers correspond to problems from Schaum's Outlines *Differential Equations* by Richard Bronson. The suggested problems are a minimum set of problems to attempt.

Concept	Relevant Problems
Frobenius Method*	28:1-4, 5-10, 12,14,16,18-20
Legendre Polynomials	27:11-13; 29:4,6,8,11,12,15
Bessel Functions	30:9,11,12,26, 27,
Gamma Functions	30:1-8, 24, 25
Substitutions	28:22-23, 34-38; 30:30,31

Click here for some handwritten solutions to many of the problems above.

Chapter 9

Systems of ODEs

This chapter covers the following ideas. When you create your lesson plan, it should contain examples which illustrate these key ideas. Before you take the quiz on this unit, meet with another student out of class and teach each other from the examples on your lesson plan.

1. Explain the basic theory of systems of linear ODEs and the Wronskian for systems.
2. Convert higher order ODEs to first order linear systems.
3. Explain how to use eigenvalues and eigenvectors to diagonalize matrices. When not possible, use generalized eigenvectors to find Jordan canonical form.
4. Find the matrix exponential of a square matrix, and use it to solve linear homogeneous and nonhomogeneous ODEs.
5. Give applications of systems of ODEs. In particular be able to setup systems of ODE related to dilution, electricity, and springs (use the computer to solve complex systems).

9.1 Bringing it all together

As you work on the problems in this section, you'll want to have a computer algebra system near by. I'll put some links to Sage worksheets in the problem set, but I strongly suggest you download the Mathematica Technology Introduction.

Our goal in this chapter is to learn how to solve systems of differential equations. We have already discussed most of the ideas in this chapter (in some context), but we have never brought all these ideas together. In this chapter, we'll try to connect everything we have done up to now. By the time we end this chapter, we'll have a tool that will solve almost every problem we have encountered. We'll see how vector fields, parametric curves, eigenvalues, eigenvectors, potentials, and power series all combine together to give a beautiful and elegant solution technique to solving ODEs.

Problem 9.1 Consider the IVP $y''+3y'+2y=0$, $y(0)=5$, $y'(0)=0$. This solution to this ODE will give the position of a mass spring system where $m=1$ kg, $c=3$ kg/s, $k=2$ kg/s^2, where the object was lifted upwards 5 cm and then let loose.

1. This is a homogeneous ODE. Find the characteristic equation, obtain a general solution, and then use the initial conditions to show that $y(t) = 10e^{-t} - 5e^{-2t}$.

2. Since we know $y(t) = 10e^{-t} - 5e^{-2t}$, we also know $y' = -10e^{-t} + 10e^{-2t}$. Write this solution as a linear combination of vectors in the form
$$\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix} e^{-t} + \begin{bmatrix} ? \\ ? \end{bmatrix} e^{-2t}.$$

3. Let $y(t)$ be the position and $v(t)$ be the velocity. This means that $y'(t) = v(t)$, and that $v'(t) + 3v(t) + 2y(t) = 0$ or $v'(t) = -2y(t) - 3v(t)$. We now have the system of ODEs $y'(t) = v(t)$ and $v'(t) = -2y(t) - 3v(t)$. We can write this as the matrix equation
$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.$$
Find the eigenvalues λ_1 and λ_2 of the coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. Then for each eigenvalue, find a corresponding eigenvector. How are the eigenvalues and eigenvectors related to the solution in part 2?

Problem 9.2 Consider again the IVP $y'' + 3y' + 2y = 0$, $y(0) = 5$, $y'(0) = 0$. We already know that $y = 10e^{-t} - 5e^{-2t}$ and $v = -10e^{-t} + 10e^{-2t}$. In this problem, you'll be constructing various graphs to visualize this solution. Use the links below to have technology do almost all of the graphing for you. Make sure you either (1) print your plots or (2) copy them down on paper.

1. Construct of graph of y verses t. On the same axes construct a graph of v versus t. You should have two graphs that show you position and velocity at any time t.

2. Now construct a graph of v versus y. We call this a velocity-position graph (the time variable is removed). Please use technology to do this (see the margin). You just need to graph the parametric curve $\vec{r}(t) = (y(t), v(t))$. You'll need to a parametric plotter.

Please check your answer with technology. You can use either Sage or Mathematica. Click on this link to get some example code that will help you with this problem. You should use this code to check your answer with all the problems in this chapter.

3. The matrix A represents a vector field $\vec{F}(y, v) = (0y + v, -2y - 3v)$. Construct a graph of this vector field in the yv plane. Put your vector field plot and your velocity-position plot on the same set of axes. Write a sentence that explains what the vector field plot tells us about the velocity-position plot.

4. Change the initial conditions to $y(0) = 0$ and $v(0) = 5$. On top of your vector field plot, draw what you think the solution should look like in the velocity-position plot. Then use software to solve the ODE, and plot your solution. Use the Sage links at the beginning of this problem to accomplish this.

Just change the initial conditions in either Sage worksheet or Mathematica notebook, and reevaluate.

5. Change the initial conditions to $y(0) = 5$ and $v(0) = -5$. On top of your vector field plot, draw what you think the solution should look like in the velocity-position plot. Explain why the solution must follow a straight line in the velocity-position plane? [Hint: What are the eigenvalues, eigenvectors?] Then state another set of initial conditions where the solution will be a straight line towards the origin.

We'll revisit the last two problems as part of every other solution we find. The next problem is a repeat of something we already solved in chapter 2.

Problem 9.3 Imagine for a moment that you have two tanks. The first tank contains 6 lbs of salt in 10 gallons of water. The second tank contains no salt in 20 gallons of water. Each tank has an inlet valve, and an outlet value. We attach hoses to the tanks, and have a pump transfer 2 gallon/minute of solution from tank 1 to tank 2, and vice versa from tank 2 to tank 1. So as time elapses, there are always 10 gallons in tank 1 and 20 gallons in tank 2. Our goal is to find the amount of salt in each tank at any time t.

1. We know there are initially 6 lbs of salt in tank 1, and no salt in tank 2. If we allow the pumps to transfer salt for enough time, explain why the salt content in tank 1 will drop to 2 lb, and the salt content in tank 2 should increase to 4 lbs.

2. Let $y_1(t)$ and $y_2(t)$ be the lbs of salt in tanks 1 and 2, respectively. Explain why
$$y_1' = -\frac{2}{10}y_1 + \frac{2}{20}y_2.$$
Then obtain a similar equation for y_2'. Write your ODEs in the form
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} -2/10 & 2/20 \\ ? & ? \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

3. Draw the vector field represented by the coefficient matrix. Sketch the solution $(y_1(t), y_2(t))$ to your IVP (start at the point $(6,0)$ and follow the field until the vectors no longer tell you to move). Show that you should stop at $(2,4)$.

4. Compute the eigenvalues and eigenvectors of the matrix A, and use them to write a general solution to this system of ODEs. Your solution should involve arbitrary constants c_1 and c_2.

   Don't forget that you can check your work with technology. Please follow this link.

5. Use the initial conditions $y_1(0) = 6$ and $y_2(0) = 0$ to solve for c_1 and c_2.

Problem 9.4 Consider the linear system of ODEs given by $y_1' = 2y_1 + y_2$ and $y_2' = 3y_1 + 4y_2$. Let $\vec{y} = (y_1, y_2)$. We can write this ODE in the form $\frac{d\vec{y}}{dt} = A\vec{y}$, where $\vec{y} = (y_1, y_2)$ and $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of the coefficient matrix A. Use them to state the general solution to this ODE. You can write your answer in the form
$$\vec{y} = c_1 \begin{pmatrix} * \\ * \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} * \\ * \end{pmatrix} e^{\lambda_2 t}.$$

2. Find a 2 by 2 matrix Q and a diagonal matrix D so that we can write this solution in the form
$$\vec{y} = Q \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = QD\vec{c},$$
where we let $D = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

   If you are struggling with this one, then think of Q as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and then expand the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. You'll then know what Q must equal.

3. We now have $\vec{y} = QD\vec{c}$. When we let $t = 0$, explain why D equals the identity matrix. This means that $\vec{y}(0) = Q\vec{c}$. Solving for $\vec{c}$ gives us $\vec{c} = Q^{-1}\vec{y}(0)$. Compute the inverse of $\vec{Q}$.

4. Since we know $\vec{y} = QD\vec{c}$ and $\vec{c} = Q^{-1}\vec{y}(0)$, this means

$$\vec{y} = QDQ^{-1}\vec{y}(0).$$

You have found Q, D, and Q^{-1}. Compute the matrix product QDQ^{-1}.

In class, we'll use your answer to give the solution to this system of ODEs if $y_1(0) = a$ and $y_2(0) = b$. Feel free to state the answer using your matrix product QDQ^{-1}.

Do you notice that in the problem above, we solved the linear system of ODEs in the form $\vec{y}' = A\vec{y}$ with initial conditions $\vec{y}(0)$ by just writing

$$\vec{y} = QDQ^{-1}\vec{y}(0).$$

The columns of Q were the eigenvectors. The nonzero entries of the diagonal matrix D contain $e^{\lambda t}$ where λ is an eigenvalue. Does this pattern work in other places?

Problem 9.5 Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$. Do so by stating Q, D, and Q^{-1}, and then perform the matrix product QDQ^{-1}. Finally, if we assume $\vec{y}(0) = (a, b)$, then give the solution to this system of IVPs by stating what $y_1(t)$ equals, and what $y_2(t)$ equals (hint, multiply out $QDQ^{-1}\vec{y}(0)$). Please use technology to perform as much of the computations as you want. Just be prepared to tell us how you got each part.

Does the pattern above continue to work if we increase the size of the matrix?

Problem 9.6 Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Do so by stating Q, D, and Q^{-1}, and then perform the matrix product QDQ^{-1}. Finally, if we assume $\vec{y}(0) = (a, b, c)$, then give the solution to this system of IVPs by stating what $y_1(t)$ equals, what $y_2(t)$ equals, and what $y_3(t)$ equals. Please use technology to perform as much of the computations as you want. Just be prepared to tell us how you got each part.

9.2 The Matrix Exponential

In the previous section, we saw that if $\vec{y}' = A\vec{y}$, then the solution is $\vec{y} = QDQ^{-1}\vec{c}$, where the initial conditions give us $\vec{c} = \vec{y}(0)$ because D is the identity matrix when $t = 0$. In the first week of class, we solved the differential equation $y' = ay$, and obtained the solution $y = e^{at}c$ where $c = y(0)$. In this section, we'll show that if we replace the constant a with a matrix of constants A, then the solution is still $\vec{y} = e^{At}\vec{c}$. To do this, we have to go back to power series.

Definition 9.1: The Matrix Exponential. We showed in the power series chapter that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots .$$

We define the matrix exponential of A to be the series

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \frac{1}{5!}A^5 + \cdots .$$

The matrix I is the identity matrix.

Problem 9.7 Use the definition above to complete the following:

Check your answer with software.

1. We know that $e^0 = 1$. If $A = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then compute e^A.
2. If $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that $e^A = \begin{bmatrix} e^1 & 0 \\ 0 & e^1 \end{bmatrix}$.
3. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then compute e^A. Make sure you show how you get your answer from the definition.
4. If $At = \begin{bmatrix} 2t & 0 & 0 \\ 0 & 3t & 0 \\ 0 & 0 & 5t \end{bmatrix}$, then compute e^{At}. You are welcome to just state an answer here.

When a matrix is diagonal, it's matrix exponential is simple to compute. Our main goal is to learn how to compute the matrix exponential of all matrices. Let's look at another type of matrix where it's easy to compute the matrix exponential.

Problem 9.8: Nilpotent Matrices Use the definition of the matrix exponential to compute the following.

Check your answer with software.

We say that a matrix A is nilpotent if A^n is the zero matrix for some n. It's easy to compute the matrix exponential of a nilpotent matrix, because the infinite series stops, and then we just have to add up finitely many terms.

1. Let $A = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ and then compute $(A)^2$ and $(A)^3$. Use this to state the matrix exponential of A. (You should get $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.)
2. Let $A = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$ and then compute $(A)^2$ and $(A)^3$. Use this to state the matrix exponential of A. Check your answer with technology by following this link.
3. Let $A = \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give the matrix exponential of A. You are welcome to guess your answer by following any pattern you saw above.
4. Let $A = \begin{bmatrix} 0 & t & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Guess the matrix exponential of A. Check your answer with technology.

With real numbers, we have the exponential rule $e^{a+b} = e^a \cdot e^b$. The exponential of a sum is the same as the product of an exponential. Does this rule work with matrices as well? Let's try it and see.

Problem 9.9 Let's write $At = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} t = \begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix} = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = Bt + Ct$, where $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

1. Use software to compute the matrix exponential of At.

   Follow this link. Please use this calculator to check your answers on the other parts of this problem, but only after you first do them by hand.

2. State the matrix exponentials of both Bt and Ct (use the patterns developed from the previous problems). We know that $At = Bt + Ct$, so how should we combine e^{Bt} and e^{Ct} to get the matrix exponential of At?

3. Without software, state the matrix exponential of $\begin{bmatrix} 3t & t & 0 & 0 \\ 0 & 3t & t & 0 \\ 0 & 0 & 3t & t \\ 0 & 0 & 0 & 3t \end{bmatrix}$.

Problem 9.10 Suppose we know that $A = QDQ^{-1}$ where D is a diagonal matrix. In this problem, we'll compute the matrix exponential of A.

1. Explain why $A^3 = (QDQ^{-1})(QDQ^{-1})(QDQ^{-1}) = QD^3Q^{-1}$. Explain why the product simplified so nicely.

2. Explain why $A^k = QD^kQ^{-1}$.

3. We know that $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$. If we replace A^k with QD^kQ^{-1}, then explain why we know $e^A = Qe^DQ^{-1}$.

4. The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has an eigenvalue 3 with corresponding eigenvector $(1, 1)$, and has an eigenvalue 1 with corresponding eigenvector $(1, -1)$. Compute the matrix exponential of A using the formula $e^A = Qe^DQ^{-1}$.

   Check your answer with software.

If we know how to compute the matrix exponential of a diagonal matrix D and $A = QDQ^{-1}$, then the previous problem showed us that $\exp(A) = Q\exp(D)Q^{-1}$. We also saw that multiplication by t doesn't affect this result, so we have

$$\exp(At) = Q\exp(Dt)Q^{-1}.$$

This is the key tool we'll use to solve systems of ODEs.

Problem 9.11 Complete each of the three parts below.

1. Consider the first order differential equation $\frac{dy}{dt} = ay$. Solve this ODE using separation of variables and show that $y = e^{at}c$. If $y(0) = P$, then what does c equal?

2. Consider the system of ODEs $y_1' = 2y_1 + y_2$ and $y_2' = y_1 + 2y_2$, which we can write in the form $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ or $\frac{d\vec{y}}{dt} = A\vec{y}$. Use eigenvalues and eigenvectors and the process right before Problem 9.5 to give a solution to this ODE.

3. Compute the matrix exponential of At (or explain where you already computed it). Show that a solution to this ODE is $\vec{y} = e^{At}\vec{c}$.

Check your answer with software.

4. What is $y_1(t)$?

Problem 9.12 Consider the system of ODEs $\frac{d\vec{y}}{dt} = A\vec{y}$ given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

1. For the coefficient matrix above, state Q and D. You should have a repeated eigenvalue, but you should also find two linearly independent eigenvectors corresponding to this eigenvalue.
2. Compute e^{At}, and state a general solution $\vec{y}(t)$.

Check your answer with software.

3. If $y_1(0) = 1, y_2(0) = -2, y_3(0) = 2$, then state $y_1(t)$, $y_2(t)$, and $y_3(t)$.

We now have the tools needed to solve every constant coefficient linear system of ODEs, whether homogeneous or not. They key to solving these problems is a formula we already developed earlier in the semester. If you have forgotten how to find an integrating factor, you may want to review some problems from chapter 2. Then tackle this problem.

Problem 9.13 Consider the first order ODE $y' - ay = f(t)$. Find an appropriate integrating factor, and then show that a general solution to this ODE is

$$y(t) = e^{at}c + e^{at}\int e^{-at}f(t)dt,$$

where c is an arbitrary constant.

If the ODE is homogeneous with $f(t) = 0$, show that $c = y(0)$.

The solutions above provides a theoretical way to solve every first order linear constant coefficient ODE. If we replace y, c, and f with vectors, and we replace a with a matrix, then the solution to $\frac{d\vec{y}}{dt} = A\vec{y}(t) + \vec{f}(t)$ is simply

$$\vec{y}(t) = e^{At}\vec{c} + e^{At}\int e^{-At}\vec{f}(t)dt.$$

This equation solves just about every linear ODE we've encountered all semester, and more. To use this solution, the system must have constant coefficients, but the function f only has to be integrable after multiplying by e^{-At}. This greatly extends our ability to solve non homogeneous ODEs.

It's possible to rework through the details of the problem above to show this is the solution. I'll leave those details to you. Solving that problem was one of my most exciting discoveries in the last 10 years of teaching. It's amazing.

Let's use this solution technique on the following problem.

Problem 9.14 Consider the linear system of ODEs

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\vec{y} + \begin{bmatrix} 0 \\ 5t \end{bmatrix},$$

with initial conditions $y_1(0) = 4$, $y_2(0) = 0$.

1. Use a computer to give the eigenvalues and an eigenvector for each eigenvalue. State Q and D so that $AQ = QD$ where D is a diagonal matrix.
2. State e^{Dt}. The perform the matrix product $e^{At} = Qe^{Dt}Q^{-1}$? Check your answer with software.
3. Compute the inverse of e^{At} and show it is the same as replacing t with $-t$.
4. Compute by hand the integral $\int e^{-At}\vec{f}(t)dt$, and then with a computer give the product $e^{At}\int e^{-At}\vec{f}(t)dt$. Matrix multiplication can be tedious, so please use software to automate it.
5. Show how to use the initial conditions to find $\vec{c}$ in $\vec{y}(t) = e^{At}\vec{c}+e^{At}\int e^{-At}\vec{f}(t)dt$.

If we can find matrices Q and D so that $AQ = QD$ where D is diagonal, then we can always find the matrix exponential of A. We can use this to solve any related differential equations. What if we can't find Q and D?

Problem 9.15 Consider the matrix $At = \begin{bmatrix} 2t & t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 & 0 \\ 0 & 0 & 2t & t & 0 \\ 0 & 0 & 0 & 2t & t \\ 0 & 0 & 0 & 0 & 2t \end{bmatrix}$. There is supposed to be a zero instead of a t in the second row. That was done on purpose.

1. Write At as the sum $At = Bt + Ct$, where Bt is a diagonal matrix, and Ct contains nonzero terms above the diagonal.
2. Compute the matrix exponential of both Bt and Ct. Then compute their product to get e^{At}. Check your answer with software.
3. Guess the matrix exponential of $\begin{bmatrix} 3t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3t & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3t & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4t & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4t \end{bmatrix}$ Check your answer with software.

We can now compute the matrix exponential of any matrix that is either diagonal, or has nonzero entries above the diagonal. We'll soon see that this means we can compute the matrix exponential of every matrix. The key is to first find the correct form. Let's apply what we just learned to solve an ODE that has a nonzero term above the diagonal.

Problem 9.16 Consider the linear system of ODEs given by

$$y_1' = -3y_1 + y_2 + 3 \quad \text{and} \quad y_2' = -3y_2 + 6,$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 0$.

1. Write this linear system in the form $\frac{d\vec{y}}{dt} = A\vec{y} + \vec{f}(t)$. [Hint: The vector $\vec{f}$ should be constant.]

2. What are the eigenvalues of A? You should get a repeated root.

3. This matrix is already in the form of the sum of a diagonal and a nilpotent matrix. You don't need eigenvectors to compute the matrix exponential. Compute e^{At} and then replace t with $-t$ to obtain e^{-At}.

Check your answer with software.

4. Compute by hand $\int e^{-At}\vec{f}(t)dt$, and then with a computer give the product $e^{At}\int e^{-At}\vec{f}(t)dt$.

Check your answer with software.

5. We know the general solution is $\vec{y}(t) = e^{At}\vec{c} + e^{At}\int e^{-At}\vec{f}(t)dt$. Use the initial conditions to find $\vec{c}$. Show us what matrix you are row reducing.

6. You should have $y_2(t) = 2 - 2e^{-3t}$. What is $y_1(t)$?

Let's focus on some application problems. You don't have to be able to compute the matrix exponential problems to solve these problems, so if you got stuck above, please do these.

$\boxed{\textbf{Problem 9.17}}$ Suppose we have three large tanks containing various amounts of salt. The tanks have volumes $V_1 = 30$ gal, $V_2 = 20$ gal, and $V_3 = 50$ gal.

We don't have to use salt. This could represent 3 different countries and products they wish to import/export. It could be sewage at a waste transfer station. We might consider three countries and the spreading of a virus. We could look at three cities and the flow of traffic. The applications are endless.

- An inlet valve pumps 5 gallons of water into tank 1 each minute. The water coming in contains 3 lbs of salt per gallon. This water is being added to the system from some external source.
- Tank 1 has an outlet value that pumps 9 gallons per minute out to tank 2. Tank 1 has an inlet value that receives 3 gallons per minute from tank 2, and 1 gallon per minute from tank 3. In all, this mean that tank 1 has 9 gallons coming in per minute, and 9 gallons going out per minute.
- Tank 2 receives 9 gallons per minute from tank 1. Of those 9 gallons, it sends 3 gallons per minute to tank 1 and 4 gallons per minute to tank 2. The other 2 gallons per minute leak out the top (through a crack).
- Tank 3 receives 4 gallons per minute from tank 2. It sends 1 gallon per minute back to tank 1, and then the remaining 3 gallons per minute are sent out a hose to some external spot.

Assume the initial salt content is zero in each tank. As time moves on, the salt that is added to tank 1 will eventually reach the other tanks. After t minutes, how much salt will be in each tank?

1. Let y_1, y_2, and y_3 be the lbs of salt in each tank after t minutes. We know that $y_1' = 5(3) - \frac{9}{30}y_1 + \frac{3}{20}y_2 + \frac{1}{50}y_3$. Obtain similar ODEs for y_2' and y_3'.

2. Write this tank mixing problem as a linear system of ODEs in the form $\frac{d\vec{y}}{dt} = A\vec{y} + \vec{f}(t)$. You should have a 3 by 3 matrix A. You should be able to explain why $\vec{f}(t) = (15, 0, 0)$.

3. Use software to solve the system. State the eigenvalues, eigenvectors, Q, D, e^{At}, and $\int e^{-At}f(t)dt$, as well as the solution $\vec{y} = (y_1(t), y_2(t), y_3(t))$. See the software link on the right.

Here's a link to Sage.

4. Let $t \to \infty$. What will be the salt content in each tank in the long run. Could you have explained this without doing any differential equations?

Problem 9.18 Consider the following mechanical system. Attach a spring to the top of the ceiling. Add an object with mass m_1 to the bottom of the spring. We'll assume the spring's mass is negligible and the spring constant is k_1. To the bottom of the first mass, we attach a second spring, and hang another object to the end of the second spring. The second object has mass m_2. The second spring has negligible mass, with spring constant k_2. We'll let $y_1(t)$ and $y_2(t)$ represent the position of the masses relative to their equilibrium positions, so if $y_1(t) = 3$, then we'd be 3 cm above the equilibrium point.

Suppose we displace the objects from equilibrium, and let them go. This means we have the initial conditions $y_1(0) = a$, $y_2(0) = b$, $y_1'(0) = 0$, and $y_2'(0) = 0$. Our goal is to predict the future, namely give the position of both springs at time t. The key is to study the forces acting on each spring.

- If we focus on the first mass, then the forces acting on this mass are the spring force above it (we'll call this F_1), and the spring force below it (we'll call this F_2).
- The force from the spring above the first mass depends on how much y_1 has moved. If the first mass has moved up y_1 units, then the force acts downwards, and we can use Hooke's law to state that $F_1 = -k_1 y_1$.
- The force from the spring below the first mass depends on both where y_1 and y_2 are located. If both moved up the same amount from equilibrium, then $F_2 = 0$. If they have moved at different amounts, then their difference $y_2 - y_1$ will either result in a contraction or expansion of the spring, and then the force is not zero.

1. Is the force from the second spring equal to $F_2 = -k_2(y_2 - y - 1)$ or $F_2 = +k_2(y_2 - y - 1)$. In other words, if we know $y_2 - y_1 > 0$ then is the spring compressed or elongated, and does this result in an upwards or downwards force? Please explain.
2. We now have an equation $m_1 y_1'' = -k_1 y_1 \pm k_2(y_2 - y_1)$ (you determined the correct sign in part 1). Obtain a similar equation for $m y_2''$ (this one should be simpler, as the only spring force acting on m_2 comes from the lower spring).
3. Let $v_1 = y_1'$ and $v_2 = y_2'$. This allows us to replace y_1'' with v_1', and y_2'' with v_2', and then we have a system of first order linear ODEs, namely

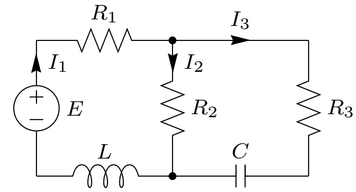
$$y_1' = v_1, y_2' = v_2, v_1' = -\frac{k_1}{m_1} y_1 \pm \frac{k_2}{m_1}(y_2 - y_1), v_2' = \ldots$$

   Write this system in the matrix form

$$\begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix}.$$

4. As a challenge (optional), how would you change your matrix if each mass were attached to a dashpot with coefficients of friction c_1 and c_2? This would add the forces $-c_1 y_1'$ and $-c_2 y_2'$ to the system.

Our last application involves modeling the current in an electrical system with two loops. A similar computation will work with any number of loops,



though the number of loops causes the size of the system to increase quite rapidly. Remember that Kirchoff's current law states that at each node, the current in equals the current out. In addition, Kirchoff's voltage laws states that along each loop, the voltage supplied equals the voltage suppressed. Each resistor contributes a voltage drop of RI ohms, each capacitor a drop of $\frac{1}{C}\int Idt$ farads, and each inductor a voltage drop of LI' Henrys.

Problem 9.19 Consider the electrical network on the right. Kirchoff's current law states that $I_1 = I_2 + I_3$. On the left loop, Kirchoff's voltage law states that $E = R_1I_1 + R_2I_2 + LI_1'$. On the right loop, Kirchoff's voltage law states that $0 = I_3R_3 + \frac{1}{C}\int I_3dt - R_2I_2$. This gives us the three equations

$$I_1 = I_2 + I_3, \quad E = R_1I_1 + R_2I_2 + LI_1', \quad 0 = I_3R_3 + \frac{1}{C}\int I_3dt - R_2I_2.$$

1. Differentiate both sides of the first and last equations so that all your equations are differential equations. The first equation becomes $I_1' = I_2' + I_3'$. What does the last equation become?

2. Solve your system of 3 ODEs for I_1', I_2', and I_3'. You should be able to write each of these three in terms of I_1, I_2, I_3, E, and the resistances.

3. Write the system of ODEs from the first part in the matrix form $\frac{d\vec{I}}{dt} = A\vec{I} + \vec{f}(t)$, i.e. in the form
$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}' = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}.$$

Once you've got the system set up, a computer can give the currents instantly using the solution $\vec{I}(t) = e^{At}\vec{c} + e^{At}\int e^{-At}\vec{f}(t)dt$. This entire problem is an algorithm that's already coded into any electrical network software package.

Let's end the chapter with one final problem.

Problem 9.20 Consider the linear system of ODEs

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\vec{y} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

1. Show that the coefficient matrix above has a repeated eigenvalue, and only one corresponding eigenvector.

2. Because there is one eigenvector, we can't make a matrix Q so that $AQ = QD$. However, if we let $Q = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, then what is J so that $AQ = QJ$?

3. Compute the matrix exponential of J, then compute e^{At} and e^{-At}. You'll want to use the ideas from problem 9.9 and 9.15.

4. Compute, by hand, $\int e^{-At}\vec{f}(t)dt$, and then with a computer give the product $e^{At}\int e^{-At}\vec{f}(t)dt$.

5. If you let $\vec{c} = \vec{0}$, then what are y_1 and y_2. This is what we called y_p in the non homogeneous ODE section.

9.3 Problems

The accompanying problems will serve as practice problems for this chapter. Handwritten solutions to most of these problems are available online (click for solutions). You can use the Mathematica technology introduction to check any answer, as well as give a step-by-step solution to any of the problems. However, on problems where the system is not diagonalizable, the matrix Q used to obtain Jordan form is not unique (so your answer may differ a little, until you actually compute the matrix exponential $Qe^{Jt}Q^{-1} = e^{At}$).

1. Solve the linear ODE $y' = ay(t) + f(t)$, where a is a constant and $f(t)$ is any function of t. You will need an integrating factor, and your solution will involve the integral of a function.

2. For each system of ODEs, solve the system using the eigenvalue approach. Find the Wronskian and compute its determinant to show that your solutions are linearly independent.

   (a) $y_1' = 2y_1 + 4y_2, y_2' = 4y_1 + 2y_2, y_1(0) = 1, y_2(0) = 4$

   (b) $y_1' = y_1 + 2y_2, y_2' = 3y_1, y_1(0) = 6, y_2(0) = 0$

   (c) $y_1' = y_1 + 4y_2, y_2' = 3y_1 + 2y_2, y_1(0) = 0, y_2(0) = 1$

   (d) $y_1' = y_2, y_2' = -3y_1 - 4y_2, y_1(0) = 1, y_2(0) = 2$

3. (Jordan Form) For each matrix A, find matrices Q, Q^{-1}, and J so that $Q^{-1}AQ = J$ is a Jordan canonical form of A.

   (a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (e) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4. For each of the following matrices A which are already in Jordan form, find the matrix exponential. Note that if t follows a matrix, that means you should multiply each entry by t.

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathrm{t}$

   (c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \mathrm{t}$

   (e) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathrm{t}$

   (f) $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \mathrm{t}$

   (g) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{t}$

   (h) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \mathrm{t}$

   (i) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{t}$

   (j) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \mathrm{t}$

5. For each of the following matrices, find the matrix exponential. You will have to find the Jordan form.

   (a) $\begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$

   (e) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   (f) $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

   (g) $\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

   (h) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

6. Set up an initial value problem in matrix format for each of the following scenarios (mixing tank, dilution problems). Solve each one with the computer.

   (a) Tank 1 contains 30 gal, tank 2 contains 40. Pumps allow 5 gal per minute to flow in each direction between the two tanks. If tank 1 initially contains 20lbs of salt, and tank 2 initially contains 120 lbs of salt, how much salt will be in each tank at any given time t. Remember, you are just supposed to set up the IVP, not actually solve it (the eigenvalues are not very pretty).

   (b) Three tanks each contain 100 gallons of water. Tank 1 contains 400lbs of salt mixed in. Pumps allow 5 gal/min to circulate in each direction between tank 1 and tank 2. Another pump allows 4 gallons of water to circulate each direction between tanks 2 and 3. How much salt is in each tank at any time t?

(c) Four tanks each contain 30 gallons. Between each pair of tanks, a set of pumps allows 1 gallon per minute to circulate in each direction (so that each tank has a total of 3 gallons leaving and 3 gallons entering). Tank 1 contains 50lbs of salt, tank 2 contains 80 lbs of salt, tank 3 contains 10 lbs of salt, and tank 4 is pure water. How much salt is in each tank at time t?

(d) Tank 1 contains 80 gallons of pure water, and tank 2 contains 50 gallons of pure water. Each minute 4 gallons of water containing 3lbs of salt per gallon are added to tank 1. Pumps allow 6 gallons per minute of water to flow from tank 1 to tank 2, and 2 gallons of water to flow from tank 2 to tank 1. A drainage pipe removes 4 gallons per minute of liquid from tank 2. How much salt is in each tank at any time t?

7. Convert each of the following high order ODEs (or systems of ODEs) to a first order linear system of ODEs. Which are homogeneous, and which are nonhomogeneous?

   (a) $y'' + 4y' + 3y = 0$
   (b) $y'' + 4y' + 3y = 4t$
   (c) $y'' + ty' - 2y = 0$
   (d) $y'' + ty' - 2y = \cos t$
   (e) $y''' + 3y'' + 3y' + y = 0$
   (f) $y'''' - 4y''' + 6y'' - 4y' + y = t$
   (g) $y_1'' = 4y_1' + 3y_2, y_2' = 5y_1 - 4y_2$.
   (h) Chapter 17, problems 1-20, in Schaum's

8. Solve the following homogeneous systems of ODEs, or higher order ODEs, with the given initial conditions.

   (a) $y_1' = 2y_1, y_2' = 4y_2, y_1(0) = 5, y_2(0) = 6$
   (b) $y_1' = 2y_1 + y_2, y_2' = 2y_2, y_1(0) = -1, y_2(0) = 3$
   (c) $y'' + 4y' + 3y = 0, y(0) = 0, y'(0) = 1$
   (d) $y'' + 2y' + y = 0, y(0) = 2, y'(0) = 0$
   (e) $y_1' = 2y_1 + y_2, y_2' = y_1 + 2y_2, y_1(0) = 2, y_2(0) = 1$
   (f) $y_1' = y_2, y_2' = -y_1, y_1(0) = 1, y_2(0) = 2$

9. Solve the following nonhomogeneous systems of ODEs, or higher order ODEs, with the given initial conditions. Use the computer to solve each of these problems, by first finding the matrix exponential and then using using the formula $\vec{y} = e^{At}\vec{c} + e^{At}\int e^{-At}\vec{f}(t)dt$. You'll have to find the matrix A and function f.

   (a) $y_1' = 2y_1 + t, y_2' = 4y_2, y_1(0) = 5, y_2(0) = 6$
   (b) $y_1' = 2y_1 + y_2, y_2' = 2y_2 - 4, y_1(0) = -1, y_2(0) = 3$
   (c) $y'' + 4y' + 3y = \cos 2t, y(0) = 0, y'(0) = 1$
   (d) $y'' + 2y' + y = \sin t, y(0) = 2, y'(0) = 0$
   (e) $y_1' = 2y_1 + y_2 - 2, y_2' = y_1 + 2y_2 + 3, y_1(0) = 2, y_2(0) = 1$
   (f) $y_1' = y_2, y_2' = -y_1 + t, y_1(0) = 1, y_2(0) = 2$

10. Mass-Spring Problems - To be added in the future.

11. Electrical Network Problems - To be added in the future.

Chapter 10

Fourier Series and PDEs

This chapter covers the following ideas. When you create your lesson plan, it should contain examples which illustrate these key ideas. Before you take the quiz on this unit, meet with another student out of class and teach each other from the examples on your lesson plan.

1. Define period, and how to find a Fourier series of a function of period 2π and $2L$.
2. Explain how to find Fourier coefficients using Euler formulas, and be able to explain why the Euler formulas are correct.
3. Give conditions as to when a Fourier series will exist, and explain the difference between a Fourier series and a function at points of discontinuity.
4. Give examples of even and odd functions, and correspondingly develop Fourier cosine and sine series. Use these ideas to discuss even and odd half-range expansions.

10.1 Fourier Series

In the power series unit, we showed how to write function $f(x)$ as a power series $\sum_{n=0}^{\infty} a_n x^n$. The radius of convergence tells us precisely for which x values the power series will converge to the function $f(x)$. We call it a power series because we use powers of x as the functions we are summing. At any finite stage, we are trying to approximate the function $f(x)$ using linear combinations of power of x. We can find the coefficients a_n with the formula $a_n = \dfrac{f^(n)(0)}{n!}$.

Are there other kinds of series? Do we have to use powers of x? In class, we discussed how to use Legendre polynomials to do the same thing. If we let $P_n(x)$ be the nth degree Legendre polynomial, then we could write $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$. We showed that on the interval $[-1, 1]$, the integral $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ if $n \neq m$. We then used this to show that we can compute the coefficients a_n using the formula

$$a_n = \frac{\int_{-1}^{1} f(x)P_n(x)dx}{\int_{-1}^{1} P_n(x)P_n(x)dx}.$$

The key to using a Legendre series is the fact that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0.$$

This integral equation is a generalization of the dot product. In the dot product, we multiply corresponding components together and then sum. That's precisely what the integral above does. Let's make a definition.

Definition 10.1: Inner Product. Let $f(x)$ and $g(x)$ be bounded functions over some interval $[a,b]$. The inner product of f and g over $[a,b]$ is the integral

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

This inner product is a generalization of the dot product. We say that two functions are orthogonal over $[a,b]$ if their inner product over $[a,b]$ is equal to zero.

Problem 10.1 Consider the function $f(x) = x$ and $g(x) = x^2$.

1. Compute the inner products $\langle f, f\rangle$, $\langle g, g\rangle$, and $\langle f, g\rangle$. [For the first, you just need to compute the integral $\int_0^1 x \cdot xdx$.]
2. Recall that $\vec{u} \cdot \vec{u} = |u|^2$. So the length of a vector is the square root of the dot product of the vector with itself. We'll define the length of a function to be the square root of the inner product of a function with itself, and write $||f|| = \sqrt{\langle f, f\rangle}$. What is the length of f and the length of g over the interval $[0,1]$. [Hint, you already did the inner products in part 1.
3. Recall that $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$, where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$. Use this idea to invent a definition for the angle between two functions, and then use your definition to compute the angle between f and g.

With the word inner product, we can now talk about the "dot product" of functions. We can use the words Length and angle when talking about functions. We obtained the Legendre series coefficients because the inner product of any two different Legendre polynomials, over the interval $[-1,1]$, is zero. The Legendre polynomials form an infinite collection of orthogonal vectors. We've got a 90 degree angle between any two functions. We can use these polynomials to approximate any other function by considering linear combinations of these orthogonal functions. This is one of the big ideas in modern science. Rather than working with complicated functions f, we can approximate them with linear combinations of simpler functions. What constitutes a "simple" function depends on the problem. Legendre polynomials show up when studying spherically symmetric problems. Bessel functions shows up when studying radially symmetric problems. Sometimes the hardest part about solving a problem is trying to determine which "simple" collection of functions to use.

Fourier was one of the first to use something other than power series to approximate a function. He did it while trying to understand how heat flowed in a cannon. Napoleon asked Fourier to study this problem, because cannons were exploding on his soldiers because the cannon balls would not leave a cannon after heat had caused the barrel to expand. Fourier discovered that you can use sine and cosine series to approximate a function. That's what we'll study in this chapter. In the work below, the variable L will stand for the length of a cannon.

Problem 10.2 Let $f_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ for $n = 0, 1, 2, \ldots$. So we have $f_0(x) = \cos\left(\frac{0\pi x}{L}\right)$, $f_1(x) = \cos\left(\frac{\pi x}{L}\right)$, $f_2(x) = \cos\left(\frac{2\pi x}{L}\right)$, $f_3(x) = \cos\left(\frac{3\pi x}{L}\right)$, etc. This is an infinite collection of functions. Our goal is to show that on the interval $[0, L]$, the functions f_n and f_m are orthogonal, provided $n \neq m$. To do this, you'll need to prove that any pair has an inner product of 0.

1. Draw the functions $f_0(x) = \cos\left(\frac{0\pi x}{L}\right)$, $f_1(x) = \cos\left(\frac{\pi x}{L}\right)$, and $f_2(x) = \cos\left(\frac{2\pi x}{L}\right)$ over the interval $[0, L]$.
2. Compute the integral $\int_0^L f_0(x) f_n(x) dx$ for $n \neq 0$. What is $\langle f_0, f_n \rangle$?
3. Now show $\int_0^L f_n(x) f_m(x) dx = 0$ for $n \neq m$, with $n, m > 0$. As a hint, you'll want to look up a product-to-sum trig identity, after which this integral is quickly doable.

We now know that the set of functions $f_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ forms an orthogonal set of functions over the interval $[0, L]$. Let's now make a Fourier cosine series for a function.

Problem 10.3 Suppose that $f(x)$ is defined on $[0, L]$. If we assume that

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

then show that

$$A_n = \frac{\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx}.$$

Then compute the bottom integral to show that for $n \geq 1$ we have

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

and if $n = 0$ then we have

$$A_0 = \frac{1}{L} \int_0^L f(x) dx.$$

[Hint: Multiply both sides by $\cos\left(\frac{m\pi x}{L}\right)$ and then integrate from 0 to L. You can use the orthogonality of cosines to solve for the coefficients. The integral of the bottom will involve a cosine half angle formula.]

We could have repeated the previous two problems with the sine function, and would have gotten very similar results. Based of the previous two problems, let's make a formal definition.

Definition 10.2: Fourier Sine and Cosine Series. Let $f(x)$ be a function defined on $[0, L]$. We define the Fourier sine series and Fourier cosine series of f (on $[-L, L]$) to be the series

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

respectively. The coefficients above are given by the formulas

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{and} \quad \begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)dx, \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Problem 10.4 Let $f(x) = x$ over the interval $[0, L]$. Draw the function f. Then compute the Fourier sine series of $f(x)$ by computing the integrals

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Write out the first 4 nonzero terms of the series by writing

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = B_1\sin\left(\frac{\pi x}{L}\right) + B_2\sin\left(\frac{2\pi x}{L}\right) + B_3\sin\left(\frac{3\pi x}{L}\right) + \cdots.$$

Be prepared to explain how you use integration by parts to compute the integrals.

Problem 10.5 Let $f(x) = x$ over the interval $[0, L]$. Draw the function f. Then compute the Fourier cosine series of $f(x)$ by computing the integrals

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \quad \text{and} \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Write out the first 4 nonzero terms of the series by writing

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = A_0 + A_1\cos\left(\frac{\pi x}{L}\right) + A_2\cos\left(\frac{2\pi x}{L}\right) + A_3\cos\left(\frac{3\pi x}{L}\right) + \cdots.$$

Be prepared to explain how you use integration by parts to compute the integrals.

Problem 10.6 In the previous two problems, you computed the Fourier cosine and Fourier sine series of $f(x) = x$ over the interval $[0, L]$. This problem will repeat the above with technology, and then you'll graph your results. To allow software to create graphs, you'll have to pick a length L that's an actual number (maybe try $L = 7$).

1. Use software to obtain the first 4 nonzero terms of the Fourier sine and Fourier cosine series of $f(x) = x$ over $[0, L]$.
2. Use software to graph both $f(x) = x$ and the first four nonzero terms of the Fourier sine series. Have the bounds of your graph be over the interval $[-3L, 3L]$.
3. Use software to graph both $f(x) = x$ and the first four nonzero terms of the Fourier cosine series. Have the bounds of your graph be over the interval $[-3L, 3L]$.
4. Make some conjectures about what you see in your graphs.

To present this in class, you should come to class with your graphs already printed out.

Definition 10.3: Piecewise Smooth. We say that function $f(x)$ is smooth on an interval $[a, b]$ if the function and its derivative are both bounded and continuous on (a, b). We say that a function $f(x)$ is piecewise smooth on an interval (a, b) if the interval can be partitioned into a finite number of pieces and on each piece the function $f(x)$ is smooth (so $f'(x)$ may not exist at finitely many points).

Definition 10.4: Fourier Series. Let $f(x)$ be a function defined on $[-L, L]$ such that the Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \text{ and} \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

exist. We define the Fourier series of f over the interval $[-L, L]$ to be the formal infinite series

$$a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right).$$

Regardless of whether or not the series converges, we will write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right).$$

Remark 10.5. In the definition above, we do not require the Fourier series to actually converge to f. The following theorem, which we will give without proof, provides the needed conditions for the series to converge. We will use this proof (which requires some real analysis) to prove other facts about Fourier series throughout this chapter.

Theorem 10.6 (Fourier's Theorem)**.** *Suppose $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$. Then the Fourier series of $f(x)$ converges to the period extension of f at the points where f is continuous. If the periodic extension of f is not continuous at a point x, then the Fourier series converges to the average of the left and right limits at x, namely $\dfrac{f(x+) + (f(x-)}{2}$.*

Problem 10.7 Let $f(x) = x$ over the interval $[-L, L]$. Compute the Fourier series of $f(x)$ by computing the integrals for a_0, a_n, and b_n. Show your integration steps.

Problem 10.8 Let $f(x) = x^2$ over the interval $[-L, L]$. Compute the Fourier series of $f(x)$ by computing the integrals for a_0, a_n, and b_n Show your integration steps.

Problem 10.9 Let $f(x) = e^x$, $g(x) = \cosh(x)$ and $h(x) = \sinh(x)$ over the interval $[-L, L]$. Use software to compute the Fourier coefficients for each function. What patterns do you see? You are welcome to just state the Fourier coefficients for each (you don't have to show your integration steps).

Notice that there are other kinds of series. In class, I showed you how to obtain a function as an infinite sum of Legendre polynomials. We also introduced the concept of orthogonality of functions, and showed that the Legendre polynomials formed an infinite collection of orthogonal functions. We also showed how to write a function f as a series of Legendre polynomials.

We can do the exact same thing with trig functions. We'll develop the cosine series, the sine series, and then the Fourier series. We'll compute these series for a few functions. After that let's look at how these were developed by Fourier, as he studied the heat equation.

Maybe we should start with why we care about something that has length L. Then get to the reason why we look at $n\pi/L$. Let's definitely do that.

Then prove orthogonality of the functions on these intervals.

Show that if we write a function as a series, and the functions are orthogonal, then we know the coefficients through an integral formula.

Obtain a formula for the cosine series coefficients and the sine series coefficients.

Find the Fourier coefficients of a cosine series. Draw the function and several terms of the series.

Find the Fourier coefficients of a sine series. Draw the function, and several terms of the series.

What are the coefficients of a Fourier series. Find the coefficients for a function.

If a function is odd or even, what can be said.

I can grab a lot of these problems from my PDE course notes.

10.2 Problems

The accompanying problems will serve as practice problems for this chapter. Handwritten solutions to most of these problems are available online (click for solutions). You can use the Mathematica technology introduction to check all your answers. I will create a solutions guide as time permits.

1. Find the fundamental period of the following functions.
   (a) $\sin x$, $\sin 2x$, $\sin\frac{x}{3}$, $\sin kx$, $\sin\frac{n\pi x}{2}$
   (b) $\tan x$, $\tan 2x$, $\tan\frac{x}{3}$, $\tan kx$, $\tan\frac{n\pi x}{2}$
2. Show $y = c$ is p-periodic for each positive p, but has no fundamental period.
3. Compute the Fourier series of each function below (assume the function is 2π-periodic). Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated Fourier series.
   (a) $f(x) = \sin 2x$
   (b) $f(x) = \cos 3x$
   (c) $f(x) = \sin 2x + \cos 3x$
   (d) $f(x) = 4$
   (e) $f(x) = 4 + 5\sin 2x - 7\cos 3x$
   (f) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$
   (g) $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$
   (h) $f(x) = \begin{cases} -2 & -\pi < x < 0 \\ 3 & 0 < x < \pi \end{cases}$
   (i) $f(x) = x$ for $-\pi < x < \pi$
   (j) $f(x) = |x|$ for $-\pi < x < \pi$
   (k) $f(x) = x$ for $0 < x < 2\pi$
   (l) $f(x) = x^2$ for $-\pi < x < \pi$
   (m) $f(x) = \begin{cases} 0 & -\pi < x < -\pi/2 \\ 1 & -\pi/2 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$
   (n) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$
   (o) $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$
4. Compute the Fourier series of each function below (assume the function is p-periodic). Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated Fourier series.
   (a) $f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$, $p = 4$
   (b) $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 2 & 0 < x < 2 \end{cases}$, $p = 4$
   (c) $f(x) = x$ for $-1 < x < 1$, $p = 2$
   (d) $f(x) = |x|$ for $-1 < x < 1$, $p = 2$
   (e) $f(x) = 1 - |x|$ for $-1 < x < 1$, $p = 2$
   (f) $f(x) = x^2$ for $-1 < x < 1$, $p = 2$
   (g) $f(x) = x^2$ for $-2 < x < 2$, $p = 4$
   (h) $f(x) = 1 - x^2$ for $-1 < x < 1$, $p = 2$
   (i) $f(x) = x^3$ for $-1 < x < 1$, $p = 2$
   (j) $f(x) = \sin(\pi x)$ for $0 < x < 1$, $p = 1$
   (k) $f(x) = \cos(\pi x)$ for $-\frac{1}{2} < x < \frac{1}{2}$, $p = 1$
   (l) $f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$, $p = 2$
   (m) $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$, $p = 4$
5. Decide if each function is even, odd, or neither.
   (a) $x, x^2, x^3, x^4, \sqrt{x}, \sqrt[3]{x}, x^2 + x + 1, x^3 + x, x^2 + 1, x^4 + x^5$.
   (b) $\sin x, \cos x, \cos 3x, \tan x, \cot x, \sec x, \csc x, \sin x \cos x, \sin^2 x, \sin x + \cos 3x$.

(c) If f is even and g is odd, $f^2, g^2, f^3, g^3, fg, f+g, 3f, xf, xg, f^n g^m$ (where n and m are integers).

6. Rewrite each function f as the sum of an even f_e and an odd f_o function, so that $f = f_e + f_0$. Make sure you show that $f_e(-x) = f_e(x)$ and $f_o(-x) = -f_o(x)$. Then plot f, f_e, and f_o on the same axes.

   (a) $f(x) = e^x$ (your answer should involve hyperbolic functions).

   (b) $f(x) = x^2 + 3x + 2$

   (c) $f(x) = \dfrac{1}{x-1}$ (since $f(1)$ is undefined, the even and odd functions are not defined at $x = -1$).

7. For each function defined on $[0, L]$, find the Fourier cosine series of the even periodic extension to $[-L, L]$. Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated series.

   (a) $f(x) = 1$ for $0 < x < 1$

   (b) $f(x) = 1$ for $0 < x < \pi$

   (c) $f(x) = x$ for $0 < x < 1$

   (d) $f(x) = x$ for $0 < x < \pi$

   (e) $f(x) = 1 - x$ for $0 < x < 1$

   (f) $f(x) = 2 - x$ for $0 < x < 2$

   (g) $f(x) = \pi - x$ for $0 < x < \pi$

   (h) $f(x) = x^2$ for $0 < x < 1$

   (i) $f(x) = x^3$ for $0 < x < 1$

   (j) $f(x) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$

   (k) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$

   (l) $f(x) = \begin{cases} 1 - x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

   (m) (Sawtooth Wave) $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \end{cases}$

8. For each function from 7, find the Fourier sine series of the odd periodic extension to $[-L, L]$. Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated series.

9. For a function $f(x)$ defined on $[0, L]$, a half wave rectifier extends f to be 0 on $[-L, 0)$, and then periodically extends the function to all real numbers. If f_e and f_o represent the even and odd periodic extentsions, then $g = \dfrac{f_e + f_o}{2}$ represents the half wave rectifier. For each function from 7, find the Fourier series of the half wave rectifier of f. Write your solution using summation notation. Then graph at least 3 periods of the half wave rectifier and compare it to a graph of a truncated series.

10. Compute the following integrals, where n and m are positive integers. You will need the product to sum trig identitites.

    (a) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx.$

    (b) $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx.$

    (c) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx.$

11. Use Fourier series to prove the following identities.

    (a) $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$

    (b) $\sin^3 x = \dfrac{3}{4}\sin x - \dfrac{1}{4}\sin(3x)$

    (c) $\sin^4 x = \dfrac{3}{8} - \dfrac{1}{2}\cos(2x) + \dfrac{1}{8}\cos(4x)$

    (d) $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos(2x)$

    (e) $\cos^3 x = \dfrac{3}{4}\cos x + \dfrac{1}{4}\cos(3x)$

    (f) $\cos^4 x = \dfrac{3}{8} + \dfrac{1}{2}\cos(2x) + \dfrac{1}{8}\cos(4x)$

12. (Gibb's Phenomenon) Pick a discontinuous function from any of the previous exercises. Use a computer to graph the partial sums

$$f_k(x) = a_0 + \sum_{n=1}^{k} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right)$$

for $k = 5, 10, 50, 100$. What do you notice happening near the point where f is discontinuous? Does increasing k make this "bump" disappear? Try letting k be 1000 (it may take little while for the computer to construct your solution).