Differential Equations with Linear Algebra

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Chapter 1

Review

This chapter covers the following ideas.

1. Graph basic functions by hand. Compute derivatives and integrals, in particular using the product rule, quotient rule, chain rule, integration by u-substitution, and integration by parts (the tabular method is useful for simplifying notation). Explain how to find a Laplace transform.
2. Explain how to verify a function is a solution to an ODE, and illustrate how to solve separable ODEs.
3. Explain how to use the language of functions in high dimensions and how to compute derivatives using a matrix. Illustrate the chain rule in high dimensions with matrix multiplication.
4. Graph the gradient of a function together with several level curves to illustrate that the gradient is normal to level curves.
5. Explain how to test if a differential form is exact (a vector field is conservative) and how to find a potential.

1.1 Basics

We need to review our ability to graph functions with multiple inputs and/or outputs. The next few problems ask you to practice some skills that will be crucial as the course progresses.

Problem 1.1 Construct graphs of the following functions. Explain how to obtain each graph by transforming and rescaling the first. Then state the amplitude and period of the function.

1. $y = \sin(x)$
2. $y = 5\sin(x) + 1$
3. $y = 4\sin(3(x - \pi)) + 2$
4. $y = 4\sin(3x - \pi) + 2$

Problem 1.2 Consider the function $f(x) = e^{-x}$.

1. Construct graphs of $y = f(x)$ and $y = 2f(-(x + 3)) - 1$.

2. State $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ from your graph.

3. Compute $\lim_{x\to\infty} xf(x)$ and $\lim_{x\to\infty} x^2 f(x)$. [Hint: L'Hopital's rule will help.]

As the semester progresses, we'll need the functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are the hyperbolic trig functions, and we say the hyperbolic sine of x when we write $\sinh x$. These functions are very similar to sine and cosine functions, and have very similarly properties.

Problem 1.3 Three useful facts about the trig functions are (1) $\frac{d}{dx}\sin x = \cos x$, (2) $\frac{d}{dx}\cos x = -\sin x$, and (3) $\cos^2 x + \sin^2 x = 1$. Use the definitions above to show the following:

1. $\frac{d}{dx}\sinh x = \cosh x$,

2. $\frac{d}{dx}\cosh x = \sinh x$, and

3. $\cosh^2 x - \sinh^2 x = 1$.

[Hint: Start by replacing the hyperbolic function with its definition in terms of exponentials. Then perform the computations.]

Problem 1.4 The three facts from the previous problem are crucial tools need to prove that $\frac{d}{dx}\tan x = \sec^2 x$.

1. Use the quotient rule to give a formula for $\frac{d}{dx}\tanh x$ in terms of hyperbolic trig functions.

2. Similarly obtain a formula for the derivative of $\operatorname{sech} x = \frac{1}{\cosh x}$.

3. What is $\frac{d}{dx}\operatorname{csch} x$?

You might ask why these function are called the hyperbolic trig functions. What does a hyperbola have to do with anything?

Problem 1.5 Each pair of parametric equations traces out a curve in the xy plane. Given a Cartesian equation of the curve by eliminating the parameter t, and then graph the curve.

1. $x = \cos t$, $y = \sin t$, $-2\pi < t < 2\pi$.

2. $x = \cosh t$, $y = \sinh t$, $-\infty < t < \infty$.

Give a reason as to why do we call cosh the hyperbolic cosine.

Problem 1.6 Use implicit differentiation to find the derivative of $y = sinh^{-1}x$. Your answer should not involve any hyperbolic trig functions, and should be in terms of x. [Hint: First write $x = sinh(y)$, and then implicitly differentiate both sides. You'll need the key identity from a few problems above to help you finish.]

The problems above asked you to review your differentiation skills. You'll want to make sure you can use the basic rules of differentiation (such as the power, product, quotient, and chain rules). The next few problems will help you review your integration techniques, and you will apply them to two new ideas.

Problem 1.7 Compute the three integrals

$$\int xe^{-x^2}dx \qquad \text{and} \qquad \int_0^1 xe^{-x^2}dx \qquad \text{and} \qquad \int_0^\infty xe^{-x^2}dx.$$

If you have never used the tabular method to perform integration-by-parts, I strongly suggest that you open the online text and read a few examples (see the bottom of page 2).

Problem 1.8 Compute $\int x\sin(5x)dx$ and $\int x^2\sin(5x)dx$.

Problem 1.9 Compute $\int \tanh^{-1} x dx$. The derivative of $\tanh^{-1} x$ is $\dfrac{1}{1-x^2}$.

1.2 Laplace Transforms

Definition 1.1: The Laplace Transform. Let $f(t)$ be a function that is defined for all $t \geq 0$. Using the function $f(t)$, we define the Laplace transform of f to be a few function F where for each s we obtain the value by computing the integral

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt.$$

The domain of F is the set of all s such that the improper integral above converges. The function $f(t)$ is called the inverse Laplace transform of $F(s)$, and we write $f(t) = \mathscr{L}^{-1}(F(s))$.

Note that the Laplace transform of a function with independent variable t is another function with a different independent variable s. After integration, all t's will be removed from $F(s)$. You can of course use any other letters besides t and s.

We will use the Laplace transform throughout the semester to help us solve many problems related to mechanical systems, electrical networks, and more. The mechanical and electrical engineers in this course will use Laplace transforms in many future courses. Our goal in the problems that follow is to practice integration-by-parts. As an extra bonus, we'll learn the Laplace transforms of some basic functions.

Problem 1.10 Compute the integral $\int_0^\infty e^{-st}dt$, and state for which s the integral converges. What is the Laplace transform of $f(t) = 1$? (If the last question seems redundant, then horray.)

Problem 1.11 Compute the Laplace transform of $f(t) = e^{2t}$, and state the domain. Then compute the Laplace transform of $f(t) = e^{3t}$ and state the domain. Finally, compute the Laplace transform of $f(t) = e^{at}$ for any a, and state the domain.

Problem 1.12 Suppose $s > 0$ and n is a positive integer. Explain why

$$\lim_{t\to\infty} \frac{t^n}{e^{st}} = 0.$$

Use this fact to prove that the Laplace transform of t^2 is

$$\mathscr{L}\{t^2\}\frac{2}{s^3}.$$

[You'll need to do integration-by-parts twice, try the tabular method.]

Problem 1.13 In the previous problems, you showed that

$$\mathscr{L}\{t^0\} = \frac{1}{s^1} \qquad \text{and} \qquad \mathscr{L}\{t^2\} = \frac{2}{s^3}.$$

Show that the Laplace transform of t is $\mathscr{L}\{t^1\} = \frac{1}{s^2}$. Then compute the Laplace transforms of t^3, t^4, and so on until you see a pattern. Use this pattern to state the Laplace transform of t^10 and t^n, provided n is a positive integer. [Hint: Try the tabular method of integration-by-parts. After evaluating at 0 and ∞, all terms but 1 will be zero.]

Theorem 1.2. *Since integration can be done term-by-term, and constants can be pulled out of the integral, we have the crucial fact that*

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$$

for functions f, g and constants a, b.

Problem 1.14 Without integrating, rather using the results above, compute the Laplace transform $L(3 + 5t^2 - 6e^{8t})$, and state the domain.

Problem 1.15 Recall that $\cosh t = \frac{e^t + e^{-t}}{2}$ and $\sinh t = \frac{e^t - e^{-t}}{2}$. Use this to prove that

$$\mathscr{L}\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad \text{and} \qquad \mathscr{L}\{\sinh at\} = \frac{a}{s^2 - a^2}.$$

1.3 Ordinary Differential Equations

A differential equation is an equation which involves derivatives (of any order) of some function. For example, the equation $y'' + xy' + \sin(xy) = xy^2$ is a differential equation. An **ordinary differential equation (ODE)** is a differential equation involving an unknown function y which depends on only one independent variable (often x or t). A partial differential equation involves an unknown function y that depends on more than one variable (such as $y(x,t)$). The order of an ODE is the order of the highest derivative in the ODE. A solution to an ODE on an interval (a, b) is a function $y(x)$ which satisfies the ODE on (a, b).

Example 1.3. The first order ODE $y'(x) = 2x$, or just $y' = 2x$, has unknown function y with independent variable x. A solution on $(-\infty, \infty)$ is the function $y = x^2 + C$ for any constant C. We obtain this solution by simply integrating both sides. Notice that there are infinitely many solutions to this ODE.

Typically a solution to an ODE involves an arbitrary constant C. There is often an entire family of curves which satisfy a differential equation, and the constant C just tells us which curve to pick. A **general solution** of an ODE is an infinite class of solutions of the ODE. A **particular solution** is one of the infinitely many solutions of an ODE.

Often an ODE comes with an **initial condition** $y(x_0) = y_0$ for some values x_0 and y_0. We can use these initial conditions to find a particular solution of the ODE. An ODE, together with an initial condition, is called an **initial value problem (IVP)**.

Example 1.4. The IVP $y' = 2x$, $y(2) = 1$, has the general solution $y = x^2 + C$ from the previous problem. Since $y = 1$ when $x = 2$, we have $1 = 2^2 + C$ which means $C = -3$. Hence the solution to our IVP is $y = x^2 - 3$.

Problem 1.16 Consider the ordinary differential equation $y'' + 9y = 0$. By computing derivatives, show that $y(t) = A\cos(3t) + B\sin(3t)$ is a general solution to the ODE, where A and B are arbitrary constants. If we know that $y(0) = 1$ and $y'(0) = 2$, determine the values of A and B.

Problem 1.17 Consider the ordinary differential equation $y\dfrac{dy}{dx} = x^2$. Find a general solution to this ODE by integrating both sides with respect to x. State an interval on which your solution is valid.

They could introduce the entire method of separation by parts without me telling them what to do. I just need to ask them to do an integral. Afterward, I could ask them to solve an ODE. Put it in the same problem.

Problem 1.18 Consider the ODE given by $y' = 4ty$. Find a general solution to this ODE. [Hint: Rewrite y' as $\dfrac{dy}{dt}$. Then put all the terms that involve y on one side of the equation, and the terms that involve t on the other. Then it should be similar to the previous problem.]

Problem 1.19 Solve the IVP given by $y' = \dfrac{x^2 - 1}{y^4 + 1}$, where $y(0) = 1$.

1.4 General Functions and Derivatives

Recall that to compute partial derivatives, we hold all but one variable constant and then differentiate with respect to that variable. Partial derivatives can be organized into a matrix Df where columns represents the partial derivative of f with respect to each variable. This matrix, called the derivative or total derivative, takes us into our study of linear algebra. Some examples of functions and their derivatives appear in Table 1.1. When the output dimension is one, the matrix has only one row and the derivative is often called the gradient of f, written ∇f.

In multivariate calculus, we focused our time on learning to graph, differentiate, and analyze each of the types of functions in the table above. The next few problems ask you to review this.

Function	Derivative
$f(x) = x^2$	$Df(x) = [2x]$
$\vec{r}(t) = (3\cos(t), 2\sin(t))$	$D\vec{r}(t) = \begin{bmatrix} -3\sin t \\ 2\cos t \end{bmatrix}$
$\vec{r}(t) = (\cos(t), \sin(t), t)$	$D\vec{r}(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$
$f(x, y) = 9 - x^2 - y^2$	$Df(x, y) = \nabla f(x, y) = [-2x \quad -2y]$
$f(x, y, z) = x^2 + y + xz^2$	$Df(x, y, z) = \nabla f(x, y, z) = \begin{bmatrix} 2x + z^2 & 1 & 2xz \end{bmatrix}$
$\vec{F}(x, y) = (-y, x)$	$D\vec{F}(x, y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\vec{F}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$	$D\vec{F}(r, \theta, z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\vec{r}(u, v) = (u, v, 9 - u^2 - v^2)$	$D\vec{r}(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & -2v \end{bmatrix}$

Table 1.1: The table above shows the (matrix) derivative of various functions. Each column of the matrix corresponds a partial derivative of the function. When the output of a function is a vector, partial derivatives are vectors which are placed in columns of the matrix. The order of the columns matches the order in which you list the variables.

Problem 1.20 Let $\vec{r}(t) = \left\langle t^2 - 1, 2t + 3 \right\rangle$. Construct a graph of $\vec{r}(t)$, and compute the derivative $D\vec{r}(t)$.

Problem 1.21 Let $f(x, y) = 4 - x^- y^2$. Construct a 3D graph of $z = f(x, y)$. Also construct a graph of several level curves. Then compute the derivative $Df(x, y)$.

Recall that a level curve of $z = f(x, y)$ is curve in the xy plane where the output z is constant.

Problem 1.22 Let $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$. Construct a 3D graph of $\vec{r}(t)$, and compute the derivative $D\vec{r}(t)$.

Problem 1.23 Let $\vec{F}(x, y) = (y, -2x)$. Construct a 2D graph of this vector field, and compute the derivative $D\vec{F}(x, y)$.

1.4.1 The General Chain Rule

The chain rule in first semester calculus states that

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

You may remember this as "the derivative of the outside function times the derivative of the inside function." In multivariable calculus, most textbooks use a tree rule to develop the formula

$$\frac{df}{dt} = f_x x_t + f_y y_t$$

for a function $f(x, y)$, where x and y depend on t (so that $\vec{r}(t) = (x(t), y(t))$ is a curve in the xy plane). Written in matrix form, the chain rule is simply

$$\frac{df}{dt} = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = Df \cdot Dr,$$

which is the (matrix) product of the derivatives, just as it was in first semester calculus. You are welcome to tackle the following problems by using the tree rule or matrix product.

Problem 1.24 Suppose that $f(x, y) = x^2 + 3xy$, where $x = t^2 + 1$ and $y = \sin t$, so we could write $\vec{r}(t) = (t^2 + 1, \sin t)$.

1. Compute $Df(x, y)$, $\dfrac{dx}{dt}$, and $D\vec{r}(t)$. (You should have two matrices.)
2. Compute $\frac{df}{dt}$.

Problem 1.25 Suppose that $f(x, y) = x + 3y$ and that $\dfrac{dx}{dt} = \cos t$ and $\dfrac{dy}{dt} = e^t$. Compute $\frac{df}{dt}$.

Problem 1.26 Suppose that $z = f(x, y)$ and that $\dfrac{\partial f}{\partial x} = 3x^2 y$ and $\dfrac{\partial f}{\partial y} = x^3 y - e^y$. Also suppose that $x = \sqrt{t}$ and $y = \ln t$. Compute $\frac{df}{dt}$.

Problem 1.27 Suppose that $z = f(x, y)$ is a differential function of two variables. Suppose that $\vec{r}(t)$ is a parametrization of a level curve of f. We can write the level curve in vector form as $\vec{r}(t) = (x(t), y(t))$, or in parametric form $x = x(t)$ and $y = y(t)$.

1. If $f(\vec{r}(0)) = 7$, then what is $f(\vec{r}(2))$?
2. Why does $\frac{df}{dt} = \nabla f(x, y) \cdot \dfrac{d\vec{r}}{dt}$?
3. Why is the gradient of f normal to level curves?

Recall that the word normal means there is a 90 degree angle between the gradient and the level curve.

Before proceeding, let's practice with an examples to visually remind us that the gradient is normal to level curves. This key fact will help us solve most of the differential equations we encounter in the course.

Problem 1.28 Consider the function $f(x, y) = x^2 - y$. Start by computing the gradient. Then construct a graph which contains several level curves of f, as well as the gradient at several points on each level curve.

1.5 Potentials of Vector Fields and Differential Forms

When the output dimension of a function is one, so we would write $f : \mathbb{R}^n \to \mathbb{R}^1$, then we call the derivative the gradient and write $\vec{\nabla} f = (f_x, f_y, f_z)$. Notice that this is a vector field. Taking a derivative gives us a vector field. Is every vector field the derivative of some function? Hopefully you remember that the answer to this question is "No."

If a vector field $\vec{F} = (M, N)$ (or in 3D $\vec{F} = (M, N, P)$) is the gradient of some some function f (so that $\vec{\nabla} f = \vec{F}$), then we say that the vector field $\vec{F}$ is a gradient field (or conservative vector field). We say that f is a potential for the vector field $\vec{F}$ when $\nabla f = \vec{F}$. In this section, we'll review how to determine if a vector field has a potential, as well as how to find a potential.

Problem 1.29 Let $\vec{F} = (M, N) = (2x + y, x + 4y)$. Find a potential for $\vec{F}$ by doing the following.

1. If we suppose $M = 2x + y$ is the partial of f with respect to x, then $f_x = 2x + y$. Find a function f whose partial with respect to x is M.
2. If we suppose $N = x + 4y$ is the partial of f with respect to y, then $f_y = x + 4y$. Find a function f whose partial with respect to y is N.
3. What is a potential for $\vec{F}$? Prove your answer is correct by computing the gradient of your answer.

By taking derivatives, there is a test that tells you if a function will have a potential. Some textbooks call it the test for a conservative field.

Problem 1.30: Test for a conservative vector field. Let's prove the test for a conservative vector field in both 2 and 3 dimensions.

The test for a conservative vector field states states more than what you showed in this problem. It states that if $\vec{F}$ is a continuously differentiable vector field on a simply connected domain, then (1) if $\vec{F}$ has potential, then certain pairs of partials must be equal, and (2) if those pairs of partial derivatives are equal, then the $\vec{F}$ has a potential. We will not prove part (2).

1. Suppose that $\vec{F}(x, y) = (M, N)$ is a continuously differentiable vector field on the entire plane. Suppose further that $\vec{F}$ has a potential f. The derivative of $\vec{F}$ is
$$D\vec{F}(x, y) = \begin{pmatrix} M_x & M_y \\ N_x & N_y \end{pmatrix}.$$
Some of the entries in this matrix must be equal? Which ones? Explain. [If you're not sure, try taking the derivative of the problem above.]
2. Suppose that $\vec{F}(x, y, z) = (M, N, P)$ is a continuously differentiable vector field on all of space. Suppose further that $\vec{F}$ has a potential f. State the derivative of $\vec{F}$, and then state which pairs of entries must be equal.

Problem 1.31 For each vector field below, either give a potential, or explain why no potential exists.

1. $\vec{F} = (4x + 5y, 5x + 6y)$
2. $\vec{F} = (2x - y, x + 3y)$
3. $\vec{F} = \left(4x + \dfrac{2y}{1 + 4x^2}, \arctan(2x)\right)$
4. $\vec{F} = (3y + 2yz, 3x + 2xz + 6z, 2xy + 6y)$

We'll finish by introducing the vocabulary of differential forms. We'll use this vocabulary throughout the semester as we study differential equations. The vocabulary of vector fields parallels the vocabulary of differential forms.

Definition 1.5: Differential Forms. Assume that f, M, N, P are all functions of three variables x, y, z. Similar definitions hold in all dimensions.

- A differential form is an expression of the form $Mdx + Ndy + Pdz$ (just as a vector field is a function $\vec{F} = (M, N, P)$).
- The differential of a function f is the expression $df = f_x dx + f_y dy + f_z dz$ (just as the gradient is $\vec{\nabla} F = (f_x, f_y, f_z)$).
- If a differential form is the differential of a function f, then the differential form is said to be exact (just as we say a vector field is a gradient field). Again, the function f is called a potential for the differential form.

A differential form is exact precisely when the corresponding vector field is a gradient field.

Notice that $Mdx + Ndy + Pdz$ is exact if and only if $\vec{F} = (M, N, P)$ is a gradient field. The language of differential forms is practically the same as the language of conservative vector fields. Why do we have different sets of words for the same idea? That happens all the time when different groups of people work on seeming different problems, only to discover years later that they have been working on the same problem. If both sets of vocabulary stick, it's often because both have advantages. We have many different notations for the derivative (such as y', $\frac{dy}{dx}$, and Df), and each notation has advantages. The language of differential forms is best suited when studying differential equations.

Problem 1.32 For each differential form below, state if the differential form is exact. If it is exact, give a potential.

1. $(2x + 3y)dx + (4x + 5y)dy$
2. $(2x - y)dx + (3y - x)dy$
3. $\left(4x + \dfrac{3y}{1 + 9x^2}\right) dx + \arctan(3x)dy$
4. $(3y + 2yz)dx + (3x + 2xz + 6z)dy + (2xy + 5y)dz$

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Linear Algebra Arithmetic

This chapter covers the following ideas.

1. Be able to use and understand matrix and vector notation, addition, scalar multiplication, the dot product, matrix multiplication, and matrix transposing.
2. Use Gaussian elimination to solve systems of linear equations. Define and use the words homogeneous, nonhomogeneous, row echelon form, and reduced row echelon form.
3. Find the rank of a matrix. Determine if a collection of vectors is linearly independent. If linearly dependent, be able to write vectors as linear combinations of the preceding vectors.
4. For square matrices, compute determinants, inverses, eigenvalues, and eigenvectors.
5. Illustrate with examples how a nonzero determinant is equivalent to having independent columns, an inverse, and nonzero eigenvalues. Similarly a zero determinant is equivalent to having dependent columns, no inverse, and a zero eigenvalue.

The next unit will focus on applications of these ideas. The main goal of this unit is to familiarize yourself with the arithmetic involved in linear algebra.

2.1 Basic Notation

Most of linear algebra centers around understanding vectors, with matrices being functions which transform vectors from one vector space into vectors in another vector space. This chapter contains a brief introduction to the arithmetic involved with matrices and vectors. The next chapter will show you many of the uses of the ideas we are learning. You will be given motivation for all of the ideas learned here, as well as real world applications of these ideas, before the end of the next chapter. For now, I want you become familiar with the arithmetic of linear algebra so that we can discuss how all of the ideas in this chapter show up throughout the course.

Definition 2.1. A matrix of size m by n has m rows and n columns. We

Matrix size is row by column.

normally write matrices using capital letters, and use the notation

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{jk}],$$

where a_{jk} is the entry in the jth row, kth column.

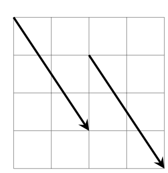
- We say two matrices A and B are equal if $a_{jk} = b_{jk}$ for all j and k.
- We add and subtract matrices of the same size entry wise. So we write $A + B = C$ where $c_{jk} = a_{jk} + b_{jk}$. If matrices do not have the same size, then we cannot add them.
- We can multiply a matrix A by a scalar C to obtain a new matrix cA. We do this multiplying every entry in the matrix A by the scalar c.
- If the number of rows and columns are equal, then we say the matrix is square.
- The main diagonal of a square ($n \times n$) matrix consists of the entries $a_{11}, a_{22}, \ldots, a_{nn}$.
- The trace of a square matrix is the sum of the entries on the main diagonal ($\sum a_{jj}$).
- The transpose of a matrix $A = [a_{jk}]$ is a new matrix $B = A^T$ formed by interchanging the rows and columns of A, so that $b_{jk} = a_{kj}$. If $A^T = A$, then we say that A is symmetric.

Problem 2.1 Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$. Compute $2A - 3B$, and find the trace of both A and B.

Problem 2.2 Write down a 3 by 2 matrix, and compute the transpose of that matrix. Then give an example of a 3 by 2 symmetric matrix, or explain why it is not possible.

Vectors represent a magnitude in a given direction. We can use vectors to model forces, acceleration, velocity, probabilities, electronic data, and more. We can use matrices to represent vectors. A row vector is a $1 \times n$ matrix. A column vector is an $m \times 1$ matrix. Textbooks often write vectors using bold face font. By hand (and in this book) we add an arrow above them. The notation $\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$ can represent either row or column vectors. Many different ways to represent vectors are used throughout different books. In particular, we can represent the vector $\langle 2, 3 \rangle$ in any of the following forms

$$\langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j} = (2, 3) = \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



Both vectors represent $\langle 2, -3 \rangle$, regardless of where we start.

The notation $(2, 3)$ has other meanings as well (like a point in the plane, or an open interval), and so when you use the notation $(2, 3)$, it should be clear from the context that you are working with a vector. To draw a vector $\langle v_1, v_2 \rangle$, one option is to draw an arrow from the origin (the tail) to the point (v_1, v_2) (the head). However, the tail does not have to be placed at the origin.

The principles of addition and subtraction of matrices apply to vectors (which can be though of as row or column matrices). We will most often think of vectors as column vectors.

Definition 2.2. The magnitude (or length) of the vector $\vec{u} = (u_1, u_2)$ is $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$. In higher dimensions we extend this as

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2 + \cdots u_n^2} = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

A unit vector is a vector with length 1. In many books unit vectors are written with a hat above them, as $\hat{\mathbf{u}}$.

A unit vector $\hat{\mathbf{u}}$ has length $|\vec{u}| = 1$

We will need to be able to find vectors of any length that point in a given direction.

$\boxed{\textbf{Problem 2.3}}$ Find a vector of length 12 that points in the same direction as the vector $\vec{v} = (1, 2, 3, 4)$. Then give a general formula for finding a vector of length c that points in the direction of $\vec{u}$.

The simplest vectors in 2D are a one unit increment in either the x or y direction, and we write these vectors in any of the equivalent forms

$$\mathbf{i} = \vec{i} = \langle 1, 0\rangle = (1, 0) \quad \text{and} \quad \mathbf{j} = \vec{j} = \langle 0, 1\rangle = (0, 1).$$

We call these the standard basis vectors in 2D. In 3D we include the vector $\mathbf{k} = \vec{j} = \langle 0, 0, 1\rangle$ as well as add a zero to both $\vec{i}$ and $\vec{j}$ to obtain the standard basis vectors. The word basis suggests that we can base other vectors on these basis vectors, and we typically write other vectors in terms of these standard basis vectors. Using only scalar multiplication and vector addition, we can obtain the other vectors in 2D from the standard basis vectors.

The standard basis vectors in 3D
$\mathbf{i} = \vec{i} = \langle 1, 0, 0\rangle = (1, 0, 0)$
$\mathbf{j} = \vec{j} = \langle 0, 1, 0\rangle = (0, 1, 0)$
$\mathbf{k} = \vec{k} = \langle 0, 0, 1\rangle = (0, 0, 1)$

$\boxed{\textbf{Problem 2.4}}$ Write the vector $(2, 3)$ in the form $(2, 3) = c_1\vec{i} + c_2\vec{j}$.
If instead we use the non-standard basis vectors $\vec{u}_1 = (1, 2)$ and $\vec{u}_2 = (-1, 4)$, then write the vector $(2, 3)$ in the form $(2, 3) = c_1\vec{u}_1 + c_2\vec{u}_2$.

Definition 2.3. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an expression of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$, where c_i is a constant for each i.

A linear combination of vectors is simply a sum of scalar multiples of the vectors. We start with some vectors, stretch each one by some scalar, and then sum the result. Much of what we will do this semester (and in many courses to come) relates directly to understanding linear combinations.

$\boxed{\textbf{Problem 2.5}}$ The force acting on an object is $\vec{F} = (-3, 2)$ N. The object is in motion and has velocity vector $\vec{v} = (1, 1)$ and acceleration vector $\vec{a} = (-1, 2)$. Write the force as a linear combination of the velocity and acceleration vectors.

$\boxed{\textbf{Problem 2.6}}$ Write the vector $(2, 3, 1)$ as a linear combination of the standard basis vectors in $\mathbb{R}^3$. Then write $(2, 3, 1)$ as a linear combination of the vectors $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$.

One of the key applications of linear combinations we will make throughout the semester is matrix multiplication. Let's introduce the idea with an example.

Example 2.4. Consider the three vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\2\\-3\end{bmatrix}$, and $\begin{bmatrix}2\\1\\1\end{bmatrix}$. Let's multiply the first vector by 2, the second by -1, and the third by 4, and then sum the result. This gives us the linear combination

$$2\begin{bmatrix}1\\0\\1\end{bmatrix} - 1\begin{bmatrix}0\\2\\-3\end{bmatrix} + 4\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}10\\2\\9\end{bmatrix}$$

We will define matrix multiplication so that multiplying a matrix on the right by a vector corresponds precisely to creating a linear combination of the columns of A. We now write the linear combination above in matrix form

$$\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\1 & -3 & 1\end{bmatrix}\begin{bmatrix}2\\-1\\4\end{bmatrix} = \begin{bmatrix}10\\2\\9\end{bmatrix}.$$

Definition 2.5: A matrix times a vector. We define the matrix product $A\vec{x}$ (a matrix times a vector) to be the linear combination of columns of A where the components of $\vec{x}$ are the scalars in the linear combination. For this to make sense, notice that the vector $\vec{x}$ must have the same number of entries as there are columns in A. We can make this definition more precise as follows. Let $\vec{v}_i$ be the ith column of A so that $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}$, and let $\vec{x} = \begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix}$. Then the matrix product is the linear combination

The product $A\vec{x}$ gives us linear combinations of the columns of A.

$$A\vec{x} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix} = \vec{a}_1x_1 + \vec{a}_2x_2 + \cdots + \vec{a}_nx_n.$$

The definition above should look like the dot product. If you think of A as a vector of vectors, then $A\vec{x}$ is just the dot product of A and $\vec{x}$.

Problem 2.7 Write down a 2 by 4 nonzero matrix, and call it A (fill the matrix with some integers of your choice). Then write down a vector $\vec{x}$ such that the matrix product $A\vec{x}$ makes sense (again, fill the vector with integers of your choice). Then use the definition above to obtain the product $A\vec{x}$.

Definition 2.6: A matrix times a matrix. Let $\vec{b}_j$ represent the jth column of B (so $B = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n\end{bmatrix}$). The product AB of two matrices $A_{m\times n}$ and $B_{n\times p}$ is a new matrix $C_{m\times p} = [c_{ij}]$ where the jth column of C is the product $A\vec{b}_j$. To summarize, the matrix product AB is a new matrix whose jth column is a linear combinations of the columns of A using the entries of the jth column of B to perform the linear combinations.

Problem 2.8 Let $A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1 & 4\\-1 & 2 & -3\end{bmatrix}$. Use the definition given above to compute both AB and BA. Be prepared to show the class how you used linear combinations to get the matrix product. (If you are used to using the row dotted by column approach, then this problem asks you to do the matrix product differently.)

We introduced matrix multiplication in terms of linear combinations of column vectors. My hope is that by doing so you immediately start thinking of linear combinations whenever you encounter matrix multiplication (as this is what it was invented to do). There are many alternate ways to think of matrix multiplication. Here are two additional methods.

1. "Row times column approach." The product AB of two matrices $A_{m\times n}$ and $B_{n\times p}$ is a new matrix $C_{m\times p} = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ is the dot product of the of the ith row of A and the jth column of B. Wikipedia has an excellent visual illustration of this approach.

2. Rephrase everything in terms of rows (instead of columns). We form linear combinations of rows using rows. The matrix product $\vec{x}B$ (notice the order is flopped) is a linear combination of the rows of B using the components of x as the scalars. For the product AB, let $\vec{a}_i$ represent the ith row of A. Then the ith row of AB is the product $\vec{a}_iB$. We'll most often use the column definition instead of this, because we use the function notation $f(x)$ from calculus, and later we will use the notation $A(\vec{x})$ instead of $(\vec{x})A$ to describe how matrices act as functions.

Problem 2.9 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 2 & -3 \end{bmatrix}$. Use the two alternate definitions above to compute AB. Be prepared to show the class how you used both alternate definitions (You'll need to show your intermediate steps).

Problem 2.10 Do each of the following:

1. Solve the system of equations $x + 2y = 3$, $4x + 5y = 6$.

2. Write the vector $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Find a vector $\vec{x}$ so that $A\vec{x} = \vec{b}$. This matrix A is called the coefficient matrix of the system in the first part.

How are these three questions related?

Prior to introducing Gaussian elimination, let's solve a system of equations using an elimination method. If $2x + 3y = 4$ and $5x + 7y = 0$, then we can eliminate x from the second equation by multiplying both sides of the first equation by 5, and both sides of the second equation by 2, and then subtracting. This would give us the equations $10x + 15y = 20$ and $10x + 14y = 1$. The first equation minus the second then gives $(10 - 10)x + (15 - 14)y = (20 - 0)$, or more simply $y = 20$. Similarly, you could multiply the first equation by 7, and the second by 3, to eliminate y.

Problem 2.11 Solve the system of equations

$$\begin{aligned} 2x + 3y - 4z &= 4 \\ 3x + 4y - 3z &= 8 \\ 7x + 12y - 12z &= 19. \end{aligned}$$

Use elimination to find your solution. Eliminate x from the 2nd and 3rd equations (which will give you two equations that do not involve x). Then use one of these simplified equations to eliminate y from the other simplified equation. At this point you should have an equation that only involves z. Then use back substitution to give y and x.

Problem 2.12 Answer the following.

1. Suppose that $ax + by = c$ and $dx + ey = f$, where a, b, c, d, e, f are all constants. This is a system of equations with 2 equations and 2 unknowns. Each equation represents a line in the plane. How many solutions are there to this system? (You should have a few different cases.)

2. Suppose that $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$ and $a_{31}x + a_{32}y + a_{33}z = b_1$, where each a_{ij} is a constant. This is a system of equations with 3 equations and 3 unknowns. Each equation represents a plane in space. How many solutions are there to this system? (You should have a few different cases.)

3. Suppose that $a_{11}x + a_{12}y + a_{13}z = b_1$ and $a_{21}x + a_{22}y + a_{23}z = b_2$, where each a_{ij} is a constant. This is a system of equations with 2 equations and 3 unknowns. Each equation represents a plane in space. How many solutions are there to this system? (You should have a few different cases.)

Definition 2.7. We say that a system of linear equation is consistent, if it has at least one solution. We say it is inconsistent if there is no solution.

2.2 Gaussian Elimination

Gaussian elimination is an efficient algorithm we will use to solve systems of equations. This is the same algorithm implemented on most computers systems. The main idea is to eliminate each variable from all but one equation/row (if possible), using the following three operations (called elementary row operations):

1. Multiply an equation (or row of a matrix) by a nonzero constant,

2. Add a nonzero multiple of any equation (or row) to another equation,

3. Interchange two equations (or rows).

These three operations are the operations learned in college algebra when solving a system using a method of elimination. Gaussian elimination streamlines elimination methods to solve generic systems of equations of any size. The process involves a forward reduction and (optionally) a backward reduction. The forward reduction creates zeros in the lower left corner of the matrix. The backward reduction puts zeros in the upper right corner of the matrix. We eliminate the variables in the lower left corner of the matrix, starting with column 1, then column 2, and proceed column by column until all variables which can be eliminated (made zero) have been eliminated. Before formally stating the algorithm, let's look at a few examples.

Example 2.8. Let's start with a system of 2 equations and 2 unknowns. I will write the augmented matrix representing the system as we proceed. To solve

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ 2x_1 - 5x_2 &= 1 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 2 & -5 & 1 \end{array}\right]$$

we eliminate the $2x_1$ in the 2nd row by adding -2 times the first row to the second row.

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ x_2 &= -7 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 1 & -7 \end{array}\right]$$

The matrix at the right is said to be in **row echelon form**. row echelon form

Definition 2.9: Row Echelon Form. We say a matrix is in row echelon form (ref) if

- each nonzero row begins with a 1 (called a leading 1),
- the leading 1 in a row occurs further right than a leading 1 in the row above, and
- any rows of all zeros appear at the bottom.

The position in the matrix where the leading 1 occurs is called a pivot. The column containing a pivot is called a pivot column. pivot column

At this point in our example, we can use "back-substitution" to get $x_2 = -7$ and $x_1 = 4+3x_2 = 4-21 = -17$. Alternatively, we can continue the elimination process by eliminating the terms above each pivot, starting on the right and working backwards. This will result in a matrix where all the pivot columns contain all zeros except for the pivot. If we add 3 times the second row to the first row, we obtain.

$$\begin{aligned} x_1 &= -17 \\ x_2 &= -7 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & -17 \\ 0 & 1 & -7 \end{array}\right]$$

The matrix on the right is said to be in **reduced row echelon form** (or just rref). We can easily read solutions to systems of equations directly from a matrix which is in reduced row echelon form.

Definition 2.10: Reduced Row Echelon Form. We say that a matrix is in reduced row echelon form (rref) if reduced row echelon form - rref

- the matrix is in row echelon form, and
- each pivot column contains all zeros except for the pivot (leading one).

Example 2.11. Let's now solve a nonhomogeneous (meaning the right side is not zero) system with 3 equations and 3 unknowns:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2 \\ x_1 - 2x_2 &= 3 \\ 4x_2 + 2x_3 &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right].$$

We'll encounter some homogeneous systems later on. To simplify the writing, we'll just use matrices this time. To keep track of each step, I will write the row operation next to the row I will replace. Remember that the 3 operations are (1)multiply a row by a nonzero constant, (2)add a multiple of one row to another, (3) interchange any two rows. If I write $R_2 + 3R1$ next to R_2, then this means I will add 3 times row 1 to row 2. If I write $2R_2 - R1$ next to R_2, then I have done two row operations, namely I multiplied R_2 by 2, and then added (-1) times $R1$ to the result (replacing $R2$ with the sum). The steps below read left to right, top to bottom. In order to avoid fractions, I wait to divide until the last step, only putting a 1 in each pivot at the very end.

$$\Rightarrow^{(1)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right] \begin{array}{c} \\ 2R_2 - R_1 \\ \\ \end{array} \quad \Rightarrow^{(2)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 4 & 2 & 1 \end{array}\right] \begin{array}{c} \\ \\ 5R_3 + 4R_2 \end{array}$$

$$\Rightarrow^{(3)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 14 & 21 \end{array}\right] \begin{array}{c} \\ \\ R_3/7 \end{array} \quad \Rightarrow^{(4)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -10 & 2 & 8 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} 2R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\Rightarrow^{(5)} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 7 \\ 0 & -10 & 0 & 5 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} \\ R_2/5 \\ \\ \end{array} \quad \Rightarrow^{(6)} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 7 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array}$$

$$\Rightarrow^{(7)} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 8 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{c} R_1/4 \\ R_2/-2 \\ R_3/2 \end{array} \quad \Rightarrow^{(8)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

Writing the final matrix in terms of a system, we have the solution $x_1 = 2, x_2 = -1/2, x_3 = 3/2$. Remember that this tells us (1) where three planes intersect, (2) how to write the 4th column $\vec{b}$ in our original augmented matrix as a linear combination of the columns of the coefficient matrix A, and (3) how to solve the matrix equation $A\vec{x} = \vec{b}$ for $\vec{x}$.

The following steps describe the Gaussian elimination algorithm that we used above. Please take a moment to compare what is written below with the example above. Most of the problems in this unit can be solved using Gaussian elimination, so we will practice it as we learn a few new ideas.

1. Forward Phase (row echelon form) - The following 4 steps should be repeated until you have mentally erased all the rows or all the columns. In step 1 or 4 you will erase a column and/or row from the matrix.

   (a) Consider the first column of your matrix. Start by interchanging rows (if needed) to place a nonzero entry in the first row. If all the elements in the first column are zero, then ignore that column in future computations (mentally erase the column) and begin again with the smaller matrix which is missing this column. If you erase the last column, then stop.

   Computer algorithms place the largest (in absolute value) nonzero entry in the first row. This reduces potential errors due to rounding that can occur in later steps.

   (b) Divide the first row (of your possibly smaller matrix) row by its leading entry so that you have a leading 1. This entry is a pivot, and the column is a pivot column. [When doing this by hand, it is often convenient to skip this step and do it at the very end so that you avoid fractional arithmetic. If you can find a common multiple of all the terms in this row, then divide by it to reduce the size of your computations.]

   (c) Use the pivot to eliminate each nonzero entry below the pivot, by adding a multiple of the top row (of your smaller matrix) to the nonzero lower row.

   (d) Ignore the row and column containing your new pivot and return to the first step (mentally cover up or erase the row and column containing your pivot). If you erase the last row, then stop.

   Ignoring rows and columns is equivalent to incrementing row and column counters in a computer program.

2. Backward Phase (reduced row echelon form - often called Gauss-Jordan elimination) - At this point each row should have a leading 1, and you should have all zeros to the left and below each leading 1. If you skipped step 2 above, then at the end of this phase you should divide each row by its leading coefficient to make each row have a leading 1.

   (a) Starting with the last pivot column. Use the pivot in that column to eliminate all the nonzero entries above it, by adding multiples of the row containing the pivot to the nonzero rows above.

   (b) Work from right to left, using each pivot to eliminate the nonzero entries above it. Nothing to the left of the current pivot column changes. By working right to left, you greatly reduce the number of computations needed to fully reduce the matrix.

Example 2.12. As a final example, let's reduce $\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 7 \\ 1 & 3 & 5 & 1 & 6 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right]$ to reduced row echelon form (rref). The first step involves swapping 2 rows. We swap row 1 and row 2 because this places a 1 as the leading entry in row 1.

(1) Get a nonzero entry in upper left

$$\Rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 7 \\ 1 & 3 & 5 & 1 & 6 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ \\ \\ \\ \end{array}$$

(2) Eliminate entries in 1st column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 + 2R_1 \end{array}$$

(3) Eliminate entries in 2nd column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & -6 & -6 & 1 & -20 \\ 0 & 7 & 7 & 2 & 17 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 6R_2 \\ R_4 - 7R_2 \end{array}$$

(4) Make a leading 1 in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & -11 & 22 \\ 0 & 0 & 0 & 16 & -32 \end{array}\right] \begin{array}{l} \\ \\ R_3/(-11) \\ R_4/16 \end{array}$$

(5) Eliminate entries in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

(6) Row Echelon Form

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

At this stage we have found a row echelon form of the matrix. Notice that we eliminated nonzero terms in the lower left of the matrix by starting with the first column and working our way over column by column. Columns 1, 2, and 4 are the pivot columns of this matrix. We now use the pivots to eliminate the other nonzero entries in each pivot column (working right to left).

Recall that a matrix is in reduced row echelon (rref) if:

1. Nonzero rows begin with a leading 1.
2. Leadings 1's on subsequent rows appear further right than previous rows.
3. Rows of zeros are at the bottom.
4. Zeros are above and below each pivot.

(7) Eliminate entries in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R3 \\ R_2 + 2R_3 \\ \\ \\ \end{array}$$

(8) Eliminate entries in 2nd column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 8 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ \\ \\ \end{array}$$

(9) Reduced Row Echelon Form

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(10) Switch to system form

$$\Rightarrow \begin{aligned} x_1 + 2x_3 &= -1 \\ x_2 + x_3 &= 3 \\ x_4 &= -2 \\ 0 &= 0 \end{aligned}$$

We have obtained the reduced row echelon form. When we write this matrix in the corresponding system form, notice that there is not a unique solution to the system. Because the third column did not contain a pivot column, we can write every variable in terms of x_3 (the redundant equation $x_3 = x_3$ allows us to write x_3 in terms of x_3). We are free to pick any value we want for x_3 and still obtain a solution. For this reason, we call x_3 a free variable, and write our infinitely many solutions in terms of x_3 as

Free variables correspond to non pivot columns. Solutions can be written in terms of free variables.

$$\begin{aligned} x_1 &= -1 - 2x_3 \\ x_2 &= 3 - x_3 \\ x_3 &= x_3 \\ x_4 &= -2 \end{aligned} \qquad \text{or by letting } x_3 = t \qquad \begin{aligned} x_1 &= -1 - 2t \\ x_2 &= 3 - t \\ x_3 &= t \\ x_4 &= -2 \end{aligned}\ .$$

By choosing a value (such as t) for x_3, we can write our solution in so called parametric form. We have now given a parametrization of the solution set, where t is an arbitrary real number.

parametric form

Problem 2.13 Each of the following augmented matrices requires one row operation to be in reduced row echelon form. Perform the required row operation, and then write the solution to the corresponding system of equations in terms of the free variables.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ -3 & -6 & 0 & 12 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 7 & 0 & 3 \\ 0 & 0 & 1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Problem 2.14 Use Gaussian elimination to solve

$$\begin{aligned} x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 + 3x_3 &= 4 \\ 4x_1 + x_2 + 4x_3 &= 5 \end{aligned}$$

by row reducing the matrix to reduced row echelon form. [Hint: Start by interchanging row 1 and row 2.]

Problem 2.15 Use Gaussian elimination to solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ -x_1 + 2x_2 + 3x_3 &= 8 \\ 2x_1 - 4x_2 + x_3 &= 5 \end{aligned}$$

by row reducing the matrix to reduced row echelon form. [Hint: You should end up with infinitely many solutions. State your solution by writing each variable in terms of the free variable(s).]

Problem 2.16 Use Gaussian elimination to solve

$$\begin{aligned} x_1 + 2x_3 + 3x_4 &= -7 \\ 2x_1 + x_2 + 4x_4 &= -7 \\ -x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 - 2x_3 - x_4 &= 4 \end{aligned}$$

by row reducing the matrix to reduced row echelon form.

2.3 Rank, Linear Independence, Inverses, and Determinants

Definition 2.13.
- The rank of a matrix is the number of pivot columns of the matrix. To find the rank of a matrix, you reduce the matrix using Gaussian elimination until you discover the pivot columns.
- The span of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is all possible linear combinations of the vectors. In terms of matrices, the span of a set of vectors is all possible vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ for some vector $\vec{x}$, where the vectors $\vec{v}_i$ are placed in the columns of A.
- We say that a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if the only solution to the homogeneous system $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. Otherwise we say the vectors are linearly dependent, and it is possible to write one of the vectors as a linear combination of the others. We say the vectors are dependent because one of them depends on (can be obtained as a linear combination of) the others.
- In terms of spans, we say vectors are linearly dependent when one of them is in the span of the other vectors.

As we complete each of the following problems in class, we'll talk about the span of the vectors, and the rank of the corresponding matrix. The key thing we need to focus on is learning to use the words "linearly independent" and "linearly dependent."

Problem 2.17 Are the vectors $\vec{v}_1 = (1, 3, 5)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (0, 3, 1)$ linearly independent? Solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Problem 2.18 Are the vectors $\vec{v}_1 = (1, 2, 0)$, $\vec{v}_2 = (2, 0, 3)$, and $\vec{v}_3 = (3, -2, 6)$ linearly independent? Solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Problem 2.19 Answer each of the following:

1. Suppose you have row reduced a 3 by 3 matrix, and discovered that the rank of the matrix is 2. Are the columns of the matrix independent or dependent? What if the rank was 3?
2. Now suppose you have row reduced a 7 by 7 matrix. If the columns are independent, what possible options do you have for the rank.
3. Now suppose you have row reduced a 7 by 5 matrix. If the columns are independent, what must the rank be.
4. Now suppose you have row reduced a 5 by 7 matrix. Explain why the columns cannot be independent.
5. If you have n vectors placed in the columns of a matrix, what must the rank of the matrix be in order to guarantee that the vectors are independent?

Problem 2.20 Is the vector $[2, 0, 1, -5]$ in the span of

$$\{[1, 0, -1, -2], [1, 2, 3, 0], [0, 1, -1, 2]\}?$$

If it is, then write it as a linear combination of the others. If it is not, then explain why it is not.

Problem 2.21 Find the reduced row echelon form of the matrix

$$B = \begin{bmatrix} 2 & -1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 3 & 3 \end{bmatrix}.$$

Use your result to answer the following questions.

1. Write both $(1, 0)$ and $(0, 1)$ as linear combinations of $(2, 1)$ and $(-1, 1)$.
2. Write $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then write $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
3. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. Find vectors $\vec{x}$ and $\vec{y}$ so that $A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
4. Find a matrix B so that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem: 21, revised Answer each of the following questions.

1. Find the reduced row echelon form of the matrix
$$B = \begin{bmatrix} 2 & -1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 3 & 3 \end{bmatrix}.$$
2. Write $(1, 0)$ as a linear combination of $(2, 1)$ and $(-1, 1)$. Remember, that when writing $c_1(2, 1) + c_2(-1, 1) = (1, 0)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
3. Write $(0, 1)$ as a linear combination of $(2, 1)$ and $(-1, 1)$. Remember, that when writing $c_1(2, 1) + c_2(-1, 1) = (0, 1)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
4. Continue to write each of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. [Hint:At some point, rather than row reducing $\begin{bmatrix} 2 & -1 & \vec{v} \\ 1 & 1 & \end{bmatrix}$, ask yourself how you could use part 1 to answer this.]

5. The following matrix row reduces to give

$$\begin{bmatrix}1&0&2&4&5&8\\0&2&5&2&-1&3\\0&-2&-1&0&2&1\end{bmatrix}\xrightarrow{\text{rref}}\begin{bmatrix}1&0&0&3&\frac{9}{2}&6\\0&1&0&-\frac{1}{4}&-\frac{9}{8}&-1\\0&0&1&\frac{1}{2}&\frac{1}{4}&1\end{bmatrix}.$$

Use this to write both $(4,2,0)$ and $(5,-1,2)$ as a linear combination of the first three columns.

Definition 2.14. The identity matrix I is a square matrix so that if A is a square matrix, then $IA = AI = A$. The identity matrix acts like the number 1 when performing matrix multiplication.

If A is a square matrix, then the inverse of A is a matrix A^{-1} where we have $AA^{-1} = A^{-1}A = I$, provided such a matrix exists.

Problem Let $A = \begin{bmatrix}1&3\\3&4\end{bmatrix}$. We now develop an algorithm for computing the inverse A^{-1}. If an inverse matrix exists, then we know it's the same size as A, so we could let $A^{-1} = \begin{bmatrix}\vec v_1 & \vec v_2\end{bmatrix}$ be the inverse matrix, where $\vec v_1$ and $\vec v_2$ are the columns of A^{-1}.

1. We know that $AA^{-1} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$. Explain why $A\vec v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $A\vec v_2 = \begin{pmatrix}0\\1\end{pmatrix}$.
2. Solve the matrix equations $A\vec v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $A\vec v_2 = \begin{pmatrix}0\\1\end{pmatrix}$. (This involves row reducing $\begin{bmatrix}1&3&1\\3&4&0\end{bmatrix}$ and $\begin{bmatrix}1&3&0\\3&4&1\end{bmatrix}$).
3. What is the reduced row echelon form of $\begin{bmatrix}1&3&1&0\\3&4&0&1\end{bmatrix}$. How is this related to your previous work.
4. State the inverse of A.

The previous problem showed you how to obtain a matrix B so that $AB = I$. You just had to row reduce that matrix $\begin{bmatrix}A & I\end{bmatrix}$ to the matrix $\begin{bmatrix}I & A^{-1}\end{bmatrix}$. The inverse shows up instantly after row reduction.

Problem 2.22 Use the algorithm describe immediately before this problem to compute the inverse of

$$A = \begin{bmatrix}3&0&3\\0&-1&1\\0&3&-4\end{bmatrix}.$$

Then use your work to write each of the standard basis vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ as a linear combination of the columns of A.

Problem 2.23 Let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$. Use Gaussian elimination to show that the inverse of A is

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}.$$

In computing the inverse of a 2 by 2 matrix, the number $ad - bc$ appears in the denominator. We call this number the determinant. If I asked you to compute the inverse of a 3 by 3 matrix, you would again see a number appear in the denominator. We call that number the determinant. This holds true in all dimensions.

Problem: Optional Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Use Gaussian elimination to find the inverse of A, and show that the common denominator is $a(ei - hf) - b(di - gf) + c(dh - ge)$.

Definition 2.15: Determinants of 2 by 2 and 3 by 3 matrices. The determinant of a 2×2 and 3×3 matrix are the numbers

$$\begin{aligned}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \\
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= a(ei - hf) - b(di - gf) + c(dh - ge)
\end{aligned}$$

We use vertical bars next to a matrix to state we want the determinant. Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

In the examples above, we obtained the determinant of a 3 by 3 matrix by computing the determinant of several 2 by 2 matrices. We obtained each 2 by 2 matrix by removing a row and column from the original 3 by 3 matrix. We now add some language to extend the definition above to all dimensions.

Definition 2.16: Minors, Cofactors, and General determinants. Let A be an n by n matrix.

- The minor M_{ij} of a matrix A is the determinant of the the matrix formed by removing row i and column j from A.
- The cofactor C_{ij} is the product of the minor M_{ij} and $(-1)^{i+j}$, so we have $C_{ij} = (-1)^{i+j} M_{ij}$. So it's either the minor, or the opposite of the minor.
- To compute the determinant, first pick a row or column. We define the determinant to be $\sum_{k=1}^{n} a_{ik} C_{ik}$ (if we chose row i) or alternatively $\sum_{k=1}^{n} a_{kj} C_{kj}$ (if we chose column j).
- You can pick ANY row or ANY column you want, and then compute the determinant by multiplying each entry of that row or column by its cofactor, and then summing the results. (The fact that this works would require proof. That proof will be left to a course in linear algebra.)
- A sign matrix keeps track of the $(-1)^{j+k}$ term in the cofactor. All you have to do is determine if the first entry of your expansion has a plus or minus, and then alternate the sign as you expand.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

sign matrix

Problem 2.24 Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 4 & 2 & 5 \end{bmatrix}$ in 3 different ways. First, use a cofactor expansion using the first row (Definition 2.15). Then use a cofactor expansion using the 2nd row. Then finally use a cofactor expansion using column 3. Which of the was the quickest, and why?

Problem 2.25 Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 5 & -1 \\ 0 & 8 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & -5 & 3 & -1 \end{bmatrix}.$$

You can make these problems really fast if you use a cofactor expansion along a row or column that contains a lot of zeros.

Problem 2.26 Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix}.$$

Then find the inverse of A (or explain why it does not exist). Are the columns of A linearly independent or linearly dependent?

Problem 2.27 Compute the determinant of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Does A have an inverse? Are the columns of A linearly independent or linearly dependent? Answer both of the previous questions without doing any row reduction. Then row reduce $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$ to confirm your answer.

After completing the previous two problems, you should see that there is a connection between the determinant, inverse, and linear independence. Make a conjecture about what this connection is. We'll learn a little more about determinants and inverses, and then you'll have a chance to state your conjecture, as well as prove it.

Problem 2.28 Start by writing the system of equations

$$\begin{cases} -2x_1 + 5x_3 &= -2 \\ -x_1 + 3x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 3 \end{cases}$$

as a matrix product $A\vec{x} = \vec{b}$. (What are A, $\vec{x}$ and $\vec{b}$?) Then find the inverse of A, and use this inverse to find $\vec{x}$. [Hint: If we just have numbesr, then to solve $ax = b$, we multiply both sides by $\frac{1}{a}$ to obtain $\frac{1}{a}ax = \frac{1}{a}b$ or just $x = \frac{1}{a}b$.]

In the next problem, you'll prove that the determinant of a 2 by 2 matrix gives the area of a parallelogram whose edges are the columns of the matrix.

Problem 2.29 To find the area of the parallelogram with vertexes $O = (0, 0)$, $U = (a, c)$, $V = (b, d)$, and $P = (a + b, c + d)$, we would find the length of OU (the base b), and multiply it by the distance from V to OU. Complete the following:

1. Find the projection of $\vec{OV}$ onto $\vec{OU}$. (You may have to look up a formula from math 215.)
2. The vector $\vec{OV} - \text{proj}_{\vec{OV}}\vec{OU}$ is called the component of $\vec{OV}$ that is orthogonal to $\vec{OU}$. The length of this vector is precisely the distance from V to OU, which we'll call h. Find the length of this vector.
3. We now have the base $b = |OU|$ and height h of a parallelogram. Compute the product, and prove it equals $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = |ad - bc|$.

The result above extends to 3 dimensions. The determinant of a 3 by 3 matrix gives the volume of a parallelepiped whose edges are the columns of the matrix. We then use determinants to define nth dimensional volume.

Problem 2.30 Answer each of the following:

1. Let $\vec{u} = (2, 3)$. If you pick a vector $\vec{v}$ that is a linear combination of $\vec{u}$, what will the determinant of $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$ equal? First explain how you know the answer (before you have even chosen a vector $\vec{v}$). Then give us an example by picking a vector that is a linear combination of $\vec{v}$.
2. Let $\vec{u} = (1, 0, 2)$ and $\vec{v} = (0, -1, 1)$. If $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, what will the determinant equal? Explain. Then show us an example to confirm your conjecture.
3. We already computed the determinant of $A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$. Swap two columns of the matrix, and then compute the determinant. How does the determinant of your matrix with swapped columns relate to the determinant of the original matrix. If you swap two columns of a matrix, what happens to the determinant?

Problem 2.31 Construct a 2 by 2 matrix whose columns are linearly independent. What is the reduced row echelon form of your matrix? Compute the rank and the determinant, and finally find the inverse (if possible).

Now construct a 2 by 2 matrix whose columns are linear dependent. What is the reduced row echelon form of your matrix? Compute the rank and the determinant, and finally find the inverse (if possible).

Make a conjecture about the connection between (1) linear dependence, (2) rref, (3) rank, (4) determinant, and (5) inverses. Then use a computer to give two 3 by 3 examples similar to the examples above. You'll be asked to show us the computations on the computer in class.

Problem 2.32 Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$. Compute the determinant of A. Then create a matrix B so that the ijth entry of B is the cofactor C_{ij} (remove row i and column j, compute the determinant, and then times by an appropriate sign). This will require that you compute nine 2 by 2 determinants. Finally, compute the inverse of A (feel free to use a computer on this part). Make a conjecture about the connection between the determinant of A, this matrix B, and the inverse of A. We'll verify your conjecture is true on a 4 by 4 matrix in class.

2.4 Eigenvalues and Eigenvectors

The final computational skill we need to tackle is to compute eigenvalues and eigenvectors. Let's start by looking at an example to motivate the language we are about to introduce.

Example 2.17. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. When we multiply this matrix by the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we obtain $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}$. Multiplication by the matrix A was miraculously the same as multiplying by the number 3. Symbolically we have $A\vec{x} = 3\vec{x}$. Not every vector $\vec{x}$ satisfies this property, as letting $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the product $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which is not a multiple of $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Our main goal in this section is to answer the following two questions:

1. For which nonzero vectors $\vec{x}$ (eigenvectors) is it possible to write $A\vec{x} = \lambda\vec{x}$?
2. Which scalars λ (eigenvalues) satisfy $A\vec{x} = \lambda\vec{x}$?

Now for some definitions.

Definition 2.18: Eigenvector and Eigenvalue. Let A be a square $n \times n$ matrix.

- An eigenvector of A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. (Matrix multiplication reduces to scalar multiplication.) We avoid letting $\vec{x}$ be the zero vector because $A\vec{0} = \lambda\vec{0}$ no matter what λ is.
- If $\vec{x}$ is an eigenvector satisfying $A\vec{x} = \lambda\vec{x}$, then we call λ and eigenvalue of A.

Problem 2.33 Use the definition above to determine with of the following are eigenvectors of $\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, (1, 4), \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1, \\ -1 \end{pmatrix}, \begin{pmatrix} -2, \\ 2 \end{pmatrix}.$$

If the vector is an eigenvector, state the corresponding eigenvalue.

The next problem gives us an algorithm for computing eigenvalues and eigenvectors.

Problem 2.34: How to compute eigenvalues and eigenvectors Let A be a square matrix.

1. If λ is an eigenvalue, explain why we can find the eigenvectors by solving the equation $(A - \lambda I)\vec{x} = \vec{0}$. This means we can subtract λ from the diagonal entries of A, and then row reduce $\begin{bmatrix} A - \lambda I & \vec{0} \end{bmatrix}$ to obtain the eigenvectors. Note that you should always obtain infinitely many solutions.
2. Explain why we can obtain the eigenvalues of A by solving for when the determinant of $(A - \lambda I)$ is zero, i.e. solving the equation

$$\det(A - \lambda I) = 0.$$

The algorithm above suggests the following definition.

Definition 2.19. If A is a square n by n matrix, then we call $\det(A - \lambda I)$ the characteristic polynomial of A. It is a polynomial in λ of degree n, and hence has n roots (counting multiplicity). These roots are the eigenvalues of A.

We now have an algorithm for finding the eigenvalues and eigenvectors of a matrix. We start by finding the characteristic polynomial of A. The zeros of this polynomial are the eigenvalues. To get the eigenvectors, we just have to row reduce the augmented matrix $\begin{bmatrix} A - \lambda I & \vec{0} \end{bmatrix}$. Finding eigenvalues and eigenvectors requires that we compute determinants, find zeros of polynomials, and then solve homogeneous systems of equations. You know you are doing the problem correctly if you get infinitely many solutions to the system $(A - \lambda I)\vec{x} = 0$ for each lambda (i.e. there is at least one row of zeros along the bottom after row reduction). As another way to check your work, the following two facts can help.

- The sum of the eigenvalues equals the trace of the matrix (the sum of the diagonal elements).
- The product of the eigenvalues equals the determinant.

The trace and determinant are equal to the sum and product of the eigenvalues.

Problem 2.35 Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$ from problem 2.33.

1. Find the characteristic polynomial of A, and then find the zeros to determine the eigenvalues.
2. For each eigenvalue, find all corresponding eigenvectors.
3. Compute the trace and determinant of A.

Problem 2.36 Consider the matrix $A = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

Problem 2.37 Consider the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

Problem 2.38 Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

2.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Linear Algebra Applications

This chapter covers the following ideas.

1. Explain the connection between vector fields and their corresponding eigenvalues and eigenvectors. Use this knowledge to apply the second derivative test.

2. Solve various problems relating to conservation laws, including stoichiometry, Kirchoff's electrical laws, and Markov Processes.

3. Use Cramer's rule to solve systems, and explain when you would choose Cramer's rule over row reduction.

4. Find interpolating polynomials, and use the transpose to solve the least squares regression problem.

5. Find the partial fraction decomposition of a rational function. Utilize this decomposition to integrate rational functions.

6. Be able to show that a function is linear, and find the kernel of a linear function.

3.1 Vector Fields

In multivariate calculus, we studied vector fields of the form $\vec{F}(x,y) = (M,N)$, where M and N are functions of x and y. The derivative of the vector field is the square matrix

$$D\vec{F}(x,y) = \begin{bmatrix} \partial M/\partial x & \partial M/\partial y \\ \partial N/\partial x & \partial N/\partial y \end{bmatrix}.$$

The eigenvalues and eigenvectors of this matrix provide us with a wealth of information about the vector field. The next few problems have you discover many of these key ideas. We'll return to these ideas throughout the semester, especially when we start studying systems of differential equations in depth.

Problem 3.1 Consider the vector field $\vec{F}(x,y) = (2x+y, x+2y)$.

1. At each of the 8 points given by $(\pm 1, \pm 1)$, $(0, \pm 1)$, $(\pm 1, 0)$, sketch the vector $\vec{F}(x,y)$ with it's base at the input point (so at point $(1,0)$, sketch $(2,1)$, a vector starting at $(1,0)$ and ending at $(3,1)$). This provides us with a rough sketch of the vector field.

2. Compute $A = D\vec{F}(x, y)$. It should be a 2 by 2 matrix.
3. Remember that we say a vector $\vec{x}$ is an eigenvector if $A\vec{x} = \lambda\vec{x}$. For any of the vectors from part 1., did you find that $A\vec{x} = \lambda\vec{x}$? Which ones (these are eigenvectors)? By how much was the vector $\vec{x}$ stretched (these are eigenvalues)?
4. Now compute the eigenvalues and eigenvectors of this matrix, using the algorithm from the previous chapter. You should obtain the same answer as part 3.

The problem above had two positive eigenvalues. In the next problem, your goal is to determine what a vector field looks like when you have both a positive and negative eigenvalue.

Problem 3.2 Complete the following:

1. For the vector field $\vec{F} = (x, 2x - y)$, compute the eigenvalues and eigenvectors of $D\vec{F}(x, y)$.
2. For the vector field $\vec{F} = (x - 4y, -6x - y)$, compute the eigenvalues and eigenvectors of $D\vec{F}(x, y)$.
3. With each vector field, use a computer to construct a vector field plot. In the plot, please show us how to see the eigenvectors, together with which eigenvector corresponds to a positive eigenvalues, and which corresponds to a negative eigenvalue. You can construct vector fields in Wolfram—Alpha by typing "vector field plot" in the input box, or just follow the link `http://www.wolframalpha.com/input/?i=vector+field+plot&lk=4&num=2`.
4. Add to your plots several trajectories, i.e. a path that a particle would follow if $\vec{F}$ represents the tangent vectors of the path. Think, "If I dropped a really light particle in this field, representing water current, where would the particle go?

Problem 3.3 The following three vector fields have imaginary eigenvalues. Compute the eigenvalues for each, construct a vector field plot, and on the plot add several trajectories (the path followed by a particle that is dropped into this field).

1. $\vec{F} = (-2y, x)$.
2. $\vec{F} = (-x + y, -x - y)$.
3. $\vec{F} = (x - y, x)$

Make a conjecture as to why one spirals in, one spirals out, and one just wraps around in ellipses. We'll address this conjecture in class.

The next problem requires that you are on a computer that can use Mathematica. These computers are available in the Ricks, Austin, Romney, and library. Alternately, you can download VMWare that will allow you to use Mathematica for free from your computer, provided you head to `https://vdiview.byui.edu/`. You can download step-by-step instructions from `http://www.byui.edu/help-desk/categories/vdivmware`. Please take a moment and make sure you can access Mathematica.

Problem 3.4 Start by downloading the Mathematica notebook VectorFields.nb (click on the link). The goal of this problem is to make a connection between a vector field and it's corresponding eigenvalues/eigenvectors. Once the notebook is open, click somewhere in the text, hold down Shift, and then press Enter. This will evaluate the commands and produce a vector field plot, with the eigenvector directions drawn in green. You can click on the bubbles with crosshairs in them to adjust the vectors (which are the columns of the matrix). Play around with the animation until you feel like you can answer each of the following questions.

1. If the vector field pushes things outwards in all directions, what do you know about the eigenvalues?
2. If the vector field pulls things inwards in all directions, what do you know about the eigenvalues?
3. How can you tell, by looking at a vector field plot, that one eigenvalue is positive and the other is negative?
4. If the vector field involves swirling motion, what do you know about the eigenvalues? What makes the difference between spiraling inwards, outwards, or just spinning in circles?
5. What happens when you have a repeated eigenvalue? This one has lots of correct answers, and it a topic for much further discussion in chapter 10. See if you can get an example of a repeated eigenvalue with a behavior that's different from the above. If you have the first 4, you can present in class. We'll have you come up to the computer and show us what you did.

3.1.1 Second Derivative Test

Vector fields and eigenvalues provide us with precisely the key information needed to locate maximums, minimums, and saddles for functions of the form $z = f(x, y)$.

Problem 3.5 Consider the function $f(x, y) = x^2 + 4xy + y^2$. The derivative (gradient) is the vector field $Df(x, y) = (2x + 4y, 4x + 2y)$. See Figure 3.1 for a graph of several level curves, together with the gradient.

1. At what point(s) does $Df(x, y) = \vec{0}$? These are the potential locations of maximums, minimums, or saddles.
2. Compute the second derivative of f, which should give you a 2 by 2 symmetric matrix. This matrix is called the Hessian.
3. By looking at the picture, are the eigenvalues of $D^2f(x, y)$ both positive, both negative, or do they differ in sign? How can you tell? Then confirm you are correct by computing the eigenvalues and eigenvectors of $D^2f(x, y)$.
4. Recall that the gradient points in the direction of greatest increase. Using this information alone, does the function have a maximum, minimum, or saddle point at $(x, y) = (0, 0)$

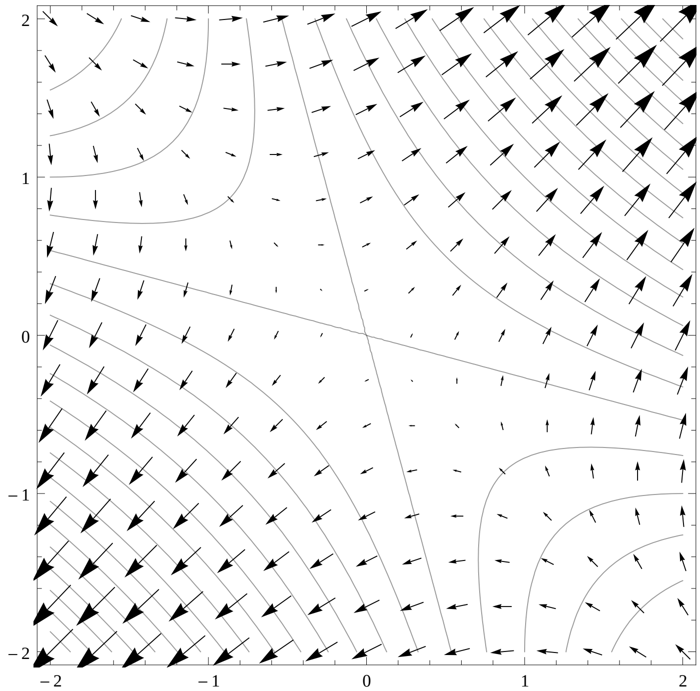


Figure 3.1: A plot of several level curves of $f(x, y) = x^2 + 4xy + y^2$ and the gradient. In one direction the gradient is pulling things towards the origin. In another direction, the gradient is pushing things away from the origin.

Theorem 3.1. *Let $f(x, y)$ be a function that is twice continuously differentiable. Suppose that $Df(x, y) = (0, 0)$ when $(x, y) = (a, b)$, so that (a, b) is a critical point. To determine if the point (a, b) corresponds to a maximum, minimum, or saddle point, we compute the eigenvalues of $D^2f(a, b)$ (the second derivative is called the Hessian).*

- *If the eigenvalues of a are all positive, then the function has a minimum at (a, b).*
- *If the eigenvalues of a are all negative, then the function has a maximum at (a, b).*
- *If there is a positive eigenvalue, and a negative eigenvalue, then the function has a saddle at (a, b).*
- *If zero is an eigenvalue, then the second derivative test fails.*

Problem 3.6 Consider the function $f(x, y) = x^3 - 3x^2 - y^2 + 2y$ See Figure 3.2 for a graph of several level curves, together with the gradient.

1. At what point(s) does $Df(x, y) = \vec{0}$? You should obtain two points. These are the potential locations of maximums, minimums, or saddles.
2. Compute the second derivative of f, which should give you a 2 by 2 symmetric matrix.
3. Pick one of the critical points. Use the vector field plot to decide if the eigenvalues of $D^2f(x, y)$ both positive, both negative, or differ in sign at that critical point, and if the function has a maximum, minimum, or saddle at that point. Then repeat with the other critical point.

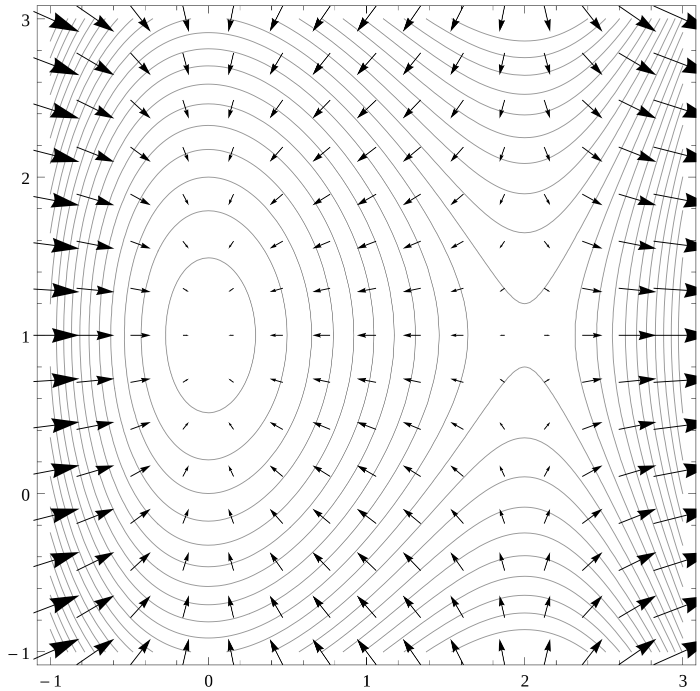


Figure 3.2: A plot of several level curves of $f(x,y) = x^3 - 3x^2 - y^2 + 2y$ and the gradient. There are two critical points. The vector field plot provides enough information to determine if the sign of the eigenvectors of the second derivative at each critical point.

4. Now compute the eigenvalues of the Hessian at each critical value. This should confirm your answer to part 3. (The matrix is diagonal, so computing eigenvalues should be quick.)

The following example adds a little more information to this discussion. I've included it to give you one additional piece of information, namely how the eigenvalues connect to the concavity of the function.

Example 3.2. For the function $f(x,y) = x^2 + xy + y^2$, the gradient is $Df = \begin{bmatrix} 2x+y & x+2y \end{bmatrix}$, which is zero only at $x = 0, y = 0$ (solve the system of equations $2x + y = 0, x + 2y = 0$). The Hessian is $D^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The eigenvalues are found by solving $0 = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = (\lambda - 3)(\lambda - 1)$, so $\lambda = 3, 1$ are the eigenvalues. Since both eigenvalues are positive, the gradient pushes things away from the origin in all direction, which means in every direction you move from the critical point, you'll increase in height. There is a minimum at $(0,0)$.

The eigenvectors of the Hessian help us understand more about the graph of the function. An eigenvector corresponding to 3 is $(1,1)$, and corresponding to 1 is $(-1,1)$. These vectors are drawn in figure 3.3, together with two parabolas whose 2nd derivatives are precisely 3 and 1. The parabola which opens upwards the most quickly has a 2nd derivative of 3. The other parabola has a second derivative of 1. In every other direction, the 2nd derivative would be between 1 and 3.

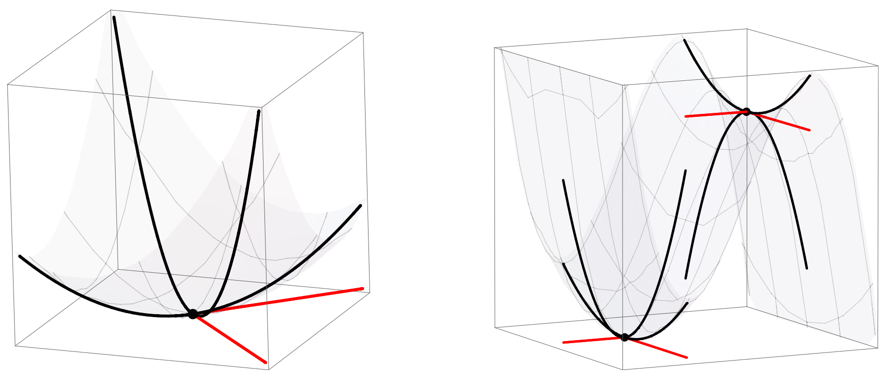


Figure 3.3: The eigenvectors of the second derivative tell you the directions in which the 2nd derivative is largest and smallest. At each critical point, two eigenvectors are drawn as well as a parabola whose second derivative (the eigenvalue) matches the second derivative of the surface in the corresponding eigenvector direction.

3.2 Conservation Laws

Many problems in nature arise from conservation laws. These laws generally focus on the principle that matter is neither is created nor destroyed, rather it is just moved, changed, or something. Any of the following could be viewed as a conservation law:

- What comes in must come out.
- Voltage supplied equals voltage suppressed.
- Atoms before equal atoms after.
- The change in a quantity is how much it increases minus how much it decreases.
- Current in equals current out.

The following problems related to some conservation law. You'll see similar laws in your future classes, regardless of your discipline.

3.2.1 Stoichiometry

Chemical reaction stoichiometry is the study balancing chemical equations. A chemical reaction will often transform reactants into by-products. The by products are generally different compounds, together with either an increase or decrease in heat. One key rule in stoichiometry is that a chemical process neither creates nor destroys matter, rather it only changes the way the matter is organized. For simple reactions (with no radioactive decay), this conservation law forces the number of atoms entering a reaction to be the same as the number leaving. The next problem asks you to use this conservation law to create a balanced chemical reaction equation.

Problem 3.7 The chemical compound hydrocarbon dodecane ($C_{12}H_{26}$) is used as a jet fuel surrogate (see Wikipedia for more info). This compound reacts with oxygen (0_2), and the chemical reaction produces carbon dioxide (CO_2), water (H_20), and heat. Suppose we expose some dodecane to oxygen, and that a chemical reaction occurs in which the dodecane is completely converted to carbon dioxide and water. Conservation requires that the number of atoms (H, C, and 0) at the beginning of the chemical reaction must be the exact same as the number at the end. We could write the chemical reaction in terms of molecules as

$$x_1C_{12}H_{26}+x_2O_2 = x_3CO_2+x_4H_2O \quad \text{or} \quad x_1C_{12}H_{26}-x_2O_2 = x_3CO_2-x_4H_2O = 0,$$

where x_1 molecules of dodecane and x_2 molecules of oxygen were converted to x_3 units of carbon dioxide and x_4 units of oxygen. If we look at each atom (carbon, hydrogen, and oxygen) individually, we obtain three equations to relate the variables x_1, x_2, x_3, x_4. The carbon equation is simply

$$x_1(12) + x_2(0) = x_3(1) + x_4(0) \quad \text{or} \quad x_1(12) + x_2(0) - x_3(1) - x_4(0) = 0.$$

Your job follows:

1. Write the other two conservation equations (for hydrogen and oxygen).
2. Solve the corresponding system of equations by row reduction. As there are only 3 equations with 4 unknowns, you should obtain infinitely many solutions. Write each variable in terms of the free variable.
3. If about 10,000 molecules of water are present at the end of the reaction, about how many molecules of dodecane were burned?

3.2.2 Kirchoff's Electrical Laws

Gustav Kirchoff discovered two laws of electricity that pertain to the conservation of charge and energy. To describe these laws, we must first discuss voltage, resistance, and current.

- Current is the flow of electricity, and often it can be compared to the flow of water.
- As a current passes across a conductor, it encounters resistance. Ohm's law states that the product of the resistance R and current I across a conductor equals the voltage V, i.e. $RI = V$. If the voltage remains constant, then a large resistance corresponds to a small current.
- A resistor is an object with high resistance which is placed in an electrical system to slow down the flow (current) of electricity. Resistors are measured in terms of ohms, and the larger the ohms, the smaller the current.

Figure 3.4 illustrates two introductory electrical systems. In this diagram, wires meet at nodes (illustrated with a dot). Batteries and voltage sources (represented by or other symbols) supply a voltage of E volts. At each node the current may change, so the arrows and letters i represent the different currents in the electrical system. The electrical current on each wire may or may not follow the arrows drawn (a negative current means that the current flows opposite the arrow). Resistors are depicted with the symbol , and the letter R represents the ohms.

Kirchoff discovered two laws. They both help us find current in a system, provided we know the voltage of any batteries, and the resistance of any resistors.

1. Kirchoff's current law states that at every node, the current flowing in equals the current flowing out (at nodes, current in = current out).

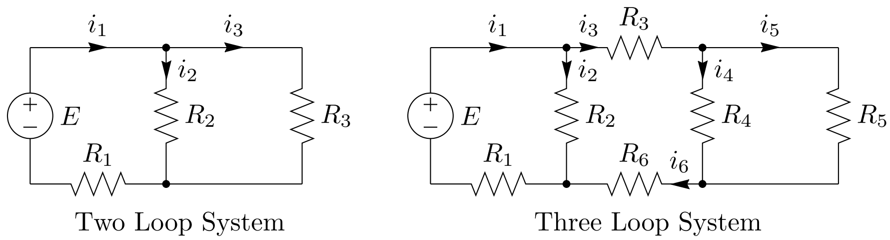


Figure 3.4: Electrical Circuit Diagrams.

2. Kirchoff's voltage law states that on any loop in the system, the directed sum of voltages supplied equals the directed sum of voltage drops (in loops, voltage in = voltage out). To use this law, pick a spot in the system. Then move around the system following a path that eventually gets you back to where you began (a closed curve). If you encounter a battery (a voltage source), then it counts as voltage in. If you encounter a resistor as you move with the current, then the voltage drop is Ri. If you encounter a resistor while moving opposite the current, then times by a negative to get a voltage drop of $-R_i$.

Let's use Kirchoff's laws to generate a system of equations for the two loop system. Remember that every time a current encounters a resistor, the voltage drop is $V = RI$, the product of the resistance and the current.

Problem 3.8 Consider the two loop system in figure 3.4. Assume that the voltage supplied from the battery E, as well as the ohms R_1, R_2, and R_3, on the resistors are known. The currents i_1, i_2, and i_3 are unknown.

1. Use Kirchoff's laws to explain how to obtain each of the equations below:

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ -i_1 + i_2 + i_3 &= 0 \\ R_1 i_1 + R_2 i_2 &= E \\ -R_2 i_2 + R_3 i_3 &= 0. \\ R_1 i_1 + R_3 i_3 &= E. \end{aligned}$$

   [Hint: If you encounter a resistor while moving backwards along a loop, then times the voltage drop becomes a voltage gain (times by a negative).]

2. Some of the equations above are linear combinations of the other equations. How could you obtain the 2nd and 5th as a linear combination of the others?

3. If $E = 12$, $R_1 = 2$, $R2 = 3$, and $R3 = 6$, then solve the system of equations above by row reducing an appropriate matrix.

Problem 3.9 Consider the three loop system in figure 3.4. Assume that the voltage supplied from the battery E and that the ohms R_j on the resistors are known. The currents are unknown.

1. There are 4 nodes in this system. Write the 4 equations we obtain by remember that the flow in at a node must equal the flow out.

2. There are three inner loops in the system above. Write the equations formed by going around each inner loop. [To get an inner loop, pick any point in the system. Then move in a clockwise fashion around the loop

3. Some of the equations above are linear combinations of the other equations. How could you obtain the 2nd and 5th as a linear combination of the others?

4. Why will row reducing the following matrix give you the unknown currents?

$$\left[\begin{array}{cccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ R_1 & R_2 & 0 & 0 & 0 & 0 & E \\ 0 & -R_2 & R_3 & R_4 & 0 & R_6 & 0 \\ 0 & 0 & 0 & -R_4 & R_5 & 0 & 0 \end{array}\right].$$

[Don't row reduce this matrix.]

Problem 3.10 Consider the three loop system in figure 3.4. If $E = 12$, $R1 = 1$, $R2 = 1$, $R3 = 1$, $R4 = 1$, $R5 = 1$, $R6 = 1$ then find the unknown currents by row reducing the matrix in part 4 above. Use a computer to check your answer. The row reduction is quite short, because the matrix is sparse (has lots of zeros).]

3.2.3 Markov Processes

Matrices can be used to model a process called a Markov Process. To fit this kind of model, a process must have specific states, and the matrix which models the process is a transition matrix which specifies how each state will change through a given transition. An example of a set of states is “open” or “closed” in an electrical circuit, or “working properly” and “working improperly” for operation of machinery at a manufacturing facility. A car rental company which rents vehicles in different locations can use a Markov Process to keep track of where their inventory of cars will be in the future. Stock market analysts use Markov processes and a generalization called stochastic processes to make predictions about future stock values.

Problem 3.11 Suppose we own a car rental company which rents cars in Idaho Falls and Rexburg. The last few weeks have shown a weekly trend that 60% of the cars which are rented in Rexburg will remain in Rexburg (the other 40% end up in Idaho Falls). About 80% of the cars which are rented in Idaho Falls will remain in Idaho Falls (the other 20% end up in Rexburg).

1. If there are currently 60 cars in Rexburg and 140 cars in IF, how many will be in each city next week? In two weeks?

2. Let R_n and I_n be the number of cars in Rexburg and Idaho Falls, respectively, at the beginning of the nth week (so $R_0 = 60$ and $I_0 = 140$). Obtain a matrix A so that $A\begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} R_1 \\ I_1 \end{pmatrix}$. Then check that $A\begin{pmatrix} R_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} R_2 \\ I_2 \end{pmatrix}$.

3. We would like to know if the number of cars will stabilize in each city. This would mean that if the current week's car totals are R and I, then we could find the next week's totals by solving the system

$$A\begin{pmatrix} R \\ I \end{pmatrix} = \begin{pmatrix} R \\ I \end{pmatrix},$$

the totals don't change. This is called a steady state solution. Find the steady state solution.

4. In the long run, what proportion of the cars will end up in Rexburg?
5. Because the system $A\begin{pmatrix}R\\I\end{pmatrix} = \begin{pmatrix}R\\I\end{pmatrix}$ had a nonzero solution, we know something about the eigenvalues of the matrix A. What is an eigenvalue of A?

(We'll answer 4 and 5 in class if you are unable. The key parts are 1-3.)

The matrix A found above is called a transition matrix. It's the matrix which tells you how to move from the current state $\vec{x}_n$ to the next state $\vec{x}_{n+1}$. This means we have

$$\begin{aligned}
\vec{x}_1 &= A\vec{x}_0\\
\vec{x}_2 &= A\vec{x}_1 = A(A\vec{x}_0) = A^2\vec{x}_0\\
\vec{x}_3 &= A\vec{x}_2 = A(A\vec{x}_1) = \cdots = A^3\vec{x}_0\\
\vec{x}_4 &= A\vec{x}_3 = A(A\vec{x}_2) = \cdots = A^4\vec{x}_0\\
&\vdots
\end{aligned}$$

You can find the nth state by computing $\vec{x}_n = A^n\vec{x}_0$, just raise the matrix to a power, and times by the initial state. Let's use this idea once more.

Problem 3.12 In a certain town, there are 3 types of land zones: residential, commercial, and industrial. The city has been undergoing growth recently, and the city has noticed the following 5 year trends.

- Every 5 years, they've notice that 10% of the residential land gets rezoned as commercial land, while 5% of the residential land gets rezoned as industrial. The other 85% of residential land remains residential.
- For commercial land, 70% remains commercial, while 10% becomes residential and 20% becomes industrial.
- For industrial land, 60% remains industrial, while 25% becomes commercial and 15% becomes residential.
- Currently the percent of land in each zone is 40% residential, 30% commercial, and 30% industrial.

Let's assume that these trends continue over an extended period of time.

1. The current state is $\vec{x}_0 = (40, 30, 30)$. After 5 years, what percentage of land will be zoned residential? Commercial? Industrial? Answering this question should give you the transition matrix A so that $\vec{x}_1 = A\vec{x}_0$.
2. Use software to find $\vec{x}_2$, $\vec{x}_3$, and $\vec{x}_4$ (the land use percentages after 10, 15, and 20 years).
3. Find the steady state solution to this Markov Process by solving $A\vec{x} = 1\vec{x}$ (i.e., the eigenvector corresponding to the eigenvalue $\lambda = 1$.)

Problem 3.13 Consider three occupations, farming, manufacturing, and clothing. Assume that goods are exchanged between the communities through barter only. Here is how the communities exchange their goods.

- The farming community keeps 1/2 of their goods, giving 1/4 to manufacturing and 1/4 to clothing.
- The manufacturing community keeps 1/3 of their goods, giving 1/3 to farming and 1/3 to clothing.
- The clothing community keeps 1/4 of their goods, giving 1/2 to farming and 1/4 to manufacturing.

Answer the following questions.

1. Suppose that all the commodities have the exact same value. If each group starts out with 12 units of their commodity, then after 1 round of bartering, how many units will each group have? Along the way you should produce a transition matrix A so that $A\begin{pmatrix}12\\12\\12\end{pmatrix}$ gives the answer.
2. Let x_1 be the value of the goods produced by farming. Let x_2 be the value of the goods produced by manufacturing. Let x_3 be the value of the goods produced by clothing. We would like to assign a value to each commodity so that each group gets a fair deal when they barter. To do this, we need to have the value of goods obtained after bartering to match the value of the goods obtained before. Explain why we can obtain this by solving the equations
$$\begin{bmatrix}\frac{1}{2}&\frac{1}{3}&\frac{1}{2}\\\frac{1}{4}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{3}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$
Then solve this equation.
3. (We'll answer this one in class. Try to come up with an answer yourself.) If the value of commodities from each group is the same, who is getting the better deal? To make sure bartering results in a fair deal for all, should farming commodities be more expensive, or less expensive than the others?

3.3 Cramer's Rule

Gabriel Cramer developed a way to solve linear systems of equations by using determinants. For small systems, the solution is extremely fast. However, for large systems, the method looses it's power because of the complexity of computing determinants. Also, when the coefficients in the system are variables, Cramer's rule provides an extremely fast algorithm for computing determinants. I'll remind you occasionally throughout the problem set to apply Cramer's rule when the problem involves variable coefficients.

Theorem 3.3 (Cramer's Rule)**.** *Consider the linear system given by $A\vec{x}=\vec{b}$, where $A=\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots\vec{v}_n\end{bmatrix}$ is an n by n matrix whose determinant is not zero. Let $D=|A|$. For each i, replace vector $\vec{v}_i$ with $\vec{b}$, and then let D_i be the determinant of the corresponding matrix. The solution to the linear system is then*
$$x_1=\frac{D_1}{D},\quad x_2=\frac{D_2}{D},\quad \cdots \quad x_n=\frac{D_n}{D}.$$

For the 2 by 2 system
$$\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix},$$

Cramer's rule states the solution is (provided $|A| \neq 0$)

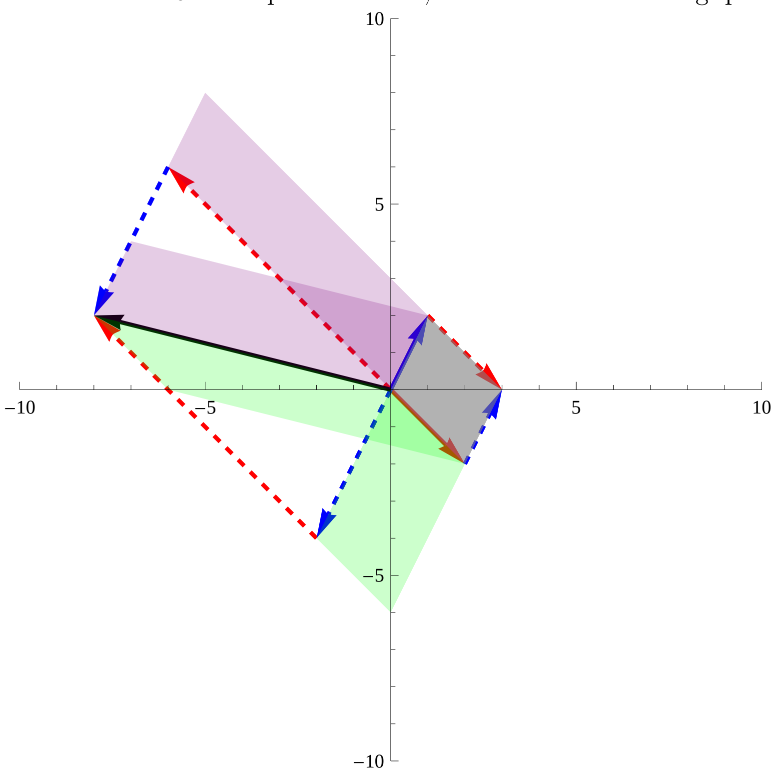
$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Problem 3.14 Consider the system of equations $x + 2y = 3, 4x + 5y = 6$. Solve this system in 2 different ways.

1. Use Cramer's rule to solve the system. You just need to compute three 2 by 2 determinants.
2. Use row reduction to solve the system. Show the steps in class.

In the next problem, you'll provide a proof of Cramer's rule in 2D. Your proof will contain the key idea needed to prove the theorem in all dimensions. The key idea is to connect determinants to areas of parallelograms.

Problem 3.15: Proof of Cramer's Rule Let $\vec{v}_1 = (2, -2)$ and $\vec{v}_2 = (1, 2)$. Let $x_1 = -3$ and $x_2 = -2$, which means that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 = (-8, 2)$. In the picture below, the solid red vector is $\vec{v}_1$, the solid blue vector is $\vec{v}_2$, and the solid black vector is $\vec{b}$. Use the picture below, to answer the following questions.



[Hint: Each question can be answered by thinking about determinants as areas.]

1. Explain why $x_1 \begin{vmatrix} \vec{v_1} & \vec{v}_2 \end{vmatrix} = \begin{vmatrix} x_1\vec{v_1} & \vec{v}_2 \end{vmatrix}$. Then explain why $\begin{vmatrix} x_1\vec{v_1} & \vec{v}_2 \end{vmatrix} = \begin{vmatrix} \vec{b_1} & \vec{v}_2 \end{vmatrix}$. Finally, solve for x_1 to show

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

2. In a similar fashion, obtain a formula for x_2.

Problem 3.16 In problem 3.8 we obtained needed to solve the system of equations

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ R_1 i_1 + R_2 i_2 &= E \\ -R_2 i_2 + R_3 i_3 &= 0. \end{aligned}$$

Write the corresponding system of equations, and then use Cramer's rule to obtain the general solution for the unknown currents. You should have i_1, i_2, and i_3 all written in terms of R_1, R_2, R_3, and E.

Cramer's rule is most useful when the coefficients in the linear system are variables, rather than numbers. Let's apply our knowledge to study the arms race (the building of armies - tanks, bombs, soldiers, etc. - between two countries). Consider two countries, country A and country B. As country B builds up their military, country A looks on and says "Hmm, we better build up our military." Similarly, as country A builds up their military, country B looks and says, "Hmm, we better build up our military." If country A has a grudge against country B, they will probably build up their military regardless of what country B does. Similarly, any past grievances and grudges that country B has against country A will increase the rate at which country B builds up their military. Building up a military costs money, so hopefully both countries have economic limitations that restrict the growth of their military. The real question behind the arms race is, "Will the two countries eventually decide they are spending enough on their military, or will their spending continue to grow without bound."

We now develop a system of differential equations that describes the above. The key principle is a general law of conservation:

> The change in a quantity equals the flow in of the quantity minus the flow out of the quantity, or more simply
>
> Change = (Flow in) - (Flow out)
>
> Change = (Increase) - (Decrease)

- Let x represent the dollar amount per year that country A spends on arms. Let y represent the dollar amount per year that country B spends on arms.

- When y is large, country A will respond by increasing their spending. We'll assume this change is proportional to y, so we see that x increases by an amount ay. Similarly, when x is large, country B responds by increasing their spending. Let's assume that y increases by an amount mx.

- The economy of each country tries to slow down the growth rate. The more money country A spends, the larger the effect of the economy. We'll assume that x decreases by an amount bx. Similarly, we'll assume y decrease by an amount ny.

- If the countries hold grudges against each other for past grievances, then they are inclined to increase their spending regardless of economic factors and the growth of the other country's army. Let c represent the amount that country A will increase their spending by, and let p represent the amount that country B will increase their spending by. These values might be zero (for example the US and Canada do not hold such grudges), but might not be zero at all (as was the cases during the cold war, between the US and USSR).

Problem 3.17 Read the arms race information above, and then answer the following questions.

1. There are three things causing x to change. The flow in (parts causing an increase) are ay and c, the response to the other country, and any grudges. The flow out (parts causing a decrease) is only bx, the economic restriction. We can write this as a differential equation
$$\frac{dx}{dt} = ay - bx + c.$$
Obtain a similar equation for $\frac{dy}{dt}$ (using the coefficients m, n, and p). Then write your system of ODEs in the form
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -b & a \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ ? \end{bmatrix}.$$

2. An equilibrium solution to the system of differential equations above is a solution that remains stable. At equilibrium, there should not be any future change in x nor y, so we should have $dx/dt = 0$ and $dy/dt = 0$. Find the equilibrium solution for the arms race problem. [Cramer's rule should make this really fast.]

3. Find the eigenvalues of the square matrix from part 1. What conditions must be met so that both eigenvalues are negative? In class, we'll pick some positive values for a, b, c, m, n, p that satisfy the conditions you tell us, and then graph the vector field $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{p}$, along with some solution curves.

3.4 Curve Fitting

3.4.1 Interpolating Polynomials

Through any two points (with different x values) there is a unique line of the form $y = mx + b$. If you know two points, then you can use them to find the values m and b. Through any 3 points (with different x values) there is a unique parabola of the form $y = ax^2 + bx + c$, and you can use the 3 points to find the values a, b, c. As you increase the number of points, there is still a unique polynomial (called an interpolating polynomial) with degree one less than the number of points, and you can use the points to find the coefficients of the polynomial. In this section we will find interpolating polynomials, and show how the solution requires solving a linear system.

To organize our work, let's first standardize the notation. Rather than writing $y = mx + b$, let's write $y = a_0 + a_1 x$ (where $a_0 = b$ and $a_1 = m$). For a parabola, let's write $y = a_0 + a_1 x + a_2 x^2 = \sum_{k=0}^{2} a_k x^k$. We can now write any polynomial in the form
$$y = a_0 + a_1 x + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k.$$
By standardizing the coefficients, we can use summation notation to express any degree polynomial by changing the n on the top of the summation sign.

Problem 3.18 Answer the following by row reducing an appropriate matrix. Please show us the steps in your row reduction. [Hint: Each point produces an equation.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1 x$ that passes through the points $(1, 2)$ and $(3, 5)$. [We could have use m and b, but I chose to use a_0 and a_1 so you can see how this generalize quickly to all dimensions.]

2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1 x^1 + a_2 x^2$ that passes through the points $(0, 1)$, $(2, 3)$, and $(1, 4)$. [Hint: The second point produces the equation $3 = a_0 + a_1(2) + a_2(2)^2$.]

Problem 3.19 Give an equation of a cubic polynomial $y = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3$ that passes through the four points $(0, 1)$, $(1, 3)$, $(1, 4)$, and $(2, 4)$. Show us the steps in your row reduction. [Hint: Each point produces an equation. You should have a linear system with 4 equations and 4 unknowns.]

Problem 3.20 Solve the following. [Hint: Because the problem involves variable points, Cramer's rule will be much faster than row reduction.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1 x$ that passes through the points (x_1, y_1) and (x_2, y_2).

2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1 x^1 + a_2 x^2$ that passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Under what conditions will your solutions above not be valid?

If we collect 2 data points, then we can usually find an equation of a line that passes through them. If we collect 3 data points, we can usually find an equation of a parabola passing through them. Continuing in this fashion, if we collect $n + 1$ data points, then we can usually find an equation of a polynomial of degree n that passes through them.

Problem 3.21 Suppose that we collect the 6 data points $(1, 1)$, $(2, 3)$, $(-1, 2)$, $(0, -1)$, $(-2, 0)$, $(3, 1)$. We would like to find a polynomial that passes through all 6 points. State the degree n of this polynomial. Then find the coefficients $a_0, a_1, \ldots, a_n$ of this polynomial. Please use technology to do your row reduction. When you present in class, show us the matrix you entered into a computer, and then show us the reduced row echelon form together with the polynomial.

3.4.2 Least Squares Regression

Interpolating polynomials give a polynomial which passes through every point listed. While they pass through every point in a set of data, the more points the polynomial must pass through, the more the polynomial may have to make large oscillations in order to pass through each point. Sometimes all we want is a simple line or parabola that passes near the points and gives a good approximation of a trend in the data. When I needed to purchase a minivan for my expanding family, I gathered mileage and price data for about 40 cars from the internet. I plotted this data and discovered an almost linear downward trend (as mileage increased, the price dropped). Using this data I was able to create a line to predict the price of a car. I then used this data to talk the dealer into dropping the price of their car by over \$1000. Finding an equation of this line, called the least squares regression line, is the content of this section. In other words, if you have 3 or more points, how do you find a line that is closest to passing through these points? The least squares regression line is used to find trends in many branches of science, in addition to haggling for lower prices when buying a car. Statistics builds upon this idea to provide powerful tools for predicting the future.

Problem 3.22 Consider the three points $(2,4)$, $(0,1)$, and $(3,5)$. We wish to find a line $y = a_0 + a_1 x$ that fits this data.

1. What 3 equations do the points and line give. Write the linear system as a matrix equation by filling in A and $\vec{b}$ below:

$$A\vec{x} = \vec{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ ? \\ ? \end{bmatrix}.$$

   The first equation $4 = a_0 + a_1(2)$ is already on the first row.

2. Row reduce the corresponding augmented matrix to show that this system has no solution. The problem is that we have more equations than we do unknowns. The system is overdetermined.

3. If we multiply both sides of the equation $A\vec{x} = \vec{b}$ by a 2 by 3 matrix C, then the product CA will be a 2 by 2 matrix. We could then solve the system $CA\vec{x} = Cb$, as it would then have 2 equations and 2 unknowns.

   The only 2 by 3 matrix in the problem is the transpose of A. So compute $A^T A$ and $A^T\vec{b}$. Then solve the system $(A^T A)\vec{x} = A^T\vec{b}$.

The previous problem suggests the following theorem. One proof of this theorem involves projecting $\vec{b}$ onto the plane spanned by the columns of A. This proof leads to the ideas behind inner product spaces, the Graham Schmidt orthogonalization process, and more, something you would study near the end of math 341 (Linear Algebra).

Theorem 3.4 (Least Squares Regression)**.** *When we collect n data points and notice the points follow a linear trend, the coefficients of the least square regression line $y = a_0 + a_1 x$ are the solutions to the equation $A^T A\vec{x} = A^T\vec{b}$, where we have*

$$\vec{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \vec{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \text{ and } A^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Problem 3.23 Suppose you collect the n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and you wish to find the least squares regression line $y = a_0 + a_1 x$. Set up the matrices A, $\vec{x}$, $\vec{b}$, and A^T. Multiply together $A^T A$ and $A^T\vec{b}$ (your result should involve sums of the form $\sum x_i$, $\sum y_i$, $\sum x_i y_i$, and $\sum x_i^2$). Then solve the equation $A^T A\vec{x} = A^T\vec{b}$ and state the coefficients a_0 and a_1. leas[Hint: Since the system involves variable coefficients, try using Cramer's rule. It will kick out the solution almost instantly.]

The key to solving the overdetermined system $A\vec{x} = \vec{b}$ is to multiply each side on the left by a matrix C, so that the produce CA is a square matrix. We then solve $CA\vec{x} = C\vec{b}$. The least square regression model comes by letting $C = A^T$. We obtain alternate data fitting models by using a matrix other than A^T (though this is a topic for another course). The next problem has you find the best fitting parabola, using the least square regression model.

Problem 3.24 Consider the 5 points $(-2, 3)$, $(-1, 1)$, $(0, -1)$, $(1, 2)$, $(2, 4)$, and . We would like to find an equation of a parabola $y = a_0 + a_1x + a_2x^2$ that approximates the trend in the data, using the least square regression model.

1. The 5 data points produce 5 equations in the three unknowns a_0, a_1, a_2. Write the linear system as a matrix equation by filling in A and $\vec{b}$ below:

$$A\vec{x} = \vec{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & -2 & 4 \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

2. Multiply both sides of the equation $A\vec{x} = \vec{b}$ by an appropriate 3 by 5 matrix C. Then solve the system $(CA)\vec{x} = C\vec{b}$. Feel free to use software to obtain your answer. In class, just show us CA, $C\vec{b}$, and the rref of $\begin{bmatrix} CA & C\vec{b} \end{bmatrix}$.

3. Plot the 5 data points and the parabola you found.

The next problem has the exact same solution as Problem 3.23, but does not require you to use a matrix transpose, nor matrix multiplication. Instead, it focuses on setting partial derivative equal to zero, which is the first step in locating minimums. You then just have to solve a system of linear equations.

Problem 3.25 Suppose you collect the n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and you wish to find the least squares regression line $y = a_0 + a_1x$. Each point (x_i, y_i) produces an error $y - y_i = (a_0 + a_1x_i) - y_i$. The least squares regression line is the line that minimized the sum of the squares of these errors, which means we need to minimize

$$f(a_0, a_1) = \sum_{i=1}^{n} \left((a_0 + a_1x_i) - y_i\right)^2.$$

1. Compute $\dfrac{\partial f}{\partial a_0}$ and $\dfrac{\partial f}{\partial a_1}$.

2. Since we seek the minimum of f, solve the system $\dfrac{\partial f}{\partial a_0} = 0$ and $\dfrac{\partial f}{\partial a_1} = 0$ for a_0 and a_1.

[Hint: Once you get each equation written in the form $(?)a_0 + (?)a_1 = ?$, use Cramer's rule to kick out the answer almost instantly.]

3.5 Partial Fraction Decompositions

A partial fraction decomposition is a method of breaking a complex rational function up into the sum of smaller simpler functions to work with. We will be using partial fraction decompositions to rapidly solve differential equations throughout the semester (using Laplace transforms). For now, we will start by gaining practice with partial fraction decompositions by integrating rational functions. To illustrate their value, let's start with an example.

Problem 3.26 Our goal is to integrate the function $f(x) = \dfrac{2x+1}{(x-2)(x-5)}$. The denominator is the product of two linear functions. Suppose we can write

$$\frac{2x+1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

for unknown constants A and B. Multiply both sides of the above equation by the denominator $(x-2)(x-5)$. Then solve for the constants A and B (try plugging in some numbers to get a system of equations). Then compute

$$\int \frac{2x+1}{(x-2)(x-5)} dx = \int \frac{A}{x-2} dx + \int \frac{B}{x-5} dx.$$

[Hint: Two lines are the same if and only if they have the same slope an intercept. You should have an equation that says two lines are equal, so equate the coefficients. Alternately, just pick two x values, plug them in, and you should get two different equations relating A and B.]

Problem 3.27 We can write

$$f(x) = \frac{2x-3}{x^2(x-3)} = \frac{Ax+B}{x^2} + \frac{C}{x-3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}.$$

Multiply both sides by the denominator $x^2(x-3)$, and then solve for the constants A, B, and C (show us the matrix you row reduced, and the rref). Then compute the integral of $f(x)$. [Hint: To get three equations, you can (1) equate the coefficients, or (2) pick 3 x values and plug them into the equation.]

Problem 3.28 We can write

$$f(x) = \frac{x^2-2}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}.$$

Multiply both sides by $(x^2+4)(x+1)$ and then solve for the constants A, B, and C (show us the matrix you row reduced, and the rref). Then compute the integral of $f(x)$ (you'll need a u-sub for one of the integrals). You'll probably want to split the numerator up, and then integrate three parts, instead two, as we can write

$$f(x) = \frac{Ax}{x^2+4} + \frac{B}{x^2+4} + \frac{C}{x+1}.$$

Problem 3.29 We can write

$$\begin{aligned} f(x) &= \frac{1}{(x+4)^3(x-3)} \\ &= \frac{A(x+4)^2 + B(x+4) + C}{(x+4)^3} + \frac{D}{x-3} \\ &= \frac{A}{(x+4)} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3} + \frac{D}{x-3}. \end{aligned}$$

Multiply both sides by the denominator of the original, and then solve for the unknown constants (show us the matrix you row reduced, and the rref). Then compute the integral of $f(x)$.

3.6 Linear Functions

We need one more bit of vocabulary before embarking on solving differential equations. You've seen the concept of a linear function in many settings, without knowing it. First let's get the definition, see an example, and then we'll discuss it more.

Definition 3.5. When the domain D and range R of a function involves quantities that can be added and multiplied by scalars, we say that the function $f : D \to R$ is linear, provided the following occurs:

1. $f(x + y) = f(x) + f(y)$ and
2. $f(cx) = cf(x)$.

The function preserves addition and scalar multiplication.

The next examples shows that the concepts of differentiation and integration are linear functions. Often when the domain is a group of functions, we'll say that the function is a linear operator (instead of a linear function).

Problem 3.30 Consider the linear operator $\frac{d}{dx}$. For simplicity, we'll assume the operator has as its domain the set of all differentiable functions that are differentiable on the entire real line. The range (or codomain) is the set of all functions. The operator $\frac{d}{dx}$ requires a differentiable function as an input, and returns a function as output.

1. Show that $\frac{d}{dx}$ is a linear operator.
2. Consider the linear operator $\int f(x)dx$. What is the domain and codomain of the integration operator? Is it linear?
3. Consider the linear operator $L\{f(t)\}$, the Laplace transform. Show it is a linear operator, and state the domain and codomain. [We'll tackle this part together in class. Have a guess for the answers.]

In calculus you learned that you can differentiate a sum by differentiating each piece separately (term-by-term differentiation), and that you can pull constants out. Similarly, you learned that you can do integration term-by-term, and constants come out. These are precisely the key properties behind a function (operator, transformation) being linear.

We'll now see that EVERY matrix represents a linear transformation, and that every linear transformation between finite dimensional vector spaces is really just a matrix product.

Problem 3.31 Suppose that we have a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$. We know that $f(1,0,0) = (1,3)$, $f(0,1,0) = (-2,4)$, and that $f(1,1,1) = (3,1)$.

1. Write $(0,0,1)$ as a linear combination of $(1,0,0)$, $(0,1,0)$, and $(1,1,1)$.
2. Use the fact that f is linear to compute $f(0,0,1)$, and then $f(x,y,z)$. [Write (x,y,z) as a linear combination of the standard basis vectors.]
3. Obtain a matrix A so that $f\begin{pmatrix}x\\y\\z\end{pmatrix} = A\begin{pmatrix}x\\y\\z\end{pmatrix}$.
4. Find the vectors $\vec{x}$ so that $A\vec{x} = \vec{0}$.

Problem 3.32 Suppose that $A = \begin{bmatrix}1 & 3 & 4\\ -2 & 0 & -2\\ 0 & 1 & 1\end{bmatrix}$, and consider the function $f\vec{x} = A\vec{x}$.

1. What are $f(1,0,0)$, $f(0,1,0)$, $f(0,0,1)$, and $f(2,3,0)$? What is $f(x,y,z)$?
2. Show that $\vec{f}$ is a linear function.
3. Find (x,y,z) such that $f(x,y) = (5,-2,1)$, or explain why it is not possible.
4. The set of possible outputs of f is an object in 3D. Describe that object.
5. Find the vectors $\vec{x}$ such that $f(\vec{x}) = \vec{0}$.

If you need to rref a matrix above, please use technology to do so. In class, only show us the matrix, and its rref.

Make sure you ask me in class to visually show you a representation of the linear functions above. There's a ton more that we could study about linear functions, and I'd like to introduce to you some of those ideas.

Matrices provide us with the key examples to understanding linear transformations. However, a matrix by nature requires that we look at functions between finite dimensional spaces. The key linear transformations we will study throughout the semester will involve infinite dimensional spaces (like the space of all differentiable functions). Most of the ideas we have learned will still be useful to us as we explore functions between infinite dimensional vector spaces. Near the end of the semester, we'll even start discussing eigenvalues and eigenfunctions of linear transformations. You'll then explore these concepts in greater detail in many of your future classes.

We now make one final definition as we wrap up this chapter. We want a word that quickly tells us which vectors are mapped to zero. We'll be using this vocabulary often as we solve differential equations. In each of the previous two problems, the last portion asked you to find the kernel of the linear transformation.

Definition 3.6: Kernel. The kernel of a linear function f is the collection of vectors that are mapped to zero, i.e. $f(\vec{x}) = \vec{0}$.

Once you have a collection of vectors in the kernel of a linear function, you can use those vectors to obtain lots of other vectors. Any linear combination of vectors in the kernel will remain in the kernel. The next problem asks you to show why.

Problem 3.33 Suppose that $\vec{x}$ and $\vec{y}$ are both in the kernel of a linear function f. Show that any linear combination of $\vec{x}$ and $\vec{y}$ are also in the kernel of f. [Hint: What is $f(\vec{x})$, $f(\vec{y})$, and then what is $f(a\vec{x}+b\vec{y})$? Be able to explain every step in your work, by telling us what definition you are using.]

The previous problem shows us that the kernel is closed under linear combinations. You can't get out of the kernel by performing linear combinations of things that are in the kernel. We now end this chapter with an example to illustrate how we will use the words "linear" and "kernel" throughout the semester. Most of the remainder of this course deals with finding the kernel of a linear function.

Problem 3.34 Consider the differential equation $y' - 3y = 0$. Let L be the operator $L(y) = y' - 3y$. With the operator notation, we can rewrite the differential equation as $L(y) = 0$ (so we need to find the kernel of L).

1. What is the domain of L?
2. Show that L is a linear operator by computing $L(y_1 + y_2)$ and $L(cy)$.
3. Solve the differential equation $y' - 3y = 0$ by using separation of variables.
4. Obtain a single solution (no unknown constants) to the ODE.
5. Using the single solution, can you obtain all solutions as a linear combination of the single solution?

The solutions to the first order ODE $y' - 3y = 0$ are linear combinations of a single solution. This is precisely because the ODE is a linear first order ODE. If we had a 2nd order linear ODE, then solution would be all linear combinations of two independent solutions. The next problem introduces this idea.

Problem 3.35 Consider the differential equation $y'' + 3y' + 2y = 0$. Let L be the operator $L(y) = y'' + 3y' + 2y$. With the operator notation, we can rewrite the differential equation as $L(y) = 0$ (so we need to find the kernel of L).

1. What is the domain of L?
2. Show that L is a linear operator by computing $L(y_1 + y_2)$ and $L(cy)$.
3. Show that both e^{-2x} and e^{-x} are in the kernel of L.
4. Are e^{-2x} and e^{-x} linearly independent? Why?
5. Why is $y = c_1 e{-}2x + c_2 e^{-x}$ a solution to the differential equation $y'' + 3y' + 2y = 0$?

3.7 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 4

First Order ODEs

This chapter covers the following ideas.

1. Be able to interpret the basic vocabulary of differential equations. In particular, interpret the terms ordinary differential equation (ODE), initial value, initial value problem (IVP), general solution, and particular solution.
2. Identify and solve separable and exact ODEs.
3. Use integrating factors and substitution to solve additional ODEs.
4. Use the three step modeling process (express, solve, and interpret) to analyze exponential growth and decay, Newton's law of cooling, mixing, and the logistics equation.
5. Use Laplace transforms to solve first order ODEs.

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the problem set, please use this list as a guideline to find related problems.

Concept	Sec.	Suggested	Relevant
Separable Review	4	42	1-8,23-45
Exact	5	5,11,26,29,34	1-13,24-40,56-65
Integrating Factors	5	21,22,41,47	21,22,41-42,47-49,51,55
Linear	6	4,13,20,32,51	1-6,9-15,20-36,43-49,50-57
Homogeneous	4,	11,12,48	11-17,46-54
Bernoulli	6	16,53	16,17,37-42,53
Applications	7 7	4[27],6[33],1[38] 10[48],17[67],7[88]	1-6 [26-44] 8-10 [45-50],16-18[65-70], 7[87-88]
Laplace Review	21	19,32,33[use table]	4-7,10-12,27-35
Inverse Transforms	22	1,2,3,6,13,15	1-3,6,15,17,20-28,42,42,45-47
Solving ODEs	24	1,14,19(parfrac)	1,2,11,14,15,19-19,22,24,25,38-42

4.1 Basic Concepts and Vocabulary

Let's start this chapter with a review problem from the first chapter.

Problem 4.1 Solve the ordinary differential equation $y' - 5y = 0$. Then use the initial condition $y(0) = 7$ to obtain the unknown constant.

Definition 4.1: Differential Equation Language. A differential equation is an equation which involves derivatives (of any order) of some function.

- An **ordinary differential equation (ODE)** is a differential equation involving a function $y(x)$ whose domain is one dimensional. The function only has ordinary derivatives.
- A **partial differential equation (PDE)** is a differential equation involving a function $y(x_1, x_2, \ldots)$ whose domain is more than one dimensional. The function has partial derivatives.
- The order of an ODE is the largest order derivative that appears in the ODE.
- A solution to an ODE on an interval (a, b) is a function $y(x)$ defined on the interval (a, b) which satisfies the ODE.

To verify that a function is a solution to an ODE, calculate derivatives and put them in the ODE. If the resulting equation is an identity for all $x \in (a, b)$, then you have verified that you have a solution.

Typically a solution to an ODE involves an arbitrary constant C. There is often an entire family of curves which satisfy a differential equation, and the constant C just tells us which curve to pick.

Definition 4.2: Initial Value Problems (IVP). Often an ODE comes with an **initial condition** $y(x_0) = y_0$ for some values x_0 and y_0.

- A **general solution** of an ODE is all possible solutions of the ODE.
- A **particular solution** is one of infinitely many solutions of an ODE.
- We can use the initial conditions to find a particular solution of the ODE.
- An ODE, together with an initial condition, is called an **initial value problem (IVP)**.

This next problem has you practice with the vocabulary above. You'll want to use separation of variables to solve this problem.

Problem 4.2 Consider the IVP $y' - 4y = 8$, where $y(0) = 3$.

1. What is the order of the ODE?
2. Obtain a general solution of the ODE. State an interval on which your general solution is valid.
3. Verify that your general solution is a solution to the ODE.
4. Solve the IVP.

One of the key uses of differential equations is their ability to model the world around us. If something is changing, then we can often use y' to represent that change. If we know a force is acting on an object, then $F = ma = my''$ allows us to build a differential equation that models the motion of the object. As the semester progresses, we'll be making these connections in each chapter, and showing how to use differential equations to model the world. We'll also see that eigenvalues and eigenvectors are the connecting piece that allows us to see, and obtain, the solution to differential equations. Many of the models we build will depend on observing that a change is proportional to something, or that a force is proportional to something. If you've forgotten what proportional means, here's a definition.

Definition 4.3: Proportional. We say that y is proportional to x if $y = kx$ for some constant k. We call the constant k the proportionality constant. When two quantities are proportional, then doubling one will double the other, tripling one will triple the other, and so on. A percentage change to one is matched by the other.

The next problem has us build our first model. Suppose you go to the doctor's office to get a strep test done. They swab the back of your throat and then put a sample of tissue from your body in a petri dish. If you have strep, then the bacteria will grow inside the petri dish, and they'll be able to see the rapid growth of the strep bacteria in a fairly short amount of time.

Problem 4.3: Exponential Growth Suppose that you place some bacteria in a petri dish. Initially, there are P mg of the bacteria in the dish, and then the bacteria starts to reproduce, so the amount of the bacteria is changing. Let $y(t)$ represent the mg of bacteria in the dish after t days. Then y' would represent the rate at which y is changing. The rate at which y grows depends on how large y is. If you were to double y, then the growth rate y' should double as well. Similarly, if you tripled y, then the growth rate y' would triple as well. It seems reasonable to assume that y' is proportional to y.

1. Express the statement "y' is proportional to y" as a differential equation. What are the initial values (if any)?

2. Solve the differential equation above, obtaining a general and particular solution.

3. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large?

4. If after 10 minutes you measure 5 mg of the bacteria, and then after 20 minutes you measure 8 mg of the bacteria, how much bacteria was present initially? [If you apply the natural logarithm to both sides of your solution, then you can solve a linear system of equations to obtain the unknowns $\ln P$ and k. You can then use Cramer's rule or RREF.]

The next problem is very similar to the previous, we'll just change the setting from growth of a bacterial culture, to growth of an investment.

Problem 4.4 Suppose you invest $P = \$10,000$ dollars in an account, and that the money accumulates interest at a constant rate. Let $A(t)$ represent the accumulated worth of your investment after the investment has had t years to grow. No new deposits are made, rather the interest is just left in the account to accumulate more interest.

1. Why is is reasonable to assume that A' is proportional to A?
2. Express the connection between A and it's growth as a differential equation. What are the initial values (if any)?
3. Solve the differential equation, obtaining a general and particular solution.
4. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large?
5. Suppose after 5 years that the value of the investment has reached \$18,000. How long will it take for the investment to reach \$100,000.

Let's look at one more application before introducing additional solution techniques. Here's the scenario. You decide to cook a turkey for Thanksgiving. You turn the oven on to 350°F, and the package says that you need to get the turkey heated up to an internal temperature of 165°F. You followed the instructions and thawed the turkey so that currently it's about 40°F. How long will it take for the turkey to heat up? If instead of heating a turkey, you wanted to heat a chicken patty, would the time vary? If you just wanted to heat a metal pan up, how would the time vary? The next problem introduces a simplistic model to examine this question. The model works best when you assume that an increase in heat is evenly distributed throughout an object (such as heating a metal pan). When you heat a turkey, the heat is not evenly distributed. This uneven heat distribution complicates the following model, and we'd need to explore PDEs to obtain a better model for heat flow. To simplify things, we'll assume that heat distributes itself evenly throughout the object.

Problem 4.5: Newton's Law of Cooling Suppose that you place an object in an oven. The oven temperature is set to A (you can use Fahrenheit, Celsius, or Kelvin). I'm using A as the temperature of the surrounding "a"tmosphere. The object's initial temperature is T_0. Let $T(t)$ represent the temperature of the object t minutes after we place the object in the oven. If $T(t)$ is really close to A, then the rate at which T increases should be pretty small, as the temperature of the object is almost the same as the temperature of the atmosphere. If T is really far from A, then the rate of temperature change should be a lot larger. Hence, it appears that T' depends on the difference $A-T$. Newton conjectured that the rate at which the temperature changes is proportional to the difference $A-T$.

1. Express the statement "the rate at which the temperature changes is proportional to the difference $A-T$" as a differential equation. What are the initial values (if any)?
2. Solve the differential equation above, obtaining a general and particular solution.
3. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large? Does this seem reasonable.

Problem 4.6 You should have obtained the solution to Newton's law of Cooling as

$$T(t) = A + (T_0 - A)e^{-kt},$$

where k is the proportionality constant. Suppose that $T_0 = 45°\text{F}$ and $A = 350°\text{F}$.

1. After 5 minutes, you check the temperature and observe $T(5) = 80°\text{F}$. What is k, and how long will it take for the object to reach 165°F.
2. After 5 minutes, you check the temperature and observe $T(5) = 120°\text{F}$. What is k, and how long will it take for the object to reach 165°F.
3. The number k depends on the material you are trying to heat. If k is large, what does that mean about the material? Think of some examples where k would be large, and where k would be small.

You've now seen a few examples of how differential equations are used to model the world around us. You will most likely find that in your future courses, you'll be taking real world phenomenon and expressing the relationships you see as differential equations. Solving those differential equations gives us mathematical models we can use to interpret the world around us. There are three parts to this process.

- Express real world phenomenon in terms of a differential equation.
- Solve the differential equation.
- Interpret the solution in the context of the problem, which often involves using the results to predict behavior.

A main focus in this class will be the second portion, "Solve." As we all come from a different background, we won't have time to develop the background material that you'll explore in your respective majors, so the "Express" portion will often come in your major courses. You may find in some future courses that they focus on the "Express" and "Interpret" portions, and then refer you to some standard reference for the "Solve" part. The goal of our course is to help you develop the key solution techniques. Along the way, we'll occasionally add some simpler problems that we can "Express" and "Interpret" without needing a lot of background.

4.2 Solution Techniques

In the review chapter, we explored finding potentials of a gradient field. We also introduced the language of differential forms. Recall the following definition.

Definition 4.4: Differential Forms. Assume that f, M, N are all functions of two variables x, y.

- A differential form is an expression of the form $Mdx + Ndy$ (just as a vector field is a function $\vec{F} = (M, N)$).
- The differential of a function f is the expression $df = f_x dx + f_y dy$ (just as the gradient is $\vec{\nabla} F = (f_x, f_y)$).

- If a differential form is the differential of a function f, then we say the differential form is exact (just as we say a vector field is a gradient field). The function f is called a potential for the differential form. Let me reiterate. We say a differential form $Mdx + Ndy$ is exact if and only if there exists a function f such that

$$df = Mdx + Ndy.$$

A differential form is exact precisely when the corresponding vector field is a gradient field.

The next problem provides the key idea need to solve almost every differential equation we'll encounter in this course. If you can rewrite the differential equation in differential form, and the differential is exact, then solving the ODE requires that you find a potential.

Problem 4.7 Consider the differential form $(2x + 3y)dx + (3x)dy$.

1. By taking derivatives, show that the differential form is exact. [See the test for a conservative vector fields, problem 1.30.] Show that a potential for this differential form is $f(x, y) = x^2 + 3xy$.
2. Rewrite the differential equation $3xy' + 3y = -2x$ in the differential form
$$Mdx + Ndy = 0.$$
What's the angle between the vectors (M, N) and (dx, dy)?
3. Explain why the solution to $Mdx+Ndy = 0$ is a level curve of the potential $f(x, y)$.
4. Give the solution to $3xy' + 3y = -2x$ if $y(2) = 1$.

I'm trying to decide on a good name for the next theorem. We'll see that this theorem is crucial to solving just about EVERY differential equation we encounter from here on out, and it also solves all the ones before now. The name below might change, but something along the lines of "the sledgehammer," or "one tool to rule them all" would work. The theorem has no official name, so we can make one up as we go. Basically, we'll show that we can reduce almost every ODE that we solve to a form which allows us to apply the following theorem.

Theorem 4.5 (The sledgehammer for ODEs - one tool to rule them all)**.** *Suppose that $Mdx + Ndy$ is an exact differential form with potential $f(x, y)$. If we can write an ordinary differential equation in the form $Mdx + Ndy = 0$, then an implicit general solution to the ODE is $f(x, y) = c$. The level curves of a potential are precisely the solutions to the ODE. Let me repeat that. The level curves of a potential are precisely the solutions to the ODE.*

Let's use the previous theorem now to solve a couple of ODEs.

Problem 4.8 Give a general solution to each of the following ODEs. You may give your solution implicitly, so don't worry about solving for y. [Hint: Use the previous theorem.]

1. $(4x + 2y)dx + (2x + y)dy = 0$
2. $(x \cos(xy) + y)y' = \sin x - y \cos xy$

4.2.1 Use integrating factor when the ODE is not exact

Let's now return to a problem we've already solved, and show how we can use the sledgehammer theorem to solve things we've already seen, provided we add one more step.

Problem 4.9 Consider the ODE $y' = -3y$ which we can write in differential as $3ydx + 1dy = 0$.

1. Show that $3ydx + 1dy$ is not exact. Then use separation of variables to solve the ODE.
2. Multiply both sides of $3ydx + 1dy = 0$ by $\frac{1}{y}$. Show that the resulting differential form is exact, and use the sledgehammer theorem to obtain a solution.
3. Multiply both sides of $3ydx + 1dy = 0$ by e^{3x}. Show that the resulting differential form is exact, and use the sledgehammer theorem to obtain a solution.

Any time we can write an ODE in the differential form $Mdx + Ndy = 0$, the zero on right hand side gives us power. Our goal will be to multiply both sides of the differential equation by some function F, called an integrating factor, so that the resulting differential is exact. The general solution to the ODE is then simply the level curves of a potential.

Definition 4.6: Integrating Factor. An integrating factor for a differential form $M(x, y)dx + N(x, y)dy$ is a function $F(x, y)$ so that the product $FMdx + FNdy$ is exact.

In Problem 4.9, I gave you two different integrating factors. Where did they come from? The next problem will show you how I obtained one of the integrating factors. There many more options.

Problem 4.10 Let $M(x, y)dx + N(x, y)dy$ be a differential form. For simplicity, we just write $Mdx + Ndy$. Suppose $F(x, y)$ is an integrating factor.

1. To be exact, explain why we must have
$$\frac{\partial F}{\partial y}M + F\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x}N + F\frac{\partial N}{\partial x}$$
2. If we assume that F only depends on x, so that $F(x, y) = F(x)$, show that a possible option for an integrating factor is
$$F(x) = e^{\int \frac{M_y - N_x}{N}dx} = \exp\left(\int \frac{M_y - N_x}{N}dx\right).$$
3. If we assume that F only depends on y, so that $F(x, y) = F(y)$, show that a possible option for an integrating factor is
$$F(y) = e^{\int \frac{N_x - M_y}{M}dy} = \exp\left(\int \frac{N_x - M_y}{M}dy\right).$$

[In class, you may omit the last part in your presentation, as it's almost an exact replica of the 2nd part.]

The problem above gives us a way of finding integrating factors for many differential equations. It will not give an integrating factor for EVERY differential equation, but it will provide an integrating factor for almost all the ODEs we tackle in this course. Let's now try using this technique on a problem we've already solved.

Problem 4.11 Consider the ODE $y' - 4y = 8$, which we solved in Problem 4.2.

1. Rewrite the ODE in differential form $Mdx + Ndy = 0$.
2. Find an integrating factor $F(x) = e^{\int \frac{M_y - N_x}{N} dx} = \exp\left(\int \frac{M_y - N_x}{N} dx\right)$.
3. Multiply both sides by the integrating factor, and then solve the ODE by applying the sledgehammer theorem.

Problem 4.12 Solve each ODE by finding an appropriate integrating factor.

1. $y' + 4xy = 3x$
2. $2ydx + (3x + 4y)dy = 0$ (Doable now)
3. $y' + 3y = e^{2x}$ (Solve for y.)
4. $y' - 4y = e^{4x}$ (Solve for y.)
5. $xy' - 4y = 2x$ (Solve for y.)

Let's now look at an additional application. We will encounter mixing model problems throughout the semester. They provide a simple way to see applications of ODEs, without requiring much background.

Problem 4.13: Mixing Model Suppose a 2000 gallon tank contains a solution of water which initially contains 50 lbs of salt. The tank has an inflow valve, and an outflow value. We would like to change the salt content, so we start pumping in 30 gallons of water (with 1/2 lb of salt per gallon) each minute. We'll assume that the mixture is evenly spread throughout the entire tank by constant stirring. At the same time, 30 gallons of the evenly stirred mixture flow through the outflow valve each minute. Let $y(t)$ represent the lbs of salt in the tank after t minutes. We currently only know $y(0) = 50$. Our goal is to determine the amount of salt $y(t)$ in the tank after t minutes.

1. (Express) The salt content changes in two ways. Salt is added through the new solution (a flow in), and salt leaves through the outlet valve (flow out). Explain how to obtain a formula for the flow in, and a formula for the flow out. Then explain why
$$y' = 15 - \frac{30}{2000}y.$$
2. (Solve) Obtain a general solution to the ODE, and then use the initial value to obtain a particular solution.
3. (Interpret) Construct a graph of your solution. As t increases, what happens to the salt content? Does your answer seem reasonable?

The mixing model problem above, as well as the exponential model and Newton's law of cooling, all belong to a special class of ODEs which we call linear ODEs.

Definition 4.7: Linear ODE. If we can write an ODE in the form $y'+p(x)y = q(x)$, then we say that the ODE is linear. This is precisely because the operator $L(y) = y' + a(x)y$ is a linear operator. If $q(x) = 0$, then we say the linear ODE is homogeneous. Otherwise, we say the linear ODE is non homogeneous.

The next problem provides a way to obtain a solution to EVERY linear ODE. Practice until you can develop this formula quickly, and then you'll have the key concepts needed for solving just about every ODE we encounter throughout the semester.

Problem 4.14: A Linear ODE Solution Consider the linear ODE $y' + p(x)y = q(x)$, where p and q are differentiable functions of x on some interval. Find an appropriate integrating factor, and then find a potential. Finish by solving for y to show that on this interval, a general solution is

$$y(x) = e^{-\int p(x)dx}C + e^{-\int p(x)dx} \int \left(e^{\int p(x)dx}q(x)\right) dx,$$

where C is an arbitrary constant. If the linear ODE is homogeneous, what is a general solution?

Problem: 14 and 1/2: Go back to problem solving ODEs by finding an integrating factor, and decide which ODEs are linear. Then pick one of the ODEs and solve it using the general solution from the previous problem.

Let's tackle a couple more application problems. As you solve them, rather than use the formula above, practice finding an appropriate integrating factor, and then find a potential.

Problem 4.15 Suppose a 50 gallon tank contains a solution of fertilizer which initially contains 10 lbs of fertilizer. We start pumping in 4 gallons per minute, where the concentration of fertilizer is 1/3 lb per gallon. Assume that the mixture is evenly spread throughout the entire tank by constant stirring. The extra At the same time, 4 gallons of the evenly stirred mixture flow through the outflow valve each minute. Let $y(t)$ represent the lbs of fertilizer in the tank after t minutes.

1. Express the mixing model as an IVP (give the ODE and the IV).
2. Solve the IVP.
3. Construct a rough graph of your solution. As t increases, what happens to the salt content? Does your answer seem reasonable?from the

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The next problem applies Newton's law of cooling to examine what happens if the temperature of the surrounding environment changes. Recall that Newton's law of cooling suggests that the rate of change of temperature of an object is proportional to the difference between the current temperature and the surrounding atmosphere, which we wrote earlier as

$$T' = k(A - T).$$

Problem 4.16 Suppose that during a summer day, the temperature outdoors fluctuates between 70°F and 110°F. We'll approximate this with a sine wave. If we let $t = 0$ be noon, then we could obtain the temperature A outdoors after t hours using the formula

$$A(t) = 20\sin(\frac{2\pi}{24}t) + 90.$$

Suppose that your air conditioner breaks at noon (your house was at 70°F at noon), and then by 6pm in the evening, the temperature had risen to 90°F.

1. Express this heating problem as an IVP.
2. Show that the ODE is linear, and then use technology to solve the ODE. You'll need to use $T(6) = 90$ to obtain the proportionality constant k.
3. Graph your solution for 3 days. In the late evenings, which is hotter, the house or the outdoors?

4.2.2 Use a substitution when you can't get an integrating factor.

We can solve most of the differential equations we tackle this semester by obtaining an integrating factor using the formulas developed in the previous section. Sometimes however, this won't work. In these cases, we often just have to make an appropriate change of coordinates (a u-substitution). Let's illustrate how this works with an example. Then we'll tackle the logistics model and introduce another application.

Problem 4.17 Consider the ODE $y' = \sin(x + y)$. There is no way that you'll get an integrating factor out of this by using our formulas for $F(x)$ and $F(y)$. The problem is the $x + y$. We now do a substitution.

1. Write the ODE in the differential form
$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = 0.$$
2. Let $x = x$ and $u = x + y$ (this is a coordinate transformation). Explain why we have
$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix}.$$
3. Show that the ODE can be written in the form $(-\sin u - 1)dx + du = 0$. Then use either separation of variables (or the sledgehammer) to solve the ODE. Don't forget to substitute back in when you're done.

The last problem introduced the key idea. If you have an ODE in the form

$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = 0,$$

then an appropriate substitution $x = x$, $y = g(x, u)$ will give us the ODE

$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_x & g_u \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix} = 0.$$

If we can find an integrating factor $F(x)$ or $F(u)$ which makes

$$F \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_x & g_u \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix}$$

exact, then we can use the sledge hammer to solve the ODE. The hard part here is finding the correct transformation. If you can find the correct transformation, then you can solve the ODE. This is not easy, and in general it can be really tough.

Problem 4.18 Consider the ODE $xyy' = 4x^2 + 2y^2$. In this situation, if you let $u = y/x$ (so $y = xu$), show that you can rewrite the ODE as

$$\frac{u}{4+u^2}du = \frac{1}{x}dx.$$

This is a separable ODE, which we can solve. Solve the ODE.

Notice that the coefficients xy, $4x^2$, and $2y^2$, all are basically second order monomial terms. When the coefficients of the ODE are monomials with the same degree, the substitution $u = y/x$ will convert the ODE into a separable ODE. I'll leave this to you to prove. You do have the tools to prove it.

Problem 4.19 Consider the ODE $(x + 2y)dx + (3x + 4y)dy = 0$. Use the substitution $u = y/x$ to convert this into a separable ODE and give a general solution.

Consider the ODE $y' + 3y = 4y^3$. This ODE is not linear (why?). We could separate variables on this ODE and solve (we'll do so in class, reminding you about partial fractions). Instead, Bernoulli noticed that if the ODE is in the form $y' + a(x)y = b(x)y^n$, then the substitution $u = y^{1-n}$ will always convert the ODE into a linear ODE, and then we can use an integrating factor to solve the ODE. It's not easy to discover the right substitution that will convert an ODE into something we can solve. We call them Bernoulli ODEs because his discovery was quite clever. The $u = y/x$ substitution above was not clever enough to get a name attached to it.

Problem 4.20: Bernoulli ODE Consider the ODE $y' + 3y = 4y^3$. Use the substitution $u = y^{1-3}$ to convert this ODE into a linear ODE, and then solve. [Hint: You know that $u = y^{-2}$. Use this to solve for y, and then compute $dy = ?du$. Then just substitute. You'll probably have a really ugly term involving $u^{-3/2}$, so multiply both sides by $u^{3/2}$ and all the ugliness will disappear.]

We'll come back to Bernoulli ODEs and see some applications of them after we review Laplace transforms.

4.3 Laplace Transforms

Recall that the Laplace transform of a function $y(t)$ defined for $t \geq 0$ is

$$Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)dt.$$

- We call the function $y(t)$ the inverse Laplace transform of $F(s)$, and we write $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.
- As a notational convenience, we describe original functions $y(t)$ using a lower case y and input variable t or x. We describe transformed functions $Y(s)$ using the same capital letter and input variable s.

$f(t)$	$F(s)$	provided
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^2	$\frac{2}{s^3}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$

$f(t)$	$F(s)$	provided
$\cos(wt)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(wt)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cosh(wt)$	$\frac{s}{s^2-\omega^2}$	$s > \lvert\omega\rvert$
$\sinh(wt)$	$\frac{\omega}{s^2-\omega^2}$	$s > \lvert\omega\rvert$
y	$\mathscr{L}\{y\} = Y$	
y'	$s\mathscr{L}\{y\} - y(0)$ $= sY - y(0)$	

Table 4.1: Table of Laplace Transforms

We've computed quite a few Laplace transforms already. For convenience, I've placed the Laplace transforms we'll use most often in Table 4.1. Feel free to use this table as you find Laplace transforms and their inverses. With practice, you will memorize this table.

We can use this table, and the linearity of the Laplace transform, to quickly compute both forward transforms and inverse transforms. The next problem asks you to do this.

Problem 4.21 Use the table of Laplace transforms to do the following:

1. Compute the Laplace transform of $y(t) = 6 + 2t + 4t^2 - 5e^{7t} + 11\cosh(3t)$.
2. Compute the inverse Laplace transform of
$$Y(s) = \frac{5}{s} + \frac{4}{s^3} + \frac{3s}{s^2+16} - \frac{2}{s^2-9}.$$
Once you have a guess for the inverse Laplace transform, verify that your guess is correct by computing the Laplace transform (using the table of course).

Problem 4.22 Find the inverse Laplace transform of $F(s) = \dfrac{2s+1}{s^2+5s+4}$. [Hint: Use a partial fraction decomposition. Start by factoring the denominator.]

Problem 4.23 Find the inverse Laplace transform of $F(s) = \dfrac{2s+1}{s^2+9} + \dfrac{5s+7}{s^2-9}$. [Hint: This can all be done using trig and hyperbolic trig functions.]

The real power behind the Laplace transform comes from the last formula in the table.

Theorem 4.8 (The Laplace Transform of a Derivative)**.** *Suppose that $y(t)$ is a differentiable function defined on $[0, \infty)$ such that $\lim_{t\to\infty} \dfrac{y(t)}{e^{st}} = 0$ for some s. We say that $y(t)$ does not grow faster than some exponential, as the function e^{st}*

grows faster that $y(t)$ (otherwise the limit would not be zero). If this is the case, then the Laplace transform of y' is

$$\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0) = sY - y(0).$$

Problem 4.24 Prove the previous theorem. In other words, show that $\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0) = sY - y(0)$. [Hint, use integration by parts once, and don't forget to use the bounds.]

Before illustrating the key value of this theorem, let's fill in the only remaining rules in our Laplace transform table that we have not yet developed.

Problem 4.25 In the table of Laplace transforms it also states that the transform of $\cos(wt)$ is $\dfrac{s}{s^2+\omega^2}$, and that the transform of $\sin(wt)$ is $\dfrac{\omega}{s^2+\omega^2}$. Pick one of these rules, and then use the definition of the Laplace transform to explain why it is true. [Hint: You'll want to use integration by parts twice. See the online text if you want more hints.]

We'll now use Theorem 4.8 to solve some ODEs. You'll see the power behind this method.

Problem 4.26 Consider the IVP $y' = 7y$, $y(0) = A$.

1. Apply the Laplace transform to both sides of this ODE. You should have an equation involving $Y(s)$.
2. Solve for Y.
3. Now find the inverse transform of Y. This is $y(t)$, the solution to the ODE.
4. We already know how to solve this ODE using either separation of variables, or by finding an integrating factor. Pick one of these methods and obtain a general solution.

Did you see the process above? Rather than integrate, we just (1) computed the Laplace transform of both sides, (2) solved an algebraic equation for Y, and then (3) obtained the inverse Laplace transform to get Y.

Here's a parable to compare to using Laplace transforms. You are inside a house that has a single door leading to the downstairs. You are on the main floor, and need to get downstairs. The door is locked and you don't know where the key is (you can't figure out how to solve the ODE). You (1) decide to walk out the front door (you apply the Laplace transform). Then you (2) walk around the house and find the back door entrance to the basement (you solve for Y, and maybe apply a partial fraction decomposition). Then (3) you walk up to the locked door and unlock it from the other side (you find the inverse transform).

The Laplace transform replaces the problem of integrating with an algebraic problem (often easier to solve). We'll be using it throughout the semester to help us see patterns, and unlock difficult problems. It works best when the ODE is linear.

Problem 4.27 Consider the IVP $y' + 3y = 5$, $y(0) = 7$. (1) Apply the Laplace transform to both sides of this ODE to obtain an equation involving $Y = \mathscr{L}\{y\}$. (2) Solve for the transformed function Y. You will need to use a partial fraction decomposition to write $Y = \dfrac{A}{s+3} + \dfrac{B}{s}$. (3) Use an inverse transform to obtain the solution $y(t)$.

You'll find that with most Laplace transform problems, we'll need a partial fraction decomposition before we can compute an inverse transform. The next problem has you practice the Laplace transform inversion process to solve multiple problems that you know the answer to using simple integration.

Problem 4.28 For each problem below, use a Laplace transform to solve the ODE. Each problem could be solved with simple integration instead. The point to this problem is to help you see how the Laplace transform gives you, in a different way, information you already know how to obtain.

1. $y' = 5t^2 + 7t + 3$, $y(0) = C$.
2. $y' = e^{at}$, $y(0) = C$.
3. $y' = \cosh(3t)$, $y(0) = 2$.
4. $y' = \sin(3t)$, $y(0) = 2$.

[Hint: You'll need a partial fraction decomposition to write $\dfrac{s}{s(s^2+9)} = \dfrac{A}{s} + \dfrac{Bx + C}{s^2+9}$ on part 4. You'll need a similar idea on 2 and 3.]

Let's end this section with two more Laplace transform problems, where the initial conditions are not given.

Problem 4.29 Solve the ODE $y' + 4y = e^{3t}$ by using a Laplace transform. No initial condition was given, so you should use something like $y(0) = C$. Because this initial condition involves an arbitrary constant, you may find that Cramer's rule helps you quickly obtain the partial fraction decomposition (rather than row reduction).

Problem 4.30 Solve the ODE $y' + 3y = \sin(2t)$ by using a Laplace transform. See the previous problem for help about what to do when no initial condition is given.

4.4 What Method Should I Use?

In this chapter, we've explored various different techniques to solve first order ODEs. Here's a list.

- Separation of variables: The easiest, if you can separate.
- Exact: The ODE has a potential. Use the sledgehammer.
- Integrating Factors: Make the ODE exact.
- Substitution: Change variables so you can make the ODE exact.
- Laplace Transforms: Dodge integration. Replace it with algebra.

The Laplace transform works nicely on linear ODEs with constant coefficients. If we're missing an initial condition, the algebra gets a little uglier, but still doable. We'll be using the Laplace transform to discover solutions to higher order ODEs as the semester progresses. However, we could have solved every one of the problems we tackled with Laplace transforms by instead using our sledge hammer tool (make the ODE exact, through substitutions and/or integrating factors, and then find a potential). The sledgehammer tool will solve a much larger range of ODEs than the Laplace transform, and near the end of the semester we'll see it's true power in terms of matrices, eigenvalues, and eigenvectors.

I'm working on writing a paper to extend the sledgehammer approach to solve just about every ODE undergraduates tackle, and provide a uniform approach to working with ODEs. I'm just waiting for an interested student to come and complete the project with me.

So which method should you use? That depends on how much work you want to do. The sledgehammer tool will solve EVERY problem we see. If an ODE is separable, it's generally much faster to just use separation of variables (which is really just using an integrating factor). If the ODE is linear, with constant coefficients, and you have an initial condition, then a Laplace transform might be faster. If all else fails, make the ODE exact and find a potential.

Problem 4.31 Which method would you use to solve each ODE below? If you opt for separation of variables, then show us how to separate. If the ODE is exact, show us how you know. If you decide to find an integrating factor, show us the integrating factor. If you will use a substitution, what substitution will you use? If you decide to use Laplace transforms, take the Laplace transform of both sides. In all cases, don't solve the ODE, rather just show us the first step in the solution process.

1. $x^2y' = 4xy^2$, $y(2) = 1$.
2. $xy' = 3y + x$, $y(2) = 1$.
3. $y' + 8y = e^x$, $y(0) = 1$.
4. $y' + 8y = y^2$, $y(0) = 1$.

Let's end the chapter by considering another application. In Problem 4.3, we considered the growth of bacteria in a petri dish. We could have applied this to any other population (such as deer in a forest, people on the Earth, cancer cells spreading through the bloodstream, number of cell phones users in Brazil, speed of computer processors, etc.) There is a problem with this model. It works great for a little while, but physical systems cannot grow exponentially forever. Eventually the growth has to slow down. In the example with the petri dish, eventually the bacteria will have gotten so large that it cannot support more growth. This is where our final problem begins.

Problem 4.32 Suppose that bacteria grow in a petri dish (if you don't like bacteria, then pick something else to put in here that interests you). From Problem 4.3, we used the model $y' = ky$ to express that the rate of growth y' is directly proportional to the size y of the population. We assumed that k was constant. Here's where we now make a change. Instead of assuming that k is constant, let's assume that as the population gets larger, that the constant k decreases. In fact, if we let M represent a theoretical maximum population, let's assume that k is proportional to the difference between the current population and this theoretical maximum.

1. (Express) Explain why $y' = -a(y - M)y$.
2. (Solve) Solve the ODE using separation of variables. You'll need to perform a partial fraction decomposition on $\frac{1}{y(y-M)}$.

3. (Interpret) Pick some constants for a, M, and the initial size of the population. Then graph your solution. You should obtain a logistics curve (please use a computer to check your work).

Problem 4.33 The ODE $y' = -a(y-M)y$ is a Bernoulli ODE. Rewrite it in the form $y' + a(x)y = b(x)y^n$, and then use Bernoulli's substitution $u = y^{1-n}$ to solve the ODE.

Question 4.9. Why can't we (yet) use a Laplace transform to solve $y' = -a(y-M)y$?

4.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Homogeneous ODEs

This chapter covers the following ideas.

1. Explain Hooke's Law in regards to mass-spring systems. Construct and solve differential equations which represent this physical model, with or without the presence of a damper.
2. Understand the vocabulary and language of higher order ODEs, such as homogeneous, linear, coefficients, superposition principle, basis, linear independence.
3. Solve homogeneous linear ODE's with constant coefficients (with and without Laplace transforms). In addition, create linear homogeneous ODE's given a basis of solutions, or the roots of the characteristic equation.
4. Explain how the Wronskian can be used to determine if a set of solutions is linear independent. Briefly mention the existence and uniqueness theorems in relation to linear ODEs, and give a reason for their importance.

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the preparation problem set, please use this list as a guideline to find related practice problems.

Concept	Sec	Suggestions	Relevant Problems
Vocabulary of ODEs	8*	33-35	1-3,33-35
2nd Order Homogeneous	9*	1,7,12,21,27,40	1-15, 17-45
nth Order Homogeneous	10*	3,7,8,9,12,18,37,41,44,49	All
IVPs (Homogeneous)	13	9	4,9,13
Applications	14	2,3,5,29,31,34,41-43	1-8,26-43
Laplace Transforms	21*	26, 54	14(c),15(b),25,26,54-58,
Inverse Transforms	22*	7, 34-36,38,read 12 and 18,44	6-10,15-19,29-30,32-53
Solving ODES	24	26,44	5,26,31,36,43,44
Wronskian and Theory	8*	9,10,18,20,43,48,53,58	5-10, 13-20, 31,36-64

*The problems in these sections are quick problems. It is important to do lots of them to learn the pattern used to solve ODEs. You may be able to finish

7 or more problems in 15 minutes or less. Please do more, so that when you encounter these kinds of problems in the future you can immediately give an answer and move forward.

5.1 Some Physical Models

In this chapter, we're going to learn how to solve a huge collection of higher order differential equations. Before diving into the details, let's make sure we know WHY we would even want to do so. If I knew you all had the same background, we could dive into lots of examples directly related to your field (you'll do that in future classes in your major). Since we have a diverse background in our class, we'll stick mostly to models that connect velocity, position, and acceleration. Before the next chapter ends, we'll add to this some information about electrical circuits.

For our first model, let's look at how we can obtain the position of an object in projectile motion from knowledge about the acceleration and velocity. You've solve this problem before, but the solution required neglecting air resistance.

Example 5.1. In multivariate calculus, we encountered the differential equation $y'' = -g$. In this differential equation, the only force $F_T = my''$ acting on an object in projectile motion is the force of gravity $F_G = -mg$. Equating these two gives us the ODE $my'' = -mg$, or just $y'' = -g$. If we have initial position $y(0) = y_0$ and initial speed $y'(0) = v_0$, then the solution is $y = -\frac{1}{2}gt^2 + v_0t + y_0$. We found that solution by integrating twice.

We don't have to neglect air resistance anymore. We could talk about sky diving (risky), dropping bombs (deadly), throwing math books off a roof (illegal), putting a satellite into geosynchronous orbit (useful), or dropping a pebble from the top of a waterfall (head to Yellowstone and try it - sounds like we need a field trip). The next problem asks you to revisit the example above, but now add in air resistance.

Problem 5.1 Joe hikes up to the top of Lower Falls in Yellowstone. His hope is to gauge the height h of the waterfall. He plans to drop a pebble from the top, and time how long it takes for the pebble to hit the ground. He'll need a model that predicts the height of the pebble at any time t.

For this to work, Joe has to make some assumptions. His assumptions might be way off, but that's how science works. We start with assumptions and then turn those assumptions into differential equations. Here's what Joe assumes:

- He assumes Newton's second law of motion, namely that $F = ma$ (the total force is the mass times the acceleration).
- He assumes that the total force is comprised of two parts.
- The first force F_G comes from a constant acceleration due to gravity. He assumes that gravity is constant $a = -g$. The negative sign comes because the acceleration causes a decrease in height.
- The second part comes from air resistance. He assumes that the faster the pebble goes, the greater this force will be. If the pebble's speed were to double, then this force should double. So he assumes that the force due to air resistance F_R is proportional to the pebble's velocity.

Let $y(t)$ represent the height, above ground, of the pebble after t seconds. Use Joe's assumptions to answer the following:

1. Rewrite Newton's second law of motion in terms of y, y', and/or y''.

2. What is the constant force F_G due to gravity?

3. Rewrite Joe's assumption about air resistance in terms of y, y' and/or y''.

4. The total force F is the sum of the two forces, i.e. we can write $F = F_G + F_R$. Use this fact, together with your answers from the previous two parts, to obtain a second order ODE. You don't have to solve the ODE, rather you just need to obtain it.

If you need any hints, try searching the web for "modeling motion if we assume that air resistance is proportional to speed."

Congrats. You've just set up your first second order ODE.

Let's now look at another position/velocity/acceleration model, but this time related to springs. We'll start by considering the following scenario. We attach an object with mass m to a spring. We place the spring horizontally, and put the mass on a frictionless track. We let go of the object, and allow it to come to rest. We'll use the function $x(t)$ to keep track of the position of the spring at any time t, with $x = 0$ corresponding to equilibrium (the mass is at rest). Robert Hooke (1635 – 1703) developed the following law, called Hooke's law:

In the next chapter, we'll hang the spring from a ceiling. In this case, we'll have an additional force $F_g = -mg$ acting on the spring.

> The force needed to extend (or compress) a spring a distance x is proportional to the distance x. Note that the force acts opposite the displacement.

Problem 5.2 Read the preceding paragraph. Then answer the following:

- Draw a picture of a horizontal track. On the left end of the track, put a wall. Put a on object, like a square block, in the center of your track and draw a spring that connects the wall to the block.

- Explain why $mx''(t) = -kx(t)$. We generally just write $mx'' = -kx$ (the t is assumed).

- If it takes 8 N $=$ 8kg m/s^2 to move the object whose mass is 4 kg about .3 m, what is the spring constant k? How far would a 12 N force cause the object to move? Does the mass of the object matter?

Hooke's law is not a perfect model for all springs, but it does a good job for most, provided the displacement is not too large. If the displacements are too large, then the spring may deform, which changes the properties of the spring in all future computations. If you take your car out onto extremely bumpy roads, and purposefully hit some nasty bumps, you could permanently damage the shocks. In this case, you would want to replace your springs.

Every linear spring has a spring constant k. This constant has many names, such as the spring modulus, Young's modulus, Young's constant, and more. The next problem shows you how to obtain the spring constant k.

Problem 5.3 You attach a spring to the ceiling. You attach a mass of 10 kg on the end, and the spring elongates 3 cm.

1. You now attach a mass of 20 kg. How long will the spring elongate?

2. What is the spring constant k? Give the units.

3. We attach a different spring, and hang the same 10 kg on the end, but this time the spring elongates 2 cm. Is the spring constant larger or smaller?

4. If a spring has really large modulus, will it be easy or hard to elongate it?

We need one more model before we start solving some ODEs. We'll use the exact same spring model as before. Place a horizontal spring whose modulus is k on a frictionless track. Attach an object whose mass is m to the end of the spring. We now place the entire mass-spring system underwater. When it was exposed to air, we neglected air resistance. Now we'll have to take resistance into account.

We don't have to place the spring underwater to get the same affect. We could use a dashpot to resist the motion. One type of dashpot is a cylindrical tube placed around a cylindrical object, so that as the object moves, it's sides come in contact with the dashpot, resulting in friction that resists motion. See Wikipedia for more info.

Problem 5.4 When we have no resistance, the mass-spring system ODE is $F_T = F_S$, or $mx'' = -km$. Assume that the liquid applies a resistive force that is proportional to the velocity of the object. If the object is resting, the liquid doesn't apply a force. If you double the speed, then the resistive force doubles. If you triple the speed, the resistive force triples. Modify the ODE $mx'' = -km$ to account for the resistive force of water.

5.2 Notation, Vocabulary, and Solutions

We can write the ODEs from the previous section as

$$my'' + ky' = -mg, \quad mx'' + ky = 0, \quad \text{and} \quad mx'' + cy' + ky = 0.$$

If we divide each ODE by m, then we can write each ODE in the general form

$$y'' + p(t)y' + q(t)y = r(t).$$

This introduces our next definition.

Definition 5.2: Linear, Constant Coefficient, and Homogeneous.

- If we can write an ODE in the form $y'' + p(t)y' + q(t)y = r(t)$, then we say the ODE is a second order linear ODE.
- The functions $p(t)$ and $q(t)$ we call the coefficients of the linear ODE.
- If the coefficients are constant, the we say the ODE is a constant coefficient linear ODE.
- If the right hand side $r(t) = 0$, then we say the linear ODE is homogeneous. Otherwise we say it is non homogeneous.
- We use the words nth order linear ODE to talk about any ODE that we can write in the form $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = r(t)$, where $y^{(n)}$ is the nth derivative of y.

We just introduces a few new words, so with each problem that follows, let's practice using those words. The next problem has you explain why we use the word "linear."

Problem 5.5 Consider the second order ODE $y'' + 7y' + 6y = 0$.

- Why is this ODE linear? Modify it so it is no longer linear, and show us in class what would make it non linear.
- Is this ODE homogeneous? Explain.

- Let $L(y) = y'' + 7y' + 6y$. Show that L is a linear operator. (See the end of chapter 3 if you need to reread the definition).
- The solutions to the ODE are the solutions to $L(y) = 0$. In the language of linear operators, what do we call the set of functions y such that $L(y) = 0$? It was another key word near the end of chapter 3. Please look it up. The set of solutions y is the ____________ of L.

To solve second order linear homogeneous ODE, we'll use the Laplace transform. In the previous chapter, we showed that

$$\mathscr{L}(y') = s\mathscr{L}(y) - y(0) = sY - y(0).$$

We need a rule for second derivatives. Repeated application of the single derivative rule will give you all the rules you need.

Problem 5.6 Show that under suitable conditions, we can compute the Laplace transform of the second derivative of y by using the formula

$$\mathscr{L}(y'') = s^2\mathscr{L}(y) - sy(0) - y'(0) = s^2Y - sy(0) - y'(0).$$

Then show that

$$\mathscr{L}(y''') = s^3\mathscr{L}(y) - s^2y(0) - sy'(0) - y''(0).$$

Conjecture a formula for the Laplace transform of the 7th derivative of y. [Hint: As stated in the paragraph before this problem, apply the rule $\mathscr{L}(y') = s\mathscr{L}(y) - y(0)$ multiple times.]

We are now ready to solve a second order ODE with Laplace transforms.

Problem 5.7 Consider the IVP $y'' + 3y' + 2y = 0$, $y(0) = 7$, $y'(0) = 5$.

1. Is the ODE linear? Is it homogeneous? Are the coefficients constant?
2. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$.
3. Use a partial fraction decomposition to show that

   $$Y = \frac{A}{s+1} + \frac{B}{s+2},$$

   where you give the constants A and B.
4. Find the solution y to this IVP by computing the inverse Laplace transform of Y.
5. How are solutions to $s^2 + 3s + 2 = 0$ connected to your solution?

Problem 5.8 Consider the IVP $y'' + 7y' + 10y = 0$, $y(0) = c$, $y'(0) = d$.

1. Is the ODE linear? Is it homogeneous? Are the coefficients constant?
2. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$.

3. If we use a partial fraction decomposition, we would write

$$Y = \frac{A}{s+2} + \frac{B}{s+5}.$$

   Why is the solution $y(t)$ a linear combination of e^{-2t} and e^{-5t}, i.e. $y(t) = Ae^{-2t} + Be^{-5t}$?

4. Now actually perform the partial fraction decomposition to obtain the constants A and B. (Since you have variables a and b in your system, you'll want to use Cramer's rule).

5. How are solutions to $s^2 + 7s + 10 = 0$ connected to your solution?

In the previous two problems, we had initial conditions. When the initial conditions are numbers, it made the partial fraction decomposition rather simple. When the initial conditions are variables, finding the constants in the partial fraction decomposition was a little trickier. The next problem has you work through a problem when we have no initial conditions.

Problem 5.9 Consider the ODE $y'' + 7y' + 12y = 0$. We would like a general solution (no initial conditions are given).

1. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$. You'll have $y(0)$ and $y'(0)$ in the numerator of your solution. It would be nice if they weren't there.

2. Factor the denominator of Y, and write your solution as $Y = \frac{A}{?} + \frac{B}{?}$. This time DO NOT solve for A and B. You don't need to.

3. Compute the inverse Laplace transform of Y. Your answer should involve the unknown constants A and B. You've found the general solution.

4. The polynomial $s^2 + 7s + 12$ showed up in your work above. How are the zeros of this polynomial connected to the solution?

In the three examples above, we took an ODE $y'' + ay' + by = 0$, applied a Laplace transform, and obtained the polynomial $s^2 + as + b$. The zeros of this polynomial seem to be intimately connected to the solution. Let's give this polynomial a name.

Definition 5.3: Characteristic Polynomial (Equation). Consider the ODE $y'' + ay' + by = 0$.

- The characteristic polynomial is $s^2 + as + b$. We could alternately use $\lambda^2 + a\lambda + b$.

- The characteristic equation is $s^2 + as + b = 0$. We could alternately use $\lambda^2 + a\lambda + b = 0$.

With this new word, we now have the correct tool to discuss solving ODEs. We noticed a pattern in the first few problems. From that pattern, we developed a new word. Now we can use that word to simplify your solution techniques.

Problem 5.10 Consider the ODE $y'' + 9y' + 20y = 0$. What is the characteristic equation of the ODE? Find the zeros of the characteristic polynomial, and then state a general solution to the ODE.

The definition of characteristic equation allows us to alternately use the variable λ instead of s. The next problem connects what we are doing to eigenvalues.

Problem 5.11 Consider the ODE $y'' + 9y' + 20y = 0$ from the previous problem. If we let $y_1 = y$ and $y_2 = y'$, then we can write the ODE in the form $y_2' + 9y_2 + 20y_1 = 0$. This becomes the system of ODEs

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -20y_1 - 9y_2. \end{aligned}$$

Write the system above in the matrix form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then find the eigenvalues of A, and use them to obtain a solution to the ODE.

Let's now tackle a problem where the characteristic equation does not have real zeros.

Problem 5.12 Consider the ODE $y'' + 16y = 0$.

1. Compute the Laplace transform of both sides of the ODE and solve for $\mathscr{L}(y) = Y$. You'll have $y(0)$ and $y'(0)$ in the numerator of your solution.
2. Compute the inverse Laplace transform of Y. Your answer will involve $y(0)$ and $y'(0)$.
3. What is the characteristic polynomial, and what are its roots?
4. If a mass of 1 kg is attached to spring with modulus 16 kg/s^2 on a frictionless track, then graph the position $x(t)$ at any time t. [What's the corresponding ODE? Didn't you already solve this ODE?]

The previous problem showed us how to tackle a problem where the roots of the characteristic polynomial are purely imaginary. What do we do if the roots repeat, or if they are complex of the form $a \pm bi$? The next problem addresses this.

Problem 5.13 Consider the ODE $y'' + 6y' + 9y = 0$.

1. What are the zeros of the characteristic equation? From these zeros, guess a general solution. (It's OK if you're wrong.)
2. Compute the Laplace transform of both sides of ODE. Then solve for Y and show that
$$Y = \frac{A(s+3)+B}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2}.$$
3. Compute the inverse Laplace transform of each part that you are able to compute, and explain why we can't perform the inverse Laplace transform of the other parts.
4. Use a computer to complete the inverse Laplace transform, and state the solution.

Problem 5.14 Consider the ODE $y'' + 4y' + 13y = 0$.

1. What are the zeros of the characteristic equation?

2. Compute the Laplace transform of both sides of ODE. Then solve for Y and complete the square to show that
$$Y = \frac{A(s+2)+B}{(s+2)^2+3^2} = \frac{A(s+2)}{(s+2)^2+3^2} + \frac{B}{(s+2)^2+3^2}.$$

3. Use the fact that $\mathscr{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}$ and that $\mathscr{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}$ to finish solving. [The next problem will show you where these came from.]

In both of the preceding problems, we encountered expressions that we could not inverse transform. The first was $\dfrac{1}{(s+3)^2}$, and the last two were $\dfrac{(s+2)}{(s+2)^2+3^2}$ and $\dfrac{1}{(s+2)^2+3^2}$. In all cases, these look like shifted versions of functions for which we know the inverse Laplace transform. For example, we know $\mathscr{L}\{\cos 3t\} = \dfrac{s}{s^2+3^9}$. The expression $\dfrac{(s+2)}{(s+2)^2+3^2}$ resembles the expression $\dfrac{(s)}{(s)^2+3^2}$, rather we just replaced s with $s+2$, which is the same as shifting s left 2. We were told that $\mathscr{L}^{-1}\left\{\dfrac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)$. What we need is a Laplace transform rule that would allow us deal with s shifting. If we know how to invert $Y(s)$, how do we invert $Y(s-a)$?

Problem 5.15: The s-shifting Theorem In this problem you'll develop a rule for the inverse transform of $Y(s-a)$.

1. We know that $Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty e^{-st}[f(t)]dt$. Replace s with $s-a$ and obtain a formula
$$Y(s-a) = \int_0^\infty e^{-st}[?]dt.$$
This gives you a formula $\mathscr{L}\{?\} = Y(s-a)$.

2. What is the inverse Laplace transform of $1/s^2$? What is the inverse Laplace transform of $1/(s-4)^2$? What is the inverse Laplace transform of $1/(s+5)^2$?

3. What is the forward Laplace transform of $\cos(bt)$? What is the forward Laplace transform of $e^{at}\cos(bt)$? What is the forward Laplace transform of $e^{7t}t^3$ and $e^{-7t}t^3$?

[Hint: The s-shifting theorem is now in Table 5.1. Try to tackle this problem without referring to the table.]

To apply the s-shifting theorem, we'll need to become good at completing the square. If we know the transform is $\dfrac{2}{s^2+4}$, then the inverse transform is

$y(t)$	$Y(s)$	provided
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
y'	$sY - y(0)$	
y''	$s^2Y - sy(0) - y'(0)$	

$y(t)$	$Y(s)$	provided
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s > \|\omega\|$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s > \|\omega\|$
$y(t)$	$Y(s)$	
$e^{at}y(t)$	$Y(s-a)$	

Table 5.1: Table of Laplace Transforms

$\sin(2t)$. If we know the transform is $\frac{2}{(s+3)^2+4}$, then the inverse transform is $e^{-3t}\sin(2t)$. However, we would normally have a characteristic polynomial in the form $s^2+6s+13$, rather than the form $(s+3)^2+4$. Once we complete the square, we can apply the s-shifting theorem.

Problem 5.16 Complete each of the following:

1. Consider the ODE $y''+2y'+5y=0$. Find the characteristic polynomial, complete the square, and state a general solution.
2. Consider the ODE $y''+6y'+9y=0$. Find the characteristic polynomial, complete the square, and state a general solution.
3. Consider the ODE $y''+4y'+3y=0$. Find the characteristic polynomial, complete the square, and state a general solution.

Before we get to far, let's practice the s shifting theorem for Laplace transforms.

Problem: 5.16 and 1/2 Complete the following:

1. Find the Laplace transform of the following:
   (a) t^3 and t^3e^{4t}
   (b) $\cos(2t)$ and $e^{-3t}\cos(2t)$
   (c) $3\sin(7t)$ and $3e^{-5t}\sin(7t)$
2. Find the inverse Laplace transform of the following:
   (a) $\frac{3}{s^4}$ and $\frac{3}{(s-5)^4}$
   (b) $\frac{s+3}{(s+3)^2+4}$ and $\frac{1}{(s+3)^2+4}$
   (c) $\frac{s}{(s+3)^2+4}$.

Problem 5.17 Consider the ODE $y'' + 6y' + 11y = 0$.

1. Find the characteristic polynomial, complete the square, and then state a general solution.
2. Find the characteristic equation, use the quadratic formula to solve the characteristic equation, and then state a general solution.
3. Solve the ODE $y'' + 5y' + 12y = 0$. Would you rather complete the square, or use the quadratic formula?

Problem 5.18 Consider the ODE $ay'' + by' + cy = 0$.

1. Obtain the characteristic equation. Complete the square. State the zeros of the characteristic equation. [When you finish this problem, you will have proved the quadratic formula.]
2. If we let $y_1 = y$ and $y_2 = y'$, we obtain the system of ODE $y_1' = y_2$ and $ay_2' + by_2 + cy_1 = 0$. Write this system in the matrix form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and obtain the eigenvalues of A.

We can now solve EVERY second order homogeneous constant coefficient ODE. All we have to do is find the characteristic equation. The zeros unlock a general solution of the ODE.

Problem 5.19 Consider the second order homogeneous constant coefficient ODE $y'' + ay' + by = 0$. Let λ_1 and λ_2 be the roots of the characteristic polynomial $s^2 + as + b$.

1. If the roots are real and $\lambda_1 \neq \lambda_2$, then $y(t) =$_______________.
2. If the roots are real and $\lambda_1 = \lambda_2$, then $y(t) =$_______________.
3. If the roots are complex where $\lambda = c \pm di$, then $y(t) =$_______________.
   If $c = 0$, then the solution is simply $y(t) =$_______________.

Problem: 5.19 Improved Suppose that we have a second order ODE, and we have already computed the roots of the characteristic polynomial to be λ_1 and λ_2.

1. If $\lambda_1 = -3$ and $\lambda_2 = -5$, then $y(t) =$_______________.
   If the roots are real and $\lambda_1 \neq \lambda_2$, then $y(t) =$_______________.
2. If $\lambda_1 = -3$ and $\lambda_2 = -3$, then $y(t) =$_______________
   If the roots are real and $\lambda_1 = \lambda_2$, then $y(t) =$_______________.
3. If $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$, then $y(t) =$_______________.
   If the roots are complex where $\lambda = a \pm bi$, then $y(t) =$_______________.
4. If $\lambda_1 = 5i$ and $\lambda_2 = -5i$, then $y(t) =$_______________.
   If the roots are purely imaginary so that $\lambda = bi$, then $y(t) =$_______________.

Have you noticed that every general solution above is a linear combination of two independent solutions? Recall that we say a differential operator is linear if $L(y_1 + y_2) = L(y_1) + L(y_2)$ and $L(cy_1) = cL(y_1)$ for functions y_1 and coefficients c.

Problem 5.20: Superposition Principle Suppose that y_1 and y_2 are both solutions to a linear differential equation $ay'' + by' + cy = 0$. Consider the linear operator $L(y) = ay'' + by' + cy$. Prove that any linear combination of y_1 and y_2 is also a solution to the ODE $L(y) = 0$. (Hint: Look at the last few problems in chapter 3, or just prove this directly.)

Many people refer to this fact as the superposition principle. To get a solution to a second order homogeneous ODE, all you need is two independent solutions. The general solution is any linear combination of them.

Now that we have a general solution, let's show how to quickly obtain the solution to an IVP. The key principle, is to first obtain a general solution. Differentiate your general solution, and then use your initial conditions to find the unknown constants.

Problem 5.21 Consider the IVP $y'' + 6y' + 5y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Plug the initial conditions into both y and y' to solve for the unknown constants in your general solution.

Problem 5.22 Consider the IVP $y'' + 6y' + 9y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Use the initial conditions to solve for the unknown constants in your general solution.

Problem 5.23 Consider the IVP $y'' + 2y' + 5y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Use the initial conditions to solve for the unknown constants in your general solution.

5.3 Mass-Spring Systems

Recall from the introductory examples that we can model the position of a spring using the ODE

$$mx'' + cx' + kx = 0$$

The constants m, c, and k are physical constants related to the mass-spring system.

- The mass of the object attached to the spring is m.
- The spring constant, or modulus, is k.
- The coefficient of friction of any attached dashpot is c. If no dashpot is attached, then we just let $c = 0$.

Problem 5.24 Suppose we attach a mass of 4 kg to a spring with modulus 12 kg/s^2. We displace the object 1 cm from the equilibrium position of the spring, and then hit the mass with a hammer. The impact causes the spring's initial velocity to be 3 cm/s back towards equilibrium. Use this information to determine the position of the spring at any time t. Construct a graph of the position. From your graph, show how you can the initial position and initial velocity.

Make sure you ask me in class to show you how the solution above graphically changes, if we alter the initial position and initial velocity.

Problem 5.25 Suppose we attach a mass of m kg to a spring with modulus k kg/s^2. We displace the object y_0 cm from the equilibrium position of the spring, and give the object an initial velocity of v_0 cm/s away from equilibrium. In the absence of friction, the mass-spring system will oscillate in a regular pattern. Determine the position of the spring at any time t. What is the period of oscillation? If you doubled the spring constant k, how would it affect the period?

Problem 5.26 Suppose we attach a mass of m kg to a spring with modulus k kg/s^2. We displace the object y_0 cm from the equilibrium position of the spring, and give the object an initial velocity of v_0 cm/s away from equilibrium. In the absence of friction, the mass-spring system will oscillate in a regular pattern. Give a formula for the amplitude of the oscillation. [Hint: If you write your solution in the form $y(t) = C\sin(\omega t + \phi)$, then you can quickly read off the amplitude. How do you write $y(t) = A\cos(bt) + B\sin(bt)$ in the form $C\sin(\omega t + \phi)$?]

Each of the problems above dealt with undamped motion, there was no friction to slow down the motion. The remaining problems include a dashpot, something placed around the mass-spring system that adds friction to the system. Wikipedia has some excellent pictures of what a dashpot could look like. I like to think of an old screen door, and the cylindrical tube at the bottom of the door that helps close the door and prevent it from smashing closed on little fingers. Ask me in class to show you a dashpot on our classroom door.

Problem 5.27 Recall from the introductory examples that we can model the position of a spring using the ODE $mx'' + cx' + kx = 0$. We now attach a mass of 1 kg to a spring. The spring is placed inside a dashpot, to add friction to the system, and the dashpot has a coefficient of friction equal to $c = 8$ kg/s. The spring is rather large, so we extend it 1 m and then release it with no initial velocity.

1. If the spring modulus is $k = 15$ kg/s^2, find the position $x(t)$ of the spring, and construct a rough sketch of x versus t.
2. If the spring modulus is $k = 16$ kg/s^2, find the position $x(t)$ of the spring, and construct a rough sketch of x versus t.
3. If the spring modulus is $k = 17$ kg/s^2, find the position $x(t)$ of the spring, and construct a rough sketch of x versus t.
4. What connection is there between c and k? If you had to describe what you saw in the examples above to someone not in this class, what would you say? You'll probably have to explain this phenomenon to a boss someday.

5.4 Higher Order ODEs

In the previous sections, we focused mainly on second order ODEs. We started by using Laplace transforms to find the exact solutions. The accompanying partial fraction decomposition was sometimes rather ugly, so we opted for guessing the form of the solution, and then taking derivatives to determine the unknown constants.

Problem 5.28 Consider the ODE $y''' + 3y'' + 3y' + y = 0$. Compute the Laplace transform of both sides. The characteristic equation is $(s + 1)^3 = 0$. Explain why the solution is $y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}$.

For the ODE $y'''' + 4y''' + 6y'' + 4y' + y = 0$, whose characteristic equation is $(s + 1)^4 = 0$, make a guess as to the solution. Then use a computer and dsolve to check that your answer is correct. (Wolfram Alpha can solve this.)

If you encounter a repeated root, what does that contribute to the solution? Explain this in a way that you and other can remember it.

Problem 5.29 You have a 6th order homogeneous ODE, and the characteristic equation factors as $(s^2 + 4)(s^2 + 9)^2 = 0$. What is the original ODE (expand the polynomial)? The roots are $\pm 2i, \pm 3i, \pm 3i$ (so $\pm 3i$ are repeated roots). Guess the general solution. Then use a computer to check if your guess was correct.

Problem 5.30 In each problem below, you'll be given the characteristic equation of an ODE. State the general solution of the ODE.

1. $(s + 3)(s + 2)(s + 1) = 0$
2. $(s + 3)(s + 3)(s + 1) = 0$
3. $(s + 3)^3(s^2 + 9) = 0$
4. $(s + 3)^2(s^2 + 9)^2 = 0$
5. $(s + 3)^2(s^2 - 9)^2 = 0$ (Note the sign change)

5.5 Existence and Uniqueness - the Wronskian

This section will be added to soon.

5.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 6

Non Homogeneous ODEs

This chapter covers the following ideas.

1. Explain Hooke's Law in regards to mass-spring systems, where there is an external force. Construct and solve differential equations which represent this physical model, with or without the presence of a damper and be able to interpret how solutions change based on changes in the model.

2. Understand the theory which relates solutions of homogeneous linear ODE's to non homogeneous ODEs.

3. Use the method of undetermined coefficients to solve non homogeneous linear ODEs.

4. Explain Kirchhoff's voltage law, Ohm's law, and how to model electrical circuits using 2nd order non homogeneous linear ODEs. Illustrate how results about circuits can be translated into results about mass-spring systems.

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the preparation problem set, please use this list as a guideline to find related practice problems.

Concept	Sec	Suggestions	Relevant Problems
Theory	8	21,65	21-23,65-67
Undetermined Coef	11	1,2,3,8,10,24,26,34,36,41,46,47,48	All
IVP	13	1,7,14	1,3,7,8,10,11,14
Applications	7	19,76	19-22,71-81
Applications	14	10,11,13,14,17,46,50,51,52,54,57	9-18,44-65