Differential Equations with Linear Algebra

Ben Woodruff[1]

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[1]Mathematics Faculty at Brigham Young University–Idaho, woodruffb@byui.edu

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Chapter 1

Review

This chapter covers the following ideas.

1. Graph basic functions by hand. Compute derivatives and integrals, in particular using the product rule, quotient rule, chain rule, integration by u-substitution, and integration by parts (the tabular method is useful for simplifying notation). Explain how to find a Laplace transform.

2. Explain how to verify a function is a solution to an ODE, and illustrate how to solve separable ODEs.

3. Explain how to use the language of functions in high dimensions and how to compute derivatives using a matrix. Illustrate the chain rule in high dimensions with matrix multiplication.

4. Graph the gradient of a function together with several level curves to illustrate that the gradient is normal to level curves.

5. Explain how to test if a differential form is exact (a vector field is conservative) and how to find a potential.

1.1 Basics

We need to review our ability to graph functions with multiple inputs and/or outputs. The next few problems ask you to practice some skills that will be crucial as the course progresses.

Problem 1.1 Construct graphs of the following functions. Explain how to obtain each graph by transforming and rescaling the first. Then state the amplitude and period of the function.

1. $y = \sin(x)$
2. $y = 5\sin(x) + 1$
3. $y = 4\sin(3(x - \pi)) + 2$
4. $y = 4\sin(3x - \pi) + 2$

Problem 1.2 Consider the function $f(x) = e^{-x}$.

1. Construct graphs of $y = f(x)$ and $y = 2f(-(x + 3)) - 1$.

2. State $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ from your graph.
3. Compute $\lim_{x\to\infty} xf(x)$ and $\lim_{x\to\infty} x^2 f(x)$. [Hint: L'Hopital's rule will help.]

As the semester progresses, we'll need the functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are the hyperbolic trig functions, and we say the hyperbolic sine of x when we write $\sinh x$. These functions are very similar to sine and cosine functions, and have very similarly properties.

$\boxed{\textbf{Problem 1.3}}$ Three useful facts about the trig functions are (1) $\frac{d}{dx}\sin x = \cos x$, (2) $\frac{d}{dx}\cos x = -\sin x$, and (3) $\cos^2 x + \sin^2 x = 1$. Use the definitions above to show the following:

1. $\frac{d}{dx}\sinh x = \cosh x$,
2. $\frac{d}{dx}\cosh x = \sinh x$, and
3. $\cosh^2 x - \sinh^2 x = 1$.

[Hint: Start by replacing the hyperbolic function with its definition in terms of exponentials. Then perform the computations.]

$\boxed{\textbf{Problem 1.4}}$ The three facts from the previous problem are crucial tools need to prove that $\frac{d}{dx}\tan x = \sec^2 x$.

1. Use the quotient rule to give a formula for $\frac{d}{dx}\tanh x$ in terms of hyperbolic trig functions.
2. Similarly obtain a formula for the derivative of $\operatorname{sech} x = \frac{1}{\cosh x}$.
3. What is $\frac{d}{dx}\operatorname{csch} x$?

You might ask why these function are called the hyperbolic trig functions. What does a hyperbola have to do with anything?

$\boxed{\textbf{Problem 1.5}}$ Each pair of parametric equations traces out a curve in the xy plane. Given a Cartesian equation of the curve by eliminating the parameter t, and then graph the curve.

1. $x = \cos t$, $y = \sin t$, $-2\pi < t < 2\pi$.
2. $x = \cosh t$, $y = \sinh t$, $-\infty < t < \infty$.

Give a reason as to why do we call cosh the hyperbolic cosine.

$\boxed{\textbf{Problem 1.6}}$ Use implicit differentiation to find the derivative of $y = sinh^{-1}x$. Your answer should not involve any hyperbolic trig functions, and should be in terms of x. [Hint: First write $x = sinh(y)$, and then implicitly differentiate both sides. You'll need the key identity from a few problems above to help you finish.]

The problems above asked you to review your differentiation skills. You'll want to make sure you can use the basic rules of differentiation (such as the power, product, quotient, and chain rules). The next few problems will help you review your integration techniques, and you will apply them to two new ideas.

Problem 1.7 Compute the three integrals

$$\int xe^{-x^2}dx \qquad \text{and} \qquad \int_0^1 xe^{-x^2}dx \qquad \text{and} \qquad \int_0^\infty xe^{-x^2}dx.$$

If you have never used the tabular method to perform integration-by-parts, I strongly suggest that you open the online text and read a few examples (see the bottom of page 2).

Problem 1.8 Compute $\displaystyle\int x\sin(5x)dx$ and $\displaystyle\int x^2\sin(5x)dx$.

Problem 1.9 Compute $\displaystyle\int \tanh^{-1} x dx$. The derivative of $\tanh^{-1} x$ is $\dfrac{1}{1-x^2}$.

1.2 Laplace Transforms

Definition 1.1: The Laplace Transform. Let $f(t)$ be a function that is defined for all $t \geq 0$. Using the function $f(t)$, we define the Laplace transform of f to be a few function F where for each s we obtain the value by computing the integral

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt.$$

The domain of F is the set of all s such that the improper integral above converges. The function $f(t)$ is called the inverse Laplace transform of $F(s)$, and we write $f(t) = \mathscr{L}^{-1}(F(s))$.

Note that the Laplace transform of a function with independent variable t is another function with a different independent variable s. After integration, all t's will be removed from $F(s)$. You can of course use any other letters besides t and s.

We will use the Laplace transform throughout the semester to help us solve many problems related to mechanical systems, electrical networks, and more. The mechanical and electrical engineers in this course will use Laplace transforms in many future courses. Our goal in the problems that follow is to practice integration-by-parts. As an extra bonus, we'll learn the Laplace transforms of some basic functions.

Problem 1.10 Compute the integral $\displaystyle\int_0^\infty e^{-st}dt$, and state for which s the integral converges. What is the Laplace transform of $f(t) = 1$? (If the last question seems redundant, then horray.)

Problem 1.11 Compute the Laplace transform of $f(t) = e^{2t}$, and state the domain. Then compute the Laplace transform of $f(t) = e^{3t}$ and state the domain. Finally, compute the Laplace transform of $f(t) = e^{at}$ for any a, and state the domain.

Problem 1.12 Suppose $s > 0$ and n is a positive integer. Explain why

$$\lim_{t\to\infty} \frac{t^n}{e^{st}} = 0.$$

Use this fact to prove that the Laplace transform of t^2 is

$$\mathscr{L}\{t^2\}\frac{2}{s^3}.$$

[You'll need to do integration-by-parts twice, try the tabular method.]

Problem 1.13 In the previous problems, you showed that

$$\mathscr{L}\{t^0\} = \frac{1}{s^1} \qquad \text{and} \qquad \mathscr{L}\{t^2\} = \frac{2}{s^3}.$$

Show that the Laplace transform of t is $\mathscr{L}\{t^1\} = \frac{1}{s^2}$. Then compute the Laplace transforms of t^3, t^4, and so on until you see a pattern. Use this pattern to state the Laplace transform of t^10 and t^n, provided n is a positive integer. [Hint: Try the tabular method of integration-by-parts. After evaluating at 0 and ∞, all terms but 1 will be zero.]

Theorem 1.2. *Since integration can be done term-by-term, and constants can be pulled out of the integral, we have the crucial fact that*

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$$

for functions f, g and constants a, b.

Problem 1.14 Without integrating, rather using the results above, compute the Laplace transform $L(3 + 5t^2 - 6e^{8t})$, and state the domain.

Problem 1.15 Recall that $\cosh t = \frac{e^t + e^{-t}}{2}$ and $\sinh t = \frac{e^t - e^{-t}}{2}$. Use this to prove that

$$\mathscr{L}\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad \text{and} \qquad \mathscr{L}\{\sinh at\} = \frac{a}{s^2 - a^2}.$$

1.3 Ordinary Differential Equations

A differential equation is an equation which involves derivatives (of any order) of some function. For example, the equation $y'' + xy' + \sin(xy) = xy^2$ is a differential equation. An **ordinary differential equation (ODE)** is a differential equation involving an unknown function y which depends on only one independent variable (often x or t). A partial differential equation involves an unknown function y that depends on more than one variable (such as $y(x,t)$). The order of an ODE is the order of the highest derivative in the ODE. A solution to an ODE on an interval (a, b) is a function $y(x)$ which satisfies the ODE on (a, b).

Example 1.3. The first order ODE $y'(x) = 2x$, or just $y' = 2x$, has unknown function y with independent variable x. A solution on $(-\infty, \infty)$ is the function $y = x^2 + C$ for any constant C. We obtain this solution by simply integrating both sides. Notice that there are infinitely many solutions to this ODE.

Typically a solution to an ODE involves an arbitrary constant C. There is often an entire family of curves which satisfy a differential equation, and the constant C just tells us which curve to pick. A **general solution** of an ODE is an infinite class of solutions of the ODE. A **particular solution** is one of the infinitely many solutions of an ODE.

Often an ODE comes with an **initial condition** $y(x_0) = y_0$ for some values x_0 and y_0. We can use these initial conditions to find a particular solution of the ODE. An ODE, together with an initial condition, is called an **initial value problem (IVP)**.

Example 1.4. The IVP $y' = 2x$, $y(2) = 1$, has the general solution $y = x^2 + C$ from the previous problem. Since $y = 1$ when $x = 2$, we have $1 = 2^2 + C$ which means $C = -3$. Hence the solution to our IVP is $y = x^2 - 3$.

Problem 1.16 Consider the ordinary differential equation $y'' + 9y = 0$. By computing derivatives, show that $y(t) = A\cos(3t) + B\sin(3t)$ is a general solution to the ODE, where A and B are arbitrary constants. If we know that $y(0) = 1$ and $y'(0) = 2$, determine the values of A and B.

Problem 1.17 Consider the ordinary differential equation $y\dfrac{dy}{dx} = x^2$. Find a general solution to this ODE by integrating both sides with respect to x. State an interval on which your solution is valid.

They could introduce the entire method of separation by parts without me telling them what to do. I just need to ask them to do an integral. Afterward, I could ask them to solve an ODE. Put it in the same problem.

Problem 1.18 Consider the ODE given by $y' = 4ty$. Find a general solution to this ODE. [Hint: Rewrite y' as $\dfrac{dy}{dt}$. Then put all the terms that involve y on one side of the equation, and the terms that involve t on the other. Then it should be similar to the previous problem.]

Problem 1.19 Solve the IVP given by $y' = \dfrac{x^2 - 1}{y^4 + 1}$, where $y(0) = 1$.

1.4 General Functions and Derivatives

Recall that to compute partial derivatives, we hold all but one variable constant and then differentiate with respect to that variable. Partial derivatives can be organized into a matrix Df where columns represents the partial derivative of f with respect to each variable. This matrix, called the derivative or total derivative, takes us into our study of linear algebra. Some examples of functions and their derivatives appear in Table 1.1. When the output dimension is one, the matrix has only one row and the derivative is often called the gradient of f, written ∇f.

In multivariate calculus, we focused our time on learning to graph, differentiate, and analyze each of the types of functions in the table above. The next few problems ask you to review this.

Function	Derivative
$f(x) = x^2$	$Df(x) = [2x]$
$\vec{r}(t) = (3\cos(t), 2\sin(t))$	$D\vec{r}(t) = \begin{bmatrix} -3\sin t \\ 2\cos t \end{bmatrix}$
$\vec{r}(t) = (\cos(t), \sin(t), t)$	$D\vec{r}(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$
$f(x, y) = 9 - x^2 - y^2$	$Df(x, y) = \nabla f(x, y) = [-2x \quad -2y]$
$f(x, y, z) = x^2 + y + xz^2$	$Df(x, y, z) = \nabla f(x, y, z) = \begin{bmatrix} 2x + z^2 & 1 & 2xz \end{bmatrix}$
$\vec{F}(x, y) = (-y, x)$	$D\vec{F}(x, y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\vec{F}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$	$D\vec{F}(r, \theta, z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\vec{r}(u, v) = (u, v, 9 - u^2 - v^2)$	$D\vec{r}(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & -2v \end{bmatrix}$

Table 1.1: The table above shows the (matrix) derivative of various functions. Each column of the matrix corresponds a partial derivative of the function. When the output of a function is a vector, partial derivatives are vectors which are placed in columns of the matrix. The order of the columns matches the order in which you list the variables.

Problem 1.20 Let $\vec{r}(t) = \left\langle t^2 - 1, 2t + 3 \right\rangle$. Construct a graph of $\vec{r}(t)$, and compute the derivative $D\vec{r}(t)$.

Problem 1.21 Let $f(x, y) = 4 - x^- y^2$. Construct a 3D graph of $z = f(x, y)$. Also construct a graph of several level curves. Then compute the derivative $Df(x, y)$.

Recall that a level curve of $z = f(x, y)$ is curve in the xy plane where the output z is constant.

Problem 1.22 Let $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$. Construct a 3D graph of $\vec{r}(t)$, and compute the derivative $D\vec{r}(t)$.

Problem 1.23 Let $\vec{F}(x, y) = (y, -2x)$. Construct a 2D graph of this vector field, and compute the derivative $D\vec{F}(x, y)$.

1.4.1 The General Chain Rule

The chain rule in first semester calculus states that

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

You may remember this as "the derivative of the outside function times the derivative of the inside function." In multivariable calculus, most textbooks use a tree rule to develop the formula

$$\frac{df}{dt} = f_x x_t + f_y y_t$$

for a function $f(x, y)$, where x and y depend on t (so that $\vec{r}(t) = (x(t), y(t))$ is a curve in the xy plane). Written in matrix form, the chain rule is simply

$$\frac{df}{dt} = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = Df \cdot Dr,$$

which is the (matrix) product of the derivatives, just as it was in first semester calculus. You are welcome to tackle the following problems by using the tree rule or matrix product.

Problem 1.24 Suppose that $f(x, y) = x^2 + 3xy$, where $x = t^2 + 1$ and $y = \sin t$, so we could write $\vec{r}(t) = (t^2 + 1, \sin t)$.

1. Compute $Df(x, y)$, $\dfrac{dx}{dt}$, and $D\vec{r}(t)$. (You should have two matrices.)
2. Compute $\frac{df}{dt}$.

Problem 1.25 Suppose that $f(x, y) = x + 3y$ and that $\dfrac{dx}{dt} = \cos t$ and $\dfrac{dy}{dt} = e^t$. Compute $\frac{df}{dt}$.

Problem 1.26 Suppose that $z = f(x, y)$ and that $\dfrac{\partial f}{\partial x} = 3x^2 y$ and $\dfrac{\partial f}{\partial y} = x^3 y - e^y$. Also suppose that $x = \sqrt{t}$ and $y = \ln t$. Compute $\frac{df}{dt}$.

Problem 1.27 Suppose that $z = f(x, y)$ is a differential function of two variables. Suppose that $\vec{r}(t)$ is a parametrization of a level curve of f. We can write the level curve in vector form as $\vec{r}(t) = (x(t), y(t))$, or in parametric form $x = x(t)$ and $y = y(t)$.

1. If $f(\vec{r}(0)) = 7$, then what is $f(\vec{r}(2))$?
2. Why does $\frac{df}{dt} = \nabla f(x, y) \cdot \dfrac{d\vec{r}}{dt}$?
3. Why is the gradient of f normal to level curves?

Recall that the word normal means there is a 90 degree angle between the gradient and the level curve.

Before proceeding, let's practice with an examples to visually remind us that the gradient is normal to level curves. This key fact will help us solve most of the differential equations we encounter in the course.

Problem 1.28 Consider the function $f(x, y) = x^2 - y$. Start by computing the gradient. Then construct a graph which contains several level curves of f, as well as the gradient at several points on each level curve.

1.5 Potentials of Vector Fields and Differential Forms

When the output dimension of a function is one, so we would write $f : \mathbb{R}^n \to \mathbb{R}^1$, then we call the derivative the gradient and write $\vec{\nabla} f = (f_x, f_y, f_z)$. Notice that this is a vector field. Taking a derivative gives us a vector field. Is every vector field the derivative of some function? Hopefully you remember that the answer to this question is "No."

If a vector field $\vec{F} = (M, N)$ (or in 3D $\vec{F} = (M, N, P)$) is the gradient of some some function f (so that $\vec{\nabla} f = \vec{F}$), then we say that the vector field $\vec{F}$ is a gradient field (or conservative vector field). We say that f is a potential for the vector field $\vec{F}$ when $\nabla f = \vec{F}$. In this section, we'll review how to determine if a vector field has a potential, as well as how to find a potential.

$\boxed{\textbf{Problem 1.29}}$ Let $\vec{F} = (M, N) = (2x + y, x + 4y)$. Find a potential for $\vec{F}$ by doing the following.

1. If we suppose $M = 2x + y$ is the partial of f with respect to x, then $f_x = 2x + y$. Find a function f whose partial with respect to x is M.
2. If we suppose $N = x + 4y$ is the partial of f with respect to y, then $f_y = x + 4y$. Find a function f whose partial with respect to y is N.
3. What is a potential for $\vec{F}$? Prove your answer is correct by computing the gradient of your answer.

By taking derivatives, there is a test that tells you if a function will have a potential. Some textbooks call it the test for a conservative field.

$\boxed{\textbf{Problem 1.30: Test for a conservative vector field.}}$ Let's prove the test for a conservative vector field in both 2 and 3 dimensions.

The test for a conservative vector field states states more than what you showed in this problem. It states that if $\vec{F}$ is a continuously differentiable vector field on a simply connected domain, then (1) if $\vec{F}$ has potential, then certain pairs of partials must be equal, and (2) if those pairs of partial derivatives are equal, then the $\vec{F}$ has a potential. We will not prove part (2).

1. Suppose that $\vec{F}(x, y) = (M, N)$ is a continuously differentiable vector field on the entire plane. Suppose further that $\vec{F}$ has a potential f. The derivative of $\vec{F}$ is
$$D\vec{F}(x, y) = \begin{pmatrix} M_x & M_y \\ N_x & N_y \end{pmatrix}.$$
Some of the entries in this matrix must be equal? Which ones? Explain. [If you're not sure, try taking the derivative of the problem above.]
2. Suppose that $\vec{F}(x, y, z) = (M, N, P)$ is a continuously differentiable vector field on all of space. Suppose further that $\vec{F}$ has a potential f. State the derivative of $\vec{F}$, and then state which pairs of entries must be equal.

$\boxed{\textbf{Problem 1.31}}$ For each vector field below, either give a potential, or explain why no potential exists.

1. $\vec{F} = (4x + 5y, 5x + 6y)$
2. $\vec{F} = (2x - y, x + 3y)$
3. $\vec{F} = \left(4x + \dfrac{2y}{1 + 4x^2}, \arctan(2x)\right)$
4. $\vec{F} = (3y + 2yz, 3x + 2xz + 6z, 2xy + 6y)$

We'll finish by introducing the vocabulary of differential forms. We'll use this vocabulary throughout the semester as we study differential equations. The vocabulary of vector fields parallels the vocabulary of differential forms.

Definition 1.5: Differential Forms. Assume that f, M, N, P are all functions of three variables x, y, z. Similar definitions hold in all dimensions.

- A differential form is an expression of the form $Mdx + Ndy + Pdz$ (just as a vector field is a function $\vec{F} = (M, N, P)$).
- The differential of a function f is the expression $df = f_x dx + f_y dy + f_z dz$ (just as the gradient is $\vec{\nabla}F = (f_x, f_y, f_z)$).
- If a differential form is the differential of a function f, then the differential form is said to be exact (just as we say a vector field is a gradient field). Again, the function f is called a potential for the differential form.

A differential form is exact precisely when the corresponding vector field is a gradient field.

Notice that $Mdx + Ndy + Pdz$ is exact if and only if $\vec{F} = (M, N, P)$ is a gradient field. The language of differential forms is practically the same as the language of conservative vector fields. Why do we have different sets of words for the same idea? That happens all the time when different groups of people work on seeming different problems, only to discover years later that they have been working on the same problem. If both sets of vocabulary stick, it's often because both have advantages. We have many different notations for the derivative (such as y', $\frac{dy}{dx}$, and Df), and each notation has advantages. The language of differential forms is best suited when studying differential equations.

Problem 1.32 For each differential form below, state if the differential form is exact. If it is exact, give a potential.

1. $(2x + 3y)dx + (4x + 5y)dy$
2. $(2x - y)dx + (3y - x)dy$
3. $\left(4x + \dfrac{3y}{1 + 9x^2}\right) dx + \arctan(3x)dy$
4. $(3y + 2yz)dx + (3x + 2xz + 6z)dy + (2xy + 5y)dz$

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Linear Algebra Arithmetic

This chapter covers the following ideas.

1. Be able to use and understand matrix and vector notation, addition, scalar multiplication, the dot product, matrix multiplication, and matrix transposing.
2. Use Gaussian elimination to solve systems of linear equations. Define and use the words homogeneous, nonhomogeneous, row echelon form, and reduced row echelon form.
3. Find the rank of a matrix. Determine if a collection of vectors is linearly independent. If linearly dependent, be able to write vectors as linear combinations of the preceding vectors.
4. For square matrices, compute determinants, inverses, eigenvalues, and eigenvectors.
5. Illustrate with examples how a nonzero determinant is equivalent to having independent columns, an inverse, and nonzero eigenvalues. Similarly a zero determinant is equivalent to having dependent columns, no inverse, and a zero eigenvalue.

The next unit will focus on applications of these ideas. The main goal of this unit is to familiarize yourself with the arithmetic involved in linear algebra.

2.1 Basic Notation

Most of linear algebra centers around understanding vectors, with matrices being functions which transform vectors from one vector space into vectors in another vector space. This chapter contains a brief introduction to the arithmetic involved with matrices and vectors. The next chapter will show you many of the uses of the ideas we are learning. You will be given motivation for all of the ideas learned here, as well as real world applications of these ideas, before the end of the next chapter. For now, I want you become familiar with the arithmetic of linear algebra so that we can discuss how all of the ideas in this chapter show up throughout the course.

Definition 2.1. A matrix of size m by n has m rows and n columns. We

Matrix size is row by column.

normally write matrices using capital letters, and use the notation

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{jk}],$$

where a_{jk} is the entry in the jth row, kth column.

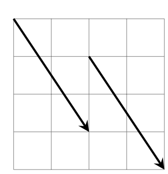
- We say two matrices A and B are equal if $a_{jk} = b_{jk}$ for all j and k.
- We add and subtract matrices of the same size entry wise. So we write $A + B = C$ where $c_{jk} = a_{jk} + b_{jk}$. If matrices do not have the same size, then we cannot add them.
- We can multiply a matrix A by a scalar C to obtain a new matrix cA. We do this multiplying every entry in the matrix A by the scalar c.
- If the number of rows and columns are equal, then we say the matrix is square.
- The main diagonal of a square ($n \times n$) matrix consists of the entries $a_{11}, a_{22}, \ldots, a_{nn}$.
- The trace of a square matrix is the sum of the entries on the main diagonal ($\sum a_{jj}$).
- The transpose of a matrix $A = [a_{jk}]$ is a new matrix $B = A^T$ formed by interchanging the rows and columns of A, so that $b_{jk} = a_{kj}$. If $A^T = A$, then we say that A is symmetric.

Problem 2.1 Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$. Compute $2A - 3B$, and find the trace of both A and B.

Problem 2.2 Write down a 3 by 2 matrix, and compute the transpose of that matrix. Then give an example of a 3 by 2 symmetric matrix, or explain why it is not possible.

Vectors represent a magnitude in a given direction. We can use vectors to model forces, acceleration, velocity, probabilities, electronic data, and more. We can use matrices to represent vectors. A row vector is a $1 \times n$ matrix. A column vector is an $m \times 1$ matrix. Textbooks often write vectors using bold face font. By hand (and in this book) we add an arrow above them. The notation $\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$ can represent either row or column vectors. Many different ways to represent vectors are used throughout different books. In particular, we can represent the vector $\langle 2, 3 \rangle$ in any of the following forms

$$\langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j} = (2, 3) = \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



Both vectors represent $\langle 2, -3 \rangle$, regardless of where we start.

The notation $(2, 3)$ has other meanings as well (like a point in the plane, or an open interval), and so when you use the notation $(2, 3)$, it should be clear from the context that you are working with a vector. To draw a vector $\langle v_1, v_2 \rangle$, one option is to draw an arrow from the origin (the tail) to the point (v_1, v_2) (the head). However, the tail does not have to be placed at the origin.

The principles of addition and subtraction of matrices apply to vectors (which can be though of as row or column matrices). We will most often think of vectors as column vectors.

Definition 2.2. The magnitude (or length) of the vector $\vec{u} = (u_1, u_2)$ is $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$. In higher dimensions we extend this as

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2 + \cdots u_n^2} = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

A unit vector is a vector with length 1. In many books unit vectors are written with a hat above them, as $\hat{\mathbf{u}}$.

A unit vector $\hat{\mathbf{u}}$ has length $|\vec{u}| = 1$

We will need to be able to find vectors of any length that point in a given direction.

Problem 2.3 Find a vector of length 12 that points in the same direction as the vector $\vec{v} = (1, 2, 3, 4)$. Then give a general formula for finding a vector of length c that points in the direction of $\vec{u}$.

The simplest vectors in 2D are a one unit increment in either the x or y direction, and we write these vectors in any of the equivalent forms

$$\mathbf{i} = \vec{i} = \langle 1, 0 \rangle = (1, 0) \quad \text{and} \quad \mathbf{j} = \vec{j} = \langle 0, 1 \rangle = (0, 1).$$

We call these the standard basis vectors in 2D. In 3D we include the vector $\mathbf{k} = \vec{j} = \langle 0, 0, 1 \rangle$ as well as add a zero to both $\vec{i}$ and $\vec{j}$ to obtain the standard basis vectors. The word basis suggests that we can base other vectors on these basis vectors, and we typically write other vectors in terms of these standard basis vectors. Using only scalar multiplication and vector addition, we can obtain the other vectors in 2D from the standard basis vectors.

The standard basis vectors in 3D
$\mathbf{i} = \vec{i} = \langle 1, 0, 0 \rangle = (1, 0, 0)$
$\mathbf{j} = \vec{j} = \langle 0, 1, 0 \rangle = (0, 1, 0)$
$\mathbf{k} = \vec{k} = \langle 0, 0, 1 \rangle = (0, 0, 1)$

Problem 2.4 Write the vector $(2, 3)$ in the form $(2, 3) = c_1\vec{i} + c_2\vec{j}$.

If instead we use the non-standard basis vectors $\vec{u}_1 = (1, 2)$ and $\vec{u}_2 = (-1, 4)$, then write the vector $(2, 3)$ in the form $(2, 3) = c_1\vec{u}_1 + c_2\vec{u}_2$.

Definition 2.3. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an expression of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$, where c_i is a constant for each i.

A linear combination of vectors is simply a sum of scalar multiples of the vectors. We start with some vectors, stretch each one by some scalar, and then sum the result. Much of what we will do this semester (and in many courses to come) relates directly to understanding linear combinations.

Problem 2.5 The force acting on an object is $\vec{F} = (-3, 2)$ N. The object is in motion and has velocity vector $\vec{v} = (1, 1)$ and acceleration vector $\vec{a} = (-1, 2)$. Write the force as a linear combination of the velocity and acceleration vectors.

Problem 2.6 Write the vector $(2, 3, 1)$ as a linear combination of the standard basis vectors in $\mathbb{R}^3$. Then write $(2, 3, 1)$ as a linear combination of the vectors $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$.

One of the key applications of linear combinations we will make throughout the semester is matrix multiplication. Let's introduce the idea with an example.

Example 2.4. Consider the three vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\2\\-3\end{bmatrix}$, and $\begin{bmatrix}2\\1\\1\end{bmatrix}$. Let's multiply the first vector by 2, the second by -1, and the third by 4, and then sum the result. This gives us the linear combination

$$2\begin{bmatrix}1\\0\\1\end{bmatrix} - 1\begin{bmatrix}0\\2\\-3\end{bmatrix} + 4\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}10\\2\\9\end{bmatrix}$$

We will define matrix multiplication so that multiplying a matrix on the right by a vector corresponds precisely to creating a linear combination of the columns of A. We now write the linear combination above in matrix form

$$\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\1 & -3 & 1\end{bmatrix}\begin{bmatrix}2\\-1\\4\end{bmatrix} = \begin{bmatrix}10\\2\\9\end{bmatrix}.$$

Definition 2.5: A matrix times a vector. We define the matrix product $A\vec{x}$ (a matrix times a vector) to be the linear combination of columns of A where the components of $\vec{x}$ are the scalars in the linear combination. For this to make sense, notice that the vector $\vec{x}$ must have the same number of entries as there are columns in A. We can make this definition more precise as follows. Let $\vec{v}_i$ be the ith column of A so that $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}$, and let $\vec{x} = \begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix}$. Then the matrix product is the linear combination

The product $A\vec{x}$ gives us linear combinations of the columns of A.

$$A\vec{x} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix} = \vec{a}_1x_1 + \vec{a}_2x_2 + \cdots + \vec{a}_nx_n.$$

The definition above should look like the dot product. If you think of A as a vector of vectors, then $A\vec{x}$ is just the dot product of A and $\vec{x}$.

Problem 2.7 Write down a 2 by 4 nonzero matrix, and call it A (fill the matrix with some integers of your choice). Then write down a vector $\vec{x}$ such that the matrix product $A\vec{x}$ makes sense (again, fill the vector with integers of your choice). Then use the definition above to obtain the product $A\vec{x}$.

Definition 2.6: A matrix times a matrix. Let $\vec{b}_j$ represent the jth column of B (so $B = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n\end{bmatrix}$). The product AB of two matrices $A_{m\times n}$ and $B_{n\times p}$ is a new matrix $C_{m\times p} = [c_{ij}]$ where the jth column of C is the product $A\vec{b}_j$. To summarize, the matrix product AB is a new matrix whose jth column is a linear combinations of the columns of A using the entries of the jth column of B to perform the linear combinations.

Problem 2.8 Let $A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1 & 4\\-1 & 2 & -3\end{bmatrix}$. Use the definition given above to compute both AB and BA. Be prepared to show the class how you used linear combinations to get the matrix product. (If you are used to using the row dotted by column approach, then this problem asks you to do the matrix product differently.)

We introduced matrix multiplication in terms of linear combinations of column vectors. My hope is that by doing so you immediately start thinking of linear combinations whenever you encounter matrix multiplication (as this is what it was invented to do). There are many alternate ways to think of matrix multiplication. Here are two additional methods.

1. "Row times column approach." The product AB of two matrices $A_{m\times n}$ and $B_{n\times p}$ is a new matrix $C_{m\times p} = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ is the dot product of the of the ith row of A and the jth column of B. Wikipedia has an excellent visual illustration of this approach.

2. Rephrase everything in terms of rows (instead of columns). We form linear combinations of rows using rows. The matrix product $\vec{x}B$ (notice the order is flopped) is a linear combination of the rows of B using the components of x as the scalars. For the product AB, let $\vec{a}_i$ represent the ith row of A. Then the ith row of AB is the product $\vec{a}_iB$. We'll most often use the column definition instead of this, because we use the function notation $f(x)$ from calculus, and later we will use the notation $A(\vec{x})$ instead of $(\vec{x})A$ to describe how matrices act as functions.

Problem 2.9 Let $A = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1 & 4\\ -1 & 2 & -3\end{bmatrix}$. Use the two alternate definitions above to compute AB. Be prepared to show the class how you used both alternate definitions (You'll need to show your intermediate steps).

Problem 2.10 Do each of the following:

1. Solve the system of equations $x + 2y = 3$, $4x + 5y = 6$.
2. Write the vector $\begin{pmatrix}3\\6\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\4\end{pmatrix}$ and $\begin{pmatrix}2\\5\end{pmatrix}$.
3. Let $A = \begin{pmatrix}1 & 2\\ 4 & 5\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}3\\6\end{pmatrix}$. Find a vector $\vec{x}$ so that $A\vec{x} = \vec{b}$. This matrix A is called the coefficient matrix of the system in the first part.

How are these three questions related?

Prior to introducing Gaussian elimination, let's solve a system of equations using an elimination method. If $2x + 3y = 4$ and $5x + 7y = 0$, then we can eliminate x from the second equation by multiplying both sides of the first equation by 5, and both sides of the second equation by 2, and then subtracting. This would give us the equations $10x + 15y = 20$ and $10x + 14y = 1$. The first equation minus the second then gives $(10 - 10)x + (15 - 14)y = (20 - 0)$, or more simply $y = 20$. Similarly, you could multiply the first equation by 7, and the second by 3, to eliminate y.

Problem 2.11 Solve the system of equations

$$\begin{aligned}2x + 3y - 4z &= 4\\ 3x + 4y - 3z &= 8\\ 7x + 12y - 12z &= 19.\end{aligned}$$

Use elimination to find your solution. Eliminate x from the 2nd and 3rd equations (which will give you two equations that do not involve x). Then use one of these simplified equations to eliminate y from the other simplified equation. At this point you should have an equation that only involves z. Then use back substitution to give y and x.

Problem 2.12 Answer the following.

1. Suppose that $ax + by = c$ and $dx + ey = f$, where a, b, c, d, e, f are all constants. This is a system of equations with 2 equations and 2 unknowns. Each equation represents a line in the plane. How many solutions are there to this system? (You should have a few different cases.)
2. Suppose that $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$ and $a_{31}x + a_{32}y + a_{33}z = b_1$, where each a_{ij} is a constant. This is a system of equations with 3 equations and 3 unknowns. Each equation represents a plane in space. How many solutions are there to this system? (You should have a few different cases.)
3. Suppose that $a_{11}x + a_{12}y + a_{13}z = b_1$ and $a_{21}x + a_{22}y + a_{23}z = b_2$, where each a_{ij} is a constant. This is a system of equations with 2 equations and 3 unknowns. Each equation represents a plane in space. How many solutions are there to this system? (You should have a few different cases.)

Definition 2.7. We say that a system of linear equation is consistent, if it has at least one solution. We say it is inconsistent if there is no solution.

2.2 Gaussian Elimination

Gaussian elimination is an efficient algorithm we will use to solve systems of equations. This is the same algorithm implemented on most computers systems. The main idea is to eliminate each variable from all but one equation/row (if possible), using the following three operations (called elementary row operations):

1. Multiply an equation (or row of a matrix) by a nonzero constant,
2. Add a nonzero multiple of any equation (or row) to another equation,
3. Interchange two equations (or rows).

These three operations are the operations learned in college algebra when solving a system using a method of elimination. Gaussian elimination streamlines elimination methods to solve generic systems of equations of any size. The process involves a forward reduction and (optionally) a backward reduction. The forward reduction creates zeros in the lower left corner of the matrix. The backward reduction puts zeros in the upper right corner of the matrix. We eliminate the variables in the lower left corner of the matrix, starting with column 1, then column 2, and proceed column by column until all variables which can be eliminated (made zero) have been eliminated. Before formally stating the algorithm, let's look at a few examples.

Example 2.8. Let's start with a system of 2 equations and 2 unknowns. I will write the augmented matrix representing the system as we proceed. To solve

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ 2x_1 - 5x_2 &= 1 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 2 & -5 & 1 \end{array}\right]$$

we eliminate the $2x_1$ in the 2nd row by adding -2 times the first row to the second row.

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ x_2 &= -7 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 1 & -7 \end{array}\right]$$

The matrix at the right is said to be in **row echelon form**. row echelon form

Definition 2.9: Row Echelon Form. We say a matrix is in row echelon form (ref) if

- each nonzero row begins with a 1 (called a leading 1),
- the leading 1 in a row occurs further right than a leading 1 in the row above, and
- any rows of all zeros appear at the bottom.

The position in the matrix where the leading 1 occurs is called a pivot. The column containing a pivot is called a pivot column. pivot column

At this point in our example, we can use "back-substitution" to get $x_2 = -7$ and $x_1 = 4+3x_2 = 4-21 = -17$. Alternatively, we can continue the elimination process by eliminating the terms above each pivot, starting on the right and working backwards. This will result in a matrix where all the pivot columns contain all zeros except for the pivot. If we add 3 times the second row to the first row, we obtain.

$$\begin{aligned} x_1 &= -17 \\ x_2 &= -7 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & -17 \\ 0 & 1 & -7 \end{array}\right]$$

The matrix on the right is said to be in **reduced row echelon form** (or just rref). We can easily read solutions to systems of equations directly from a matrix which is in reduced row echelon form.

Definition 2.10: Reduced Row Echelon Form. We say that a matrix is in reduced row echelon form (rref) if reduced row echelon form - rref

- the matrix is in row echelon form, and
- each pivot column contains all zeros except for the pivot (leading one).

Example 2.11. Let's now solve a nonhomogeneous (meaning the right side is not zero) system with 3 equations and 3 unknowns:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2 \\ x_1 - 2x_2 &= 3 \\ 4x_2 + 2x_3 &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right].$$

We'll encounter some homogeneous systems later on. To simplify the writing, we'll just use matrices this time. To keep track of each step, I will write the row operation next to the row I will replace. Remember that the 3 operations are (1)multiply a row by a nonzero constant, (2)add a multiple of one row to another, (3) interchange any two rows. If I write $R_2 + 3R1$ next to R_2, then this means I will add 3 times row 1 to row 2. If I write $2R_2 - R1$ next to R_2, then I have done two row operations, namely I multiplied R_2 by 2, and then added (-1) times $R1$ to the result (replacing $R2$ with the sum). The steps below read left to right, top to bottom. In order to avoid fractions, I wait to divide until the last step, only putting a 1 in each pivot at the very end.

$$\Rightarrow^{(1)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ \\ \end{array} \quad \Rightarrow^{(2)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 4 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 + 4R_2 \end{array}$$

$$\Rightarrow^{(3)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 14 & 21 \end{array}\right] \begin{array}{l} \\ \\ R_3/7 \end{array} \quad \Rightarrow^{(4)} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -10 & 2 & 8 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} 2R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\Rightarrow^{(5)} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 7 \\ 0 & -10 & 0 & 5 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} \\ R_2/5 \\ \\ \end{array} \quad \Rightarrow^{(6)} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 7 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ \\ \end{array}$$

$$\Rightarrow^{(7)} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 8 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} R_1/4 \\ R_2/-2 \\ R_3/2 \end{array} \quad \Rightarrow^{(8)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

Writing the final matrix in terms of a system, we have the solution $x_1 = 2, x_2 = -1/2, x_3 = 3/2$. Remember that this tells us (1) where three planes intersect, (2) how to write the 4th column $\vec{b}$ in our original augmented matrix as a linear combination of the columns of the coefficient matrix A, and (3) how to solve the matrix equation $A\vec{x} = \vec{b}$ for $\vec{x}$.

The following steps describe the Gaussian elimination algorithm that we used above. Please take a moment to compare what is written below with the example above. Most of the problems in this unit can be solved using Gaussian elimination, so we will practice it as we learn a few new ideas.

1. Forward Phase (row echelon form) - The following 4 steps should be repeated until you have mentally erased all the rows or all the columns. In step 1 or 4 you will erase a column and/or row from the matrix.

   (a) Consider the first column of your matrix. Start by interchanging rows (if needed) to place a nonzero entry in the first row. If all the elements in the first column are zero, then ignore that column in future computations (mentally erase the column) and begin again with the smaller matrix which is missing this column. If you erase the last column, then stop.

   Computer algorithms place the largest (in absolute value) nonzero entry in the first row. This reduces potential errors due to rounding that can occur in later steps.

   (b) Divide the first row (of your possibly smaller matrix) row by its leading entry so that you have a leading 1. This entry is a pivot, and the column is a pivot column. [When doing this by hand, it is often convenient to skip this step and do it at the very end so that you avoid fractional arithmetic. If you can find a common multiple of all the terms in this row, then divide by it to reduce the size of your computations.]

   (c) Use the pivot to eliminate each nonzero entry below the pivot, by adding a multiple of the top row (of your smaller matrix) to the nonzero lower row.

   (d) Ignore the row and column containing your new pivot and return to the first step (mentally cover up or erase the row and column containing your pivot). If you erase the last row, then stop.

   Ignoring rows and columns is equivalent to incrementing row and column counters in a computer program.

2. Backward Phase (reduced row echelon form - often called Gauss-Jordan elimination) - At this point each row should have a leading 1, and you should have all zeros to the left and below each leading 1. If you skipped step 2 above, then at the end of this phase you should divide each row by its leading coefficient to make each row have a leading 1.

   (a) Starting with the last pivot column. Use the pivot in that column to eliminate all the nonzero entries above it, by adding multiples of the row containing the pivot to the nonzero rows above.

   (b) Work from right to left, using each pivot to eliminate the nonzero entries above it. Nothing to the left of the current pivot column changes. By working right to left, you greatly reduce the number of computations needed to fully reduce the matrix.

Example 2.12. As a final example, let's reduce $\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 7 \\ 1 & 3 & 5 & 1 & 6 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right]$ to reduced row echelon form (rref). The first step involves swapping 2 rows. We swap row 1 and row 2 because this places a 1 as the leading entry in row 1.

(1) Get a nonzero entry in upper left

$$\Rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 7 \\ 1 & 3 & 5 & 1 & 6 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ \\ \\ \\ \end{array}$$

(2) Eliminate entries in 1st column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 2 & 0 & 4 & 3 & -8 \\ -2 & 1 & -3 & 0 & 5 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 + 2R_1 \end{array}$$

(3) Eliminate entries in 2nd column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & -6 & -6 & 1 & -20 \\ 0 & 7 & 7 & 2 & 17 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 6R_2 \\ R_4 - 7R_2 \end{array}$$

(4) Make a leading 1 in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & -11 & 22 \\ 0 & 0 & 0 & 16 & -32 \end{array}\right] \begin{array}{l} \\ \\ R_3/(-11) \\ R_4/16 \end{array}$$

(5) Eliminate entries in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

(6) Row Echelon Form

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

At this stage we have found a row echelon form of the matrix. Notice that we eliminated nonzero terms in the lower left of the matrix by starting with the first column and working our way over column by column. Columns 1, 2, and 4 are the pivot columns of this matrix. We now use the pivots to eliminate the other nonzero entries in each pivot column (working right to left).

Recall that a matrix is in reduced row echelon (rref) if:

1. Nonzero rows begin with a leading 1.
2. Leadings 1's on subsequent rows appear further right than previous rows.
3. Rows of zeros are at the bottom.
4. Zeros are above and below each pivot.

(7) Eliminate entries in 4th column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 6 \\ 0 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R3 \\ R_2 + 2R_3 \\ \\ \\ \end{array}$$

(8) Eliminate entries in 2nd column

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 8 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ \\ \\ \end{array}$$

(9) Reduced Row Echelon Form

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(10) Switch to system form

$$\Rightarrow \begin{aligned} x_1 + 2x_3 &= -1 \\ x_2 + x_3 &= 3 \\ x_4 &= -2 \\ 0 &= 0 \end{aligned}$$

We have obtained the reduced row echelon form. When we write this matrix in the corresponding system form, notice that there is not a unique solution to the system. Because the third column did not contain a pivot column, we can write every variable in terms of x_3 (the redundant equation $x_3 = x_3$ allows us to write x_3 in terms of x_3). We are free to pick any value we want for x_3 and still obtain a solution. For this reason, we call x_3 a free variable, and write our infinitely many solutions in terms of x_3 as

Free variables correspond to non pivot columns. Solutions can be written in terms of free variables.

$$\begin{aligned} x_1 &= -1 - 2x_3 \\ x_2 &= 3 - x_3 \\ x_3 &= x_3 \\ x_4 &= -2 \end{aligned} \quad \text{or by letting } x_3 = t \quad \begin{aligned} x_1 &= -1 - 2t \\ x_2 &= 3 - t \\ x_3 &= t \\ x_4 &= -2 \end{aligned}.$$

By choosing a value (such as t) for x_3, we can write our solution in so called parametric form. We have now given a parametrization of the solution set, where t is an arbitrary real number.

parametric form

Problem 2.13 Each of the following augmented matrices requires one row operation to be in reduced row echelon form. Perform the required row operation, and then write the solution to the corresponding system of equations in terms of the free variables.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ -3 & -6 & 0 & 12 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 7 & 0 & 3 \\ 0 & 0 & 1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Problem 2.14 Use Gaussian elimination to solve

$$\begin{aligned} x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 + 3x_3 &= 4 \\ 4x_1 + x_2 + 4x_3 &= 5 \end{aligned}$$

by row reducing the matrix to reduced row echelon form. [Hint: Start by interchanging row 1 and row 2.]

Problem 2.15 Use Gaussian elimination to solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ -x_1 + 2x_2 + 3x_3 &= 8 \\ 2x_1 - 4x_2 + x_3 &= 5 \end{aligned}$$

by row reducing the matrix to reduced row echelon form. [Hint: You should end up with infinitely many solutions. State your solution by writing each variable in terms of the free variable(s).]

Problem 2.16 Use Gaussian elimination to solve

$$\begin{aligned} x_1 + 2x_3 + 3x_4 &= -7 \\ 2x_1 + x_2 + 4x_4 &= -7 \\ -x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 - 2x_3 - x_4 &= 4 \end{aligned}$$

by row reducing the matrix to reduced row echelon form.

2.3 Rank, Linear Independence, Inverses, and Determinants

Definition 2.13.

- The rank of a matrix is the number of pivot columns of the matrix. To find the rank of a matrix, you reduce the matrix using Gaussian elimination until you discover the pivot columns.
- The span of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is all possible linear combinations of the vectors. In terms of matrices, the span of a set of vectors is all possible vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ for some vector $\vec{x}$, where the vectors $\vec{v}_i$ are placed in the columns of A.
- We say that a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if the only solution to the homogeneous system $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. Otherwise we say the vectors are linearly dependent, and it is possible to write one of the vectors as a linear combination of the others. We say the vectors are dependent because one of them depends on (can be obtained as a linear combination of) the others.
- In terms of spans, we say vectors are linearly dependent when one of them is in the span of the other vectors.

As we complete each of the following problems in class, we'll talk about the span of the vectors, and the rank of the corresponding matrix. The key thing we need to focus on is learning to use the words "linearly independent" and "linearly dependent."

Problem 2.17 Are the vectors $\vec{v}_1 = (1, 3, 5)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (0, 3, 1)$ linearly independent? Solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Problem 2.18 Are the vectors $\vec{v}_1 = (1, 2, 0)$, $\vec{v}_2 = (2, 0, 3)$, and $\vec{v}_3 = (3, -2, 6)$ linearly independent? Solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ to answer this question. If they are dependent, then write one of the vectors as a linear combination of the others.

Problem 2.19 Answer each of the following:

1. Suppose you have row reduced a 3 by 3 matrix, and discovered that the rank of the matrix is 2. Are the columns of the matrix independent or dependent? What if the rank was 3?
2. Now suppose you have row reduced a 7 by 7 matrix. If the columns are independent, what possible options do you have for the rank.
3. Now suppose you have row reduced a 7 by 5 matrix. If the columns are independent, what must the rank be.
4. Now suppose you have row reduced a 5 by 7 matrix. Explain why the columns cannot be independent.
5. If you have n vectors placed in the columns of a matrix, what must the rank of the matrix be in order to guarantee that the vectors are independent?

Problem 2.20 Is the vector $[2, 0, 1, -5]$ in the span of

$$\{[1, 0, -1, -2], [1, 2, 3, 0], [0, 1, -1, 2]\}?$$

If it is, then write it as a linear combination of the others. If it is not, then explain why it is not.

Problem 2.21 Find the reduced row echelon form of the matrix

$$B = \begin{bmatrix} 2 & -1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 3 & 3 \end{bmatrix}.$$

Use your result to answer the following questions.

1. Write both $(1, 0)$ and $(0, 1)$ as linear combinations of $(2, 1)$ and $(-1, 1)$.
2. Write $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then write $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
3. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. Find vectors $\vec{x}$ and $\vec{y}$ so that $A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
4. Find a matrix B so that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem: 21, revised Answer each of the following questions.

1. Find the reduced row echelon form of the matrix
$$B = \begin{bmatrix} 2 & -1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 3 & 3 \end{bmatrix}.$$
2. Write $(1, 0)$ as a linear combination of $(2, 1)$ and $(-1, 1)$. Remember, that when writing $c_1(2, 1) + c_2(-1, 1) = (1, 0)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
3. Write $(0, 1)$ as a linear combination of $(2, 1)$ and $(-1, 1)$. Remember, that when writing $c_1(2, 1) + c_2(-1, 1) = (0, 1)$, you must solve for the unknown constants. Feel free to row reduce the augmented matrix $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
4. Continue to write each of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. [Hint:At some point, rather than row reducing $\begin{bmatrix} 2 & -1 & \vec{v} \\ 1 & 1 & \end{bmatrix}$, ask yourself how you could use part 1 to answer this.]

5. The following matrix row reduces to give

$$\begin{bmatrix}1 & 0 & 2 & 4 & 5 & 8\\ 0 & 2 & 5 & 2 & -1 & 3\\ 0 & -2 & -1 & 0 & 2 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0 & 3 & \frac{9}{2} & 6\\ 0 & 1 & 0 & -\frac{1}{4} & -\frac{9}{8} & -1\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 1\end{bmatrix}.$$

   Use this to write both $(4, 2, 0)$ and $(5, -1, 2)$ as a linear combination of the first three columns.

Definition 2.14. The identity matrix I is a square matrix so that if A is a square matrix, then $IA = AI = A$. The identity matrix acts like the number 1 when performing matrix multiplication.

If A is a square matrix, then the inverse of A is a matrix A^{-1} where we have $AA^{-1} = A^{-1}A = I$, provided such a matrix exists.

Problem Let $A = \begin{bmatrix}1 & 3\\ 3 & 4\end{bmatrix}$. We now develop an algorithm for computing the inverse A^{-1}. If an inverse matrix exists, then we know it's the same size as A, so we could let $A^{-1} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$ be the inverse matrix, where $\vec{v}_1$ and $\vec{v}_2$ are the columns of A^{-1}.

1. We know that $AA^{-1} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Explain why $A\vec{v}_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$ and $A\vec{v}_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$.
2. Solve the matrix equations $A\vec{v}_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$ and $A\vec{v}_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$. (This involves row reducing $\begin{bmatrix}1 & 3 & 1\\ 3 & 4 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 3 & 0\\ 3 & 4 & 1\end{bmatrix}$).
3. What is the reduced row echelon form of $\begin{bmatrix}1 & 3 & 1 & 0\\ 3 & 4 & 0 & 1\end{bmatrix}$. How is this related to your previous work.
4. State the inverse of A.

The previous problem showed you how to obtain a matrix B so that $AB = I$. You just had to row reduce that matrix $\begin{bmatrix}A & I\end{bmatrix}$ to the matrix $\begin{bmatrix}I & A^{-1}\end{bmatrix}$. The inverse shows up instantly after row reduction.

Problem 2.22 Use the algorithm describe immediately before this problem to compute the inverse of

$$A = \begin{bmatrix}3 & 0 & 3\\ 0 & -1 & 1\\ 0 & 3 & -4\end{bmatrix}.$$

Then use your work to write each of the standard basis vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ as a linear combination of the columns of A.

Problem 2.23 Let $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Use Gaussian elimination to show that the inverse of A is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$$

In computing the inverse of a 2 by 2 matrix, the number $ad - bc$ appears in the denominator. We call this number the determinant. If I asked you to compute the inverse of a 3 by 3 matrix, you would again see a number appear in the denominator. We call that number the determinant. This holds true in all dimensions.

Problem: Optional Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Use Gaussian elimination to find the inverse of A, and show that the common denominator is $a(ei - hf) - b(di - gf) + c(dh - ge)$.

Definition 2.15: Determinants of 2 by 2 and 3 by 3 matrices. The determinant of a 2×2 and 3×3 matrix are the numbers

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge) \end{aligned}$$

We use vertical bars next to a matrix to state we want the determinant. Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

In the examples above, we obtained the determinant of a 3 by 3 matrix by computing the determinant of several 2 by 2 matrices. We obtained each 2 by 2 matrix by removing a row and column from the original 3 by 3 matrix. We now add some language to extend the definition above to all dimensions.

Definition 2.16: Minors, Cofactors, and General determinants. Let A be an n by n matrix.

- The minor M_{ij} of a matrix A is the determinant of the the matrix formed by removing row i and column j from A.
- The cofactor C_{ij} is the product of the minor M_{ij} and $(-1)^{i+j}$, so we have $C_{ij} = (-1)^{i+j} M_{ij}$. So it's either the minor, or the opposite of the minor.
- To compute the determinant, first pick a row or column. We define the determinant to be $\sum_{k=1}^{n} a_{ik} C_{ik}$ (if we chose row i) or alternatively $\sum_{k=1}^{n} a_{kj} C_{kj}$ (if we chose column j).
- You can pick ANY row or ANY column you want, and then compute the determinant by multiplying each entry of that row or column by its cofactor, and then summing the results. (The fact that this works would require proof. That proof will be left to a course in linear algebra.)
- A sign matrix keeps track of the $(-1)^{j+k}$ term in the cofactor. All you have to do is determine if the first entry of your expansion has a plus or minus, and then alternate the sign as you expand.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

sign matrix

Problem 2.24 Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 4 & 2 & 5 \end{bmatrix}$ in 3 different ways. First, use a cofactor expansion using the first row (Definition 2.15). Then use a cofactor expansion using the 2nd row. Then finally use a cofactor expansion using column 3. Which of the was the quickest, and why?

Problem 2.25 Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 5 & -1 \\ 0 & 8 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & -5 & 3 & -1 \end{bmatrix}.$$

You can make these problems really fast if you use a cofactor expansion along a row or column that contains a lot of zeros.

Problem 2.26 Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix}.$$

Then find the inverse of A (or explain why it does not exist). Are the columns of A linearly independent or linearly dependent?

Problem 2.27 Compute the determinant of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Does A have an inverse? Are the columns of A linearly independent or linearly dependent? Answer both of the previous questions without doing any row reduction. Then row reduce $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$ to confirm your answer.

After completing the previous two problems, you should see that there is a connection between the determinant, inverse, and linear independence. Make a conjecture about what this connection is. We'll learn a little more about determinants and inverses, and then you'll have a chance to state your conjecture, as well as prove it.

Problem 2.28 Start by writing the system of equations

$$\begin{cases} -2x_1 + 5x_3 &= -2 \\ -x_1 + 3x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 3 \end{cases}$$

as a matrix product $A\vec{x} = \vec{b}$. (What are A, $\vec{x}$ and $\vec{b}$?) Then find the inverse of A, and use this inverse to find $\vec{x}$. [Hint: If we just have numbesr, then to solve $ax = b$, we multiply both sides by $\frac{1}{a}$ to obtain $\frac{1}{a}ax = \frac{1}{a}b$ or just $x = \frac{1}{a}b$.]

In the next problem, you'll prove that the determinant of a 2 by 2 matrix gives the area of a parallelogram whose edges are the columns of the matrix.

Problem 2.29 To find the area of the parallelogram with vertexes $O = (0, 0)$, $U = (a, c)$, $V = (b, d)$, and $P = (a + b, c + d)$, we would find the length of OU (the base b), and multiply it by the distance from V to OU. Complete the following:

1. Find the projection of $\vec{OV}$ onto $\vec{OU}$. (You may have to look up a formula from math 215.)
2. The vector $\vec{OV} - \text{proj}_{\vec{OV}}\vec{OU}$ is called the component of $\vec{OV}$ that is orthogonal to $\vec{OU}$. The length of this vector is precisely the distance from V to OU, which we'll call h. Find the length of this vector.
3. We now have the base $b = |OU|$ and height h of a parallelogram. Compute the product, and prove it equals $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = |ad - bc|$.

The result above extends to 3 dimensions. The determinant of a 3 by 3 matrix gives the volume of a parallelepiped whose edges are the columns of the matrix. We then use determinants to define nth dimensional volume.

Problem 2.30 Answer each of the following:

1. Let $\vec{u} = (2, 3)$. If you pick a vector $\vec{v}$ that is a linear combination of $\vec{u}$, what will the determinant of $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$ equal? First explain how you know the answer (before you have even chosen a vector $\vec{v}$). Then give us an example by picking a vector that is a linear combination of $\vec{v}$.
2. Let $\vec{u} = (1, 0, 2)$ and $\vec{v} = (0, -1, 1)$. If $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, what will the determinant equal? Explain. Then show us an example to confirm your conjecture.
3. We already computed the determinant of $A = \begin{bmatrix} 2 & 1 & -6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$. Swap two columns of the matrix, and then compute the determinant. How does the determinant of your matrix with swapped columns relate to the determinant of the original matrix. If you swap two columns of a matrix, what happens to the determinant?

Problem 2.31 Construct a 2 by 2 matrix whose columns are linearly independent. What is the reduced row echelon form of your matrix? Compute the rank and the determinant, and finally find the inverse (if possible).

Now construct a 2 by 2 matrix whose columns are linear dependent. What is the reduced row echelon form of your matrix? Compute the rank and the determinant, and finally find the inverse (if possible).

Make a conjecture about the connection between (1) linear dependence, (2) rref, (3) rank, (4) determinant, and (5) inverses. Then use a computer to give two 3 by 3 examples similar to the examples above. You'll be asked to show us the computations on the computer in class.

Problem 2.32 Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$. Compute the determinant of A. Then create a matrix B so that the ijth entry of B is the cofactor C_{ij} (remove row i and column j, compute the determinant, and then times by an appropriate sign). This will require that you compute nine 2 by 2 determinants. Finally, compute the inverse of A (feel free to use a computer on this part). Make a conjecture about the connection between the determinant of A, this matrix B, and the inverse of A. We'll verify your conjecture is true on a 4 by 4 matrix in class.

2.4 Eigenvalues and Eigenvectors

The final computational skill we need to tackle is to compute eigenvalues and eigenvectors. Let's start by looking at an example to motivate the language we are about to introduce.

Example 2.17. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. When we multiply this matrix by the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we obtain $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}$. Multiplication by the matrix A was miraculously the same as multiplying by the number 3. Symbolically we have $A\vec{x} = 3\vec{x}$. Not every vector $\vec{x}$ satisfies this property, as letting $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the product $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which is not a multiple of $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Our main goal in this section is to answer the following two questions:

1. For which nonzero vectors $\vec{x}$ (eigenvectors) is it possible to write $A\vec{x} = \lambda\vec{x}$?
2. Which scalars λ (eigenvalues) satisfy $A\vec{x} = \lambda\vec{x}$?

Now for some definitions.

Definition 2.18: Eigenvector and Eigenvalue. Let A be a square $n \times n$ matrix.

- An eigenvector of A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. (Matrix multiplication reduces to scalar multiplication.) We avoid letting $\vec{x}$ be the zero vector because $A\vec{0} = \lambda\vec{0}$ no matter what λ is.
- If $\vec{x}$ is an eigenvector satisfying $A\vec{x} = \lambda\vec{x}$, then we call λ and eigenvalue of A.

Problem 2.33 Use the definition above to determine with of the following are eigenvectors of $\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, (1, 4), \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1, \\ -1 \end{pmatrix}, \begin{pmatrix} -2, \\ 2 \end{pmatrix}.$$

If the vector is an eigenvector, state the corresponding eigenvalue.

The next problem gives us an algorithm for computing eigenvalues and eigenvectors.

Problem 2.34: How to compute eigenvalues and eigenvectors Let A be a square matrix.

1. If λ is an eigenvalue, explain why we can find the eigenvectors by solving the equation $(A-\lambda I)\vec{x}=\vec{0}$. This means we can subtract λ from the diagonal entries of A, and then row reduce $\begin{bmatrix} A-\lambda I & \vec{0} \end{bmatrix}$ to obtain the eigenvectors. Note that you should always obtain infinitely many solutions.
2. Explain why we can obtain the eigenvalues of A by solving for when the determinant of $(A-\lambda I)$ is zero, i.e. solving the equation
$$\det(A-\lambda I)=0.$$

The algorithm above suggests the following definition.

Definition 2.19. If A is a square n by n matrix, then we call $\det(A-\lambda I)$ the characteristic polynomial of A. It is a polynomial in λ of degree n, and hence has n roots (counting multiplicity). These roots are the eigenvalues of A.

We now have an algorithm for finding the eigenvalues and eigenvectors of a matrix. We start by finding the characteristic polynomial of A. The zeros of this polynomial are the eigenvalues. To get the eigenvectors, we just have to row reduce the augmented matrix $\begin{bmatrix} A-\lambda I & \vec{0} \end{bmatrix}$. Finding eigenvalues and eigenvectors requires that we compute determinants, find zeros of polynomials, and then solve homogeneous systems of equations. You know you are doing the problem correctly if you get infinitely many solutions to the system $(A-\lambda I)\vec{x}=0$ for each lambda (i.e. there is at least one row of zeros along the bottom after row reduction). As another way to check your work, the following two facts can help.

- The sum of the eigenvalues equals the trace of the matrix (the sum of the diagonal elements).
- The product of the eigenvalues equals the determinant.

The trace and determinant are equal to the sum and product of the eigenvalues.

Problem 2.35 Consider the matrix $A=\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$ from problem 2.33.

1. Find the characteristic polynomial of A, and then find the zeros to determine the eigenvalues.
2. For each eigenvalue, find all corresponding eigenvectors.
3. Compute the trace and determinant of A.

Problem 2.36 Consider the matrix $A=\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

Problem 2.37 Consider the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

Problem 2.38 Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Find the characteristic polynomial and eigenvalues of A. Then for each eigenvalue, find all corresponding eigenvectors. (Check your work by computing the trace and determinant of A.)

2.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Linear Algebra Applications

This chapter covers the following ideas.

1. Explain the connection between vector fields and their corresponding eigenvalues and eigenvectors. Use this knowledge to apply the second derivative test.
2. Solve various problems relating to conservation laws, including stoichiometry, Kirchoff's electrical laws, and Markov Processes.
3. Use Cramer's rule to solve systems, and explain when you would choose Cramer's rule over row reduction.
4. Find interpolating polynomials, and use the transpose to solve the least squares regression problem.
5. Find the partial fraction decomposition of a rational function. Utilize this decomposition to integrate rational functions.
6. Be able to show that a function is linear, and find the kernel of a linear function.

3.1 Vector Fields

In multivariate calculus, we studied vector fields of the form $\vec{F}(x,y) = (M,N)$, where M and N are functions of x and y. The derivative of the vector field is the square matrix

$$D\vec{F}(x,y) = \begin{bmatrix} \partial M/\partial x & \partial M/\partial y \\ \partial N/\partial x & \partial N/\partial y \end{bmatrix}.$$

The eigenvalues and eigenvectors of this matrix provide us with a wealth of information about the vector field. The next few problems have you discover many of these key ideas. We'll return to these ideas throughout the semester, especially when we start studying systems of differential equations in depth.

Problem 3.1 Consider the vector field $\vec{F}(x,y) = (2x+y, x+2y)$.

1. At each of the 8 points given by $(\pm 1, \pm 1)$, $(0, \pm 1)$, $(\pm 1, 0)$, sketch the vector $\vec{F}(x,y)$ with it's base at the input point (so at point $(1,0)$, sketch $(2,1)$, a vector starting at $(1,0)$ and ending at $(3,1)$). This provides us with a rough sketch of the vector field.

2. Compute $A = D\vec{F}(x, y)$. It should be a 2 by 2 matrix.

3. Remember that we say a vector $\vec{x}$ is an eigenvector if $A\vec{x} = \lambda\vec{x}$. For any of the vectors from part 1., did you find that $A\vec{x} = \lambda\vec{x}$? Which ones (these are eigenvectors)? By how much was the vector $\vec{x}$ stretched (these are eigenvalues)?

4. Now compute the eigenvalues and eigenvectors of this matrix, using the algorithm from the previous chapter. You should obtain the same answer as part 3.

The problem above had two positive eigenvalues. In the next problem, your goal is to determine what a vector field looks like when you have both a positive and negative eigenvalue.

Problem 3.2 Complete the following:

1. For the vector field $\vec{F} = (x, 2x - y)$, compute the eigenvalues and eigenvectors of $D\vec{F}(x, y)$.

2. For the vector field $\vec{F} = (x - 4y, -6x - y)$, compute the eigenvalues and eigenvectors of $D\vec{F}(x, y)$.

3. With each vector field, use a computer to construct a vector field plot. In the plot, please show us how to see the eigenvectors, together with which eigenvector corresponds to a positive eigenvalues, and which corresponds to a negative eigenvalue. You can construct vector fields in Wolfram—Alpha by typing "vector field plot" in the input box, or just follow the link http://www.wolframalpha.com/input/?i=vector+field+plot&lk=4&num=2.

4. Add to your plots several trajectories, i.e. a path that a particle would follow if $\vec{F}$ represents the tangent vectors of the path. Think, "If I dropped a really light particle in this field, representing water current, where would the particle go?

Problem 3.3 The following three vector fields have imaginary eigenvalues. Compute the eigenvalues for each, construct a vector field plot, and on the plot add several trajectories (the path followed by a particle that is dropped into this field).

1. $\vec{F} = (-2y, x)$.

2. $\vec{F} = (-x + y, -x - y)$.

3. $\vec{F} = (x - y, x)$

Make a conjecture as to why one spirals in, one spirals out, and one just wraps around in ellipses. We'll address this conjecture in class.

The next problem requires that you are on a computer that can use Mathematica. These computers are available in the Ricks, Austin, Romney, and library. Alternately, you can download VMWare that will allow you to use Mathematica for free from your computer, provided you head to https://vdiview.byui.edu/. You can download step-by-step instructions from http://www.byui.edu/help-desk/categories/vdivmware. Please take a moment and make sure you can access Mathematica.

Problem 3.4 Start by downloading the Mathematica notebook VectorFields.nb (click on the link). The goal of this problem is to make a connection between a vector field and it's corresponding eigenvalues/eigenvectors. Once the notebook is open, click somewhere in the text, hold down Shift, and then press Enter. This will evaluate the commands and produce a vector field plot, with the eigenvector directions drawn in green. You can click on the bubbles with crosshairs in them to adjust the vectors (which are the columns of the matrix). Play around with the animation until you feel like you can answer each of the following questions.

1. If the vector field pushes things outwards in all directions, what do you know about the eigenvalues?
2. If the vector field pulls things inwards in all directions, what do you know about the eigenvalues?
3. How can you tell, by looking at a vector field plot, that one eigenvalue is positive and the other is negative?
4. If the vector field involves swirling motion, what do you know about the eigenvalues? What makes the difference between spiraling inwards, outwards, or just spinning in circles?
5. What happens when you have a repeated eigenvalue? This one has lots of correct answers, and it a topic for much further discussion in chapter 10. See if you can get an example of a repeated eigenvalue with a behavior that's different from the above. If you have the first 4, you can present in class. We'll have you come up to the computer and show us what you did.

3.1.1 Second Derivative Test

Vector fields and eigenvalues provide us with precisely the key information needed to locate maximums, minimums, and saddles for functions of the form $z = f(x, y)$.

Problem 3.5 Consider the function $f(x, y) = x^2 + 4xy + y^2$. The derivative (gradient) is the vector field $Df(x, y) = (2x + 4y, 4x + 2y)$. See Figure 3.1 for a graph of several level curves, together with the gradient.

1. At what point(s) does $Df(x, y) = \vec{0}$? These are the potential locations of maximums, minimums, or saddles.
2. Compute the second derivative of f, which should give you a 2 by 2 symmetric matrix. This matrix is called the Hessian.
3. By looking at the picture, are the eigenvalues of $D^2f(x, y)$ both positive, both negative, or do they differ in sign? How can you tell? Then confirm you are correct by computing the eigenvalues and eigenvectors of $D^2f(x, y)$.
4. Recall that the gradient points in the direction of greatest increase. Using this information alone, does the function have a maximum, minimum, or saddle point at $(x, y) = (0, 0)$

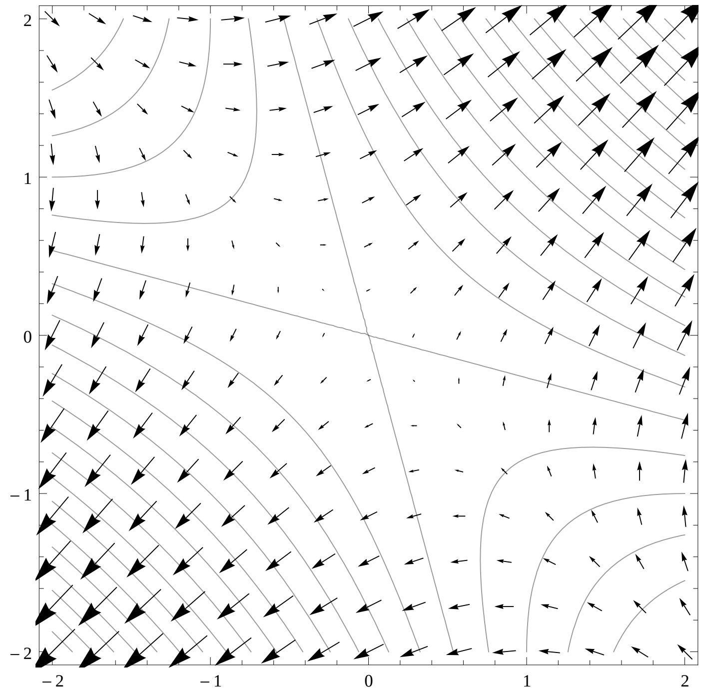


Figure 3.1: A plot of several level curves of $f(x,y) = x^2 + 4xy + y^2$ and the gradient. In one direction the gradient is pulling things towards the origin. In another direction, the gradient is pushing things away from the origin.

Theorem 3.1. *Let $f(x,y)$ be a function that is twice continuously differentiable. Suppose that $Df(x,y) = (0,0)$ when $(x,y) = (a,b)$, so that (a,b) is a critical point. To determine if the point (a,b) corresponds to a maximum, minimum, or saddle point, we compute the eigenvalues of $D^2f(a,b)$ (the second derivative is called the Hessian).*

- *If the eigenvalues of a are all positive, then the function has a minimum at (a,b).*
- *If the eigenvalues of a are all negative, then the function has a maximum at (a,b).*
- *If there is a positive eigenvalue, and a negative eigenvalue, then the function has a saddle at (a,b).*
- *If zero is an eigenvalue, then the second derivative test fails.*

Problem 3.6 Consider the function $f(x,y) = x^3 - 3x^2 - y^2 + 2y$ See Figure 3.2 for a graph of several level curves, together with the gradient.

1. At what point(s) does $Df(x,y) = \vec{0}$? You should obtain two points. These are the potential locations of maximums, minimums, or saddles.
2. Compute the second derivative of f, which should give you a 2 by 2 symmetric matrix.
3. Pick one of the critical points. Use the vector field plot to decide if the eigenvalues of $D^2f(x,y)$ both positive, both negative, or differ in sign at that critical point, and if the function has a maximum, minimum, or saddle at that point. Then repeat with the other critical point.

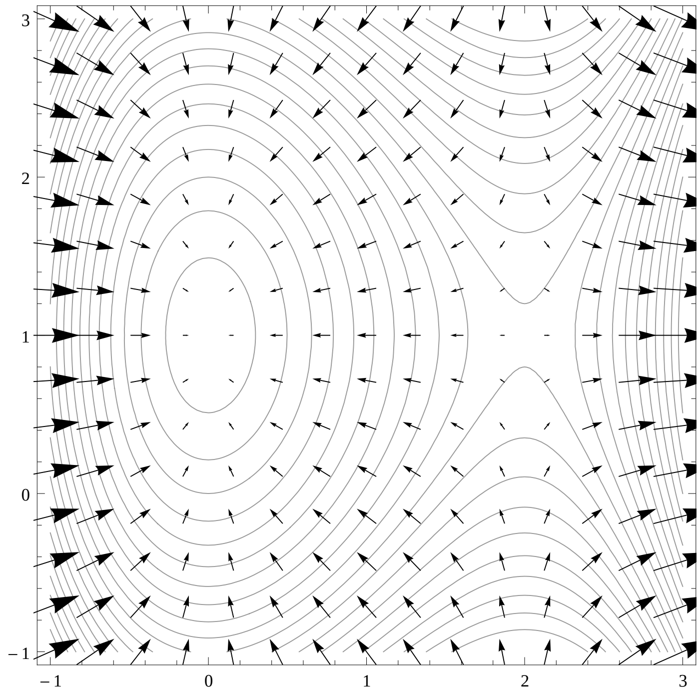


Figure 3.2: A plot of several level curves of $f(x,y) = x^3 - 3x^2 - y^2 + 2y$ and the gradient. There are two critical points. The vector field plot provides enough information to determine if the sign of the eigenvectors of the second derivative at each critical point.

4. Now compute the eigenvalues of the Hessian at each critical value. This should confirm your answer to part 3. (The matrix is diagonal, so computing eigenvalues should be quick.)

The following example adds a little more information to this discussion. I've included it to give you one additional piece of information, namely how the eigenvalues connect to the concavity of the function.

Example 3.2. For the function $f(x,y) = x^2 + xy + y^2$, the gradient is $Df = \begin{bmatrix} 2x+y & x+2y \end{bmatrix}$, which is zero only at $x = 0, y = 0$ (solve the system of equations $2x + y = 0, x + 2y = 0$). The Hessian is $D^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The eigenvalues are found by solving $0 = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = (\lambda - 3)(\lambda - 1)$, so $\lambda = 3, 1$ are the eigenvalues. Since both eigenvalues are positive, the gradient pushes things away from the origin in all direction, which means in every direction you move from the critical point, you'll increase in height. There is a minimum at $(0,0)$.

The eigenvectors of the Hessian help us understand more about the graph of the function. An eigenvector corresponding to 3 is $(1,1)$, and corresponding to 1 is $(-1,1)$. These vectors are drawn in figure 3.3, together with two parabolas whose 2nd derivatives are precisely 3 and 1. The parabola which opens upwards the most quickly has a 2nd derivative of 3. The other parabola has a second derivative of 1. In every other direction, the 2nd derivative would be between 1 and 3.

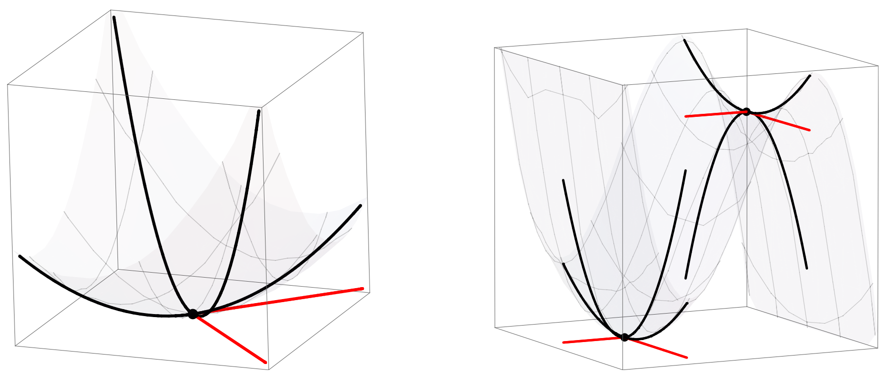


Figure 3.3: The eigenvectors of the second derivative tell you the directions in which the 2nd derivative is largest and smallest. At each critical point, two eigenvectors are drawn as well as a parabola whose second derivative (the eigenvalue) matches the second derivative of the surface in the corresponding eigenvector direction.

3.2 Conservation Laws

Many problems in nature arise from conservation laws. These laws generally focus on the principle that matter is neither is created nor destroyed, rather it is just moved, changed, or something. Any of the following could be viewed as a conservation law:

- What comes in must come out.
- Voltage supplied equals voltage suppressed.
- Atoms before equal atoms after.
- The change in a quantity is how much it increases minus how much it decreases.
- Current in equals current out.

The following problems related to some conservation law. You'll see similar laws in your future classes, regardless of your discipline.

3.2.1 Stoichiometry

Chemical reaction stoichiometry is the study balancing chemical equations. A chemical reaction will often transform reactants into by-products. The by products are generally different compounds, together with either an increase or decrease in heat. One key rule in stoichiometry is that a chemical process neither creates nor destroys matter, rather it only changes the way the matter is organized. For simple reactions (with no radioactive decay), this conservation law forces the number of atoms entering a reaction to be the same as the number leaving. The next problem asks you to use this conservation law to create a balanced chemical reaction equation.

Problem 3.7 The chemical compound hydrocarbon dodecane ($C_{12}H_{26}$) is used as a jet fuel surrogate (see Wikipedia for more info). This compound reacts with oxygen (0_2), and the chemical reaction produces carbon dioxide (CO_2), water (H_20), and heat. Suppose we expose some dodecane to oxygen, and that a chemical reaction occurs in which the dodecane is completely converted to carbon dioxide and water. Conservation requires that the number of atoms (H, C, and 0) at the beginning of the chemical reaction must be the exact same as the number at the end. We could write the chemical reaction in terms of molecules as

$$x_1C_{12}H_{26}+x_2O_2 = x_3CO_2+x_4H_2O \quad \text{or} \quad x_1C_{12}H_{26}-x_2O_2 = x_3CO_2-x_4H_2O = 0,$$

where x_1 molecules of dodecane and x_2 molecules of oxygen were converted to x_3 units of carbon dioxide and x_4 units of oxygen. If we look at each atom (carbon, hydrogen, and oxygen) individually, we obtain three equations to relate the variables x_1, x_2, x_3, x_4. The carbon equation is simply

$$x_1(12) + x_2(0) = x_3(1) + x_4(0) \quad \text{or} \quad x_1(12) + x_2(0) - x_3(1) - x_4(0) = 0.$$

Your job follows:

1. Write the other two conservation equations (for hydrogen and oxygen).
2. Solve the corresponding system of equations by row reduction. As there are only 3 equations with 4 unknowns, you should obtain infinitely many solutions. Write each variable in terms of the free variable.
3. If about 10,000 molecules of water are present at the end of the reaction, about how many molecules of dodecane were burned?

3.2.2 Kirchoff's Electrical Laws

Gustav Kirchoff discovered two laws of electricity that pertain to the conservation of charge and energy. To describe these laws, we must first discuss voltage, resistance, and current.

- Current is the flow of electricity, and often it can be compared to the flow of water.
- As a current passes across a conductor, it encounters resistance. Ohm's law states that the product of the resistance R and current I across a conductor equals the voltage V, i.e. $RI = V$. If the voltage remains constant, then a large resistance corresponds to a small current.
- A resistor is an object with high resistance which is placed in an electrical system to slow down the flow (current) of electricity. Resistors are measured in terms of ohms, and the larger the ohms, the smaller the current.

Figure 3.4 illustrates two introductory electrical systems. In this diagram, wires meet at nodes (illustrated with a dot). Batteries and voltage sources (represented by or other symbols) supply a voltage of E volts. At each node the current may change, so the arrows and letters i represent the different currents in the electrical system. The electrical current on each wire may or may not follow the arrows drawn (a negative current means that the current flows opposite the arrow). Resistors are depicted with the symbol , and the letter R represents the ohms.

Kirchoff discovered two laws. They both help us find current in a system, provided we know the voltage of any batteries, and the resistance of any resistors.

1. Kirchoff's current law states that at every node, the current flowing in equals the current flowing out (at nodes, current in = current out).

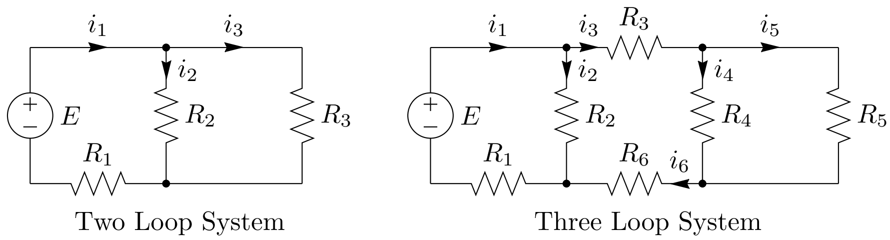


Figure 3.4: Electrical Circuit Diagrams.

2. Kirchoff's voltage law states that on any loop in the system, the directed sum of voltages supplied equals the directed sum of voltage drops (in loops, voltage in = voltage out). To use this law, pick a spot in the system. Then move around the system following a path that eventually gets you back to where you began (a closed curve). If you encounter a battery (a voltage source), then it counts as voltage in. If you encounter a resistor as you move with the current, then the voltage drop is Ri. If you encounter a resistor while moving opposite the current, then times by a negative to get a voltage drop of $-R_i$.

Let's use Kirchoff's laws to generate a system of equations for the two loop system. Remember that every time a current encounters a resistor, the voltage drop is $V = RI$, the product of the resistance and the current.

Problem 3.8 Consider the two loop system in figure 3.4. Assume that the voltage supplied from the battery E, as well as the ohms R_1, R_2, and R_3, on the resistors are known. The currents i_1, i_2, and i_3 are unknown.

1. Use Kirchoff's laws to explain how to obtain each of the equations below:

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ -i_1 + i_2 + i_3 &= 0 \\ R_1 i_1 + R_2 i_2 &= E \\ -R_2 i_2 + R_3 i_3 &= 0. \\ R_1 i_1 + R_3 i_3 &= E. \end{aligned}$$

   [Hint: If you encounter a resistor while moving backwards along a loop, then times the voltage drop becomes a voltage gain (times by a negative).]

2. Some of the equations above are linear combinations of the other equations. How could you obtain the 2nd and 5th as a linear combination of the others?

3. If $E = 12$, $R_1 = 2$, $R2 = 3$, and $R3 = 6$, then solve the system of equations above by row reducing an appropriate matrix.

Problem 3.9 Consider the three loop system in figure 3.4. Assume that the voltage supplied from the battery E and that the ohms R_j on the resistors are known. The currents are unknown.

1. There are 4 nodes in this system. Write the 4 equations we obtain by remember that the flow in at a node must equal the flow out.

2. There are three inner loops in the system above. Write the equations formed by going around each inner loop. [To get an inner loop, pick any point in the system. Then move in a clockwise fashion around the loop

3. Some of the equations above are linear combinations of the other equations. How could you obtain the 2nd and 5th as a linear combination of the others?

4. Why will row reducing the following matrix give you the unknown currents?

$$\left[\begin{array}{cccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ R_1 & R_2 & 0 & 0 & 0 & 0 & E \\ 0 & -R_2 & R_3 & R_4 & 0 & R_6 & 0 \\ 0 & 0 & 0 & -R_4 & R_5 & 0 & 0 \end{array}\right].$$

[Don't row reduce this matrix.]

Problem 3.10 Consider the three loop system in figure 3.4. If $E = 12$, $R1 = 1$, $R2 = 1$, $R3 = 1$, $R4 = 1$, $R5 = 1$, $R6 = 1$ then find the unknown currents by row reducing the matrix in part 4 above. Use a computer to check your answer. The row reduction is quite short, because the matrix is sparse (has lots of zeros).]

3.2.3 Markov Processes

Matrices can be used to model a process called a Markov Process. To fit this kind of model, a process must have specific states, and the matrix which models the process is a transition matrix which specifies how each state will change through a given transition. An example of a set of states is "open" or "closed" in an electrical circuit, or "working properly" and "working improperly" for operation of machinery at a manufacturing facility. A car rental company which rents vehicles in different locations can use a Markov Process to keep track of where their inventory of cars will be in the future. Stock market analysts use Markov processes and a generalization called stochastic processes to make predictions about future stock values.

Problem 3.11 Suppose we own a car rental company which rents cars in Idaho Falls and Rexburg. The last few weeks have shown a weekly trend that 60% of the cars which are rented in Rexburg will remain in Rexburg (the other 40% end up in Idaho Falls). About 80% of the cars which are rented in Idaho Falls will remain in Idaho Falls (the other 20% end up in Rexburg).

1. If there are currently 60 cars in Rexburg and 140 cars in IF, how many will be in each city next week? In two weeks?

2. Let R_n and I_n be the number of cars in Rexburg and Idaho Falls, respectively, at the beginning of the nth week (so $R_0 = 60$ and $I_0 = 140$). Obtain a matrix A so that $A\begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} R_1 \\ I_1 \end{pmatrix}$. Then check that $A\begin{pmatrix} R_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} R_2 \\ I_2 \end{pmatrix}$.

3. We would like to know if the number of cars will stabilize in each city. This would mean that if the current week's car totals are R and I, then we could find the next week's totals by solving the system

$$A\begin{pmatrix} R \\ I \end{pmatrix} = \begin{pmatrix} R \\ I \end{pmatrix},$$

the totals don't change. This is called a steady state solution. Find the steady state solution.

4. In the long run, what proportion of the cars will end up in Rexburg?
5. Because the system $A\begin{pmatrix} R \\ I \end{pmatrix} = \begin{pmatrix} R \\ I \end{pmatrix}$ had a nonzero solution, we know something about the eigenvalues of the matrix A. What is an eigenvalue of A?

(We'll answer 4 and 5 in class if you are unable. The key parts are 1-3.)

The matrix A found above is called a transition matrix. It's the matrix which tells you how to move from the current state $\vec{x}_n$ to the next state $\vec{x}_{n+1}$. This means we have

$$\begin{aligned}
\vec{x}_1 &= A\vec{x}_0 \\
\vec{x}_2 &= A\vec{x}_1 = A(A\vec{x}_0) = A^2\vec{x}_0 \\
\vec{x}_3 &= A\vec{x}_2 = A(A\vec{x}_1) = \cdots = A^3\vec{x}_0 \\
\vec{x}_4 &= A\vec{x}_3 = A(A\vec{x}_2) = \cdots = A^4\vec{x}_0 \\
&\vdots
\end{aligned}$$

You can find the nth state by computing $\vec{x}_n = A^n\vec{x}_0$, just raise the matrix to a power, and times by the initial state. Let's use this idea once more.

Problem 3.12 In a certain town, there are 3 types of land zones: residential, commercial, and industrial. The city has been undergoing growth recently, and the city has noticed the following 5 year trends.

- Every 5 years, they've notice that 10% of the residential land gets rezoned as commercial land, while 5% of the residential land gets rezoned as industrial. The other 85% of residential land remains residential.
- For commercial land, 70% remains commercial, while 10% becomes residential and 20% becomes industrial.
- For industrial land, 60% remains industrial, while 25% becomes commercial and 15% becomes residential.
- Currently the percent of land in each zone is 40% residential, 30% commercial, and 30% industrial.

Let's assume that these trends continue over an extended period of time.

1. The current state is $\vec{x}_0 = (40, 30, 30)$. After 5 years, what percentage of land will be zoned residential? Commercial? Industrial? Answering this question should give you the transition matrix A so that $\vec{x}_1 = A\vec{x}_0$.
2. Use software to find $\vec{x}_2$, $\vec{x}_3$, and $\vec{x}_4$ (the land use percentages after 10, 15, and 20 years).
3. Find the steady state solution to this Markov Process by solving $A\vec{x} = 1\vec{x}$ (i.e., the eigenvector corresponding to the eigenvalue $\lambda = 1$.)

Problem 3.13 Consider three occupations, farming, manufacturing, and clothing. Assume that goods are exchanged between the communities through barter only. Here is how the communities exchange their goods.

- The farming community keeps 1/2 of their goods, giving 1/4 to manufacturing and 1/4 to clothing.
- The manufacturing community keeps 1/3 of their goods, giving 1/3 to farming and 1/3 to clothing.
- The clothing community keeps 1/4 of their goods, giving 1/2 to farming and 1/4 to manufacturing.

Answer the following questions.

1. Suppose that all the commodities have the exact same value. If each group starts out with 12 units of their commodity, then after 1 round of bartering, how many units will each group have? Along the way you should produce a transition matrix A so that $A\begin{pmatrix}12\\12\\12\end{pmatrix}$ gives the answer.
2. Let x_1 be the value of the goods produced by farming. Let x_2 be the value of the goods produced by manufacturing. Let x_3 be the value of the goods produced by clothing. We would like to assign a value to each commodity so that each group gets a fair deal when they barter. To do this, we need to have the value of goods obtained after bartering to match the value of the goods obtained before. Explain why we can obtain this by solving the equations
$$\begin{bmatrix}\frac{1}{2}&\frac{1}{3}&\frac{1}{2}\\\frac{1}{4}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{3}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$
Then solve this equation.
3. (We'll answer this one in class. Try to come up with an answer yourself.) If the value of commodities from each group is the same, who is getting the better deal? To make sure bartering results in a fair deal for all, should farming commodities be more expensive, or less expensive than the others?

3.3 Cramer's Rule

Gabriel Cramer developed a way to solve linear systems of equations by using determinants. For small systems, the solution is extremely fast. However, for large systems, the method looses it's power because of the complexity of computing determinants. Also, when the coefficients in the system are variables, Cramer's rule provides an extremely fast algorithm for computing determinants. I'll remind you occasionally throughout the problem set to apply Cramer's rule when the problem involves variable coefficients.

Theorem 3.3 (Cramer's Rule). *Consider the linear system given by $A\vec{x}=\vec{b}$, where $A=\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots\vec{v}_n\end{bmatrix}$ is an n by n matrix whose determinant is not zero. Let $D=|A|$. For each i, replace vector $\vec{v}_i$ with $\vec{b}$, and then let D_i be the determinant of the corresponding matrix. The solution to the linear system is then*
$$x_1=\frac{D_1}{D},\quad x_2=\frac{D_2}{D},\quad\cdots\quad x_n=\frac{D_n}{D}.$$

For the 2 by 2 system
$$\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix},$$

Cramer's rule states the solution is (provided $|A| \neq 0$)

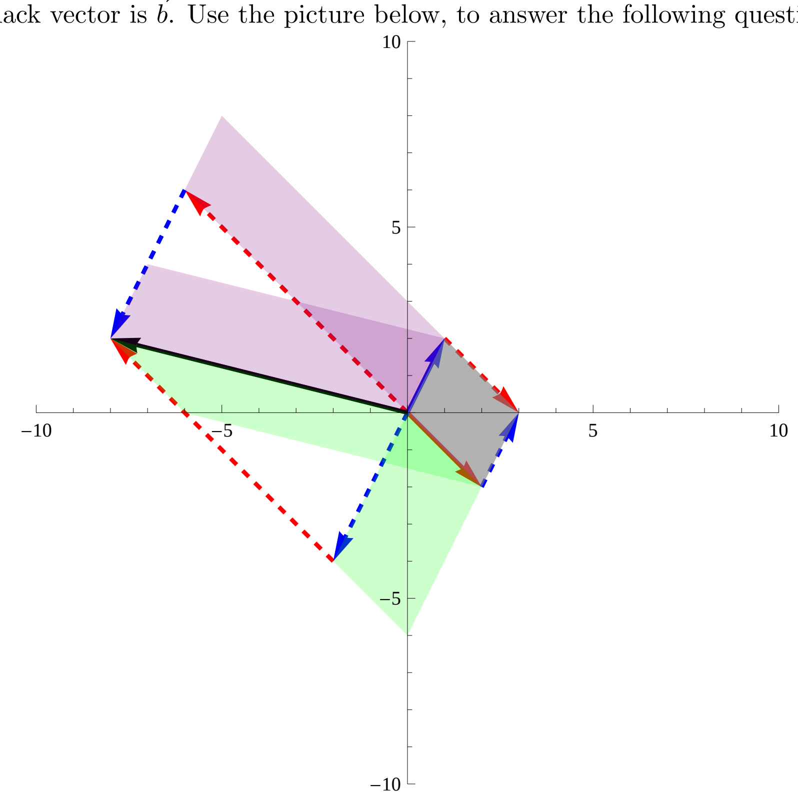
$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Problem 3.14 Consider the system of equations $x + 2y = 3, 4x + 5y = 6$. Solve this system in 2 different ways.

1. Use Cramer's rule to solve the system. You just need to compute three 2 by 2 determinants.
2. Use row reduction to solve the system. Show the steps in class.

In the next problem, you'll provide a proof of Cramer's rule in 2D. Your proof will contain the key idea needed to prove the theorem in all dimensions. The key idea is to connect determinants to areas of parallelograms.

Problem 3.15: Proof of Cramer's Rule Let $\vec{v}_1 = (2, -2)$ and $\vec{v}_2 = (1, 2)$. Let $x_1 = -3$ and $x_2 = -2$, which means that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 = (-8, 2)$. In the picture below, the solid red vector is $\vec{v}_1$, the solid blue vector is $\vec{v}_2$, and the solid black vector is $\vec{b}$. Use the picture below, to answer the following questions.



[Hint: Each question can be answered by thinking about determinants as areas.]

1. Explain why $x_1 \begin{vmatrix} \vec{v_1} & \vec{v}_2 \end{vmatrix} = \begin{vmatrix} x_1\vec{v_1} & \vec{v}_2 \end{vmatrix}$. Then explain why $\begin{vmatrix} x_1\vec{v_1} & \vec{v}_2 \end{vmatrix} = \begin{vmatrix} \vec{b_1} & \vec{v}_2 \end{vmatrix}$. Finally, solve for x_1 to show

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

2. In a similar fashion, obtain a formula for x_2.

Problem 3.16 In problem 3.8 we obtained needed to solve the system of equations

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ R_1 i_1 + R_2 i_2 &= E \\ -R_2 i_2 + R_3 i_3 &= 0. \end{aligned}$$

Write the corresponding system of equations, and then use Cramer's rule to obtain the general solution for the unknown currents. You should have i_1, i_2, and i_3 all written in terms of R_1, R_2, R_3, and E.

Cramer's rule is most useful when the coefficients in the linear system are variables, rather than numbers. Let's apply our knowledge to study the arms race (the building of armies - tanks, bombs, soldiers, etc. - between two countries). Consider two countries, country A and country B. As country B builds up their military, country A looks on and says "Hmm, we better build up our military." Similarly, as country A builds up their military, country B looks and says, "Hmm, we better build up our military." If country A has a grudge against country B, they will probably build up their military regardless of what country B does. Similarly, any past grievances and grudges that country B has against country A will increase the rate at which country B builds up their military. Building up a military costs money, so hopefully both countries have economic limitations that restrict the growth of their military. The real question behind the arms race is, "Will the two countries eventually decide they are spending enough on their military, or will their spending continue to grow without bound."

We now develop a system of differential equations that describes the above. The key principle is a general law of conservation:

> The change in a quantity equals the flow in of the quantity minus the flow out of the quantity, or more simply
>
> Change = (Flow in) - (Flow out)
>
> Change = (Increase) - (Decrease)

- Let x represent the dollar amount per year that country A spends on arms. Let y represent the dollar amount per year that country B spends on arms.

- When y is large, country A will respond by increasing their spending. We'll assume this change is proportional to y, so we see that x increases by an amount ay. Similarly, when x is large, country B responds by increasing their spending. Let's assume that y increases by an amount mx.

- The economy of each country tries to slow down the growth rate. The more money country A spends, the larger the effect of the economy. We'll assume that x decreases by an amount bx. Similarly, we'll assume y decrease by an amount ny.

- If the countries hold grudges against each other for past grievances, then they are inclined to increase their spending regardless of economic factors and the growth of the other country's army. Let c represent the amount that country A will increase their spending by, and let p represent the amount that country B will increase their spending by. These values might be zero (for example the US and Canada do not hold such grudges), but might not be zero at all (as was the cases during the cold war, between the US and USSR).

Problem 3.17 Read the arms race information above, and then answer the following questions.

1. There are three things causing x to change. The flow in (parts causing an increase) are ay and c, the response to the other country, and any grudges. The flow out (parts causing a decrease) is only bx, the economic restriction. We can write this as a differential equation
$$\frac{dx}{dt} = ay - bx + c.$$
Obtain a similar equation for $\frac{dy}{dt}$ (using the coefficients m, n, and p). Then write your system of ODEs in the form
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -b & a \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ ? \end{bmatrix}.$$

2. An equilibrium solution to the system of differential equations above is a solution that remains stable. At equilibrium, there should not be any future change in x nor y, so we should have $dx/dt = 0$ and $dy/dt = 0$. Find the equilibrium solution for the arms race problem. [Cramer's rule should make this really fast.]

3. Find the eigenvalues of the square matrix from part 1. What conditions must be met so that both eigenvalues are negative? In class, we'll pick some positive values for a, b, c, m, n, p that satisfy the conditions you tell us, and then graph the vector field $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{p}$, along with some solution curves.

3.4 Curve Fitting

3.4.1 Interpolating Polynomials

Through any two points (with different x values) there is a unique line of the form $y = mx + b$. If you know two points, then you can use them to find the values m and b. Through any 3 points (with different x values) there is a unique parabola of the form $y = ax^2 + bx + c$, and you can use the 3 points to find the values a, b, c. As you increase the number of points, there is still a unique polynomial (called an interpolating polynomial) with degree one less than the number of points, and you can use the points to find the coefficients of the polynomial. In this section we will find interpolating polynomials, and show how the solution requires solving a linear system.

To organize our work, let's first standardize the notation. Rather than writing $y = mx + b$, let's write $y = a_0 + a_1 x$ (where $a_0 = b$ and $a_1 = m$). For a parabola, let's write $y = a_0 + a_1 x + a_2 x^2 = \sum_{k=0}^{2} a_k x^k$. We can now write any polynomial in the form
$$y = a_0 + a_1 x + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k.$$
By standardizing the coefficients, we can use summation notation to express any degree polynomial by changing the n on the top of the summation sign.

Problem 3.18 Answer the following by row reducing an appropriate matrix. Please show us the steps in your row reduction. [Hint: Each point produces an equation.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1x$ that passes through the points $(1, 2)$ and $(3, 5)$. [We could have use m and b, but I chose to use a_0 and a_1 so you can see how this generalize quickly to all dimensions.]

2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1x^1 + a_2x^2$ that passes through the points $(0, 1)$, $(2, 3)$, and $(1, 4)$. [Hint: The second point produces the equation $3 = a_0 + a_1(2) + a_2(2)^2$.]

Problem 3.19 Give an equation of a cubic polynomial $y = a_0+a_1x^1+a_2x^2+a_3x^3$ that passes through the four points $(0, 1)$, $(1, 3)$, $(1, 4)$, and $(2, 4)$. Show us the steps in your row reduction. [Hint: Each point produces an equation. You should have a linear system with 4 equations and 4 unknowns.]

Problem 3.20 Solve the following. [Hint: Because the problem involves variable points, Cramer's rule will be much faster than row reduction.]

1. Find the intercept a_0 and slope a_1 of a line $y = a_0 + a_1x$ that passes through the points (x_1, y_1) and (x_2, y_2).

2. Find the coefficients a_0, a_1, and a_2 of a parabola $y = a_0 + a_1x^1 + a_2x^2$ that passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Under what conditions will your solutions above not be valid?

If we collect 2 data points, then we can usually find an equation of a line that passes through them. If we collect 3 data points, we can usually find an equation of a parabola passing through them. Continuing in this fashion, if we collect $n + 1$ data points, then we can usually find an equation of a polynomial of degree n that passes through them.

Problem 3.21 Suppose that we collect the 6 data points $(1, 1)$, $(2, 3)$, $(-1, 2)$, $(0, -1)$, $(-2, 0)$, $(3, 1)$. We would like to find a polynomial that passes through all 6 points. State the degree n of this polynomial. Then find the coefficients $a_0, a_1, \ldots, a_n$ of this polynomial. Please use technology to do your row reduction. When you present in class, show us the matrix you entered into a computer, and then show us the reduced row echelon form together with the polynomial.

3.4.2 Least Squares Regression

Interpolating polynomials give a polynomial which passes through every point listed. While they pass through every point in a set of data, the more points the polynomial must pass through, the more the polynomial may have to make large oscillations in order to pass through each point. Sometimes all we want is a simple line or parabola that passes near the points and gives a good approximation of a trend in the data. When I needed to purchase a minivan for my expanding family, I gathered mileage and price data for about 40 cars from the internet. I plotted this data and discovered an almost linear downward trend (as mileage increased, the price dropped). Using this data I was able to create a line to predict the price of a car. I then used this data to talk the dealer into dropping the price of their car by over \$1000. Finding an equation of this line, called the least squares regression line, is the content of this section. In other words, if you have 3 or more points, how do you find a line that is closest to passing through these points? The least squares regression line is used to find trends in many branches of science, in addition to haggling for lower prices when buying a car. Statistics builds upon this idea to provide powerful tools for predicting the future.

Problem 3.22 Consider the three points $(2,4)$, $(0,1)$, and $(3,5)$. We wish to find a line $y = a_0 + a_1 x$ that fits this data.

1. What 3 equations do the points and line give. Write the linear system as a matrix equation by filling in A and $\vec{b}$ below:

$$A\vec{x} = \vec{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ ? \\ ? \end{bmatrix}.$$

   The first equation $4 = a_0 + a_1(2)$ is already on the first row.

2. Row reduce the corresponding augmented matrix to show that this system has no solution. The problem is that we have more equations than we do unknowns. The system is overdetermined.

3. If we multiply both sides of the equation $A\vec{x} = \vec{b}$ by a 2 by 3 matrix C, then the product CA will be a 2 by 2 matrix. We could then solve the system $CA\vec{x} = Cb$, as it would then have 2 equations and 2 unknowns.

   The only 2 by 3 matrix in the problem is the transpose of A. So compute $A^T A$ and $A^T\vec{b}$. Then solve the system $(A^T A)\vec{x} = A^T\vec{b}$.

The previous problem suggests the following theorem. One proof of this theorem involves projecting $\vec{b}$ onto the plane spanned by the columns of A. This proof leads to the ideas behind inner product spaces, the Graham Schmidt orthogonalization process, and more, something you would study near the end of math 341 (Linear Algebra).

Theorem 3.4 (Least Squares Regression)**.** *When we collect n data points and notice the points follow a linear trend, the coefficients of the least square regression line $y = a_0 + a_1 x$ are the solutions to the equation $A^T A\vec{x} = A^T\vec{b}$, where we have*

$$\vec{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \vec{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \text{ and } A^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Problem 3.23 Suppose you collect the n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and you wish to find the least squares regression line $y = a_0 + a_1 x$. Set up the matrices A, $\vec{x}$, $\vec{b}$, and A^T. Multiply together $A^T A$ and $A^T\vec{b}$ (your result should involve sums of the form $\sum x_i$, $\sum y_i$, $\sum x_i y_i$, and $\sum x_i^2$). Then solve the equation $A^T A\vec{x} = A^T\vec{b}$ and state the coefficients a_0 and a_1. leas[Hint: Since the system involves variable coefficients, try using Cramer's rule. It will kick out the solution almost instantly.]

The key to solving the overdetermined system $A\vec{x} = \vec{b}$ is to multiply each side on the left by a matrix C, so that the produce CA is a square matrix. We then solve $CA\vec{x} = C\vec{b}$. The least square regression model comes by letting $C = A^T$. We obtain alternate data fitting models by using a matrix other than A^T (though this is a topic for another course). The next problem has you find the best fitting parabola, using the least square regression model.

Problem 3.24 Consider the 5 points $(-2,3)$, $(-1,1)$, $(0,-1)$, $(1,2)$, $(2,4)$, and . We would like to find an equation of a parabola $y = a_0 + a_1x + a_2x^2$ that approximates the trend in the data, using the least square regression model.

1. The 5 data points produce 5 equations in the three unknowns a_0, a_1, a_2. Write the linear system as a matrix equation by filling in A and $\vec{b}$ below:

$$A\vec{x} = \vec{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & -2 & 4 \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

2. Multiply both sides of the equation $A\vec{x} = \vec{b}$ by an appropriate 3 by 5 matrix C. Then solve the system $(CA)\vec{x} = C\vec{b}$. Feel free to use software to obtain your answer. In class, just show us CA, $C\vec{b}$, and the rref of $\begin{bmatrix} CA & C\vec{b} \end{bmatrix}$.

3. Plot the 5 data points and the parabola you found.

The next problem has the exact same solution as Problem 3.23, but does not require you to use a matrix transpose, nor matrix multiplication. Instead, it focuses on setting partial derivative equal to zero, which is the first step in locating minimums. You then just have to solve a system of linear equations.

Problem 3.25 Suppose you collect the n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and you wish to find the least squares regression line $y = a_0 + a_1x$. Each point (x_i, y_i) produces an error $y - y_i = (a_0 + a_1x_i) - y_i$. The least squares regression line is the line that minimized the sum of the squares of these errors, which means we need to minimize

$$f(a_0, a_1) = \sum_{i=1}^{n} \left((a_0 + a_1x_i) - y_i\right)^2.$$

1. Compute $\dfrac{\partial f}{\partial a_0}$ and $\dfrac{\partial f}{\partial a_1}$.

2. Since we seek the minimum of f, solve the system $\dfrac{\partial f}{\partial a_0} = 0$ and $\dfrac{\partial f}{\partial a_1} = 0$ for a_0 and a_1.

[Hint: Once you get each equation written in the form $(?)a_0 + (?)a_1 = ?$, use Cramer's rule to kick out the answer almost instantly.]

3.5 Partial Fraction Decompositions

A partial fraction decomposition is a method of breaking a complex rational function up into the sum of smaller simpler functions to work with. We will be using partial fraction decompositions to rapidly solve differential equations throughout the semester (using Laplace transforms). For now, we will start by gaining practice with partial fraction decompositions by integrating rational functions. To illustrate their value, let's start with an example.

Problem 3.26 Our goal is to integrate the function $f(x) = \dfrac{2x+1}{(x-2)(x-5)}$. The denominator is the product of two linear functions. Suppose we can write

$$\frac{2x+1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

for unknown constants A and B. Multiply both sides of the above equation by the denominator $(x-2)(x-5)$. Then solve for the constants A and B (try plugging in some numbers to get a system of equations). Then compute

$$\int \frac{2x+1}{(x-2)(x-5)} dx = \int \frac{A}{x-2} dx + \int \frac{B}{x-5} dx.$$

[Hint: Two lines are the same if and only if they have the same slope an intercept. You should have an equation that says two lines are equal, so equate the coefficients. Alternately, just pick two x values, plug them in, and you should get two different equations relating A and B.]

Problem 3.27 We can write

$$f(x) = \frac{2x-3}{x^2(x-3)} = \frac{Ax+B}{x^2} + \frac{C}{x-3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}.$$

Multiply both sides by the denominator $x^2(x-3)$, and then solve for the constants A, B, and C (show us the matrix you row reduced, and the rref). Then compute the integral of $f(x)$. [Hint: To get three equations, you can (1) equate the coefficients, or (2) pick 3 x values and plug them into the equation.]

Problem 3.28 We can write

$$f(x) = \frac{x^2-2}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}.$$

Multiply both sides by $(x^2+4)(x+1)$ and then solve for the constants A, B, and C (show us the matrix you row reduced, and the rref). Then compute the integral of $f(x)$ (you'll need a u-sub for one of the integrals). You'll probably want to split the numerator up, and then integrate three parts, instead two, as we can write

$$f(x) = \frac{Ax}{x^2+4} + \frac{B}{x^2+4} + \frac{C}{x+1}.$$

Problem 3.29 We can write

$$\begin{aligned} f(x) &= \frac{1}{(x+4)^3(x-3)} \\ &= \frac{A(x+4)^2+B(x+4)+C}{(x+4)^3} + \frac{D}{x-3} \\ &= \frac{A}{(x+4)} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3} + \frac{D}{x-3}. \end{aligned}$$

Multiply both sides by the denominator of the original, and then solve for the unknown constants (show us the matrix you row reduced, and the rref). Then compute the integral of $f(x)$.

3.6 Linear Functions

We need one more bit of vocabulary before embarking on solving differential equations. You've seen the concept of a linear function in many settings, without knowing it. First let's get the definition, see an example, and then we'll discuss it more.

Definition 3.5. When the domain D and range R of a function involves quantities that can be added and multiplied by scalars, we say that the function $f : D \to R$ is linear, provided the following occurs:

1. $f(x+y) = f(x) + f(y)$ and
2. $f(cx) = cf(x)$.

The function preserves addition and scalar multiplication.

The next examples shows that the concepts of differentiation and integration are linear functions. Often when the domain is a group of functions, we'll say that the function is a linear operator (instead of a linear function).

Problem 3.30 Consider the linear operator $\frac{d}{dx}$. For simplicity, we'll assume the operator has as its domain the set of all differentiable functions that are differentiable on the entire real line. The range (or codomain) is the set of all functions. The operator $\frac{d}{dx}$ requires a differentiable function as an input, and returns a function as output.

1. Show that $\frac{d}{dx}$ is a linear operator.
2. Consider the linear operator $\int f(x)dx$. What is the domain and codomain of the integration operator? Is it linear?
3. Consider the linear operator $L\{f(t)\}$, the Laplace transform. Show it is a linear operator, and state the domain and codomain. [We'll tackle this part together in class. Have a guess for the answers.]

In calculus you learned that you can differentiate a sum by differentiating each piece separately (term-by-term differentiation), and that you can pull constants out. Similarly, you learned that you can do integration term-by-term, and constants come out. These are precisely the key properties behind a function (operator, transformation) being linear.

We'll now see that EVERY matrix represents a linear transformation, and that every linear transformation between finite dimensional vector spaces is really just a matrix product.

Problem 3.31 Suppose that we have a linear transformation $f:\mathbb{R}^3\to\mathbb{R}^2$. We know that $f(1,0,0)=(1,3)$, $f(0,1,0)=(-2,4)$, and that $f(1,1,1)=(3,1)$.

1. Write $(0,0,1)$ as a linear combination of $(1,0,0)$, $(0,1,0)$, and $(1,1,1)$.
2. Use the fact that f is linear to compute $f(0,0,1)$, and then $f(x,y,z)$. [Write (x,y,z) as a linear combination of the standard basis vectors.]
3. Obtain a matrix A so that $f\begin{pmatrix}x\\y\\z\end{pmatrix}=A\begin{pmatrix}x\\y\\z\end{pmatrix}$.
4. Find the vectors $\vec{x}$ so that $A\vec{x}=\vec{0}$.

Problem 3.32 Suppose that $A=\begin{bmatrix}1&3&4\\-2&0&-2\\0&1&1\end{bmatrix}$, and consider the function $f\vec{x}=A\vec{x}$.

1. What are $f(1,0,0)$, $f(0,1,0)$, $f(0,0,1)$, and $f(2,3,0)$? What is $f(x,y,z)$?
2. Show that $\vec{f}$ is a linear function.
3. Find (x,y,z) such that $f(x,y)=(5,-2,1)$, or explain why it is not possible.
4. The set of possible outputs of f is an object in 3D. Describe that object.
5. Find the vectors $\vec{x}$ such that $f(\vec{x})=\vec{0}$.

If you need to rref a matrix above, please use technology to do so. In class, only show us the matrix, and its rref.

Make sure you ask me in class to visually show you a representation of the linear functions above. There's a ton more that we could study about linear functions, and I'd like to introduce to you some of those ideas.

Matrices provide us with the key examples to understanding linear transformations. However, a matrix by nature requires that we look at functions between finite dimensional spaces. The key linear transformations we will study throughout the semester will involve infinite dimensional spaces (like the space of all differentiable functions). Most of the ideas we have learned will still be useful to us as we explore functions between infinite dimensional vector spaces. Near the end of the semester, we'll even start discussing eigenvalues and eigenfunctions of linear transformations. You'll then explore these concepts in greater detail in many of your future classes.

We now make one final definition as we wrap up this chapter. We want a word that quickly tells us which vectors are mapped to zero. We'll be using this vocabulary often as we solve differential equations. In each of the previous two problems, the last portion asked you to find the kernel of the linear transformation.

Definition 3.6: Kernel. The kernel of a linear function f is the collection of vectors that are mapped to zero, i.e. $f(\vec{x})=\vec{0}$.

Once you have a collection of vectors in the kernel of a linear function, you can use those vectors to obtain lots of other vectors. Any linear combination of vectors in the kernel will remain in the kernel. The next problem asks you to show why.

Problem 3.33 Suppose that $\vec{x}$ and $\vec{y}$ are both in the kernel of a linear function f. Show that any linear combination of $\vec{x}$ and $\vec{y}$ are also in the kernel of f. [Hint: What is $f(\vec{x})$, $f(\vec{y})$, and then what is $f(a\vec{x}+b\vec{y})$? Be able to explain every step in your work, by telling us what definition you are using.]

The previous problem shows us that the kernel is closed under linear combinations. You can't get out of the kernel by performing linear combinations of things that are in the kernel. We now end this chapter with an example to illustrate how we will use the words "linear" and "kernel" throughout the semester. Most of the remainder of this course deals with finding the kernel of a linear function.

Problem 3.34 Consider the differential equation $y' - 3y = 0$. Let L be the operator $L(y) = y' - 3y$. With the operator notation, we can rewrite the differential equation as $L(y) = 0$ (so we need to find the kernel of L).

1. What is the domain of L?
2. Show that L is a linear operator by computing $L(y_1 + y_2)$ and $L(cy)$.
3. Solve the differential equation $y' - 3y = 0$ by using separation of variables.
4. Obtain a single solution (no unknown constants) to the ODE.
5. Using the single solution, can you obtain all solutions as a linear combination of the single solution?

The solutions to the first order ODE $y' - 3y = 0$ are linear combinations of a single solution. This is precisely because the ODE is a linear first order ODE. If we had a 2nd order linear ODE, then solution would be all linear combinations of two independent solutions. The next problem introduces this idea.

Problem 3.35 Consider the differential equation $y'' + 3y' + 2y = 0$. Let L be the operator $L(y) = y'' + 3y' + 2y$. With the operator notation, we can rewrite the differential equation as $L(y) = 0$ (so we need to find the kernel of L).

1. What is the domain of L?
2. Show that L is a linear operator by computing $L(y_1 + y_2)$ and $L(cy)$.
3. Show that both e^{-2x} and e^{-x} are in the kernel of L.
4. Are e^{-2x} and e^{-x} linearly independent? Why?
5. Why is $y = c_1e{-}2x + c_2e^{-x}$ a solution to the differential equation $y'' + 3y' + 2y = 0$?

3.7 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 4

First Order ODEs

This chapter covers the following ideas.

1. Be able to interpret the basic vocabulary of differential equations. In particular, interpret the terms ordinary differential equation (ODE), initial value, initial value problem (IVP), general solution, and particular solution.
2. Identify and solve separable and exact ODEs.
3. Use integrating factors and substitution to solve additional ODEs.
4. Use the three step modeling process (express, solve, and interpret) to analyze exponential growth and decay, Newton's law of cooling, mixing, and the logistics equation.
5. Use Laplace transforms to solve first order ODEs.

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the problem set, please use this list as a guideline to find related problems.

Concept	Sec.	Suggested	Relevant
Separable Review	4	42	1-8,23-45
Exact	5	5,11,26,29,34	1-13,24-40,56-65
Integrating Factors	5	21,22,41,47	21,22,41-42,47-49,51,55
Linear	6	4,13,20,32,51	1-6,9-15,20-36,43-49,50-57
Homogeneous	4,	11,12,48	11-17,46-54
Bernoulli	6	16,53	16,17,37-42,53
Applications	7 7	4[27],6[33],1[38] 10[48],17[67],7[88]	1-6 [26-44] 8-10 [45-50],16-18[65-70], 7[87-88]
Laplace Review	21	19,32,33[use table]	4-7,10-12,27-35
Inverse Transforms	22	1,2,3,6,13,15	1-3,6,15,17,20-28,42,42,45-47
Solving ODEs	24	1,14,19(parfrac)	1,2,11,14,15,19-19,22,24,25,38-42

4.1 Basic Concepts and Vocabulary

Let's start this chapter with a review problem from the first chapter.

Problem 4.1 Solve the ordinary differential equation $y' - 5y = 0$. Then use the initial condition $y(0) = 7$ to obtain the unknown constant.

Definition 4.1: Differential Equation Language. A differential equation is an equation which involves derivatives (of any order) of some function.

- An **ordinary differential equation (ODE)** is a differential equation involving a function $y(x)$ whose domain is one dimensional. The function only has ordinary derivatives.
- A **partial differential equation (PDE)** is a differential equation involving a function $y(x_1, x_2, \ldots)$ whose domain is more than one dimensional. The function has partial derivatives.
- The order of an ODE is the largest order derivative that appears in the ODE.
- A solution to an ODE on an interval (a, b) is a function $y(x)$ defined on the interval (a, b) which satisfies the ODE.

To verify that a function is a solution to an ODE, calculate derivatives and put them in the ODE. If the resulting equation is an identity for all $x \in (a, b)$, then you have verified that you have a solution.

Typically a solution to an ODE involves an arbitrary constant C. There is often an entire family of curves which satisfy a differential equation, and the constant C just tells us which curve to pick.

Definition 4.2: Initial Value Problems (IVP). Often an ODE comes with an **initial condition** $y(x_0) = y_0$ for some values x_0 and y_0.

- A **general solution** of an ODE is all possible solutions of the ODE.
- A **particular solution** is one of infinitely many solutions of an ODE.
- We can use the initial conditions to find a particular solution of the ODE.
- An ODE, together with an initial condition, is called an **initial value problem (IVP)**.

This next problem has you practice with the vocabulary above. You'll want to use separation of variables to solve this problem.

Problem 4.2 Consider the IVP $y' - 4y = 8$, where $y(0) = 3$.

1. What is the order of the ODE?
2. Obtain a general solution of the ODE. State an interval on which your general solution is valid.
3. Verify that your general solution is a solution to the ODE.
4. Solve the IVP.

One of the key uses of differential equations is their ability to model the world around us. If something is changing, then we can often use y' to represent that change. If we know a force is acting on an object, then $F = ma = my''$ allows us to build a differential equation that models the motion of the object. As the semester progresses, we'll be making these connections in each chapter, and showing how to use differential equations to model the world. We'll also see that eigenvalues and eigenvectors are the connecting piece that allows us to see, and obtain, the solution to differential equations. Many of the models we build will depend on observing that a change is proportional to something, or that a force is proportional to something. If you've forgotten what proportional means, here's a definition.

Definition 4.3: Proportional. We say that y is proportional to x if $y = kx$ for some constant k. We call the constant k the proportionality constant. When two quantities are proportional, then doubling one will double the other, tripling one will triple the other, and so on. A percentage change to one is matched by the other.

The next problem has us build our first model. Suppose you go to the doctor's office to get a strep test done. They swab the back of your throat and then put a sample of tissue from your body in a petri dish. If you have strep, then the bacteria will grow inside the petri dish, and they'll be able to see the rapid growth of the strep bacteria in a fairly short amount of time.

Problem 4.3: Exponential Growth Suppose that you place some bacteria in a petri dish. Initially, there are P mg of the bacteria in the dish, and then the bacteria starts to reproduce, so the amount of the bacteria is changing. Let $y(t)$ represent the mg of bacteria in the dish after t days. Then y' would represent the rate at which y is changing. The rate at which y grows depends on how large y is. If you were to double y, then the growth rate y' should double as well. Similarly, if you tripled y, then the growth rate y' would triple as well. It seems reasonable to assume that y' is proportional to y.

1. Express the statement "y' is proportional to y" as a differential equation. What are the initial values (if any)?

2. Solve the differential equation above, obtaining a general and particular solution.

3. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large?

4. If after 10 minutes you measure 5 mg of the bacteria, and then after 20 minutes you measure 8 mg of the bacteria, how much bacteria was present initially? [If you apply the natural logarithm to both sides of your solution, then you can solve a linear system of equations to obtain the unknowns $\ln P$ and k. You can then use Cramer's rule or RREF.]

The next problem is very similar to the previous, we'll just change the setting from growth of a bacterial culture, to growth of an investment.

Problem 4.4 Suppose you invest $P = \$10,000$ dollars in an account, and that the money accumulates interest at a constant rate. Let $A(t)$ represent the accumulated worth of your investment after the investment has had t years to grow. No new deposits are made, rather the interest is just left in the account to accumulate more interest.

1. Why is is reasonable to assume that A' is proportional to A?
2. Express the connection between A and it's growth as a differential equation. What are the initial values (if any)?
3. Solve the differential equation, obtaining a general and particular solution.
4. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large?
5. Suppose after 5 years that the value of the investment has reached \$18,000. How long will it take for the investment to reach \$100,000.

Let's look at one more application before introducing additional solution techniques. Here's the scenario. You decide to cook a turkey for Thanksgiving. You turn the oven on to 350°F, and the package says that you need to get the turkey heated up to an internal temperature of 165°F. You followed the instructions and thawed the turkey so that currently it's about 40°F. How long will it take for the turkey to heat up? If instead of heating a turkey, you wanted to heat a chicken patty, would the time vary? If you just wanted to heat a metal pan up, how would the time vary? The next problem introduces a simplistic model to examine this question. The model works best when you assume that an increase in heat is evenly distributed throughout an object (such as heating a metal pan). When you heat a turkey, the heat is not evenly distributed. This uneven heat distribution complicates the following model, and we'd need to explore PDEs to obtain a better model for heat flow. To simplify things, we'll assume that heat distributes itself evenly throughout the object.

Problem 4.5: Newton's Law of Cooling Suppose that you place an object in an oven. The oven temperature is set to A (you can use Fahrenheit, Celsius, or Kelvin). I'm using A as the temperature of the surrounding "a"tmosphere. The object's initial temperature is T_0. Let $T(t)$ represent the temperature of the object t minutes after we place the object in the oven. If $T(t)$ is really close to A, then the rate at which T increases should be pretty small, as the temperature of the object is almost the same as the temperature of the atmosphere. If T is really far from A, then the rate of temperature change should be a lot larger. Hence, it appears that T' depends on the difference $A - T$. Newton conjectured that the rate at which the temperature changes is proportional to the difference $A - T$.

1. Express the statement "the rate at which the temperature changes is proportional to the difference $A - T$" as a differential equation. What are the initial values (if any)?
2. Solve the differential equation above, obtaining a general and particular solution.
3. Interpret your solution in the context of the original problem. What does a typical graph of your solution look like (it's got some constants in it, but you can show the general shape). If your solution is correct, what will happen as t gets large? Does this seem reasonable.

Problem 4.6 You should have obtained the solution to Newton's law of Cooling as

$$T(t) = A + (T_0 - A)e^{-kt},$$

where k is the proportionality constant. Suppose that $T_0 = 45°\text{F}$ and $A = 350°\text{F}$.

1. After 5 minutes, you check the temperature and observe $T(5) = 80°\text{F}$. What is k, and how long will it take for the object to reach 165°F.
2. After 5 minutes, you check the temperature and observe $T(5) = 120°\text{F}$. What is k, and how long will it take for the object to reach 165°F.
3. The number k depends on the material you are trying to heat. If k is large, what does that mean about the material? Think of some examples where k would be large, and where k would be small.

You've now seen a few examples of how differential equations are used to model the world around us. You will most likely find that in your future courses, you'll be taking real world phenomenon and expressing the relationships you see as differential equations. Solving those differential equations gives us mathematical models we can use to interpret the world around us. There are three parts to this process.

- Express real world phenomenon in terms of a differential equation.
- Solve the differential equation.
- Interpret the solution in the context of the problem, which often involves using the results to predict behavior.

A main focus in this class will be the second portion, "Solve." As we all come from a different background, we won't have time to develop the background material that you'll explore in your respective majors, so the "Express" portion will often come in your major courses. You may find in some future courses that they focus on the "Express" and "Interpret" portions, and then refer you to some standard reference for the "Solve" part. The goal of our course is to help you develop the key solution techniques. Along the way, we'll occasionally add some simpler problems that we can "Express" and "Interpret" without needing a lot of background.

4.2 Solution Techniques

In the review chapter, we explored finding potentials of a gradient field. We also introduced the language of differential forms. Recall the following definition.

Definition 4.4: Differential Forms. Assume that f, M, N are all functions of two variables x, y.

- A differential form is an expression of the form $Mdx + Ndy$ (just as a vector field is a function $\vec{F} = (M, N)$).
- The differential of a function f is the expression $df = f_x dx + f_y dy$ (just as the gradient is $\vec{\nabla} F = (f_x, f_y)$).

- If a differential form is the differential of a function f, then we say the differential form is exact (just as we say a vector field is a gradient field). The function f is called a potential for the differential form. Let me reiterate. We say a differential form $Mdx + Ndy$ is exact if and only if there exists a function f such that
$$df = Mdx + Ndy.$$

A differential form is exact precisely when the corresponding vector field is a gradient field.

The next problem provides the key idea need to solve almost every differential equation we'll encounter in this course. If you can rewrite the differential equation in differential form, and the differential is exact, then solving the ODE requires that you find a potential.

Problem 4.7 Consider the differential form $(2x + 3y)dx + (3x)dy$.

1. By taking derivatives, show that the differential form is exact. [See the test for a conservative vector fields, problem 1.30.] Show that a potential for this differential form is $f(x, y) = x^2 + 3xy$.
2. Rewrite the differential equation $3xy' + 3y = -2x$ in the differential form
$$Mdx + Ndy = 0.$$
What's the angle between the vectors (M, N) and (dx, dy)?
3. Explain why the solution to $Mdx+Ndy = 0$ is a level curve of the potential $f(x, y)$.
4. Give the solution to $3xy' + 3y = -2x$ if $y(2) = 1$.

I'm trying to decide on a good name for the next theorem. We'll see that this theorem is crucial to solving just about EVERY differential equation we encounter from here on out, and it also solves all the ones before now. The name below might change, but something along the lines of "the sledgehammer," or "one tool to rule them all" would work. The theorem has no official name, so we can make one up as we go. Basically, we'll show that we can reduce almost every ODE that we solve to a form which allows us to apply the following theorem.

Theorem 4.5 (The sledgehammer for ODEs - one tool to rule them all)**.** *Suppose that $Mdx + Ndy$ is an exact differential form with potential $f(x, y)$. If we can write an ordinary differential equation in the form $Mdx + Ndy = 0$, then an implicit general solution to the ODE is $f(x, y) = c$. The level curves of a potential are precisely the solutions to the ODE. Let me repeat that. The level curves of a potential are precisely the solutions to the ODE.*

Let's use the previous theorem now to solve a couple of ODEs.

Problem 4.8 Give a general solution to each of the following ODEs. You may give your solution implicitly, so don't worry about solving for y. [Hint: Use the previous theorem.]

1. $(4x + 2y)dx + (2x + y)dy = 0$
2. $(x \cos(xy) + y)y' = \sin x - y \cos xy$

4.2.1 Use integrating factor when the ODE is not exact

Let's now return to a problem we've already solved, and show how we can use the sledgehammer theorem to solve things we've already seen, provided we add one more step.

Problem 4.9 Consider the ODE $y' = -3y$ which we can write in differential as $3ydx + 1dy = 0$.

1. Show that $3ydx + 1dy$ is not exact. Then use separation of variables to solve the ODE.
2. Multiply both sides of $3ydx + 1dy = 0$ by $\frac{1}{y}$. Show that the resulting differential form is exact, and use the sledgehammer theorem to obtain a solution.
3. Multiply both sides of $3ydx + 1dy = 0$ by e^{3x}. Show that the resulting differential form is exact, and use the sledgehammer theorem to obtain a solution.

Any time we can write an ODE in the differential form $Mdx + Ndy = 0$, the zero on right hand side gives us power. Our goal will be to multiply both sides of the differential equation by some function F, called an integrating factor, so that the resulting differential is exact. The general solution to the ODE is then simply the level curves of a potential.

Definition 4.6: Integrating Factor. An integrating factor for a differential form $M(x,y)dx + N(x,y)dy$ is a function $F(x,y)$ so that the product $FMdx + FNdy$ is exact.

In Problem 4.9, I gave you two different integrating factors. Where did they come from? The next problem will show you how I obtained one of the integrating factors. There many more options.

Problem 4.10 Let $M(x,y)dx + N(x,y)dy$ be a differential form. For simplicity, we just write $Mdx + Ndy$. Suppose $F(x,y)$ is an integrating factor.

1. To be exact, explain why we must have
$$\frac{\partial F}{\partial y}M + F\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x}N + F\frac{\partial N}{\partial x}$$
2. If we assume that F only depends on x, so that $F(x,y) = F(x)$, show that a possible option for an integrating factor is
$$F(x) = e^{\int \frac{M_y - N_x}{N} dx} = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$
3. If we assume that F only depends on y, so that $F(x,y) = F(y)$, show that a possible option for an integrating factor is
$$F(y) = e^{\int \frac{N_x - M_y}{M} dy} = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

[In class, you may omit the last part in your presentation, as it's almost an exact replica of the 2nd part.]

The problem above gives us a way of finding integrating factors for many differential equations. It will not give an integrating factor for EVERY differential equation, but it will provide an integrating factor for almost all the ODEs we tackle in this course. Let's now try using this technique on a problem we've already solved.

Problem 4.11 Consider the ODE $y' - 4y = 8$, which we solved in Problem 4.2.

1. Rewrite the ODE in differential form $Mdx + Ndy = 0$.
2. Find an integrating factor $F(x) = e^{\int \frac{M_y - N_x}{N} dx} = \exp\left(\int \frac{M_y - N_x}{N} dx\right)$.
3. Multiply both sides by the integrating factor, and then solve the ODE by applying the sledgehammer theorem.

Problem 4.12 Solve each ODE by finding an appropriate integrating factor.

1. $y' + 4xy = 3x$
2. $2ydx + (3x + 4y)dy = 0$ (Doable now)
3. $y' + 3y = e^{2x}$ (Solve for y.)
4. $y' - 4y = e^{4x}$ (Solve for y.)
5. $xy' - 4y = 2x$ (Solve for y.)

Let's now look at an additional application. We will encounter mixing model problems throughout the semester. They provide a simple way to see applications of ODEs, without requiring much background.

Problem 4.13: Mixing Model Suppose a 2000 gallon tank contains a solution of water which initially contains 50 lbs of salt. The tank has an inflow valve, and an outflow value. We would like to change the salt content, so we start pumping in 30 gallons of water (with 1/2 lb of salt per gallon) each minute. We'll assume that the mixture is evenly spread throughout the entire tank by constant stirring. At the same time, 30 gallons of the evenly stirred mixture flow through the outflow valve each minute. Let $y(t)$ represent the lbs of salt in the tank after t minutes. We currently only know $y(0) = 50$. Our goal is to determine the amount of salt $y(t)$ in the tank after t minutes.

1. (Express) The salt content changes in two ways. Salt is added through the new solution (a flow in), and salt leaves through the outlet valve (flow out). Explain how to obtain a formula for the flow in, and a formula for the flow out. Then explain why
$$y' = 15 - \frac{30}{2000}y.$$
2. (Solve) Obtain a general solution to the ODE, and then use the initial value to obtain a particular solution.
3. (Interpret) Construct a graph of your solution. As t increases, what happens to the salt content? Does your answer seem reasonable?

The mixing model problem above, as well as the exponential model and Newton's law of cooling, all belong to a special class of ODEs which we call linear ODEs.

Definition 4.7: Linear ODE. If we can write an ODE in the form $y'+p(x)y = q(x)$, then we say that the ODE is linear. This is precisely because the operator $L(y) = y' + a(x)y$ is a linear operator. If $q(x) = 0$, then we say the linear ODE is homogeneous. Otherwise, we say the linear ODE is non homogeneous.

The next problem provides a way to obtain a solution to EVERY linear ODE. Practice until you can develop this formula quickly, and then you'll have the key concepts needed for solving just about every ODE we encounter throughout the semester.

Problem 4.14: A Linear ODE Solution Consider the linear ODE $y' + p(x)y = q(x)$, where p and q are differentiable functions of x on some interval. Find an appropriate integrating factor, and then find a potential. Finish by solving for y to show that on this interval, a general solution is

$$y(x) = e^{-\int p(x)dx}C + e^{-\int p(x)dx}\int \left(e^{\int p(x)dx}q(x)\right) dx,$$

where C is an arbitrary constant. If the linear ODE is homogeneous, what is a general solution?

Problem: 14 and 1/2: Go back to problem solving ODEs by finding an integrating factor, and decide which ODEs are linear. Then pick one of the ODEs and solve it using the general solution from the previous problem.

Let's tackle a couple more application problems. As you solve them, rather than use the formula above, practice finding an appropriate integrating factor, and then find a potential.

Problem 4.15 Suppose a 50 gallon tank contains a solution of fertilizer which initially contains 10 lbs of fertilizer. We start pumping in 4 gallons per minute, where the concentration of fertilizer is 1/3 lb per gallon. Assume that the mixture is evenly spread throughout the entire tank by constant stirring. The extra At the same time, 4 gallons of the evenly stirred mixture flow through the outflow valve each minute. Let $y(t)$ represent the lbs of fertilizer in the tank after t minutes.

1. Express the mixing model as an IVP (give the ODE and the IV).
2. Solve the IVP.
3. Construct a rough graph of your solution. As t increases, what happens to the salt content? Does your answer seem reasonable?from the

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The next problem applies Newton's law of cooling to examine what happens if the temperature of the surrounding environment changes. Recall that Newton's law of cooling suggests that the rate of change of temperature of an object is proportional to the difference between the current temperature and the surrounding atmosphere, which we wrote earlier as

$$T' = k(A - T).$$

Problem 4.16 Suppose that during a summer day, the temperature outdoors fluctuates between 70°F and 110°F. We'll approximate this with a sine wave. If we let $t = 0$ be noon, then we could obtain the temperature A outdoors after t hours using the formula

$$A(t) = 20\sin(\frac{2\pi}{24}t) + 90.$$

Suppose that your air conditioner breaks at noon (your house was at 70°F at noon), and then by 6pm in the evening, the temperature had risen to 90°F.

1. Express this heating problem as an IVP.

2. Show that the ODE is linear, and then use technology to solve the ODE. You'll need to use $T(6) = 90$ to obtain the proportionality constant k.

3. Graph your solution for 3 days. In the late evenings, which is hotter, the house or the outdoors?

4.2.2 Use a substitution when you can't get an integrating factor.

We can solve most of the differential equations we tackle this semester by obtaining an integrating factor using the formulas developed in the previous section. Sometimes however, this won't work. In these cases, we often just have to make an appropriate change of coordinates (a u-substitution). Let's illustrate how this works with an example. Then we'll tackle the logistics model and introduce another application.

Problem 4.17 Consider the ODE $y' = \sin(x + y)$. There is no way that you'll get an integrating factor out of this by using our formulas for $F(x)$ and $F(y)$. The problem is the $x + y$. We now do a substitution.

1. Write the ODE in the differential form
$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = 0.$$

2. Let $x = x$ and $u = x + y$ (this is a coordinate transformation). Explain why we have
$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix}.$$

3. Show that the ODE can be written in the form $(-\sin u - 1)dx + du = 0$. Then use either separation of variables (or the sledgehammer) to solve the ODE. Don't forget to substitute back in when you're done.

The last problem introduced the key idea. If you have an ODE in the form

$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = 0,$$

then an appropriate substitution $x = x$, $y = g(x, u)$ will give us the ODE

$$\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_x & g_u \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix} = 0.$$

If we can find an integrating factor $F(x)$ or $F(u)$ which makes

$$F\begin{bmatrix}M & N\end{bmatrix}\begin{bmatrix}1 & 0\\ g_x & g_u\end{bmatrix}\begin{bmatrix}dx\\ du\end{bmatrix}$$

exact, then we can use the sledge hammer to solve the ODE. The hard part here is finding the correct transformation. If you can find the correct transformation, then you can solve the ODE. This is not easy, and in general it can be really tough.

Problem 4.18 Consider the ODE $xyy' = 4x^2 + 2y^2$. In this situation, if you let $u = y/x$ (so $y = xu$), show that you can rewrite the ODE as

$$\frac{u}{4+u^2}du = \frac{1}{x}dx.$$

This is a separable ODE, which we can solve. Solve the ODE.

Notice that the coefficients xy, $4x^2$, and $2y^2$, all are basically second order monomial terms. When the coefficients of the ODE are monomials with the same degree, the substitution $u = y/x$ will convert the ODE into a separable ODE. I'll leave this to you to prove. You do have the tools to prove it.

Problem 4.19 Consider the ODE $(x + 2y)dx + (3x + 4y)dy = 0$. Use the substitution $u = y/x$ to convert this into a separable ODE and give a general solution.

Consider the ODE $y' + 3y = 4y^3$. This ODE is not linear (why?). We could separate variables on this ODE and solve (we'll do so in class, reminding you about partial fractions). Instead, Bernoulli noticed that if the ODE is in the form $y' + a(x)y = b(x)y^n$, then the substitution $u = y^{1-n}$ will always convert the ODE into a linear ODE, and then we can use an integrating factor to solve the ODE. It's not easy to discover the right substitution that will convert an ODE into something we can solve. We call them Bernoulli ODEs because his discovery was quite clever. The $u = y/x$ substitution above was not clever enough to get a name attached to it.

Problem 4.20: Bernoulli ODE Consider the ODE $y' + 3y = 4y^3$. Use the substitution $u = y^{1-3}$ to convert this ODE into a linear ODE, and then solve. [Hint: You know that $u = y^{-2}$. Use this to solve for y, and then compute $dy = ?du$. Then just substitute. You'll probably have a really ugly term involving $u^{-3/2}$, so multiply both sides by $u^{3/2}$ and all the ugliness will disappear.]

We'll come back to Bernoulli ODEs and see some applications of them after we review Laplace transforms.

4.3 Laplace Transforms

Recall that the Laplace transform of a function $y(t)$ defined for $t \geq 0$ is

$$Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)dt.$$

- We call the function $y(t)$ the inverse Laplace transform of $F(s)$, and we write $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.
- As a notational convenience, we describe original functions $y(t)$ using a lower case y and input variable t or x. We describe transformed functions $Y(s)$ using the same capital letter and input variable s.

$f(t)$	$F(s)$	provided
1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^2	$\frac{2}{s^3}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$

$f(t)$	$F(s)$	provided
$\cos(wt)$	$\frac{s}{s^2+\omega^2}$	$s>0$
$\sin(wt)$	$\frac{\omega}{s^2+\omega^2}$	$s>0$
$\cosh(wt)$	$\frac{s}{s^2-\omega^2}$	$s>\lvert\omega\rvert$
$\sinh(wt)$	$\frac{\omega}{s^2-\omega^2}$	$s>\lvert\omega\rvert$
y	$\mathscr{L}\{y\}=Y$	
y'	$s\mathscr{L}\{y\}-y(0)$ $=sY-y(0)$	

Table 4.1: Table of Laplace Transforms

We've computed quite a few Laplace transforms already. For convenience, I've placed the Laplace transforms we'll use most often in Table 4.1. Feel free to use this table as you find Laplace transforms and their inverses. With practice, you will memorize this table.

We can use this table, and the linearity of the Laplace transform, to quickly compute both forward transforms and inverse transforms. The next problem asks you to do this.

Problem 4.21 Use the table of Laplace transforms to do the following:

1. Compute the Laplace transform of $y(t)=6+2t+4t^2-5e^{7t}+11\cosh(3t)$.
2. Compute the inverse Laplace transform of
$$Y(s)=\frac{5}{s}+\frac{4}{s^3}+\frac{3s}{s^2+16}-\frac{2}{s^2-9}.$$
Once you have a guess for the inverse Laplace transform, verify that your guess is correct by computing the Laplace transform (using the table of course).

Problem 4.22 Find the inverse Laplace transform of $F(s)=\dfrac{2s+1}{s^2+5s+4}$. [Hint: Use a partial fraction decomposition. Start by factoring the denominator.]

Problem 4.23 Find the inverse Laplace transform of $F(s)=\dfrac{2s+1}{s^2+9}+\dfrac{5s+7}{s^2-9}$. [Hint: This can all be done using trig and hyperbolic trig functions.]

The real power behind the Laplace transform comes from the last formula in the table.

Theorem 4.8 (The Laplace Transform of a Derivative)**.** *Suppose that $y(t)$ is a differentiable function defined on $[0,\infty)$ such that $\lim\limits_{t\to\infty}\dfrac{y(t)}{e^{st}}=0$ for some s. We say that $y(t)$ does not grow faster than some exponential, as the function e^{st}*

grows faster that $y(t)$ (otherwise the limit would not be zero). If this is the case, then the Laplace transform of y' is

$$\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0) = sY - y(0).$$

Problem 4.24 Prove the previous theorem. In other words, show that $\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0) = sY - y(0)$. [Hint, use integration by parts once, and don't forget to use the bounds.]

Before illustrating the key value of this theorem, let's fill in the only remaining rules in our Laplace transform table that we have not yet developed.

Problem 4.25 In the table of Laplace transforms it also states that the transform of $\cos(wt)$ is $\dfrac{s}{s^2+\omega^2}$, and that the transform of $\sin(wt)$ is $\dfrac{\omega}{s^2+\omega^2}$. Pick one of these rules, and then use the definition of the Laplace transform to explain why it is true. [Hint: You'll want to use integration by parts twice. See the online text if you want more hints.]

We'll now use Theorem 4.8 to solve some ODEs. You'll see the power behind this method.

Problem 4.26 Consider the IVP $y' = 7y$, $y(0) = A$.

1. Apply the Laplace transform to both sides of this ODE. You should have an equation involving $Y(s)$.
2. Solve for Y.
3. Now find the inverse transform of Y. This is $y(t)$, the solution to the ODE.
4. We already know how to solve this ODE using either separation of variables, or by finding an integrating factor. Pick one of these methods and obtain a general solution.

Did you see the process above? Rather than integrate, we just (1) computed the Laplace transform of both sides, (2) solved an algebraic equation for Y, and then (3) obtained the inverse Laplace transform to get Y.

Here's a parable to compare to using Laplace transforms. You are inside a house that has a single door leading to the downstairs. You are on the main floor, and need to get downstairs. The door is locked and you don't know where the key is (you can't figure out how to solve the ODE). You (1) decide to walk out the front door (you apply the Laplace transform). Then you (2) walk around the house and find the back door entrance to the basement (you solve for Y, and maybe apply a partial fraction decomposition). Then (3) you walk up to the locked door and unlock it from the other side (you find the inverse transform).

The Laplace transform replaces the problem of integrating with an algebraic problem (often easier to solve). We'll be using it throughout the semester to help us see patterns, and unlock difficult problems. It works best when the ODE is linear.

Problem 4.27 Consider the IVP $y' + 3y = 5$, $y(0) = 7$. (1) Apply the Laplace transform to both sides of this ODE to obtain an equation involving $Y = \mathscr{L}\{y\}$. (2) Solve for the transformed function Y. You will need to use a partial fraction decomposition to write $Y = \dfrac{A}{s+3} + \dfrac{B}{s}$. (3) Use an inverse transform to obtain the solution $y(t)$.

You'll find that with most Laplace transform problems, we'll need a partial fraction decomposition before we can compute an inverse transform. The next problem has you practice the Laplace transform inversion process to solve multiple problems that you know the answer to using simple integration.

Problem 4.28 For each problem below, use a Laplace transform to solve the ODE. Each problem could be solved with simple integration instead. The point to this problem is to help you see how the Laplace transform gives you, in a different way, information you already know how to obtain.

1. $y' = 5t^2 + 7t + 3$, $y(0) = C$.
2. $y' = e^{at}$, $y(0) = C$.
3. $y' = \cosh(3t)$, $y(0) = 2$.
4. $y' = \sin(3t)$, $y(0) = 2$.

[Hint: You'll need a partial fraction decomposition to write $\frac{s}{s(s^2+9)} = \frac{A}{s} + \frac{Bx+C}{s^2+9}$ on part 4. You'll need a similar idea on 2 and 3.]

Let's end this section with two more Laplace transform problems, where the initial conditions are not given.

Problem 4.29 Solve the ODE $y' + 4y = e^{3t}$ by using a Laplace transform. No initial condition was given, so you should use something like $y(0) = C$. Because this initial condition involves an arbitrary constant, you may find that Cramer's rule helps you quickly obtain the partial fraction decomposition (rather than row reduction).

Problem 4.30 Solve the ODE $y' + 3y = \sin(2t)$ by using a Laplace transform. See the previous problem for help about what to do when no initial condition is given.

4.4 What Method Should I Use?

In this chapter, we've explored various different techniques to solve first order ODEs. Here's a list.

- Separation of variables: The easiest, if you can separate.
- Exact: The ODE has a potential. Use the sledgehammer.
- Integrating Factors: Make the ODE exact.
- Substitution: Change variables so you can make the ODE exact.
- Laplace Transforms: Dodge integration. Replace it with algebra.

The Laplace transform works nicely on linear ODEs with constant coefficients. If we're missing an initial condition, the algebra gets a little uglier, but still doable. We'll be using the Laplace transform to discover solutions to higher order ODEs as the semester progresses. However, we could have solved every one of the problems we tackled with Laplace transforms by instead using our sledge hammer tool (make the ODE exact, through substitutions and/or integrating factors, and then find a potential). The sledgehammer tool will solve a much larger range of ODEs than the Laplace transform, and near the end of the semester we'll see it's true power in terms of matrices, eigenvalues, and eigenvectors.

I'm working on writing a paper to extend the sledgehammer approach to solve just about every ODE undergraduates tackle, and provide a uniform approach to working with ODEs. I'm just waiting for an interested student to come and complete the project with me.

So which method should you use? That depends on how much work you want to do. The sledgehammer tool will solve EVERY problem we see. If an ODE is separable, it's generally much faster to just use separation of variables (which is really just using an integrating factor). If the ODE is linear, with constant coefficients, and you have an initial condition, then a Laplace transform might be faster. If all else fails, make the ODE exact and find a potential.

Problem 4.31 Which method would you use to solve each ODE below? If you opt for separation of variables, then show us how to separate. If the ODE is exact, show us how you know. If you decide to find an integrating factor, show us the integrating factor. If you will use a substitution, what substitution will you use? If you decide to use Laplace transforms, take the Laplace transform of both sides. In all cases, don't solve the ODE, rather just show us the first step in the solution process.

1. $x^2y' = 4xy^2$, $y(2) = 1$.
2. $xy' = 3y + x$, $y(2) = 1$.
3. $y' + 8y = e^x$, $y(0) = 1$.
4. $y' + 8y = y^2$, $y(0) = 1$.

Let's end the chapter by considering another application. In Problem 4.3, we considered the growth of bacteria in a petri dish. We could have applied this to any other population (such as deer in a forest, people on the Earth, cancer cells spreading through the bloodstream, number of cell phones users in Brazil, speed of computer processors, etc.) There is a problem with this model. It works great for a little while, but physical systems cannot grow exponentially forever. Eventually the growth has to slow down. In the example with the petri dish, eventually the bacteria will have gotten so large that it cannot support more growth. This is where our final problem begins.

Problem 4.32 Suppose that bacteria grow in a petri dish (if you don't like bacteria, then pick something else to put in here that interests you). From Problem 4.3, we used the model $y' = ky$ to express that the rate of growth y' is directly proportional to the size y of the population. We assumed that k was constant. Here's where we now make a change. Instead of assuming that k is constant, let's assume that as the population gets larger, that the constant k decreases. In fact, if we let M represent a theoretical maximum population, let's assume that k is proportional to the difference between the current population and this theoretical maximum.

1. (Express) Explain why $y' = -a(y - M)y$.
2. (Solve) Solve the ODE using separation of variables. You'll need to perform a partial fraction decomposition on $\frac{1}{y(y-M)}$.

3. (Interpret) Pick some constants for a, M, and the initial size of the population. Then graph your solution. You should obtain a logistics curve (please use a computer to check your work).

Problem 4.33 The ODE $y' = -a(y - M)y$ is a Bernoulli ODE. Rewrite it in the form $y' + a(x)y = b(x)y^n$, and then use Bernoulli's substitution $u = y^{1-n}$ to solve the ODE.

Question 4.9. Why can't we (yet) use a Laplace transform to solve $y' = -a(y - M)y$?

4.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Homogeneous ODEs

This chapter covers the following ideas.

1. Explain Hooke's Law in regards to mass-spring systems. Construct and solve differential equations which represent this physical model, with or without the presence of a damper.

2. Understand the vocabulary and language of higher order ODEs, such as homogeneous, linear, coefficients, superposition principle, basis, linear independence.

3. Solve homogeneous linear ODE's with constant coefficients (with and without Laplace transforms). In addition, create linear homogeneous ODE's given a basis of solutions, or the roots of the characteristic equation.

4. Explain how the Wronskian can be used to determine if a set of solutions is linear independent. Briefly mention the existence and uniqueness theorems in relation to linear ODEs, and give a reason for their importance.

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the preparation problem set, please use this list as a guideline to find related practice problems.

Concept	Sec	Suggestions	Relevant Problems
Vocabulary of ODEs	8*	33-35	1-3,33-35
2nd Order Homogeneous	9*	1,7,12,21,27,40	1-15, 17-45
nth Order Homogeneous	10*	3,7,8,9,12,18,37,41,44,49	All
IVPs (Homogeneous)	13	9	4,9,13
Applications	14	2,3,5,29,31,34,41-43	1-8,26-43
Laplace Transforms	21*	26, 54	14(c),15(b),25,26,54-58,
Inverse Transforms	22*	7, 34-36,38,read 12 and 18,44	6-10,15-19,29-30,32-53
Solving ODES	24	26,44	5,26,31,36,43,44
Wronskian and Theory	8*	9,10,18,20,43,48,53,58	5-10, 13-20, 31,36-64

*The problems in these sections are quick problems. It is important to do lots of them to learn the pattern used to solve ODEs. You may be able to finish

7 or more problems in 15 minutes or less. Please do more, so that when you encounter these kinds of problems in the future you can immediately give an answer and move forward.

5.1 Some Physical Models

In this chapter, we're going to learn how to solve a huge collection of higher order differential equations. Before diving into the details, let's make sure we know WHY we would even want to do so. If I knew you all had the same background, we could dive into lots of examples directly related to your field (you'll do that in future classes in your major). Since we have a diverse background in our class, we'll stick mostly to models that connect velocity, position, and acceleration. Before the next chapter ends, we'll add to this some information about electrical circuits.

For our first model, let's look at how we can obtain the position of an object in projectile motion from knowledge about the acceleration and velocity. You've solve this problem before, but the solution required neglecting air resistance.

Example 5.1. In multivariate calculus, we encountered the differential equation $y'' = -g$. In this differential equation, the only force $F_T = my''$ acting on an object in projectile motion is the force of gravity $F_G = -mg$. Equating these two gives us the ODE $my'' = -mg$, or just $y'' = -g$. If we have initial position $y(0) = y_0$ and initial speed $y'(0) = v_0$, then the solution is $y = -\frac{1}{2}gt^2 + v_0t + y_0$. We found that solution by integrating twice.

We don't have to neglect air resistance anymore. We could talk about sky diving (risky), dropping bombs (deadly), throwing math books off a roof (illegal), putting a satellite into geosynchronous orbit (useful), or dropping a pebble from the top of a waterfall (head to Yellowstone and try it - sounds like we need a field trip). The next problem asks you to revisit the example above, but now add in air resistance.

Problem 5.1 Joe hikes up to the top of Lower Falls in Yellowstone. His hope is to gauge the height h of the waterfall. He plans to drop a pebble from the top, and time how long it takes for the pebble to hit the ground. He'll need a model that predicts the height of the pebble at any time t.

For this to work, Joe has to make some assumptions. His assumptions might be way off, but that's how science works. We start with assumptions and then turn those assumptions into differential equations. Here's what Joe assumes:

- He assumes Newton's second law of motion, namely that $F = ma$ (the total force is the mass times the acceleration).
- He assumes that the total force is comprised of two parts.
- The first force F_G comes from a constant acceleration due to gravity. He assumes that gravity is constant $a = -g$. The negative sign comes because the acceleration causes a decrease in height.
- The second part comes from air resistance. He assumes that the faster the pebble goes, the greater this force will be. If the pebble's speed were to double, then this force should double. So he assumes that the force due to air resistance F_R is proportional to the pebble's velocity.

Let $y(t)$ represent the height, above ground, of the pebble after t seconds. Use Joe's assumptions to answer the following:

1. Rewrite Newton's second law of motion in terms of y, y', and/or y''.

2. What is the constant force F_G due to gravity?

3. Rewrite Joe's assumption about air resistance in terms of y, y' and/or y''.

4. The total force F is the sum of the two forces, i.e. we can write $F = F_G + F_R$. Use this fact, together with your answers from the previous two parts, to obtain a second order ODE. You don't have to solve the ODE, rather you just need to obtain it.

If you need any hints, try searching the web for "modeling motion if we assume that air resistance is proportional to speed."

Congrats. You've just set up your first second order ODE.

Let's now look at another position/velocity/acceleration model, but this time related to springs. We'll start by considering the following scenario. We attach an object with mass m to a spring. We place the spring horizontally, and put the mass on a frictionless track. We let go of the object, and allow it to come to rest. We'll use the function $x(t)$ to keep track of the position of the spring at any time t, with $x = 0$ corresponding to equilibrium (the mass is at rest). Robert Hooke (1635 – 1703) developed the following law, called Hooke's law:

In the next chapter, we'll hang the spring from a ceiling. In this case, we'll have an additional force $F_g = -mg$ acting on the spring.

> The force needed to extend (or compress) a spring a distance x is proportional to the distance x. Note that the force acts opposite the displacement.

Problem 5.2 Read the preceding paragraph. Then answer the following:

- Draw a picture of a horizontal track. On the left end of the track, put a wall. Put a on object, like a square block, in the center of your track and draw a spring that connects the wall to the block.

- Explain why $mx''(t) = -kx(t)$. We generally just write $mx'' = -kx$ (the t is assumed).

- If it takes 8 N $=$ 8kg m/s^2 to move the object whose mass is 4 kg about .3 m, what is the spring constant k? How far would a 12 N force cause the object to move? Does the mass of the object matter?

Hooke's law is not a perfect model for all springs, but it does a good job for most, provided the displacement is not too large. If the displacements are too large, then the spring may deform, which changes the properties of the spring in all future computations. If you take your car out onto extremely bumpy roads, and purposefully hit some nasty bumps, you could permanently damage the shocks. In this case, you would want to replace your springs.

Every linear spring has a spring constant k. This constant has many names, such as the spring modulus, Young's modulus, Young's constant, and more. The next problem shows you how to obtain the spring constant k.

Problem 5.3 You attach a spring to the ceiling. You attach a mass of 10 kg on the end, and the spring elongates 3 cm.

1. You now attach a mass of 20 kg. How long will the spring elongate?

2. What is the spring constant k? Give the units.

3. We attach a different spring, and hang the same 10 kg on the end, but this time the spring elongates 2 cm. Is the spring constant larger or smaller?

4. If a spring has really large modulus, will it be easy or hard to elongate it?

We need one more model before we start solving some ODEs. We'll use the exact same spring model as before. Place a horizontal spring whose modulus is k on a frictionless track. Attach an object whose mass is m to the end of the spring. We now place the entire mass-spring system underwater. When it was exposed to air, we neglected air resistance. Now we'll have to take resistance into account.

We don't have to place the spring underwater to get the same affect. We could use a dashpot to resist the motion. One type of dashpot is a cylindrical tube placed around a cylindrical object, so that as the object moves, it's sides come in contact with the dashpot, resulting in friction that resists motion. See Wikipedia for more info.

Problem 5.4 When we have no resistance, the mass-spring system ODE is $F_T = F_S$, or $mx'' = -km$. Assume that the liquid applies a resistive force that is proportional to the velocity of the object. If the object is resting, the liquid doesn't apply a force. If you double the speed, then the resistive force doubles. If you triple the speed, the resistive force triples. Modify the ODE $mx'' = -km$ to account for the resistive force of water.

5.2 Notation, Vocabulary, and Solutions

We can write the ODEs from the previous section as

$$my'' + ky' = -mg, \quad mx'' + ky = 0, \quad \text{and} \quad mx'' + cy' + ky = 0.$$

If we divide each ODE by m, then we can write each ODE in the general form

$$y'' + p(t)y' + q(t)y = r(t).$$

This introduces our next definition.

Definition 5.2: Linear, Constant Coefficient, and Homogeneous.

- If we can write an ODE in the form $y'' + p(t)y' + q(t)y = r(t)$, then we say the ODE is a second order linear ODE.
- The functions $p(t)$ and $q(t)$ we call the coefficients of the linear ODE.
- If the coefficients are constant, the we say the ODE is a constant coefficient linear ODE.
- If the right hand side $r(t) = 0$, then we say the linear ODE is homogeneous. Otherwise we say it is non homogeneous.
- We use the words nth order linear ODE to talk about any ODE that we can write in the form $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = r(t)$, where $y^{(n)}$ is the nth derivative of y.

We just introduces a few new words, so with each problem that follows, let's practice using those words. The next problem has you explain why we use the word "linear."

Problem 5.5 Consider the second order ODE $y'' + 7y' + 6y = 0$.

- Why is this ODE linear? Modify it so it is no longer linear, and show us in class what would make it non linear.
- Is this ODE homogeneous? Explain.

- Let $L(y) = y'' + 7y' + 6y$. Show that L is a linear operator. (See the end of chapter 3 if you need to reread the definition).
- The solutions to the ODE are the solutions to $L(y) = 0$. In the language of linear operators, what do we call the set of functions y such that $L(y) = 0$? It was another key word near the end of chapter 3. Please look it up. The set of solutions y is the ______________ of L.

To solve second order linear homogeneous ODE, we'll use the Laplace transform. In the previous chapter, we showed that

$$\mathscr{L}(y') = s\mathscr{L}(y) - y(0) = sY - y(0).$$

We need a rule for second derivatives. Repeated application of the single derivative rule will give you all the rules you need.

Problem 5.6 Show that under suitable conditions, we can compute the Laplace transform of the second derivative of y by using the formula

$$\mathscr{L}(y'') = s^2\mathscr{L}(y) - sy(0) - y'(0) = s^2Y - sy(0) - y'(0).$$

Then show that

$$\mathscr{L}(y''') = s^3\mathscr{L}(y) - s^2y(0) - sy'(0) - y''(0).$$

Conjecture a formula for the Laplace transform of the 7th derivative of y. [Hint: As stated in the paragraph before this problem, apply the rule $\mathscr{L}(y') = s\mathscr{L}(y) - y(0)$ multiple times.]

We are now ready to solve a second order ODE with Laplace transforms.

Problem 5.7 Consider the IVP $y'' + 3y' + 2y = 0$, $y(0) = 7$, $y'(0) = 5$.

1. Is the ODE linear? Is it homogeneous? Are the coefficients constant?
2. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$.
3. Use a partial fraction decomposition to show that
$$Y = \frac{A}{s+1} + \frac{B}{s+2},$$
where you give the constants A and B.
4. Find the solution y to this IVP by computing the inverse Laplace transform of Y.
5. How are solutions to $s^2 + 3s + 2 = 0$ connected to your solution?

Problem 5.8 Consider the IVP $y'' + 7y' + 10y = 0$, $y(0) = c$, $y'(0) = d$.

1. Is the ODE linear? Is it homogeneous? Are the coefficients constant?
2. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$.

3. If we use a partial fraction decomposition, we would write
$$Y = \frac{A}{s+2} + \frac{B}{s+5}.$$
Why is the solution $y(t)$ a linear combination of e^{-2t} and e^{-5t}, i.e. $y(t) = Ae^{-2t} + Be^{-5t}$?

4. Now actually perform the partial fraction decomposition to obtain the constants A and B. (Since you have variables a and b in your system, you'll want to use Cramer's rule).

5. How are solutions to $s^2 + 7s + 10 = 0$ connected to your solution?

In the previous two problems, we had initial conditions. When the initial conditions are numbers, it made the partial fraction decomposition rather simple. When the initial conditions are variables, finding the constants in the partial fraction decomposition was a little trickier. The next problem has you work through a problem when we have no initial conditions.

Problem 5.9 Consider the ODE $y'' + 7y' + 12y = 0$. We would like a general solution (no initial conditions are given).

1. Compute the Laplace transform of both sides and solve for $\mathscr{L}(y) = Y$. You'll have $y(0)$ and $y'(0)$ in the numerator of your solution. It would be nice if they weren't there.

2. Factor the denominator of Y, and write your solution as $Y = \frac{A}{?} + \frac{B}{?}$. This time DO NOT solve for A and B. You don't need to.

3. Compute the inverse Laplace transform of Y. Your answer should involve the unknown constants A and B. You've found the general solution.

4. The polynomial $s^2 + 7s + 12$ showed up in your work above. How are the zeros of this polynomial connected to the solution?

In the three examples above, we took an ODE $y'' + ay' + by = 0$, applied a Laplace transform, and obtained the polynomial $s^2 + as + b$. The zeros of this polynomial seem to be intimately connected to the solution. Let's give this polynomial a name.

Definition 5.3: Characteristic Polynomial (Equation). Consider the ODE $y'' + ay' + by = 0$.

- The characteristic polynomial is $s^2 + as + b$. We could alternately use $\lambda^2 + a\lambda + b$.

- The characteristic equation is $s^2 + as + b = 0$. We could alternately use $\lambda^2 + a\lambda + b = 0$.

With this new word, we now have the correct tool to discuss solving ODEs. We noticed a pattern in the first few problems. From that pattern, we developed a new word. Now we can use that word to simplify your solution techniques.

Problem 5.10 Consider the ODE $y'' + 9y' + 20y = 0$. What is the characteristic equation of the ODE? Find the zeros of the characteristic polynomial, and then state a general solution to the ODE.

The definition of characteristic equation allows us to alternately use the variable λ instead of s. The next problem connects what we are doing to eigenvalues.

Problem 5.11 Consider the ODE $y'' + 9y' + 20y = 0$ from the previous problem. If we let $y_1 = y$ and $y_2 = y'$, then we can write the ODE in the form $y_2' + 9y_2 + 20y_1 = 0$. This becomes the system of ODEs

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -20y_1 - 9y_2. \end{aligned}$$

Write the system above in the matrix form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then find the eigenvalues of A, and use them to obtain a solution to the ODE.

Let's now tackle a problem where the characteristic equation does not have real zeros.

Problem 5.12 Consider the ODE $y'' + 16y = 0$.

1. Compute the Laplace transform of both sides of the ODE and solve for $\mathscr{L}(y) = Y$. You'll have $y(0)$ and $y'(0)$ in the numerator of your solution.
2. Compute the inverse Laplace transform of Y. Your answer will involve $y(0)$ and $y'(0)$.
3. What is the characteristic polynomial, and what are its roots?
4. If a mass of 1 kg is attached to spring with modulus 16 kg/s^2 on a frictionless track, then graph the position $x(t)$ at any time t. [What's the corresponding ODE? Didn't you already solve this ODE?]

The previous problem showed us how to tackle a problem where the roots of the characteristic polynomial are purely imaginary. What do we do if the roots repeat, or if they are complex of the form $a \pm bi$? The next problem addresses this.

Problem 5.13 Consider the ODE $y'' + 6y' + 9y = 0$.

1. What are the zeros of the characteristic equation? From these zeros, guess a general solution. (It's OK if you're wrong.)
2. Compute the Laplace transform of both sides of ODE. Then solve for Y and show that
$$Y = \frac{A(s+3)+B}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2}.$$
3. Compute the inverse Laplace transform of each part that you are able to compute, and explain why we can't perform the inverse Laplace transform of the other parts.
4. Use a computer to complete the inverse Laplace transform, and state the solution.

Problem 5.14 Consider the ODE $y'' + 4y' + 13y = 0$.

1. What are the zeros of the characteristic equation?

2. Compute the Laplace transform of both sides of ODE. Then solve for Y and complete the square to show that
$$Y = \frac{A(s+2)+B}{(s+2)^2+3^2} = \frac{A(s+2)}{(s+2)^2+3^2} + \frac{B}{(s+2)^2+3^2}.$$

3. Use the fact that $\mathscr{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}$ and that $\mathscr{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}$ to finish solving. [The next problem will show you where these came from.]

In both of the preceding problems, we encountered expressions that we could not inverse transform. The first was $\dfrac{1}{(s+3)^2}$, and the last two were $\dfrac{(s+2)}{(s+2)^2+3^2}$ and $\dfrac{1}{(s+2)^2+3^2}$. In all cases, these look like shifted versions of functions for which we know the inverse Laplace transform. For example, we know $\mathscr{L}\{\cos 3t\} = \dfrac{s}{s^2+3^9}$. The expression $\dfrac{(s+2)}{(s+2)^2+3^2}$ resembles the expression $\dfrac{(s)}{(s)^2+3^2}$, rather we just replaced s with $s+2$, which is the same as shifting s left 2. We were told that $\mathscr{L}^{-1}\left\{\dfrac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)$. What we need is a Laplace transform rule that would allow us deal with s shifting. If we know how to invert $Y(s)$, how do we invert $Y(s-a)$?

Problem 5.15: The s-shifting Theorem In this problem you'll develop a rule for the inverse transform of $Y(s-a)$.

1. We know that $Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty e^{-st}[f(t)]dt$. Replace s with $s-a$ and obtain a formula
$$Y(s-a) = \int_0^\infty e^{-st}[?]dt.$$
This gives you a formula $\mathscr{L}\{?\} = Y(s-a)$.

2. What is the inverse Laplace transform of $1/s^2$? What is the inverse Laplace transform of $1/(s-4)^2$? What is the inverse Laplace transform of $1/(s+5)^2$?

3. What is the forward Laplace transform of $\cos(bt)$? What is the forward Laplace transform of $e^{at}\cos(bt)$? What is the forward Laplace transform of $e^{7t}t^3$ and $e^{-7t}t^3$?

[Hint: The s-shifting theorem is now in Table 5.1. Try to tackle this problem without referring to the table.]

To apply the s-shifting theorem, we'll need to become good at completing the square. If we know the transform is $\dfrac{2}{s^2+4}$, then the inverse transform is

$y(t)$	$Y(s)$	provided
1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
y'	$sY-y(0)$	
y''	$s^2Y-sy(0)-y'(0)$	

$y(t)$	$Y(s)$	provided
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s>0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s>0$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s>\lvert\omega\rvert$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s>\lvert\omega\rvert$
$y(t)$	$Y(s)$	
$e^{at}y(t)$	$Y(s-a)$	

Table 5.1: Table of Laplace Transforms

$\sin(2t)$. If we know the transform is $\dfrac{2}{(s+3)^2+4}$, then the inverse transform is $e^{-3t}\sin(2t)$. However, we would normally have a characteristic polynomial in the form $s^2+6s+13$, rather than the form $(s+3)^2+4$. Once we complete the square, we can apply the s-shifting theorem.

Problem 5.16 Complete each of the following:

1. Consider the ODE $y''+2y'+5y=0$. Find the characteristic polynomial, complete the square, and state a general solution.
2. Consider the ODE $y''+6y'+9y=0$. Find the characteristic polynomial, complete the square, and state a general solution.
3. Consider the ODE $y''+4y'+3y=0$. Find the characteristic polynomial, complete the square, and state a general solution.

Before we get to far, let's practice the s shifting theorem for Laplace transforms.

Problem: 5.16 and 1/2 Complete the following:

1. Find the Laplace transform of the following:
   (a) t^3 and t^3e^{4t}
   (b) $\cos(2t)$ and $e^{-3t}\cos(2t)$
   (c) $3\sin(7t)$ and $3e^{-5t}\sin(7t)$
2. Find the inverse Laplace transform of the following:
   (a) $\dfrac{3}{s^4}$ and $\dfrac{3}{(s-5)^4}$
   (b) $\dfrac{s+3}{(s+3)^2+4}$ and $\dfrac{1}{(s+3)^2+4}$
   (c) $\dfrac{s}{(s+3)^2+4}$.

Problem 5.17 Consider the ODE $y'' + 6y' + 11y = 0$.

1. Find the characteristic polynomial, complete the square, and then state a general solution.
2. Find the characteristic equation, use the quadratic formula to solve the characteristic equation, and then state a general solution.
3. Solve the ODE $y'' + 5y' + 12y = 0$. Would you rather complete the square, or use the quadratic formula?

Problem 5.18 Consider the ODE $ay'' + by' + cy = 0$.

1. Obtain the characteristic equation. Complete the square. State the zeros of the characteristic equation. [When you finish this problem, you will have proved the quadratic formula.]
2. If we let $y_1 = y$ and $y_2 = y'$, we obtain the system of ODE $y_1' = y_2$ and $ay_2' + by_2 + cy_1 = 0$. Write this system in the matrix form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and obtain the eigenvalues of A.

We can now solve EVERY second order homogeneous constant coefficient ODE. All we have to do is find the characteristic equation. The zeros unlock a general solution of the ODE.

Problem 5.19 Consider the second order homogeneous constant coefficient ODE $y'' + ay' + by = 0$. Let λ_1 and λ_2 be the roots of the characteristic polynomial $s^2 + as + b$.

1. If the roots are real and $\lambda_1 \neq \lambda_2$, then $y(t) =$______________.
2. If the roots are real and $\lambda_1 = \lambda_2$, then $y(t) =$______________.
3. If the roots are complex where $\lambda = c \pm di$, then $y(t) =$______________.
   If $c = 0$, then the solution is simply $y(t) =$______________.

Problem: 5.19 Improved Suppose that we have a second order ODE, and we have already computed the roots of the characteristic polynomial to be λ_1 and λ_2.

1. If $\lambda_1 = -3$ and $\lambda_2 = -5$, then $y(t) =$______________.
   If the roots are real and $\lambda_1 \neq \lambda_2$, then $y(t) =$______________.
2. If $\lambda_1 = -3$ and $\lambda_2 = -3$, then $y(t) =$______________
   If the roots are real and $\lambda_1 = \lambda_2$, then $y(t) =$______________.
3. If $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$, then $y(t) =$______________.
   If the roots are complex where $\lambda = a \pm bi$, then $y(t) =$______________.
4. If $\lambda_1 = 5i$ and $\lambda_2 = -5i$, then $y(t) =$______________.
   If the roots are purely imaginary so that $\lambda = bi$, then $y(t) =$______________.

Have you noticed that every general solution above is a linear combination of two independent solutions? Recall that we say a differential operator is linear if $L(y_1 + y_2) = L(y_1) + L(y_2)$ and $L(cy_1) = cL(y_1)$ for functions y_1 and coefficients c.

Problem 5.20: Superposition Principle Suppose that y_1 and y_2 are both solutions to a linear differential equation $ay'' + by' + cy = 0$. Consider the linear operator $L(y) = ay'' + by' + cy$. Prove that any linear combination of y_1 and y_2 is also a solution to the ODE $L(y) = 0$. (Hint: Look at the last few problems in chapter 3, or just prove this directly.)

Many people refer to this fact as the superposition principle. To get a solution to a second order homogeneous ODE, all you need is two independent solutions. The general solution is any linear combination of them.

Now that we have a general solution, let's show how to quickly obtain the solution to an IVP. The key principle, is to first obtain a general solution. Differentiate your general solution, and then use your initial conditions to find the unknown constants.

Problem 5.21 Consider the IVP $y'' + 6y' + 5y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Plug the initial conditions into both y and y' to solve for the unknown constants in your general solution.

Problem 5.22 Consider the IVP $y'' + 6y' + 9y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Use the initial conditions to solve for the unknown constants in your general solution.

Problem 5.23 Consider the IVP $y'' + 2y' + 5y = 0$, with $y(0) = 4$ and $y'(0) = 5$. Obtain a general solution. Then compute $y'(t)$. Use the initial conditions to solve for the unknown constants in your general solution.

5.3 Mass-Spring Systems

Recall from the introductory examples that we can model the position of a spring using the ODE

$$mx'' + cx' + kx = 0$$

The constants m, c, and k are physical constants related to the mass-spring system.

- The mass of the object attached to the spring is m.
- The spring constant, or modulus, is k.
- The coefficient of friction of any attached dashpot is c. If no dashpot is attached, then we just let $c = 0$.

Problem 5.24 Suppose we attach a mass of 4 kg to a spring with modulus 12 kg/s^2. We displace the object 1 cm from the equilibrium position of the spring, and then hit the mass with a hammer. The impact causes the spring's initial velocity to be 3 cm/s back towards equilibrium. Use this information to determine the position of the spring at any time t. Construct a graph of the position. From your graph, show how you can the initial position and initial velocity.

Make sure you ask me in class to show you how the solution above graphically changes, if we alter the initial position and initial velocity.

Problem 5.25 Suppose we attach a mass of m kg to a spring with modulus k kg/s^2. We displace the object y_0 cm from the equilibrium position of the spring, and give the object an initial velocity of v_0 cm/s away from equilibrium. In the absence of friction, the mass-spring system will oscillate in a regular pattern. Determine the position of the spring at any time t. What is the period of oscillation? If you doubled the spring constant k, how would it affect the period?

Problem 5.26 Suppose we attach a mass of m kg to a spring with modulus k kg/s^2. We displace the object y_0 cm from the equilibrium position of the spring, and give the object an initial velocity of v_0 cm/s away from equilibrium. In the absence of friction, the mass-spring system will oscillate in a regular pattern. Give a formula for the amplitude of the oscillation. [Hint: If you write your solution in the form $y(t) = C\sin(\omega t + \phi)$, then you can quickly read off the amplitude. How do you write $y(t) = A\cos(bt) + B\sin(bt)$ in the form $C\sin(\omega t + \phi)$?]

Each of the problems above dealt with undamped motion, there was no friction to slow down the motion. The remaining problems include a dashpot, something placed around the mass-spring system that adds friction to the system. Wikipedia has some excellent pictures of what a dashpot could look like. I like to think of an old screen door, and the cylindrical tube at the bottom of the door that helps close the door and prevent it from smashing closed on little fingers. Ask me in class to show you a dashpot on our classroom door.

Problem 5.27 Recall from the introductory examples that we can model the position of a spring using the ODE $mx'' + cx' + kx = 0$. We now attach a mass of 1 kg to a spring. The spring is placed inside a dashpot, to add friction to the system, and the dashpot has a coefficient of friction equal to $c = 8$ kg/s. The spring is rather large, so we extend it 1 m and then release it with no initial velocity.

1. If the spring modulus is $k = 15$ kg/s^2, find the position $x(t)$ of the spring, and construct a rough sketch of x versus t.
2. If the spring modulus is $k = 16$ kg/s^2, find the position $x(t)$ of the spring, and construct a rough sketch of x versus t.
3. If the spring modulus is $k = 17$ kg/s^2, find the position $x(t)$ of the spring, and construct a rough sketch of x versus t.
4. What connection is there between c and k? If you had to describe what you saw in the examples above to someone not in this class, what would you say? You'll probably have to explain this phenomenon to a boss someday.

5.4 Higher Order ODEs

In the previous sections, we focused mainly on second order ODEs. We started by using Laplace transforms to find the exact solutions. The accompanying partial fraction decomposition was sometimes rather ugly, so we opted for guessing the form of the solution, and then taking derivatives to determine the unknown constants.

Problem 5.28 Consider the ODE $y''' + 3y'' + 3y' + y = 0$. Compute the Laplace transform of both sides. The characteristic equation is $(s+1)^3 = 0$. Explain why the solution is $y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}$.

For the ODE $y'''' + 4y''' + 6y'' + 4y' + y = 0$, whose characteristic equation is $(s+1)^4 = 0$, make a guess as to the solution. Then use a computer and dsolve to check that your answer is correct. (Wolfram Alpha can solve this.)

If you encounter a repeated root, what does that contribute to the solution? Explain this in a way that you and other can remember it.

Problem 5.29 You have a 6th order homogeneous ODE, and the characteristic equation factors as $(s^2+4)(s^2+9)^2 = 0$. What is the original ODE (expand the polynomial)? The roots are $\pm 2i, \pm 3i, \pm 3i$ (so $\pm 3i$ are repeated roots). Guess the general solution. Then use a computer to check if your guess was correct.

Problem 5.30 In each problem below, you'll be given the characteristic equation of an ODE. State the general solution of the ODE.

1. $(s+3)(s+2)(s+1) = 0$
2. $(s+3)(s+3)(s+1) = 0$
3. $(s+3)^3(s^2+9) = 0$
4. $(s+3)^2(s^2+9)^2 = 0$
5. $(s+3)^2(s^2-9)^2 = 0$ (Note the sign change)

5.5 Existence and Uniqueness

One of the key theorems pertaining to ODEs is that under sufficient conditions (see Wikipedia) an ODE must have a solution, and that solution must be unique. This means that if two people come up with a solution in different ways, then those solutions must agree. Let's use this to build some connections between trigonometry and complex exponentials.

Problem 5.31 Consider the ODE $y'' + 9y = 0$. This models the undamped motion of a spring where $m = 1$ and $k = 9$.

1. Show that $y_r(t) = c_1 \cos 3t + c_2 \sin 3t$ and $y_c(t) = c_3e^{3it} + c_4e^{-3it}$ are both general solutions of this ODE.
2. If $y(0) = 1$ and $y'(0) = 0$, then find c_1, c_2, c_3, c_4.
3. If $y(0) = 0$ and $y'(0) = 3$, then find c_1, c_2, c_3, c_4.
4. Show that $\cos 3t + i \sin 3t = e^{3it}$.

5.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 6

Non Homogeneous ODEs

This chapter covers the following ideas.

1. Explain Hooke's Law in regards to mass-spring systems, where there is an external force. Construct and solve differential equations which represent this physical model, with or without the presence of a damper and be able to interpret how solutions change based on changes in the model.
2. Understand the theory which relates solutions of homogeneous linear ODE's to non homogeneous ODEs.
3. Use the method of undetermined coefficients to solve non homogeneous linear ODEs.
4. Explain Kirchhoff's voltage law, Ohm's law, and how to model electrical circuits using 2nd order non homogeneous linear ODEs. Illustrate how results about circuits can be translated into results about mass-spring systems.

The problems below come from Schaum's Outlines *Differential Equations* by Richard Bronson. If you are struggling with a topic from the preparation problem set, please use this list as a guideline to find related practice problems.

Concept	Sec	Suggestions	Relevant Problems
Theory	8	21,65	21-23,65-67
Undetermined Coef	11	1,2,3,8,10,24,26,34,36,41,46,47,48	All
IVP	13	1,7,14	1,3,7,8,10,11,14
Applications	7	19,76	19-22,71-81
Applications	14	10,11,13,14,17,46,50,51,52,54,57	9-18,44-65

6.1 Non Homogeneous Linear Systems

In the previous chapter, we focused on solving homogeneous ODEs of the form $L(y) = 0$. We need to learn how to solve ODEs if the right hand side is not zero, namely $L(y) = r(t)$. Before solving these kinds of ODEs, let's revisit solving matrix equations of the form $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$. As we compare the solutions to these matrix equations (where solving is much easier), we just might glean some patterns that help us with solving non homogeneous ODEs.

I highly suggest you complete this review problem, and check your answer, to make sure you are comfortable with how to express infinitely many solutions in vector form.

Review You want to solve the system $A\vec{x} = \vec{b}$. You row reduced the augmented matrix and obtain

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -3 & 4 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the solution $\vec{x}$? See [1] for an answer.

If the vector $\vec{b}$ above had been $\vec{b} = 0$, what would the right most column have been after row reduction? If you said, "A column of all zeros," then congrats. In this case, how would it have affected the solution? If you want a confirmation that your answer is correct, ask me in class.

Problem 6.1 Consider the homogeneous linear system $\begin{cases} x + 2y - 3z = 0 \\ 2x + 4y - 6z = 0 \\ -x - 2y + 3z = 0 \end{cases}$.

1. Solve this homogeneous system. Show us your rref, and then your solution (in vector form). For matrix reduction problem, show the rref.
2. Solve the non homogeneous system $\begin{cases} x + 2y - 3z = 4 \\ 2x + 4y - 6z = 8 \\ -x - 2y + 3z = -4 \end{cases}$.
3. Compare and contrast the two solutions.

Problem 6.2 Consider the matrix equation $\begin{bmatrix} 1 & 0 & 1 & -2 \\ -1 & 3 & 5 & 5 \\ 2 & 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

1. Solve this homogeneous matrix equation. Your answer will involve free variables.
2. Now solve $\begin{bmatrix} 1 & 0 & 1 & -2 \\ -1 & 3 & 5 & 5 \\ 2 & 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \\ 1 \end{bmatrix}$.
3. Compare and contrast the two solutions.

---

[1] The free variables are x_3 and x_4 (as these columns don't have a pivot). The three nonzero rows of our matrix are the equations $x_1 - 3x_3 + 4x_4 = 5$, $x_2 + 2x_3 + x_4 = -2$, and $x_5 = 3$. We solve for each variable in terms of the free variable to obtain

$$\begin{aligned} x_1 &= 3x_3 - 4x_4 + 5 \\ x_2 &= -2x_3 - 4x_4 - 2 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= x_3 \end{aligned} \qquad \text{or in vector form} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

Problem 6.3 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 5 \\ 2 & 1 & 4 \end{bmatrix}$. Consider the linear function defined by $L(\vec{x}) = A\vec{x}$. Let $\vec{b} = (-1, 10, 1)$. We wish to solve the linear equation $L(\vec{x}) = \vec{b}$.

1. Find the kernel ($L(\vec{x}) = 0$) of this linear function. Your answer should involve a free variable.
2. One solution to $L(\vec{x}) = \vec{b}$ is $\vec{x}_1 = (-1, 3, 0)$. Give another solution, which we'll call $\vec{x}_2$.
3. One solution to $L(\vec{x}) = \vec{b}$ is $\vec{x}_1 = (-1, 3, 0)$. Give all solutions. How does part 1 help?

Problem 6.4 Consider the matrix equation $A\vec{x} = \vec{b}$. Suppose that $\vec{x}_1$ and $\vec{x}_2$ are both solutions to this non homogeneous equation.

1. Why is $\vec{x}_1 - \vec{x}_2$ a solution to the homogeneous equation $A\vec{x} = 0$?
2. If $\vec{x}_p$ is a single particular solution to $A\vec{x} = \vec{b}$, and we know that all solutions to $A\vec{x} = \vec{0}$ are given by $\vec{x}_h$, then give all solutions to $A\vec{x} = b$.
3. If we replace $A\vec{x}$ with any linear function $L(y)$, does this result still hold?

Look back at the last 4 problems. Do you notice how we solve a few problems using matrices and noticed a pattern, namely that the solution to $A\vec{x} = \vec{b}$ is simply $\vec{x}_h + \vec{x}_p$, where $\vec{x}_h$ is the general solution to the homogeneous equation $A\vec{x} = \vec{0}$, and $\vec{x}_p$ is a single solution to the original equation. The last problem showed that this pattern continued for ANY linear function. So if we can show something is linear, then the solution follows the same technique.

The next few problems have you practice finding the kernel of a linear function, by asking you to find eigenvectors. You'll also see how to use the eigenvalues and eigenvectors of a matrix to get a solution to a homogeneous ODE.

Problem 6.5 Consider the ODE $L(y) = y'' + 5y' + 6y = 0$.

1. Find a general solution y_h to the ODE. Is $y_p = 0$ a solution to $L(y) = 0$?
2. If we let $y_1 = y$ and $y_2 = y'$, explain why we can write this ODE in the matrix form
$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$
3. Find the eigenvalues and eigenvectors of the coefficient matrix $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$.

Check your answer with this link to Sage. You can use this link to check your answers for ANY matrix.

4. A solution is
$$\begin{aligned} y &= c_1 e^{\lambda_1 t} + c_1 e^{\lambda_2 t}, && \text{which means} \\ y' &= c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{aligned}$$
Write this as a vector equation $\begin{pmatrix} y \\ y' \end{pmatrix} = c_1 \begin{pmatrix} ? \\ ? \end{pmatrix} e^{\lambda_1 t} + \cdots$. Make a conjecture about how to use the eigenvalues and eigenvectors to obtain a solution.

Problem 6.6 This problem requires you have completed the previous. Consider a horizontal mass-spring system with $m = 2$, $c = 12$, and $k = 10$ (the units agree). Suppose the spring has been extended 4 units and released from rest.

1. State the IVP that corresponds to this system.
2. Write the second order ODE as a system of first order ODEs (let $y_1 = y$ and $y_2 = y'$). Write the system in the matrix form
$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$
3. Find the eigenvalues and eigenvectors of A, and use them to state a general solution for y and y'.
4. Use the initial conditions to find c_1 and c_2.

Check your answer with this link to Sage. You can use this link to check your answers for ANY matrix.

Problem 6.7 Consider the third order ODE $y''' + 7y'' + 14y' + 8y = 0$. The characteristic polynomial factors as $(\lambda + 1)(\lambda + 2)(\lambda + 4)$.

1. If we let $y_1 = y$, $y_2 = y'$, and $y_3 = y''$, then show how to rewrite this third order ODE as the linear system
$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$
2. Compute the eigenvalues (how does the polynomial factor ... look above).
3. For each eigenvalue, give all the eigenvectors (what matrix did you rref, what is the rref, and how did you obtain the solution). As there are infinitely many eigenvectors, you'll probably find the vectors are easiest to work with if you scale them to get rid of all fractions.
4. Use the eigenvalues and eigenvectors to state a general solution for y, y', and y''.

Check your answer with this link to Sage. You can use this link to check your answers for ANY matrix.

6.2 Solving Non Homogeneous ODEs

From the last section, we now know that the solutions to a non homogeneous ODE $L(y) = r(t)$ must look like $y = y_h + y_p$ where y_h is the general solution to the homogeneous ODE, and y_p is a single particular solution. We've already seen a couple of ODEs that are non homogeneous in the previous section. We have the tools to solve them with Laplace transforms. Let's look at a few examples, discover some patterns, and then speed things up.

Review We'll occasionally need to solve inverse transforms with rather ugly coefficients. Let's review this. Find the inverse Laplace transform of $\dfrac{fs^2 + gs + h}{s^2(s+3)}$. See [2] for an answer.

[2] We use a partial fraction decomposition to write
$$\frac{fs^2 + gs + h}{s^2(s+3)} = \frac{As + B}{s^2} + \frac{C}{s+3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}.$$
The inverse transform is $A + Bt + Ce^{-3t}$. We can obtain A, B, and C in terms of f, g, and h. Since $fs^2 + gs + h = (As + B)(s + 3) + Cs^2$, we would solve the system $f = A + C$, $g = 3A + B$, $h = 3B$, which gives $B = h/3$, $A = g/3 - h/9$, and $C = h/9 - g/3$.

Problem 6.8 Consider the ODE $my'' = -ky' - mg$ from Problem 5.1. This ODE models the position of a pebble (or any other object) as it falls through the air. With this problem, we assumed that gravity pulls the object down (the $-mg$ term), and that air resistance is proportional to velocity (the $-ky'$ terms).

Please use dsolve with WolframAlpha to check EVERY problem you do in this chapter. Class will go much better if you do.

1. For the homogeneous ODE $my'' + ky' = 0$, what are the zeros of the characteristic polynomial. Give a general solution y_h to this homogeneous ODE.
2. For simplicity, let's have $m = 1$, $k = 5$, and $g = -32$. Use Laplace transforms to solve the ODE $y'' + 5y' = -32$ if $y(0) = 0$ and $y'(0) = 0$. What part of your solution is not a term in y_h? See WolframAlpha
3. Now use Laplace transforms to solve the ODE $y'' + 5y' = -32$ if $y(0) = h$ and $y'(0) = v$. Again what part of your solution is not a term in y_h? See WolframAlpha

For our mass spring systems in the last chapter, we assumed they were placed the system horizontally so that we could ignore the force due to gravity. Let's now hang the mass-spring system vertically.

Problem 6.9 Consider a vertical mass-spring system inside a dashpot. The object's mass is m kg. The dashpot applied a frictional force proportional to the velocity and has a coefficient of friction equal to c kg/s. The spring constant is k kg/s^2.

1. Use Newton's second law of motion to explain why $my'' = -cy' - ky - mg$, or $my'' + cy' + ky = -mg$.
2. For simplicity, suppose $m = 1$ kg, $c = 5$ kg/s, and $k = 4$ kg/s^2. Give the solution y_h to the homogeneous ODE $y'' + 5y' + 4y = 0$.
3. Using the same conditions, use Laplace transforms to solve the non homogeneous ODE $y'' + 5y' + 4y = -32$. [You might want to use $y(0) = h$ and $y'(0) = v$, but remember they are arbitrary.] See WolframAlpha
4. As $t \to \infty$, what happens to $y(t)$?

Review Find the inverse Laplace transform of $\dfrac{1}{s(s^2+9)}$. See [3] for an answer.

Problem 6.10 Consider a vertical mass-spring system without friction, so the corresponding ODE is $my'' = -0y' - ky - mg$, or $my'' + ky = -mg$. Suppose $m = 1$ kg and $k = 16$ kg/s^2.

1. Suppose $m = 1$ kg and $k = 4$ kg/s^2. Solve the homogeneous ODE $y'' + 4y = 0$.
2. Using the same conditions, use Laplace transforms to solve the non homogeneous ODE $y'' + 4y = -32$. [You might want to use $y(0) = h$ and $y'(0) = v$, but remember they are arbitrary.] See WolframAlpha

[3] The quadratic $s^2 + 9$ does not factor over the reals, so we write

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9},$$

whose inverse transform is $A + B\cos(3t) + \frac{C}{3}\sin(3t)$. Multiplying both sides by $s(s^2 + 9)$ gives $1 = A(s^2 + 9) + (Bs + C)(s)$. Equating coefficients gives the system $0 = A + B$, $0 = C$, $1 = 9A$. Solving this system yields $A = 1/9$, $B = -1/9$, and $C = 0$.

3. Compare your solutions to the homogeneous and non homogeneous ODEs.

In all three problems above, we applied an external force to the physical system. This constant force $-mg$ showed up on the right had side of the ODE. Whenever we apply an external force to a problem, it shows up on the right hand side of an ODE. If all the forces are internal, then we are solving a homogeneous ODE. Any external forces change it to a non homogeneous ODE.

The forces above are all constant. What do we do with a non constant force? The same thing! It just might get messier. Because we know how to compute Laplace transforms of polynomials, exponentials, cosines and sines, and products of these, we'll focus our attention on external forces that involve these kinds of functions. In a future chapter, we'll explore how to solve problems involving arbitrary external forces.

6.2.1 Learning to Guess Appropriately

In the last chapter, we discovered that we can solve homogeneous ODEs by simply finding the zeros of a polynomial. We gleaned all this information by studying Laplace transforms. In this section, let's tackle a few problems and look for patterns that should greatly simply our ability to solve non homogeneous ODEs.

In all these problems, we'll be solving second order ODEs of the form $L(y) = r(t)$.

Problem 6.11 Consider the ODE $y'' + 5y' + 4y = r(t)$, which we could write as $L(y) = r(t)$, where $L(y)$ is a linear operator.

1. Find y_h, a general solution to the homogeneous ODE $y'' + 5y' + 4y = 0$.

2. Let $r(t) = 3t$. Laplace transform both sides, and show that

$$Y = \frac{sy(0) + y'(0) + 5y(0)}{s^2 + 5s + 4} + \frac{3}{s^2(s^2 + 5s + 4)}.$$

   Explain why a solution to $y'' + 5y' + 4y = 3t$ must be

$$y = c_1 e^{-4t} + c_2 e^{-t} + At + B,$$

   where c_1 and c_2 are arbitrary (depending on the initial conditions), but A and B could be determined by doing a partial fraction decomposition (just set the decompositions up, don't spend time finding the constants).

3. Since c_1 and c_2 are arbitrary constants, let them be zero. This means a particular solution to our ODE is $y_p = At + B$. Substitute $y_p = At + B$, $y_p' = A$ and $y_p'' = 0$ into the ODE $y'' + 5y' + 4y = 3t$ to get $0 + 5(A) + 4(At + B) = 3t + 0$. Use this to find A and B.

4. We now have y_h and y_p. State a general solution to the ODE. See WolframAlpha

5. If $r(t) = 7t^3 - 4t$, make a guess as to what form y_p would take. Use dsolve to check if you are correct. See WolframAlpha

Did you notice that if the external force is a polynomial, then a particular solution y_p is a polynomial of the same degree?

Problem 6.12 Again consider the ODE $y'' + 5y' + 4y = r(t)$, or $L(y) = r(t)$. We know the solution to $L(y) = 0$ (the kernel of L) is $y_h = c_1 e^{-4t} + c_2 e^{-t}$.

1. Let $r(t) = 2\cos 3t$. Laplace transform both sides, and show that
$$Y = \frac{sy(0) + y'(0) + 5y(0)}{s^2 + 5s + 4} + \frac{s}{(s^2 + 9)(s^2 + 5s + 4)}.$$
Explain why a solution to $y'' + 5y' + 4y = 2\cos 3t$ must be
$$y = c_1 e^{-4t} + c_2 e^{-t} + A\cos(3t) + B\sin(3t),$$
where c_1 and c_2 are arbitrary (depending on initial conditions), but A and B could be determined by doing a partial fraction decomposition (just set the decompositions up, don't spend time finding the constants).

2. Since c_1 and c_2 are arbitrary constants, let them be zero. This means a particular solution to our ODE is $y_p = A\cos 3t + B\sin 3t$. Substitute y_p, y_p' and y_p'' into the ODE to get
$$(A\cos 3t + B\sin 3t)'' + 5(A\cos 3t + B\sin 3t)' + 4(A\cos 3t + B\sin 3t) = 2\cos 3t.$$
Use this equation to show $A = -\frac{1}{25}$ and $B = \frac{3}{25}$. [Hint: The right hand side is $2\cos 3t + 0\sin 3t$.]

3. We now have y_h and y_p. State a general solution to the ODE. See WolframAlpha

4. If $r(t) = \sin(7t)$, make a guess as to what form y_p would take. Use dsolve to check if you are correct.

Did you notice that if the external force is a trig function, then a particular solution will be a linear combination of trig functions?

Let's look at one more, but this time let's have $r(t)$ involve exponentials. Something different happens when $r(t)$ is actually part of the kernel of L.

Problem 6.13 Again consider the ODE $y'' + 5y' + 4y = r(t)$, or $L(y) = r(t)$. We know the kernel of L is $y_h = c_1 e^{-4t} + c_2 e^{-t}$.

1. Let $r(t) = 7e^{-3t}$. Compute the Laplace transform of both sides, and solve for Y. Explain why a solution to $y'' + 5y' + 4y = 7e^{-3t}$ must be $y = c_1 e^{-4t} + c_2 e^{-t} + Ae^{-3t}$, where c_1 and c_2 are arbitrary, but A could be determined by doing a partial fraction decomposition (just set the decompositions up, don't spend time finding the constants).

2. Since c_1 and c_2 are arbitrary constants, let them be zero. This means a particular solution to our ODE is $y_p = Ae^{-3t}$. Substitute y_p, y_p' and y_p'' into the ODE to get
$$(Ae^{-3t})'' + 5(Ae^{-3t})' + 4(Ae^{-3t}) = 7e^{-3t}.$$
Use this equation to find A.

3. We now have y_h and y_p. State a general solution to the ODE. See WolframAlpha

4. If $r(t) = e^{-2t}$, make a guess as to what form y_p would take. Use dsolve to check if you are correct.

5. If $r(t) = e^{-4t}$, make a guess as to what form y_p would take. Use dsolve to check if you are correct. You should notice that this answer is different than the others. What makes it different? Why do you think this difference occurred?

Did you notice that when the external force is an exponential, then a particular solution is an exponential. Also, did you notice that if the external force matched the solutions to the homogeneous solution, then our particular solution was multiplied by t?

The previous three problems developed some key ideas we need to expand. Whenever we need to solve an ODE of the form $L(y) = r(t)$, we have a few things to consider.

- First, we need to get the solution y_h to the homogeneous ODE $L(y) = 0$. This is just the kernel of the linear function L.
- Then we need to find a particular solution y_p to $L(y) = r(t)$. The previous 3 problems suggested that we can guess a form for y_p (based off r), and then use our guess to determine the value of any coefficients.
- The last problem suggested that if $r(t)$ is actually part of the homogeneous solution, then we have to modify our guess slightly (multiply by t) to get y_p.

We need to make sure this pattern works on more problems. Here's where software comes in handy. We can use dsolve with WolframAlpha to kick out solutions to ODEs really fast. What we need is to practice guessing a particular solution with lots of ODEs, and make sure we build up the right patterns. Then, we can start tackling every non homogeneous ODE in a consistent, fast, clean, way. The next problem asks you to do this. Here's a pattern of what I expect.

- For the ODE $y'' + 5y' + 6y = t + e^{-7t}$, the characteristic equation is $\lambda^2 + 5y + 6 = (\lambda + 2)(\lambda + 3) = 0$. This gives $y_h = c_1e^{-2t} + c_2e^{-3t}$. Since $r(t) = t + e^{-7t}$, I'm going to guess $y_p = (At + B) + (Ce^{-7t})$. I'm guessing this because the Laplace transforms of t and e^{-7t} would put $\frac{1}{s^2}$ and $\frac{1}{s+7}$ in the problem. When setting up the partial fraction decomposition, we'd have something like

  $$Y = \frac{c_1}{s+2} + \frac{c_2}{s+3} + \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+7}.$$

  Checking my guess with WolframAlpha shows I'm correct, where $A = 1/6$, $B = -5/36$ and $C = 1/20$. See WolframAlpha

Problem 6.14 Consider the ODE $y'' + 6y' + 9y = \sin(4t) + e^{-t}$.

1. Find the kernel of $L(y) = y'' + 6y' + 9y$ (i.e. state y_h).
2. Make a guess for y_p (with undetermined coefficients), and explain why you made this guess based on what the Laplace transform would yield.
3. Use software to find y_p, and update your guess above (with any reasons for the changes needed). Then state a general solution. See WolframAlpha
4. Repeat parts 2 and 3 if instead you needed to solve $y'' + 6y' + 9y = e^{-3t}$. What makes this different?

Problem 6.15 For each problem below, (1) state y_h, (2) make a guess for y_p (your guess will involve undetermined coefficients), and then (3) check your answer using software (giving the value of any coefficients in your guess). If your guess was wrong, please tell us your original guess, and why it was wrong.

1. $y'' + 3y' + 2y = t + e^{-2t} + e^{-5t}$ See WolframAlpha
2. $y'' + 4y = 4t^2 - 3\cos(3t) + 5\sin(2t)$ (Some constants could be zero.) See WolframAlpha
3. $y'' + 7y' + 10y = 5e^{-2t}\cos(3t) - 6e^{-5t}$ See WolframAlpha
4. $y'' + 6y' + 25y = 7e^{-3t} - 2\cos(4t) + 6e^{-3t}\cos(4t)$ (Wolfram's solution is factored. Expand it.) See WolframAlpha

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Let's try to summarize the patterns we've seen.

Problem 6.16 Suppose that $L(y) = r(t)$ is a 2nd order constant coefficient ODE.

1. If $r(t)$ is in the table below, what would you guess for y_p?

Form of $r(t)$	Guess for y_p
ke^{at}	
kt^n	
$k\cos(\omega t)$	
$k\sin(\omega t)$	
$ke^{at}\cos(\omega t)$	
$ke^{at}\sin(\omega t)$	

2. If $r(t)$ involves a sum of terms in the table above, what do you guess for y_p? (So if $r(t) = 7\cos(2t) - 3\sin(2t) + 8e^{3t}\cos(2t)$, then $y_p = ?$)
3. If part of your guess is in the kernel of L, how should you modify your guess? (So if y_h involves c_1e^{-3t}, and $r(t) = 7e^{-3t} + t^3$, then $y_p = ?$)

The ideas above work with higher order ODEs as well. Let's try this on a 5th order ODE.

Problem 6.17 Suppose we have a constant coefficient linear differential equation of the form $L(y) = r(t)$. It's a 5th order ODE, and the characteristic equation has zeros $2, -3, -3, -4 + 5i, -4 - 5i$.

1. What do you guess for y_p if $r(t) = 3e^{-2t} - 7e^{2t} + 5e^{-3t} + \cos(5t) - 12e^{-4t}\sin(5t)$.
2. Use software to expand $(\lambda - 2)(\lambda + 3)^2(\lambda - (-4 + 5i))(\lambda - (-4 - 5i))$, and then state a 5th order ODE that would have this characteristic polynomial.
3. Use software to check if your guess is correct. WolframAlpha won't solve this one, so please use Mathematica or Maple to solve it, and check if your guess is correct. We'll open up Mathematica in class and solve this.

At this point, we've got the basic idea to solve $L(y) = r(t)$, provided L is a constant coefficient linear operator. We find y_h, and then guess a particular solution y_p, with undetermined coefficients. We then take a couple derivatives to figure out the unknown constants. This is called "The Method of Undetermined Coefficients." Here's a few observations:

- We could just solve all the ODEs using Laplace transforms. The problem is that if we need a general solution, the solution might get ugly really fast. Laplace transforms work best when we have initial conditions.
- If we have initial conditions, maybe we should just do a Laplace transform flat out. No guessing is needed. We'll have to decide which is faster.
- These ideas work on higher order ODEs, in the exact same way they work on second order ODEs.

6.3 Applications

At this point, we need to practice the method of undetermined coefficients. Rather than just solve a bunch of ODEs with no application, let's connect each one to a physical problem.

Problem 6.18 Consider a vertical mass spring system with $m = 1$ kg, $c = 5$ kg/s, and $k = 6$ kg/s^2. Let's assume $g = -10$ m/s^2 (it makes the arithmetic simpler). The 1 kg object is a magnetic brick. We turn on an electromagnet underneath the brick, and then slowly ramp up the force of the magnet be $2t$ N. When the magnet was turned on, the brick was motionless. How far has the bricked dropped after t seconds.

1. Solve the ODE $y'' + 5y' + 6y = -10$. Use this to state how much the brick gets elongated by the magnet. See WolframAlpha
2. Solve the IVP $y'' + 5y' + 6y = -10 - 2t$, where $y(0)$ is the position from the first part, and $y'(0) = 0$. See WolframAlpha
3. Solve the IVP $y'' + 5y' + 6y = -2t$, where $y(0) = 0$ and $y'(0) = 0$. What does this have to do with the problem above? See WolframAlpha

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Problem 6.19 Solve the following IVPs. (This problem originally had some other stuff at the beginning, but it all got changed - this is where my migraine hit from dehydration.)

1. Assume $r(t) = 20e^{-t}$. Solve the IVP $y'' + 5y' + 6y = 20e^{-t}$, $y(0) = 0$, $y'(0) = 0$.
2. Assume $r(t) = 20e^{-2t}$. Solve the IVP $y'' + 5y' + 6y = 20e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.
3. Check both your answers with Wolfram Alpha. You may have to expand WolframAlpha's solution to get it to match yours. See WolframAlpha

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Problem 6.20 We build a rocket and attach an engine. In free fall, we already know the ODE which models the motion is $my'' + ky' = -mg$. The engine adds an external force $r(t)$ to this system. Because the engine burns fuel as it propels upwards, the mass $m(t)$ now depends on time. This gives us the ODE $m(t)y'' + ky' = -m(t)g + r(t)$. If we fire the rocket in space, then we could neglect the $-m(t)g$ part (but then k would probably also be zero). We need a good model for engine thrust.

Most rocket engines have a three part thrust. The engine first ramps up (linearly) to some constant thrust, stays at that constant thrust for a time, and then ramps down linearly. We'll revisit this again in the next chapter, when we have some powerful tools for working with piecewise defined functions.

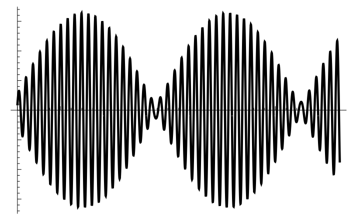
Let's fire a toy rocket from the earth's surface. Suppose $m = .2$ kg and $k = .6$ kg/s. For simplicity, use $g = 10$m/s^2. Let's assume the rocket thrust starts out fast, and drops to zero exponentially. We'll also assume that the fuel is extremely light, so that we can assume $m(t)$ is just the constant .2 kg. This gives us an external force $f(t) = ae^{bt}$ for some known constants a and b.

1. In this chapter, we are studying linear constant coefficient non homogeneous ODEs. If we allowed m to change with t, why does the material in this chapter no longer apply?
2. Why are the initial conditions $y(0) = 0$ and $y'(0) = 0$.
3. If $r(t) = 7e^{-5t}$, determine the rocket's height $y(t)$ after t seconds. Here I gave you some specific numbers to work with. This is often the key to working on a problem with symbols.
4. If $r(t) = ae^{bt}$, determine the rocket's height $y(t)$ after t seconds.

For the next problem, let's imagine attaching the left end of a spring to a wheel, and then rotate the wheel. We'll keep the mass-spring system horizontal, so we can neglect gravity. Please look at the links to

- LearnersTV or
- Wolfram Demonstrations Project

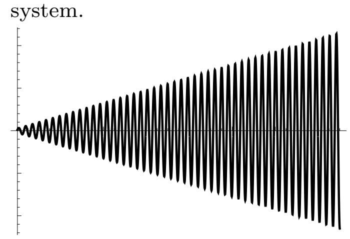
to see a picture of such a system. This rotating wheel applies a periodic external force to the mass-spring system. This force is often called a driving force.



Notice the beats. In this example, $\omega = 2$, $m = 1.1$, $c \approx 0$, and $k = 4$. Since $\omega_0 = \sqrt{4/1.1} \approx 2 = \omega$, the solution results in large periodic oscillations. If the oscillations are too large, they will destroy the system.

Problem 6.21 Let's attach a mass-spring system to a wheel. Suppose $m = 1$ and $k = 4$ (with no dashpot). The driving force, $r(t)$, is periodic.

1. Assume the driving force is $r(t) = 7\sin(5t)$. Solve the IVP $y'' + 4y = 7\sin(5t)$, where $y(0) = 0$ and $y'(0) = 0$.
2. Assume m and k are now some known constant, and the driving force is $r(t) = A\sin(\omega t)$, where $\omega \neq \sqrt{k/m}$. Solve the IVP $my'' + ky = F\sin(\omega t)$, where $y(0) = 0$ and $y'(0) = 0$. [Because the coefficients are variables, you might want to use Cramer's rule when solving for any unknown constants.]



When $\omega_0 = \omega$ and friction is negligible, a system will oscillate with an amplitude that grows without bound. Beware of this situation, as any mechanical system which undergoes this kind of oscillation will self destruct.

Problem 6.22 Again, let's attach a mass-spring system to a wheel. Suppose $m = 1$ and $k = 4$ (with no dashpot). The driving force, $r(t)$, is periodic.

1. Assume the driving force is $r(t) = 7\sin(2t)$. Solve the IVP $y'' + 4y = 7\sin(2t)$, where $y(0) = 0$ and $y'(0) = 0$.
2. Assume m and k are now some constant, and the driving force is $r(t) = A\sin(\omega t)$, where $\omega = \sqrt{k/m}$. Solve the IVP $my'' + ky = F\sin(\omega t)$, where $y(0) = 0$ and $y'(0) = 0$.

Make sure you ask me in class to graph your two solutions above in Mathematica. The plots get interesting when $\omega \approx \sqrt{k/m}$, and the solution produces steady beats. You should see a rhythmic rise and fall in the amplitude of the solution. When $\omega = \sqrt{k/m}$, the solution grows without bound, which many people call resonance. The following YouTube videos show the collapse of the Tacoma Narrows bridge, and airplane flutter. The points to these videos is to show you the dangerous bad things that can (and do) happen to a structure when the designer forgets to take into account how external driving forces might interact with the internal frequencies of the mechanical system.

- The Tacoma Narrows bridge collapses.
- The tail of a small airplane begins to flutter.

- Watch a collection of flutter examples.
- An RC airplane looses its wing midflight.

In both problems above, we assumed there was no friction ($c = 0$). Can we produce beats or resonance when there is friction? Let's analyze this problem and show that YES, bad things can still happen when friction is involved. To discover when disaster might occur, we have to work with symbolic answers and then ask, "What would it take to produce large oscillations?" Let's analyze this first with some specific numbers (to notice patterns) and then we'll analyze what happens symbolically.

Problem 6.23 Consider the ODE $y'' + 2y' + 5y = 5\sin(3t)$.

1. Find a general solution to this ODE. See WolframAlpha
2. As t increases, what happens to the homogeneous solution?
3. If $y(0) = 0$ and $y'(0) = 0$, solve the IVP.

When friction enters a mass spring system, the homogeneous solution will always die out over time. The particular solution y_p is called "the steady-state" solution, or "steady periodic solution." As time moves on, friction will damp out all oscillation except for the steady-state solution, y_p.

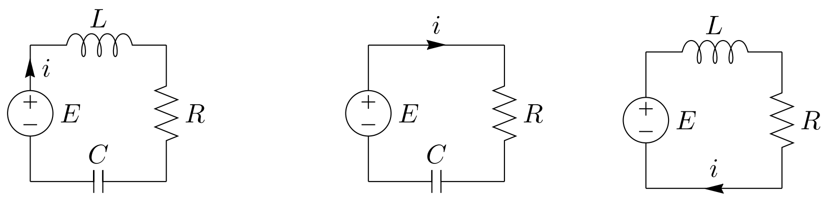
Problem 6.24 Consider the ODE $my'' + cy' + ky = F\sin(\omega t)$.

1. What are the roots of the characteristic polynomial?
2. We guess the steady-state solution (particular solution) is $y_p = A\cos(\omega t) + B\sin(\omega t)$. Why do we never have to multiplying the guess by t?
3. Find the steady-state solution. As a hint, you'll probably find Cramer's useful when solving for A and B (because you'll get a linear system with variables as coefficients). See WolframAlpha
4. What would it take to get a really large amplitude for the steady-state solution (thus destroying the mechanical system)?

6.3.1 Electric circuits

Remember that Kirchoff's voltage law states that the voltage (electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop. We've summarized this by saying "voltage in equals voltage out." Because we've been using complex numbers in our work, we'll use $I(t)$ to represent the current in a loop instead of $i(t)$. We've already used Kirchoff's voltage law in connection with resistors. Let's now add an inductor and a capacitor to a single loop. Each element (resistor, inductor, capacitor) produces the voltage drop given in the table below.

Component	Voltage drop	Other
Resistor	RI	Ohm's law, where R is in ohms
Inductor	$LI' = L(dI/dt)$	L is in henrys
Capacitor	$\dfrac{Q}{C} = \frac{1}{C}\frac{dI}{dt}$	Q is in coulombs, C in farads.



An RLC-circuit	An RC-circuit	An RL-circuit
$LI' + RI + \frac{1}{C}\int I(t)dt = E(t)$	$RI + \frac{1}{C}\int I(t)dt = E(t)$	$LI' + RI = E(t)$
$LQ'' + RQ' + \frac{1}{C}Q = E(t)$	$RQ' + \frac{1}{C}Q = E(t)$	
$LI'' + RI' + \frac{1}{C}I = E'(t)$	$RI' + \frac{1}{C}I = E'(t)$	

Table 6.1: Typical diagrams of RCL, RC, and RL circuits, and their corresponding ODEs. The first row is an integro-differential equation for the current $I(t)$. The second row is the ODE for the charge Q on the capacitor. The third row is the derivative of the first row.

The charge Q on a capacitor is related to the current by $I(t) = \frac{dQ}{dt}$, or $Q = \int I(t)dt$.

We'll be studying RC, RLC, and RL circuits in the next few chapters. Table 6.1 shows the differential equations corresponding to each type of circuit.

In a circuit with one resistor, one inductor, and one capacitor (an RLC circuit), if the electromotive force is $E(t)$, then Kirchoff's Voltage law gives the integro-differential equation

$$LI' + RI + \frac{1}{C}Q(t) = E(t) \quad \text{or} \quad LI' + RI + \frac{1}{C}\int I(t)dt = E(t).$$

Differentiating both sides removes the integral and gives

$$LI'' + RI' + \frac{1}{C}I(t) = E'(t),$$

which is a 2nd order non homogeneous linear differential equation with constant coefficients. However, the initial conditions are in terms of initial charge $Q(0)$ and initial current $I(0)$. To solve the differential equation for I, we need $I'(0,$ which you can get from the equation $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$. Problem 14.13 in Schaum's provides an excellent example that summarizes the solution technique.

Problem 6.25 Consider an RLC circuit with $L = 1/2$, $R = 2$, and $C = 2/3$. Let's plug the circuit into an alternating current power source (like a wall outlet), which means we might have something like $E(t) = 2\cos(3t)$. Initially, assume that the current is zero and the charge on the capacitor is zero. We'd like to find the current at any time t in the circuit.

1. Explain why the current satisfies $I'' + 4I' + 3I = -12\sin(3t)$. Find a general solution to this ODE. See WolframAlpha

2. We know that $I(0) = 0$ and $Q(0) = 0$. Use the equation $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$ to find $I'(0)$.

3. Find the current in the wire at any time t by solving the corresponding IVP. Use the initial conditions you found in the previous part.. See WolframAlpha

4. What is the steady-state current? (Which part of your solution above does not vanish after sufficient time has passed? This would be the current flowing through the circuit after the initial conditions have died out.)

Problem 6.26 Consider an RLC circuit with $L = 1$, $R = 8$, and $C = \frac{1}{25}$. Let's plug the circuit into a 12 V battery, so we have $E(t) = 12$. Initially, assume that the current is zero and the charge on the capacitor is zero. We'd like to find the current at any time t in the circuit.

1. State the IVP whose solution would give the current at any time t. (What's the ODE, and what are the initial conditions $I(0)$ and $I'(0)$). [Hint: Use the equation $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$ to find $I'(0)$.]
2. Find the current in the wire at any time t. Check your answer with WolframAlpha (you'll want to use y instead of I).
3. What's the steady-state current?

Observation 6.1. Mechanical models are expensive to build. Electrical models are fairly simple to build and measure. If you need to create a mechanical system, it may prove beneficial financially to start with an electrical model. Engineers spend another semester on this idea in system dynamics. Hydraulic systems are also very closely related. In bridging between mechanical and electrical systems, we compare the following variables.

Mechanical System	m	c	k	$r(t) = F_0 \cos \omega t$	$y(t)$
Electrical System	L	R	$1/C$	$E'(t) = E_0 \omega \cos \omega t$	$I(t)$

Solving a problem in one system (either mechanical or electrical) can provide useful results in the other.

6.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 7

Laplace Transforms

This chapter covers the following ideas.

1. Explain how to compute Laplace transforms and inverse Laplace transforms. Explain and use both shifting theorems, and be able to prove them.

2. Use Laplace transforms to solve IVPs.

3. Describe the Dirac delta function, and use it to solve ODEs. Illustrate what the Dirac delta function does to a system by applying it to examples in mass-spring systems and electrical networks.

4. Explain what a convolution is, and how it relates to Laplace transforms.

You can find additional practice problems in Schaum's Outlines *Differential Equations* by Richard Bronson. You'll find relevant problems in chapters 21 -24, as well as some extra practice problems at the end of this chapter. Do enough of each type that you feel comfortable with the ideas.

$f(t)$	$F(s)$	provided
1	$\frac{1}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
y'	$sY - y(0)$	
y''	$s^2Y - sy(0) - y'(0)$	
$e^{at}f(t)$	$F(s-a)$	
$f(t) * g(t)$	$F(s)G(s)$	

$f(t)$	$F(s)$	provided
$\cos(wt)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(wt)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cosh(wt)$	$\frac{s}{s^2-\omega^2}$	$s > \|\omega\|$
$\sinh(wt)$	$\frac{\omega}{s^2-\omega^2}$	$s > \|\omega\|$
$u(t-a)$	$\frac{1}{s}e^{-as}$	
$\delta(t-a)$	e^{-as}	
$f(t-a)u(t-a)$ $f(t)u(t-a)$	$\mathscr{L}(f(t))e^{-as}$ $\mathscr{L}(f(t+a))e^{-as}$	

Table 7.1: Table of Laplace Transforms. Note that the s shifting theorem $\mathscr{L}(e^{at}f(t)) = F(s-a)$ has a positive a in the exponent, while the t shifting theorem $\mathscr{L}(f(t-a)u(t-a)) = \mathscr{L}(f(t))e^{-as}$ has a negative a in the exponent.

You must practice lots of problems to gain a feel for patterns. Many of the problems in 21-23 are fast. Please take a few minutes every day to just flat out practice with the basics (kind of like when you were learning the times tables - they get really fast if you just practice them). When you feel like you have the basics down, see if you can complete chapters 21 and 22 in less than an hour. If one stumps you, skip it and come back later.

Once you feel confident, chapters 23 (on convolutions and the heaviside function) and 24 (solving IVPS) will help you use the Laplace transforms to solve ODEs. At the end of this chapter are some additional problems to help you cement your understanding. Table 7.1 summarizes the transforms we use most often.

7.1 Review

Our real goal in this chapter is to learn how to handle non differentiable changes in an ODE. We'll find that Laplace transforms provide us with extremely nice tools to solve problems of this type. Before we jump in, let's review how to solve a couple ODEs with Laplace transforms, and perhaps make some connections that we haven't yet made.

Review Compute the inverse Laplace transform of $Y = \frac{3s+8}{(s+1)^2+16}$. See [1].

Problem 7.1 Compute the inverse Laplace transform of

$$Y = \frac{5}{(s+3)^3} + \frac{2s+3}{(s+4)^2+9} + \frac{3s+1}{(s+2)^2-49}.$$

Use the rules for cosh and sinh to tackle the last terms, rather than doing a partial fraction decomposition. The goal of this problem is to make sure you have the s-shifting theorem mastered.

We've solve inverse transforms such as this one multiple times. If you need to refresh, please head to chapters 21 and 22 in Schaum's, and just practice the problems where answers are provided.

Problem 7.2 Consider the IVP $y'' + 6y' + 8y = 0$, $y(0) = 2$, $y'(0) = 3$.

1. Use Laplace transforms to solve the IVP. Write your answer as a linear combination of e^{-2t} and e^{-4t}.

2. Use Laplace transforms to solve the IVP. Note that we could complete the square to write $s^2 + 6s + 8 = (s+3)^2 - 1$. Write your answer as a linear combination of $e^{-3t}\cosh(t)$ and $e^{-3t}\sinh(t)$.

3. Remember that $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$ and $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$. Use this to show how your second solution is really the same as your first.

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[1] We can rewrite Y as

$$Y = \frac{3(s+1) - 3 + 8}{(s+1)^2 - 16} = \frac{3(s+1)}{(s+1)^2 - 16} + \frac{5}{(s+1)^2 - 16}\frac{4}{4}.$$

The inverse Laplace transform is then

$$y(t) = 3e^{-t}\cosh(4t) + \frac{5}{4}e^{-t}\sinh(4t).$$

Problem 7.3 Consider the IVP $y'' + 7y' + 10y = 0$, $y(0) = 4$, $y'(0) = -3$.

1. Use Laplace transforms to solve the IVP. Write your answer as a linear combination of e^{-2t} and e^{-5t}.
2. Complete the square on $s^2 + 7s + 10$ and use the Laplace transform with cosh and sinh rules to solve the IVP.
3. Which is easier?

The last two problems should have reviewed the main ideas used in solving Laplace transforms. In addition, I hope you see how useful it is to know the transforms for $\cosh \omega t$ and $\sinh \omega t$. They can greatly simplify some computations.

7.2 Piecewise Defined Functions

We now turn to perhaps the key reason Laplace transforms were invented. We can use them to obtain quick solutions to problems with discontinuous external forces. Let's start by examining a RL circuit with a battery, because it keeps the computations simple.

Problem 7.4 Consider an RL circuit with $R = 1$ ohms and $L = 1$ Henry. At time zero, there is no battery in the system. After 2 seconds, we connect a battery $E = 12V$ to the circuit. Two seconds after connecting the battery, we disconnect it. Our goal is to determine the current in the wire exactly 2 second after we disconnect the battery.

1. During the first two seconds, we need to solve the IVP $I' + I = 0$ where $I(0) = 0$. Solve this IVP and use your solution to show that the current after 2 seconds is $I(2) = 0$.
2. Between $t = 2$ and $t = 4$, we know $E = 12$. Solve the IVP $I' + I = 12$, $I(2) = 0$. What is $I(4)$, the current right when the battery gets removed? Your answer will involve the constant e^2.
3. When we remove the battery, the ODE is $I' + I = 0$. We know the initial condition $I(4)$ from the last part. Solve the IVP, and state $I(6)$.
4. We can now predict the current at any time t. Use piecewise function notation to state the current in the form

$$I(t) = \begin{cases} 0 & 0 \leq t < 2 \\ 12 - 12e^2e^{-t} & 2 \leq t < 4 \\ ? & 4 \leq t \end{cases}.$$

In the previous problem, we found the current using the initial conditions $I(0) = 0$, $I(2) = 0$, and $I(4) = ?$. Another way to tackle this problem is to move our reference frame, letting $t = 0$ correspond to the beginning of each change. The computations are often simpler, and we then just have to shift the reference frame back when we finish the problem.

Problem 7.5 Consider again the same RL circuit with $R = 1$ ohms and $L = 1$ Henry. At time zero, there is no battery in the system. After 2 seconds, we connect a battery $E = 12V$ to the circuit. Two seconds after connecting the battery, we disconnect it. Our goal is to determine the current in the wire exactly 2 second after we disconnect the battery. We'll solve this problem by always making $t = 0$ the start of each IVP.

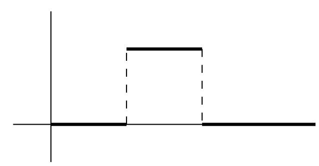
1. During the first two seconds, we need to solve the IVP $I' + I = 0$ where $I(0) = 0$. Solve this IVP and use your solution to show that the current after 2 seconds is $I(2) = 0$.

2. Between 2 and 4 seconds, we know $E = 12$. Letting $t = 0$ correspond to 2 seconds, solve the IVP $I' + I = 12$, $I(0) = 0$. What is $I(2)$, the current right when the battery gets removed?

3. When we remove the battery, the ODE is $I' + I = 0$. Let $t = 0$ correspond to 4 seconds, and then we know the initial condition $I(0)$ from the last part. Solve the IVP, and state the current $I(2)$ after 6 seconds.

4. Use piecewise function notation to state the current at any time t. Remember to shift your solutions from the 2nd and 3rd part. Your answer will look like
$$I(t) = \begin{cases} 0 & 0 \leq t < 2 \\ 12 - 12e^{-(t-2)} & 2 \leq t < 4 \\ ? & 4 \leq t \end{cases}.$$

The electromotive force in the previous problem ($E(t)$) was a piecewise defined force. It was zero, then 12, then 0. Using piecewise notation, we would write this as
$$E(t) = \begin{cases} 0 & 0 \leq t < 2 \\ 12 & 2 \leq t < 4 \\ 0 & 4 \leq t \end{cases},$$



and we could graph the function $E(t)$ using the figure to the right. We need a nice clean way to work with piecewise defined external forces. We also need to become comfortable graphing and working with these kinds of forces.

Problem 7.6 Consider the functions

$$f(t) = \begin{cases} 2t & 0 \leq t < 2 \\ -t^2 + 4t & 2 \leq t < 4 \\ 18 - 3t & 4 \leq t < 6 \\ t^2 - 12t + 35 & 6 \leq t < 8 \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 2t & 0 \leq t < 2 \\ 4 - (t-2)^2 & 2 \leq t < 4 \\ 6 - 3(t-4) & 4 \leq t < 6 \\ (t-6)^2 - 1 & 6 \leq t < 8 \end{cases}.$$

1. Graph $f(t)$ in the ty plane.

2. Graph $g(t)$ in the ty plane.

3. Graph $y = 2t$, $y = 4 - t^2$, $y = 6 - 3t$, and $y = t^2 - 1$ in the ty plane. What does this have to do with the above?

7.2.1 The Heaviside function $u(t - a)$

We now define the key function that allows us to work with piecewise defined functions. Some people call this the Heaviside function, some call it the unit step function. This is a simple function that jumps up a single unit at a specified value of t.

Definition 7.1: Heaviside or Unit Step Function. We define the Heaviside, or unit step function, to be the function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \end{cases}.$$

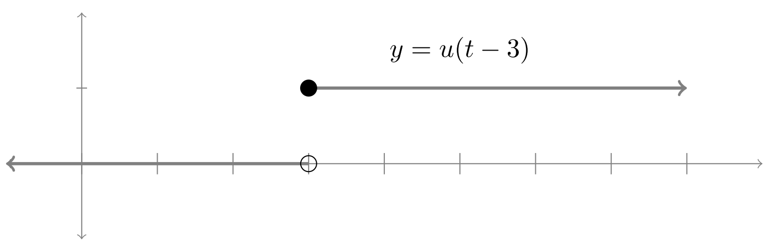
In our work, it won't matter what we define $u(0)$ to equal. Here, we define $u(0) = 1$, but we could have just as easily define $u(0) = 0$ or $u(0) = 1/2$. This last option, the $1/2$, comes in handy when working with Fourier series.

We'll most often shift this function right a units (so we replace t with $t - a$, which means we could write

$$u(t-a) = \begin{cases} 0 & t-a < 0 \\ 1 & 0 \le t-a \end{cases} = \begin{cases} 0 & t < a \\ 1 & a \le t \end{cases}.$$

Why does this function matter. It's like an on/off function. If you multiply $f(t)$ by $u(t-a)$, then the function $f(t)u(t-a)$ is zero to the left of a, and is equal to $f(t)$ after a. If $a = 3$, then look below for the graph of $y = u(t-3)$.

Mathematica uses the name "HeavisideTheta" for the Heaviside function. You'll see the symbol θ show up as the name of a function.



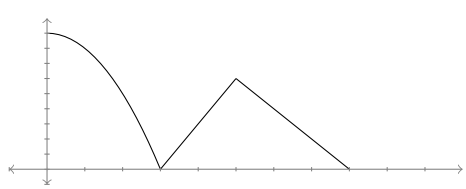
Problem 7.7 Construct a graph of each of the following:

1. $f(t) = u(t-4) - u(t-7)$
2. $g(t) = (10-t)(u(t-4) - u(t-7))$
3. $h(t) = (10-(t-4))(u(t-4) - u(t-7))$ (How does this differ from the previous?)
4. $k(t) = t^2(u(t-3) - u(t-5))$
5. $l(t) = (t-3)^2(u(t-3) - u(t-5))$ (How does this differ from the previous?)

WolframAlpha and I are having issues when it comes to plotting Heavisides. I can plot the first one just fine, but as soon as I times it by $(10-t)$, it tries to plot a surface. As such, please use this link to Sage to check your work. Make sure you can explain how the graphs are made (not just give them).

Make sure you check your solution by following the link to Sage.

Problem 7.8 The graphs of $f(t) = 9 - t^2$ for $0 \le t \le 3$, and $g(t) = 3t$ for $0 \le t \le 2$, and $h(t) = 6 - 2t$ for $0 \le t \le 2$ are connected together (when one ends, the others starts) to give the following graph.



1. Write this function using piecewise function notation.
2. Write this function using Heaviside notation. You'll want to use the idea that $u(t-a) - u(t-b)$ turns a function on at a and off at b.

3. When you think you have the function, use this Sage link to check if you are correct (you'll have to type in your function).

   Feel free to use Mathematica instead, if you have downloaded and installed it. Remember that BYU-I students can now install Mathematica on their personal machines for free. Please head to I-Learn for instructions. You can then download the Mathematica Technology Introduction, and you'll see how to code HeavisideTheta functions in Mathematica.

Did you notice in your work above that it was a lot easier to graph a piecewise defined function when everything was shifted to the starting point. It's much easier to graph $f(t-a)u(t-a)$ than it is to graph $f(t)u(t-a)$. We'll find that this remains true as well, when we start applying Laplace transforms.

It's time to look at the Laplace transform of the Heaviside function. The next problem is the key to why Laplace transforms work so nicely with piecewise defined functions. We'll compute the Laplace transform of both $f(t-a)u(t-a)$ and $f(t)u(t-a)$. Then we'll practice on a few problems.

Problem 7.9: t-shifting Theorem Suppose that $y = f(t)$ is a function for which you can find the Laplace transform. Show, using the definition of the Laplace transform, that

$$\mathscr{L}\{f(t-a)u(t-a)\} = \mathscr{L}\{f(t)\}e^{-as}.$$

In particular, this means that $\mathscr{L}\{1u(t-a)\} = \frac{1}{s}e^{-as}$. Then show that

$$\mathscr{L}\{f(t)u(t-a)\} = \mathscr{L}\{f(t+a)\}e^{-as}.$$

[Hint: Just write down the definition of the Laplace transform. You'll have to do a u substitution, but you'll probably want to use a different variable, like w. Remember that $u(t-a) = 0$ if you are below a, which should allow you to remove $u(t-a)$ from any integral, after updating the bounds.]

Problem 7.10 Compute the Laplace transforms of each of the following functions.

See Schaum's chapter 22 for lots more practice. Please do a bunch of these until you feel like you have the idea down. Each problem takes just a tiny bit of time. Unless you practice this a bunch, you'll be lost and spend gobs of time on the upcoming problems.

1. $f(t) = 3u(t-4)$
2. $f(t) = 3(t-4)\,u(t-4)$
3. $f(t) = 3t\,u(t-4)$ [Hint: $3t = 3(t-4) + 12$]
4. $f(t) = (t-3)^2u(t-3)$
5. $f(t) = t^2(u(t-2) - u(t-5))$

Problem 7.11 Compute the inverse Laplace transform of each of the following functions.

1. $\dfrac{4}{s^3}e^{-2s}$ 2. $\dfrac{4}{(s+5)^3}e^{-2s}$ 3. $\dfrac{2s+1}{s^2+9}e^{-\pi s/6}$ 4. $\dfrac{3s+4}{(s+2)^2+16}e^{-5s}$

We are now ready to solve Laplace transform problems with the Heaviside function. The simplest example is an RL circuit. To get to RC and RLC circuits, we'll need to discuss the derivative of the Heaviside function (which technically doesn't exist).

Problem 7.12 Consider an RL circuit with $R = 4$ and $L = 1$. At $t = 0$ there is no current in the wire. Two seconds in, we connect a 9V battery to the circuit. Three seconds later ($t = 5$) we remove the battery. This gives us the electromotive force as $E(t) = 9(u(t-2) - u(t-5))$. We need to solve the IVP

$$1I' + 4I = 9(u(t-2) - u(t-5)), \quad I(0) = 0.$$

Use Laplace transforms to predict the current $I(t)$ at any time t. You'll want to ignore the e^{-as} terms when you perform any needed partial fraction decompositions.

Problem 7.13 Consider an RL circuit with $R = 5$ and $L = 1$. We connected a variable voltage source to the circuit and start to ramp up the power. Our electromotive force is $E(t) = t$ volts. After 12 seconds, we loose power and $E(t)$ drops to zero. Solve for the current in the wire at any time t.

[Hint: The electromotive force is $E(t) = t - tu(t-12)$, which means we solve

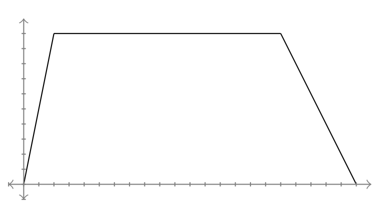
$$I' + 5I = t - tu(t-12), \quad I(0) = 0.$$

Use Laplace transforms and the t-shifting theorem to complete this.]

Problem 7.14 Consider a vertical mass spring system, where we attach a magnetic brick to the bottom of the spring. Let $y = 0$ be the equilibrium height of the brick after accounting for gravity (so we can ignore the force of gravity). Suppose that $m = 1$, $c = 0$, and $k = 4$. The spring is pushed upwards 1 unit, and then let go from rest. After 3 seconds, an electromagnet pulls down on the spring with a force of 7 (the units all agree). We can model this using $r(t) = -7u(t-3)$. Solve the ODE $y'' + 4y = -7u(t-3)$. Check your solution with Mathematica or Sage.

Problem 7.15 We decide to launch a rocket. We attach an engine to the rocket, and light it. Neglect the mass of the fuel, or think "it's a battery powered rocket.") The mass of the rocket and engine we'll assume is 4 kg. Let's launch the rocket in space, so we can ignore the force due to gravity. Since we are in space, we don't have air resistance, so instead let's assume that we've put some Jello out in space, and the rocket plans to fly through the Jello (something to slow it down). Assume that the force due to the Jello's resistance is proportional to the velocity of the rocket, with proportionality constant 8 kg/s. When we light the engine, for the first 2 seconds the force ramps up, following a linear path until it gives a force of 10 N after 2 seconds. Then for 15 more seconds the rocket maintains a force of 10 N. The force then starts to ramp down linearly, taking an additional 5 seconds until the force drops to zero. A picture of this force is below.

Ask me in class to show you how to modify this to add in gravity, or come by and show me what you would do. It's a pretty fun switch.)



1. Set up an initial value problem whose solutions is the position of the rocket after any time t. The right hand side should be written in terms of Heaviside functions, where the discontinuities occur after 2, 17, and 22 seconds.
2. Use Laplace transforms and technology to solve your IVP. Let the computer do all the solving. The goal is to just GET a solution, and then we'll interpret it.
3. The rocket will eventually get stuck in the Jello. How far will it travel before the rocket stops moving?

7.2.2 The Dirac-Delta distribution $\delta(t-a)$ and impulses.

We've been solving mass-spring problems with magnets. What if we instead hit the system with a hammer? We could place a magnet underneath a magnet brick, and pull the brick down. We'd have to leave the magnet on for a while to get the brick to come down. Alternately, we could hit the brick with a hammer. The blow occurs almost instantly, and yet could result in the exact same downward pull as the magnet. The next problem has you develop the connection between magnets and hammer blows. You should see that a hammer blow is like having an infinitely strong magnet on for no time.

Problem 7.16 Consider a mass-spring system with $m = 1$ kg, $c = 3$ kg/s, and $k = 2$ kg/s^2. Initially the system is at rest (so $y(0) = 0$ and $y'(0) = 0$). We turn on a magnet underneath the brick. We'll vary the strength of the magnet, and the time we leave the magnet on, and then solve the IVP $y'' + 3y' + 2y = r(t)$, $y(0) = 0$, $y'(0) = 0$.

1. Solve the IVP if the magnet pulls down with a force of 10 N for 1 second so that $r(t) = -10 + 10u(t-1)$).
2. Solve the IVP if the magnet pulls down with a force of 20 N for 1/2 seconds so that $r(t) = -20 + 20u(t-1/2)$).
3. Solve the IVP if the magnet pulls down with a force of 40 N for 1/4 seconds so that $r(t) = -40 + 40u(t-1/4)$).
4. Use a computer to graph $y(t)$ for each problem above for $0 \leq t \leq 10$. On each graph, estimate how far down the spring moves before bouncing back up and coming to rest.

You can solve all of the IVPs in this problem all at once, if you are willing to work with symbols. You just have to solve

$$y'' + 3y' + 2y = -\frac{10}{h} + \frac{10}{h}u(t-h).$$

because all the forces are constant, the forward transforms are quite quick. Let $s = 0, 1, 2$ to get the coefficients of any partial fraction decompositions.

[Hint: If you use the Mathematica introduction, then you can quickly check all your answers, and see the graphs instantly. Just open the Laplace transform section, expand the "Springs" section, and then type in your IVP. You'll see all the steps involved in solving this by hand, as well as the plot of $y(t)$. The by-hand solution is not bad at all, but graphing by hand could be.]

Problem 7.17 Consider the function $f_h(t) = \begin{cases} \frac{1}{h} & 0 \leq t < h \\ 0 & \text{otherwise} \end{cases}$. This function represents a pulse of strength $1/h$ for h seconds. We would like to examine what happens to this function as $h \to 0$ (so that we have a really strong pulse held for almost no time).

1. For $h = 1$, draw the function $f_1(t)$ for $0 \leq t \leq 2$ and compute $\int_0^\infty f_1(t)dt$.
2. Repeat the previous part if $h = 1/2$, $h = 1/4$, and $h = 1/10$.
3. If $u(t)$ is the Heaviside function, then compute both $\dfrac{u(h) - 0}{h}$ and $u(h)$ at $h = 1, 1/2, 1/4, 1/10$.
4. What patterns do you see? If someone asked you to compute $\lim_{h\to 0} f_h(t)$ and $\lim_{h\to 0} \int_0^\infty f_h(t)dt$, what would you give as answers?

Because of our work above, let's make a definition

Definition 7.2: Dirac Delta Distribution (or function). We define the Dirac delta distribution to be the "function"

$$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}.$$

It's not really a function, because the output is ∞ at a single point. We'll require that the Dirac Delta distribution satisfies the integral sifting properties

$$\int_0^\infty \delta(t-a)dt = 1 \qquad \text{and} \qquad \int_0^\infty g(t)\delta(t-a)dt = g(a).$$

When you time a function by the dirac delta and integrate, you eliminate everything except the function at that single point.

Problem 7.18: $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ Prove that the Laplace transform of the Dirac delta distribution is $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ (look at the definition above).

Then consider the IVP $y' = \delta(t-5), y(0) = 0$. Use $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ to solve this IVP and find the function whose derivative is $\delta(t-5)$. Use your solution to give a function whose derivative is $5\delta(t-3)$.

From the previous problem, you should have observed that the derivative of the Heaviside function is the Dirac delta function.

Theorem 7.3. *If $u(t-a)$ is the Heaviside function, and $\delta(t-a)$ is the Dirac delta function, then*

$$\frac{d}{dt}u(t-a) = \delta(t-a).$$

The derivative of a Heaviside is a Dirac delta.

We now have all the tools we need to solve some pretty cool electricity and mass-spring problems. We couldn't tackle any capacitors in our problem before now, because we need have to compute $E'(t)$. Now that we have derivatives of Heavisides, we can compute Laplace transforms.

Problem 7.19 Consider an RC circuit with $R = 2$ and $C = 1/8$. Suppose that capacitor has no charge on it, and there is no current flowing at $t = 0$. At $t = 2$, we connect a 12 V battery to the circuit. Then at $t = 7$ we remove the battery. Set up an IVP that would give the current at any time t, and then solve the IVP. Use software to construct a graph of your solution. [Hint: There are no partial fraction decompositions in this one, so it should be REALLY fast.]

Problem 7.20 Consider a mass spring system with no friction. Let $m = 1$ and $k = 9$. We'll examine what happens if a hammer hits the system. Suppose the spring is initially at rest at equilibrium. After 3 seconds, a hammer hits the spring downwards with a force of 10 N (so $r(t) = -10\delta(t-3)$). The mass-spring system starts to oscillate. Set up and solve an IVP what would give the position of the spring at any time t, and state the amplitude of the oscillation. [Again, there are no partial fraction decompositions on this problem, so it should go quite fast.]

Problem 7.21 Consider an RLC circuit, with $R = 6$, $L = 1$, and $C = 1/10$. Suppose that at $t = 0$ there is no charge on the capacitor, and no current in the wire. We attach a variable voltage source to the RLC circuit, $E(t) = 2t$, and ramp up the power for the first 6 seconds. After 6 seconds, the voltage drops to zero. This gives us $E(t) = 2t(1 - u(t-6))$. Set up and solve an IVP that would give the current in the wire after t seconds.

Problem If you have a beam, and you know there is a distributed load on the beam, with a point force applied at another spot on the beam, can you compute the shear stress and the moment at any point in the beam? This is exactly the same question as finding the Velocity and position of an object, provided you know the acceleration on the rocket (some external driving force turned on/off based on time) and the rocket is hit by hammers, or meteors (dirac delta) at various points along its path.

Come see me if you'd like to see this application. We won't have time to discuss this (we lost a day last Friday because so many of you were busy with other exams).

If you have had strength of materials, then this problem connects how you use use Heaviside and Dirac delta to tackle all the problems in strengths. You can also use this to solve problems in Statics. You can use it to solve any problem where you have a distributed load (use Heavisides to turn it on and off), and a point force (Dirac delta).

7.3 Convolutions

We've already seen that the Laplace transform of the product $f \cdot g$ of two functions is not the product of the Laplace transform of each ($L(fg) \neq L(f)L(g)$). Is there some kind of product rule for transforms? This question lead mathematicians to invent what we now call a convolution. They discovered that the Laplace inverse of $H(s) = L(f)L(g)$ is equal to the quantity $h(t) = f * g(t) = \int_0^t f(p)g(t-p)dp$. We call this the convolution of f and g. The last problem of this chapter has you prove why. The variable p is a dummy variable of integration, and we could call it anything else (some books use τ, but I find it really hard to distinguish between t and τ when I'm writing on paper, so I use p instead).

Definition 7.4: Convolutions. If $f(t)$ and $g(t)$ are function, then we define the convolution of f and g to be

$$(f * g)(t) = \int_0^t f(p)g(t-p)dp.$$

Theorem 7.5 (The Convolution Theorem). *If $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$, then the inverse Laplace transform of $F(s)G(s)$ is the convolution*

$$\mathscr{L}\{F(s)G(s)\} = (f * g)(t).$$

This is as close as we get to an inverse product rule for Laplace transforms.

We need to practice the convolution theorem. It's just an integral.

Problem 7.22 Do the following:

1. Show that $1*1=t$. You'll need to compute $\int_0^t f(p)g(t-p)dp = \int_0^t 1\cdot 1 dp$.
2. Compute $1*t$ and $t*1$. You'll need to compute $\int_0^t f(p)g(t-p)dp = \int_0^t 1\cdot(t-p)dp$ and $\int_0^t f(p)g(t-p)dp = \int_0^t p\cdot 1dp$.
3. Compute $t*t$. Compare this to the inverse transform of $\frac{1}{s^2}\frac{1}{s^2}$.
4. Compute $\sin(t)*t$.

Problem 7.23 Compute $t*e^{-3t}$ and $e^{-3t}*t$. Which is easier? What is the Laplace inverse of $\dfrac{1}{s^2}\dfrac{1}{s+3}$?

Problem 7.24 Complete each of the following:

1. Compute the convolution $\sin(2t)*1$. Then solve the IVP $y''+4y=1$, $y(0)=0$, $y'(0)=0$.
2. Compute the convolution $\sin(3t)*t$. Then solve the IVP $y''+9y=t$, $y(0)=0$, $y'(0)=0$.

Problem 7.25 We've been using the modification rule when we have double complex roots. Use convolutions to find the Laplace inverse of

$$\frac{1}{(s^2+9)^2}=\frac{1}{s^2+9}\frac{1}{s^2+9}.$$

Problem 7.26 Consider the IVP $y''+5y'+6y=0$, $y(0)=0$, and $y'(0)=1$. Solve this IVP in three ways.

1. Laplace both sides, and then use a convolution (no partial fraction decomposition) to obtain a solution.
2. Laplace both sides, but use a partial fraction decomposition.
3. Obtain the homogeneous solution using the characteristic equation, and then use the initial conditions to obtain the constants.

We'll compare your three solutions in class, and discuss why someone would care about the convolution approach (if you don't see why it's so cool as you use it).

If you'd like to know WHY the convolution theorem works, please complete the next problem. It requires that you can swap the order of integration on a double integral.

Problem 7.27 Prove the convolution theorem (Theorem 7.5). Here are some hints.

- Let $F(s) = \int_0^\infty f(t)e^{-st}dt$ and then use $G(s) = \int_0^\infty g(w)e^{-sw}dw$. (Why can I use w instead of t?)
- Explain why
$$F(s)G(s) = \int_0^\infty \int_0^\infty f(t)g(w)e^{-s(t+w)}\,dw\,dt.$$
- Do a p substitution $p = t + w$ on the inside integral. You should have something like
$$F(s)G(s) = \int_0^\infty \int_t^\infty ?e^{-s(p)}\,dp\,dt.$$
- Swap the order of integration so that t is inside and p is outside. This will require that you draw the region of integration in the tp plane.
- Show that
$$F(s)G(s) = \int_0^\infty \left[\int_0^p f(t)g(t-p)\,dt\right] e^{-sp}\,dp.$$
Why does this complete the theorem?

7.4 Extra Problems

Make sure you try some of each type of problem from chapters 21-24 (except for the last set of problems in 23). The new ideas involve convolutions and the Heaviside (unit step) function in 23. Once you have tried some of each of these, use this page to give you more practice.

I Find the Laplace transform of each of the following, and use Mathematica to check your answer.

1. $f(x) = 8e^{-3x}\cos 2x - 4e^{4x}\sin 5x + 3e^{7x}x^5$
2. $f(x) = xu(x-4) + \delta(x-6)$
3. $f(x) = e^{3x}u(x-2) + 7\delta(x-4)$

II Find the inverse Laplace transform of each of the following, and use Mathematica to check your answer. Many of these will require you to use a partial fraction decomposition.

4. $\dfrac{s}{(s+3)^2+25} + \dfrac{2}{(s-2)^4}e^{-3s}$
5. $\dfrac{s}{s^2+4s+13}e^{-4s}$
6. $\dfrac{1}{s(s^2+1)}e^{-5s}$
7. $\dfrac{1}{s^2(s^2+1)}e^{-3s}$

8. $\dfrac{2s+1}{(s-1)^2(s+1)}e^{-7s}$

9. $\dfrac{1}{(s-1)(s+2)(s-3)}e^{-4s}$

III Use Laplace transforms to find the position $y(t)$ of an object or current current $I(t)$ in each of the following scenarios. I will give you the constants m, c, k and the driving force $r(t)$, or I will give you the inductance L, resistance R, capacitance C, and voltage source $E(t)$, as well as any relevant initial conditions. Your job is to use Laplace transforms to find the solution. Use Mathematica to check your solution, and draw the graph of $y(t)$ or $I(t)$ and the steady-state (steady periodic) solution to see how the Heaviside and Dirac delta functions affect the graph. The point here is to see these two new functions affect solutions. I suggest that you do all of these problems with the computer, so you can quickly see the effects of a Heaviside function or Dirac delta distribution.

10. $m = 1, c = 0, k = 4, r(t) = u(t-1), y(0) = 1, y'(0) = 0$

11. $m = 1, c = 0, k = 4, r(t) = \delta(t-3), y(0) = 1, y'(0) = 0$

12. $m = 1, c = 0, k = 4, r(t) = 7u(t-3), y(0) = 1, y'(0) = 0$

13. $m = 1, c = 0, k = 4, r(t) = 7u(t-3) + 11\delta(t-5), y(0) = 1, y'(0) = 0$

14. $m = 1, c = 0, k = 4, r(t) = 7tu(t-3), y(0) = 1, y'(0) = 0$

15. $m = 1, c = 0, k = 4, r(t) = 7, y(0) = 1, y'(0) = 0$

16. $m = 1, c = 0, k = 4, r(t) = 7, y(\pi) = 1, y'(\pi) = 0$

17. $m = 1, c = 3, k = 2, r(t) = u(t-2), y(0) = 0, y'(0) = 0$

18. $m = 1, c = 3, k = 2, r(t) = \delta(t-2), y(0) = 0, y'(0) = 0$

19. $m = 1, c = 3, k = 2, r(t) = 4u(t-1), y(0) = 0, y'(0) = 0$

20. $m = 1, c = 3, k = 2, r(t) = 4u(t-1) + 10\delta(t-2), y(0) = 0, y'(0) = 0$

21. $m = 1, c = 3, k = 2, r(t) = 4tu(t-1), y(0) = 0, y'(0) = 0$

22. $L = 0, R = 2, C = 1/5, E(t) = 12u(t-2), Q(0) = 0$ (use first order ODE)

23. $L = 1, R = 2, C = 0, E(t) = 12u(t-2), I(0) = 0$ (use first order ODE)

24. $L = 1, R = 2, C = 1/5, E(t) = 12, Q(0) = 0, I(0) = 0$ (first find $I'(0)$.)

25. $L = 1, R = 2, C = 1/5, E(t) = 12u(t-2), Q(0) = 0, I(0) = 0$

26. $L = 1, R = 2, C = 1/5, E(t) = e^{3t}u(t-2), Q(0) = 0, I(0) = 0$

27. $L = 1, R = 2, C = 1/5, E(t) = 4\cos(3t), Q(0) = 0, I(0) = 0$

28. $L = 1, R = 4, C = 1/4, E(t) = u(t-3), Q(0) = 0, I(0) = 0$

29. $L = 1, R = 4, C = 1/4, E(t) = e^{-2t}, Q(0) = 0, I(0) = 0$

7.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 8

Power Series

This chapter covers the following ideas. When you create your lesson plan, it should contain examples which illustrate these key ideas. Before you take the quiz on this unit, meet with another student out of class and teach each other from the examples on your lesson plan.

1. Compute MacLaurin series for various common functions, either directly by taking derivatives, or by solving ODEs.
2. Use the power series method to solve ODEs where $x = 0$ is an ordinary point.
3. Explain how to use the ratio test to find the radius of convergence of a power series.
4. Explain the Frobenius method and use it to solve ODEs where $x = 0$ is a regular singular point.
5. Be able to classify $x = 0$ as an ordinary, singular, and/or regular singular point of an ODE.
6. Define the Gamma function and show how it generalizes the factorial. Be able to compute the Gamma function at any multiple of $\frac{1}{2}$.

You'll find extra practice problems at the end of this chapter. You can use these to gain practice with the ideas. Handwritten solutions are available online. Click for solutions.

8.1 MacLaurin Series

8.1.1 MacLaurin Series and ODEs

As we proceed in this unit, we'll be looking for patterns. When you are looking for patterns, one key rule is to avoid simplifying. Instead of writing $2 \cdot 3 = 6$, just leave it as $2 \cdot 3$. If notice a pattern, like $2 \cdot 3 \cdot 4 \cdot 5$, then write $5!$ instead of 120. If you will resist the urge to simplify, you'll find a lot of patterns immediately pop out.

Problem 8.1 Consider the function $f(x) = e^x$. In this problem we would like to approximate $f(x)$ using various polynomials. We'd like to make sure that the function and its derivatives match the polynomial and its derivatives.

1. Let's approximate $f(x)$ using a 4th degree polynomial. Write the polynomial as
$$P_4(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4,$$
where the coefficients $a_0, a_1, \ldots, a_4$ are unknown (we'll discover them in a bit). Compute the first 4 derivatives of $P_4(x)$ and the first 4 derivatives of $f(x)$. As there are 5 unknowns, we need 5 equations. Let's require that f and P_4, together with their derivatives, match at $x = 0$. This gives us the 5 equations
$$\begin{aligned} f(0) &= P_4(0), \\ f'(0) &= P_4'(0), \\ f''(0) &= P_4''(0), \\ f'''(0) &= P_4'''(0), \text{ and} \\ f''''(0) &= P_4''''(0). \end{aligned}$$
Use these equations to solve for the unknown constants.

2. If you wanted a 7th degree polynomial, what should the coefficients a_5, a_6, and a_7 equal?

In this chapter, our goal is to solve ODEs where the coefficients are no longer constant. We'll learn how to solve a mass-spring problem where the mass is changing, the spring constant errodes over time, or the friction coefficient increases as we tighten a dashpot. We'll also gain the key ideas need to deal with rocket problems where the mass decreases because fuel burns up. To solve these problems, we're going to start approximating functions with polynomials. We'll be using really large polynomials. We'll then solving the problems using these polynomials. The only catch is that we'll start using polynomials that are arbitrarily large. These polynomials are called Taylor polynomials. When we consider an infinitely long polynomial, we call it a Taylor series, or MacLaurin series. We'll get a formal definition in a bit.

Problem 8.2 We already know the solution to the ODE $y' - y = 0$ (see part 1). Let's find this solution using a series approach. Suppose we write
$$y = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots,$$
where the polynomial continues on for as long as we want (why not forever). We'll use this polynomial to find a solution.

1. Solve the ODE $y' - y = 0$ by any method you would like. The characteristic equation might make this really fast.

2. Now consider the series (infinitely long polynomial) above. Compute y' by computing the derivative (so $y' = 0 + a_1 + 2a_2x + \cdots$). Write out the first 7 terms or so.

3. Now subtract y from y'. You can combine the two infinite sums by adding coefficients that are multiplied by collecting the coefficients on the same powers of x. You'll get an infinitely long sum of the form
$$(a_1 - a_0) + (2a_2 - a_1)x + (?)x^2 + (?)x^3 + \cdots.$$
Carry this out 7 terms. What pattern do you see?

4. Because $y' - y = 0$, and $0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$, you should now have an infinitely large system of equations by equating coefficients. The first two equations are $a_1 - a_0 = 0$ and $2a_2 - a_1 = 0$. If you let $a_0 = c$, then solve for a_1, a_2, a_3, and so on in terms of c. What is a_n in terms of c?

The last two problems dealt with the function e^x. Let's now turn our attention to $\cos x$ and $\sin x$.

Problem 8.3 Let $f(x) = \cos x$.

1. Find a 6th degree polynomial to approximate cosine. So let

$$P(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6.$$

   Now require that f and P have the same values at $x = 0$, and that the first 6 derivatives of both f and P have the same values at $x = 0$. You might want to organize your work in table (keep track of f, its first 6 derivatives, and their values at $x = 0$, as well as P, its first 6 derivatives, and their values at $x = 0$). What pattern do you see?

2. Guess what the 20th degree polynomial would be.

3. If $x = 2$, use a calculator to compute $\cos(2)$ as well as $P(2)$ for your 6th degree polynomial. We'll compute $P(2)$ for your 20th degree polynomial in class.

Problem 8.4 We know the solution to the IVP $y'' + y = 0$, $y(0) = c$, $y'(0) = d$ is $y(t) = c\cos(t) + d\sin(t)$. Suppose that

$$y = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots.$$

1. Compute both y' and y'' by taking the derivative, term-by-term, of the infinitely long series. REMEMBER, DO NOT SIMPLIFY. So you should have something like

$$y'' = 2 \cdot 1 a_2 + 3 \cdot 2 a_3 x + \cdots.$$

   Continue this out 7 terms.

2. We want to solve $y'' + y = 0$, so add together y'' and y. Group together terms that are multiplied by the same power of x, so your answer will look something like

$$y'' + y = (2 \cdot 1 a_2 + a_0) + (3 \cdot 2 a_3 + a_1)x + (?)x^2 + (?)x^3 \cdots.$$

   Carry this out 7 terms.

3. If $y(0) = c$, then what is a_0? If $y'(0) = d$, then what is a_1?

4. Write a_2, a_3, a_4, and so on, in terms of c and d. Can you guess the Taylor series for $\sin x$?

Problem 8.5: MacLaurin Series Suppose we write $f(x)$ as the series

$$f(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots.$$

1. Compute the first 4 derivatives of f and evaluate them at 0. What pattern do you see? State the nth derivative of f evaluated at $x = 0$, which we write as $f^{(n)}(0)$.

2. Solve for the coefficient a_n in terms of the nth derivative of f.

3. Let $f(x) = \sin x$. Compute the first 8 derivatives of $\sin x$ and evaluate each at $x = 0$. Then use the pattern you see to state what a_n equals for each n if we write

$$\sin x = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots .$$

Carry out your sum until you hit x^9. If you continue forever, we call this infinite polynomial the MacLaurin series of $\sin(x)$.

Based on your results to the previous problem, we make the following definitions.

Definition 8.1: MacLaurin Series. Let $f(x)$ be a function. We define the MacLaurin series of $f(x)$ to be the infinite series

$$a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = \dfrac{f^{(n)}(0)}{n!}$. We use the notation $f^{(n)}(x)$ to denote the nth derivative. Note that $0! = 1$, and that $f^{(0)}(x)$ is the 0th derivative (so original function). With this notation, we could write the MacLaurin series as

$$f(0)+f'(0)x^1+\frac{f''(0)}{2!}x^2+\frac{f''''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Definition 8.2: Power Series. A power series is an expression of the form

$$a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where a_n is any real number. It's a power series because we create an infinite series using powers of x.

A MacLaurin series is a power series. We'll often start with a power series, and then look for the function $f(x)$ whose MacLaurin series is the power series we started with.

The MacLaurin series of a function depends on the value of the function and its derivatives at $x = 0$. Sometimes, you would rather compute the function and its derivatives at another spot. We won't have much use for doing this in our course, but for completeness, you should see the full definition of a Taylor series centered at $x = c$.

Definition 8.3: Taylor Series centered at $x = c$. Let $f(x)$ be a function. We define the Taylor series of $f(x)$ centered at $x = c$ to be the infinite series

$$a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + a_(x-c)x^4 + \cdots = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where $a_n = \dfrac{f^{(n)}(c)}{n!}$. The MacLaurin series is the Taylor series centered at $x = 0$.

Let's compute a few more MacLaurin series.

Problem 8.6: MacLaurin series for $\cosh x$ and $\sinh x$ Obtain the first 10 terms of the MacLaurin series for both $\cosh x$ and $\sinh x$. Do so by using the formula $a_n = \dfrac{f^{(n)}(0)}{n!}$. Write out the two series. What patterns do you see. Write down a formula for the coefficient a_n for both $\cosh x$ and $\sinh x$.

The next problem shows you how to obtain the MacLaurin series for $\cosh x$ and $\sinh x$ in a different way.

Problem 8.7 Consider the IVP given by $y'' - y = 0$, with $y(0) = A$ and $y'(0) = B$.

1. Show, using Laplace transforms, that the solution to this IVP is $y(x) = A\cosh x + B\sinh x$.

2. We'll now obtain the solution using power series. Suppose
$$y = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$
Compute y' and y''. Substitute y and y'' into the ODE $y'' - y = 0$, and group together terms that are multiplied by the same power of x. You should have something of the form
$$(2a_2 - a_0) + (3 \cdot 2a_3 - a_1)x + (?)x^2 + \cdots = 0 + 0x + 0x^2.$$

3. Use the initial conditions to explain why $a_0 = A$ and $a_1 = B$. Then solve for a_2, a_3, and so on, in terms of A and B. Keep going until you see a pattern for a_n.

4. You now have the solution y. Some of the coefficients depend on A. Some depend on B. Group together the terms that involve A and the terms that involve B, and write your solution in the form
$$y = A(1 + \frac{1}{2!}x^2 + \cdots) + B(x + \frac{1}{3!}x^3 + \cdots).$$
Please carry out each series at least 5 terms.

We have so far developed the following MacLaurin series:

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \\
\cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots \\
\cosh(x) &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \\
\sinh(x) &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots.
\end{aligned}$$

Problem 8.8: Euler's Formula Use the MacLaurin series above to show that we can express e^{ix} in the form
$$e^{ix} = \cos x + i \sin x$$
and that
$$\cosh(ix) = \cos x.$$
Then use the first fact to compute $e^{i\pi}$. What does $\sinh(ix)$ equal, if written in terms of $\sin x$?

The equation
$$e^{ix} = \cos x + i \sin x$$
is called Euler's formula.

We'll return to Euler's formula in a minute. Before we do so, let's examine a different function.

Problem 8.9 Consider the IVP $(x+1)y' = 1$, $y(0) = 0$. Is it linear? Is it homogeneous? Does it have constant coefficients? Solve the ODE first by using separation of variables. Then solve the ODE using a power series (assume $y = a_0 + a_1x + a_2x^2 + \cdots$, compute y', plug these into the ODE, and then solve for the unknown constants a_0, a_1, a_2, etc.). From your answer, what are the first 5 terms in the the MacLaurin series of $\ln(x+1)$?

Problem 8.10 Find a 9th degree polynomial to approximate $\ln(x+1)$. Do this by computing the first 9 derivatives of $\ln(x+1)$ or by computing the first 4 and noticing a pattern when you plug in 0. The formula $a_n = \frac{f^{(n)}(0)}{n!}$ will give you to coefficients. Then use your polynomial to estimate $\ln 1.2$, $\ln(1-.8) = \ln(.2)$ and $\ln 2.5$. Use a computer to draw $\ln(x+1)$ and your 9th degree polynomial. For which values of x do you think the polynomial will do a poor job approximating $\ln(x+1)$. Why do you think this? Will increasing the degree of your approximation ever help you approximate $\ln 2.5$?

We've now turned multiple functions into power series. Polynomials are extremely easy to differentiate and integrate. What happens if we differentiate or integrate a power series. Do we get the power series of the derivative of the function?

Problem 8.11 For each function below, compute the derivative of the function, and the derivative of the power series. Write your solution in summation notation. Then answer the question at the end.

1. $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$
2. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$
3. $\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}x^n.$

What function has the MacLaurin series $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$. [Hint: If you modify the derivative on part 3 slightly, you should get this.]

Problem 8.12 In the last problem you showed that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$. Let's obtain the MacLaurin series for $\arctan x$.

1. Compute the first 3 derivatives of $f(x) = \arctan x$. Use this to obtain the third degree Taylor polynomial of $\arctan x$, centered at $x = 0$.
2. We know that $\int \frac{1}{1+x^2}dx = \arctan x$. Replace each x with x^2 in $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$, and then integrate to obtain a power series for $\arctan x$. Write your answer with summation notation. Does it match your first answer?

3. We know that $\arctan(1) = \pi/4$. Plug in $x = 1$ to the 15th degree polynomial to obtain an approximation of π. If you have software, put $x = 1$ into the 1000th degree polynomial to obtain an approximation for π. Don't forget to multiply by 4, as $\arctan(1) = \pi/4$.

Problem 8.13 Let's solve the IVP $(x^2+1)y' = 1$, $y(0) = 0$ in two ways.

1. Use the power series method. Let $y = \sum_{n=0}^{\infty} a_n x^n$, compute y', plug these into the ODE, collect coefficients of the same powers of x, and then solve for the unknowns a_n.
2. Use separation of variables.

Compare your solution here with the previous problem.

Problem 8.14 Solve the ODE $y' + 2xy = 0$ by using power series. Your initial condition will just be $y(0) = a_0$. After you have a solution, look at the table of known power series and try to match the solution you got to one of our known power series (you might have to replace x with something). Then use separation of variables to solve the ODE, and check if you are correct.

In the previous problem, the power series solution results in a series that we can match with a series we already recognize. We might have to replace x with x^2, but the power series is still quite manageable. Things won't always be this nice.

Problem 8.15 Solve the ODE $y'' + 2xy' + y = 0$ by using power series. Your initial conditions are $y(0) = a_0$ and $y'(0) = a_1$. When you're done, write your solution as

$$y(x) = a_0(y_1(x)) + a_1(y_2(x))$$

You won't find either y_1 or y_2 on the list of power series we recognize.

where y_1 and y_2 are power series. Just give the first 4 terms of y_1 and y_2, together with a rule that would allow us to compute more terms if needed (so how could I find a_{10} if I knew a_8 and a_9, or better yet, how could I find a_{n+2} if I knew a_n and a_{n+1}).

The next problem suggests a sigma notation way to solve all the power series problems. Some of you will immediately recognize the value of using this approach, and start using it exclusively. Some of you would rather not use this approach. I'm fine either way, though by the end of the chapter you'll see the need for this approach.

Problem 8.16 Complete each of the following:

1. Compute both $\sum_{n=4}^{6}(n-3)^2$ and $\sum_{s=1}^{3} s^2$. Which is easier?
2. Consider the sum $\sum_{n=3}^{10} x^{n-2}$. If we let $s = n-2$, then rewrite the sum as $\sum_{s=?}^{?} x^s$ (find the ?). Check your answer by writing out the first few terms, and the last term, of both series. We call this index shifting.

3. Rewrite the sum $\sum_{n=2}^{\infty} n(n-1)x^{n-2}$ so that x^{n-2} is replaced with x^s (i.e. let $n-2=s$ and then shift the index). Check you are correct by writing out the first few terms of both.

Let's introduce this method with a problem you've already solved using power series.

Problem 8.17 Consider the ODE $y''+4y=0$. We know the solution is $y(x) = A\cos(2x) + \frac{B}{2}\sin(2x)$. However, let's solve this using power series, summation notation, and index shifting. We start by assuming $y = \sum_{n=0}^{\infty} a_n x^n$.

1. Compute both y' and y'' using summation notation. Show that the second derivative is

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=1}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

   Why can we allow the sum to start at 0, 1, or 2? We'll most often have it start at 2.

2. Plugging our sums into the ODE $y''+4y=0$ gives the equation

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = 0.$$

   If all the powers of x in this equation were the same, we could easily collect the coefficients of like powers of x. So let $s=n-2$ for the first sum, and let $s=n$ for the second sum. Rewrite your summation formula now in terms of s, giving

$$\sum_{s=?}^{\infty} ?a_? x^s + \sum_{s=0}^{\infty} 4a_s x^s = 0 + 0x + 0x^2 + \cdots.$$

3. When $s=0$, we obtain the coefficients in front of x^0 in both sums. Use this to show $a_2 = -\frac{4}{2}$. Let $s=1$ to show $a_3 = -\frac{4}{3\cdot 2}$. [Remember that right hand side is zero.]

4. For any $s \geq 0$, show that $a_{s+2} = -\frac{4}{(s+2)(s+1)}a_s$. This is called a recurrence relation. Then let $s = 2,3,4,5,6,\ldots$ to rapidly find $a_4, a_5, a_6, \ldots$.

5. Write your solution for y in the form $y = a_0(y_1) + a_1(y_2)$, where y_1 and y_2 are power series. Show the first 4 terms of each series.

Let's now solve a problem that we've never tackled before.

Problem 8.18 Consider the ODE $y'' + 2xy = 0$. Solve this ODE using a power series. We start by assuming $y = \sum_{n=0}^{\infty} a_n x^n$.

1. Compute y' and y'' as you did in the previous problem. Plug them into your ODE and obtain an equation with sigma notation. Your powers of x will not match, so we index shift.

2. Your left sum should have an $n-2$ as a power of x. Your right sum should have an $n+1$. Let's shift $n-2$ to become $s+1$. So in the left sum, let $n-2 = s+1$, and then write your equation in the form

$$\sum_{s=?}^{\infty} ?a_? x^{s+1} + \sum_{n=0}^{\infty} 2a_s x^{s+1} = 0 + 0x + 0x^2 + \cdots .$$

3. When $s = -1$, the left sum should contribute a term, but the right sum does not. Use this to find a_2.

4. When $s \geq 0$, both sums contribute a term. Give a formula for a_{s+3} in terms of a_s (called a recurrence relation). Use this formula to compute a_3, a_4, a_5, a_6, a_7, a_8.

5. Write out the first 6 nonzero terms of your series solution.

Did you notice the pattern in the previous two problems? We assume y is a power series. We then differentiate the series and plug in the derivatives to our ODE. We then index shift so that each sum has the same power above x. Often, it's easiest if you index shift so that the lowest ones all match the largest. Once the powers all match, we can start finding coefficients. If one series starts at a different spot than another, we take care of those cases first. Once we start getting a term from each series, we obtain a recurrence relation and use it to get a_n for as many n as we want. Then we can write out as many terms of the solution as requested.

Problem 8.19 Consider the ODE $y'' + 3xy' + 2y = 0$. Solve this ODE using power series methods. Write your answer by give the first 8 nonzero terms of the series, and make sure you state a recurrence relation that will give more coefficients of the series. [Hint: You'll probably need to use $s = n-2$ for one series, and $s = n$ for the others.]

Problem 8.20 Consider the ODE $(1+x^3)y'' + 3x^2y' + 2xy = 0$. Solve this ODE using power series methods. Write your answer by give the first 6 nonzero terms of the series. State a recurrence relation that will give more coefficients of the series. What is a_{62}, the coefficient in front of x^{62}? [Hint: I'd let $s+1 = n-2$ somewhere above.]

8.2 Radius of Convergence

You've now seen the power series technique used to solve many problems. Sometimes the power series does a good job of approximating the function. Sometimes, it does a really bad job. How can we determine the difference between when a power series does a good job, and when it does a bad job? That's the content of this section.

Problem 8.21: Geometric Series Consider the infinite series

This series is called a geometric series. You can obtain the next terms in the series by multiplying by the common ration r.

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n.$$

If we add up the first k terms, we obtain the kth partial sum

$$S_k = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{k-1} = \sum_{n=0}^{k-1} ar^n.$$

Our goal on this problem is to determine for which a and r we can compute the limit as $k \to \infty$ of the partial sums, and obtain a value.

1. Show that $S_k = \frac{a(1-r^k)}{1-r}$. [Hint: Consider the difference $s_k - rs_k$. Just write out each. Lots should cancel.]

2. Compute $\lim_{k\to\infty} S_k$. For which a and r does this limit exist, and for which does it not exist. Explain.

3. We have seen the power series given by

$$1 + x + x^2 + x^3 + x^4 + \cdots .$$

If we want to know what this infinite sum approaches, we could compute the partial sums and then find the limit of the partial sums. The partial sums are $S_1 = 1$, $S_2 = 1 + x$, $S_3 = 1 + x + x^2$, etc. What is the limit as $k \to \infty$ of these partials sum? For which x does this limit not exist? [Hint: What are a and r from part 2. You already did this in part 2.]

We now have a way to determine the sum of an infinite series. We just look at the partial sums, and then compute their limit (provided it exists). This means we can go back to all the power series we created and ask, "Which of these power series actually have sum that matches the function we started with." Let's make some definitions.

Definition 8.4: Converges and Diverges. Consider the infinite series

$$b_0 + b_1 + b_2 + b_3 + b_4 + \cdots = \sum_{n=0}^{\infty} b_n.$$

- The kth partial sum of this series is the sum of the first k terms. So we have $S_1 = b_0$, $S_2 = b_0 + b_1$, $S_3 = b_0 + b_1 + b_2$, etcetera.
- We say the series converges if $\lim_{k\to\infty} S_k$ exists. In this case, we say the series converges to this limit.
- We say the series diverges if $\lim_{k\to\infty} S_k$ does not exist.

This next problem helps you discover the Ratio test, which is one of the most powerful tests for determining if a power series converges or diverges.

Problem 8.22 Consider again the geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^k + \cdots = \sum_{k=0}^{\infty} ar^k.$$

1. If we write this series in the form $\sum_{k=0}^{\infty} b_k$, then what is b_k? Then compute the quotient $\dfrac{b_{k+1}}{b_k}$ and explain why the series converges precisely when $\left|\dfrac{b_{k+1}}{b_k}\right| < 1$.

For the geometric series, the number r is called the common ratio.

2. Consider now the infinite series

$$-1^2\frac{1}{2}+2^2\left(\frac{1}{2}\right)^2-3^2\left(\frac{1}{2}\right)^3+\cdots+n^2\left(\frac{-1}{2}\right)^n+\cdots=\sum_{n=1}^{\infty}n^2\left(\frac{-1}{2}\right)^n.$$

Compute $\left|\frac{b_{n+1}}{b_n}\right|$ for $n=1,2,3$, and then show $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|=\frac{1}{2}$.

3. Do you think this series will converge or diverge? Use the Mathematica code "Sum[n^2*(-1/2)^n, {n, 0, 10}] // N" to examine the 10th partial sum, and then the 20th, and so on. Does the series appear to converge or diverge?

As me in class to show you how to obtain this result exactly, by taking derivatives of known power series.

If you've forgotten how to compute limits, use this to review.

Review Compute each of the following limits.

1. $\lim_{n\to\infty}\frac{3n^2+5n+4}{2n^2+8n+7}$ 2. $\lim_{n\to\infty}\frac{(1/2)^{2(n+1)}}{(1/2)^{2n}}$ 3. $\lim_{n\to\infty}\frac{1/(n+1)!}{1/n!}$

See[1] for answers.

We'll generalize the problem above into a powerful test used to determine when a power series converges or diverges.

Theorem 8.5 (The Ratio Test)**.** *Consider the infinite series*

$$b_0+b_1+b_2+b_3+b_4+\cdots=\sum_{n=0}^{\infty}b_n.$$

Compute the limit $L=\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$, *which represents the limiting ratio of consecutive terms.*

- *If the ratio L is smaller than 1, then the terms in the series eventually start shrinking so quickly that the series converges.*
- *If the ratio L is greater than 1, then the terms in the series eventually start growing so quickly that the series diverges.*
- *If the limit L does not exist, or if the limiting ratio is 1, then the ratio test fails.*

[1]

1. The polynomials have the same degree, so you just have to divide their leading coefficients. This gives the limit as $\frac{3}{2}$.
2. We compute

$$\frac{(1/2)^{2(n+1)}}{(1/2)^{2n}}=\frac{(2)^{2n}}{(2)^{2(n+1)}}=\frac{(2)^{2n}}{(2)^{2n+2}}=\frac{(2)^{2n}}{(2)^{2n}2^2}=\frac{1}{4}.$$

Computing a limit gives 1/4.

3. We write

$$\frac{1/(n+1)!}{1/n!}=\frac{(n)!}{(n+1)!}=\frac{n(n-1)\cdots3\cdot2\cdot1}{(n+1)n(n-1)\cdots3\cdot2\cdot1}=\frac{1}{n+1}.$$

The limit is 0.

The ratio test basically says that as long as the ratio of consecutive terms eventually stays below 1, then the series will converge. So if we have a power series of the form $\sum_{n=0}^{\infty} a_n x^n$, then all we need to do is find for which x we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1.$$

Problem 8.23 For each power series below, use the ratio test to determine for which x the series with converge.

1. $\sum_{n=0}^{\infty} \frac{n2^n}{3^{n+1}}x^n$
2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$
3. $\sum_{n=1}^{\infty} \frac{3^n}{n^2 4^n}x^{2n}$
4. $\sum_{n=0}^{\infty} \frac{n!}{10^n}x^n$

Each answer above should result in an interval, or single point, which we call the interval of convergence. If the interval of convergence is $(-3, 3)$, then we say that the radius of convergence is $R = 3$. If the interval is $(-\infty, \infty)$, then the radius of convergence is $R = \infty$. If you ended up with a single point, then the radius is $R = 0$. The radius of convergence is half the width of the interval of convergence.

Definition 8.6: Interval and Radius of Convergence. Analytic versus Singular. Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ centered at $x = c$.

- The values of x for which the series converges is called the interval of convergence. This interval may be the single point $x = c$, or it will be an interval of real numbers whose center is at $x = c$, or it may be all real numbers.
- The radius of convergence is half the width of the interval of convergence.
- We say that a function is analytic at $x = c$ if it has a power series representation centered at $x = c$ with a nonzero radius of convergence. If a function is not analytic at $x = c$, then we say the function is singular at $x = c$.

Problem 8.24 Find the interval and radius of convergence for each power series below. Is the function analytic or singular at $x = c$.

Make sure you read the definition above. This problem has the exact same instructions as the the previous, rather it just uses the new vocabulary from the definition above.

1. $\sum_{n=0}^{\infty} \frac{n^2+1}{n+2}\left(\frac{x}{3}\right)^{2n}$, $c = 0$
2. $\sum_{n=0}^{\infty} \frac{1}{n}(x-2)^n$, $c = 2$
3. $\sum_{n=0}^{\infty} (2n)!\,(x-1)^n$, $c = 0$
4. $\sum_{n=0}^{\infty} n\,(2x-3)^{4n}$, $c = 3/2$

Problem 8.25 Find the radius of convergence of the MacLaurin series for each of the following functions.

1. $\cos x$ 2. e^{3x} 3. $\arctan(x)$ 4. $\ln(1+2x)$

8.3 Special Functions

We've seen the power series method work for many problems now where the coefficients are not constants. Will this method work for every problem with variable coefficients? No. Why would it fail? Let's considering a few more power series problems, and discover why it would fail. For some, the power series method will work. For others, it will not. By the time we're done with this section, we'll know what to look for. It all has to do with certain coefficients being analytic at $x = 0$.

Problem 8.26: Legendre Polynomials Consider the ODE

$$(1-x^2)y'' - 2xy' + 20y = 0.$$

1. Use the power series method to solve this ODE. Give all the coefficients up to a_6. What is a_{20}?
2. Write your solution in the form $y(x) = a_0y_1(x) + a_1y_2(x)$. If $y(0) = 1$ and $y'(0) = 0$, what is the solution?
3. Now solve the IVP $(1-x^2)y'' - 2xy' + 6y = 0$, $y(0) = 0$, $y'(0) = 1$. Show that the solution is a polynomial.

Legendre's ODE is

$$(1-x^2)y''-2xy'+(n)(n+1)y = 0.$$

This ODE shows up when solving Laplace's equation in spherical coordinates (studying heat, waves, gravity, and/or electric/static potentials). When n is an integer, one of the solutions will terminate in a polynomial of degree n. These polynomials are called Legendre polynomials.

As in the problem above, sometimes the power series method gives you a polynomial, because the series stops. In the next problem, the power series method will fail, but you should find that with a slight modification (multiply the power series by x^λ), you quickly get two solutions that each have only one term. The entire solution should then be a linear combination of these two solutions.

Review Suppose that a 2nd order homogeneous ODE has a solution $y_1(x) = e^{-3x}$. Suppose that another solution is $y_2(x) = e^{-2x}$. State a general solution to this ODE. See [2].

Problem 8.27: Euler-Cauchy Equation Consider the ODE

$$2x^2y'' + 5xy' + y = 0.$$

1. Let's first try the power series, so suppose $y = \sum_{n=0}^{\infty} a_0x^n$. Compute both derivatives and plug them into the ODE. Use this to explain why the only solution that the power series method will get you is $y = 0$.
2. Earlier in the semester we noticed that sometimes to get a solution, we

Any ODE of the form $ax^2y'' + bxy' + cy = 0$, where a, b, c are constants, is called an Euler-Cauchy ODE.

Frobenius suggested that we multiply a power series by x^λ to get a solution. He also gave conditions on the ODE that state when this method is needed, and when it will succeed.

[2] If the ODE is homogeneous, then the solution is a linear combination of two linearly independent solutions, namely

$$y(x) = c_1e^{-3x} + c_2e^{-2x}.$$

The solutions y_1 and y_2 are linearly independent, because the only solution to $c_1e^{-3x} + c_2e^{-2x} = 0$ is $c_1 = c_2 = 0$. This is because it is impossible to write one of the functions as a multiple of the other. We obtain solutions by summing together linearly independent solutions.

had to multiply by a power of x. Let's see if this works with power series as well. Suppose instead that

$$y = x^\lambda \sum_{n=0}^{\infty} a_0 x^n = \sum_{n=0}^{\infty} a_0 x^{n+\lambda}.$$

Compute both derivatives and plug them into the ODE. Make sure you explain why you cannot change your sum so that it starts at $n = 1$, as we did in the power series method. With each derivative, your sums will still start at 0.

3. If $n = 0$, you should get that either $a_0 = 0$, or that a polynomial equals zero. If we set this polynomial equal to zero, we call the corresponding equation the indicial equation. Find the values of λ that solve the indicial equation. You should get two values for λ. Let's call the largest value λ_1, and the smallest value λ_2.

4. If you replace each λ with λ_1, show that $a_n = 0$ for every $n \geq 1$. Then repeat with $\lambda = \lambda_2$ and show that $a_n = 0$ for $n \geq 1$.

5. You should now have two solutions to this ODE. Use the superposition principle to state a solution to the ODE. Make sure you check your work with the link to WolframAlpha, or use Mathematica.

See Problem 5.20 if you forgot the superposition principle. You can check your work with Mathematica, or here's a link to WolframAlpha.

Why did the power series method fail in the previous problem? The answer lies in a quick computation. If we take the ODE $2x^2y'' + 5xy' + y = 0$ and divide by the leading coefficient of y'', we obtain

$$y'' + \frac{5}{2x}y' + \frac{1}{2x^2}y = 0.$$

The coefficients of the ODE, namely $\frac{5}{2x}$ and $\frac{1}{2x^2}$ are now not defined at $x = 0$, hence not analytic at $x = 0$. To guaranteed that the power series method will succeed and give the entire general solution, these coefficients must be analytic at $x = 0$. Let's try one more, and then introduce some vocabulary.

Problem 8.28: Bessel Equation Consider the ODE

$$x^2y'' + xy' + (x^2 - 9)y = 0.$$

1. Rewrite the ODE so that the coefficient in front of y'' is a one. Then state the other coefficients, and show that they are not analytic at $x = 0$. [Hint: See the previous paragraph.]

2. Since the power series method may not give both solutions, let's multiply the series by x^λ (Frobenius's idea) and suppose that

$$y = x^\lambda \sum_{n=0}^{\infty} a_0 x^n = \sum_{n=0}^{\infty} a_0 x^{n+\lambda}.$$

Compute both derivatives and plug them into the ODE. Multiply the coefficients x^2, x, and $x^2 - 9$ into the sums, splitting the $x^2 - 9$ product into two sums. You'll want to index shift one sum, as you'll have an $x^{n+\lambda+2}$ in one spot.

3. When $n = 0$, the equation (with coefficients a_0) gives you the indicial equation. Show that $\lambda = \pm 3$. We'll let $\lambda_1 = 3$ and $\lambda_2 = -3$. Frobenius always chose λ_1 to be the larger of these roots.

4. Let $\lambda = 3$, and then solve for the other coefficients a_1, a_2, a_3, a_4, etc. State the solution, making sure to list the first 4 nonzero terms.

5. If you let $\lambda = -3$, show that all the coefficients are zero.

In the previous problem, we were only able to obtain one solution y_1 to the ODE. Frobenius showed how to obtain another linearly independent solution, and gave an algorithm for obtaining that solution. If the roots of the indicial equation have a difference that is not an integer, then our current method will give the second solution. However with Bessel's equation above, we got the roots to be ± 3, which differ by the integer 6. This is why we did not find a second solution. You are welcome to study this topic more on your own, if and when you need it.

Let's now focus on when we should use the power series method, and when we should use the Frobenius method. Let's introduce some vocabulary, and state some facts without proof.

Definition 8.7: Ordinary, Singular, Regular Singular. Consider the ODE

$$y'' + b(x)y' + c(x)y = 0.$$

Notice that the coefficient in front of y'' is one. Some people call this form of an ODE the standard form.

- As a reminder, we say that a function is analytic at $x = c$ if it has a power series solution centered at $x = c$ with a positive radius of convergence. Polynomials, exponentials, trig function, and rational functions whose denominator is not zero at $x = c$ are all analytic.

- If $b(x)$ and $c(x)$ are both analytic at $x = 0$, then the solution y to the ODE is analytic. We say that $x = 0$ is an ordinary point of the ODE. The power series method will yield a complete solution.

- If either $b(x)$ or $c(x)$ are not analytic at $x = 0$, then we say that $x = 0$ is a singular point of the ODE. The power series method is not guaranteed to work. You can try it, and you might get lucky.

- If $x = 0$ is a singular point, and both $xb(x)$ and $x^2c(x)$ are analytic, then we say $x = 0$ is a regular singular point of the ODE. We can use the Frobenius method to solve ODEs at regular singular points. The big idea is to guess a solution of the form $y = x^\lambda \displaystyle\sum_{n=0}^{\infty} a_n x^n$ and then solve for λ and the remaining coefficients as in the power series method.

- The indicial equation is the first equation resulting from matching coefficients in the Frobenius method. It's roots λ_1 and λ_2 are sometimes called the exponents of the ODE.

We could also define ordinary, singular, and regular singular points at $x = c$ by considering power series representations centered at $x = c$ instead of $x = 0$.

The next problem asks you to use the vocabulary above to determine which method you should use to solve the ODE.

Problem 8.29 For each ODE, write the ODE in standard form. Then determine if the point $x = 0$ is an ordinary point or a singular point. If it is an ordinary point of the ODE, determine if it is a regular singular point. To solve the ODE, should you use the power series method, the Frobenius method, or neither?

1. $x^2y'' + xy' + (x^2 - v^2)y = 0$ where v is a constant.
2. $x^2y'' + ax^3y' + x^2e^{bx}y = 0$ where a and b are constants.
3. $x^2y'' + cy' + x^ny = 0$ where c is a constant, and n is a positive integer.
4. $\frac{d}{dx}\left((1-x^2)y'\right) = -\lambda y$ where λ is a constant. [Hint: Use the product rule to expand the derivative. Then write the ODE in standard form.]

Ask me in class about how this relates to eigenvalues and Legendre's equation.

Let's end this section with one final problem. In this problem, the difference between the roots of the indicial equation will

Problem 8.30 Consider the ODE

$$8x^2y'' + 10xy' + (x-1)y = 0.$$

1. Show that $x = 0$ is a regular singular point of this ODE.
2. State the indicial equation, and obtain the zeros. You should have $\lambda_1 = 1/4$. What is λ_2?
3. When $\lambda = \lambda_1$, obtain the first 3 nonzero terms of the solution, which we'll call y_1.
4. When $\lambda = \lambda_2$, obtain the first 3 nonzero terms of the solution, which we'll call y_2.
5. State the general solution to this ODE.

The Mathematica technology introduction will help you check your work. Just look in the Special Functions section.

There are a lot of special functions that we have not even touched on. You could spend years studying all the special functions that have already been discovered and classified. This section gave you an introduction to the techniques needed to solve these ODEs.

8.3.1 The Gamma Function

We'll end this chapter with one last special function, the Gamma function $\Gamma(x)$. This function generalizes the factorial. We've already learned that the Laplace transform of t^n is $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$. This formula only works if we require n to be an integer. So what about the Laplace transform of something like $\sqrt{t}$? Once we've defined the Gamma function, we'll have the formula

The symbol Γ is the uppercase greek letter Gamma. That's why we capitalize the "G" in the Gamma function.

$$\mathscr{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}.$$

Definition 8.8: The Gamma Function $\Gamma(t)$. We define the Gamma function to be

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$

As x is a dummy variable, we could have also written $\Gamma(t) = \int_0^\infty p^{t-1}e^{-p}dp$ or $\Gamma(x) = \int_0^\infty p^{x-1}e^{-p}dp$.

Problem 8.31 Do the following:

1. Show that $\Gamma(1) = 1$. You are welcome to skip to the last part right now.

2. Show that $\Gamma(2) = 1$ and that $\Gamma(3) = 2$.
3. Compute $\Gamma(4)$ and then make a conjecture for $\Gamma(5)$, $\Gamma(6)$, and $\Gamma(7)$. Use software to check if you are correct.
4. Now show that for any n, we know that $\Gamma(n+1) = n\Gamma(n)$. Now use this rule to repeat parts 2 and 3 above.

The Gamma function is a generalization of the factorial function. In order to evaluate the gamma function at non integers, we would need to compute the integral that defines the Gamma function. This is in general a very nontrivial task. The next problem shows you how to do this. If you've forgotten how to

Problem 8.32: $\Gamma(1/2) = \sqrt{\pi}$ In this problem we'll prove that $\Gamma(1/2) = \sqrt{pi}$. First, notice that by definition we have

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^x dx.$$

1. Let $u = x^{1/2}$. Use this u-substitution to explain why

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx = 2\int_0^\infty e^{-u^2} du = 2I.$$

   If we could compute the integral $I = \int_0^\infty e^{-u^2} du$, we'd be done. There is no way however to compute this integral exactly, unless we employ higher dimensional tools.

2. Explain why we can write

$$I^2 = \left(\int_0^\infty e^{-u^2} du\right)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

3. Convert this integral to a double integral in polar coordinates (what is the Jacobian) and then evaluate the integral. This gives you I^2. Solve for $\Gamma(1/2)$.

Problem 8.33 We know that $\Gamma(1/2) = \sqrt{\pi}$, and we know that $\Gamma(n+1) = n\Gamma(n)$. Use this to compute $\Gamma(3/2)$, $\Gamma(5/2)$, and $\Gamma(11/2)$. Then state the Laplace transform of $t^{9/2}$. [Hint: You may have to repeatedly apply the rule $\Gamma(n+1) = n\Gamma(n)$, as we have $3/2 = 1/2 + 1$, and $5/2 = 3/2 + 1$, and so on.]

8.4 Extra Practice Problems

Extra homework for this unit is right here. Make sure you try a few of each type of problem, ASAP. I suggest that the first night you try one of each type of problem. It's OK if you get stuck and don't know what to do, as long as you decide to learn how to do it and then return to the ones where you got stuck. Eventually do enough of each type to master the ideas. The only section in Schaum's with relevant problems is chapter 27. Handwritten solutions are available online. Click for solutions.

Most engineering textbooks assume you have seen Taylor series and power series before (in math 113), but many of you have not. If you have your old Math 215 book, you can find many relevant problems and explanations in the section on the Ratio Test and Taylor Series.

Here are a few key functions and their Taylor series centered at $x = 0$ (their MacLaurin series).

$f(x)$	MacLaurin Series	Radius	$f(x)$	MacLaurin Series	Radius
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$R = \infty$	$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$R = 1$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$R = \infty$	$\cosh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$R = \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$R = \infty$	$\sinh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$R = \infty$

(I) For each of the following, find a Taylor polynomial of degree n centered at $x = c$ of the function $f(x)$.

1. $e^{4x}, n = 3, c = 0$
2. $\cos(x), n = 4, c = \pi$
3. $\cos(2x), n = 4, c = 0$
4. $\sin(\frac{1}{2}x), n = 5, c = 0$
5. $\frac{1}{x}, n = 3, c = 1$
6. $\ln x, n = 3, c = 1$
7. $\ln(1 - x), n = 4, c = 0$
8. $\ln(1 + x), n = 4, c = 0$

(II) Find the radius of convergence of each power series.

9. $\sum_{n=0}^{\infty} \frac{1}{3^n} x^n$
10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^n$
11. $\sum_{n=0}^{\infty} \frac{n}{2^n} x^{3n}$
12. $\sum_{n=0}^{\infty} \frac{3n+1}{n^2+4} x^n$
13. $\sum_{n=0}^{\infty} \frac{(-4)^n n}{n^2+1} x^{2n}$
14. $\sum_{n=0}^{\infty} \frac{n}{2^n} x^{2n}$
15. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$
16. $\sum_{n=0}^{\infty} \frac{n!}{10^n} x^{2n}$

(III) For each function, find the MacLaurin series and state the radius of convergence.

17. $f(x) = e^x$
18. $f(x) = \cos x$
19. $f(x) = \sin x$
20. $f(x) = \dfrac{1}{1-x}$
21. $f(x) = \dfrac{1}{1+x}$
22. $f(x) = \cosh x$
23. $f(x) = \sinh x$

(IV) Prove the following formulas are true by considering power series. These formulas will allow us to eliminate complex numbers in future sections.

24. $e^{ix} = \cos x + i \sin x$ (called Euler's formula)
25. $\cosh(ix) = \cos x$
26. $\cos(ix) = \cosh x$
27. $\sinh(ix) = i \sin x$
28. $\sin(ix) = i \sinh x$

(V) Use MacLaurin series of known functions to find the MacLaurin series of these functions (by integrating, differentiating, composing, or multiplying together two power series). Then state the radius of convergence.

29. $f(x) = x^2 e^{3x}$
30. $f(x) = \frac{x^2}{e^{3x}}$ [hint, use negative exponents]
31. $f(x) = \cos 4x$
32. $f(x) = x \sin(2x)$
33. $f(x) = \frac{x}{1+x}$

34. $f(x) = \frac{1}{1+x^2}$

35. $f(x) = \arctan x$ [hint, integrate the previous]

36. $f(x) = \arctan(3x)$

(VI) Shift the indices on each sum so that it begins at $n = 0$.

37. $\sum_{n=3}^{6} n + 2$

38. $\sum_{n=2}^{8} n^2$

39. $\sum_{n=4}^{\infty} 2^n$

40. $\sum_{n=2}^{\infty} x^n$

41. $\sum_{n=1}^{\infty} n a_n x^n$

42. $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

(VII) Solve the following ODEs by the power series method. With some, initial conditions are given (meaning you know $y(0) = a_0$ and $y'(0) = a_1$). Identify the function whose MacLaurin series equals the power series you obtain.

43. $y' = 3y$

44. $y' = 2xy$

45. $y'' + 4y = 0$

46. $y'' - 9y = 0, y(0) = 2, y'(0) = 3$

47. $y'' + 4y' + 3y = 0, y(0) = 1, y'(0) = -1$

(VII) Determine whether the given values of x are ordinary points or singular points of the given ODE.

48. Chapter 27, problems 26-34 (these are really quick).

(VIII) Solve the following ODEs by the power series method. State the recurrence relation used to generate the terms of your solution, and write out the first 5 nonzero terms of your solution.

49. Chapter 27, problems 35-47 (or from the worked problems).

8.5 Extra Practice Solutions

Handwritten solutions are available online. Click for solutions.

8.6 Special Functions

Here are some extra practice problems related to the Frobenius method and other special functions. Section numbers correspond to problems from Schaum's Outlines *Differential Equations* by Richard Bronson. The suggested problems are a minimum set of problems to attempt.

Concept	Relevant Problems
Frobenius Method*	28:1-4, 5-10, 12,14,16,18-20
Legendre Polynomials	27:11-13; 29:4,6,8,11,12,15
Bessel Functions	30:9,11,12,26, 27,
Gamma Functions	30:1-8, 24, 25
Substitutions	28:22-23, 34-38; 30:30,31

Click here for some handwritten solutions to many of the problems above.

8.7 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 9

Systems of ODEs

This chapter covers the following ideas. When you create your lesson plan, it should contain examples which illustrate these key ideas. Before you take the quiz on this unit, meet with another student out of class and teach each other from the examples on your lesson plan.

1. Explain the basic theory of systems of linear ODEs and the Wronskian for systems.
2. Convert higher order ODEs to first order linear systems.
3. Explain how to use eigenvalues and eigenvectors to diagonalize matrices. When not possible, use generalized eigenvectors to find Jordan canonical form.
4. Find the matrix exponential of a square matrix, and use it to solve linear homogeneous and nonhomogeneous ODEs.
5. Give applications of systems of ODEs. In particular be able to setup systems of ODE related to dilution, electricity, and springs (use the computer to solve complex systems).

9.1 Bringing it all together

As you work on the problems in this section, you'll want to have a computer algebra system near by. I'll put some links to Sage worksheets in the problem set, but I strongly suggest you download the Mathematica Technology Introduction.

Our goal in this chapter is to learn how to solve systems of differential equations. We have already discussed most of the ideas in this chapter (in some context), but we have never brought all these ideas together. In this chapter, we'll try to connect everything we have done up to now. By the time we end this chapter, we'll have a tool that will solve almost every problem we have encountered. We'll see how vector fields, parametric curves, eigenvalues, eigenvectors, potentials, and power series all combine together to give a beautiful and elegant solution technique to solving ODEs.

Problem 9.1 Consider the IVP $y'' + 3y' + 2y = 0$, $y(0) = 5$, $y'(0) = 0$. This solution to this ODE will give the position of a mass spring system where $m = 1$ kg, $c = 3$ kg/s, $k = 2$ kg/s^2, where the object was lifted upwards 5 cm and then let loose.

1. This is a homogeneous ODE. What is the characteristic equation? State a general solution, and then use the initial conditions to get the solution. (This is review.)

2. Let $y(t)$ be the position and $v(t)$ be the velocity, so $v(t) = y'(t)$. This means $v'(t) + 3v(t) + 2y(t) = 0$. Notice that this is a first order ODE with two different functions y and v that are unknown. Explain how to write this as the matrix equation

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.$$

3. Find the eigenvalues λ_1 and λ_2 of the coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. Then for each eigenvalue, find a corresponding eigenvector, which we'll call $\vec{x}_1$ and $\vec{x}_2$.

4. Look back at your solution on part 1. Compute y' and write your solution from part 1 in the form

$$\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix} e^{-t} + \begin{bmatrix} ? \\ ? \end{bmatrix} e^{-2t}.$$

   What is the connection between the solution to the ODE and eigenvalues/eigenvectors?

5. Suppose you needed to find a general solution to the system of ODEs

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = A \begin{bmatrix} y \\ v \end{bmatrix},$$

   and you knew the eigenvalues λ_1 and λ_2 and a corresponding eigenvector for each. Make a guess for what a general solution to the ODE is.

Problem 9.2 Consider again the IVP $y'' + 3y' + 2y = 0$, $y(0) = 5$, $y'(0) = 0$. We already have the solution above. In this problem, you'll be constructing various graphs.

1. Construct of graph of y verses t and a graph of v versus t. You should have two graphs that show you position and velocity at any time t.

2. Now construct a graph of v versus y. We could call this a velocity-position graph. Please use technology to do this. You just need to graph the parametric curve $\vec{r}(t) = (y(t), y'(t))$. You'll need to use a parametric plotter.

Please check your answer with technology. You can use either Sage or Mathematica. Click on this link to get some example code that will help you with this problem. You should use this code to check your answer with the previous problem.

3. The matrix A represents a vector field $\vec{F}(y, v) = (0y + v, -2y - 3v)$. Construct a graph of this vector field in the yv plane.

4. Put your vector field plot, and your velocity-position plot, on the same set of axes. What does the vector field plot tell you about the velocity-position plot?

5. Change the initial conditions to $y(0) = 0$ and $v(0) = 5$. On top of your vector field plot, draw what you think the solution should look like in the velocity-position plot. Then use software to solve the ODE, and plot your solution.

Just change the initial conditions in either Sage worksheet or Mathematica notebook, and reevaluate.

6. Change the initial conditions to $y(0) = 5$ and $v(0) = -5$. On top of your vector field plot, draw what you think the solution should look like in the velocity-position plot. Explain why the solution must follow a straight line in the velocity-position plane? [Hint: What are the eigenvalues, eigenvectors?] Then state another set of initial conditions where the solution will be a straight line towards the origin.

We'll revisit the last two problems as part of every other solution find. Let's introduce a new type of application that shows us the need for linear systems of ODEs.

Problem 9.3 Imagine for a moment that you have two tanks. The first tank contains 6 lbs of salt in 10 gallons of water. The second tank contains no salt in 20 gallons of water. Each tank has an inlet valve, and an outlet value. We attach hoses to the tanks, and have a pump transfer 2 gallon/minute of solution from tank 1 to tank 2, and vice versa from tank 2 to tank 1. So as time elapses, there are always 10 gallons in tank 1 and 20 gallons in tank 2. Our goal is to find the amount of salt in each tank at any time t.

1. We know there are initially 6 lbs of salt in tank 1, and no salt in tank 2. If we allow the pumps to transfer salt for enough time, explain why the salt content in tank 1 will drop to 2 lb, and the salt content in tank 2 should increase to 4 lbs.

2. Let $y_1(t)$ and $y_2(t)$ be the lbs of salt in tanks 1 and 2, respectively. Explain why
$$y_1' = -\frac{2}{10}y_1 + \frac{2}{20}y_2.$$

3. Obtain a similar equation for y_2'. Write your ODEs in the matrix form
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} -2/10 & 2/20 \\ ? & ? \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

4. Draw the vector field represented by the coefficient matrix. Sketch the solution $(y_1(t), y_2(t))$ to your IVP (start at the point $(6, 0)$ and follow the field until the vectors no longer tell you to move). Show that you should stop at $(2, 4)$.

5. Compute the eigenvalues and eigenvectors of the matrix A, and draw two lines through the origin to represent the eigenvector directions.

Don't forget that you can check your work with technology. Please following this link

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Problem 9.4 Again consider the mixing tank problem from before, with the system of ODEs
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} -2/10 & 2/20 \\ ? & ? \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Our goal is to determine the amount of salt in each tank at any time t.

1. Compute the eigenvalues and eigenvectors of this matrix. Use them to write a general solution to this system of ODEs. Your solution should involve arbitrary constants c_1 and c_2.

2. Use the initial conditions $y_1(0) = 6$ and $y_2(0) = 0$ to solve for c_1 and c_2.

3. Construct a graph that contains the the vector field representing the coefficient matrix and the parametric plot $(y_1(t), y_2(t))$ of your solution.

Don't forget that you can check your work with technology. Please following this link

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Problem 9.5 Two tanks are connected with hoses and pumps so that 3 gallons/second flows back and forth between the tanks. The first tank is a 60 gallon tank, with 2 lbs of salt inside. The second tank is a 90 gallon tank with 23 lbs of salt in it. Please find the amount of salt in each tank at any time t.

1. Write a linear system of ODEs in the form
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$
whose solution will give the amount of salt in each tank at any time t.
2. Compute the eigenvalues and eigenvectors of A, and then write a general solution to this system of ODEs. Your solution should involve arbitrary constants c_1 and c_2.
3. Use the initial conditions to solve for c_1 and c_2.
4. Construct a graph that contains the the vector field representing the coefficient matrix and the parametric plot $\vec{y}(t) = (y_1(t), y_2(t))$ of your solution.

Don't forget that you can check your work with technology. Please following this link

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Problem 9.6 Consider the linear system of ODEs given by $y_1' = 2y_1 + y_2$ and $y_2' = 3y_1 + 4y_2$. Let $\vec{y} = (y_1, y_2)$. We can write this ODE in the form $\vec{y}' = A\vec{y}$, where $\vec{y} = (y_1, y_2)$.

1. Find the eigenvalues and eigenvectors of the coefficient matrix A.
2. We know that we can write a general solution to this system of ODEs as
$$\vec{y} = c_1\vec{x}_1e^{\lambda_1 t} + c_2\vec{x}_2e^{\lambda_2 t}.$$
Find a 2 by 2 matrix Q so that we can write this solution in the form
$$\vec{y} = Q \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = QD\vec{c},$$
where we have the diagonal matrix $D = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ and the vector $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.
3. We now have $\vec{y} = QJ\vec{c}$. When we let $t = 0$, explain why D equals the identity matrix. This means that $\vec{y}(0) = Q\vec{c}$. Using an inverse matrix, we have $\vec{c} = Q^{-1}\vec{y}(0)$. Compute the inverse of $\vec{Q}$.
4. Since we know $\vec{y} = QJ\vec{c}$ and $\vec{c} = Q^{-1}\vec{y}(0)$, this means
$$\vec{y} = QDQ^{-1}\vec{y}(0).$$
You have found Q, D, and Q^{-1}. If we let $y_1(0) = a$ and $y_2(0) = b$ which means $\vec{y}(0) = (a, b)$, then multiply out the matrix product $QDQ^{-1}\vec{y}(0)$ and state the solution to this IVP.

Do you notice that in the problem above, we solved the linear system of ODEs in the form $\vec{y}' = A\vec{y}$ with initial conditions $\vec{y}(0)$ by just writing

$$\vec{y} = QDQ^{-1}\vec{y}(0).$$

The columns of Q were the eigenvectors. The nonzero entries of the diagonal matrix D contain $e^{\lambda t}$ where λ is an eigenvalue. Does this pattern work in other places?

Problem 9.7 Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$. Do so by stating Q, D, and Q^{-1}, and then perform the matrix product QDQ^{-1}. Finally, if we assume $\vec{y}(0) = (a, b)$, then give the solution to this system of IVPs by stating what $y_1(t)$ equals, and what $y_2(t)$ equals (hint, multiply out $QDQ^{-1}\vec{y}(0)$). Please use technology to perform as much of the computations as you want. Just be prepared to tell us how you got each part.

Does the pattern above continue to work if we increase the size of the matrix?

Problem 9.8 Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Do so by stating Q, D, and Q^{-1}, and then perform the matrix product QDQ^{-1}. Finally, if we assume $\vec{y}(0) = (a, b, c)$, then give the solution to this system of IVPs by stating what $y_1(t)$ equals, what $y_2(t)$ equals, and what $y_3(t)$ equals. Please use technology to perform as much of the computations as you want. Just be prepared to tell us how you got each part.

Does the pattern even work if the eigenvalues are complex?

Problem 9.9 Solve the system of ODEs $\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

1. Find the eigenvalues by hand. Then for each eigenvalue, compute an eigenvector by hand. State Q and D from this information. You are welcome to use e^{it} in your work as needed.

2. Compute Q^{-1} by hand. Then compute the product QDQ^{-1} by hand.

3. Use Euler's formula $e^{it} = \cos t + i \sin t$ to simplify the product QDQ^{-1}. You should be able to simplify the product to remove all complex terms.

4. If $y_1(0) = 5$ and $y_2(0) = 7$, then what are $y_1(t)$ and $y_2(t)$. Then use software to check your answer. You should see that the solution in the y_1y_2 plane, along with the appropriate vector field, gives circular motion.

Problem 9.10 Suppose that $\frac{d\vec{y}}{dt} = A\vec{y}$ is a linear system of ODEs. Also suppose that $\vec{y} = \vec{x}e^{ct}$ is a nonzero solution to this system. Explain, using the definition of eigenvalues and eigenvectors, why we must have that c is an eigenvalue, and $\vec{x}$ is an eigenvector corresponding to c. [Hint: Look up the definition of eigenvalues and eigenvectors. If you compute $\frac{d\vec{y}}{dt}$ and then place both $\vec{y}$ and $\frac{d\vec{y}}{dt}$ into the system $\frac{d\vec{y}}{dt} = A\vec{y}$, you should see the definition appear.]

9.2 The Matrix Exponential

In the previous section, we saw that if $\vec{y}' = A\vec{y}$, then the solution is $\vec{y} = QDQ^{-1}\vec{c}$, where the initial conditions give us $\vec{c} = \vec{y}(0)$ because D is the identity matrix when $t = 0$. In the first week of class, we solved the differential equation $y' = ay$, and obtained the solution $y = e^{at}c$ where $c = y(0)$. In this section, we'll show that if we replace the constant a with a matrix of constants A, then the solution is still $\vec{y} = e^{At}\vec{c}$. To do this, we have to go back to power series.

Definition 9.1: The Matrix Exponential. We showed in the power series chapter that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots .$$

We define the matrix exponential of A to be the series

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \frac{1}{5!}A^5 + \cdots .$$

The matrix I is the identity matrix.

$\boxed{\textbf{Problem 9.11}}$ Use the definition above to complete the following:

1. We know that $e^0 = 1$. If $A = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then compute e^A.
2. If $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that $e^A = \begin{bmatrix} e^1 & 0 \\ 0 & e^1 \end{bmatrix}$.
3. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then compute e^A. Make sure you show how you get your answer from the definition.
4. If $At = \begin{bmatrix} 2t & 0 & 0 \\ 0 & 3t & 0 \\ 0 & 0 & 5t \end{bmatrix}$, then compute e^{At}. You are welcome to just state an answer here.

When a matrix is diagonal, it's matrix exponential is simple to compute. Our main goal is to learn how to compute the matrix exponential of all matrices. Let's look at another type of matrix where it's easy to compute the matrix exponential.

$\boxed{\textbf{Problem 9.12: Nilpotent Matrices}}$ Use the definition of the matrix exponential to compute the following.

We say that a matrix A is nilpotent if A^n is the zero matrix for some n. It's easy to compute the matrix exponential of a nilpotent matrix, because the infinite series stops, and then we just have to add up finitely many terms.

1. Let $At = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ and then compute $(At)^2$ and $(At)^3$. Use this to state the matrix exponential of At.
2. Let $At = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$ and then compute $(At)^2$ and $(At)^3$. Use this to state the matrix exponential of At.

3. Let $At = \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give the matrix exponential of At. You are welcome to guess your answer by following any pattern you saw above.

4. Let $At = \begin{bmatrix} 0 & t & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Guess the matrix exponential of At.

You can check your answer with technology. Follow this link.

With real numbers, we have the exponential rule $e^{a+b} = e^a \cdot e^b$. The exponential of a sum is the same as the product of an exponential. Does this rule work with matrices as well? Let's try it and see.

Problem 9.13 Let's write $At = \begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix} = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = Bt + Ct$, where $Bt = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix}$ and $Ct = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$.

1. Use software to compute the matrix exponential of At.

2. State the matrix exponentials of both Bt and Ct (use the patterns developed from the previous problems). We know that $At = Bt + Ct$, so how should we combine e^{Bt} and e^{Ct} to get the matrix exponential of At?

3. Without software, state the matrix exponential of $\begin{bmatrix} 3t & t & 0 & 0 \\ 0 & 3t & t & 0 \\ 0 & 0 & 3t & t \\ 0 & 0 & 0 & 3t \end{bmatrix}$.

Follow this link. Please use this calculator to check your answers on the other parts of this problem, but only after you first do them by hand.

Problem 9.14 Consider the matrix $At = \begin{bmatrix} 2t & t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 & 0 \\ 0 & 0 & 2t & t & 0 \\ 0 & 0 & 0 & 2t & t \\ 0 & 0 & 0 & 0 & 2t \end{bmatrix}$. There is supposed to be a zero instead of a t in the second row. That was done on purpose.

1. Write At as the sum $At = Bt + Ct$, where Bt is a diagonal matrix, and Ct contains nonzero terms above the diagonal.

2. Compute the matrix exponential of both Bt and Ct. Then compute their product to get e^{At}. Check your answer with software.

3. Guess the matrix exponential of $\begin{bmatrix} 3t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3t & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3t & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4t & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4t \end{bmatrix}$

Check your answer with software.

We can now compute the matrix exponential of any matrix that is either diagonal, or has nonzero entries above the diagonal. We'll soon see that this means we can compute the matrix exponential of every matrix. The key is to first find the correct form.

Problem 9.15 Consider the matrix $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$.

1. Find the eigenvalues of A. For each eigenvalue, find a corresponding eigenvector.

2. Let Q be a matrix whose columns are the eigenvectors from the previous part. Compute AQ by hand.

3. If $\vec{x}$ is an eigenvector, then by definition we must have $A\vec{x} = \lambda x$. Use this to write $AQ = QJ$ where J is a diagonal matrix. What is the matrix J? Show that $Q^{-1}AQ = J$, and that $QJQ^{-1} = A$.

4. Now suppose that A is a matrix with eigenvalues 2 and 3 and corresponding eigenvectors $(1, 3)$ and $(-1, 2)$. Use this to state Q, J, and A.

If you're struggling with guessing what J should be, then compute $Q^{-1}AQ$. Then try to explain why J should be what you see from the equation $AQ = QJ$.

Did you notice on this last part that you could construct A solely from knowledge about eigenvalues and eigenvectors.

Problem 9.16 Suppose we know that $A = QJQ^{-1}$ where J is a diagonal matrix. In this problem, we'll compute the matrix exponential of A.

1. Explain why $A^k = QJ^kQ^{-1}$.

2. Use the definition of the matrix exponential to explain why $e^A = Qe^JQ^{-1}$.

3. The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has an eigenvalue 3 with corresponding eigenvector $(1, 1)$, and has an eigenvalue 1 with corresponding eigenvector $(1, -1)$. Compute the matrix exponential of A.

4. Solve the system of ODEs $y_1' = 2y_1 + y_2$ and $y_2' = y_1 + 2y_2$. State $y_1(t)$.

If we know how to compute the matrix exponential of J and $A = QJQ^{-1}$, then the previous problem showed us that $\exp(A) = Q\exp(J)Q^{-1}$. We also saw that multiplication by t doesn't affect this result, so we have

$$\exp(At) = Q\exp(Jt)Q^{-1}.$$

This is the key tool we'll use to solve systems of ODEs.

Problem 9.17 Consider the system of ODEs $\dfrac{d\vec{y}}{dt} = A\vec{y}$ given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

1. For the coefficient matrix above, state Q and J. You should have a repeated eigenvalue, but you should also find two linearly independent eigenvectors corresponding to this eigenvalue.

2. Compute e^{At}, and state a general solution $\vec{y}(t)$.

3. If $y_1(0) = 1, y_2(0) = -2, y_3(0) = 2$, then state $y_1(t)$, $y_2(t)$, and $y_3(t)$.

Problem 9.18 Consider the system of ODEs $y_1' = y_2$ and $y_2' = -9y_1 - 6y_2$.

1. Write this system in the form $\frac{d\vec{y}}{dt} = A\vec{y}$. Find the eigenvalues of A. State all the eigenvectors of A.
2. Why are there not enough eigenvectors to form an invertible matrix Q?
3. The system of ODEs above is equivalent to the ODE $y'' + 6y' + 9y = 0$. Explain why.
4. State a general solution to this ODEs using methods from before.

We've now seen two examples with repeated eigenvalues. Sometimes when we see a repeated eigenvalue, we'll be able to get enough linearly independent eigenvectors to form an invertible matrix Q. In this case, we'll be able to write $A = QJQ^{-1}$ where J is diagonal. If we can't get enough linearly independent eigenvectors, then we'll have to do something else. We'll show that we can always write $A = QJQ^{-1}$, where the only nonzero terms off the diagonal are perhaps a few 1's directly above the diagonal. Luckily, we know how to compute the matrix exponential of this kind of matrix. The matrices Q and J are called a Jordan decomposition for A.

Problem 9.19 Consider again the system of ODEs $y_1' = y_2$ and $y_2' = -9y_1 - 6y_2$.

1. We already know the only eigenvalue of the coefficient matrix is $\lambda = -3$. To find the eigenvectors corresponding to $\lambda = -3$, we solve $(A + 3I)\vec{x} = \vec{0}$. Solve this system, and state the solution.
2. Because the eigenvalue $\lambda = -3$ was repeated, we were hoping to find two linearly independent eigenvalues, but we did not. Instead of solving $(A - \lambda I)\vec{x} = \vec{0}$, let's solve $(A - \lambda I)^2\vec{x} = \vec{0}$.

   Find the solutions to $(A - \lambda I)^2\vec{x} = \vec{0}$. Show by hand how to solve this system, and state the solution as linear combination of two independent vectors.
3. Pick a nonzero vector from the previous part that is not an eigenvector, and call it $\vec{v}_2$. Compute $\vec{v}_1 = (A - \lambda I)\vec{v}_2$, and show that $\vec{v}_1$ is an eigenvector.
4. Let $Q = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$. Compute by hand AQ and show that it equals QJ where $J = \begin{bmatrix}-3 & 1\\ 0 & -3\end{bmatrix}$.
5. Now that we have Q and J, compute e^{At}. The solution to the ODE is $\vec{y} = e^{At}\vec{c}$, which you can now compare with your previous problem.

In three previous problem, we considered the linear system of equations $(A - \lambda I)^2\vec{x} = \vec{0}$. This system produced an extra vector we could use to get an invertible matrix Q. The solutions to this system provide us with what we now call generalized eigenvectors.

Definition 9.2: Generalized Eigenvectors. The nonzero solutions to the linear system $(A - \lambda I)^1\vec{x} = \vec{0}$ are the eigenvectors of the matrix A.

1. If a vector $\vec{x}$ satisfies $(A-\lambda I)^2\vec{x}=\vec{0}$ but does not satisfy $(A-\lambda I)^1\vec{x}=\vec{0}$, then we say that $\vec{x}$ is generalized eigenvector of order 2.

2. If a vector $\vec{x}$ satisfies $(A-\lambda I)^k\vec{x}=\vec{0}$ but does not satisfy $(A-\lambda I)^{k-1}\vec{x}=\vec{0}$, then we say that $\vec{x}$ is generalized eigenvector of order k.

Problem 9.20 Suppose that A has an eigenvalue λ and has a generalized eigenvector of order 3 which we'll call $\vec{v}_3$. Let $v_2=(A-\lambda I)\vec{v}_3$ and $\vec{v}_1=(A-\lambda I)\vec{v}_2$.

1. Explain why $\vec{v}_1$ is an eigenvector of A and $\vec{v}_2$ is a generalized eigenvector of order 2.

2. Let $T=\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}$. Show that

$$AT=T\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}.$$

Remember that $(A-\lambda I)\vec{v}_3=\vec{v}_2$, so we know that $A\vec{v}_3=\lambda\vec{v}_3+\vec{v}_2$. How will this help you show what is asked for?

3. Let $S=\begin{bmatrix}\vec{v}_3 & \vec{v}_2 & \vec{v}_1\end{bmatrix}$. Show that

$$AS=S\begin{bmatrix}\lambda & 0 & 0\\ 1 & \lambda & 0\\ 0 & 1 & \lambda\end{bmatrix}.$$

Definition 9.3: Jordan Canonical Form. Let A be a square matrix. Suppose that $AQ=QJ$, where Q is an invertible matrix and J is a matrix whose only nonzero entries off the diagonal are potentially 1's above the diagonal. If there is a one in J above the diagonal, then the entries below and to the left of the 1 must be the same. Under these conditions, we call J a Jordan canonical form for A.

The 1's appear in Jordan canonical form precisely when

It is a theorem that every matrix A admits a Jordan canonical form. The matrix Q consists of eigenvalues and generalized eigenvalues. Once we have a Jordan canonical form for A, the matrix exponential of At is $e^{At}=Qe^{Jt}Q^{-1}$.

Problem 9.21 Consider the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$. Our goal is to find a Jordan canonical form for A by stating Q and J so that $AQ=QJ$.

1. Show that the eigenvalues of A are $\lambda=1$ and $\lambda=3$. The eigenvalue 3 is repeated. Find an eigenvector corresponding to $\lambda=1$, and call it $\vec{u}$.

2. Solve $(A-3I)\vec{x}=\vec{0}$ to find the eigenvectors corresponding to $\lambda=3$. We're hoping for two independent vectors, but show you only get one.

3. Now solve $(A-3I)^2\vec{x}=\vec{0}$. Pick a solution that is not an eigenvector and call it $\vec{v}_2$. Then compute $\vec{v}_1=(A-3I)\vec{v}_2$.

4. The vectors $\vec{u}$, $\vec{v}_1$, and $\vec{v}_2$ need to be placed in matrix Q. How should you place them? Make an educated guess, and then compute $Q^{-1}AQ$ to verify if it equals J. If it does not, then try a different order on Q. You know you've got the right J when it's almost diagonal with potentially a 1 above and to the right of a repeated eigenvalue.

Have you noticed the pattern for finding Jordan form? Find the eigenvalues. Then find the eigenvectors. If you don't have enough linearly independent eigenvectors, then you look for generalized eigenvectors of order 2. Continue finding generalized eigenvectors as needed until you get enough linearly independent vectors. When you select a higher order generalized eigenvector $\vec{v}$, make sure you compute $(A-\lambda I)^k\vec{v}$ for $k = 1, 2, ...$ and use those as your lower order vectors. This will get you Jordan form. I'll demo some of this in class.

We've shown that you can start with A, and from it determine Q and J. We know $AQ = QJ$, which means if we had Q and J, we could obtain A. The next problem starts with a Q and J. You'll then obtain A. After obtaining A, we'll find another Q and J so that $AQ = QJ$. The Q and J are not unique.

Problem 9.22 Let $J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

Use software to help you complete this problem.

1. Compute $A = QJQ^{-1}$ (make sure you use software). Why are the eigenvalues of A equal to 2 and 3? What's the multiplicity of each eigenvalue? For each eigenvalue, give all possible eigenvectors. (Show the appropriate matrices you would rref, state the rref, and then state the eigenvectors.)

2. The eigenvalue 2 shows up three times, but only contributes two linearly independent eigenvectors. Find a generalized eigenvector, which we'll call $\vec{v}_2$, of degree 2 for this eigenvalue (remember to solve $(A - \lambda I)^2\vec{x} = \vec{0}$). Then compute $\vec{v}_1 = (A - 2I)\vec{v}_2$, which should be an eigenvector. This gives you two vectors corresponding to $\lambda = 2$. For $\vec{v}_3$, pick an eigenvector corresponding to $\lambda = 2$ that is not a multiple of $\vec{v}_1$. Let $\vec{v}_4$ be an eigenvector corresponding to the other eigenvalue.

3. You now have enough information to state Q. It will mostly likely be different than the one you started with. State Q and then compute $Q^{-1}AQ$ (with a computer) to get J. You should have an almost diagonal matrix with a single 1 above the diagonal.

4. State e^{At} (feel free to use a computer to perform any needed matrix multiplications). Check your answer with a computer.

9.3 Solving Non Homogeneous ODEs

We now have the tools needed to solve every constant coefficient linear system of ODEs, whether homogeneous or not. They key to solving these problems is a formula we already developed earlier in the semester. If you have forgotten how to find an integrating factor, you may want to review some problems from chapter 4. Then tackle this problem.

Problem 9.23 Consider the first order ODE $y' - ay = f(t)$. Find an appropriate integrating factor, and then show that a general solution to this ODE is

$$y(t) = e^{at}c + e^{at}\int_0^t e^{-at}f(t)dt,$$

where c is an arbitrary constant. If the ODE is homogeneous with $f(t) = 0$, show that $c = y(0)$.

The solutions above provides a theoretical way to solve every first order linear constant coefficient ODE. If we replace y, c, and f with vectors, and we replace a with a matrix, then the solution to $\frac{d\vec{y}}{dt} = A\vec{y}(t) + \vec{f}(t)$ is simply

$$\vec{y}(t) = e^{At}\vec{c} + e^{At}\int_0^t e^{-At}\vec{f}(t)dt.$$

This equation solves just about every ODE we've encountered all semester, and more. To use this solution, the system must have constant coefficients, but the function f only has to be integrable after multiplying by e^{-At}. This greatly extends our ability to solve non homogeneous ODEs.

It's possible to rework through the details of the problem above to show this is the solution. I'll leave those details to you. Solving that problem was one of my most exciting discoveries in the last 10 years of teaching. It's amazing.

Problem 9.24 Consider the linear system of ODEs given by

$$y_1' = -3y_1 + y_2 + 3 \quad \text{and} \quad y_2' = -3y_2 + 6,$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 0$.

1. Write this linear system in the form $\frac{d\vec{y}}{dt} - A\vec{y} = \vec{f}(t)$. [Hint: The right hand side might be constant.]
2. Explain how to compute e^{At} and e^{-At}. What are $[e^{At}]^{-1}$ and $e^{A(-t)}$?
3. Compute, by hand, $\int_0^t e^{-At}\vec{f}(t)dt$, and then with a computer give the product $e^{At}\int_0^t e^{-At}\vec{f}(t)dt$.
4. We know the general solution is $\vec{y}(t) = e^{At}\vec{c} + e^{At}\int_0^t e^{-At}\vec{f}(t)dt$. Use the initial conditions to find $\vec{c}$. Show us what matrix you are row reducing.
5. You should have $y_2(t) = 2 - 2e^{-3t}$. What is $y_1(t)$?

Problem 9.25 Consider the linear system of ODEs

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\vec{y} + \begin{bmatrix} 0 \\ 5t \end{bmatrix},$$

with initial conditions $y_1(0) = 4$, $y_2(0) = 0$.

1. Use a computer to give the eigenvalues and an eigenvector for each eigenvalue. State Q and J so that $AQ = QJ$ where J is a Jordan form for A.
2. State e^{Jt}. What matrix product would you compute to get e^{At}? State the matrices you would multiply, and then state both e^{At} and e^{-At}. Please use a computer to do all this.
3. Compute, by hand, $\int_0^t e^{-At}\vec{f}(t)dt$, and then with a computer give the product $e^{At}\int_0^t e^{-At}\vec{f}(t)dt$.
4. Show how to use the initial conditions to find $\vec{c}$ in $\vec{y}(t) = e^{At}\vec{c}+e^{At}\int_0^t e^{-At}\vec{f}(t)dt$.

Problem 9.26 Consider the linear system of ODEs

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \vec{y} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

1. Find Q and J so that J is a Jordan form for A. You'll need to get a generalized eigenvector. Show how you did this.
2. State e^{At} and e^{-At}.
3. Compute, by hand, $\int_0^t e^{-At}\vec{f}(t)dt$, and then with a computer give the product $e^{At}\int_0^t e^{-At}\vec{f}(t)dt$.
4. If you let $\vec{c} = \vec{0}$, then what are y_1 and y_2. This is what we called y_p in the non homogeneous ODE section.

You've now seen the key ideas needed to solve any kind of system. The computations can get quite intense, but we can program a computer to take care of all the computations.

9.4 Applications

Let's finish this chapter with some application problems. You don't have to be able to compute the matrix exponential problems to solve these problems, so if you got stuck above, please do these.

Problem 9.27 Suppose we have three large tanks containing various amounts of salt. The tanks have volumes $V_1 = 30$ gal, $V_2 = 20$ gal, and $V_3 = 50$ gal.

We don't have to use salt. This could represent 3 different countries and products they wish to import/export. It could be sewage at a waste transfer station. We might consider three countries and the spreading of a virus. We could look at three cities and the flow of traffic. The applications are endless.

- An inlet valve pumps 5 gallons of water into tank 1 each minute. The water coming in contains 3 lbs of salt per gallon. This water is being added to the system from some external source.
- Tank 1 has an outlet value that pumps 9 gallons per minute out to tank 2. Tank 1 has an inlet value that receives 3 gallons per minute from tank 2, and 1 gallon per minute from tank 3. In all, this mean that tank 1 has 9 gallons coming in per minute, and 9 gallons going out per minute.
- Tank 2 receives 9 gallons per minute from tank 1. Of those 9 gallons, it sends 3 gallons per minute to tank 1 and 4 gallons per minute to tank 2. The other 2 gallons per minute leak out the top (through a crack).
- Tank 3 receives 4 gallons per minute from tank 2. It sends 1 gallon per minute back to tank 1, and then the remaining 3 gallons per minute are sent out a hose to some external spot.

Assume the initial salt content is zero in each tank. As time moves on, the salt that is added to tank 1 will eventually reach the other tanks. After some time has elapsed, how much salt will be in each tank?

1. Let y_1, y_2, and y_3 be the lbs of salt in each tank after t minutes. Write this tank mixing problem as a linear system of ODEs in the form $\frac{d\vec{y}}{dt} - A\vec{y} = \vec{f}(t)$.
2. Use software to completely solve the system. State $y_1(t)$, $y_2(t)$, and $y_3(t)$.
3. Let $t \to \infty$. What will be the salt content in each tank in the long run.

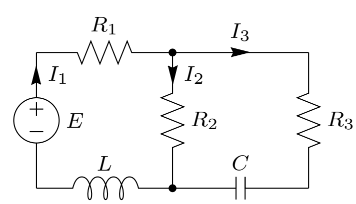
Problem 9.28 Consider the following mechanical system. Attach a spring to the top of the ceiling. Add an object with mass m_1 to the bottom of the spring. We'll assume the spring's mass is negligible. The spring constant is k_1, and the coefficient of friction is c_1. To the bottom of the first mass, we attach a second spring, and hang another object to the end of the second spring. The second object has mass m_2. The second spring has negligible mass, with spring constant k_2 and coefficient of friction c_2. Assume that this mechanical system is currently stabilized, so neither mass is moving. We'll let $y_1(t)$ be the position of the first mass, relative to this equilibrium position (so if $y_1(t) = 3$, then we'd be 3 cm above the equilibrium point). We'll let $y_2(t)$ be the position of the second mass relative to its equilibrium position. We displace the objects from equilibrium, and let them go, so we have the initial conditions $y_1(0) = a$, $y_2(0) = b$, $y_1'(0) = 0$, and $y_2'(0) = 0$. Our goal is to predict the future, namely give the position of both springs at time t.

1. Start by assuming that both $c_1 = 0$ and $c_2 = 0$. Explain why $m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$. Obtain a similar equation for my_2''.

2. Let $v_1 = y_1'$ and $v_2 = y_2'$. This allows us to replace y_1'' with v_1', and y_2'' with v_2', and then we have a system of first order linear ODEs. Write this system in the matrix form
$$\begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix}' = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix}.$$

3. Now assume that c_1 and c_2 are not zero. How does this change the ODEs from part 1? What's the corresponding 4 by 4 matrix in part 2?

Our last application involves modeling the current in an electrical system with two loops. A similar computation will work with any number of loops, though the number of loops causes the size of the system to increase quite rapidly. Remember that Kirchoff's current law states that at each node, the current in equals the current out. In addition, Kirchoff's voltage laws states that along each loop, the voltage supplied equals the voltage suppressed. Each resistor contributes a voltage drop of RI ohms, each capacitor a drop of $\frac{1}{C}\int I dt$ farads, and each inductor a voltage drop of LI' Henrys.



Problem 9.29 Consider the electrical network on the right. Kirchoff's current law states that $I_1 = I_2 + I_3$. On the left loop, Kirchoff's voltage law states that $E = R_1 I_1 + R_2 I2 + LI_1'$. On the right loop, Kirchoff's voltage law states that $0 = I_3 R_3 + \frac{1}{C}\int I_3 dt - R_2 I_2$. Solve this system for I_1', I_2', and I_3', and then write the system in matrix form $\dfrac{d\vec{I}}{dt} = A\vec{I} + \vec{f}(t)$, i.e. in the form
$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}' = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}.$$

You'll need to differentiate the first and last equation to get an I_1' and I_3'. Once you've got the system set up, a computer will give the currents almost instantly. This algorithm is coded into any electrical network software package.

9.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

9.6 Problems

The accompanying problems will serve as practice problems for this chapter. Handwritten solutions to most of these problems are available online (click for solutions). You can use the Mathematica technology introduction to check any answer, as well as give a step-by-step solution to any of the problems. However, on problems where the system is not diagonalizable, the matrix Q used to obtain Jordan form is not unique (so your answer may differ a little, until you actually compute the matrix exponential $Qe^{Jt}Q^{-1} = e^{At}$).

1. Solve the linear ODE $y' = ay(t) + f(t)$, where a is a constant and $f(t)$ is any function of t. You will need an integrating factor, and your solution will involve the integral of a function.

2. For each system of ODEs, solve the system using the eigenvalue approach. Find the Wronskian and compute its determinant to show that your solutions are linearly independent.

   (a) $y_1' = 2y_1 + 4y_2, y_2' = 4y_1 + 2y_2, y_1(0) = 1, y_2(0) = 4$

   (b) $y_1' = y_1 + 2y_2, y_2' = 3y_1, y_1(0) = 6, y_2(0) = 0$

   (c) $y_1' = y_1 + 4y_2, y_2' = 3y_1 + 2y_2, y_1(0) = 0, y_2(0) = 1$

   (d) $y_1' = y_2, y_2' = -3y_1 - 4y_2, y_1(0) = 1, y_2(0) = 2$

3. (Jordan Form) For each matrix A, find matrices Q, Q^{-1}, and J so that $Q^{-1}AQ = J$ is a Jordan canonical form of A.

   (a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (e) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4. For each of the following matrices A which are already in Jordan form, find the matrix exponential. Note that if t follows a matrix, that means you should multiply each entry by t.

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$t

   (c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$t

   (e) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$t

   (f) $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$t

   (g) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$t

   (h) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$t

   (i) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$t

   (j) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$t

5. For each of the following matrices, find the matrix exponential. You will have to find the Jordan form.

   (a) $\begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$

   (e) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   (f) $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

   (g) $\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

   (h) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

6. Set up an initial value problem in matrix format for each of the following scenarios (mixing tank, dilution problems). Solve each one with the computer.

   (a) Tank 1 contains 30 gal, tank 2 contains 40. Pumps allow 5 gal per minute to flow in each direction between the two tanks. If tank 1 initially contains 20lbs of salt, and tank 2 initially contains 120 lbs of salt, how much salt will be in each tank at any given time t. Remember, you are just supposed to set up the IVP, not actually solve it (the eigenvalues are not very pretty).

   (b) Three tanks each contain 100 gallons of water. Tank 1 contains 400lbs of salt mixed in. Pumps allow 5 gal/min to circulate in each direction between tank 1 and tank 2. Another pump allows 4 gallons of water to circulate each direction between tanks 2 and 3. How much salt is in each tank at any time t?

(c) Four tanks each contain 30 gallons. Between each pair of tanks, a set of pumps allows 1 gallon per minute to circulate in each direction (so that each tank has a total of 3 gallons leaving and 3 gallons entering). Tank 1 contains 50lbs of salt, tank 2 contains 80 lbs of salt, tank 3 contains 10 lbs of salt, and tank 4 is pure water. How much salt is in each tank at time t?

(d) Tank 1 contains 80 gallons of pure water, and tank 2 contains 50 gallons of pure water. Each minute 4 gallons of water containing 3lbs of salt per gallon are added to tank 1. Pumps allow 6 gallons per minute of water to flow from tank 1 to tank 2, and 2 gallons of water to flow from tank 2 to tank 1. A drainage pipe removes 4 gallons per minute of liquid from tank 2. How much salt is in each tank at any time t?

7. Convert each of the following high order ODEs (or systems of ODEs) to a first order linear system of ODEs. Which are homogeneous, and which are nonhomogeneous?

   (a) $y'' + 4y' + 3y = 0$
   (b) $y'' + 4y' + 3y = 4t$
   (c) $y'' + ty' - 2y = 0$
   (d) $y'' + ty' - 2y = \cos t$
   (e) $y''' + 3y'' + 3y' + y = 0$
   (f) $y'''' - 4y''' + 6y'' - 4y' + y = t$
   (g) $y_1'' = 4y_1' + 3y_2, y_2' = 5y_1 - 4y_2$.
   (h) Chapter 17, problems 1-20, in Schaum's

8. Solve the following homogeneous systems of ODEs, or higher order ODEs, with the given initial conditions.

   (a) $y_1' = 2y_1, y_2' = 4y_2, y_1(0) = 5, y_2(0) = 6$
   (b) $y_1' = 2y_1 + y_2, y_2' = 2y_2, y_1(0) = -1, y_2(0) = 3$
   (c) $y'' + 4y' + 3y = 0, y(0) = 0, y'(0) = 1$
   (d) $y'' + 2y' + y = 0, y(0) = 2, y'(0) = 0$
   (e) $y_1' = 2y_1 + y_2, y_2' = y_1 + 2y_2, y_1(0) = 2, y_2(0) = 1$
   (f) $y_1' = y_2, y_2' = -y_1, y_1(0) = 1, y_2(0) = 2$

9. Solve the following nonhomogeneous systems of ODEs, or higher order ODEs, with the given initial conditions. Use the computer to solve each of these problems, by first finding the matrix exponential and then using using the formula $\vec{y} = e^{At}\vec{c} + e^{At}\int e^{-At}\vec{f}(t)dt$. You'll have to find the matrix A and function f.

   (a) $y_1' = 2y_1 + t, y_2' = 4y_2, y_1(0) = 5, y_2(0) = 6$
   (b) $y_1' = 2y_1 + y_2, y_2' = 2y_2 - 4, y_1(0) = -1, y_2(0) = 3$
   (c) $y'' + 4y' + 3y = \cos 2t, y(0) = 0, y'(0) = 1$
   (d) $y'' + 2y' + y = \sin t, y(0) = 2, y'(0) = 0$
   (e) $y_1' = 2y_1 + y_2 - 2, y_2' = y_1 + 2y_2 + 3, y_1(0) = 2, y_2(0) = 1$
   (f) $y_1' = y_2, y_2' = -y_1 + t, y_1(0) = 1, y_2(0) = 2$

10. Mass-Spring Problems - To be added in the future.

11. Electrical Network Problems - To be added in the future.

Chapter 10

Fourier Series and PDEs

This chapter covers the following ideas. When you create your lesson plan, it should contain examples which illustrate these key ideas. Before you take the quiz on this unit, meet with another student out of class and teach each other from the examples on your lesson plan.

1. Define period, and how to find a Fourier series of a function of period 2π and $2L$.
2. Explain how to find Fourier coefficients using Euler formulas, and be able to explain why the Euler formulas are correct.
3. Give conditions as to when a Fourier series will exist, and explain the difference between a Fourier series and a function at points of discontinuity.
4. Give examples of even and odd functions, and correspondingly develop Fourier cosine and sine series. Use these ideas to discuss even and odd half-range expansions.

10.1 Fourier Series

In the power series unit, we showed how to write function $f(x)$ as a power series $\sum_{n=0}^{\infty} a_n x^n$. The radius of convergence tells us precisely for which x values the power series will converge to the function $f(x)$. We call it a power series because we use powers of x as the functions we are summing. At any finite stage, we are trying to approximate the function $f(x)$ using linear combinations of power of x. We can find the coefficients a_n with the formula $a_n = \frac{f^(n)(0)}{n!}$.

Are there other kinds of series? Do we have to use powers of x? In class, we discussed how to use Legendre polynomials to do the same thing. If we let $P_n(x)$ be the nth degree Legendre polynomial, then we could write $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$. We showed that on the interval $[-1, 1]$, the integral $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ if $n \neq m$. We then used this to show that we can compute the coefficients a_n using the formula

$$a_n = \frac{\int_{-1}^{1} f(x)P_n(x)dx}{\int_{-1}^{1} P_n(x)P_n(x)dx}.$$

The key to using a Legendre series is the fact that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0.$$

This integral equation is a generalization of the dot product. In the dot product, we multiply corresponding components together and then sum. That's precisely what the integral above does. Let's make a definition.

Definition 10.1: Inner Product. Let $f(x)$ and $g(x)$ be bounded functions over some interval $[a, b]$. The inner product of f and g over $[a, b]$ is the integral

$$\langle f, g\rangle = \int_{a}^{b} f(x)g(x)dx.$$

This inner product is a generalization of the dot product. We say that two functions are orthogonal over $[a, b]$ if their inner product over $[a, b]$ is equal to zero.

Problem 10.1 Consider the function $f(x) = x$ and $g(x) = x^2$.

1. Compute the inner products $\langle f, f\rangle$, $\langle g, g\rangle$, and $\langle f, g\rangle$. [For the first, you just need to compute the integral $\int_0^1 x \cdot x dx$.]
2. Recall that $\vec{u} \cdot \vec{u} = |u|^2$. So the length of a vector is the square root of the dot product of the vector with itself. We'll define the length of a function to be the square root of the inner product of a function with itself, and write $||f|| = \sqrt{\langle f, f\rangle}$. What is the length of f and the length of g over the interval $[0, 1]$. [Hint, you already did the inner products in part 1.
3. Recall that $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$, where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$. Use this idea to invent a definition for the angle between two functions, and then use your definition to compute the angle between f and g.

With the word inner product, we can now talk about the "dot product" of functions. We can use the words Length and angle when talking about functions. We obtained the Legendre series coefficients because the inner product of any two different Legendre polynomials, over the interval $[-1, 1]$, is zero. The Legendre polynomials form an infinite collection of orthogonal vectors. We've got a 90 degree angle between any two functions. We can use these polynomials to approximate any other function by considering linear combinations of these orthogonal functions. This is one of the big ideas in modern science. Rather than working with complicated functions f, we can approximate them with linear combinations of simpler functions. What constitutes a "simple" function depends on the problem. Legendre polynomials show up when studying spherically symmetric problems. Bessel functions shows up when studying radially symmetric problems. Sometimes the hardest part about solving a problem is trying to determine which "simple" collection of functions to use.

Fourier was one of the first to use something other than power series to approximate a function. He did it while trying to understand how heat flowed in a cannon. Napoleon asked Fourier to study this problem, because cannons were exploding on his soldiers because the cannon balls would not leave a cannon after heat had caused the barrel to expand. Fourier discovered that you can use sine and cosine series to approximate a function. That's what we'll study in this chapter. In the work below, the variable L will stand for the length of a cannon.

Problem 10.2 Let $f_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ for $n = 0, 1, 2, \ldots$. So we have $f_0(x) = \cos\left(\frac{0\pi x}{L}\right)$, $f_1(x) = \cos\left(\frac{\pi x}{L}\right)$, $f_2(x) = \cos\left(\frac{2\pi x}{L}\right)$, $f_3(x) = \cos\left(\frac{3\pi x}{L}\right)$, etc. This is an infinite collection of functions. Our goal is to show that on the interval $[0, L]$, the functions f_n and f_m are orthogonal, provided $n \neq m$. To do this, you'll need to prove that any pair has an inner product of 0.

1. Draw the functions $f_0(x) = \cos\left(\frac{0\pi x}{L}\right)$, $f_1(x) = \cos\left(\frac{\pi x}{L}\right)$, and $f_2(x) = \cos\left(\frac{2\pi x}{L}\right)$ over the interval $[0, L]$.
2. Compute the integral $\int_0^L f_0(x) f_n(x) dx$ for $n \neq 0$. What is $\langle f_0, f_n \rangle$?
3. Now show $\int_0^L f_n(x) f_m(x) dx = 0$ for $n \neq m$, with $n, m > 0$. As a hint, you'll want to look up a product-to-sum trig identity, after which this integral is quickly doable.

We now know that the set of functions $f_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ forms an orthogonal set of functions over the interval $[0, L]$. Let's now make a Fourier cosine series for a function.

Problem 10.3 Suppose that $f(x)$ is defined on $[0, L]$. If we assume that

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

then show that

$$A_n = \frac{\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx}.$$

Then compute the bottom integral to show that for $n \geq 1$ we have

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

and if $n = 0$ then we have

$$A_0 = \frac{1}{L} \int_0^L f(x) dx.$$

[Hint: Multiply both sides by $\cos\left(\frac{m\pi x}{L}\right)$ and then integrate from 0 to L. You can use the orthogonality of cosines to solve for the coefficients. The integral of the bottom will involve a cosine half angle formula.]

We could have repeated the previous two problems with the sine function, and would have gotten very similar results. Based of the previous two problems, let's make a formal definition.

Definition 10.2: Fourier Sine and Cosine Series. Let $f(x)$ be a function defined on $[0, L]$. We define the Fourier sine series and Fourier cosine series of f (on $[-L, L]$) to be the series

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

respectively. The coefficients above are given by the formulas

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{and} \quad \begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)dx, \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Problem 10.4 Let $f(x) = x$ over the interval $[0, L]$. Draw the function f. Then compute the Fourier sine series of $f(x)$ by computing the integrals

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Write out the first 4 nonzero terms of the series by writing

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = B_1 \sin\left(\frac{\pi x}{L}\right) + B_2 \sin\left(\frac{2\pi x}{L}\right) + B_3 \sin\left(\frac{3\pi x}{L}\right) + \cdots.$$

Be prepared to explain how you use integration by parts to compute the integrals.

Problem 10.5 Let $f(x) = x$ over the interval $[0, L]$. Draw the function f. Then compute the Fourier cosine series of $f(x)$ by computing the integrals

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \quad \text{and} \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Write out the first 4 nonzero terms of the series by writing

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = A_0 + A_1 \cos\left(\frac{\pi x}{L}\right) + A_2 \cos\left(\frac{2\pi x}{L}\right) + A_3 \cos\left(\frac{3\pi x}{L}\right) + \cdots.$$

Be prepared to explain how you use integration by parts to compute the integrals.

Problem 10.6 In the previous two problems, you computed the Fourier cosine and Fourier sine series of $f(x) = x$ over the interval $[0, L]$. This problem will repeat the above with technology, and then you'll graph your results. To allow software to create graphs, you'll have to pick a length L that's an actual number (maybe try $L = 7$).

1. Use software to obtain the first 4 nonzero terms of the Fourier sine and Fourier cosine series of $f(x) = x$ over $[0, L]$.
2. Use software to graph both $f(x) = x$ and the first four nonzero terms of the Fourier sine series. Have the bounds of your graph be over the interval $[-3L, 3L]$.
3. Use software to graph both $f(x) = x$ and the first four nonzero terms of the Fourier cosine series. Have the bounds of your graph be over the interval $[-3L, 3L]$.
4. Make some conjectures about what you see in your graphs.

To present this in class, you should come to class with your graphs already printed out.

Definition 10.3: Piecewise Smooth. We say that function $f(x)$ is smooth on an interval $[a, b]$ if the function and its derivative are both bounded and continuous on (a, b). We say that a function $f(x)$ is piecewise smooth on an interval (a, b) if the interval can be partitioned into a finite number of pieces and on each piece the function $f(x)$ is smooth (so $f'(x)$ may not exist at finitely many points).

Definition 10.4: Fourier Series. Let $f(x)$ be a function defined on $[-L, L]$ such that the Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \text{ and} \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

exist. We define the Fourier series of f over the interval $[-L, L]$ to be the formal infinite series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Regardless of whether or not the series converges, we will write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Remark 10.5. In the definition above, we do not require the Fourier series to actually converge to f. The following theorem, which we will give without proof, provides the needed conditions for the series to converge. We will use this proof (which requires some real analysis) to prove other facts about Fourier series throughout this chapter.

Theorem 10.6 (Fourier's Theorem)**.** *Suppose $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$. Then the Fourier series of $f(x)$ converges to the period extension of f at the points where f is continuous. If the periodic extension of f is not continuous at a point x, then the Fourier series converges to the average of the left and right limits at x, namely $\dfrac{f(x+) + (f(x-)}{2}$.*

$\boxed{\textbf{Problem 10.7}}$ Let $f(x) = x$ over the interval $[-L, L]$. Compute the Fourier series of $f(x)$ by computing the integrals for a_0, a_n, and b_n. Show your integration steps.

$\boxed{\textbf{Problem 10.8}}$ Let $f(x) = x^2$ over the interval $[-L, L]$. Compute the Fourier series of $f(x)$ by computing the integrals for a_0, a_n, and b_n Show your integration steps.

Problem 10.9 Let $f(x) = e^x$, $g(x) = \cosh(x)$ and $h(x) = \sinh(x)$ over the interval $[-L, L]$. Use software to compute the Fourier coefficients for each function. What patterns do you see? You are welcome to just state the Fourier coefficients for each (you don't have to show your integration steps).

Notice that there are other kinds of series. In class, I showed you how to obtain a function as an infinite sum of Legendre polynomials. We also introduced the concept of orthogonality of functions, and showed that the Legendre polynomials formed an infinite collection of orthogonal functions. We also showed how to write a function f as a series of Legendre polynomials.

We can do the exact same thing with trig functions. We'll develop the cosine series, the sine series, and then the Fourier series. We'll compute these series for a few functions. After that let's look at how these were developed by Fourier, as he studied the heat equation.

Maybe we should start with why we care about something that has length L. Then get to the reason why we look at $n\pi/L$. Let's definitely do that.

Then prove orthogonality of the functions on these intervals.

Show that if we write a function as a series, and the functions are orthogonal, then we know the coefficients through an integral formula.

Obtain a formula for the cosine series coefficients and the sine series coefficients.

Find the Fourier coefficients of a cosine series. Draw the function and several terms of the series.

Find the Fourier coefficients of a sine series. Draw the function, and several terms of the series.

What are the coefficients of a Fourier series. Find the coefficients for a function.

If a function is odd or even, what can be said.

I can grab a lot of these problems from my PDE course notes.

10.2 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, upload your work to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

10.3 Problems

The accompanying problems will serve as practice problems for this chapter. Handwritten solutions to most of these problems are available online (click for solutions). You can use the Mathematica technology introduction to check all your answers. I will create a solutions guide as time permits.

1. Find the fundamental period of the following functions.
   (a) $\sin x$, $\sin 2x$, $\sin \frac{x}{3}$, $\sin kx$, $\sin \frac{n\pi x}{2}$
   (b) $\tan x$, $\tan 2x$, $\tan \frac{x}{3}$, $\tan kx$, $\tan \frac{n\pi x}{2}$
2. Show $y = c$ is p-periodic for each positive p, but has no fundamental period.
3. Compute the Fourier series of each function below (assume the function is 2π-periodic). Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated Fourier series.
   (a) $f(x) = \sin 2x$
   (b) $f(x) = \cos 3x$
   (c) $f(x) = \sin 2x + \cos 3x$
   (d) $f(x) = 4$
   (e) $f(x) = 4 + 5\sin 2x - 7\cos 3x$
   (f) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$
   (g) $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$
   (h) $f(x) = \begin{cases} -2 & -\pi < x < 0 \\ 3 & 0 < x < \pi \end{cases}$
   (i) $f(x) = x$ for $-\pi < x < \pi$
   (j) $f(x) = |x|$ for $-\pi < x < \pi$
   (k) $f(x) = x$ for $0 < x < 2\pi$
   (l) $f(x) = x^2$ for $-\pi < x < \pi$
   (m) $f(x) = \begin{cases} 0 & -\pi < x < -\pi/2 \\ 1 & -\pi/2 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$
   (n) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$
   (o) $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$
4. Compute the Fourier series of each function below (assume the function is p-periodic). Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated Fourier series.
   (a) $f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$, $p = 4$
   (b) $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 2 & 0 < x < 2 \end{cases}$, $p = 4$
   (c) $f(x) = x$ for $-1 < x < 1$, $p = 2$
   (d) $f(x) = |x|$ for $-1 < x < 1$, $p = 2$
   (e) $f(x) = 1 - |x|$ for $-1 < x < 1$, $p = 2$
   (f) $f(x) = x^2$ for $-1 < x < 1$, $p = 2$
   (g) $f(x) = x^2$ for $-2 < x < 2$, $p = 4$
   (h) $f(x) = 1 - x^2$ for $-1 < x < 1$, $p = 2$
   (i) $f(x) = x^3$ for $-1 < x < 1$, $p = 2$
   (j) $f(x) = \sin(\pi x)$ for $0 < x < 1$, $p = 1$
   (k) $f(x) = \cos(\pi x)$ for $-\frac{1}{2} < x < \frac{1}{2}$, $p = 1$
   (l) $f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$, $p = 2$
   (m) $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$, $p = 4$
5. Decide if each function is even, odd, or neither.
   (a) $x, x^2, x^3, x^4, \sqrt{x}, \sqrt[3]{x}, x^2 + x + 1, x^3 + x, x^2 + 1, x^4 + x^5$.
   (b) $\sin x, \cos x, \cos 3x, \tan x, \cot x, \sec x, \csc x, \sin x \cos x, \sin^2 x, \sin x + \cos 3x$.

(c) If f is even and g is odd, $f^2, g^2, f^3, g^3, fg, f+g, 3f, xf, xg, f^n g^m$ (where n and m are integers).

6. Rewrite each function f as the sum of an even f_e and an odd f_o function, so that $f = f_e + f_0$. Make sure you show that $f_e(-x) = f_e(x)$ and $f_o(-x) = -f_o(x)$. Then plot f, f_e, and f_o on the same axes.

   (a) $f(x) = e^x$ (your answer should involve hyperbolic functions).

   (b) $f(x) = x^2 + 3x + 2$

   (c) $f(x) = \dfrac{1}{x-1}$ (since $f(1)$ is undefined, the even and odd functions are not defined at $x = -1$).

7. For each function defined on $[0, L]$, find the Fourier cosine series of the even periodic extension to $[-L, L]$. Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated series.

   (a) $f(x) = 1$ for $0 < x < 1$

   (b) $f(x) = 1$ for $0 < x < \pi$

   (c) $f(x) = x$ for $0 < x < 1$

   (d) $f(x) = x$ for $0 < x < \pi$

   (e) $f(x) = 1 - x$ for $0 < x < 1$

   (f) $f(x) = 2 - x$ for $0 < x < 2$

   (g) $f(x) = \pi - x$ for $0 < x < \pi$

   (h) $f(x) = x^2$ for $0 < x < 1$

   (i) $f(x) = x^3$ for $0 < x < 1$

   (j) $f(x) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$

   (k) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$

   (l) $f(x) = \begin{cases} 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

   (m) (Sawtooth Wave) $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$

8. For each function from 7, find the Fourier sine series of the odd periodic extension to $[-L, L]$. Write your solution using summation notation. Then graph at least 3 periods of the function and compare the graph of the function with a graph of a truncated series.

9. For a function $f(x)$ defined on $[0, L]$, a half wave rectifier extends f to be 0 on $[-L, 0)$, and then periodically extends the function to all real numbers. If f_e and f_o represent the even and odd periodic extentsions, then $g = \dfrac{f_e + f_o}{2}$ represents the half wave rectifier. For each function from 7, find the Fourier series of the half wave rectifier of f. Write your solution using summation notation. Then graph at least 3 periods of the half wave rectifier and compare it to a graph of a truncated series.

10. Compute the following integrals, where n and m are positive integers. You will need the product to sum trig identitites.

    (a) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx.$

    (b) $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx.$

    (c) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx.$

11. Use Fourier series to prove the following identities.

    (a) $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$

    (b) $\sin^3 x = \dfrac{3}{4}\sin x - \dfrac{1}{4}\sin(3x)$

    (c) $\sin^4 x = \dfrac{3}{8} - \dfrac{1}{2}\cos(2x) + \dfrac{1}{8}\cos(4x)$

    (d) $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos(2x)$

    (e) $\cos^3 x = \dfrac{3}{4}\cos x + \dfrac{1}{4}\cos(3x)$

    (f) $\cos^4 x = \dfrac{3}{8} + \dfrac{1}{2}\cos(2x) + \dfrac{1}{8}\cos(4x)$

12. (Gibb's Phenomenon) Pick a discontinuous function from any of the previous exercises. Use a computer to graph the partial sums

$$f_k(x) = a_0 + \sum_{n=1}^{k} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right)$$

for $k = 5, 10, 50, 100$. What do you notice happening near the point where f is discontinuous? Does increasing k make this "bump" disappear? Try letting k be 1000 (it may take little while for the computer to construct your solution).