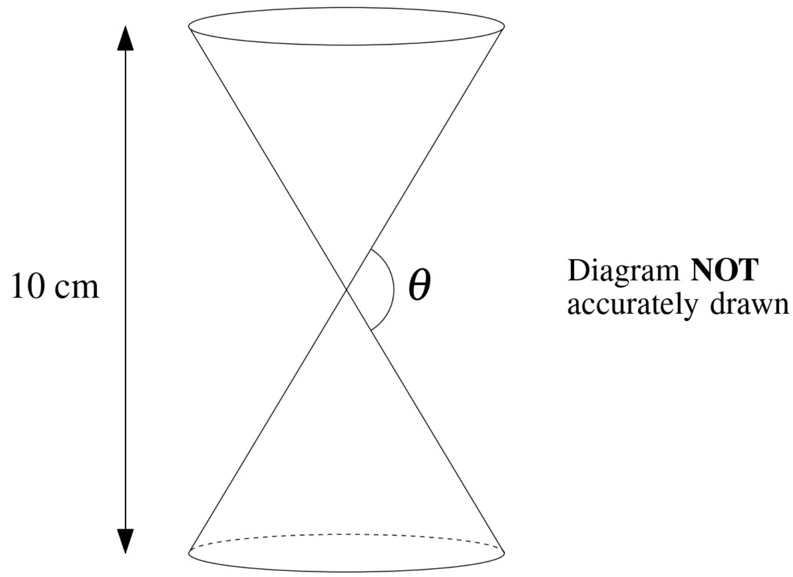
1 3D Shapes

1. The diagram below shows a simple model of an hourglass consisting of two hollow glass cones.



The bases of the two cones are parallel and the height of the hourglass is 10 cm.
The hourglass contains 10000 spherical sand grains, each of radius 0.7 mm.
Exactly 27% of the total volume of the hourglass is occupied by the sand grains.
Determine the value of angle θ to 1 decimal place.

[7 marks]

2. Cylinder **C** has radius r and height h.
C is cut in half to produce two smaller cylinders.
The total surface area of the two smaller cylinders is 20% larger than the surface area of **C**.
Find the ratio $r : h$, giving your answer in its simplest form.

[5 marks]

3. Some spherical plastic pellets are to be melted down and shaped into a single cube.
During the manufacturing process, exactly 6% of the total volume of plastic is lost.
The number of plastic pellets is estimated as 1100, correct to the nearest hundred.
The surface area of each pellet is $15\,\text{mm}^2$, correct to the nearest square millimetre.
Find the lower bound on the surface area of the cube.
Give your answer in cm^2 and to 3 significant figures.

[8 marks]

2 Arithmetic Sequences

1. An arithmetic sequence has nth term $12n - 104$.
   The sum of the first k terms is 588.
   Find the sum of the first $2k$ terms.

   [5 marks]

2. The first three terms of an arithmetic sequence are x, $5\sqrt{2}$ and $\frac{1}{x}$, in that order.
   The second term is smaller than the first term.
   Find the value of the common difference of the sequence, d.

   [6 marks]

3. The 50th term of an arithmetic sequence is -184.
   The sum of its first 16 terms is 18 times larger than its 8th term.
   Is -500 a term of this sequence?
   You must give reasons for your answer.

   [6 marks]

4. An arithmetic sequence has first term a and common difference d.
   Given that

   the 5th term : the 13th term = 2 : 5

   find the ratio $a : d$.
   Express your answer as a fully simplified ratio of integers.

   [5 marks]

3 Proportion

1. x is inversely proportional to $\sqrt{y}$.
   y is directly proportional to z^3.
   Given that $z = 5$ when $x = 2$, find a formula for x in terms of z.
   Express your answer in the form $x = kz^a$, where k is a simplified surd and a is a simplified fraction.
   [5 marks]

4 Circular Sectors & Segments

1. The diagram below shows rhombus $ABCD$.

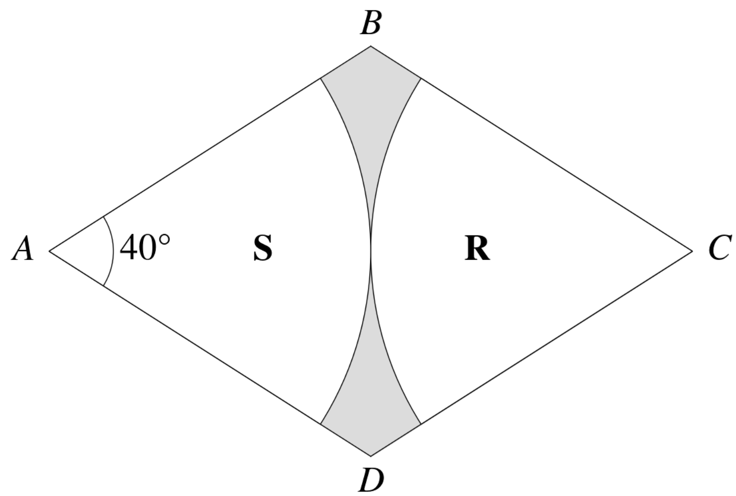


Diagram **NOT** accurately drawn

S and **R** are congruent circular sectors that touch at the centre of $ABCD$.

Angle $BAD = 40°$

Find the shaded area as a percentage of the area of $ABCD$.

Give your answer to 3 significant figures.

[6 marks]

2. The diagram below shows a circle with centre O.

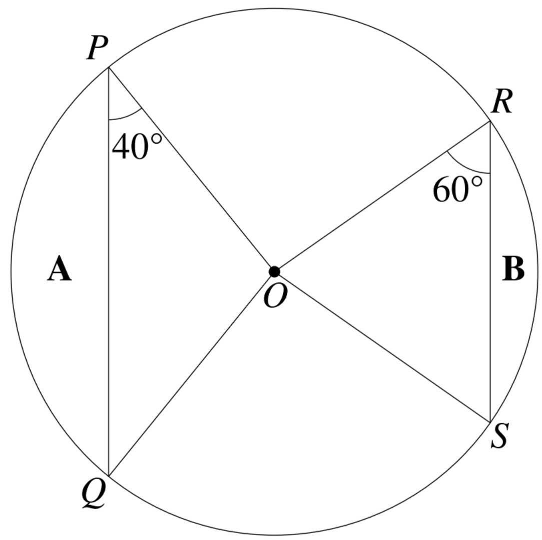


Diagram **NOT** accurately drawn

P, Q, R and S are points on the circumference of the circle.

A and **B** are minor segments of the circle.

Angle $OPQ = 40°$

Angle $ORS = 60°$

The area of segment **B** is $10\,\text{cm}^2$.

Find the perimeter of segment **A**.

[7 marks]

3. The diagram below shows circular sector **S** and right-angled isosceles triangle **T**.

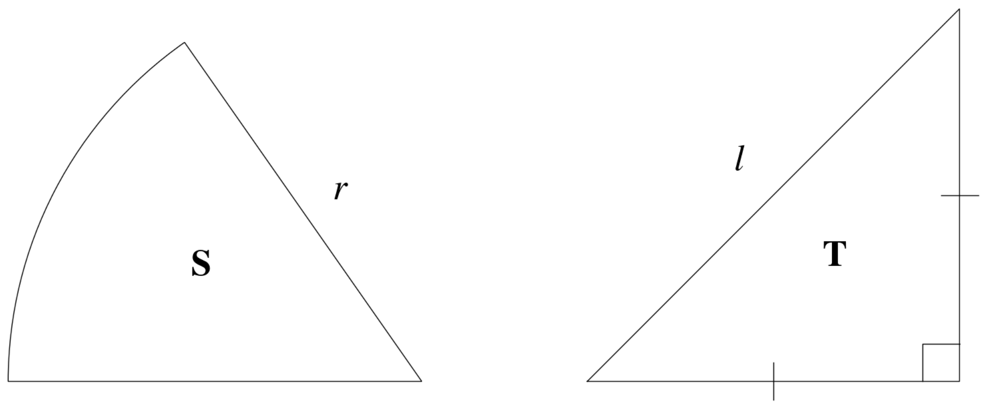


Diagram **NOT** accurately drawn

The radius of **S** is r and the length of the hypotenuse of **T** is l.
The perimeter of **S** is kr.
S and **T** have equal areas.

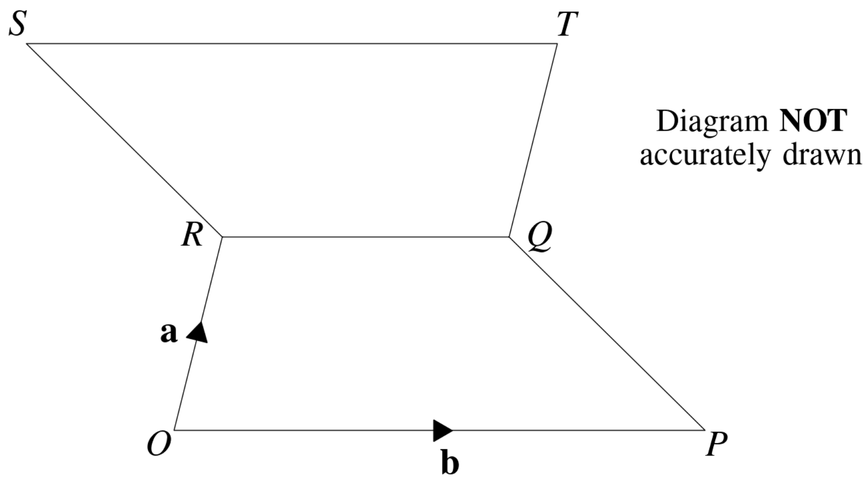
(a) State the range of possible values of k, giving your answer in the form $a < k < b$.
[2 marks]

(b) By considering the areas of **S** and **T**, show that $l = r\sqrt{2(k-2)}$.
[6 marks]

(c) Hence, or otherwise, find the value of k if r and l are equal.
[2 marks]

5 Vectors

1. The diagram below shows two congruent trapezia, $OPQR$ and $QRST$.



$OR = QT$ and $PQ = RS$

$\overrightarrow{OR} = \mathbf{a}$ and $\overrightarrow{OP} = \mathbf{b}$

$RQ : OP = 3 : 5$

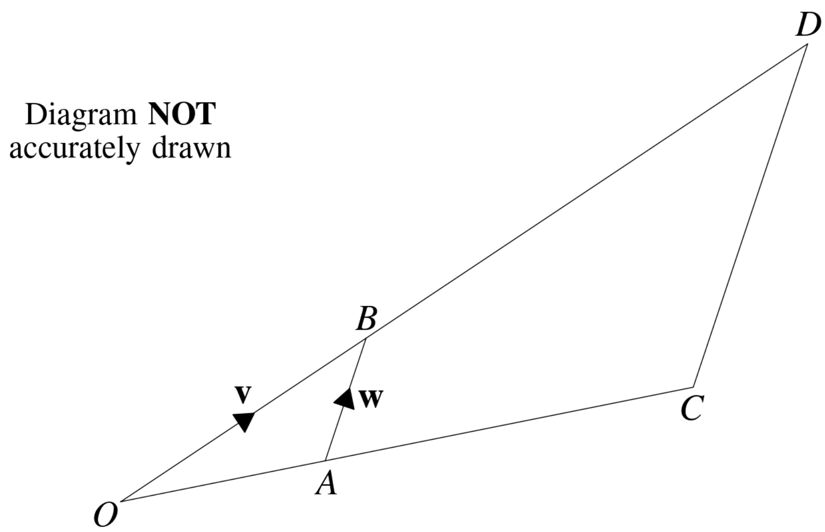
X is the point on OP such that $OX = k\,OP$.

Y is the midpoint of PQ.

Given that XY is parallel to OT, use a vector method to find the value of k.

[7 marks]

2.



OAC and OBD are straight lines.

CD is parallel to AB.

$\overrightarrow{OB} = \mathbf{v}$ and $\overrightarrow{AB} = \mathbf{w}$

$BD = 2\,OB$

X is a point on line segment AB such that $\frac{AX}{AB} = p$.

Y is a point on line segment CD such that $\frac{CY}{CD} = q$.

(a) Express $\overrightarrow{XY}$ in terms of p, q, $\mathbf{v}$ and $\mathbf{w}$.

[3 marks]

(b) Given that XY is parallel to OD, find a formula for q in terms of p.

[2 marks]

(c) Hence, or otherwise, find the minimum possible value of q.

[2 marks]

6 Proof

1. (a) Three different positive integers, p, q and r, are each one more than a multiple of 6.
   Prove that the mean of p, q and r is an odd integer.

   [3 marks]

   (b) Another three different positive integers, x, y and z, are each one more than a multiple of 5.
   Explain whether the mean of x, y and z is always an integer.

   [2 marks]

   (c) k is a positive integer.
   a, b, c, d and e are five consecutive multiples of k.
   Prove that the median of a, b, c, d and e is equal to the mean of these numbers.

   [3 marks]

2. Prove that the side lengths of a right-angled isosceles triangle cannot all be integers.
   You may assume without proof that a surd cannot be written as a fraction whose numerator and denominator are both integers.

   [2 marks]

3. $N = 1000a + 100b + 10c + d$, where a, b, c and d are integers between 0 and 9 inclusive.
   N is a multiple of 3.

   (a) Write down an expression for the sum of the digits of N.

   [1 mark]

   (b) Hence prove that the sum of the digits of N is a multiple of 3.

   [3 marks]

   (c) m is an integer where $2 \leq m \leq 9$ and $m \neq 3$.
   The sum of the digits of any four-digit multiple of m is also a multiple of m.
   Find the value of m, giving clear reasons for your answer.

   [1 mark]

7 Differentiation

1. The curve with equation $y = x^3 - 7x^2 + 8x - 1$ has two turning points, T and S.
   Find the equation of the straight line that passes through T and S.
   Give your answer in the form $ax + by = c$, where a, b and c are integers.

   [6 marks]

2. Curve **C** has equation $y = \dfrac{32 + 30x^3}{x^2}$
   At the point where $x = a$, the tangent to **C** is perpendicular to the line with equation $x + 3y = 6$.
   Find the value of a.

   [5 marks]

3. A sphere made of clay has radius 3 cm.
   The sphere is to be reshaped into a cylinder of radius r and surface area S.

   (a) By considering the volume of the clay, find a formula for S in terms of r.

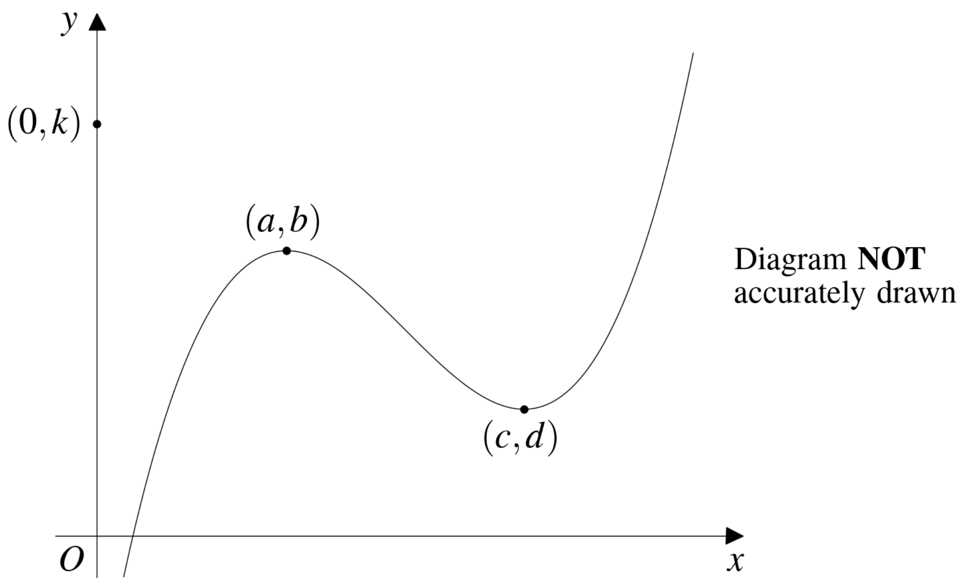
   [3 marks]

   (b) Find the exact value of r that gives the smallest possible value of S.
   Give your answer in the form $n^{1/3}$ cm, where n is an integer.

   [3 marks]

8 Advanced Graphs

1. The diagram below shows part of the curve with equation $y = \mathrm{f}(x)$ and the point $(0, k)$.



The curve has exactly two turning points, with coordinates (a, b) and (c, d).
$0 < a < c$ and $0 < d < b < k$

(a) $\mathrm{g}(x) = p\mathrm{f}(x)$, where p is a positive number.
Given that the equation $\mathrm{g}(x) = k$ has more than one solution for x, find expressions for the smallest and largest possible values of p.

[2 marks]

(b) $\mathrm{h}(x) = \mathrm{f}(x) + q$, where q is a positive number.
The equation $\mathrm{h}(x) = k$ has more than one solution for x.
Given that the set of possible values of q is the same as the set of possible values of p, find expressions for b and d in terms of k.

[4 marks]

(c) Hence explain fully why k must be greater than 4.

[2 marks]

(d) How many solutions for x does the equation $\mathrm{f}(x + k) = k$ have?
You must give reasons for your answer.

[1 mark]

2. (a) By completing the square, find expressions in terms of a and b for the values of x that satisfy the equation $x^2 - 2ax - b = 0$.

[2 marks]

(b) Curve **C** has equation $y = x^2 - 6x - 5$ and line **L** has equation $2x - y + 3 = 0$.
C and **L** intersect at points P and Q.
The solutions of the equation $x^2 - 2ax - b = 0$ are the x coordinates of points P and Q.
Find the values of a and b.

[2 marks]

(c) Using your answers to parts (a) and (b), write down the x coordinates of points P and Q.
Give your answers in an exact and fully simplified form.

[1 mark]

(d) Find the exact length of line segment PQ.

[3 marks]