Gravitational & Electromagnetic Fields: Week 2

Solutions by Ben Yelverton, May 2020

7. Consider building up the star out of spherical shells of gas. When we add a shell of radius r and thickness $\mathrm{d}r$ to a sphere of radius r, an energy given by the standard expression $\frac{Gm_1m_2}{r}$ is released, where m_1 and m_2 are the masses of the sphere and the shell. It's fine to use this expression that you're probably used to seeing for point masses, because the sphere's gravitational field (outside the radius r) is the same as that of a point mass, and the mass of the shell is all located at the same radius and uniformly distributed across the surface. So, the energy change associated with this process is

$$\begin{aligned}
\mathrm{d}U &= -\frac{G \cdot \frac{4}{3}\pi r^3 \rho \cdot 4\pi r^2 \mathrm{d}r \rho}{r} \\
&= -\frac{16\pi^2 G \rho^2 r^4 \mathrm{d}r}{3} \,.
\end{aligned}$$

To obtain the total energy, we can simply sum up the contributions of all the shells by integrating:

$$\begin{aligned}
U &= -\int_0^R \frac{16\pi^2 G \rho^2 r^4 \mathrm{d}r}{3} \\
&= -\frac{16\pi^2 G \rho^2 r^4}{3} \cdot \frac{R^5}{5} \\
&= -\frac{3G}{5R}\left(\frac{4}{3}\pi R^3\right)^2 \\
&= -\frac{3GM^2}{5R} \,.
\end{aligned}$$

Thus, for a Sun-like star, $\sim 2.3 \times 10^{41}$ J of energy is released. We can calculate the average temperature of the star as follows. The number of hydrogen atoms, N, is given by

$$\begin{aligned}
N &= \frac{M_\odot}{M_\mathrm{H}} \\
&\approx 1.2 \times 10^{57} \,,
\end{aligned}$$

with M_H the mass of one atom. Note that I'm assuming here (reasonably!) that the temperature is sufficiently high that the hydrogen will exist in monatomic form. Then, given the expression for thermal energy per atom in the question, we know that

$$\frac{U}{N} = \frac{3}{2} k_\mathrm{B} T \,.$$

This implies that $T \approx 9.2 \times 10^6$ K. Despite the simple model, this is actually a decent order of magnitude estimate of the Sun's core temperature.

8. Kepler's third law states that

$$T^2 = \frac{4\pi^2 a^3}{GM}\,. \tag{1}$$

When asked to check a power law dependency, it's always best to make a log-log plot since you can then get the actual index from the gradient. Additionally, the semi-major axes and orbital periods here span several orders of magnitude, and in such cases logarithmic plots are more appropriate as the data will be more uniformly spread out. Taking the logarithm of (1) gives:

$$\ln T = \frac{1}{2}\ln\left(\frac{4\pi^2}{GM}\right) + \frac{3}{2}\ln a\,. \tag{2}$$

So, if we plot $\ln T$ against $\ln a$ and Kepler's third law holds, we expect a gradient of $3/2$. Using the Solar System data from lectures, we do indeed find a gradient of 1.5. The Solar mass can be derived from the intercept I; from equation (2), we have

$$M_\odot = \frac{4\pi^2}{Ge^{2I}}\,.$$

This should yield $\sim 2.0 \times 10^{30}$ kg. The period of an orbit doesn't depend on its eccentricity – only its semi-major axis – and so Eris fits nicely along the line.

Note that the data table from the lectures states that the mean distance from the Sun is equal to the semi-major axis. This is actually not true – eccentric particles spend most of their time near apocentre, i.e. at larger distances than the semi-major axis. The numbers given in the table are in fact semi-major axes rather than average distances, which explains why this works!

Aside: any thoughts on the origin of Eris' high eccentricity?

9. The electric field and potential are given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\,,$$
$$V = \frac{q}{4\pi\epsilon_0 r}\,.$$

(a) Using the provided distance r and the electronic charge gives $V = 27.2$ V.

(b) By definition of an electron-volt, the electric potential energy in this configuration is $E_p = -27.2$ eV (negative because the proton and electron are oppositely charged).

(c) To find the kinetic energy E_k we need the orbital speed of the electron. Equating electrostatic and centripetal forces gives

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r}\,,$$

and thus

$$\begin{aligned} E_k &= \frac{1}{2}mv^2 \\ &= \frac{e^2}{8\pi\epsilon_0 r}\,, \end{aligned}$$

which evaluates to 13.6 eV. The total energy is then $E_p + E_k = -13.6$ eV.

(d) To ionise the atom, we need to give it a total energy ≥ 0, so that its trajectory becomes unbound. Thus, we need to put in 13.6 eV.

10. A carbon nucleus has charge $6e$. By Coulomb's law, the (scalar) force exerted on the nucleus of interest by each of the other three is

$$F = \frac{(6e)^2}{4\pi\epsilon_0 a^2}\,,$$

where a is the side length. These forces are of course directed along different edges of the tetrahedron, and so we need to add them as vectors. Placing the nucleus of interest at the origin, and noting that the other three corners of the tetrahedron are coincident with the corners of a cube of side $a/\sqrt{2}$ (google 'tetrahedron in a cube' if you can't picture this), we find

$$\begin{aligned} \mathbf{F}_{\text{tot}} &= -\frac{(6e)^2}{4\pi\epsilon_0 a^2}\left[\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right] \\ &= -\frac{(6e)^2}{4\pi\epsilon_0 a^2}\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix} \\ \therefore \quad F_{\text{tot}} &= \frac{(6e)^2}{4\pi\epsilon_0 a^2}\sqrt{6}\,. \end{aligned}$$

Given $a = 0.154$ nm, this yields $F_{\text{tot}} = 8.56 \times 10^{-7}$ N.

11. (a) This question is all about the principle of superposition in the context of continuous charge distributions. We're asked to sum up the fields produced by each element of charge, which means an integral is required. The field from a small element of the ring is given by Coulomb's law:

$$\begin{aligned}\mathbf{E} &= \int \mathrm{d}\mathbf{E} \\ &= \int \frac{\mathrm{d}q}{4\pi\epsilon_0 d^2} \cdot \frac{\mathbf{d}}{d}\,,\end{aligned}$$

where $\mathbf{d}$ is the vector from charge element $\mathrm{d}q$ to a point on the z axis. From geometry (draw a diagram!) this is given by

$$\mathbf{d} = \begin{pmatrix} -r\cos\phi \\ -r\sin\phi \\ z \end{pmatrix}\,,$$

where r, ϕ and z are the usual cylindrical polar coordinates. This also tells us that $d = \sqrt{r^2 + z^2}$. Therefore,

$$\begin{aligned}\mathbf{E} &= \int \mathrm{d}\mathbf{E} \\ &= \int \frac{1}{4\pi\epsilon_0 \left(r^2+z^2\right)^{3/2}} \begin{pmatrix} -r\cos\phi \\ -r\sin\phi \\ z \end{pmatrix} \mathrm{d}q \\ &= \frac{1}{4\pi\epsilon_0 \left(r^2+z^2\right)^{3/2}} \int_0^{2\pi} \begin{pmatrix} -r\cos\phi \\ -r\sin\phi \\ z \end{pmatrix} \cdot \frac{q\,\mathrm{d}\phi}{2\pi} \\ &= \frac{1}{4\pi\epsilon_0 \left(r^2+z^2\right)^{3/2}} \begin{pmatrix} 0 \\ 0 \\ qz \end{pmatrix} \\ &= \frac{qz}{4\pi\epsilon_0 \left(r^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}}\,. \end{aligned} \tag{3}$$

Aside: how does this result relate to question 10?

(b) Now we're summing up fields due to concentric rings (as derived above) to build up a solid disc. For a ring of radius r and thickness $\mathrm{d}r$, the charge is $2\pi r\sigma\,\mathrm{d}r$. So, let's replace the q in equation (3) with this expression, and integrate from 0 to a:

$$\begin{aligned}\mathbf{E} &= \int_0^a \frac{2\pi r\sigma\,\mathrm{d}r}{4\pi\epsilon_0 \left(r^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}} \\ &= \frac{z\sigma}{2\epsilon_0}\int_0^a r\left(r^2+z^2\right)^{-3/2}\mathrm{d}r\,\hat{\mathbf{z}} \\ &= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{\mathbf{z}}\,, \end{aligned} \tag{4}$$

since $r\left(r^2+z^2\right)^{-3/2}$ nicely integrates to $-\left(r^2+z^2\right)^{-1/2}$.

(c) We can derive these results in an alternative way – summing up the potentials due to individual elements then taking the gradient. For the ring, given that the potential associated with a point charge Q at a distance R is $\frac{Q}{4\pi\epsilon_0 R}$ we find

$$\begin{aligned} V &= \int \frac{\mathrm{d}q}{4\pi\epsilon_0\sqrt{r^2+z^2}} \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0\sqrt{r^2+z^2}} \cdot \frac{q\,\mathrm{d}\phi}{2\pi} \\ &= \frac{q}{4\pi\epsilon_0\sqrt{r^2+z^2}}\,. \end{aligned} \tag{5}$$

Then, given that $\mathbf{E} = -\nabla V$, the x and y components are zero since V doesn't depend on those coordinates, and the z component is just $-\frac{\partial V}{\partial z}$. Thus, we recover the result previously derived by adding electric fields, i.e. equation (3).

For the solid disc, we sum up the potentials associated with each constituent ring, using equation (5):

$$\begin{aligned} V &= \int_0^{a} \frac{2\pi r\sigma\,\mathrm{d}r}{4\pi\epsilon_0\sqrt{r^2+z^2}} \\ &= \frac{\sigma}{2\epsilon_0}\int_0^a r\left(r^2+z^2\right)^{-1/2}\mathrm{d}r \\ &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+z^2}-z\right)\,. \end{aligned}$$

Again, there is no x or y dependence, so $\mathbf{E} = -\frac{\partial V}{\partial z}\hat{\mathbf{z}}$, which gives the same result as equation (4).

(d) When $z \ll a$, we should write (4) in terms of the small parameter $\frac{z}{a}$ to get an approximate form. So let's do that:

$$\begin{aligned} \mathbf{E} &= \frac{\sigma}{2\epsilon_0}\left(1 - z\left(a^2+z^2\right)^{-1/2}\right)\hat{\mathbf{z}} \\ &= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{a}\left(1+\frac{z^2}{a^2}\right)^{-1/2}\right)\hat{\mathbf{z}}\,. \end{aligned}$$

So, to lowest (i.e. zeroth) order, $\mathbf{E} \approx \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$. This is the field of an infinite slab of charge, which makes sense because when you're very close to the surface, the edge of the disc will appear far away.

If instead $z \gg a$, we should write the field in terms of $\frac{a}{z}$:

$$
\begin{aligned}
\mathbf{E} &= \frac{\sigma}{2\epsilon_0}\left(1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right)\hat{\mathbf{z}} \\
&\approx \frac{\sigma}{2\epsilon_0}\left[1-\left(1-\frac{a^2}{2z^2}\right)\right]\hat{\mathbf{z}} \\
&= \frac{\sigma a^2}{4\epsilon_0 z^2}\hat{\mathbf{z}} \\
&= \frac{Q}{4\pi\epsilon_0 z^2}\hat{\mathbf{z}}\,,
\end{aligned}
$$

where $Q = \pi a^2 \sigma$. This is the field of a point charge, as we'd expect because from a large distance we cannot see the detailed structure of the disc.

12. (a) The standard procedure for finding the capacitance of a capacitor with some particular geometry is to first calculate the electric field in the capacitor, then integrate to find the voltage between the plates, and finally use the definition of capacitance (i.e. charge stored per unit voltage).

For this cylindrical capacitor, let's use Gauss' law,

$$
\oiint_{\partial V} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0}\iiint_V \rho\,\mathrm{d}V\,,
$$

to find the field (note that ρ here is the *charge* density, not the mass density). Choosing a cylindrical Gaussian surface of radius r and length l, since by symmetry the electric field must be purely radial, we find that

$$
\begin{aligned}
E\cdot 2\pi r l &= \frac{2\pi a l \sigma}{\epsilon_0} \\
\therefore \quad E &= \frac{a\sigma}{\epsilon_0 r}\,.
\end{aligned} \tag{6}
$$

Next, we use this to find the potential difference:

$$
\begin{aligned}
V(b)-V(a) &= -\int_a^b \mathbf{E}\cdot \mathrm{d}\mathbf{r} \\
&= -\frac{a\sigma}{\epsilon_0}\int_a^b \frac{\mathrm{d}r}{r} \\
&= -\frac{a\sigma}{\epsilon_0}\ln\left(\frac{b}{a}\right)\,.
\end{aligned} \tag{7}
$$

The minus sign simply tells us that the inner plate (radius a) is at higher potential than the outer (radius b) if $\sigma > 0$, as expected. Of course, the polarity with which we charge the capacitor doesn't matter, and we simply take the modulus of the potential difference when calculating the capacitance:

$$C = \frac{2\pi a l \sigma}{\frac{a\sigma}{\epsilon_0} \ln\left(\frac{b}{a}\right)}$$
$$\therefore \quad \frac{C}{l} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)},$$

where $\frac{C}{l}$ is the capacitance per unit length.

(b) The maximum field E_{max} is attained at $r = a$, since the field is monotonically decreasing with r. From equation (6), the field there is $\frac{\sigma}{\epsilon_0}$. Then, from equation (7) we can write $V = |V(b) - V(a)| = aE_{\mathrm{max}} \ln\left(\frac{b}{a}\right)$.

(c) To maximise the potential difference V, we can just differentiate with respect to a and set the derivative to zero. Using the product rule:

$$\frac{\partial V}{\partial a} = aE_{\mathrm{max}} \cdot \frac{-b/a^2}{b/a} + E_{\mathrm{max}} \ln\left(\frac{b}{a}\right)$$
$$= E_{\mathrm{max}} \left[\ln\left(\frac{b}{a}\right) - 1\right].$$

This goes to zero when $\ln\left(\frac{b}{a}\right) = 1$, i.e. $a = b/e = 3.7$ mm. The maximum possible voltage is aE_{max}, where E_{max} is the breakdown electric field of air, 3×10^6 Vm^{-1}. Thus, $V = 11$ kV. At higher voltages, there will be a spark between the plates, so that a current flows between them and they become neutral.