Gravitational & Electromagnetic Fields: Week 1

Solutions by Ben Yelverton, May 2020

1. Given the spherical symmetry of the mass distribution, the natural starting point for finding its gravitational field is Gauss' law,

$$\oiint_{\partial V} \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G \iiint_V \rho \, \mathrm{d}V, \tag{1}$$

where ∂V is the surface bounding the volume V. By symmetry, $\mathbf{g}$ is pointing radially inwards everywhere, so that $\mathbf{g} \cdot \mathrm{d}\mathbf{S} = -g \, \mathrm{d}S$. Taking V to be a sphere of radius r, so that we can evaluate the integrals, we find that

$$-4\pi r^2 g = -4\pi G \times \frac{4}{3}\pi r^3 \rho,$$

since g does not vary across the surface and ρ is constant throughout the volume. Simplifying and putting this in vector form (recalling that the field is in the $-\hat{\mathbf{r}}$ direction) gives

$$\mathbf{g} = -\frac{4\pi}{3} G\rho r \, \hat{\mathbf{r}},$$

as required. For the next part, applying Newton's second law to the dropped particle (of mass m) gives

$$\begin{aligned} m\ddot{r} &= -mg \\ \therefore \quad \ddot{r} &= -\frac{4\pi}{3} G\rho r, \end{aligned}$$

which is the equation of motion of SHM with $\omega^2 = \frac{4\pi G\rho}{3}$. Thus, the period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= 2\pi \sqrt{\frac{3}{4\pi G\rho}}, \end{aligned} \tag{2}$$

which evaluates to $\sim$84 min given the density provided in the question. For a satellite orbiting the Earth, we can equate the gravitational and centripetal forces to obtain:

$$\begin{aligned} \frac{mv^2}{r} &= \frac{GM_\oplus m}{r^2} \\ \therefore \quad v &= \sqrt{\frac{GM_\oplus}{r}}. \end{aligned}$$

The satellite travels a distance $2\pi r$ in one orbital period P, so that

$$P = \frac{2\pi r}{v}$$
$$= 2\pi\sqrt{\frac{r^3}{GM_\oplus}}\,. \tag{3}$$

For a low Earth orbit $r \approx R_\oplus$, and since $M_\oplus = \frac{4\pi R_\oplus^3 \rho}{3}$ the RHS of equations (2) and (3) are the same, so that $P = T$. Why is that? Consider projecting the satellite's motion along the axis of the tunnel, which I'll call the y axis. The component of the force on the satellite along this axis is, simply from trigonometry,

$$F_y = -\frac{GM_\oplus m}{R_\oplus^2} \times \frac{y}{R_\oplus}$$
$$= -\frac{GM_\oplus my}{R_\oplus^3}$$
$$= -\frac{4\pi}{3}G\rho my.$$

But this is exactly the same as the equation of motion we found for the particle in the tunnel earlier! So, interestingly, the satellite's motion projected onto the tunnel is exactly the same as the motion of the particle in the tunnel. Note that this result depends on the assumption of uniform density, which we found led to SHM. If we accounted for the fact that the Earth is more dense towards the core, the particle's motion would no longer be SHM, while the satellite's projected motion will always be SHM because the satellite doesn't care about the radial mass distribution inside the Earth.

2. This can be approached using the principle of superposition, which tells us that

$$\mathbf{g} + \mathbf{g}_{a/4} = \mathbf{g}_a, \tag{4}$$

where $\mathbf{g}$ is the field of the object we're interested in, $\mathbf{g}_{a/4}$ is the field that would be produced by a homogeneous sphere of radius $a/4$ at the position of the cavity, and $\mathbf{g}_a$ is the field that would be produced by a sphere of radius a without the cavity. At points A and B, we are inside the large sphere (so that its field is given by the equation from question 1) but outside the imaginary smaller sphere (so that the standard GM/r^2 field is appropriate). Calling the distance from the centre of the large sphere r_1, the distance from the centre of the small sphere r_2, and the unit vector along the line connecting the centres of the two spheres $\hat{\mathbf{r}}$, equation (4) then becomes

$$
\begin{aligned}
\mathbf{g} &= -\frac{4\pi}{3}G\rho r_1\hat{\mathbf{r}} + \frac{G \times \frac{4\pi}{3}\left(\frac{a}{4}\right)^3\rho}{r_2^2}\,\hat{\mathbf{r}} \\
&= \frac{4\pi G\rho}{3}\left(\frac{a^3}{64r_2^2} - r_1\right)\hat{\mathbf{r}} \\
\therefore \quad g &\propto \frac{a^3}{64r_2^2} - r_1 . \qquad (5)
\end{aligned}
$$

At point A, $r_1 = a$ while $r_2 = 3a/8$. At point B, r_1 is still a but $r_2 = 2a - 3a/8 = 13a/8$. Thus from equation (5) we get $g_\mathrm{A} = \mathrm{const} \times \frac{8}{9}$ and $g_\mathrm{B} = \mathrm{const} \times \frac{168}{169}$, so that $g_\mathrm{A}/g_\mathrm{B} = 169/189$, as required.

3. (a) This is a standard dimensional analysis problem. Let's start by assuming that

$$\Delta g \propto G^\alpha d^\beta \rho^\gamma .$$

Comparing dimensions gives

$$[L][T]^{-2} = \left([F][L]^2[M]^{-2}\right)^\alpha \left([L]\right)^\beta \left([M][L]^{-3}\right)^\gamma .$$

Recalling that $[F] = [M][L][T]^{-2}$, the above equation reduces to

$$[L][T]^{-2} = [M]^{\gamma-\alpha}[L]^{3\alpha-3\gamma+\beta}[T]^{-2\alpha} ,$$

and the requirement for both sides to have the same dimensions gives three simultaneous equations that can be solved to yield $\alpha = \beta = \gamma = 1$, as required.

(b) Given that the room is extremely large, let's model the layer of lead on the floor as an infinite slab. By symmetry, the slab's gravitational field $\mathbf{\Delta g}$ must be normal to the slab everywhere and cannot vary with horizontal position. We can use Gauss' law, equation (1), with the surface chosen for convenience to be a cylinder, centered on the slab and with its axis normal to the slab, and with a height larger than d. Denoting the surface area of the curved face as A, we obtain

$$
\begin{aligned}
-A\Delta g \times 2 &= -4\pi G \times Ad\rho \\
\therefore \quad \Delta g &= 2\pi G d\rho .
\end{aligned}
$$

The surface integral on the LHS only gets a contribution from the circular faces because $\mathbf{\Delta g}$ is parallel to the curved faces; the minus sign arises because the vector surface area points outward by definition while $\mathbf{\Delta g}$ points inward. The RHS is simply the necessary pre-factor multiplied by the mass in the slab enclosed by the cylinder. Note that the height of the cylinder doesn't come into this equation, implying that the field is uniform everywhere.

(c) Since $T = 2\pi\sqrt{l/g}$, increasing g will decrease T. Thus, the pendulum clock will have oscillated through more cycles than it should have done, and appears fast. The period in the presence of the lead floor is

$$\begin{aligned}
T + \Delta T &= 2\pi\sqrt{\frac{l}{g + \Delta g}} \\
&= 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\Delta g}{g}\right)^{-1/2} \\
&= T\left(1 + \frac{\Delta g}{g}\right)^{-1/2} \\
&\approx T\left(1 - \frac{\Delta g}{2g}\right) \\
\therefore \quad \frac{\Delta T}{T} &\approx -\frac{\Delta g}{2g} \\
2\pi G d\rho &\approx -2g\frac{\Delta T}{T} \\
d &\approx -\frac{g}{\pi G\rho}\frac{\Delta T}{T}\,.
\end{aligned}$$

Given that $\Delta T = 1$ s and $T = 1$ yr, and using the density provided in the question, we find that $d = 13$ cm.

4. The gravitational potential energy of the system is the work done against gravity in bringing the system into its current configuration, starting from infinite separation so that the masses are initially not interacting. For the two-mass system in this question,

$$\begin{aligned}
\Phi &= -\int_{\infty}^{r} -\frac{Gm_1m_2}{r^2}\,\mathrm{d}r \\
&= -\frac{Gm_1m_2}{r}\,.
\end{aligned}$$

We need to be careful about the signs here: the minus sign in front of the integral is there because we want the work done *against* rather than *by* gravity, and the one in the integrand is there because the gravitational force points inward, in the $-\hat{\mathbf{r}}$ direction. Φ is negative because we define the zero of potential to be at infinite separation by convention, and gravity is always attractive so energy is always released when two masses are brought together.

(a) The total potential energy follows from the principle of superposition - we can just add the contributions from the two masses. The distances to the masses at $(\pm a,\, 0,\, 0)$ are $|a \mp x|$, so that the potential energies due to each mass individually are

$$\Phi_\pm = -\frac{GmM}{|a \mp x|}\,,$$

and Φ is simply $\Phi_+ + \Phi_-$. To sketch Φ, note that it's simply the sum of two shifted $-1/|x|$ potentials, so you can begin by sketching the two terms individually and then joining up the curve at $x = 0$, where $\Phi = -2GmM/a$. Make sure to sketch on the asymptotes, mark on the axis crossing value and label your axes as always!

To find the derivative, we need to use the chain rule as follows, noting that we can usefully rewrite $|a \pm x|$ as $\sqrt{(a \pm x)^2}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}|a \pm x|^{-1} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(a \pm x)^2\right]^{-1/2} \\
&= -\frac{1}{2}\left[(a \pm x)^2\right]^{-3/2} \times 2(a \pm x) \times (\pm 1) \\
&= \mp\frac{(a \pm x)}{|a \pm x|^3} \\
\therefore \quad \frac{\mathrm{d}\Phi}{\mathrm{d}x} &= -GmM\left[-\frac{a + x}{|a + x|^3} + \frac{a - x}{|a - x|^3}\right].
\end{aligned}$$

At $x = 0$ we find that this is zero, hence it's an equilbrium point. Note that if you ignore the modulus signs, you'll get an answer that works for $|x| < a$, and is thus fine for the purposes of this question, but has the wrong sign for $|x| > a$. Now let's find the second derivative. From above, and using the chain and product rules:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{(a \pm x)}{|a \pm x|^3} &= \left[(a \pm x)\frac{\mathrm{d}}{\mathrm{d}x}\left[(a \pm x)^2\right]^{-3/2}\right] \pm \frac{1}{|a \pm x|^3} \\
&= \left[(a \pm x) \times -\frac{3}{2}\left[(a \pm x)^2\right]^{-5/2} \times 2(a \pm x) \times (\pm 1)\right] \pm \frac{1}{|a \pm x|^3} \\
&= \mp 3\frac{(a \pm x)^2}{|a \pm x|^5} \pm \frac{1}{|a \pm x|^3} \\
\therefore \quad \frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} &= -GmM\left[3\frac{(a + x)^2}{|a + x|^5} - \frac{1}{|a + x|^3} + 3\frac{(a - x)^2}{|a - x|^5} - \frac{1}{|a - x|^3}\right] \\
&= -2GmM\left[\frac{1}{|a + x|^3} + \frac{1}{|a - x|^3}\right].
\end{aligned}$$

At $x = 0$, this evaluates to $-4GmM/a^3$, which is less than 0, hence the stationary point there is a maximum and the equilibrium is unstable.

(b) If the particle is now constrained to move along the y axis, it is always a distance $\sqrt{a^2 + y^2}$ from both of the masses, so that the potential is simply

$$\Phi = -\frac{2GmM}{\sqrt{a^2 + y^2}}\,.$$

To sketch this, take limits at small and large y. When $|y| \gg a$, $\Phi \propto 1/|y|$. When $|y| \ll a$, a binomial expansion finds that $\Phi \approx \frac{2GmM}{a}(\frac{y^2}{2a^2} - 1)$, i.e. the curve is approximately parabolic near the origin. The equilibrium is thus stable, since there's a minimum in Φ. The physical reason for the stability is that the y components of the forces from both masses always point towards the origin, i.e. the force is restoring. Previously, when the motion was along the x axis, the equilibrium was unstable because the particle, once displaced, would feel a net force towards the mass it is closer to rather than back towards the origin.

5. I'm going to rename the question's parameter a to α here, to avoid confusion with the semi-major axis!

From Kepler's first law, the orbit must be an ellipse with the centre of the moon at one focus. The impulse doesn't change the direction of motion, so the velocity is still perpendicular to the radius vector afterwards. Thus, the orientation of the orbit is such that the apocentre (point of greatest distance from the centre of the moon) is where the impulse was applied. The pericentre is at the surface of the moon. You should make labelled sketches of the initial and final orbits here for clarity.

Since we're asked for the shape of the orbit, we should really give its eccentricity e. To do this, recall that the apocentre and pericentre distances (usually called Q and q respectively) of an elliptical orbit with semi-major axis a are given by $a(1 \pm e)$. Here, we know that $Q = \alpha r$ and $q = r$ from the discussion above. Thus, $a(1 + e) = \alpha r$ and $a(1 - e) = r$. We can divide these two equations to eliminate r, and rearrange to find $e = \frac{\alpha-1}{\alpha+1}$.

To find α, let's first consider the initial circular orbit. Equating the centripetal and gravitational forces gives

$$\frac{mv^2}{\alpha r} = \frac{GMm}{(\alpha r)^2}\,,$$

where the symbols have their usual meanings. Thus,

$$\alpha = \frac{GM}{rv^2}\,. \tag{6}$$

We can find an expression for v^2 by considering the final orbit, given (from the question) that the speed at apocentre is $v/2$. The usual approach for orbit problems is to conserve energy and angular momentum at apocentre and pericentre. If we denote the pericentre speed by w, energy conservation gives

$$\frac{1}{2}m\left(\frac{v}{2}\right)^2 - \frac{GMm}{\alpha r} = \frac{1}{2}mw^2 - \frac{GMm}{r}\,, \tag{7}$$

while angular momentum conservation gives

$$m\alpha r\frac{v}{2} = mrw\,. \tag{8}$$

So, to find v^2 we can use (8) to substitute for w in (7), then rearrange to get

$$v^2 = \frac{8GM}{r\alpha(1+\alpha)}\,. \tag{9}$$

Finally, put this into equation (6) and rearrange to find $\alpha = 7$ (and hence $e = 0.75$).

6. Kepler's third law relates periods and semi-major axes as follows:

$$T^2 = \frac{4\pi^2}{GM}\,a^3.$$

This question tells us T and a, so we can rearrange the above equation for M, finding that it's $\sim 8.1 \times 10^{36}$ kg or $\sim 4.1 \times 10^6\ M_\odot$. To find the maximum velocity, remember that this velocity will always be attained at pericentre (closest approach) where the distance $r = a(1-e)$. As usual, let's equate the energies at apocentre and pericentre:

$$\frac{1}{2}mv_a^2 - \frac{GMm}{a(1+e)} = \frac{1}{2}mv_p^2 - \frac{GMm}{a(1-e)}\,, \tag{10}$$

and the angular momenta:

$$mv_a a(1+e) = mv_p a(1-e)\,, \tag{11}$$

where v_a and v_p are the velocities at apocentre at pericentre respectively. We can use (11) to substitute for v_a in (10), then rearrange to make v_p the subject, finding

$$v_p = \sqrt{\frac{GM}{a}\cdot\frac{1+e}{1-e}}\,,$$

which given the numbers in the question and the mass we just calculated evaluates to $7600\ \mathrm{km\,s^{-1}}$.

Exam question: 2013 A2

- The astronaut has a gravitational potential energy of $-\frac{GMm}{r}$, where the symbols have their usual meanings. So, to escape the asteroid she needs to gain this much energy by jumping. The energy she gains from the jump can be estimated from the height h of a typical jump on Earth, which is simply mgh. Additionally, note that $g = GM_\oplus/r_\oplus^2$ (a standard result that can be derived quickly from Gauss' law). Finally, since the asteroid's mean density is equal to Earth's, we know that its mass $M = M_\oplus(r/r_\oplus)^3$. Thus, we infer that

$$\begin{aligned} mgh &> \frac{GMm}{r} \\ \frac{GM_\oplus mh}{r_\oplus^2} &> \frac{GMm}{r} \\ \frac{M_\oplus h}{r_\oplus^2} &> \frac{M}{r} \\ \frac{M_\oplus h}{r_\oplus^2} &> \frac{M_\oplus r^2}{r_\oplus^3} \\ \therefore \quad r &< \sqrt{hr_\oplus} \,. \end{aligned}$$

The 'official' answer provided is 2.5 km, and given that $r_\oplus = 6400$ km this suggests that the examiners used $h \approx 1$ m, which seems a bit high. However, any sensible (i.e. not 1 cm and not 10 m) estimate should gain full marks.

Exam question: 2017 A4

- We know that $T = 2\pi\sqrt{l/g}$, so let's first find g at a distance $r \leq r_\oplus$ from the centre of the Earth. In fact, we've already done this (using Gauss' law) in question 1 of the course examples: $g = \frac{4\pi G\rho r}{3}$. Since $T \propto 1/\sqrt{g}$ and $g \propto r$, we have $T \propto 1/\sqrt{r}$. Thus,

  $$T_{\text{mine}} = \sqrt{\frac{6371}{6370}} T_{\text{surface}} \,.$$

  The clock in the mine thus oscillates fewer times than that on the surface, so that it appears to be behind. How do we calculate the time difference after an hour? The apparent time t_{app} shown by each clock will simply be proportional to the number of oscillations N since $t = 0$, i.e. $t_{\text{app}} \propto N$. Additionally, $N = t/T$, i.e. $N \propto 1/T$, so $t_{\text{app}} \propto 1/T$. Putting all this together gives

  $$\begin{aligned} \frac{t_{\text{app, mine}}}{t_{\text{app, surface}}} &= \frac{T_{\text{surface}}}{T_{\text{mine}}} \\ t_{\text{app, mine}} &= \sqrt{\frac{6370}{6371}} t_{\text{app, surface}} \\ \therefore \quad t_{\text{app, mine}} - t_{\text{app, surface}} &= \left(\sqrt{\frac{6370}{6371}} - 1\right) t_{\text{app, surface}} \,. \end{aligned}$$

  When $t_{\text{app, surface}} = 1$ h, we find that this evaluates to -0.28 s, i.e. the clock in the mine is 0.28 s behind.