Gravitational & Electromagnetic Fields: Week 4

Solutions by Ben Yelverton, May 2020

19. The inductance L is defined as $L = \frac{\Phi}{I}$, where Φ is the magnetic flux linked through a circuit and I is the current flowing through it. For this question, let's assume that the currents in the wires are going in opposite directions so that they form a circuit (albeit one that extends out to infinity). From Ampère's law, we know that the field produced by a single straight wire (outside the wire itself) is simply

$$B_{\text{wire}} = \frac{\mu_0 I}{2\pi r} \,.$$

To find Φ in this case, we need to integrate the B field due to both wires over the area enclosed by a unit length of the circuit. Since we're told that $2d \gg a$, let's ignore the contribution to the flux from the region inside the wires themselves. By superposition, the total field at a distance r from the centre of one of the wires is

$$B(r) = \frac{\mu_0 I}{2\pi}\left[\frac{1}{r} + \frac{1}{2d - r}\right] ,$$

which means that the flux through a length l of the circuit is given by

$$\begin{aligned}
\Phi &= \int_a^{2d-a} B(r) l \, \mathrm{d}r \\
\therefore \quad \frac{\Phi}{l} &= \frac{\mu_0 I}{2\pi} \int_a^{2d-a} \left[\frac{1}{r} + \frac{1}{2d - r}\right] \mathrm{d}r \\
&= \frac{\mu_0 I}{2\pi}\Big[\ln(r) - \ln(2d - r)\Big]_a^{2d-a} \\
&= \frac{\mu_0 I}{2\pi} \times 2\ln\left(\frac{2d - a}{a}\right) \\
&\approx \frac{\mu_0 I}{\pi} \ln\left(\frac{2d}{a}\right) ,
\end{aligned}$$

so that the inductance per unit length is

$$\frac{L}{l} = \frac{\Phi}{Il} \approx \frac{\mu_0}{\pi} \ln\left(\frac{2d}{a}\right) .$$

20. (a) Let's begin by finding the magnetic field inside the solenoid. To do this, draw a rectangular Ampèrian loop such that the bottom side (of length x) is inside the solenoid and the top is outside. By symmetry, inside the solenoid the field B is constant and directed along the axis of the solenoid. Thus, when we evaluate the Ampèrian integral $\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l}$, we get Bx from the bottom side, zero from the perpendicular sides (since the dot product is zero along those sides), and we can take the top side arbitrarily far from the solenoid to where any field must be negligibly small. If there are n turns per unit length, our loop encloses nx currents that are all equal to I. Thus, Ampère's law gives

$$\begin{aligned} Bx &= \mu_0 n x I \\ \therefore \quad B &= \mu_0 n I \,. \end{aligned}$$

To get the self-inductance, note that the magnetic flux is linked through the solenoid nl times:

$$\begin{aligned} L &= \frac{\Phi}{I} \\ &= \frac{nl \times \mu_0 n I \times \pi r^2}{I} \\ &= \mu_0 \pi r^2 n^2 l \,. \end{aligned}$$

With the values given in the question this evaluates to 85 μH. The rate of change of current required to produce a particular emf $\mathcal{E}$ is given by Faraday's law (ignoring signs):

$$\begin{aligned} \mathcal{E} &= \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ &= L\frac{\mathrm{d}I}{\mathrm{d}t} \\ \therefore \quad \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\mathcal{E}}{L} \,, \end{aligned}$$

which evaluates to 7.0×10^6 As^{-1}.

(b) The energy W stored in an inductor can be calculated by considering the work that had to be done against the inductor's back emf $\mathcal{E}$ to build up a current I in increments dI. The work done in moving a charge dq through this emf is

$$\begin{aligned} \mathrm{d}W &= \mathcal{E} \times \mathrm{d}q \\ &= L\frac{\mathrm{d}I}{\mathrm{d}t} \times I\mathrm{d}t \\ &= LI\,\mathrm{d}I \\ \therefore \quad W &= \int_0^I LI'\,\mathrm{d}I' \\ &= \frac{1}{2}LI^2 \,. \end{aligned}$$

In general, magnetic energy density $u = \frac{B^2}{2\mu_0}$. Therefore, given the B fields from question 17, we can simply write down that

$$u = \begin{cases} \dfrac{\mu_0 I^2 r^2}{8\pi^2 a^4} & \text{for } 0 < r < a\,; \\ \dfrac{\mu_0 I^2}{8\pi^2 r^2} & \text{for } a < r < b\,; \\ \dfrac{\mu_0 I^2}{8\pi^2 r^2}\left(\dfrac{c^2 - r^2}{c^2 - b^2}\right)^2 & \text{for } b < r < c\,; \\ 0 & \text{for } r > c\,. \end{cases}$$

Then, the magnetic energy stored in a unit-length cylindrical annulus (for $0 < r < b$, as requested) is as follows, since the volume of the annulus is $2\pi r \mathrm{d}r \times 1$:

$$\mathrm{d}W = \begin{cases} \dfrac{\mu_0 I^2 r^3 \mathrm{d}r}{4\pi a^4} & \text{for } 0 < r < a\,; \\ \dfrac{\mu_0 I^2 \mathrm{d}r}{4\pi r} & \text{for } a < r < b\,. \end{cases}$$

So, the magnetic energy stored per unit length in this region is

$$\begin{aligned} W &= \int_0^a \frac{\mu_0 I^2 r^3 \mathrm{d}r}{4\pi a^4} + \int_a^b \frac{\mu_0 I^2 \mathrm{d}r}{4\pi r} \\ &= \frac{\mu_0 I^2}{16\pi}\left[1 + 4\ln\frac{b}{a}\right]. \end{aligned}$$

Equating this with the general expression $W = \frac{1}{2}LI^2$ derived earlier shows that the self-inductance per unit length is

$$L = \frac{\mu_0}{8\pi}\left[1 + 4\ln\frac{b}{a}\right].$$

21. (a) The flux through the loop in the diagram is decreasing, since $\theta < 90°$ and is increasing with time, and $\Phi = \mathbf{B}\cdot\mathbf{A}$. Lenz's law tells us that the magnetic field produced by the induced current will try to resist the change in flux that induced it. Here, that means that the horizontal component of the induced field must be pointing to the right, so that the field tries to restore the lost flux. From the right hand rule, the current therefore flows in the direction QPTS. From $\mathbf{F} = I\,\mathbf{l}\times\mathbf{B}$, the force on side QP acts upwards, PT out of the page, TS downwards, and SQ into the page. Thus, there is a net torque which is clockwise about the vertical axis – i.e. it opposes the direction of ω, as we would expect from Lenz's law. Finally, the energy that keeps the coil rotating at constant ω must come from some external agent (e.g. the coil could be attached to a motor that guarantees constant angular velocity, or someone could be manually turning it round).

(b) $\Phi = \mathbf{B}\cdot\mathbf{A}$, where A is the vector area of the loop. Thus, here we have

$$\begin{aligned} \Phi &= Ba^2 \cos\theta \\ &= Ba^2 \cos\omega t \\ \therefore \quad \mathcal{E} &= \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right| = \omega a^2 B \sin\omega t \\ \therefore \quad I &= \frac{\mathcal{E}}{R} = \frac{\omega a^2 B \sin\omega t}{R}\,, \end{aligned}$$

as required.

(c) On sides QS and PT, the field is perpendicular to the current, so that

$$\begin{aligned} F &= IaB \\ &= \frac{\omega a^3 B^2 \sin\omega t}{R} \\ \therefore \quad G &= 2 \times \frac{a}{2}\sin\omega t \times F \\ &= \frac{\omega a^4 B^2 \sin^2\omega t}{R}\,. \end{aligned}$$

In order to keep the coil rotating at constant angular speed, an equal and opposite couple must be applied.

(d) The power produced by this couple is

$$\begin{aligned} P &= G\omega \\ &= \frac{\omega^2 a^4 B^2 \sin^2\omega t}{R}\,. \end{aligned}$$

The power dissipated as heat in the coil is

$$\begin{aligned} P' &= I^2 R \\ &= \left[\frac{\omega a^2 B \sin\omega t}{R}\right]^2 \times R \\ &= \frac{\omega^2 a^4 B^2 \sin^2\omega t}{R}\,. \end{aligned}$$

Thus, $P = P'$, as is required for a constant angular velocity.

22. First let's find the speed v of the ions as they enter the magnetic field. This comes from consideration of energy:

$$qV = \frac{1}{2}mv^2$$
$$\therefore \quad v = \sqrt{\frac{2qV}{m}}\,.$$

The required relationship between the ions' masses m and positions along the screen x comes from considering the circular motion that takes place in the magnetic field. Noting that the radius of the circular path is $x/2$, we find that

$$\frac{mv^2}{x/2} = qvB$$
$$\therefore \quad \frac{2mv}{x} = qB\,.$$

Then we simply take the expression for v derived above, substitute and rearrange to find that

$$m = \frac{B^2q}{8V}\,x^2\,.$$

Clearly this implies that

$$x = \sqrt{\frac{8mV}{B^2q}}$$
$$\therefore \quad \Delta x = \sqrt{\frac{8V}{B^2q}}\left(\sqrt{m_2} - \sqrt{m_1}\right),$$

where the m_i are the masses of the two types of ion. Given the values in the question, we find $\Delta x = 7.91$ mm.

23. The fact that the electrons are not deflected means that $\mathbf{F}_{\text{elec}} = -\mathbf{F}_{\text{mag}}$. Thus, to draw a diagram of $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{v}$, we need to think about the vector equations $\mathbf{F}_{\text{elec}} = q\mathbf{E}$ and $\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}$. So, for example, $\mathbf{E}$ could be pointing down, $\mathbf{B}$ into the page and $\mathbf{v}$ to the right. The speed of the electrons can be found from $qE = qvB$, which gives $v = E/B = 1.0 \times 10^7$ ms^{-1}. We can then get the charge-to-mass ratio from the radius R of the circular motion in the absence of the $\mathbf{E}$ field:

$$\frac{mv^2}{R} = qvB$$
$$\therefore \quad \frac{q}{m} = \frac{v}{BR}\,,$$

which given the speed we just calculated is 1.8×10^{11} C kg^{-1}.

If we pass protons through instead, then if they have the same velocity as the electrons then they will still move in the same straight line because the condition $qE = qvB$ still holds. If instead they have the same kinetic energy as the electrons, their speed will be lower (since their mass is higher), which means the beam will be deflected towards the direction of the $\mathbf{E}$ field.

24. To verify the given solutions for $\mathbf{E}$ and $\mathbf{B}$, we just need to show that they satisfy Maxwell's equations in free space. So let's do that:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \partial_x \left[E_x \cos(kz - \omega t)\right] + 0 + 0 \\ &= 0 \quad \checkmark\end{aligned}$$

$$\begin{aligned}\nabla \times \mathbf{E} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ E_x \cos(kz - \omega t) & 0 & 0 \end{vmatrix} \\ &= \partial_z \left[E_x \cos(kz - \omega t)\right] \hat{\mathbf{j}} \\ &= -kE_x \sin(kz - \omega t)\, \hat{\mathbf{j}}\end{aligned}$$

$$\begin{aligned}-\frac{\partial \mathbf{B}}{\partial t} &= -\partial_t \left[B_y \cos(kz - \omega t)\right] \hat{\mathbf{j}} \\ &= -\omega B_y \sin(kz - \omega t)\, \hat{\mathbf{j}}\end{aligned}$$

$$\therefore \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{if} \quad kEx = \omega B_y \quad \text{i.e.} \quad \frac{E_x}{B_y} = \frac{\omega}{k} \quad \checkmark$$

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 + \partial_y \left[B_y \cos(kz - \omega t)\right] + 0 \\ &= 0 \quad \checkmark\end{aligned}$$

$$\begin{aligned}\nabla \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y \cos(kz - \omega t) & 0 \end{vmatrix} \\ &= -\partial_z \left[B_y \cos(kz - \omega t)\right] \hat{\mathbf{i}} \\ &= kB_y \sin(kz - \omega t)\, \hat{\mathbf{i}}\end{aligned}$$

$$\begin{aligned}\mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0\epsilon_0\, \partial_t \left[E_x \cos(kz - \omega t)\right] \hat{\mathbf{i}} \\ &= \mu_0\epsilon_0\, \omega E_x \sin(kz - \omega t)\, \hat{\mathbf{i}}\end{aligned}$$

$$\therefore \quad \nabla \times \mathbf{B} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{if} \quad kB_y = \mu_0\epsilon_0\, \omega E_x \quad \text{i.e.} \quad \frac{\omega^2}{k^2} = \frac{1}{\mu_0\epsilon_0} \quad \checkmark$$

Physically, this solution represents a transverse electromagnetic wave travelling in the $+z$ direction with speed $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$. The electric and magnetic fields oscillate in phase in two perpendicular planes. If the arguments of the cos terms were instead $kz + \omega t$, the solution would represent a wave travelling in the $-z$ direction, whose properties are otherwise the same.

Exam question: 2013 E14

- Ampère's and Faraday's laws are

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enclosed}}\,,$$

$$|\mathcal{E}| = \left| \frac{\mathrm{d}\Phi}{\mathrm{d}t} \right|\,,$$

where $\mathbf{B}$ is the magnetic field, $\mathrm{d}\mathbf{l}$ is a line element, μ_0 is the permeability of free space, I_{enclosed} is the current flowing through a surface enclosed by the integration path, $\mathcal{E}$ is an induced emf, Φ is the magnetic flux linked through a circuit and t is time. For a long straight wire, Ampère's law immediately gives

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}\,,$$

where $\hat{\boldsymbol{\phi}}$ is the unit vector in the azimuthal direction. Finally, the force on a wire element is

$$\mathrm{d}\mathbf{F} = I\,\mathrm{d}\mathbf{l} \times \mathbf{B}\,.$$

The force is perpendicular to both the direction of current flow and the magnetic field, and has magnitude $I\,|\mathrm{d}\mathbf{l}|\,|\mathbf{B}|\sin\theta$, where θ is the angle between $\mathrm{d}\mathbf{l}$ and $\mathbf{B}$.

- The total magnetic flux through the coil is simply the surface integral of the magnetic field over the surface area covered by the coil, multiplied by the number of turns:

$$\begin{aligned}\Phi &= N \iint \mathbf{B} \cdot \mathrm{d}\mathbf{S} \\ &= N \int_s^{s+a} \frac{\mu_0 I}{2\pi r}\, a\,\mathrm{d}r \\ &= \frac{\mu_0 N a I}{2\pi} \ln\left(\frac{s+a}{s}\right)\end{aligned}$$

- The current I_c in the coil can be obtained from Faraday's law and then Ohm's law. It could be argued that the equation given in the question should have an extra minus sign from Lenz's law, but the sign of the current ultimately depends on whether we define clockwise currents as positive or negative, and no such definition is given in the question.

$$\begin{aligned}
I_c &= \frac{\mathcal{E}}{R} \\
&= \frac{1}{R}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= \frac{\mu_0 N a \partial_t \left[I_0 \sin\omega t\right]}{2\pi R} \ln\left(\frac{s+a}{s}\right) \\
&= \frac{\mu_0\, \omega N a I_0 \cos\omega t}{2\pi R} \ln\left(\frac{s+a}{s}\right),
\end{aligned}$$

as required.

- Since there are N turns, the magnitude of the force on the top side of the coil is

$$\begin{aligned}
F_{\text{top}} &= N I_c a B(s) \\
&= N \times \frac{\mu_0\, \omega N a I_0 \cos\omega t}{2\pi R} \ln\left(\frac{s+a}{s}\right) \times a \times \frac{\mu_0 I_0 \sin\omega t}{2\pi s} \\
&= \frac{\mu_0^2\, \omega N^2 a^2 I_0^2 \sin\omega t \cos\omega t}{4\pi^2 R s} \ln\left(\frac{s+a}{s}\right).
\end{aligned}$$

The net force on the two sides normal to the wire is zero because these sides are always subjected to the same magnetic field, and have equal but opposite currents and thus feel equal but opposite forces. To get the total force on the coil, note that the force on the bottom side is almost the same as that on the top, except in the opposite direction (since the current is opposite) and with a factor of $\frac{1}{s+a}$ instead of $\frac{1}{s}$ from the magnetic field term:

$$\begin{aligned}
F_{\text{total}} &= N I_c a B(s) - N I_c a B(s+a) \\
&= \frac{\mu_0^2\, \omega N^2 a^2 I_0^2 \sin\omega t \cos\omega t}{4\pi^2 R} \ln\left(\frac{s+a}{s}\right)\left[\frac{1}{s} - \frac{1}{s+a}\right] \\
&= \frac{\mu_0^2\, \omega N^2 a^3 I_0^2 \sin\omega t \cos\omega t}{4\pi^2 R s(s+a)} \ln\left(\frac{s+a}{s}\right).
\end{aligned}$$

- The average power is

$$\begin{aligned}
\langle P \rangle &= \langle I_c^2 \rangle R \\
&= \frac{\mu_0^2\, \omega^2 N^2 a^2 I_0^2 \langle \cos^2\omega t\rangle}{4\pi^2 R^2} \left[\ln\left(\frac{s+a}{s}\right)\right]^2 \times R \\
&= \frac{\mu_0^2\, \omega^2 N^2 a^2 I_0^2}{8\pi^2 R} \left[\ln\left(\frac{s+a}{s}\right)\right]^2.
\end{aligned}$$