Gravitational & Electromagnetic Fields: Week 3

Solutions by Ben Yelverton, May 2020

13. First, we need to establish the capacitance of a capacitor with circular plates. To do this, we can approximate the plates as infinite in extent and apply Gauss' law as we would for an infinite slab of charge. Writing the area of the end part of the Gaussian surface as A and the field due to *one* plate as E, we get:

$$\begin{aligned} 2AE &= \frac{A\sigma}{\epsilon_0} \\ \therefore \quad E &= \frac{\sigma}{2\epsilon_0} \\ &= \frac{Q}{2\pi r^2 \epsilon_0}\,. \end{aligned}$$

By superposition, the field between the plates is constant and equal to $2E$, so the potential difference is simply $V = 2Ed$. Thus,

$$\begin{aligned} C &= \frac{Q}{V} \\ &= Q \cdot \frac{2\pi r^2 \epsilon_0}{2Qd} \\ &= \frac{\epsilon_0 \pi r^2}{d}\,. \end{aligned}$$

The capacitances of parallel capacitors add, so we can write:

$$\begin{aligned} C_{\text{tot}} &= C + C' \\ \frac{Q}{V} &= \frac{\epsilon_0 \pi r^2}{d} + C' \\ \frac{1}{V} &= \frac{\epsilon_0 \pi r^2}{Q} \cdot \frac{1}{d} + \frac{C'}{Q}\,. \end{aligned}$$

Thus, a plot of $1/V$ against $1/d$ should give a straight line, as indeed it does. The gradient is $\frac{\epsilon_0 \pi r^2}{Q}$, from which we can calculate $Q \approx 2.6$ nC, while the intercept is $\frac{C'}{Q}$, which leads to $C' \approx 77$ pF.

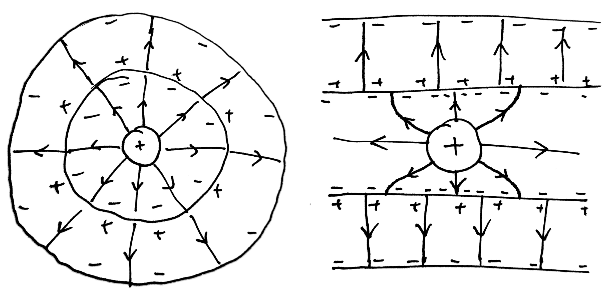
14. (a) From Gauss' law, the field of a charged sphere is $\frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$. Since $\mathbf{E} = -\nabla V$, the potential V on the surface of the sphere is

$$
\begin{aligned}
V &= -\int_{\infty}^{R} \mathbf{E} \cdot \mathrm{d}\mathbf{r} \\
&= -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^{R} r^{-2} \mathrm{d}r \\
&= \frac{Q}{4\pi\epsilon_0 R} \, .
\end{aligned}
$$

Then, $C = Q/V = 4\pi\epsilon_0 R$, as required.

(b) These diagrams require some careful thought. The key features are:

- The surface of a conductor is an equipotential, which means $\mathbf{E}$ has to be normal to all surfaces. In particular this means that the field lines are curved for $r < a$.
- On the inner cylinder, there's no *net* charge, but the inner surface and outer surfaces have opposite charges.
- The inner surface of the inner cylinder will have a maximum in charge density at the point closest to the sphere, since the sphere's field is strongest there.
- Because the electrons distribute themselves along the inner surface in such a way as to cancel the field of the sphere, the electric field is zero inside the conductor. Thus, the outer surface cannot communicate with the inner, and its charge distribution is uniform.
- On the outer cylinder, the inner surface has the opposite charge to the sphere. However, since this cylinder is Earthed (i.e. electrons are free to move between the cylinder and the Earth), the outer surface will be neutral.



The field strength for $a < r < b$ follows from Gauss' law, with a Gaussian cylinder:

$$
\begin{aligned}
2\pi r l E &= \frac{Q}{\epsilon_0} \\
&= \frac{4\pi\epsilon_0 R V}{\epsilon_0} \\
\therefore \quad E &= \frac{2RV}{rl} \, .
\end{aligned}
$$

To find the potential of the inner cylinder, note that

$$
\begin{aligned}
V(b) - V(a) &= -\int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{r} \\
0 - V(a) &= -\int_a^b \frac{2RV}{rl} \mathrm{d}r \\
\therefore \quad V(a) &= \frac{2RV}{l} \ln\left(\frac{b}{a}\right) .
\end{aligned}
$$

If the sphere were moved off-axis, the electric field for $r < a$ would no longer be axisymmetric. However, because the electrons would redistribute themselves across the inner cylinder's inner surface so as to cancel this field, the charge distribution on the outer surface would remain uniform, so that the situation for $r > a$, and hence our answers to this question, would remain unchanged.

15. Here we simply model the tip of the whisker as a sphere. The electric field close to the tip is then simply $E = \frac{Q}{4\pi\epsilon_0 R^2}$. What's Q? From the capacitance derived in question 14(a), $Q = CV = 4\pi\epsilon_0 RV$. Thus, we have $E = V/R = 10^6$ Vm^{-1}.

16. We need to add the electric fields of the two charges together. Denoting the vector from the charge $+q$ to a point in space as $\mathbf{r}_1$, and that from the charge $-q$ as $\mathbf{r}_2$:

$$
\mathbf{E} = \frac{q}{4\pi\epsilon_0 r_1^3}\mathbf{r}_1 - \frac{q}{4\pi\epsilon_0 r_2^3}\mathbf{r}_2 .
$$

At the point $(0, 0, d)$, we have:

$$
\begin{aligned}
\mathbf{r}_1 &= \left(0, 0, d - \frac{a}{2}\right) , \\
\mathbf{r}_2 &= \left(0, 0, d + \frac{a}{2}\right) ,
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\left(d - \frac{a}{2}\right)^2} - \frac{1}{\left(d + \frac{a}{2}\right)^2} \right] \hat{\mathbf{z}} \\
&= \frac{q}{4\pi\epsilon_0 d^2} \left[\left(1 - \frac{a}{2d}\right)^{-2} - \left(1 + \frac{a}{2d}\right)^{-2} \right] \hat{\mathbf{z}} \\
&\approx \frac{q}{4\pi\epsilon_0 d^2} \left[\left(1 + \frac{a}{d}\right) - \left(1 - \frac{a}{d}\right) \right] \hat{\mathbf{z}} \\
&\approx \frac{qa}{2\pi\epsilon_0 d^3} \, \hat{\mathbf{z}} .
\end{aligned}
$$

If instead we look at the dipole from the point $(d, 0, 0)$, we have:

$$\begin{aligned}
\mathbf{r}_1 &= \left(d, 0, -\frac{a}{2}\right), \\
\mathbf{r}_2 &= \left(d, 0, +\frac{a}{2}\right), \\
\therefore \quad r_1 = r_2 &= \sqrt{d^2 + \left(\frac{a}{2}\right)^2},
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0 \left[d^2 + \left(\frac{a}{2}\right)^2\right]^{3/2}} \left[\left(d, 0, -\frac{a}{2}\right) - \left(d, 0, +\frac{a}{2}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0 d^3} \left[1 + \left(\frac{a}{2d}\right)^2\right]^{-3/2} \times -a\hat{\mathbf{z}} \\
&\approx -\frac{qa}{4\pi\epsilon_0 d^3}\,\hat{\mathbf{z}}\,.
\end{aligned}$$

For the more general case of $(d\sin\theta, 0, d\cos\theta)$, the procedure is the same as above, just with slightly more complicated $\mathbf{r}_i$ vectors.

17. To calculate the current at a particular radius, we can use Ampère's law:

$$\begin{aligned}
\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} &= \mu_0 \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \\
&= \mu_0 I_{\text{enclosed}}\,,
\end{aligned}$$

where C is a closed curve and S is any surface bounded by C. Given the cylindrical symmetry of the cable in this problem, the magnetic field $\mathbf{B}$ must be purely azimuthal (i.e. it wraps around the central axis in circles), so the integral on the left hand side can be evaluated easily. Let's consider the different radial ranges separately. For $0 < r < a$:

$$\begin{aligned}
2\pi r B &= \mu_0 I \frac{\pi r^2}{\pi a^2} \\
\therefore \quad B &= \frac{\mu_0 I r}{2\pi a^2}\,,
\end{aligned}$$

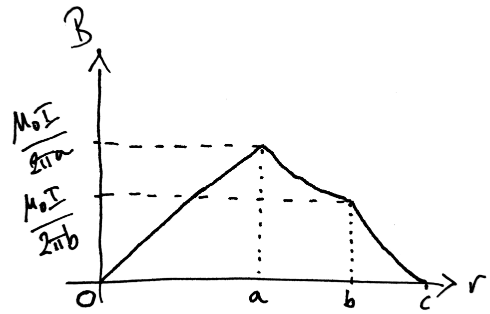
since there is a total current I in the inner conductor, which is uniformly distributed over area πa^2. Then, for $a < r < b$:

$$\begin{aligned}
2\pi r B &= \mu_0 I \\
\therefore \quad B &= \frac{\mu_0 I}{2\pi r}\,.
\end{aligned}$$

For $b < r < c$, make sure to take account of the fact that the current in the outer conductor is flowing in the opposite direction to that in the inner:

$$2\pi r B = \mu_0 \left[I - I \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)} \right]$$
$$\therefore \quad B = \frac{\mu_0 I}{2\pi r} \cdot \frac{c^2 - r^2}{c^2 - b^2}.$$

Finally, for $r > c$ the net current enclosed – and hence the field – is zero, because there are currents of I and $-I$ flowing through the inner and outer conductors. Thus, $B(r)$ looks as follows:

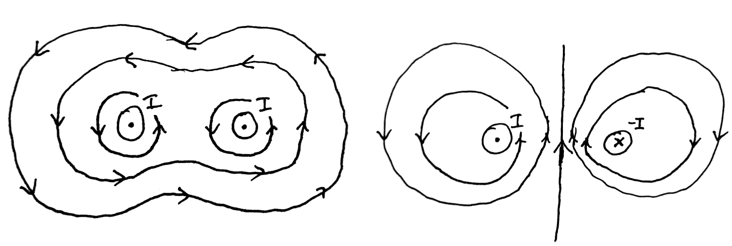


18. (a) From symmetry and Ampère's law, for a long straight wire we have

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}.$$

So, if $I = 1$ A and $r = 1$ m, then $B = 2 \times 10^{-7}$ T. The field takes a value of 5×10^{-5} T (similar to the Earth's field) at a distance of 4 mm. The field lines are simply circles wrapping around the wire, in a direction given by the right hand rule.

(b) For $I_1 = I_2 = +1$ A, or $I_1 = I_2 = -1$ A, $B = 0$ at the midpoint since the individual fields are equal and opposite. If the currents instead have opposite signs, then B is twice the field at 0.5 m due to a single wire, by superposition: $B = 2 \times \frac{\mu_0 \times 1\,\mathrm{A}}{2\pi \times 0.5\,\mathrm{m}} = 8 \times 10^{-7}$ T. The field lines either add destructively or constructively at the midpoint, so that the diagrams are as follows:



The force between the wires follows from $\mathbf{F} = I\,\mathbf{l} \times \mathbf{B}$. Here the $\mathbf{l}$ and $\mathbf{B}$ vectors are perpendicular, so the force per unit length is $F/l = BI = 2 \times 10^{-7}$ T $\times$ 1 A =

2×10^{-7} Nm^{-1}. Consideration of the direction of the cross product shows that the parallel currents attract, while the antiparallel currents repel.

Exam question: 2010 E14

- $\mathbf{E}$ is the electric field, d$\mathbf{S}$ is an element of vector surface area (i.e. normal to the integration surface, with a magnitude equal to the area of the element), Q is the charge enclosed by the integration surface, and ϵ_0 is the permittivity of free space.

  Gauss' law always applies, but is only useful for solving problems analytically when the charge distribution (and hence the electric field) has a symmetry (e.g. spherical or cylindrical) that enables the integral on the left hand side to be calculated. In such cases, we can choose a surface over which the electric field is constant in magnitude and normal to the surface, so that the integral evaluates simply to EA, where E is the field magnitude and A the area of the surface. Where this is possible, it is easier than summing up electric field contributions from charge elements (i.e. using superposition).

- We're asked to show that the capacitance of a conducting sphere of radius a is $4\pi\epsilon_0 a$. Conveniently, we've already done this in question 14(a) of the examples – see above!

- We're given that the energy density in the field is $\frac{1}{2}\epsilon_0 E^2$. Since $E = \frac{Q}{4\pi\epsilon_0 r^2}$ depends on the distance r from the centre of the sphere, so does the energy density. Thus, to find the total energy in the field we need to sum up the energies stored in each volume element in space. Given the spherical symmetry of the field, it will be most straightforward to split the space up into spherical shells of thickness dr. Denoting the energy stored in a particular shell as dU, we have

$$\begin{aligned} \mathrm{d}U &= \frac{1}{2}\epsilon_0 E^2 \cdot 4\pi r^2 \mathrm{d}r \\ &= \frac{Q^2}{8\pi\epsilon_0 r^2}\,\mathrm{d}r \\ \therefore \quad U &= \int_a^\infty \frac{Q^2}{8\pi\epsilon_0 r^2}\,\mathrm{d}r \\ &= \frac{Q^2}{8\pi\epsilon_0 a} \end{aligned}$$

- We can find the potential of the final (merged) droplet given its charge and its capacitance. To find its charge, note that the charge stored on each initial droplet is, from the capacitance of a sphere, $4\pi\epsilon_0 a V_1$. Thus, by conservation of charge, the final droplet must have charge $8\pi\epsilon_0 a V_1$. To find its capacitance, we need its radius R (see question 14a). This can be found from conservation of volume (we don't lose any mercury during the merging process):

$$\begin{aligned} \frac{4}{3}\pi R^3 &= 2 \times \frac{4}{3}\pi a^3 \\ \therefore \quad R &= 2^{1/3} a\,. \end{aligned}$$

  Thus, from the definition of capacitance, the final potential is

$$\begin{aligned} V_2 &= \frac{8\pi\epsilon_0 a V_1}{4\pi\epsilon_0 \cdot 2^{1/3} a} \\ &= 2^{2/3} V_1 \,. \end{aligned}$$

- The ratio of initial to final energies follows from the expression for U that we derived earlier.

$$\begin{aligned} \frac{U_\mathrm{i}}{U_\mathrm{f}} &= \frac{2 \times \frac{Q^2}{8\pi\epsilon_0 a}}{\left[\frac{(2Q)^2}{8\pi\epsilon_0 \cdot 2^{1/3} a}\right]} \\ &= 2^{-2/3} \,, \end{aligned}$$

as required. So, the energy in the electric field decreases – but energy is of course conserved, so where did the extra energy go? The answer is that work had to be done in order to bring the droplets together, since they both have the same charge.