Rotational Mechanics & Special Relativity: Week 4

Solutions by Ben Yelverton, April 2020

19. (a) In reference frame S,

$$E = 5 \text{ GeV}, \quad p = 3 \text{ GeV}/c.$$

E and p are given by γmc^2 and γmu respectively, so taking their ratio gives

$$p/E = u/c^2.$$

Thus, we get $u = 3c/5$ and hence $\gamma = (1 - u^2/c^2)^{-1/2} = 5/4$.

(b) Use for example $m = E/\gamma c^2$ to get 4 GeV$/c^2$.

(c) In the new frame of reference, S', we can repeat part (a) to get $u' = p'c^2/E' = c/\sqrt{2}$. Then we have u and u', and want to find the relative speed of the frames, v, so we need to use the speed addition formula:

$$u' = \frac{u - v}{1 - uv/c^2},$$

and rearrange for v. This gives $v = -0.186c$, i.e. frame S sees frame S' moving in the negative x direction with speed $0.186c$.

20. Need to provide a kinetic energy of $K = (\gamma - 1)mc^2$, where γ corresponds to a speed of $c/2$, i.e. $\gamma = 2/\sqrt{3}$. This gives 79 keV. Non-relativistically, simply use $K = \frac{1}{2}mv^2$ to get 64 keV, so that $K_{\text{relativistic}}/K_{\text{classical}} \approx 1.2$. Thus at a speed of $c/2$ the classical approximation is good for an order-of-magnitude estimate, but there's a non-negligible 20% difference. Relativistic effects should therefore be considered significant at $c/2$.

21. Using $K = (\gamma - 1)mc^2$:

$$\begin{aligned}
\text{LHS} &= K^2 + 2mKc^2 \\
&= \left[(\gamma - 1)^2 + 2(\gamma - 1)\right] m^2c^4 \\
&= (\gamma^2 - 1)m^2c^4 \\
&= \left(\frac{1}{1 - v^2/c^2} - 1\right) m^2c^4 \\
&= \left(\frac{v^2/c^2}{1 - v^2/c^2}\right) m^2c^4 \\
&= \gamma^2m^2v^2c^2.
\end{aligned}$$

This gives the required result since $p = \gamma m v$. For the next part, we can get a hint that we should use the E–p invariant, $\left(\sum E\right)^2 - \left(\sum p\right)^2 c^2 =$ constant, because there's a p^2c^2 term in the equation we just derived. So let's do that:

$$\left[(K + mc^2) + (2mc^2)\right]^2 - (K^2 + 2mKc^2) = (\sqrt{17}mc^2)^2.$$

Here I evaluated the LHS before the collision and in the lab frame, and the RHS after the collision and in the rest frame of the composite particle. Then we just have to expand the LHS; the K^2 terms cancel and we're left with $K = 2mc^2$.

22. We need to solve for K_1 and K_2, the kinetic energies of the two fragments, which means we need two independent equations containing these quantities. The obvious one is conservation of energy:

$$K_1 + K_2 = \left[M - (M/2 + M/4)\right] c^2 = Mc^2/4, \tag{1}$$

since the KE of the fragments must come from the liberated rest mass energy. We can also use conservation of momentum to get a condition on the K values, since the result from question 21 relates K directly to p. The total momentum is zero here, so that

$$\begin{aligned} p_1 &= p_2 \\ p_1^2c^2 &= p_2^2c^2 \\ K_1^2 + 2\frac{M}{2}K_1c^2 &= K_2^2 + 2\frac{M}{4}K_2c^2. \end{aligned}$$

Then substitute e.g. $K_2 = Mc^2/4 - K_1$ from equation (1), and solve for K_1 to get $3Mc^2/32$, and subsequently $K_2 = 5Mc^2/32$.

23. (a) The protons have *just enough* energy for the reaction to occur. This means that in the ZMF all of the product particles will be stationary, since there is no 'excess' energy available, only enough to produce the π^0 meson.

In the ZMF, the two protons approach each other with equal and opposite speeds v_{ZMF} (since one of them is stationary in the lab frame). The particles in the end state are stationary in the ZMF, which means that in the lab frame they all travel with speed $v = v_{\text{ZMF}}$ in the direction of motion of the originally moving proton.

Conserving energy in the ZMF gives

$$\begin{aligned} 2\gamma m_{\text{p}}c^2 &= (2m_{\text{p}} + m_\pi)c^2 \\ \gamma &= \frac{2m_{\text{p}} + m_\pi}{2m_{\text{p}}}. \end{aligned}$$

Given that $m_\pi = 0.144m_{\text{p}}$, we get $\gamma = 1.072$ and hence $v = 0.36c$.

(b) The two photons must travel at angles $\pm\theta$ to the π^0 direction, to conserve momentum perpendicular to that direction. Recalling that for a photon $E = pc$, conservation of momentum along the line of motion of the π^0 particle gives

$$\gamma m_\pi v = 2\frac{E}{c}\cos\theta,$$

while conservation of energy gives

$$\gamma m_\pi c^2 = 2E,$$

where E is the energy of one photon. Dividing these two conditions eliminates E and yields $\cos\theta = v/c$. The opening angle is simply $2\theta = 138°$.

(c) We have two 'events', the birth and death of the π^0 particle. Given the proper lifetime $\Delta t'$, we can use a Lorentz transformation to find that $\Delta t = \gamma\Delta t'$ in the lab frame, since $\Delta x' = 0$. The distance travelled in the lab frame is then $\gamma v\Delta t' =$ 9.7 nm.

24. The kinetic energy gained by a particle of charge q as it is accelerated through a voltage V is qV, so its total energy is $E = mc^2 + qV$. Determining the voltage (as requested by the question) is thus equivalent to determining the energy. We know the masses of the α particle and proton, and their momentum ratio is specified in the question, so we probably want to use an equation relating the energy, momentum and mass of a single particle. This is of course $E^2 - p^2c^2 = m^2c^4$. Using our expression for E in terms of V yields

$$q^2V^2 + 2qVmc^2 - p^2c^2 = 0 \tag{2}$$

in general. Now, for a proton $m = m_\mathrm{p}$ and $q = e$, while for an α particle $m = 4m_\mathrm{p}$ and $q = 2e$. Let's denote the momentum of the proton as p; the question then tells us that the α particle's momentum is $\sqrt{6}p$. Putting the values for each particle into equation (2) gives the simultaneous equations:

$$e^2V^2 + 2m_\mathrm{p}eVc^2 - p^2c^2 = 0, \tag{3}$$

$$4e^2V^2 + 16m_\mathrm{p}eVc^2 - 6p^2c^2 = 0. \tag{4}$$

To solve for V, we can eliminate p by taking $6\times(3) - (4)$, to get $V = 2m_\mathrm{p}c^2/e = 1.88$ GV. Note that there's also an uninteresting solution with $V = 0$ and $p = 0$.

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- $E = \gamma mc^2$ and $p = \gamma mu$, where $\gamma = (1 - u^2/c^2)^{-1/2}$. Using these definitions, we can evaluate

$$\begin{aligned}
E^2 - p^2c^2 &= \gamma^2 m^2 c^4 (1 - u^2/c^2) \\
&= \gamma^2 m^2 c^4 \gamma^{-2} \\
&= m^2 c^4,
\end{aligned}$$

giving $E^2 = p^2c^2 + m^2c^4$ as required. For photons ($m = 0$) this reduces to $E = pc$. Substituting $E = hf$ and $p = h/\lambda$ gives

$$\begin{aligned}
hf &= hc/\lambda \\
c &= f\lambda,
\end{aligned}$$

i.e. the well-known wavelength-frequency relationship for a photon is recovered. Since $c = f\lambda$ clearly holds for any photon, we find that $E = pc$ is consistent with the de Broglie relations.

- For the photon, $p = E/c = hf/c$, i.e. $p \propto f$, so that momenta transform in the same way as frequencies. From the given Doppler shift equation,

$$\begin{aligned}
p'_\gamma &= p_\gamma \sqrt{\frac{1 - v/c}{1 + v/c}} \\
&= \frac{E_0}{c} \sqrt{\frac{1 - v/c}{1 + v/c}}. \qquad (5)
\end{aligned}$$

The electron, which is stationary in the lab frame, acquires a velocity $-v$ in the moving frame. Thus its momentum is

$$p'_\mathrm{e} = \frac{-mv}{\sqrt{1 - v^2/c^2}}. \qquad (6)$$

In the ZMF, where $v = v_\mathrm{ZMF}$, $p'_\gamma + p'_\mathrm{e}$ must be zero. So, we use equations (5) and (6), along with the fact that $E_0 = mc^2$, to solve for v_ZMF, finding $v_\mathrm{ZMF} = c/2$.

- Clearly the electron must move in the opposite direction to the photon in the ZMF, to conserve momentum. To find its final speed w, let's first conserve energy:

$$E'_0 + \gamma_v mc^2 = E'_1 + \gamma_w mc^2, \qquad (7)$$

where E'_1 is the photon energy in the ZMF after the collision. Additionally, conserving momentum gives

$$\gamma_w mw = E'_1/c. \qquad (8)$$

Substituting (8) into (7) yields

$$E'_0 + \gamma_v mc^2 = \gamma_w mc(w + c).$$

The only unknown here is w, since we found previously that $v = c/2$ for this frame, implying that $\gamma_v = 2/\sqrt{3}$ and (from the Doppler shift equation) $E_0' = E_0/\sqrt{3}$. After some algebra, we find that $w = c/2$, i.e. the electron's velocity in the ZMF is $(u_x', u_y') = (0, c/2)$, where I defined the axes such that the photon is initially moving in the x direction and finally in the $-y$ direction. To transform this back to the lab frame, note that we need the inverse of the transformations given in the questions, which can be simply obtained by reversing the sign of v and swapping the primed and unprimed quantities, i.e.

$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2}, \quad u_y = \frac{u_y'}{\gamma(1 + u_x' v/c^2)}.$$

Putting in all the values gives $(u_x, u_y) = (c/2, \sqrt{3}c/4)$, i.e. a speed of $\sqrt{7}c/4$ at an angle of 41° to the x axis.

- To get the photon energy in the lab frame, we can't simply use the Doppler shift equation, since the photon is moving perpendicular to the frame itself. We can instead use an energy-momentum Lorentz transformation:

$$\begin{aligned} E &= \gamma(E' + vp_x') \\ \therefore E_1 &= \gamma_v(\gamma_w mwc + 0) \\ &= \frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}\frac{mc^2}{2}\right) \\ &= \frac{2}{3}mc^2, \end{aligned}$$

where equation (8) was used in the second line.