Rotational Mechanics & Special Relativity: Week 3

Solutions by Ben Yelverton, April 2020

13. This is one of the more difficult questions on the examples sheet, in my opinion, and is fairly typical of exam problems on relativistic kinematics. Such questions can usually be solved using Lorentz transformations; this is mathematically simple but usually requires some careful thought. The key to answering them successfully is to define and label the events clearly, translate the information given in the question into appropriate mathematical notation (e.g. Δt_{AB}, $\Delta x'_{\mathrm{CD}}$), then identify which quantity you need to solve for and in what frame. It should then become clear which Lorentz transformation you need to use.

(a) We have two events:

(A) The ship passes the Earth,

(B) The ship passes the navigation station.

Let's denote the Earth's frame by S and the ship's frame by S', as is conventional. The question tells us that $\Delta t'_{\mathrm{AB}} = \frac{1}{2}$ h. We also know the ship's velocity $v = 4c/5$ and hence $\gamma = [1-(4/5)^2]^{-1/2} = 5/3$, so we can simply use a Lorentz transformation to find that:

$$\begin{aligned}\Delta t_{\mathrm{AB}} &= \gamma(\Delta t'_{\mathrm{AB}} + v\Delta x'_{\mathrm{AB}}/c^2) \\ &= \frac{5}{3}\left(\frac{1}{2}\text{ h} + 0\right) \\ &= \frac{5}{6}\text{ h},\end{aligned}$$

i.e. the time on the navigation station clock has advanced by $\frac{5}{6}$ h to 12:50.

(b) The Earth–station distance in the Earth's frame is Δx_{AB}. Similarly to part (a), this can be calculated using a Lorentz transformation as follows:

$$\begin{aligned}\Delta x_{\mathrm{AB}} &= \gamma(\Delta x'_{\mathrm{AB}} + v\Delta t'_{\mathrm{AB}}) \\ &= \frac{5}{3}\left(0 + \frac{4c}{5} \times \frac{1}{2}\text{ h}\right) \\ &= \frac{2c}{3}\text{ h},\end{aligned}$$

i.e. $\frac{2}{3}$ light-hours, or 7.2×10^{11} m.

(c) Radio waves travel at speed c, and thus take a time of $\frac{2}{3}$ h to travel the required distance of $\frac{2c}{3}$ h. Thus, in the Earth's frame, the signal arrives $\frac{2}{3}$ h after 12:50, i.e. at 13:30.

(d) At event (B) as defined above, the ship sends its signal towards the Earth. There are two more events that are relevant here:

(C) The Earth receives the ship's signal and sends a reply,

(D) The ship receives the Earth's reply.

We want to find $t'_{\rm D}$, which can be expressed as $t'_{\rm A} + \Delta t'_{\rm AB} + \Delta t'_{\rm BC} + \Delta t'_{\rm CD}$. We know from the question that $t'_{\rm A}$ is 12:00, and that $\Delta t'_{\rm AB} = \frac{1}{2}$ h. So let's find $\Delta t'_{\rm BC}$ next, using a Lorentz transformation as usual. We know that $\Delta x_{\rm BC} = -\Delta x_{\rm AB} = -\frac{2c}{3}$ h, and from part (c), $\Delta t_{\rm BC} = \frac{2}{3}$ h. So:

$$\begin{aligned}\Delta t'_{\rm BC} &= \gamma(\Delta t_{\rm BC} - v\Delta x_{\rm BC}/c^2)\\ &= \frac{5}{3}\left(\frac{2}{3} - \frac{4}{5c} \times -\frac{2c}{3}\right)\ \text{h}\\ &= 2\ \text{h}.\end{aligned}$$

How about $\Delta t'_{\rm CD}$? To calculate this using a Lorentz transformation, we first need $\Delta t_{\rm CD}$ and $\Delta x_{\rm CD}$, i.e. when and where the Earth's signal reaches the ship according to the Earth. We can figure these out using the equations of motion of the ship and the signal in the Earth's frame. For the ship,

$$x = x_0 + \frac{4c}{5}(t - t_{\rm C}), \tag{1}$$

where x_0 is the position of the ship at the instant when the Earth sends its signal out, which must be given by

$$x_0 = \frac{2c}{3}\ \text{h} + \frac{4c}{5} \times \frac{2}{3}\ \text{h} = \frac{6c}{5}\ \text{h}, \tag{2}$$

because the ship was at $x = \frac{2c}{3}$ h when it sent its signal, and then travelled at speed $4c/5$ for $\frac{2}{3}$ h while the signal travelled towards the Earth. The Earth's signal simply obeys

$$x = c(t - t_{\rm C}). \tag{3}$$

Putting (2) into (1) and then equating (1) to (3) gives $\Delta t_{\rm CD} = 6$ h and hence $\Delta x_{\rm CD} = 6c$ h. Finally, we get

$$\begin{aligned}\Delta t'_{\rm CD} &= \gamma(\Delta t_{\rm CD} - v\Delta x_{\rm CD}/c^2)\\ &= \frac{5}{3}\left(6 - \frac{4}{5c} \times 6c\right)\ \text{h}\\ &= 2\ \text{h},\end{aligned}$$

so that

$$\begin{aligned}t'_{\rm D} &= t'_{\rm A} + \Delta t'_{\rm AB} + \Delta t'_{\rm BC} + \Delta t'_{\rm CD}\\ &= 12{:}00 + \frac{1}{2}\ \text{h} + 2\ \text{h} + 2\ \text{h}\\ &= 16{:}30.\end{aligned}$$

14. The relevant events here are:

(A) Flash emitted at the tail,

(B) Flash reflected at the nose,

(C) Flash received back at the tail.

Let the Earth's frame be S and the rocket's frame be S'. We want to find $\Delta t_{\rm AB}$ and then $\Delta t_{\rm BC}$. In frame S' the problem is very simple, there's just a pulse of light travelling a distance l_0 at speed c, first in the $+x$ direction and then in the $-x$ direction after reflection. We can thus immediately write down:

$$\begin{aligned}\Delta x'_{\rm AB} &= l_0, \\ \Delta x'_{\rm BC} &= -l_0, \\ \Delta t'_{\rm AB} &= \Delta t'_{\rm BC} = l_0/c.\end{aligned}$$

Then we use a Lorentz transformation as follows:

$$\begin{aligned}\Delta t_{\rm AB} &= \gamma(\Delta t'_{\rm AB} + v\Delta x'_{\rm AB}/c^2) \\ &= \gamma\left(\frac{l_0}{c} + \frac{vl_0}{c^2}\right) \\ &= \frac{l_0}{c}\frac{1+v/c}{\sqrt{1-v^2/c^2}} \\ &= \frac{l_0}{c}\sqrt{\frac{c+v}{c-v}}.\end{aligned}$$

The calculation for $\Delta t_{\rm BC}$ is almost identical; the only difference is that there is a minus instead of a plus in the second line above. This leads to the result

$$\Delta t_{\rm BC} = \frac{l_0}{c}\sqrt{\frac{c-v}{c+v}}.$$

15. (a) The train is Lorentz contracted in the station frame S, so that its length in that frame is L_0/γ. For this to be exactly equal to the platform's length we simply set $L_0/\gamma = L$, and solve for v, finding that $v = c\sqrt{1 - L^2/L_0^2}$.

(b) (i) According to an observer on the platform, the train was of length L when the dents were made. Once the train has come to a stop, however, it has expanded back to its proper length L_0, so that the dents are now a distance L_0 apart.

(ii) According to a passenger on the train, the dents must have always been a distance L_0 apart (since the train's length doesn't change), despite the fact that the porters are a distance L/γ apart. The explanation is that the kicks are not simultaneous. The kick at the front of the train occurs before the kick at the back. You can see this by drawing a diagram of the platform moving towards the train in the train's rest frame at several successive times – this should make it clear that the front end of the train coincides with the front end of the platform *before* the back end of the train coincides with the back end of the platform.

(c) From Lorentz contraction, $L' = L/\gamma$, but also from part (a) we know that $L = L_0/\gamma$. Thus, $L' = L_0/\gamma^2$.

16. To find the relative velocity, we can transform into the frame of reference of either particle. The answer we get by transforming into the other particle's frame would be equal in magnitude but opposite in direction. Let's transform into the frame S' of the particle moving at a speed of $v = 3c/5$. In the lab frame, the other particle has a velocity $u_x = -2c/5$, and we want to find u'_x. Thus we use the velocity transformation formula:

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \\ &= \frac{-\frac{2c}{5} - \frac{3c}{5}}{1 - \frac{1}{c^2}\left(\frac{-2c}{5}\right)\left(\frac{3c}{5}\right)} \\ &= -\frac{25}{31}c, \end{aligned}$$

i.e. the speed is $25c/31$. For the next part, consider a frame S'' moving with speed w. The velocity of the first particle in that frame is

$$\frac{\frac{3c}{5} - w}{1 - \frac{1}{c^2}\left(\frac{3c}{5}\right)w},$$

while that of the second particle is

$$\frac{-\frac{2c}{5} - w}{1 - \frac{1}{c^2}\left(\frac{-2c}{5}\right)w}.$$

We want to find the w that makes these two velocities equal and opposite, which means we have to solve

$$\frac{\frac{3c}{5} - w}{1 - \frac{1}{c^2}\left(\frac{3c}{5}\right)w} = -\left[\frac{-\frac{2c}{5} - w}{1 - \frac{1}{c^2}\left(\frac{-2c}{5}\right)w}\right].$$

Rearranging this gives the quadratic equation $5w^2 - 38cw + 5c^2 = 0$, which has solutions $w = 7.47c$ and $w = 0.134c$. As w must of course be below c, the second solution is the answer we need.

17. In frame S', the two photons must travel in exactly opposite directions to conserve momentum. So, photon 1 travels at an angle $\theta'_1 = 60°$ to the x' axis, while photon 2 travels at $\theta'_2 = -120°$. Since the speeds of both are c, we know the x' and y' components of both of their velocities:

$$\begin{aligned} u'_{x,1} &= c\cos\theta'_1 = c/2 \\ u'_{y,1} &= c\sin\theta'_1 = \sqrt{3}c/2 \\ u'_{x,2} &= c\cos\theta'_2 = -c/2 \\ u'_{y,2} &= c\sin\theta'_2 = -\sqrt{3}c/2. \end{aligned} \tag{4}$$

To transform these back to the lab frame we just have to use the standard velocity addition formulae:

$$(u_x, u_y) = \left(\frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}, \frac{u'_y}{\gamma[1 + \frac{u'_x v}{c^2}]}\right). \tag{5}$$

Putting the values from (4) into (5), with $v = 3c/4$ and hence $\gamma = 4/\sqrt{7}$, gives

$$\begin{aligned} (u_{x,1}, u_{y,1}) &= \left(\frac{10}{11}c, \frac{\sqrt{21}}{11}c\right), \\ (u_{x,2}, u_{y,2}) &= \left(\frac{2}{5}c, -\frac{\sqrt{21}}{5}c\right). \end{aligned}$$

The corresponding angles to the x axis are $\theta_1 = \tan^{-1}\left(\frac{\sqrt{21}}{10}\right) = 24.6°$ and $\theta_2 = \tan^{-1}\left(-\frac{\sqrt{21}}{2}\right) = -66.4°$.

18. (a) The question is asking for the standard derivation of the relativistic Doppler shift. Let's define events A and B to be the emission of two consecutive pulses by the rocket. We're given that $\Delta t'_{\mathrm{AB}} = 1/f_0$, and clearly it's also true that $\Delta x'_{\mathrm{AB}} = 0$. Lorentz transforming the time difference into the Earth's frame gives:

$$\begin{aligned} \Delta t_{\mathrm{AB}} &= \gamma(\Delta t'_{\mathrm{AB}} + v\Delta x'_{\mathrm{AB}}/c^2) \\ &= \gamma/f_0, \end{aligned} \tag{6}$$

and for the spatial coordinates:

$$\begin{aligned} \Delta x_{\mathrm{AB}} &= \gamma(\Delta x'_{\mathrm{AB}} + v\Delta t'_{\mathrm{AB}}) \\ &= \gamma v/f_0. \end{aligned} \tag{7}$$

The time difference between the *reception* of the two pulses on Earth, i.e. $1/f$, is equal to $\Delta t_{\mathrm{AB}} + \Delta x_{\mathrm{AB}}/c$, where the second term arises because the second pulse emitted has to travel a distance Δx_{AB} more than the first one. From equations (6) and (7) we therefore get

$$\begin{aligned} 1/f &= \gamma/f_0 + \frac{\gamma v}{cf_0} \\ &= \frac{\gamma}{f_0}\left(1+\frac{v}{c}\right) \\ &= \frac{1}{f_0}\frac{1+\frac{v}{c}}{\sqrt{1-\frac{v}{c}}} \\ &= \frac{1}{f_0}\sqrt{\frac{c+v}{c-v}} \\ \therefore \quad f &= f_0\sqrt{\frac{c-v}{c+v}}. \end{aligned} \tag{8}$$

(b) The pulses received on Earth directly have a frequency given by equation (8). The pulses that reach the distant planet have a frequency given by a similar expression, but with v replaced with $-v$, because the rocket is moving *towards* that planet. These pulses don't change their frequency upon reflection, and so when they reach the Earth they have a higher frequency than those that arrived directly. The frequency ratio is observed to be $1:2$, which means that

$$\sqrt{\frac{c+v}{c-v}} = 2\sqrt{\frac{c-v}{c+v}},$$

which can be solved to give $v = c/3$.

(c) The two relevant events here are:

- (E) The rocket leaves Earth,
- (P) The rocket arrives at the distant planet,

and we need to calculate Δx_{EP}. The Lorentz transformation tells us that $\Delta x_{\mathrm{EP}} = \gamma(\Delta x'_{\mathrm{EP}} + v\Delta t'_{\mathrm{EP}})$, and clearly $\Delta x'_{\mathrm{EP}} = 0$. We can write $\Delta t'_{\mathrm{EP}} = N/f_0$, where $N = 10^4$ is the number of pulses received directly, which must also be the same as the total number of pulses sent. Thus, we find $\Delta x_{\mathrm{EP}} = \gamma v N/f_0$. We know from part (b) that $v = c/3$, and from the question $f_0 = 1$ Hz; putting in the numbers gives 1.06×10^{12} m.